The Structure of Diagonally Constructed ASH Algebras

by

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Abstract

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We introduce a class of recursive subhomogeneous algebras which are constructed using a type of diagonal map similar to those previously defined for homogeneous algebras. We call these diagonal subhomogeneous (DSH) algebras. Using homomorphisms that also exhibit a kind of diagonal structure, we study certain limits of DSH algebras. Our first result is that a simple limit of DSH algebras with diagonal maps has stable rank one. As an application we show that whenever X is a compact Hausdorff space and σ is a minimal homeomorphism thereof, the crossed product algebra $C^*(\mathbb{Z}, X, \sigma)$ has stable rank one. We also define mean dimension in the context of these limits. Our second result is that mean dimension zero implies $\mathcal{Z}$-stability for simple separable limits of DSH algebras with diagonal maps. We also show that the tensor product of any two simple separable limit algebras of this kind is $\mathcal{Z}$-stable.

Contents

Introduction

In this work we investigate two properties, stable rank one and $\mathcal{Z}$-stability, that have played central roles in the classification of unital, simple, separable, nuclear C^*-algebras. The classification project was concluded in the case of $\mathcal{Z}$-stable algebras when in [9] and [5] the authors showed that such algebras are classified by their K-theory data.

A C^*-algebra is subhomogeneous if all of its irreducible representations are finite dimensional, and it is homogeneous if all the representations are of equal dimension. In this paper, we are interested in a certain subclass of limits of subhomogeneous algebras, and it is within this class that we seek to identify the presence of stable rank one and $\mathcal{Z}$-stability.

A C^*-algebra A is said to have stable rank one if the invertible elements of A are dense in A. This property was first introduced by Rieffel in [17] and has played a prominent role in the study of stably finite C^*-algebras. This has been of particular interest for limits of subhomogeneous algebras, where in [6] and [7] it was used to identify a classifiable family of algebras.

The Jiang-Su algebra was first introduced in [11] and is denoted by $\mathcal{Z}$. A C^*-algebra A is said to be $\mathcal{Z}$-stable if A is isomorphic to the tensor product $A \otimes \mathcal{Z}$. Such algebras are understood to be well-behaved from the perspective of classification, as their K-theory invariants have additional structure not seen in the general case. Additionally, it was shown by Rørdam in [19] that unital, simple, stably finite $\mathcal{Z}$-stable algebras have stable rank one. In light of classification, it is desirable to know when a C^*-algebra is $\mathcal{Z}$-stable.

Given maps $\sigma_1, \ldots, \sigma_k : Y \to X$, where X, Y are compact Hausdorff spaces, we may readily construct a homomorphism from $C(X, M_n)$ to $C(Y, M_{nk})$ by sending $f \in (X, M_n)$ to $\operatorname{diag}(f \circ \sigma_1, \ldots, f \circ \sigma_k)$, and we say this map is diagonal. Diagonal maps and their generalizations have played a prominent role in constructing examples of limits of homogeneous algebras, including the Goodearle construction in [10] and Villadsen's examples [21] and [22]. This latter result, which used highly non-diagonal maps, gave the first example of a simple approximately homogeneous (AH) algebra that did not have stable rank one. It was shown in [4] that any simple AH algebra constructed from diagonal maps has stable rank one. In [14], a notion of mean dimension was introduced for AH algebras constructed from diagonal maps and mean dimension zero was shown to imply Jiang-Su stability. Our present work uses the techniques developed in those papers and their antecedents and applies them to a certain class of approximately subhomogeneous (ASH) algebras. Maps between these algebras cannot be said to be diagonal in the above sense; we do not have any analogue of the single valued eigenvalue maps σ_i. However, by restricting our attention to a class of subhomogeneous algebras in which there is a rigid notion of a diagonal, we can readily define a class of maps which send each point in the spectrum of the range algebra to an ordered list of eigenvalues in the domain algebra. Thus we give a notion of diagonal maps, and it turns out these are enough to prove stable rank one and construct a dimension theory. The DSH class of algebras is inspired by the orbit breaking algebras introduced in [12] following from work in [16]. Using work in [1], we are able to show that dynamical cross products which come from a single minimal homeomorphism of a compact Hausdorff space have stable rank one.

This main body of this thesis is constructed as follows: in our first chapter, we develop the class of diagonal subhomogeneous (DSH) algebras and derive some elementary properties of these structures. We then pursue two independent avenues of investigation. In the second chapter, we show that appropriate limits of these algebras have stable rank one. In the third, we give a sufficient condition for such algebras to be $\mathcal{Z}$-stable.

Chapter 1

Diagonal Subhomogeneous Algebras

We define a well-behaved class of unital subhomogeneous algebras that behaves like a diagonal version of recursive subhomogeneous (RSH) algebras as defined in [15].

Definition 1.0.1. The class of *recursive subhomogeneous algebras* is the smallest one that satisfies the following conditions.

1. If X is a compact Hausdorff space and $n \in \mathbb{N}$ then $C(X, M_n)$ is RSH.

2. If A is RSH and X is a compact Hausdorff space with a closed subset Y and $\phi : A \to C(Y, M_n)$ is any unital homomorphism and $\rho : C(X, M_n) \to C(Y, M_n)$ is the restriction homomorphism then

$$A \oplus_{C(Y,M_n)} C(X, M_n) = \{(a, f) \in A \oplus C(X, M_n) : \phi(a) = \rho(f)\}$$

   is RSH.

It follows that we can write any RSH algebra A in terms of a finite composition sequence using spaces $X_1, \ldots, X_l$ that contain closed subspaces $Y_1, \ldots, Y_l$; integers $n_1, \ldots, n_l$;

and unital homomorphisms $\phi_1, \ldots, \phi_{l-1}$ using the above notation

$$A = \Big[\cdots [[C_1 \oplus_{C_2'} C_2] \oplus_{C_3'} C_3] \cdots \Big] \oplus_{C_l'} C_l$$

where $C_i = C(X_i, M_{n_i})$ and $C_i' = C(Y_i, M_{n_i})$. We refer to l as the length of the composition sequence.

Given a subalgebra $A \subseteq \bigoplus_{i=1}^{l} C(X_i, M_{n_i})$ and any $f \in A$, we will denote by f_i the summand of f in $C(x_i, M_{n_i})$.

Definition 1.0.2. Suppose we have compact Hausdorff spaces $X_1, \ldots, X_l$, closed subspaces $Y_i \subseteq X_i$, dimensions $n_1, \ldots, n_l$, C^*-algebras $A_i \subseteq \bigoplus_{j=1}^{i} C(X_j, M_{n_j})$ such that $A_1 = C(X_1, M_{n_1})$ and homomorphisms $\phi_i : A_i \to C(Y_{i+1}, M_{n_{i+1}})$ for $i \in \{1, \ldots, l-1\}$ such that for each $y \in Y_{i+1}$ there are corresponding points $x_1, \ldots, x_t$ where $x_j \in X_{i_j}$ such that $\phi_i(f)_{i+1}(y) = \operatorname{diag}(f_{i_1}(x_1), \ldots, f_{i_t}(x_t))$ and $A_{i+1} = A_i \oplus_{C(Y_{i+1}, M_{n_{i+1}})} C(X_{i+1}, M_{n_{i+1}})$, then we say A_l is *diagonal subhomogeneous* (DSH).

Note that we can always assume that when $x \in Y_i$, each of the corresponding points $x_1, \ldots, x_t$ lies in $X_{i'} \setminus Y_{i'}$ for some $i' < i$. We can do this by taking the list given by diagonality $x_1', \ldots x_t'$, and whenever $x_i \in Y_{i'}$, we can replace it with the sequence of points for x_i in the construction of $\phi_{i'-1}$. Iterating, we eventually obtain a list as desired. Furthermore, we can check in the definition that if Y_i has a non-empty interior, we may delete $\operatorname{int}(Y_i)$ without changing the algebra. That is, if $X_{i+1}' = X_{i+1} \setminus \operatorname{int}(Y_i)$ and $Y_i' = Y \setminus \operatorname{int}(Y_i)$, then

$$\{(a, f) \in A \oplus C(X_{i+1}', M_{n_{i+1}}) : \forall j, y \in Y_i', f_{i+1}(y) = \operatorname{diag}(a_{i_1}(x_{y_1}), \ldots, a_{i_s}(x_{y_s}))\}$$

is isomorphic to the algebra constructed in the definition. Therefore we may assume each set Y_i has an empty interior.

We will show that the orbit breaking algebras of [12] are examples of DSH algebras.

In the DSH construction, for each space $i \in \{1, \ldots, l\}$ and each $k \in 1, \ldots, n_i$, we may define $B_{i,k}$ to be the subset of X_i such that a new representation in the diagonal decomposition begins at line k. Every point in X_i lies in $B_{i,1}$. For $k > 1$, $B_{i,k} \subseteq Y_i$.

When $y \in Y_i$ with corresponding points $x_1, \ldots, x_s$ in $X_{i_1}, \ldots, X_{i_s}$, $y \in B_{i,k}$ for $k = n_{i_1}+1, n_{i_1}+n_{i_2}+1$ and so on. Note that n_1, the smallest dimension of any representation of A, also gives a restriction on the sets $B_{i,k}$ in which a given point can appear. If $y \in B_{i,k}$, then $y \notin B_{i,k+1}, \ldots, B_{i,k+n_1-1}$.

If C is an $n \times n$ matrix, let us say C has a block point at position k if $C_{i,j} = 0$ whenever either $i \geq k$ and $j < k$ or $i < k$ and $j \geq k$.

Lemma 1.0.3. *Suppose that A is a DSH algebra. For each i, k, $x \in X_i$ lies in $B_{i,k}$ if and only if for every $f \in A$, $f_i(x)$ has a block point at position k. Every such set $B_{i,k}$ is closed.*

Proof. It is clear that if $x \in B_{i,k}$, then every $f_i(x)$ has a block point at position k for every f. Suppose $x \in X_i$ does not lie in $B_{i,k}$. If $x \notin Y_i$, then because Y_i is closed, we may easily find a function $f \in A$ for which $f_i(x)$ does not have a block point at k. Simply let $f_{i'}$ be 0 on $X_{i'}$ for all $i' < i$ and choose an appropriate function supported on $X_i \setminus Y_i$. Otherwise $x \in Y_i$ where we assume the that k lies in the middle of some block beginning at $k' < k$, so there exists a point $x' \in X_{i'}$ such that for every $f \in A$, $f_i(x)$ has a block point at position $k - k'$, where $x' \notin B_{i',k-k'}$. We may therefore always reduce this to the case that $x \notin Y_i$. Since the set of points $x \in X_i$ at which any $f_i(x)$ has blockpoints at position k is evidently closed, it follows that $B_{i,k}$ is closed. □

We highlight two ways of constructing new DSH algebras from existing ones.

Example 1.0.4. Suppose that A is a DSH algebra and $\psi : A \to B$ is a surjective homomorphism. We define $\widehat{\psi}$ to be the injective, single-valued map from $\widehat{B} \to \widehat{A}$, the spectra of irriducible representations of B and A. For each i, we have that subset of $\widehat{A}$ of representations of dimension n_i is homeomorphic to $X_i \setminus Y_i$. Then for $i = 1, \ldots, l$, define X_i' to be the closure of $X_i \bigcap \widehat{\psi}(\widehat{B})$, and Y_i' to be $X_i' \bigcap Y_i$. Since $\widehat{B}$ is compact, it follows that $\widehat{\psi}(\widehat{B})$

is closed, and therefore if $y \in Y_i'$, each corresponding point $x_1, \ldots, x_t$ in the diagonal decomposition of ϕ_{i-1} lies in some X_j' for $j < i$. Define $A(1)'$ to be $C(X_1', M_{n_1})$, and define a diagonal homomorphism $\phi_1' : A(1)' \to C(Y_2', M_{n_2})$ using the same diagonal decomposition as ϕ_1 on Y_i restricted to Y_i'. Define $A(2)'$ to be $A(1)' \oplus_{C(Y_2', M_{n_2})} C(X_2', M_{n_2})$. Similarly, for $i = 2, \ldots, l-1$, we define $\phi_i' : A(i)' \to C(Y_{i+1}', M_{n_{i+1}})$ via the restriction of the diagonal decomposition of ϕ_i to Y_{i+1}' and $A(i+1)'$ as $A(i)' \oplus_{C(Y_{i+1}', M_{n_{i+1}})} C(X_{i+1}', M_{n_{i+1}})$.

Lemma 1.0.5. *Every quotient of a DSH algebra is DSH.*

Proof. We prove that the DSH algebra $A(l)'$ is isomorphic to B. For each $\pi \in \widehat{B}$, we choose a unique representation $R[\pi]$ from the equivalence class π such that for every $f \in B$, $R[\pi](f) = \tilde{f}_i(\widehat{\psi}(\pi))$ where $\tilde{f}$ is any preimage of f and $\widehat{\psi}(\pi) \in X_i$. Then for each element of $f \in B$ we can define an element $\theta(f) \in A(l)'$ by $\theta(f)_i(x) = R[\widehat{\psi}^{-1}(x)](f)$. We can then check that this is a homomorphism and a bijection. □

Example 1.0.6. Suppose we have a DSH algebra A with spaces $X_1, \ldots, X_l$, compact Hausdorff spaces $\Delta_1, \ldots, \Delta_l$ and a continuous surjective map σ on $\bigsqcup_i X_i$ composed of maps $\sigma_i : X_i \to \Delta_i$ subject to the condition that if $y_1, y_2 \in Y_i$ and $\sigma_i(y_1) = \sigma_i(y_2)$ then the sequences of corresponding points $x_{y_1,1}, \ldots, x_{y_1,s}$ and $x_{y_2,1}, \ldots, x_{y_2,s}$ are the same length and for $j = 1, \ldots, s$ we have that $x_{y_1,j}$ and $x_{y_2,s}$ lie in the same space $X_{i'}$ and $\sigma_{i'}(y_1) = \sigma_{i'}(y_2)$. We may then define a DSH algebra B on the spaces $\Delta_1, \ldots, \Delta_l$, with restriction spaces $\sigma_1(Y_1), \ldots, \sigma_l(Y_l)$, the same dimensions $n_1, \ldots, n_l$ of A and diagonal homomorphisms ϕ_i' such that if the points corresponding to y are $x_1, \ldots, x_t$ then the points corresponding to $\sigma(y)$ are $\sigma(x_1), \ldots, \sigma(x_t)$. Moreover we can identify B with a subalgebra of A by sending $f \in B$ to the element $\tilde{f}$ of A such that $\tilde{f}_i(x) = f_i(\sigma_i(x))$

where $x \in X_i$.

In the DSH construction, each point $y \in Y_i$ has a sequence of corresponding points $x_{y,1}, \ldots, x_{y,s}$. We will call the set of points in the sequence $\Phi(y)$, and we write $\Phi^{-1}(x) = \{y \in \bigcup_i X_i : x \in \Phi(y)\}$. Moreover, if $x_{y,j} \in X_{i_j}$ then the representation corresponding to $x_{y,j}$ begins at the line $1 + \sum_{k=1}^{j-1} n_{i_j}$. The set of points in Y_i that have representations beginning at line k is exactly $B_{i,k}$. Let K_y be the sequence of indices for which $y \in B_{i,k}$ and write this as $(k_1, \ldots, k_s)$. Denote by $b(y,k)$ the greatest element in K_y which is $\leq k$. Then for $y \in Y_i$ and $1 \leq k \leq n_i$ we define $\Phi_k(y)$ to be the point $x_{y,a}$, where $k_a = b(y,k)$. We extend this to X_i by saying that if $x \in X_i \setminus Y_i$ then $\Phi_k(x) = x$ for every k. We write $\tilde{\Phi}(y) = (\Phi_1(y), \ldots, \Phi_{n_i}(y))$.

Observe that, like in the homogeneous case, we may conclude that each X_i is a metric space under the assumption that A is separable. We do this under the assumption that for distinct $y_1, y_2 \in Y_i$, $\tilde{\Phi}(y_1) \neq \tilde{\Phi}(y_2)$. Therefore there is always element $f \in A$ such that $f_i(y_1) \neq f_i(y_2)$. Let $a(1), a(2), \ldots$ be a dense sequence in the unit ball of A. We obtain a sequence of pseudometrics d_n given by $d_n(x,y) = \|a(n)_i(x) - a(n)_i(y)\|$ and define a metric d on X_i by $d(x,y) = \sum_{n=1}^{\infty} 2^{-n} d_n(x,y)$.

Chapter 2

Stable Rank

In this chapter, devoted to study of stable rank, we first recall the families of unitary paths between permutation matrices which were used in [4]. In section 2, we show how to construct unitary elements within DSH algebras using the matrices of section 1. We then detail in section 3 how the assumption of simplicity allows us to ensure that any non-invertible element in a limit of such algebras is eventually sent to a function very near to one which has a non-invertible image in any representation. We are able use the unitaries we have constructed to find a conjugate element with a certain regular structure. In section 4 we prove our main theorem, that simple diagonal limits of diagonal subhomogenous algebras have stable rank one. We do this by considering any non-invertible element of such an algebra. We can approximate this by an element of a sequence algebra. Using the results of the previous sections we can then manually construct a nearby element which is unitarily equivalent to a block diagonal element with many zeroes down its diagonal. After applying more unitaries, we obtain an element which is nilpotent and therefore arbitrarily close to an invertible element. This proves that the algebra has stable rank one. In section 5 we show that the inclusion map between nested orbit breaking algebras is diagonal. It then follows from [1] and our main theorem that dynamical cross products coming from minimal homeomorphisms on compact spaces have stable rank one.

2.1 Permutation unitaries

As in [4], given a permutation $\sigma \in S_n$, denote by $U[\sigma]$ the permutation matrix corresponding to σ. When σ is a transposition, let $u_\sigma : [0,1] \to \mathcal{U}(M_n)$ be a continuous map such that

1. $u_\sigma(0) = 1_n$,
2. $u_\sigma(1) = U[\sigma]$,
3. if either i or j is fixed by σ, then $u_\sigma(x)_{i,j} = \delta_{i,j}$ for all x,

where $\delta_{i,j}$ is the Kronecker delta function.

We may choose u_σ such that for every t, if $B = u_\sigma(t)Au_\sigma(t)^*$, then $B_{i,j}$ is a linear combination of the entries $A_{i,j}, A_{\sigma(i),j}, A_{i,\sigma(j)}$, and $A_{\sigma(i),\sigma(j)}$.

Indeed, we can choose functions $g_1, g_2, g_3, g_4 : [0,1] \to \mathbb{C}$ such that for any $k_1 < k_2$, $u_{(k_1\ k_2)}(t)_{i,j} = \delta_{i,j}$ for all t if either i or j is not in $\{k_1, k_2\}$ and $u_{(k_1\ k_2)}(t)_{k_1,k_1} = g_1(t)$,$u_{(k_1\ k_2)}(t)_{k_1,k_2} = g_2(t)$,$u_{(k_1\ k_2)}(t)_{k_2,k_1} = g_3(t)$,$u_{(k_1\ k_2)}(t)_{k_2,k_2} = g_4(t)$. By defining u in this way we have that whenever $k_3 > k_2 > k_1$, we have

$$U[(k_2\ k_3)]u_{(k_1\ k_2)}(t)U[(k_2\ k_3)] = u_{(k_1\ k_3)}(t). \tag{2.1}$$

Definition 2.1.1. We say an $n \times n$ matrix A has a *zero cross* at position k if $A_{k,j} = 0$ and $A_{i,k} = 0$ for all i, j.

Lemma 2.1.2. *Suppose that A is an $n \times n$ matrix and zero crosses at distinct positions $z_1, \ldots, z_m$. Let $t_1, \ldots, t_m \in [0,1]$.*

For some $k \in \{1, \ldots, n\} \setminus \{z_1, \ldots, z_m\}$, consider the matrix

$$B = u_{(k\ z_m)}(t_m) \cdots u_{(k\ z_1)}(t_1) A u_{(k\ z_1)}(t_1)^* \cdots u_{(k\ z_m)}(t_m)^*.$$

1. *If i and j are not in $\{k, z_1, \ldots, z_m\}$ then $B_{i,j} = A_{i,j}$.*

2. *If i is in $\{k, z_1, \ldots, z_m\}$ and j is not then $B_{i,j}$ is non-zero only if $A_{k,j}$ is non-zero.*

3. *If j is in $\{k, z_1, \ldots, z_m\}$ and i is not then $B_{i,j}$ is non-zero only if $A_{i,k}$ is non-zero.*

4. *If any of $t_1, \ldots, t_m = 1$ then B has a zero cross at position k.*

Proof. We do this by induction on m. Consider $u = u_{(k\ z)}(t)$ where A has a zero cross at position z. If i and j are not in $\{k, z\}$, then for $B = uAu^*$,

$$\begin{aligned} B_{i,j} &= \sum_{x=1}^{n}\sum_{y=1}^{n} u_{i,x}A_{x,y}u^*_{y,j} \\ &= \sum_{x=1}^{n}\sum_{y=1}^{n} \delta_{i,x}A_{x,y}\delta_{y,j} \\ &= A_{i,j}. \end{aligned}$$

If $i \in \{k, m\}$ and j is not, then

$$\begin{aligned} B_{i,j} &= \sum_{x=1}^{n}\sum_{y=1}^{n} u_{i,x}A_{x,y}u^*_{y,j} \\ &= u_{i,k}A_{k,j} + u_{i,z}A_{z,j} \\ &= u_{i,k}A_{k,j} \end{aligned}$$

since $A_{z,j}$ is zero. This establishes (2). The case wherein $j \in \{k, m\}$ and i is not is similar. Finally, when $t = 1$, $B_{i,j} = A_{\sigma(i),\sigma(j)}$. Hence we have $B_{i,j} = 0$ whenever i or j is equal to k.

Suppose all four claims hold for a given m. Suppose A has a zero cross at position z_{m+1}. We first show that $B = u_{(k\ z_m)}(t_m) \cdots u_{(k\ z_1)}(t_1) A u_{(k\ z_1)}(t_1)^* \cdots u_{(k\ z_m)}(t_m)^*$ also has a zero cross at position z_{m+1}. By (1), $B_{i,z_{m+1}} = A_{i,z_{m+1}} = 0$ whenever $i \notin \{k, z_1, \ldots, z_m\}$. By (2), when $i \in \{k, z_1, \ldots, z_m\}$, $B_{i,z_{m+1}} = 0$ because $A_{k,z_{m+1}} = 0$. Hence $B_{i,z_{m+1}} = 0$ for all i and similarly $B_{z_{m+1},j} = 0$. Consider $B' = u_{(k\ z_m)}(t_m) B u_{(k\ z_m)}(t_m)^*$. If $i, j \notin \{k, z_1, \ldots, z_{m+1}\}$ then, as above, $B'_{i,j} = B_{i,j}$. By assumption this is the same as $A_{i,j}$. Suppose i is in $\{k, z_1, \ldots, z_{m+1}\}$ and j is not. If $i \in \{k, z_{m+1}\}$, then $B'_{i,j}$ is non-zero only if $B_{i,k}$ is non-zero. If i is in $\{z_1, \ldots, z_m\}$, then $B'_{i,j} = B_{i,j}$. In either case, $B'_{i,j}$ is non-zero only if $A_{i,k}$ is non-zero. Claim (3) is similar.

Suppose at least one value in $\{t_1, \ldots, t_{m+1}\}$ is one. It is immediate that if $t_{m+1} = 1$ then B' has a zero cross in position k. If any of $t_1, \ldots t_m$ is 1, then by assumption B has a zero cross at position k as well as at z_{m+1}. It follows that B' has a zero cross in position k. □

Lemma 2.1.3. *Suppose we have an $n \times n$ matrix A and $\Delta \in [0,1]^n$ such that $\Delta_i > 0$ only if A has a zero cross at position i. Suppose that for some k, $M \leq n - k + 1$, we have that $\Delta_i = 1$ for some $i \in \{k, \ldots, k + M - 1\}$. Then for*

$$V = u_{(k\ k+1)}(\Delta_{k+1}) \cdots u_{(k\ k+M-1)}(\Delta_{k+M-1}),$$

VFV^* has a zero cross at position k.

Proof. If $\Delta_j = 0$ then $u_{(k\ j)}(\Delta_j)$ is the identity, thus we may rewrite

$$V = u_{(k\ k+1)}(\Delta_{k+1}) \cdots u_{(k\ k+M-1)}(\Delta_{k+M-1})$$

$$= u_{(k\ z_1)}(\Delta_{z_1}) \cdots u_{(k\ z_m)}(\Delta_{z_m})$$

where $\Delta_{z_j} > 0$ for each j and thus A has a zero cross in position z_j. If $\Delta_i = 0$ for all $i > k$ then this product is vacuous. If this is the case then $\Delta_k = 1$ and $A = VAV^*$ already has a zero cross in position k. Otherwise, apply Lemma 2.1.2 (4) to conclude that VAV^* has a zero cross in position k. □

If $A_{i,j} = 0$ whenever $|i - j| \geq r$, let us say A has diagonal radius r. Denote the smallest integer such that this holds as $r(A)$.

Lemma 2.1.4. *Suppose $\Delta_i = 1$ for some $i \in \{k, \ldots, k + M - 1\}$ for $k = k_1, \ldots, k_N$ where $k_{j+1} - k_j \geq M$ and $k_N \leq n - M + 1$. Set*

$$V_i = u_{(k_i\ k_i+1)}(\Delta_{k_i+1}) \cdots u_{(k_i\ k_i+M-1)}(\Delta_{k_i+M-1}).$$

For $j = 0, \ldots, N$, set $A(j) = V_j \cdots V_1 A V_1^* \cdots V_j^*$. Then $A(N)$ has zero crosses at $k_1, \ldots, k_N$ and $r(A(N)) \leq r(A) + M - 1$.

Proof. We now show that $A(N)$ has zero crosses at $k_1, \ldots, k_N$. First note that if B has a zero cross at position z, and $z \notin \{k_i, \ldots, k_i + M - 1\}$, then $V_i B V_i^*$ also has a zero cross at position z. This follows from Lemma 2.1.2 (1),(2), and (3). For each j, define the set $K_j = \{k_j, \ldots, k_j + M - 1\}$ and observe that since $k_{j+1} - k_j \geq M$, these sets are disjoint.

Therefore, since A has a zero cross at position i for some $i \in K_j$, this is also true of $V_{j-1} \cdots V_1 F V_1^* \cdots V_{j-1}^*$. Therefore we may apply the previous lemma and conclude that $V_j \cdots V_1 F V_1^* \cdots V_j^*$ has zero crosses at positions $k_1, \ldots, k_j$. Hence $A(N)$ has the desired property.

To bound the diagonal radius of $A(N)$, for each $s, t \in \{1, \ldots, n\}$ consider $A(j)_{s,t}$. If neither s nor t lies in K_{j+1}, then $A(j+1)_{s,t} = A(j)_{s,t}$. Since each of s and t lies in at most one set K_j, the sequence $A(1)_{s,t}, \ldots,$ $A(N)_{s,t}$ takes on at most three values. We will show that if $|s-t| \geq r(A) + M$ then $A(N)_{s,t} = 0$. Since$|s-t| \geq r(A) + M \geq M$, s and t cannot lie in the same set K_j. Suppose that $s \in K_{j_s}$ and $t \in K_{j_t}$. $A(j_s)_{s,t}$ only differs from $A(j_{s-1})_{s,t}$ if $A(j_{s-1})$ has a zero cross at position s or if $s = k_{j_s}$. In either case, $A(j_s)_{s,t}$ is non-zero only if $A(j_{s-1})_{k_{j_s},t}$ is also non-zero. It follows, then, that if $A_{s,t}, A_{k_s,t}, A_{s,k_t}$ and A_{k_s,k_t} are all zero then $A(N)_{s,t}$ is also zero. If s and t differ by at least $r + M - 1$ then each of these co-ordinate pairs differs by at least r, and hence all four terms are zero. The cases in which either s or t or both are not in any set K_j are similar. □

Lemma 2.1.5. *Suppose $A \in M_n$ has zero crosses at positions $z_1, \ldots, z_m$. Then there exists a unitary $V \in C([0,1], M_n)$ such that $VAV^*(1)$ has zero crosses at positions $1, \ldots, m$, $V(0)$ is the identity and for all $\theta \in [0,1]$, $r(VAV^*(\theta)) \leq r(A) + 2$.*

Proof. For $0 < i \leq j$, let $\delta_j^i : [0,1] \to [0,1]$ be a continuous increasing function with $\delta_j^i(\theta) = 0$ for $\theta \leq \frac{i-1}{j}$ and $\delta_j^i(\theta) = 1$ for $\theta \geq \frac{i}{j}$.

Construct $u_j^i \in C([0,1], M_n)$ by

$$u_j^i(\theta) = u_{(i\; i+1)} \circ \delta_{j-i}^{j-i}(\theta) \cdots u_{(j-1\; j)} \circ \delta_{j-i}^{1}(\theta).$$

For any θ, there exists at most one value k such that $\delta_{j-i}^k(\theta) \in (0,1)$. Hence

$$\begin{aligned} u_j^i(\theta) &= u_{(j-k\; j-k-1)}(\delta_{j-i}^k(\theta)) u_{(j-k+1\; j-k)}(1) \cdots u_{(j\; j-1)}(1) \\ &= u_{(j-k\; j-k-1)}(\delta_{j-i}^k(\theta)) U[(j-k\; j-k+1 \cdots j)]. \end{aligned}$$

Let us show that for any $A \in M_n$ with a zero cross at position j, $r(u_j^i(1) A u_j^i(1)^*) \leq r(A) + 1$, and when $i = 1$, $r(u_j^1(1) A u_j^1(1)^*) \leq r(A)$.

Consider $B = u_j^1(1) A u_j^1(1)^* = U[(1\; 2 \cdots j)] A U[(1\; 2 \cdots j)]^*$. If $A_{s,t}$ is non-zero then neither s nor t is equal to j, and $A_{s,t}$ sits in one of four regions, each of which is either fixed (if $s, t > j$), shifted parallel to the diagonal (if $s, t < j$), or shifted toward the diagonal (if $s < j < t$ or $t < j < s$). No non-zero entry in A is shifted away from the diagonal; hence $r(B) \leq r(A)$.

The case where $i > 1$ is similar except that if $i \leq s < j$ and $t < i$ or $i \leq t < j$ and $s < i$ then this possible non-zero entry is shifted by one away from the diagonal. Hence $r(u_j^i(1) A u_j^i(1)^*) \leq r(A) + 1$.

For a fixed k we let $B = U[(j-k\; j-k+1 \cdots j)] A U[(j-k\; j-k+1 \cdots j)]^*$ and we have just shown that $r(B) \leq r(A) + 1$. We can rewrite $u^i_j(\theta)Au^i_j(\theta)^*$ as

$$u_{(j-k\ j-k-1)}(\delta^k_{j-i}(\theta))Bu_{(j-k\ j-k-1)}(\delta^k_{j-i}(\theta))^*$$

Let $C = u_{(j-k\ j-k-1)}(\delta^k_{j-i}(\theta))Bu_{(j-k\ j-k-1)}(\delta^k_{j-i}(\theta))$. It then follows from Lemma 2.1.4 with $N = 1$ and $M = 2$ that $r(C) \leq r(B) + 1$. Hence $r(u^i_j(\theta)Au^{i*}_j(\theta)) \leq r(A) + 2$.

Suppose then that we have $A \in M_n$ with zero crosses at positions $z_1, \ldots, z_m$. Set $V = u^1_{z_1} \circ \delta^1_m \cdots u^1_{z_m} \circ \delta^m_m$. For each t, there exists a $k \in \{1, \ldots, m\}$ such that

$$V(\theta) = u^1_{z_1}(1) \cdots u^1_{z_{k-1}}(1)u^1_{z_k}(\delta^k_m(\theta)).$$

Then

$$r(u^1_{z_{k-1}}(1) \cdots u^1_{z_1}(1)Au^1_{z_1}(1)^* \cdots u^1_{z_{k-1}}(1)^*) = r(A)$$

and after conjugating by $u^1_{z_k}(\delta^k_m(\theta))$ and applying the above estimate, we have

$$r(VAV^*(\theta)) \leq r(A) + 2.$$

□

Fix $N < n$. For any $k \leq n$, denote by $\gamma_{k,n}$ the cycle $(k\ k{+}1 \cdots n)$. For any k such that $N \leq k \leq n$, denote by $\eta_{k,n}$ the product of transpositions $(k - N + 1\ n - N + 1) \cdots (k\ n)$, which exchanges the N entries up to and including k with the final N entries in $\{1, \ldots, n\}$. We will use $u_{\eta_{k,n}}(\theta)$ to denote $u_{(k-N+1\ n-N+1)}(\theta) \cdots u_{(k\ n)}(\theta)$. Note that whenever $k \leq n - N$ these terms all commute and if $\theta = 0$ then they are all trivial.

Lemma 2.1.6. *If $k \leq i - N$ and $i \leq n - N$ then*

$$u_{\eta_{n,k}}(\theta) = U[\eta_{i,n}]u_{\eta_{i,k}}(\theta)U[\eta_{i,n}]. \tag{2.2}$$

Proof. By definition

$$U[\eta_{i,n}]u_{\eta_{i,k}}(\theta) = (\prod_{j=-(N-1)}^{0} U[(i+j\ n+j)])(\prod_{j=-(N-1)}^{0} u_{\eta_{k+j\ i+j}}(\theta)).$$

Since $i \leq n$, $i+j$ cannot equal $n+j'$ unless $j' \leq j$. Since $k \leq i-N$, $k+j$ is never equal to $i+j'$ or $n+j'$ for any $j' \in \{-(N-1), \ldots, 0\}$. It follows that $u_{\eta_{(k+j\ i+j)}}$ commutes with every $U[(i+j'\ n+j')]$ such that $j' > j$. Hence

$$U[\eta_{i,n}]u_{\eta_{i,k}}(\theta) = \prod_{j=-(N-1)}^{0} U[(i+j\ n+j)]u_{\eta_{k+j\ i+j}}(\theta)$$

Since $i \leq n-N$, all the terms $U[(i+j\ n+j)]$ commute with one another. We therefore have

$$U[\eta_{i,n}]u_{\eta_{i,k}}(\theta)U[\eta_{i,n}] = \prod_{j=-(N-1)}^{0} U[(i+j\ n+j)]u_{\eta_{k+j\ i+j}}(\theta)U[(i+j\ n+j)].$$

Then by Eq.(2.1) we have

$$U[\eta_{i,n}]u_{\eta_{i,k}}(\theta)U[\eta_{i,n}] = \prod_{j=-(N-1)}^{0} u_{\eta_{k+j\ n+j}}(\theta) = u_{\eta_{n,k}}(\theta).$$

□

The following calculation is elementary.

Lemma 2.1.7. *When $i \leq n - N$, we have*

$$U[\gamma_{1,n}]^N U[\eta_{i-1,n}] = U[\gamma_{1,i-1}]^N U[\gamma_{i,n}]^N.$$

We can now construct the final class of unitaries we require.

Lemma 2.1.8. *For some n, let $\Theta \in [0,1]^{(n)}$ and suppose the first entry is 1, the final N entries are all zero and at most one of any consecutive N entries is non-zero. We will write Θ_k to denote the kth entry of Θ. Consider the unitary*

$$V_n(\Theta) = U[\gamma_{1,n}]^N \Big(\prod_{k=N}^{n-1} u_{\eta_{k,n}}(\Theta_{k+1}) \Big).$$

If $\Theta_k = 1$ for $k \in K = \{k_1, \ldots, k_m\}$, where $k_1 = 1$ and let $k_{m+1} = n + 1$, then

$$V_n(\Theta) = \mathrm{diag}(V_{k_2-k_1}(\Theta_{k_1}, \ldots, \Theta_{k_2-1}), \ldots, V_{k_{m+1}-k_m}(\Theta_{k_m}, \ldots, \Theta k_{m+1} - 1)). \tag{2.3}$$

Proof. If $\Theta_i = 1$ then $i \leq n - N$. Note that for $k \in 1, \ldots, i-2$, $u_{\eta_{k,n}}(\Theta_{k+1}) = U[\eta_{i,n}] u_{\eta_{k,i}}(\Theta_{k+1}) U[\eta_{i,n}]$. This holds by Eq. (2.2) for $k \leq i - N$, and holds because $\Theta_{k+1} = 0$ for $i - N \leq k < i - 1$. We can then compute that

$$\prod_{k=N}^{i-2} u_{\eta_{k,n}}(\Theta_{k+1}) = \prod_{k=N}^{i-2} U[\eta_{i-1,n}] u_{\eta_{i-1,k}}(\Theta_{k+1}) U[\eta_{i-1,n}]$$

$$= U[\eta_{i-1,n}]\Big(\prod_{k=N}^{i-2} u_{\eta_{k,i-1}}(\Theta_{k+1})\Big)U[\eta_{i-1,n}].$$

It then follows from the previous lemma that

$$U[\gamma_{1,n}]^N\Big(\prod_{k=N}^{i-2} u_{\eta_{k,n}}(\Theta_{k+1})\Big) = U[\gamma_{1,i-1}]^N U[\gamma_{i,n}]^N\Big(\prod_{k=N}^{i-2} u_{\eta_{k,i-1}}(\Theta_{k+1})\Big)U[\eta_{i-1,n}].$$

This latter term can be written as

$$\operatorname{diag}(U[\gamma_{1,i-1}]^N \prod_{k=N}^{i-2} u_{\eta_{k,i-1}}(\Theta_{k+1}), U[\gamma_{i,n}]^N U[\eta_{i-1,n}]).$$

And we note that

$$V_{i-1}((\Theta_1,\ldots,\Theta_{i-1})) = U[\gamma_{1,i-1}]^N \prod_{k=N}^{i-2} u_{\eta_{k,i-1}}(\Theta_{k+1}).$$

Using the fact that $\Theta_{k_2} = 1$, can rewrite

$$V_n(\Theta) = U[\gamma_{1,n}]^N\Big(\prod_{k=1}^{k_2-2} u_{\eta_{k,n}}(\Theta_{k+1})\Big)U[\eta_{i-1,n}]\Big(\prod_{k=k_2}^{n-1} u_{\eta_{k,n}}(\Theta_{k+1})\Big)$$

$$= U[\gamma_{1,k_2-1}]^N\Big(\prod_{k=N}^{k_2-2} u_{\eta_{k,k_2-1}}(\Theta_{k+1})\Big)U[\gamma_{k_2,n}]^N U[\eta_{k_2-1,n}]U[\eta_{k_2-1,n}]\Big(\prod_{k=k_2}^{n-1} u_{\eta_{k,n}}(\Theta_{k+1})\Big)$$

$$= U[\gamma_{1,k_2-1}]^N\Big(\prod_{k=N}^{k_2-2} u_{\eta_{k,k_2-1}}(\Theta_{k_2+1})\Big)U[\gamma_{k_2,n}]^N\Big(\prod_{k=k_2}^{n-1} u_{\eta_{k,n}}(\Theta_{k+1})\Big)$$

Using the fact that $\Theta_{k+1} = 0$ for $k_2 \leq k < k_2 + N - 1$ we have

$$V_n(\Theta) = U[\gamma_{1,k_2-1}]^N \Big(\prod_{k=1}^{k_2-2} u_{\eta_{k,k_2-1}}(\Theta_{k+1}) \Big) U[\gamma_{k_2,n}]^N \Big(\prod_{k=k_2+N-1}^{n} u_{\eta_{k,n}}(\Theta_{k+1}) \Big)$$

$$= \operatorname{diag}(V_{k_2-k_1}(\Theta_{k_1}, \ldots, \Theta_{k_2-1}), V_{k_{m+1}-k_2}(\Theta_{k_2}, \ldots, \Theta_{k_{m+1}-1})).$$

We may then repeat this on $V_{k_{m+1}-k_2}(\Theta_{k_2}, \ldots, \Theta_{k_{m+1}-1})$ using k_3 in place of k_2 and Eq. 2.3 follows. □

Lemma 2.1.9. *If A is an $n \times n$ matrix with $r(A) \leq N$, Θ is as above, and whenever $\Theta_k > 0$, A has zero crosses at positions $k, \ldots, k + N - 1$, then AV_n is a strictly lower triangular matrix.*

Proof. Suppose A is an $n \times n$ matrix with $r(A) \leq N$, and whenever $\Theta_k > 0$, A has zero crosses at positions $k, \ldots, k + N - 1$. Then $AU[\gamma_{1,n}]^N$ is strictly lower triangular and whenever $\Theta_k > 0$, $AU[\gamma_{1,n}]^N$ has zero columns at positions $k - N, \ldots, k - 1$ as well as at positions $n - N + 1, \ldots, n$. Therefore, for every k such that $\Theta_k > 0$, every column that $u_{\eta_{k,i}}(\Theta_k)$ does not fix is already zero. Therefore AV_n is also strictly lower triangular. □

2.2 Unitaries in DSH Algebras

We will need the following approximation result.

Lemma 2.2.1. *Suppose that $f \in A$ where A is DSH. Then for any ϵ there exist $f' \in A$ such that $|f - f'| < \epsilon$ and open sets $U_{i,k}$ such that $B_{i,k} \subseteq U_{i,k}$ and whenever $x \in U_{i,k}$, $f'_i(x)$ has a block point at position k. Moreover, f' retains any zero crosses of f and $r(f'_i(x)) \leq r(f_i(x))$ for all i and $x \in X_i$.*

Proof. Let $\epsilon > 0$. We construct f' by simultaneously modifying f on every X_i. We do this by considering each function $f_i(x)_{s,t}$ for indices $s, t \in \{1, \dots n_i\}$ as a function in $C(X_i)$. For $\delta > 0$ let $g_\delta \in C(\mathbb{C})$ be given by $g_\delta(z) = z\max(0, |z| - \delta)/|z|$. For any space X and any $f \in C(X)$ and any $x \in X$ such that $f(x) = 0$, $g_\delta \circ f$ is zero on an open set containing x. Since each space X_i is compact, we may choose a δ such that $|f_i(x)_{s,t} - g_\delta(f_i(x)_{s,t})| < \epsilon/n_l^2$, where n_l is the largest dimension of any irreducible representation of A.

For $x \in X_i$, we define $f_i'(x)$ to be the matrix given by the entries $f_i'(x)_{s,t} = g_\delta(f_i(x)_{s,t})$. Then for each x, $|f_i'(x) - f_i(x)| < \epsilon$, hence $|f' - f| < \epsilon$. Since we chose a single value of δ and modified every entry in a uniform manner, it is easy to see that when $y \in Y_i$, $f_i'(y) = \mathrm{diag}(f_{i_1}'(x_{y_1}), \dots, f_{i_s}'(x_{y_s}))$, hence $f' \in A$.

For every i, x, s, t, if $f_i(x)_{s,t} = 0$ then $f_i'(x)_{s,t} = 0$. Therefore f_i' has a zero cross wherever f_i does and $r(f_i'(x)) \leq r(f_i(x))$. For $k \in \{1, \dots, n_i\}$.

Fix i. Define $P(k) \subseteq \{1, \dots, n_i\}^2$ by $P(k) = \{(s,t) : a < k \text{ and } t \geq k \text{ or } t < k \text{ and } s \geq k\}$. For any k, when $x \in B_{i,k}$, $f_i(x)_{s,t} = 0$ whenever $(s,t) \in P(k)$. For each $(s,t) \in P(k)$, there is an open set $O_{s,t}$ containing $B_{i,k}$ on which $f_i'(x)_{s,t} = 0$. Taking $U_{i,k}$ to be the intersection $\bigcap_{(s,t)\in P(k)} O_{s,t}$, $f_i'(x)$ has a block point at position k for all $x \in U_{i,k}$. □

We will need to construct certain indicator functions in order to implement the unitaries defined above.

Lemma 2.2.2. *Suppose that A is a DSH algebra, $M \in \mathbb{N}$ and a set of integers $K = \{K_1, \dots, K_m\}$ such that $K_{t+1} - K_t \geq M$, $K_1 \geq 0$, and $K_m \leq n_1 - M$. For each i, k let $F_{i,k} \subset X_i$ be a closed set separated from $B_{i,k-K_t}$ for each $t \in \{1, \dots, m\}$. Then there is a* *function* $\Theta \in A$ *such that*

1. *for every* $x \in X_i$, $\Theta_i(x)$ *is a diagonal matrix with entries in* $[0, 1]$.

2. *For any* M *consecutive entries on the diagonal, at most one is non-zero.*

3. $\Theta_i(x)_{k,k} = 0$ *for all* $k \in \{n_i - (M-1), \ldots, n_i\}$.

4. *For each* i, k, $\Theta_i(x)_{k,k} = 0$ *for all* $x \in F_{i,k}$.

5. *For each* i, k, *there is an open subset* $U_{i,k} \subset X_i$ *such that* $B_{i,k} \subseteq U_{i,k}$ *and if* $x \in U_{i,k}$ *then* $\Theta_i(x)_{k+K_t,k+K_t} = 1$ *for all* $t \in \{1, \ldots, m\}$.

Proof. We claim it is sufficient to find a $\Theta \in A$ for which conditions (1), (2), and (3) hold and furthermore $\Theta_i(x)_{k,k} = 1$ if and only if there exists a j and a $t \in \{1, \ldots, m\}$ such that $k = j + K_t$ and $x \in B_{i,j}$. We call this condition (6). Given such a Θ, we will modify it to satisfy conditions (4) and (5). Let $g : [0, 1] \to [0, 1]$ be any function for which $g(x) = 0$ on some neighbourhood of 0 and $g(x) = 1$ on some neighbourhood of 1.

On each X_i and for each k, we consider $\Theta_i(x)_{k,k}$ to be a function in $C(X_i, [0, 1])$, and we refer to this function as $\Theta_{i,k}$. Considering $\Theta_i(x) = \text{diag}(\Theta_{i,1}(x), \ldots, \Theta_{i,n_i}(x))$, we define $\Theta'_i \in C(X_i, M_{n_i})$ by $\Theta'_i(x) = \text{diag}(g \circ \Theta_{i,1}(x), \ldots, g \circ \Theta_{i,n_i}(x))$. Letting $\Theta' = \oplus_i \Theta'_i$, we may check that this lies in A. It is clear that Θ' satisfies condition (1), and since g preserves 0, it also satisfies (2), and (3).

Since each $F_{i,k}$ is separated from $B_{i,k-K_t}$ for each $t \in \{1, \ldots, m\}$, condition (6) tells us that $\Theta_{i,k}(x) < 1$ whenever $x \in F_{i,k}$. Since each set $F_{i,k}$ is compact, there exists a maximum value $\delta < 1$ such that for every i, k, $\Theta_{i,k}(x) \leq \delta$ whenever $x \in F_{i,k}$. Therefore we may choose g such that $g(x) = 0$ on $[0, \delta]$, so that $g \circ \Theta_{i,k}(x) = 0$ whenever $x \in F_{i,k}$.

Finally, it follows from the definition of g that for each any j and $k \in K$, $\Theta_i(x)_{j+k,j+k} = 1$ on an open set O_k containing $B_{i,j}$. We take $U_{i,k}$ to be the intersection $\bigcap_{k\in K} O_k$, and Θ' therefore satisfies the lemma, proving our claim.

To construct Θ, we proceed recursively on i.

For $i = 1$, we simply let $\Theta_1 = \operatorname{diag}(\chi_K(1), \chi_K(2), \ldots, \chi_K(n_1))$ for all x where χ_K is the characteristic function for K. Suppose we have already defined $\Theta_{i'}$ for $i' < i$ and that for each $k \leq n_1$, $\Theta_{i',k}$ is the constant function $\chi_K(k)$ and for $k \geq n_{i'} - n_1 + 1$, $\Theta_{i',k} = 0$.

Then Θ_i is determined on Y_i. Fixing $y \in Y_i$, we check that conditions (1)-(3) and (6) are satisfied by $\Theta_i(y) = \operatorname{diag}(\Theta_{i_1}(x_1), \ldots, \Theta_{i_s}(x_s))$. Condition (1) is trivial. (2) needs only to be checked when we consider M consecutive entries spanning distinct components $\Theta_{i_t}(x_{y_t})$. Since $M < n_1$, it suffice to consider two consecutive blocks $\Theta_{i_t}(x_{y_t}) \oplus \Theta_{i_{t+1}}(x_{y_{t+1}})$. Condition (3) applied to $\Theta_{i_t}(x_{y_t})$ gives us condition (2). Condition (3) for $\Theta_i(y)$ follows from the same for $\Theta_{i_s}(x_{y_s})$. To check condition (6), fix k and consider $y \in B_{i,k} \cap Y_i$ with corresponding points $x_{y,1}, \ldots x_{y,s}$, where $x_{y,t} \in X_{i_t}$, and suppose $k = n_{i_1} + \cdots + n_{i_t} + k'$ where $k' \leq n_{i_{t+1}}$. Then $\Theta_{i,k}(y) = \Theta_{i_{t+1},k'}(x_{t+1})$. $y \in B_{i,j}$ if and only if $j \in \{1, n_{i_1} + 1, n_{i_1} + n_{i_2} + 1, \ldots\}$, and the only value of j such that $y \in B_{i,j}$ for which there can exist a $K_t \in K$ such that $k = j + K_t$ is $j = n_{i_1} + \cdots + n_{i_t} + 1$. Hence there exists a j and a $t \in \{1, \ldots, m\}$ such that $k = j + K_t$ and $x \in B_{i,j}$ if and only if $k'K$. We can assume x_{t+1} lies in $X_{i_{t+1}} \setminus Y_{i_{t+1}}$, so x_{t+1} lies in $B_{i_{t+1},j}$ if and only if $j = 1$. Therefore, applying condition (6) to $\Theta_{i_{t+1}}$, $\Theta_{i_{t+1},k'}(x_{t+1}) = 1$ if and only if $k' \in K$. This shows that Θ_i satisfies (6) on Y_i.

We extend $\Theta_{i,k}$ to X_i as the constant function $\chi_K(k)$ for every $k \leq n_1$ and the final $n_1 - 1$ functions as the constant zero function.

Suppose we have defined $\Theta_{i,1},\ldots,\Theta_{i,k-1}$ on all of X_i and that k does not fall into one of the sets for which $\Theta_{i,k}$ has already been given. Assume $\mathrm{supp}(\Theta_{i,k})\subset Y_i$ is disjoint from $\bigcup_{t=1}^{M-1}\overline{\mathrm{supp}(\Theta_{i,k-t})}$. We first set $f=\Theta_{i,k}$ on Y_i and $f=0$ on $\bigcup_{t=1}^{M-1}\overline{\mathrm{supp}(\Theta_{i,k-t})}$. We set $f=0$ closed set $f_{i,k}$ that is disjoint from Y_i. We can then extend f to a function that is strictly less than 1 on $X_i\setminus Y_i$.

We define $g=\Theta_{i,k}-\sum_{t=1}^{M-1}\Theta_{i,k+t}$ on Y_i, we may extend to the entirety of X_i. Let $g'=\max(g,0)$. g' is therefore 0 on an open set U containing $\bigcup_{t=1}^{M-1}\mathrm{supp}(\Theta_{i,k+t})$. Hence, each of these sets is separated from $\overline{\mathrm{supp}(\Theta_{i,k})}$. Moreover, g' agrees with $\Theta_{i,k}$ on Y_i.

Finally, we let $\Theta_{i,k}=\min(f,g')$. $\Theta_{i,k}$ is therefore 0 on $\bigcup_{t=1}^{M-1}\overline{\mathrm{supp}(\Theta_{n-t})}\cup f$, 1 on $U_1\cap U_2$, and $\overline{\mathrm{supp}(\Theta_{i,k})}$ is disjoint from $\bigcup_{t=1}^{M-1}\mathrm{supp}(\Theta_{i,n+t})$. We may check that $\Theta_i(x)$ then satisfies conditions (1)-(3) and (6) for every $x\in X_i$. □

Lemma 2.2.3. *Suppose that A is a DSH algebra, $M,N\in\mathbb{N}$ with $MN<n_1$, and $f\in A$, is a function such that for every i,k, there is an open set $U_{i,k}$ containing $B_{i,k}$ such that when $x\in U_{i,k}$, $f_i(x)$ has zero crosses in positions $k,k+M,k+2M,\ldots,k+(N-1)M$ and a block point at position k. Then there exists a unitary $V\in A$ such that $r(Vf_iV^*(x))\leq r(f_i(x))+2$ for all i and $x\in X_i$ and open sets $U'_{i,k}\supset B_{i,k}$ such that when $x\in U'_{i,k}$, $Vf_iV^*(x)$ has zero crosses in positions $k,k+1,k+2,\ldots,k+(N-1)$.*

Proof. We utilize Lemma 2.2.2 with $K=\{0\}$ to obtain $\Theta\in A$ such that for each i,k, $\Theta_{i,k}=0$ on the complement of the open set $U_{i,k}$. There exist open sets $U'_{i,k}$ such that $U_{i,k}\supseteq\overline{U'_{i,k}}\supseteq B_{i,k}$ on which $\Theta_{i,k}=1$. For any consecutive NM entries on the diagonal of any $\Theta_i(x)$, at most one is non-zero and the final $NM-1$ entries are all 0.

For an index $k\leq n_i-NM$, let $u_k\in C(X_i,M_{n_i})$ be the operator $1_{k-1}\oplus V(\Theta_{i,k}(x))\oplus$

$1_{n_i-(NM+k-1)}$ where $V : [0,1] \to M_{NM}$ is the operator given by Lemma 2.1.5 for $z_1 = 1, z_2 = M+1, \ldots z_N = (N-1)M+1$. Let u_k be the identity for $k > n_i - NM$.

By hypothesis, whenever $\Theta_{i,k}(x)$ is non-zero, $f_i(x)$ has zero crosses at positions $k, k+M, k+2M, \ldots, k+(N-1)M$ and a block point at position k. It follows then, from Lemma 2.1.5, that $(u_k f_i u_k^*)(x)$ has zero crosses at positions $k, k+1, \ldots, k+(N-1)$ whenever $x \in U'_{i,k}$.

Fix $x \in X_i$ and let $K(x) = \{k : \Theta_{i,k}(x) > 0\}$ and write this set as $\{k_1, \ldots, k_s\}$. Note that $k_1 = 1$ and $k_{t+1} - k_t \geq NM$. Consider the operator $V_i' \in C(X_i, M_{n_i})$ given by $V_i'(x) = \prod_{k=1}^{n_i} u_k(x)$. Since $u_k(x)$ is the identity whenever $k \notin K(x)$, we can rewrite this as $V_i'(x) = \prod_{t=1}^{s} u_{k_t}$. Since the distance between successive k_t is greater than the dimension of V, we can give a direct sum decomposition of V_i'.

$$V_i'(x) = \operatorname{diag}(V(\Theta_{i,k_1}(x)), 1_{d_1}, V(\Theta_{i,k_2}(x)), 1_{d_2}, \ldots, V(\Theta_{i,k_s}(x)), 1_{d_s})$$

where $d_t = k_{t+1} - (k_t + NM)$ for $t < m$ and $d_s = n_i - (k_s + NM)$.

Also, because $f_i(x)$ has a block point at position k_t for each t, $f_i(x)$ can be written as $Z_1 \oplus Z_2 \oplus \ldots$ where Z_t is a $k_{t+1} - k_t$ dimensional matrix. Then

$$u_{k_t} f_i u_{k_t}^*(x) = \operatorname{diag}(Z_1, \ldots, Z_{t-1}, (V(\Theta_{k_t}(x)) \oplus 1_{d_t}) Z_t (V(\Theta_{k_t}(x))^* \oplus 1_{d_t}), \ldots).$$

It then follows from Lemma 2.1.5 applied for each t that $(V_i' f_i V_i'^*)(x)$ has zero crosses at positions $k, k+1, \ldots, k+(N-1)$ whenever $x \in U'_{i,k}$. We also find that $(V(\Theta_{k_t}) \oplus 1_{d_t}) Z_t (V(\Theta_{k_t}) \oplus 1_{d_t})^*$ has diagonal radius at most $r(Z_t) + 2$. Therefore $r((V_i' f_i V'^*)(x)) \leq$

$r(f_i(x)) + 2$.

It remains to show that $V' = \oplus_i V_i' \in A$. It suffices to show that whenever $y \in Y_i \subset X_i$ with corresponding points $x_{y,1}, \ldots x_{y,s}$, where $x_{y,t} \in X_{i_t}$, $V_i'(y) = \text{diag}(V_{i_1}'(x_{y,1}), \ldots, V_{i_s}'(x_{y,s}))$. Let $B(x) = \{k : x \in B_{i,k}\}$ and note that $B(x) \subseteq K(x)$. Since $\Theta \in A$, $\Theta_i(y) = \text{diag}(\Theta_{i_1}(x_{y,1}), \ldots, \Theta_{i_s}(x_{y,s}))$. Therefore if k_{t_1} and k_{t_2} are successive elements of $B(x)$, (they need not be successive in $K(x)$), then

$$\text{diag}(V(\Theta_{i,k_{t_1}}(y)), 1_{d_{t_1}}, \ldots, V(\Theta_{i,k_{t_2-1}}(y)), 1_{d_{t_2-1}})$$

is exactly $V_{i_{t_1}}'(x_{y,t_1})$ so $V_i'(x)$ decomposes as necessary.

□

Lemma 2.2.4. *Suppose that A is a DSH algebra, $N \in \mathbb{N}$ with $N < n_1$, and $f \in A$ is a function such that for every i, k, there is an open set $U_{i,k}$ containing $B_{i,k}$ such that when $x \in U_{i,k}$, $f_i(x)$ has zero crosses in positions $k, k+1, \ldots, k+N-1$, and $r(f_i(x)) < N$. Then there exists a unitary $V \in A$ such that $(fV)_i(x)$ is strictly lower triangular for every x.*

Proof. We utilize Lemma 2.2.2 with $K = \{0\}$ to obtain $\Theta \in A$ such that for each i, k, $\Theta_{i,k} = 0$ on $U_{i,k}^C$, $\Theta_{i,k} = 1$ on $B_{i,k}$ and for any consecutive N entries on the diagonal, at most one is non-zero. For each i we then apply Lemma 2.1.8 to obtain $V_i \in C([0,1]^{n_i}, M_{n_i})$, and let $V' = \bigoplus_i V_i \circ \Theta_i$, where we consider $\Theta_i(x)$ to be in $[0,1]^{n_i}$. V_i' is then a unitary in $C(X_i, M_{n_i})$.

Suppose that $y \in Y_i \subset X_i$ with corresponding points $x_{y,1}, \ldots x_{y,s}$, where $x_{y,t} \in X_{i_t}$. Consider the set $B = \{k_1 = 1, \ldots, k_s\}$ of k such that $y \in B_{i,k}$. By our choice of Θ,

$\Theta_{i,k}(y) = 1$ for every $k \in B$, and by the DSH decomposition, $k_{t+1} - k_t = n_{i_t}$. Since $\Theta \in A$, $\Theta_i(y) = \mathrm{diag}(\Theta_1(x_{y,1}), \ldots, \Theta_s(x_{y,s}))$. Therefore by (2.3),

$$V_i(\Theta_i(x)) = \mathrm{diag}(V_{i_{y,s}}(\Theta_{i_1}(x_{y,1})), \ldots, V_{i_s}(\Theta_{i_s}(x_{y,s}))),$$

and similarly

$$V_i'(y) = \mathrm{diag}(V_{i_1}'(x_{y,1}), \ldots, V_{i_s}'(x_{y,s})),$$

Therefore, $V' \in A$. Finally, we apply Lemma 2.1.9 at each point x to conclude that $(fV)_i(x)$ is an lower triangular matrix for every i and x. □

2.3 Spectral Considerations

The unitary equivalence classes of irreducible representations of A under the kernel-hull topology will be denoted by $\widehat{A}$. The topology is defined by

$$\overline{X} = \{\rho : \ker(\rho) \supseteq \bigcap_{\pi \in X} \ker(\pi)\}.$$

For any cardinal number n, $\widehat{A}(n)$ will denote the subspace of n-dimensional representation classes.

Let A and B be unital C^*-algebras, and suppose that $\phi : A \to B$ is a homomorphism. Then every irreducible representation of B decomposes into a direct sum of a unique set of irreducible representations of A. We define – via decomposition into irreducible representations – a set-valued map $\widehat{\phi} : \widehat{B} \to \mathcal{P}(\widehat{A})$, which ignores multiplicity.

We can characterize simplicity of an inductive limit algebra $A = \lim(A_n, \phi_n)$ in terms of dual maps. This is almost certainly known; however we have been unable to find a reference. It generalizes Proposition 2.1 of [2], and the proof is very similar. For $n' > n$, we will denote by $\phi_{n',n}$ the map $\phi_{n'-1} \circ \cdots \circ \phi_n$.

Theorem 2.3.1. *Let $A = \lim(A_n, \phi_n)$ be a unital C^*-algebra and suppose each ϕ_n is* *injective. Then the following statements are equivalent*

1. *A is simple.*

2. *$\forall i \in \mathbb{N}$, open $U \subset \widehat{A}_i$, $\exists j > i$ such that $\widehat{\phi}_{j,i}(\pi) \cap U \neq \varnothing$ for all $\pi \in \widehat{A}_j$.*

3. *For any non-zero $a \in A_i$, there exists $j > i$ such that $\pi(\phi_{j,i}(a)) \neq 0$ for all $\pi \in \widehat{A}_j$.*

Proof. Fix i and let $U \subset \widehat{A}_i$ be open. For each $j \geq i$, let $F_j = \{\pi \in \widehat{A}_j : \widehat{\phi}_{j,i}(\pi) \cap U = \varnothing\}$. Consider $\rho \in \widehat{A}_j$ and suppose $\widehat{\phi}_{j,i}(\rho)$ contains a point $\pi \in U$. Then we may find $f \in A_i$ such that $\pi(f) \neq 0$ and $\sigma(f) = 0$ for all $\sigma \in \widehat{A}_i \setminus U$. Therefore $\ker(\rho)$ does not contain $\bigcap_{\sigma \in F_j} \ker(\sigma)$ since $\pi_{j,i}(f)$ lies in the latter but not the former. It follows that F_j is closed. Suppose F_j is non-empty for each $j \geq i$. Then for each j, we may construct a closed non-zero ideal $I_j = \{a \in A_j : \pi(a) = 0 \ \forall \pi \in F_j\}$. For any $k \geq j$, $\widehat{\phi}_{k,j} \circ \widehat{\phi}_{j,i} = \widehat{\phi}_{k,i}$. It follows that if $\pi(a) = 0$ for all $\pi \in F_j$ then $\pi(\phi_{k,j}(a)) = 0$ for every $\pi \in F_k$, as $\widehat{\phi}_{k,j}(\pi) \subseteq F_j$. Therefore, $\phi_{k,j}(I_j) \subseteq I_k$. We let $I = \lim(I_n, \phi_n)$, and since 1_A cannot lie in I, I is a proper ideal of A. Therefore (1) implies (2). For any non-zero $a \in A_i$, let $U \subset \widehat{A}_i$ be the set $\{\pi : \pi(a) \neq 0\}$. Applying condition (2) immediately implies condition (3). When (3) holds, it is clear that for any non-zero $a \in A_i$, there exists $j > i$ such that $\phi_{j,i}(a)$ does not lie in any proper ideal of A_j. Therefore any ideal of A which contains a non-zero element of A_i must contain all of A_j for some j. Since A_j contains 1_A, the ideal contains all of A. □

We restate Lemma 2.1 of [15] in our notation.

Lemma 2.3.2. *Suppose that A is a DSH algebra. For $n_i \in \{n_1, \ldots, n_l\}$, $\widehat{A}(n_i)$ is homeomorphic to $X_i \setminus Y_i$. For $n \notin \{n_1, \ldots, n_l\}$, $\widehat{A}(n)$ is empty.*

The following lemma allows us to modify a non-invertible element to produce one with a zero cross.

Lemma 2.3.3. *Let A be a DSH algebra, let $\epsilon > 0$ and suppose $f \in A$ is non-invertible. Then there exists $f' \in A$ such that $|f - f'| < \epsilon$ and a unitary $v \in A$ such that for some i, $(vf')_i$ has a zero cross in position 1 everywhere in some open set $U \subseteq \widehat{A}$ with $\overline{U} \subseteq (X_i \setminus Y_i)$. Moreover, there exists $\Delta \in A$ such that for every x, $\Delta(x)$ is a diagonal matrix with entries in $[0,1]$ where $\Delta(x)_{k,k} > 0$ implies f has a zero cross at position k and $\Delta(x)_{1,1} = 1$ for all $x \in U$.*

Proof. Since f is non-invertible, there exists X_i and a point $x \in X_i$ such that $f(x)$ is a non-invertible matrix. Reducing to a non-invertible summand if necessary, we may assume $x \in X_i \setminus Y_i$. Let i' be the greatest integer such that x lies in the closure of $\widehat{A}(n_{i'})$. Choose an open neighbourhood $U_1 \subseteq \widehat{A}$ of x such that $U_1 \cap \widehat{A}(n_{i''}) = \emptyset$ for all $i'' > i'$. Since x lies in the closure of $(X_{i'} \setminus Y_{i'}) \subset \widehat{A}$, there exists a point $y \in Y_{i'}$ such that $f_{i'}(y) = M_1 \oplus f_i(x) \oplus M_2$ for some matrices M_1, M_2. We may find a nearby point $x' \in U_1 \cap (X_{i'} \setminus Y_{i'})$ such that $|f_{i'}(x') - M_1 \oplus f(x) \oplus M_2| < \frac{\epsilon}{2}$.

It follows from Proposition 3.6.3 of [3] that $\widehat{A}_{n_{i'}} \bigcap U_1$ is open in U_1.

Since U_1 does not lie in the closure of $X_{i''} \setminus Y_{i''}$ for any $i'' > i'$, we may, by assuming that each $Y_{i''}$ has empty interior, conclude that there is no point in any $Y_{i''}$ has a corresponding point $x_{y,t} \in U_1$. We may then perturb f on $U_1 \cap (X_{i'} \setminus Y_{i'})$ without altering $f_{i''}$ for any $i'' > i'$. In particular, we can obtain a function f' such that $|f - f'| < \epsilon$ which is equal to $M_1 \oplus f_i(x) \oplus M_2$ on a neighbourhood U_2 of x' with $\overline{U_2} \subseteq (X_{i'} \setminus Y_{i'}) \cap U_1$. Then there exist a unitary matrix V such that $V(M_1 \oplus f(y) \oplus M_2)$ has a zero cross in position 1. By considering a path of unitaries in $U(n_{i'})$ between V and the identity, we may extend V

to a unitary v in A which is constantly equal to V on U_2 and is the identity everywhere outside of $(X_{i'} \setminus Y_{i'}) \cap U_1$.

To construct Δ, begin by setting $\Delta_{i''} = 0$ for $i'' < i'$. On $X_{i'} \setminus Y_{i'}$, we let $\Delta_{i'}(x)$ be zero everywhere except on U_2 and always zero in every coordinate of $\Delta_{i'}$ except $\Delta_{i'}(x)_{1,1}$. We consider $\Delta_{i'}(x)_{1,1}$ to be in $C(X_{i'}, [0,1])$ and choose a function such that $\Delta_{i'}(x)_{1,1} = 1$ everywhere on some non-empty open set $U_3 \subseteq U_2$ and 0 everywhere outside U_2. Since $\overline{U_2}$ is separated from $X_{i''}$ for any $i'' > i'$, we may set $\Delta_{i''}$ to be 0 on $X_{i''}$.

We find that f',v, Δ, and U_3 then have the desired properties. □

Note that the set U_3 is, using the bijection we have established between $(X_i \setminus Y_i)$ and $\widehat{A}(n_i)$, also an open set in $\widehat{A}$.

When discussing a sequence $\lim(A_j, \phi_j)$ of DSH algebras, we will identify the spaces in the composition sequence of A_j as X_i^j and the dimensions as n_i^j. We may think of $\widehat{\phi}_{j',j}$ as a map sending points in $\bigsqcup_i X_i^{j'}$ to finite sets of points in $\bigsqcup_i X_i^j$.

We say that a map $\phi_{j_2,j_1} : A_{j_1} \to A_{j_2}$ is diagonal if for every $x \in X_i^{j_2}$ there are points $x_1, \ldots, x_t$, where $x_k \in X_{i_k}^{j_1}$ such that for every $f \in A_{j_1}$, $\phi_{j_2,j_1}(f)_i(x) = \mathrm{diag}(f_{i_1}(x_1), \ldots, f_{i_t}(x_t))$.

In the following, we must eliminate the case in which the representations of A_j do not grow in dimension. By 2.3.1, when the sequence is injective, such a limit can only be simple if the algebras A_j are all of finite spectrum. Therefore it is enough to assume each A_j is infinite dimensional.

Lemma 2.3.4. *Suppose that $A = \lim(A_j, \phi_{j+1,j})$ where A_j is infinite dimensional and DSH, $\phi_{j+1,j}$ is diagonal and A is simple. Suppose that f is a non-invertible element in A_j and let $\epsilon > 0$. Then there exists $f' \in A_j$ with $|f - f'| < \epsilon$ and a constant M such that for any $N \in \mathbb{N}$, there exists $j' > j$ such that $A_{j'}$, unitaries $V_1, V_2 \in A_{j'}$ such for every $X_i^{j'}, k \leq n_i^{j'}$, there is an open set $U_{i,k} \subset$ containing $B_{i,k} \subset X_i^{j'}$, such that for each $x \in U_{i,k}$, $V_1 \phi_{i',i}(f'(x)) V_2$ has zero crosses in positions $k, M+k, 2M+k, \ldots, (N-1)M+k$ and $r(V_1 \phi_{i',i}(f'(x)) V_2) < R + M - 1$, where R is the size of the largest representation of*

A_j.

Proof. Since f is non-invertible, we may find $f', v, \Delta \in A_j$ and U, as in Lemma 2.3.3, where U is an open set in $\widehat{A_j}$. By Theorem 2.3.1, we may choose j'' such that for every point $x \in \widehat{A}_{j''}$, $\widehat{\phi}_{j'',j}(x)$ contains a point in U. If n is the the largest dimension of the irreducible representations of $A_{j''}$ then we choose $M = 2n$.

Consider Δ'', the image of Δ in $A_{j''}$. Because $\phi_{i'',i}$ is diagonal, $\Delta''_i(x)$ is a diagonal matrix for every point x in every $X_i^{j''}$, and its entries are in $[0,1]$. It follows from our choice of j'' that for every $x \in X_i^{j''}$, $\Delta''_i(x)$ has an entry that is equal to 1 somewhere on the diagonal coming from the point in $\widehat{\phi}_{j'',j}(x)$ that lies in U.

Let $N \in \mathbb{N}$ be arbitrary. We can choose j' such that n_1, the smallest dimension of a representation of $A_{j'}$, is arbitrarily large. In particular, we choose n_1 to be at least NM.

Let $\Delta' = \phi_{j',j}(\Delta)$, and consider Δ' as the diagonal image of Δ''. As before, we condense our notation, writing $\Delta'_{i,k}(x)$ rather than $\Delta'_i(x)_{k,k}$. For any i, $x \in X_i^{j'}$ and any entry $k < n_i^{j'} - M$, there must be an entry $k' \in \{k, k+1, \ldots, k+M-1\}$ for which $\Delta'_{i,k}(x) = 1$.

Let $g = \phi_{j',j}(vf')$. Since $\phi_{j',j}$ is diagonal, $g_i(x)$ is block diagonal for every x in every $X_i^{j'}$, and the maximum block size is R, so $r(G(x)) \leq R$. Moreover, whenever $\Delta'_{i,k}(x) > 0$, $g(x)$ has a zero cross at position k.

We will need a second set of indicator functions. We construct $\Theta \in A_{j'}$ as in Lemma 2.2.2 for our already given value of M and $K = \{0, M, \ldots, (N-1)M\}$. For each $k, X_i^{j'}$,, there is an open set $U_{i,k}$ containing $B_{i,k}$, such that for each $x \in U_{i,k}$, $\Theta_{i,k}(x) = \Theta_{i,k+M}(x) = \cdots = \Theta_{i,k+(N-1)M}(x) = 1$. For $k \in 0, \ldots, M-1$, Θ_{i,n_i-k} is the constant zero function. Finally, for arbitrary x, k, at most one of $\Theta_{i,k}(x), \ldots, \Theta_{i,k+M-1}(z)$ is non-zero.

We can now construct a unitary element V such that VGV^* has zero crosses where required and a bounded diagonal radius. On $X_i^{j'}$, we let $u_k(x) \in C(X_i^{j'}, M_{n_i})$ be given by

$$u_k(z) = u_{(k\ k+1)}(\Theta_{i,k}(x)\Delta'_{k+1}(x)) \cdots u_{(k\ k+M-1)}(\Theta_{i,k}(x)\Delta'_{k+M-1}).$$

We then define $V_i \in C(X_i^{j'}, M_{n_i^{j'}})$ by $V_i = \prod_{k=1}^{n_i-(M-1)} u_k$. When $\Theta_k(x) = 0$, $u_i(x)$ is the identity, so V_i can be rewritten at each x as a product of $u_{k_1}, \ldots, u_{k_n}$ where, by the definition of Θ, $k_{i+1} - k_i \geq M$. When $x \in U_{i,k}$, $\Theta_k(x) = \Theta_{k+M}(x) = \cdots = \Theta_{k+(N-1)M}(x) = 1$ and at least one value in $\Delta'_{k+aM}(x), \ldots, \Delta'_{k+(a+1)M-1}(x)$ is equal to 1 for $a \in \{0, \ldots N-1\}$, hence by Lemma 2.1.3, $V_i g_i V_i^*(x)$ has zero crosses at positions $k, k+M, \ldots, k+(N-1)M$. Also by Lemma 2.1.4, we have that $r(V_i g_i V_i^*(x)) \leq R+M-1$ for all x.

It remains to show that $V = \bigoplus_i V_i \in A_{j'}$. For $x \in Y_i^{j'} \subseteq X_i^{j'}$, with corresponding points $x_1, \ldots x_s$, where $x_t \in X_{i_t}^{j'}$ and let $K = \{k_1 = 1, \ldots, k_s\}$ be the set of k such that $x \in B_{i,k}$. We decompose the product $V_i = \prod_{k=1}^{n_i-(M-1)} u_k$ into $\prod_{m=1}^{s} \prod_{k=k_m}^{k_{m+1}-(M-1)} u_k$ where we let $k_{s+1} = n_i + 1$. Consider then the product $v_m(x) = \prod_{k=k_m}^{k_{m+1}-(M-1)} u_k$,

$$v_m(x) = \prod_{k=k_m}^{k_{m+1}-(M)} \prod_{a=1}^{M-1} u_{(k\ k+a)}(\Theta_{i,k}(x)\Delta'_{i,k+a}(x)),$$

and note first that because $\Theta_{i,k}(x) = 0$ for $k \in k_m - M + 1, \ldots, k_m - 1$, this is the same as taking the product in k to $k_{m+1}-1$. $v_m(x)$ is block diagonal of the form $1_{k_m} \oplus B \oplus 1_{n_i - k_{m+1}}$

where B is a square $k_{m+1} - k_m$ dimensional matrix. Furthermore, the matrix B is given by

$$u_{(k\ k+1)}(\Theta_{i,k_m+k}(x)\Delta_{i,k_m+k+1}(x)) \cdots u_{(k\ k+M-1)}(\Theta_{i,k}(x)\Delta'_{i,k+M-1}(x)).$$

Since we have constructed Θ and Δ' as elements of $A_{j'}$, $\Theta_{i,k_t+k}(x) = \Theta_{i_t,k}(x_t)$ and $\Delta'_{i,k_t+k}(x) = \Delta_{i_t,k}(x_t)'$ where $k < k_{m+1} - k_m$. We have that $B = V_{i_t}(x_t)$ and $V_i = V_{i_1}(x_1) \oplus \cdots \oplus V_{i_s}(x_s)$. Therefore $V \in A_{j'}$.

Finally, consider the unitaries $V\phi_{j',j}(v)$ and V^*. These unitaries satisfy the lemma, since $V\phi_{j',j}(v)\phi(f')V^* = VGV^*$. □

2.4 The Main Theorem

Theorem 2.4.1. *Suppose $A = \lim(A_j, \phi_j)$ is a simple injective limit of DSH algebras with diagonal maps. Then A has stable rank one.*

Proof. If A is a limit of finite dimensional algebras then we are done, so we may assume all A_j are infinite dimensional. Let $\epsilon > 0$ and $a \in A$. We will show that there is an invertible element $a' \in A$ such that $|a - a'| < \epsilon$. We find A_j and an element $f \in A_j$ such that $|\iota(f) - a| < \epsilon/4$, where ι is the inclusion of A_j into A. If f is invertible we are done, so we assume it is not. Denote by R the largest dimension of a representation of A_j. By 2.3.4, there is a $f' \in A_j$ such that $|f - f'| < \epsilon/4$ and a constant M such that for any $N \in \mathbb{N}$, there exists a $j' > j$, unitaries $V_1, V_2 \in A_{j'}$ such that in the construction of $A_{j'} \subset \bigoplus_i C(X_i, M_{n_i})$, for every i, k, there is an open set $U_{i,k}$ containing $B_{i,k}$, such that for each $x \in U_{i,k}$, $V_1\phi(f'(x))V_2$ has zero crosses in positions $k, M+k, 2M+k, \ldots, (N-1)M+k$

and $r(V_1\phi(f_i'(x))V_2) < R+M$. In particular, we choose $N > R+M+2$.

Hence we let $G = V_1\phi(f')V_2 \in A_{j'}$ and apply Lemma 2.2.1 to obtain $G' \in A_{j'}$ such that $|G - G'| < \epsilon/4$, and there exist open sets $U'_{i,k}$ containing each $B_{i,k}$ such that for each $x \in U_{i,k}$, $G(x)$ has zero crosses in positions $k, M+k, 2M+k, \ldots, (N-1)M+k$ and a block point at position k. Moreover, for every $x \in X_i$, $r(G'(x)) < R+M$. We apply Lemma 2.2.3 to obtain V_3 such that there exist open sets $U''_{i,k}$ on which $V_3G'V_3^*$ has zero crosses at positions $1, \ldots, N$ where $N > R+M+2$ and for every $x \in X_i$, $r(V_3G_i'V_3^*(x)) < R+M+2$. Then by Lemma 2.2.4 there exists a unitary $V_4 \in A_{j'}$ such that $(V_3G'V_3^*(x)V_4)_i$ is lower triangular at every $x \in X_i$. Therefore $V_3G'V_3^*V_4$ is nilpotent and by [18] there exists an invertible element $G'' \in A_{j'}$ such that $|V_3G'V_3^*V_4 - G''| < \epsilon/4$. Then $|\phi(f) - V_1^*V_3^*G'V_4^*V_3V_2^*| < 3\epsilon/4$, so $|a - \iota(V_1^*V_3^*G''V_4^*V_3V_2^*)| < \epsilon$ and the latter is invertible. □

2.5 Dynamical Crossed Products

Let X be an infinite compact Hausdorff space and let $\sigma : X \to X$ be a minimal homeomorphism. We denote also by σ the automorphism of $C(X)$ given by $\sigma(f) = f \circ \sigma^{-1}$. We let u be a unitary such that $ufu^* = \sigma(f)$.

Let $Y \subseteq X$ be the closure of an open set in X and consider the algebra A_Y generated by $\{f, ug : g(x)|_Y = 0\}$. By results of Q. Lin in [12], this algebra is subhomogeneous and can be described explicitly as follows.

For $y \in Y$, write $R(y) = \min\{n > 0 : \sigma^n(y) \in Y\}$. This is the first return time for y. By compactness $R(Y)$ is a finite set, and we list its values as $n_1 \ldots, n_l$. Moreover, we write $X_i = \overline{R^{-1}(n^i)}$ for $i = 1, \ldots, l$. Then A_Y can be written as a subalgebra of $\bigoplus_{i=1}^l M_{n_i}(C(X_i))$.

Define $Y_i = X_i \setminus R^{-1}(n_i)$. Whenever $y \in Y_i$, there exists $t_1, \ldots t_s$ and $y' \in X_{t_1}$ such that $n_{t_1} + \cdots + n_{t_s} = n_i$ and $\sigma^{n_{t_1}+\cdots+n_{t_j}}(y') \in Y$ for $1 \le j \le s$. In this case when $f \in A_Y$, $f_i(y) = \mathrm{diag}(f_{t_1}(y'), f_{t_2}(\sigma^{r_{t_1}}(y')), \ldots, f_{t_s}(\sigma^{r_{t_1}+\cdots+r_{t_{s-1}}}(y')))$.

For $x \in X$, we write A_x rather than $A_{\{x\}}$. A_x is simple and by [1], is a centrally large subalgebra of the crossed product. We aim to show A_x has stable rank one.

The algebra A_Y is a DSH algebra with a composition sequence of length l, spaces $X_1, \ldots, X_l$, restriction subspaces Y_i and dimensions $n_1, \ldots, n_k$. The points corresponding to $x \in X_i$ are determined by the set of values in $k \in \{1, \ldots, n_i - 1\}$ for which $\sigma^k(y) \in Y$. Writing this set as $K = \{k_1, \ldots, k_s\}$, these points are given by $x, \sigma^{k_1}(x), \ldots, \sigma^{k_s}(x)$.

It is clear from the generators that if $Z \subset Y$ then $A_Y \subset A_Z$ and we let ϕ be the natural embedding.

Lemma 2.5.1. *ϕ is a diagonal map between DSH algebras.*

Proof. We will give an explicit diagonal description of ϕ. We define the return time of a point $z \in Z$ to Z as $R_Z(z)$. Z decomposes into sets $Z_i = \overline{R_Z^{-1}(q_i)}$ where $q_1, \ldots, q_k$ is the list of possible return times. Since every point in $Z_i \setminus R_Z^{-1}(q_i)$ lies in Z_j for some $j < i$, it suffices to describe $\phi(f)$ at an arbitrary point in $R_Z^{-1}(q_i)$. Suppose $z \in R_Z^{-1}(q_i)$. There exist $n_{t_1}, \ldots n_{t_s}$ such that $n_{t_1} + \cdots + n_{t_s} = q_i$ and $\sigma^k(z) \in Y$ if and only if $k = n_{t_1} + \cdots + n_{t_j}$ for some $0 \leq j \leq s$. Let $\sigma^{n_{t_1} + \cdots + n_{t_j}}(z) \in X_{i_j}$. Then for $f \in A_Y$,

$$\phi(f)_i(z) = \operatorname{diag}(f_{i_1}(z), f_{i_2}(\sigma^{n_{t_1}}(z)), \ldots, f_{i_s}(\sigma^{n_{t_1} + \cdots + n_{t_{s-1}}}(z))). \tag{2.4}$$

Since ϕ is evidently a diagonal homomorphism, it suffices to prove (2.4) for the generators f and ug. Suppose $f \in C(X)$, and denote by ι_Y the function on $\sqcup X_i$ in our presentation of A_Y and similarly with ι_Z. By [12], $\iota_Y(f) = \bigoplus_{i=1}^{l} \operatorname{diag}(f \circ \sigma, \ldots, f \circ \sigma^{n_i})$. Suppose then that for some j, $\sigma^j(z) \in X_k$. Then $\iota_Y(f)(\sigma^j(z)) = \operatorname{diag}(f \circ \sigma^{j+1}, \ldots, f \circ \sigma^{i+j_k})$. We may decompose the sequence $1, \ldots, q_i$ into $1, \ldots, n_{t_1}$ followed by $(\sum_{i=1}^{j} n_{t_i}) + 1, \ldots, (\sum_{i=1}^{j} n_{t_i}) + n_{t_{j+1}}$

for $j = 1, \ldots, s-1$. For each of these subsequences

$$\mathrm{diag}(f(\sigma^{(\sum_{i=1}^{j} n_{t_i})+1}(z)), \ldots, f(\sigma^{(\sum_{i=1}^{j} n_{t_i})+n_{t_{j+1}}}(z))) = \iota_Y(f)_{i_j}(\sigma^{n_{t_1}+\cdots+n_{t_j}}(z)).$$

We then have that

$$\phi(\iota_Y(f)_{i_j}(z)) = \mathrm{diag}(\iota_Y(f)_{i_0}(z), \ldots, \iota_Y(f)_{i_{s-1}}(\sigma^{n_{t_1}+\cdots+n_{t_{s-1}}}(z)))$$

$$= \mathrm{diag}(f \circ \sigma(z), \ldots, f \circ \sigma^{n_{t_1}}(z), \ldots, f \circ \sigma^{(\sum_{i=1}^{s-1} n_{t_i})+1}(z), \ldots, f \circ \sigma^{\sum_{i=1}^{s} n_{t_i}}(z))$$

$$= \mathrm{diag}(f \circ \sigma(z), \ldots, f \circ \sigma^{q_i}(z))$$

$$= \iota_Z(f)_i(z).$$

Suppose $g \in C(X)$ and $g(y) = 0$ for all $y \in Y$. Then $ug \in A_Y \subset A_Z$ and we denote its two presentations by $\iota_Y(ug)$ and $\iota_Z(ug)$. For $z \in Z_i$,

$$\iota_Z(ug)_k(z) = \begin{bmatrix} 0 & & & \\ g \circ \sigma(z) & 0 & & \\ & \ddots & \ddots & \\ & & g \circ \sigma^{q_i - 1}(z) & 0 \end{bmatrix}.$$

Moreover, since $\sigma^{n_{t_1}+\cdots+n_{t_j}}(z) \in X_{i_j}$ for $0 \le j \le s$, $ug \circ \sigma^{n_{t_1}+\cdots+n_{t_j}}(z) = 0$. Thus $\iota_Z(ug)_i(z)$ can be decomposed into s diagonal blocks of size $n_{t_1}, \ldots, n_{t_s}$. Each such block can be identified with a value of $\iota_Y(ug)_{i_j}(\sigma^{n_{t_1}+\cdots+n_{t_j}}(z))$ in our presentation of A_Y via the equation

$$\iota_Y(ug)_{i_j}(\sigma^{n_{t_1}+\cdots+n_{t_j}}(z)) = \begin{bmatrix} 0 & & & \\ g\circ\sigma^{R_j+1}(z) & 0 & & \\ & \ddots & \ddots & \\ & & g\circ\sigma^{R_j+n_{t_{j+1}}-1}(z) & 0 \end{bmatrix}$$

where $R_j = n_{t_1} + \cdots + n_{t_j}$. Hence $\mathrm{diag}(\iota_Y(ug)_{i_0}(z), \ldots, \iota_Y(ug)_{i_s} \circ \sigma^{n_{t_1}+\cdots+n_{t_s}}(z)) = \iota_Z(ug)_k(z) = \phi(\iota_Y(ug))_k(z)$. □

Corollary 2.5.2. *For any compact Hausdorff space and any minimal homeomorphism $\sigma : X \to X$, the crossed product $C^*(\mathbb{Z}, X, \sigma)$ has stable rank one.*

Proof. Lemma 2.4 shows that A_y can be constructed as a limit of DSH algebras via diagonal maps. It follows from Theorem 2.4.1 that A_y has stable rank one. It is shown in [1] that A_y is centrally large in $C^*(\mathbb{Z}, X, \sigma)$. By Theorem 6.3 of [1], any infinite dimensional simple separable unital C^*-algebra with a centrally large subalgebra of stable rank one has stable rank one. □

Chapter 3

Mean Dimension and $\mathcal{Z}$-stability

Dimension theories have long played a prominent role in the study of sub-homogeneous algebras. In particular, a handful of related conditions for dimension growth have been used to identify a class of well-behaved limits of homogeneous and subhomogeneous algebras. $\mathcal{Z}$-stability is the condition that we will use to identify well-behaved algebras. This condition, along with finite nuclear dimension, has recently been used to identify a family of nuclear separable simple $C*$-algebras which are classifiable by their K-theory and tracial data. Since the $\mathcal{Z}$-stability results we will use require separability, we henceforth assume that all algebras under discussion are separable.

In the homogeneous context, dimension is a quantity that compares the topological dimension of a space X to the algebraic dimension of matrices n in, for example, a matrix algebra $C(X, M_n)$. This quantity has a natural generalization to subhomogeneous algebras by considering the subquotients associated with the subspace $\text{prim}_n(A)$ of the spectrum given by a particular size of representation of A. It was shown in [20] that when the ratio $\lim_i \sup_n \text{prim}_n(A_i)/n$ goes to zero in a limit of subhomogeneous algebras A_i, the corresponding limit is $\mathcal{Z}$-stable.

In [14], this notion is generalized in the case of diagonal AH algebras. That is, while the above limit may be non-zero, it is possible that the image of one algebra A_i in a subsequent algebra A_j may be contained in an intermediate homogeneous algebra with a dimension ration smaller than either A_i or A_j. In the diagonal AH context this can be done by direct study of the corresponding eigenvalue maps in the diagonal decomposition. This bears a close resemblance to the notion of mean dimension for dynamical systems as defined in [13].

In this chapter, we generalize this notion a class of limits of subhomogeneous algebras with a particular diagonal structure. This chapter is laid out as follows. In the first section, we develop the notion of a layering, which are families of open covers associated with the spaces in the construction of a DSH algebra. These also identify certain DSH subalgebras within the larger algebra. We then develop the notion of a stratification of a layering will eventually allow us to reduce the dimension of the base spaces within these subalgebras. In section 2, we show how to compute dimension ratio for a stratification. We also show that, given a diagonal map between two DSH algebra, a layering of the first may be used to generate a layering for the second, and that the corresponding subalgebra in the second then contains the image of the first algebra. We then consider joins of layerings, which are layerings that have larger corresponding subalgebras but which have, we show, stratifications with smaller dimension ratios. In section 3 we define the mean dimension of a diagonal limit of DSH algebras and show that this is in a agreement with previously defined notions of mean dimension. Finally, in section 4 we develop the notion of internal approximations corresponding to stratifications, which give subalgebras with base spaces that are smaller than those which are originally given. We then give the final results concerning $\mathcal{Z}$-stability of simple limits.

3.1 Layerings

Definition 3.1.1. When A is a DSH algebra, a *layering* for A is a disjoint union α of open covers $\alpha(i,j,k)$ where $1 \le i \le l$, $1 \le k < k+j \le n_i+1$, every $U \in \alpha(i,j,k)$ is a subset of X_i and furthermore

1. If $k_1 < k_2 < k_3 < k_4$ and there exist $U_1 \in \alpha(i,k_3-k_1,k_1), U_2 \in \alpha(i,k_4-k_2,k_2)$, then $U_1 \cap U_2$ is empty.

2. For every $U \in \alpha(i,j,k)$, $U \cap \bigcup_{m=k+1}^{k+j-1} B_{i,m}$ is empty.

Each $U \in \alpha$ has co-ordinates $I(U)$,$J(U)$ and $K(U)$ such that $U \in \alpha(I(U),J(U),K(U))$. A refinement β of a layering α is a layering such that for every i,j,k, $\beta(i,j,k)$ is a refinement of $\alpha(i,j,k)$.

We will denote the set of compositions of a number n by $\mathcal{C}(n)$. For a composition $c \in \mathcal{C}(n)$ which we write out as $c = (c_1, \ldots, c_{|c|})$, we define $a_1, \ldots, a_{|c|}$ as $a_j = \sum_{k=1}^{j-1} c_k$. We will require the notion of a tower, which we think of as a finite family of sets stacked on top of one another. For i, j, k, we define the towers of height j at line k in space X_i,

$$T_\alpha(i,j,k) = \bigcup_{c \in \mathcal{C}(j)} \{(U_1, \ldots, U_{|c|}) : U_t \in \alpha(i, c_t, k + a_t)\}.$$

We will suppress the subscript α when there is no ambiguity about the layering in question. If $T = (U_1, \ldots, U_s) \in T_\alpha(i,j,k)$ is a tower, then we write $c(T) = (J(U_1), \ldots, J(U_s))$ and $\pi(T) = \bigcap_{t=1}^{s} U_t$. We write T_α to denote all towers derived from α. A subtower of T is a tower $T(U_m, \ldots, U_n)$ where $1 \leq m \leq n \leq s$ and we write $T' \leq T$.

Given a layering α we define $\alpha|_i$ to be the refinement such that $\alpha|_i(i', j, k) = \alpha(i', j, k)$ for $i' \leq i$ and $\alpha|_i(i', j, k) = \emptyset$ for $i' < i$.

Definition 3.1.2. We say a layering α for A is *full* if for every $x \in X_i$, there exists $T \in T_\alpha(i, n_i, 1)$ such that $x \in \pi(T)$. We say α is *full at stage* i' if it satisfies this condition for every $x \in X_i$ for $i \leq i'$ and $\alpha(i,j,k)$ is empty for every j, k when $i > i'$.

Definition 3.1.3. A *spectral structure (at stage i)* $\mathcal{S}$ for α, a layering, is a family of disjoint sets in α such that each element $U \in \alpha$ (with $I(U) \leq i$) lies in exactly one S, which we denote $S(U)$. If $U, V \in S \in \mathcal{S}$ then $J(U) = J(V)$. Let $S \in \mathcal{S}$ and suppose $y \in Y_{i'}$ (where $i' \leq i$), $\Phi_k(y) = x \in X_{i''}$. Then there exists $U \in S$ with $y \in U$ and $U \in \alpha(i', j, k)$ if and only if there exists $V \in S$ with $x \in V$ and $V \in \alpha(i'', j, k - b(y,k) + 1)$. When this occurs we write $\Phi(U, y) = (V, x)$. If α has a spectral structure (at stage i) then we say α is spectral (at stage i).

Since J takes a common value on S we will refer to this value as $J(S)$. We may assume that for each i, j, k and $S \in \mathcal{S}$ there is at most one $U \in S$ such that $U \in \alpha(i,j,k)$. We do this by taking the union if there are more than one. When $S \in \mathcal{S}$, we let $S(i,k)$ denote the unique element $U \in S$ such that $U \in \alpha(i, J(S), k)$. If no such set exists, we define $S(i,k)$ to be the empty set.

Note that when $y \in \pi(T) \bigcap Y_i$ for $T = (U_1, \ldots, U_s) \in T(i,j,k)$ and $\Phi_{K(U_t)}(y)$ is the same for each $t \in \{1, \ldots, s\}$ then we can identify as $\Phi(T, y)$ the unique tower $T' = (V_1, \ldots, V_s)$ where $\Phi(U_t, y) = (V_t, \Phi_k(y))$.

Our standard procedure for producing a refinement α' that is spectral of a spectral layering α is to iterate through i, and when modifying a set $U \in \alpha(i,j,k)$ we will not alter $U \bigcap Y_i$. However, when we remove a point x from U, we check every set in $V \in S(U)$ and if $\Phi(V, y) = (U, x)$ for some $y \in V$ then we delete this point from V. If α' continues to satisfy the fullness condition at x then it will also continue to do so at y.

For a composition $c \in \mathcal{C}(j)$, define the set of spectral towers

$$ST_\alpha(c) = \{(S_1, \ldots, S_{|c|}) : S_t \in \mathcal{S}, J(S_t) = c_t\}.$$

If $\mathcal{T} = (S_1, \ldots, S_s) \in ST_\alpha(c)$, we write $c(\mathcal{T}) = c$, $J(\mathcal{T}) = j$, and we define $\mathcal{T}(i,k)$ to be $(S_1(i,k), \ldots, S_s(i, k + a_s))$. Finally, we will write ST_α to denote the union of $ST_\alpha(c)$ across all $c \in \bigcup_{j=1}^{n_l} \mathcal{C}(j)$.

We will denote by $M_\alpha(i,j,k)$ the set $\bigcup_{U \in \alpha(i,j,k)} U$. Note that (1) together with fullness implies that each x in each X_i has a unique coarsest composition $c_\alpha(x) \in \mathcal{C}(n_i)$ such that there exists $(V_1, \ldots, V_{|c|}) \in T(i, n_i, 1)$ with $x \in \pi(T)$. When α is full at stage i', for each $i \leq i', k \leq n_i, j \leq n_i - k$, let $L_\alpha(i,j,k)$ be the set of $x \in X_i$ such that j is an entry $c(x)_n$ in $c(x)$ and $c(x)_1 + \cdots + c(x)_{n-1} = k - 1$. Let $W_\alpha(i,j,k) = \overline{L_\alpha(i,j,k)} \setminus L_\alpha(i,j,k)$. We define the set $N_\alpha(i,j,k)$ to be the set of points in X_i such that $x \in M_\alpha(i, j', k')$ such that $k' \leq k < k + j \leq k' + j'$. We will omit the subscripts on these four sets when there is no ambiguity about the layering. Clearly $W(i,j,k) \subseteq L(i,j,k) \subseteq M(i,j,k) \subseteq N(i,j,k)$.

When α is full at stage i', $i \leq i'$ and $T \in T(i,j,k)$, let $\pi'(T) = \pi(T) \bigcap \overline{L(i,j,k)}$.

Definition 3.1.4. A *stratification* α' of a layering α for A that is full and spectral at stage i' consists of open covers $\alpha'(i,j,k)$ of $\overline{L(i,j,k)}$ for each $i \leq i'$ and j,k that refine $\overline{\alpha}(i,j,k) = \{\pi'(T) : T \in T(i,j,k)\}$. For each tower we define an open set $\alpha'_T \subset \pi'(T)$ such that if $x \in W(i,j,k) \bigcap L(i,j',k') \bigcap \pi(T)$ where $T \in T(i,j,k)$ with $k \leq k' < k' + j' \leq k + j$ then $x \in \alpha'_{T'}$ for the unique $T' \leq T$ such that $T' \in T(i,j',k')$. Also, if $\mathcal{T} \in ST_\alpha$, $x \in \overline{L(i, J(\mathcal{T}), k)}$,$y \in \overline{L(i', J(\mathcal{T}), k')}$, $\Phi_k(x) = \Phi_{k'}(y)$ and $k - b(x,k) = k' - b(y,k')$ then $x \in \alpha'_{\mathcal{T}(i,k)}$ if and only if $y \in \alpha'_{\mathcal{T}(i',k')}$.

We denote by $\mathcal{A}(\alpha)$ the set of stratifications α. Note that every full spectral layering has a trivial stratification wherein $\alpha'_T = \pi'(T)$. When $x \in X_i$ we write $\mathrm{ind}(x) = i$.

Definition 3.1.5. A *partition of unity* for a DSH algebra A that is subordinate to a stratification α' is a family of continuous functions $\psi_T : \overline{L(i,j,k)} \to [0,1]$ where $T \in T(i,j,k)$ such that

(i) ψ_T is supported on α'_T for each $T \in T(i,j,k)$.

(ii) If $T \in T(i,j,k)$, $y \in Y_i \bigcap \pi'(T) \bigcap \overline{L(i,j,k)}$, and

$$\Phi_k(y) \in \overline{L(\mathrm{ind}(\Phi_k(y)), j, k - b(y,k) + 1)}$$

then $\psi_T(y) = \psi_{\Phi(T,y)}(\Phi_k(y))$.

(iii) If $x \in W(i,j,k) \bigcap \overline{L(i,j',k')} \bigcap \alpha'_T$ where $T \in T(i,j',k')$ with $k \le k' < k'+j' \le k+j$ then

$$\psi_T(x) = \sum_{T' \in S} \psi_{T'}(x) \tag{3.1}$$

where S is the set of towers in $T(i,j,k)$, which contain T as a subtower.

(iv) For every $x \in \overline{L(i,j,k)}$, $\sum_{T \in T(i,j,k)} \psi_T(x) = 1$.

Definition 3.1.6. We say that a layering α at stage i' has *compatible boundaries* if for every $i \le i'$ and j,k, $\overline{N(i,j,k)} \bigcap Y_i = \overline{N(i,j,k) \cap Y_i}$.

Lemma 3.1.7. *If A is a DSH algebra and α' is a stratification of a finite full spectral layering α with compatible boundaries then A has a partition of unity that is subordinate to α'.*

Proof. We define ψ_T for each $T \in T(i,j,k)$ by increasing i and for each i we begin with $T(i,j,k)$ where j is the largest value for which $L(i,j,k)$ is non-empty for some k. For fixed i, j distinct values of k can be addressed in any order.

Suppose we have defined ψ_T for every $T \in T(i',j',k')$ for all j' and k' when $i' < i$ and all $j' > j$ and all k' when $i = i'$. If $T \in T(i,j,k)$ then ψ_T is determined by conditions (ii) and (iii) on $S = \overline{L(i,j,k)} \bigcap (Y_i \cup \bigcup_{(j',k')\in K} W(i,j',k'))$ where K is the set of (j',k') such that $k' \leq k \leq k+j \leq k'+j'$. We must ensure that these values are consistent. Since S is closed, if the values are consistent then we may always extend each ψ_T to create a partition of unity on that is subordinate to $\alpha'(i,j,k)$.

When $k' \leq k \leq k+j \leq k'+j'$ and $T \in T(i,j,k)$, let $T(i,j',k')[T]$ be the set of towers $T' \in T(i,j',k')$ such that $T \leq T'$.

If $x \in \overline{L(i,j,k)} \bigcap \overline{L(i,j',k')} \bigcap \overline{L(i,j'',k'')} \bigcap \pi(T)$ where $T \in T(i,j'',k'')$ and $k \leq k' \leq k'' < k''+j'' \leq k'+j' \leq k+j$. Then we must show that

$$\psi_T(x) = \sum_{T' \in T(i,j',k')[T]} \psi_{T'}(x) = \sum_{T' \in T(i,j,k)[T]} \psi_{T'}(x). \tag{3.2}$$

Since $x \in W(i,j,k) \bigcap W(i,j',k')$, by assumption, for each $T' \in T(i,j',k')$ we have

$$\psi'_T(x) = \sum_{T'' \in T(i,j,k)[T']} \psi_{T''}(x).$$

Every tower T'' in $T(i,j,k)$ such that $\pi(T'')$ intersects $W(i,j',k')$ must contain a subtower $T' \in T(i,j',k')$ and we therefore have a bijection between $T(i,j,k)[T]$ and the union $\bigcup_{T'\in T(i,j',k')[T]} T(i,j,k)[T']$. Hence

$$\sum_{T'\in T(i,j,k)[T]} \psi_{T'}(x) = \sum_{T'\in T(i,j',k')[T]} \sum_{T''\in T(i,j,k)[T']} \psi_{T''}(x).$$

If $y \in \overline{L(i,j,k)} \bigcap L(i,j',k') \bigcap Y_i \bigcap \alpha'_T$ where $T \in T(i,j',k')$ and $k \leq k' \leq k'+j' \leq k+j$ then we must show that

$$\psi_{\Phi(T,y)}(\Phi_{k'}(y)) = \sum_{T'\in T(i,j,k)[T]} \psi_{T'}(y).$$

The point y is in $\overline{N(i,j,k)} \cap Y_i$, so by compatible boundaries we may take a sequence of points y_n approaching y in $N(i,j,k)\cap Y_i$, and for any $T' \in T(i,j,k)$ such that $\psi_{T'}(y)$ is non-zero, y_n eventually lies in $\pi(T')$. By the definition of $N(i,j,k)$, y_n lies in some $L(i,j_n,k_n)$ where $k_n \leq k < k_j \leq k_n + j_n$. There are only finitely many possibilities for k_n and j_n as n varies, so by taking a subsequence we can assume these sequences are constant with values $k^* = k_n$ and $j^* = j_n$ for all n. We have already checked that $\sum_{T'\in T(i,a,b)[T]} \psi_{T'}(y)$ is the same for any values of a,b such that $y \in \overline{L(i,a,b)}$ and $b \leq k < k+j \leq b+a$. It is enough to show that

$$\psi_{\Phi(T,y)}(x) = \sum_{T'\in T(i,j^*,k^*)[T]} \psi_{T'}(y).$$

$\Phi(T',y_n)$ can only take on finitely many values as n varies, so we may assume we have a subsequence of y_n where this is constant. Suppose that none of these constant subsequences y_{n_t} of y_n gives us $\Phi(T',y_n) = \Phi(T,y)$. This may only occur if $\Phi_k(y_{n_t})$ is a sequence which converges in some $X_{i'}$ to a point in $y' \in Y_{i'}$. Since $y \notin L(i,j,k)$, $y' \notin L(i',j,k-(y,k)+1)$ while $\Phi_k(y_{n_t})$ does lie in this set. Therefore $y' \in W(i',j,k-b(y,k)+1) \bigcap L(i',j',k'-b(y,k)+1)$ and hence $\lim \psi_{\Phi(T',y_n)}(\Phi_k(y_{n_t})) = \psi_{\Phi(T,y)}(y')$. Hence for every $T' \in T(i,j,k)$, $\psi_{T'}(y) = \psi_{\Phi(T,y)}(y')$.

By our inductive hypothesis, $\psi_{\Phi(T,y)}(y') = \sum_{T^* \in S'} \psi_{T^*}(y')$ where S' is the set of $T^* \in T(i',j,k-b(y,k)+1)$ such that $T^* \geq \Phi(T,y)$. Every $T^* \in S'$ such that $\psi_{T^*}(y')$ is nonzero comes from some $T' \in S$ because if $\Phi(T,y) \leq T^*$ then we can compute some T' for which $\Phi(T',y_n) = (T^*, \Phi_k(y_n))$ holds for all n. Hence

$$\psi_{\Phi(T,y)}(y') = \sum_{T^* \in S'} \psi_{T^*}(y') = \sum_{T' \in S} \psi_{T'}(y).$$

Hence the values are consistent.

Now we can obtain for each k functions ψ_T for each $T \in T(i,j,k)$. □

A refinement α'' of a stratification α' of a full spectral layering is a stratification such that $\alpha''(i,j,k) \succ \alpha'(i,j,k)$ for each i,j,k.

Corollary 3.1.8. *Any $\gamma \in \mathcal{A}(\alpha)$ for a full spectral layering α has a refinement $\alpha' = \{\alpha'_T : T \in T_\alpha\}$ such that whenever $T \in T(i,j,k)$ and $x \in \alpha'_T \bigcap W(i,j',k')$ for some $k' \leq k < k+j \leq k'+j'$, there exists a $T' \in T(i,j',k')$ such that $T < T'$ and $x \in \alpha'_{T'}$.*

Proof. When α has compatible boundaries we may construct a partition of unity $\{\psi_T : T \in T_\alpha\}$ subordinate to γ and simply let $\alpha'_T \in \alpha'$ be equal to the support of ψ_T.

First note that since Y_1 is always empty, $\alpha|_i$ always has compatible boundaries, and therefore we may assume the desired property for γ holds here. We may then pass to a refinement α'' on which it holds for any $y \in Y_2$. We may then construct a partition of unity subordinate to α'' defined only for towers in $T(2, j, k)$ where we do not require condition (ii) of a partition of unity. We can then pass to a refinement α'' on which the property holds for any $x \in X_2$. We can then iterate this procedure and the result follows. □

When α is a layering and $1 \leq k_1 < k_2 \leq n_i$, let

$$L[i, k_1, k_2] = \bigcup_{j_1, j_2} (L(i, j_1, k_1) \bigcap L(i, j_2, k_2)).$$

When $x \in L[i, k_1, k_2]$, let $c(x)[k_1, k_2]$ be the composition in $\mathcal{C}(k_2 - k_1)$ such that $c(x)$ decomposes into compositions $c_1, c(x)[k_1, k_2], c_2$ where $c_1 \in \mathcal{C}(k_1)$ and $c_2 \in \mathcal{C}(n_i - k_2)$.

Suppose α is a full spectral layering of a DSH algebra A and fix $J \in \mathbb{N}$. Let c be a composition in $c(J)$ define $\alpha[c]$ be the set of spectral layerings β with compatible boundaries such that for any i, k, $\beta(i, j, k)$ is only non-empty for $j = J$ and $\beta(i, J, k)$ consists of at most one non-empty set $U_{i,k} \subseteq L_\alpha[i, k, k + J]$ such that for any $x \in U_{i,k}$, $c(x)[k, k + J] \in \{c, (J)\}$. We construct a full layering α^β with spectral structure $\mathcal{S}^\beta$. We call α^β a *direct join* of α. We first set $\alpha^\beta(i, j, k) = \alpha(i, j, k)$ whenever $j \neq J$. We set

$$\alpha^\beta(i, J, k) = \alpha(i, J, k) \bigcup \{\pi(T) \bigcap U_{i,k} : T \in T(i, n_1, k), c(T) = c\}$$

and

$$\mathcal{S}^\beta = \mathcal{S} \cup \bigcup_{\mathcal{T} \in \mathcal{T}_\alpha(c)} S_{\mathcal{T}}$$

where $S_{\mathcal{T}} = \bigcup_{i,k} \pi(\mathcal{T}(i, k)) \bigcap U_{i,k}$.

Lemma 3.1.9. *For any full spectral layering α of a DSH algebra A, $J \in \mathbb{N}$, $c \in \mathcal{C}(J)$, and any $\beta \in \alpha[c]$, α^β is a full spectral layering and $\mathcal{S}^\beta$ is a spectral structure for α^β. If α has compatible boundaries then so does α^β.*

Proof. Since α is full and α^β contains α, it follows that α^β is full. Furthermore, α^β is a refinement of $\alpha \bigcup \beta$, and since each satisfies (2) and the union satisfies (1), it follows that

α^β satisfies (2) and (1). Every element $U \in \alpha^\beta(i,J,k)\backslash\alpha$ comes from some $T \in T_\alpha(i,J,k)$. $T = \mathcal{T}(i,k)$ for some $\mathcal{T}$, so U lies in $S_\mathcal{T} \in \mathcal{S}^\beta$. Furthermore, if $y \in Y_i \bigcap U$ then $y \in \pi(T)$ and therefore there exist T' such that $\Phi(T,y) = T'$. Since β satisfies (2), it follows that $\Phi_k(y) = \Phi_{k+j}(y)$ for $j < J$. We call this point x, and the spectral nature of β ensures that $x \in \Phi(U_{i,k}, y) \in \beta$ and therefore $x \in \Phi(U_{i,k}, y) \bigcap \pi(T)'$, which again lies in $S_\mathcal{T}$. This ensures that α^β is spectral. Assume that α has compatible boundaries. If $j > J$ then $N_{\alpha^\beta}(i,j,k) = N_\alpha(i,j,k)$ for all i,k, hence each of these satisfies the compatible boundary condition. For $j = J$, $N_{\alpha^\beta}(i,j,k) = N_\alpha(i,j,k) \bigcup U_{i,k}$. Therefore if $y \in \overline{N_{\alpha^\beta}(i,j,k)} \bigcap Y_i$ then either $y \in \overline{N_\alpha(i,j,k)}$ or $y \in \overline{U_{i,k}}$. It follows that $y \in \overline{N_{\alpha^\beta}(i,j,k) \bigcap Y_i}$. For $j < J$, $N_{\alpha^\beta}(i,j,k) = N_\alpha(i,j,k) \cup \bigcup_{k'=k+j-J}^{k} U_{i,k'}$. If $y \in \overline{N_{\alpha^\beta}(i,j,k)} \bigcap Y_i$ and $y \notin \overline{N_{\alpha^\beta}(i,j,k) \bigcap Y_i}$ then $y \in \overline{U_{i,k'}}$ for some $k+j-J \leq k' \leq k$. But by the compatible boundaries of β, this implies that $y \in \overline{N_{\alpha^\beta}(i,j,k) \bigcap Y_i}$. Hence α^β has compatible boundaries. □

We define the set of joins, $\vee(\alpha)$, of a full spectral layering α to be the smallest class of full spectral layerings which contains α and is closed under taking direct joins.

For any $F \subset X_i$, define $c(F)$ to be $\{c \in \mathcal{C}(n_i) : c(x) = c \text{ for some } x \in F\}$. Observe that $(n_i) \notin c(Y_i)$.

Lemma 3.1.10. *If α is a full spectral layering for A with compatible boundaries then $\vee(\alpha)$ contains a spectral layering β such that the smallest j to appear as an entry in $c_\beta(x)$ for any x in any X_i is n_1.*

Proof. Suppose that α is full and spectral with compatible boundaries and for a fixed i, the smallest j to appear as an entry in $c_\beta(x)$ for some x in some $X_{i'}$ with $i' < i$ is n_1. We will construct a layering in $\vee(\alpha)$ such that this also holds for $i' = i$.

Since every point $y \in Y_i$ has its composition $c(y)$ determined by the points in $\Phi(y)$, it follows that $c_\alpha(y)$ contains no entry $j < n_1$. We can then find an open set V containing Y_i such that no composition in $c_\alpha(\overline{V})$ contains an entry $j < n_1$.

For any $c \in C(n_i)$, we define $L_c = \{x \in X_i : c_\alpha(x) = c\}$. Let $c = (c_1, \ldots, c_{|c|})$ be a proper composition in $\mathcal{C}(n_i)$ in $c(X_i)$ such that the only coarser composition in $c(X_i \setminus \overline{V})$ is (n_i). Then the set $L_{(n_i)} \bigcup L_c$ is open. Then we will construct a spectral layering in $\gamma \in \alpha[c]$ by first letting $U_{i,1} = (L_{(n_i)} \bigcup L_c) \setminus \overline{V}$. Once this is complete, the set $c_{\alpha^\gamma}(X_i \setminus \overline{V})$ will be equal to $c_\alpha(X_i \setminus \overline{V}) \setminus \{c\}$, hence under iteration we will obtain a join β for which $c_\beta(X_i \setminus \overline{V}) = \{(n_i)\}$ and this has the desired property.

Suppose we have a γ that is spectral at stage i' such that for any $i'' < i'$,k, $\beta(i'', j, k)$ is only non-empty for $j = J$ and $\beta(i'', J, k)$ consists of at most one non-empty set $U_{i'',k} \subseteq L_\alpha[i'', k, k+J]$ such that for any $x \in U_{i'',k}$, $c(x)[k, k+J] \in \{c, (J)\}$ and $U_{i,1} = (L_{(n_i)} \bigcup L_c) \setminus \overline{V}$. Then for each k, define $V_{i',l}$ to be the set of points $y \in Y_i$ such that $\Phi_k(y) \in U_{I(\Phi_k(y)),k'}$ where $k' = k - b(y, k) + 1$. Then, whenever $y \in V_{i,k}$, $y \in L_\alpha[i', k, k+n_i]$ and $c(y)[k, k+n_i] \in \{c, (n_i)\}$. $V_{i,k}$ is open in Y_i, and since α has compatible boundaries, we may find an open set $U_{i,k}$ that is contained in $L_\alpha[i', k, k+n_i]$, for which $c(x)[k, k+n_i] \in \{c, (n_i)\}$ for every $x \in U_{i',k}$ and furthermore $U_{i',k} \bigcap Y_{i'} = V_{i',k}$. We can to this for each k and furthermore guarantee that U_{i',k_1} and U_{i',k_2} are disjoint whenever $|k_1 - k_2| < n_i$, which guarantees that γ satisfies (1). γ is then a spectral layering with compatible boundaries in $\alpha[c]$. $\square$

3.2 Dimension Ratio

When α is an open cover of a space X, define

$$\operatorname{ord}(\alpha) = -1 + \max_{x \in X} |\{U \in \alpha : x \in U\}| .$$

When an open cover β refines α we will write $\beta \succ \alpha$. We may then define $\mathcal{D}(\alpha) = \min_{\beta \succ \alpha} \mathrm{ord}(\beta)$. For a pair of covers α,β we define $\alpha \wedge \beta = \{U \bigcap V : U \in \alpha, V \in \beta\}$. We then have that $\mathcal{D}(\alpha \vee \beta) \leq \mathcal{D}(\alpha) + \mathcal{D}(\beta)$. We write $\mathcal{C}(X)$ to denote the open covers of X.

When $\alpha \in \mathcal{C}(X)$, a map $f : X \to Y$ is α-compatible if Y has an open cover β such that for every $U \in \beta$ there is a $V \in \alpha$ such that $f^{-1}(U) \subseteq V$. We then have the following result from [13].

Lemma 3.2.1. *When $\alpha \in \mathcal{C}(X)$, $\mathcal{D}(\alpha) \leq d$ if and only if there exists an α-compatible map from X into a d-dimensional simplex.*

For a DSH algebra A, we can define the dimension ratio of A as $\mathrm{dimRatio}(A) = \max_i \mathrm{dim}(X_i)/n_i$. We also give a natural notion of a dimension ratio for α', a stratification of a full spectral layering for A. We write $\mathrm{dimRatio}(\alpha') = \max_{i,j,k} \mathrm{ord}(\alpha'(i,j,k))/j$.

Lemma 3.2.2. *Suppose α is a full spectral layering with compatible boundaries and $\gamma \in \vee(\alpha)$ and $\alpha' \in \mathcal{A}(\alpha)$. Then γ has a stratification with* $\mathrm{dimRatio}(\gamma') \leq \mathrm{dimRatio}(\alpha')$.

Proof. It is sufficient to prove this when γ is a direct join α^{β} for some J, $c = (c_1, \ldots, c_s) \in \mathcal{C}(J)$ and spectral layering $\beta \in \alpha[c]$.

When $j \neq J$, both α and α' give us open covers of the sets $L_{\gamma}(i,j,k)$. If $T \in T(i,j,k)$ then let $[T] = \pi(T) \bigcap L_{\gamma}(i,j,k)$ and $[T]' = \alpha'_T$.

We construct a partition of unity subordinate α'. For $j < J$, we then have maps $p[i,j,k] : L_{\gamma}(i,j,k) \to \Gamma(i,j,k)$ where $\Gamma(i,j,k)$ is $\mathrm{ord}(\alpha'(i,j,k))$-dimensional simplicial complex. For $j = J$, we construct a map $p'[i,j,k] : L_{\alpha}(i,j,k) \to \Gamma'(i,j,k)$ where $\Gamma'(i,j,k)$ is simplex of dimension $\mathrm{ord}(\alpha'(i,j,k))$.

Define $Z_{i,k} = \overline{U_{i,k} \setminus L_{\alpha}(i,J,k)}$.

We have a total of $2s$ open covers of $Z_{i,k}$ that come from α' and α, for each $T \in T(i,J,k)$ we can write $T = (T_1, \ldots, T_s)$ where $T_t \in T(i, c_t, k + a_t)$, we label $\pi(T_t) \bigcap Z_{i,k}$

as $[T]_t$ and $\alpha'_{T_t} \bigcap Z_{i,k}$ as $[T]'_t$. We then define $\beta(t)_{i,k}$ to be the set of sets $[T]_t$ and $\beta(t)'_{i,k}$ to be the set of $[T]'_t$.

Using our partitions of unity, we have s maps $q(t)_{i,k} : Z_{i,k} \to \Lambda(t)_{i,k}$ for $t = 1, \ldots, s$ where $\Lambda(t)_{i,k}$ is a $\mathrm{ord}(\beta(t)'_{i,k})$ dimensional simplex and $q(t)_{i,k}$ is a restriction of $p[i, c_t, k + a_t]$. We combine these into a map $q_{i,k} = q(1)_{i,k} \times \cdots \times q(s)_{i,k} : Z_{i,k} \to \Lambda_{i,k} = \Lambda(1)_{i,k} \times \ldots \times \Lambda(s)_{i,k}$.

If $x, y \in Z_{i,k} \bigcap \overline{W_\alpha(i, J, k)}$ and $p'[i, J, k](x) = p'[i, J, k](y)$ then $\psi_T(x) = \psi_T(y)$ for every $T \in T(i, J, k)$. Then Eq.(3.1) implies that $p[i, c_t, k + a_t](x) = p[i, c_t, k + a_t](y)$ for all t and therefore $q_{i,k}(x) = q_{i,k}(y)$.

We may define a map

$$s[i, k] : p'[i, J, k](Z_{i,k} \bigcap \overline{L_\alpha(i, J, k)}) \to \Gamma(i, J, k)$$

such that $q_{i,k} = s[i, k] \circ p'[i, J, k]$.

We define a map $p[i, J, k] : L_\gamma(i, J, k) \to \Gamma(i, J, k)$ where $\Gamma(i, J, k)$ is the quotient space of $\Gamma'(i, J, k) \bigcup \Lambda_{i,k}$ formed by identifying points in the domain of $s[i, k]$ with their images such that $p[i, J, k]$ agrees with $s[i, k] \circ p'[i, J, k]$ and $q_{i,k}$ on their domains. The space $\Gamma(i, j, k)$ then has dimension less than or equal to $\max(\sum_{t=1}^{s} \mathrm{ord}(\beta(t)'_{i,k}), \mathrm{ord}(\alpha'(i, j, k)))$.

Suppose $x, y \in Y_i \bigcap L_\gamma(i, j, k)$, $\mathrm{ind}(\Phi_k(x)) = \mathrm{ind}(\Phi_k(y)) = i'$ and $b(x, k) = b(y, k)$ and $p[i, j, k](x) = p[i, j, k](y)$. Let $k' = k - b(x, k) + 1$. Then $\Phi_k(x), \Phi_k(y) \in L_\gamma(i', j, k')$ and for every $T \in T(i, j, k)$, $\psi_T(x) = \psi_T(y)$. Hence $\psi_{\Phi(T,x)}(\Phi_k(x)) = \psi_{\Phi(T,y)}(\Phi_k(y))$ so

$p[i',j,k'](\Phi_k(x)) = p[i',j,k'](\Phi_k(y))$. Therefore we may define a map

$$\Phi_{i,j,k}^{i',k'} : p[i,j,k](Y_i) \to \Gamma(i',j,k')$$

such that

$$p[i',j,k'] \circ \Phi_k(x) = \Phi_{i,j,k}^{i',k'} \circ p[i,j,k](x).$$

If $T \in T(i,j,k)$ and $\Phi(T,x) = T'$ then we write $\Phi'(T, p[i,j,k](x)) = T'$.

We will now construct a stratification γ' for γ with the desired properties. For $j = c_t \in \{c_1, \ldots, c_s\}$, $T \in T(i,j,k)$, let

$$\gamma'_T = \alpha'_T \setminus \bigcup_{k'=k+j-J}^{k} U_{i,k'-a_{t'}}.$$

For other values of $j < J$ and every value $j > J$, $T \in T(i,j,k)$, let $\gamma'_T = \alpha'_T$.

We will first construct open covers on the space $\Gamma(i,J,k)$ and then pull these back to obtain γ'.

For each tower in $T_\gamma(i,J,k)$, we define a set $g(T) \subseteq L_\gamma(i,j,k)$. If $[T] \subseteq L_\alpha(i,j,k)$ then let $g(T) = [T]'$. Otherwise $T = (T_1, \ldots, T_s)$ as above, and we write $g(T) = \bigcap_{t=1}^{s} [T_t]'_t$. We call the cover composed of these sets $G(i,j,k)$. We can then show that that map $p[i,j,k]$ is $G(i,j,k)$-compatible. With this in hand, we may iteratively construct in terms of increasing i covers $G'(i,j,k)$ and refinements $G''(i,j,k)$ of $\Gamma(i,j,k)$ composed of sets $g'(T)$ and $g''(T)$ for $T \in T(i,j,k)$. Suppose that for all $i' < i$ and all values of k' we have defined $G'(i',j,k')$ and $G''(i',j,k')$ and that $\mathrm{ord}(G''(i',j,k')) \leq \dim(\Gamma(i',j,k'))$. Suppose further that if $i_1, i_2 < i$, $T_1 \in T(i_1,j,k)$, $x \in \Gamma(i_1,j,k)$ and $\Phi'(T_1,x) = T_2$ then $x \in g''(T_1)$

if and only if $\Phi_k(x) \in g''(T_2)$.

Then for each $T \in T(i,j,k)$ we define $g'(T)$ to be the largest subset of $\{x \in \Gamma(i,j,k) : p[i,j,k]^{-1}(x) \subseteq g(T)\}$ such that if $x \in p[i,j,k](Y_i)$ and $\Phi'(T,x) = T' \in T(i',j,k')$ then $\Phi^{i',k'}_{i,j,k}(x) \in g'(T')$. We then define $g'(i,j,k)$ to be the cover composed of the sets $g'(T)$.

If $x \in p[i,j,k](Y_i)$ and $x \in g'(T)$ and $T' = \Phi(T,x) \in T(i',k')$ then $\Phi^{i',k'}_{i,j,k}(x) \in g''(T')$, so

$$|\{U \in g'(i,j,k) : x \in U\}| - 1 \leq \max_{i',k'} \dim(\Gamma(i',j,k')).$$

We can then pass to a refinement of $G'(i,j,k)$ on which this holds on a neighborhood U of $p[i,j,k](Y_i)$. We can then refine again on the complement of U to obtain $G''(i,j,k)$ such that $\mathrm{ord}(G''(i,j,k)) \leq \max_{i',k'} \dim(\Gamma(i',j,k'))$. Since we have not modified $G'(i,j,k)$ on $p[i,j,k](Y_i)$, it follows that if $i' < i$, $T \in T(i,j,k)$, $x \in \Gamma(i,j,k)$ and $\Phi'(T,x) = T'$ then $x \in g''(T)$ if and only if $\Phi_k(x) \in g''(T')$. Hence we can construct $G'(i,j,k)$ for every i,j,k.

Finally, we pull back each of the covers $G''(i,j,k)$ using $p[i,j,k]$ to a cover $\gamma'(i,j,k)$ on $L_\gamma(i,j,k)$. We can check that these form a stratification γ' of γ. For $j < J$, $\mathrm{ord}(\gamma'(i,j,k)) \leq \mathrm{ord}(\alpha'(i,j,k))$. For $j = J$,

$$\mathrm{ord}(\gamma'(i,j,k)) \leq \max(\mathrm{ord}(\alpha'(i,j,k)), \sum_{t=1}^{s} \mathrm{ord}(\beta'_t(i,c_t,k+a_t))).$$

For $j > J$, $\mathrm{ord}(\gamma'(i,j,k)) = \mathrm{ord}(\alpha'(i,j,k))$. We then have that

$$\mathrm{dimRatio}(\gamma') \leq \mathrm{dimRatio}(\alpha')$$

. □

We give the dimension ratio for a full spectral layering as

$$\text{dimRatio}(\alpha) = \inf_{\alpha' \in \mathcal{A}(\vee(\alpha))} \text{dimRatio}(\alpha').$$

For an $n \times n$ matrix B, let $P[n, j, k](B)$ be the $j \times j$ cutdown onto rows and columns k to $k + j - 1$.

Definition 3.2.3. When F is a finite subset of A and $\epsilon > 0$, we say that α, a full spectral layering at stage i is F, ϵ-compatible if for any $f \in F$ there exists an f' with $|f - f'| < \epsilon$ and

- for any $U, V \in S \in \mathcal{S}$, and any $x \in U$, $y \in V$, we have

$$\left\| P[n_{I(U)}, J(U), K(U))](f'_{I(U)}(x)) - P[n_{I(V)}, J(V), K(V)](f'_{I(V)}(y)) \right\| < \epsilon$$

- and for every $U \in \alpha$, $x \in U$,

$$P[n_{I(U)}, J(U), K(U)](f'_{I(U)}(x)) - (f'_{I(U)}(x))P[n_{I(U)}, J(U), K(U)] = 0.$$

We will consider $\varnothing$ to be a full spectral layering at stage 0.

Lemma 3.2.4. *Suppose we have a DSH algebra A, finite set $F \subset A$, $\epsilon > 0$, $i \in \{1, \ldots, l-1\}$ and α, a full spectral layering at stage i with compatible boundaries that is F, ϵ-compatible and let $\alpha' \in \mathcal{A}(\alpha)$. Then there exists a full spectral layering at stage $i + 1$, β with compatible boundaries that is F, ϵ-compatible and $\beta|_i = \alpha$. Furthermore* *there exists* $\beta' \in \mathcal{A}(\beta)$ *such that*

$$\mathrm{dimRatio}(\beta') \leq \max(\mathrm{dimRatio}(\alpha'), \dim(X_{i+1})/n_{i+1}).$$

Proof. Suppose we have α, a full spectral layering at stage i with compatible boundaries and a stratification α' consisting of a set α'_T for each $T \in T(i', j, k)$ for any $i' \leq i$ and any j, k.

For $k \in \{1, \ldots, n_i\}$ and some $S \in \mathcal{S}$, we define $V_{S,k}$ to be the set of points

$$\bigcup_{U \in S} \{y \in Y_{i+1} : \Phi_k(y) \in U \text{ and } K(U) = k - b_{i,k}(y) + 1\}.$$

This set is easily seen to be open in Y_{i+1}.

Since α is F, ϵ-compatible we have for each $f \in F$ an appropriate $f' \in A$. It follows that if $J(S) = j$ then

$$P[n_{i+1}, j, k](f'_{i+1}(x)) - (f'_{i+1}(x))P[n_{i+1}, j, k] = 0$$

for $x \in V_{S,k}$. Since $|f - f'| < \epsilon$ we may find f'' which agrees with f' on $X_{i'}$ for $i' < i+1$ such that $|f'' - f'| < \epsilon - |f - f'|$ and f'' commutes with $P[n_{i+1}, j, k]$ everywhere in an open set $O_{S,k}$ in X_{i+1} that contains $V_{S,k}$. Let F'' be the set of f''.

By assumption, for any $U, V \in S$, and any $x \in U$, $y \in V$, we have

$$\left\| P[n_{I(U)}, J(U), K(U)](f_{I(U)}(x)) - P[n_{I(V)}, J(V), K(V)](f_{I(V)}(y)) \right\| < \epsilon$$

for every $f \in F''$. Therefore when $x \in U \in S$ and $y \in V_{S,k}$,

$$\left\| P[n_{I(U)}, J(S), K(U)](f_{I(U)}(x)) - P[n_{i+1}, J(S), k] f_{i+1}(y) \right\| < \epsilon$$

so there exists an open set $U_{S,k} \subseteq O_{S,k}$ such that $V_{S,k} = U_{S,k} \bigcap Y_i$ and the above holds for all $y \in U_{S,k}$ and $f \in F''$. We may then construct γ, a well-drawn spectral layering at stage i with spectral structure $\mathcal{S}''$ by first setting $\gamma(i', j, k) = \alpha(i', j, k)$ for each $i' \leq i$ and j, k. For $j < n_i$ we let $\gamma(i+1, j, k) = \bigcup_{S \in \mathcal{S}: J(S)=j} U_{S,k}$ and we let $\gamma(i+1, n_i, k)$ be empty for each k. Finally, $S'' = \{S \cup \bigcup_k U_{S,k} : S \in \mathcal{S}\}$. Note that γ is F, ϵ-compatible.

Having done this, we consider the towers $T_\gamma(i+1, j, k)$. We may now compute for every j, k and $T \in T_\gamma(i+1, j, k)$ a set

$$V_T = \{y \in \pi(T) \cap Y_{i+1} : y \in \pi(T), \Phi(T, y) = (T', x) \text{ for some } T' \in T_\alpha, x \in \alpha'_{T'} \in \alpha'\}.$$

We now construct sets α'_T of which we require several properties.

(i) $\alpha'_T \subseteq \pi'(T)$.

(ii) $\overline{\alpha'_T} \bigcap Y_i = \overline{V_T}$.

(iii) For any s and any choice of $T_1, \ldots, T_s$, if the sets $V_{T_1}, \ldots, V_{T_s}$ have no common intersection, neither do $\alpha'_{T_1}, \ldots, \alpha'_{T_s}$.

Note that if we delete some relatively closed subset of any α'_T that does not intersect Y_{i+1} then (ii) and (iii) continue to be satisfied. We may divide Y_{i+1} into three sets: $D = V_T$, $\overline{V_T} \setminus V_T$ and $E = Y_{i+1} \setminus \overline{V_T}$. For some compatible metric ρ on X_{i+1}, let

$\alpha'_T \subseteq X_{i+1}$ be the set $\{x \in X_{i+1} : \rho(x, D) < \rho(x, E)\} \bigcap \pi(T)$. This set satisfies (ii).

Furthermore, if P is some set of towers such that a given V_{T^*} is disjoint from $\bigcap_{(T)\in P} V_T$, then we will show that α'_{T^*} is disjoint from $\bigcap_{(T)\in P} \alpha'_T$. Since Y_{i+1} is compact, every point x in α'_T satisfies the condition that all the points in Y_i of minimum distance from x are in V_T. Therefore $x \in \bigcap_{T\in P} \alpha'_T$ only if all the points in Y_{i+1} of minimum distance from x are in $\bigcap_{T\in P} V_T$. Since this intersection is disjoint from V_{T^*}, x cannot lie in α'_{T^*}. Therefore our choice of sets satisfies (iii).

Let $T_1, \dots, T_t$ be a set of towers in $T_\gamma(i+1, j, k)$ and suppose that $\bigcap_{m=1}^{t} \alpha'_{T_j}$ is non-empty. Then by (iii), $\bigcap_{m=1}^{t} V_{T_j}$ is non-empty. Let y be some point in this set, and let T'_j be the unique tower such that $\Phi(T_j, y) = (T'_j, \Phi_k(y))$. Then $x \in \bigcap_{m=1}^{t} \alpha'_{T'_j}$. It follows then that $\text{ord}(\alpha'(i+1, j, k)) \leq \max_{i'<i+1,k'} \text{ord}(\alpha'(i', j, k'))$.

We may readily verify that fullness and (2) are satisfied by $\bigcup_{S,k} U_{S,k}$ on Y_{i+1}. There is then an ϵ' such that whenever $\rho(x, Y_{i+1}) \leq \epsilon'$, fullness is satisfied at x. Let S be the set $\{x \in X_{i+1} : \rho(x, Y_{i+1}) = \epsilon'\}$. We then construct a dense sequence of points $x(n)$ in $X_{i+1} \backslash B(Y_{i+1}, \epsilon)$ and for each $x(n)$ we find a set $U_{x(n)}$ on which $\|f_{i+1}(x) - f_{i+1}(x(n))\| < \epsilon$ for all $x \in U_x$. We can then take a finite subset $N \subset \mathbb{N}$ such that $\bigcup_{n\in N} U_{x(n)} = X_{i+1}$. We then construct a new layering β by setting $\beta(i, n_i, 1) = \{U_{x(n)} : n \in N\}$ and $\beta(i', j, k) = \gamma(i', j, k)$ for all other values of i', j, k. Let $\mathcal{S}' = \mathcal{S}'' \cup \bigcup_n \{U_{x(n)}\}$. We set $J(U_{x(n)}) = n_{i+1}$. We now find that β satisfies (1) and (2), is full and spectral at stage $i+1$ and is again F, ϵ-compatible

We may now compute $L_\beta(i+1, j, k)$, $W_\beta(i+1, j, k)$ and $T_\beta(i+1, j, k)$ for all j, k. We find that $L(i+1, n_{i+1}, 1) = X_{i+1} \backslash \overline{B(Y_{i+1}, \epsilon)}$. By our choice of ϵ, every point in $W(i+1, n_{i+1}, 1)$ lies in $\pi(T)$ for some tower $T \in T(i+1, n_{i+1}, 1)$. Therefore $\{\pi(T) : T \in T(i+1, n_{i+1}, 1)\}$

is an open cover of $X_{i+1} \setminus B(Y_{i+1}, \epsilon)$. We may choose $\beta'(i+1, n_{i+1}, 1)$ to be an arbitrary subcover with $\mathrm{ord}(\beta'(i+1, n_{i+1}, 1)) \leq \dim(X_{i+1})$. We have chosen sets α'_T for every tower $T \in T(i+1, j, k)$ for $j < n_{i+1}$ and by (iii) we have that $\mathrm{ord}(\beta'(i+1, j, k)) \leq \max_{i'<i+1,k}\mathrm{ord}(\alpha'(i', j, k')$. Therefore we set $\beta'(i+1, j, k) = \{\alpha'_T : T \in T(i+1, j, k)\}$ and we set $\beta'(i', j, k) = \alpha(i', j, k)$ whenever $i' < i+1$. Therefore

$$\mathrm{dimRatio}(\beta') = \max(\mathrm{dimRatio}(\alpha'), \dim(X_{i+1})/n_{i+1}).$$

Finally, each set in $\beta(i+1, j, k)$ was chosen so that $\overline{U} \bigcap Y_{i+1} = \overline{U \bigcap Y_{i+1}}$. Therefore β has compatible boundaries. □

Corollary 3.2.5. *For any DSH algebra A, finite subset $F \subset A$, $\epsilon > 0$ there exists a full spectral layering α that is F, ϵ-compatible with a stratification α' such that* $\mathrm{dimRatio}(\alpha') \leq \mathrm{dimRatio}(A)$.

We say that a map $\phi_{j_2,j_1} : A_{j_1} \to A_{j_2}$ between DSH algebras is diagonal if for every $x \in X_i^{j_2}$ there are points $x_1, \ldots, x_t$, where $x_k \in X_{i_k}^{j_1}$ such that for every $f \in A_{j_1}$, $\phi_{j_2,j_1}(f)_i(x) = \mathrm{diag}(f_{i_1}(x_1), \ldots, f_{i_t}(x_t))$.

Suppose A and B are DSH and α is a full spectral layering for A and $\phi : A \to B$ is a diagonal map. Let A be constructed on spaces $X_1, \ldots, X_r$ with dimensions m_i boundary space Y_i and B be constructed on $Z_1, \ldots, Z_s$ with dimensions n_i and boundary space W_i. We may construct a full spectral layering $\phi(\alpha)$ with spectral structure $\phi(\mathcal{S})$ for B in the following manner. For and each i, k we may construct a continuous function $\widehat{\phi}_k(z) : Z_i \to \widehat{A}$ by way of the pointwise decomposition. We define $\widehat{\phi}_k(z)$ so that it is never a point in any boundary space Y_i. In particular, if $f \in A$ and $z \in Z_i \setminus W_i$ then $\phi(f)(z) = \mathrm{diag}(f(x_1), \ldots, f(x_a))$ where $x_j \in X_{i_j}$ and $k = m_{i_1} + \cdots + m_{i_t} + \widehat{k}(z,k)$ for $0 < \widehat{k}(z,k) \leq m_{i_{t+1}}$ then $\widehat{\phi}_k(z) = x_{t+1}$. $\widehat{k}(z,k)$ depends on z and k and we may extend both $\widehat{k}$ and $\widehat{\phi}_k$ to $w \in W_i$ by defining $\widehat{k}(w,k) = \widehat{k}(\Phi_k(w), k - b(w,k) + 1)$ and $\widehat{\phi}_k(w) = \widehat{\phi}_{k-b(w,k)+1}(\Phi_k(w))$. We also write $\widehat{i}(z,k) = \mathrm{ind}(\widehat{\phi}_k(z))$. For each i, k and $C \in S$, we define

$$U(S, i, k) = \{z \in Z_i : \widehat{\phi}_k(z) \in S(\widehat{i}(z,k), \widehat{k}(z,k))\}.$$

We set $J(U(S,i,k)) = J(S)$. For each $S \in \mathcal{S}$, we create a corresponding set $\phi(S)$ composed of the sets $U(S,i,k)$ and we call the collection of these $\phi(\mathcal{S})$.

We additionally wish to push forward stratifications. This is somewhat complicated by the fact that when $z \in \overline{L_{\phi(\alpha)}(i,j,k)}$ it does not follow that $\widehat{\phi}_k(z) \in \overline{L_\alpha(\widehat{i}(z,k),j,\widehat{k}(z,k))}$. When $\mathcal{T} = (S_1,\dots,S_t) \in ST_\alpha$, we set $\pi(\mathcal{T}) = (\phi(S_1),\dots,\phi(S_t))$ and observe that this is a spectral tower in $ST_{\phi(\alpha)}$. For any $\alpha' \in \mathcal{A}(\alpha)$ and each $\mathcal{T} \in ST_\alpha$, we first define

$$\tilde{U}(\mathcal{T},i,k) = \{z \in L(i,J(\mathcal{T}),k) : \widehat{\phi}_k(z) \in \alpha'_{\mathcal{T}(\widehat{i},\widehat{k})}\}.$$

We then define $\phi(\alpha')_{\mathcal{T}(i,k)}$ the set of points $x \in \overline{L_{\phi(\alpha)}(i,j,k)}$ such that there exists an open set $U \subseteq \overline{L_{\phi(\alpha)}(i,j,k)}$ containing x such that $U \bigcap L_{\phi(\alpha)}(i,j,k) \subseteq \tilde{U}(\mathcal{T},i,k)$.

Lemma 3.2.6. *$\phi(\alpha)$ is a full spectral layering.* $\mathrm{dimRatio}(\phi(\alpha)) \leq \mathrm{dimRatio}(\alpha)$.

Proof. If $\Phi_k(w) = z \in Z_j$ and $x \in U(S,i,k)$ for some $S \in \mathcal{S}$ then by definition $\widehat{\phi}_k(w) \in S(j',\hat{k}(w,k))$ for some j', $\widehat{\phi}_k(w) = \widehat{\phi}_{k-b(w,k)+1}(z)$, and $\hat{k}(w,k) = \hat{k}(z,k-b(w,k)+1)$. Therefore $\widehat{\phi}_{k-b(w,k)+1}(z) \in S(j',\hat{k}(w,k-b(w,k)+1))$ so $z \in U(C,i,k-b(w,k)+1)$. Hence $\phi(\alpha)$ is a spectral layering.

It suffices to prove fullness on $Z_i \setminus W_i$. If $\phi(f)(z) = \mathrm{diag}(f(x_1),\dots,f(x_a))$ where $x_j \in X_{i_j}$, then for each x_j there is a tower $(U_{j,1},\dots,U_{j,s_{c(j)}}) \in T(i_j,m_{i_j},1)$ such that $x_j \in \bigcap_{t=1}^{s_{c(j)}} U_{j,t}$ and $c(j) \in \mathcal{C}(m_{i_j})$. Then for each j,t there is a corresponding set $V_{j,t} \in \phi(\alpha)(i,J(U_{j,t}),K(j,t))$ where $K(j,t) = m_{i_1} + \cdots m_{i_{j-1}} + a_t + 1$ such that

$$(V_{1,1},\dots,V_{1,s_{c(1)}},V_{2,1},\dots,V_{a,s_{c(a)}}) \in T(i,n_i,1).$$

Therefore fullness is satisfied at z. Suppose $z \in V \in \phi(\alpha)(i,j,k)$ and $k+j > n_i+1$. We have a decomposition such that $\phi(f)(z) = \mathrm{diag}(f(x_1),\dots,f(x_a))$ where $x_t \in X_{i_t}$. Then by the definition of $\phi(\alpha)$, there is a $V' \in \alpha(i_{t^*},J(V),\hat{k}(z,k))$. Since $n_i = \sum_{t=1}^a m_{i_t}$ and

$k = \hat{k}(z,k) + \sum_{t=1}^{t^*-1} m_{i_t}$ it follows that $\hat{k}(z,k) + J(V') > m_{i_{t^*}}$ so V' violates (2) for α. Therefore $\phi(\alpha)$ satisfies (2).

Suppose that there exists a point $z \in Z_i$ and sets U_1, U_2 such that $z \in U_1 \bigcap U_2$ and there exist $U_1 \in \phi(\alpha)(i, k_3 - k_1, k_1), U_2 \in \phi(\alpha)(i, k_4 - k_2, k_2)$ where $k_1 < k_2 < k_3 < k_4$. We write $\phi(f)(z) = \mathrm{diag}(f(x_1), \ldots, f(x_a))$ where $x_t \in X_{i_t}$. Let $j_t = 1 + n_{i_1} + \cdots n_{i_{t-1}}$. If there exists a t such that $j_t \leq k_1 < k_2 < j_{t+1}$ then there exist $V_1 \in \alpha(i_t, k_3 - k_1, k_1 - j_t + 1)$ and $V_2 \in \alpha(i_t, k_4 - k_2, k_2 - j_t + 1)$ such that $x_t \in V_1 \bigcap V_2$, which contradicts (1) for α. If $k_1 < j_t \leq k_2$ then for any $U \in \phi(\alpha)(i, j, k_1)$, (2) for α implies that $j \leq j_t - k_1$, so no violation of (1) is possible. Therefore $\phi(\alpha)$ is a full layering.

Suppose $\alpha' \in \mathcal{A}(\alpha)$. We first show that $\phi(\alpha') \in \mathcal{A}(\phi(\alpha))$. Suppose $z \in \overline{L_{\phi(\alpha)}(i,j,k)}$. Suppose $\widehat{\phi}_k(z) \notin \overline{L_\alpha(\hat{i}(k,z), j, \hat{k}(z,k))}$. Then there must exist an i'', k'' and a point $y \in Y_{i''}$ such that $y \in \overline{L_\alpha(i'', j, k'')}$ and $\Phi_{k''}(y) = \widehat{\phi}_k(z)$. Therefore there is a $T \in T(i'', j, k'')$ such that $y \in \alpha'_T \in \alpha'$. We let $T = \mathcal{T}(i'', k'')$ where $\mathcal{T} \in ST_\alpha$. Then $z \in \phi(\alpha')_{\phi(\mathcal{T})(i,k)}$.

The subtower property for $\phi(\alpha')$ follows immediately from the subtower property for $\alpha\prime$. Suppose that $z_1 \in \overline{L_{\phi(\alpha)}(i_1, j, k_1)}, z_2 \in \overline{L_{\phi(\alpha)}(i_2, j, k_2)}$, $\Phi_{k_1}(z_1) = \Phi_{k_2}(z_2)$ and $b_{i_1,k_1}(z_1) = b_{i_2,k_2}(z_2)$. Then $\widehat{\phi}_{k_1}(z_1) = \widehat{\phi}_{k_2}(z_2)$ so for any $\mathcal{T} \in ST_\alpha$, $z_1 \in U(\mathcal{T}, i_1, k_1)$ if and only if $z_2 \in U(\mathcal{T}, i_2, k_2)$.

Finally, for any $\alpha' \in \mathcal{A}(\alpha)$, $\mathrm{dimRatio}(\phi(\alpha')) \leq \mathrm{dimRatio}(\alpha')$. We can check that any join of α pulls back to a join of $\phi(\alpha)$, hence $\mathrm{dimRatio}(\phi(\alpha)) \leq \mathrm{dimRatio}(\alpha)$. $\square$

3.3 Mean Dimension

Definition 3.3.1. If A is a limit of DSH algebras A_n by diagonal maps $\phi_{n_1,n_2} : A_{n_1} \to A_{n_2}$, we define

$$\gamma_n(A) = \lim_{n' \to \infty} \sup_{\alpha \in \mathcal{C}(A_n)} \text{dimRatio}(\phi_{n,n'}(\alpha)).$$

Finally, we define $\text{mdim}_{DSH}(A) = \lim_n \gamma_n(A)$ to be the DSH mean dimension of A. We will omit the subscript when there is no ambiguity about the context.

Example 3.3.2. In [14], Niu defines diagonal maps between piecewise homogeneous algebras and gives a notion of mean dimension on AH algebras constructed from diagonal maps. We will show that this notion agrees with ours. Piecewise homogeneous algebras are clearly DSH. Suppose we have two such algebras $A = \oplus_{i=1}^r C(X_i, M_{n_i})$ and $B = \oplus_{i=1}^s C(Z_i, M_{m_i})$ and $\phi : A \to B$ is diagonal. For some $z^* \in Z_i$ there is a set of points $x_1, \ldots, x_t$ where $x_j \in X_{i_j}$ such that $\phi(f)_i(z^*) = \text{diag}(f_{i_1}(x_1), \ldots, f_{i_t}(x_t))$. It follows that on a connected component C of Z_i containing z^* we must have t maps $\lambda_j : C \to X_{i_j}$ such that

$$\phi(f)_i(z) = \text{diag}(f_{i_1}(\lambda_1(z)), \ldots, f_{i_t}(\lambda_{i_t}(z))).$$

Therefore this map is diagonal in the AH sense.

Suppose we have α, some open cover of $\bigcup_{i'=1}^r X_{i'}$, and assume that each $U \in \alpha$ is a subset of some individual $X_{i'}$. Then we may consider a layering $\tilde{\alpha}$ consisting of the sets of α such that if $U \in \alpha$ is a subset of $X_{i'}$ then $I(U) = i'$, $J(U) = n_{i'}$, and $K(U) = 1$. We can also give this a trivial spectral structure in which every set of $\mathcal{S}$ is a singleton. We can then consider $\phi(\tilde{\alpha})$. If $U \subset X_{i'}$ then let $S_U = \{U\} \in \mathcal{S}$. For

$i \in \{1, \dots, s\}$ and $k \in \{1, \dots, n_j\}$ we then have a set $U(S_u, j, k) \in \phi(\tilde{\alpha})(i, n_{i'}, k)$. We partition this into sets $U(S_u, j, k)[C]$ where C is a connected component of Z_i. We then write $\phi(\tilde{\alpha})(i, j, k)[C] = \{U[C] : U \in \phi(\tilde{\alpha})(i, j, k)\}$.

We have maps $\lambda_j : C \to X_{i_j}$, for $j \in \{1, \dots, t\}$, and we then find that $\phi(\tilde{\alpha})(i, j', k)[C]$ is an open cover of C if $j' = n_{i_j}$ and $k = n_{i_1} + \cdots + n_{i_{j-1}}$ for some j and the empty set otherwise. We note that $c(x) = (n_{i_1}, \dots, n_{i_{s(C)}})$ for every $x \in C$ and we call this composition c. We then have that $\phi(\tilde{\alpha})(i, j, a_t)[C] = \lambda_t^{-1}(\alpha)$ for each $t \in \{1, \dots, s(C)\}$. We then define a layering β such that $\beta(i, m_i, 1) = \bigcup_{C \subseteq X_i} \bigvee_t \phi(\tilde{\alpha})(i, c_t, a_t)[C]$ and letting $\beta(i, j, k)$ be empty for other values of j, k. Therefore $\{U \cap C : U \in \beta(i, n_i, 1)\} = \lambda_1^{-1}(\alpha) \vee \cdots \vee \lambda_{s(C)}^{-1}(\alpha)$ for each i, C. We then find that $\mathrm{dimRatio}(\beta) = \max_{i,C} \mathcal{D}(\lambda_1^{-1}(\alpha) \vee \cdots \vee \lambda_{t_C}^{-1}(\alpha)\gamma)/n_i$. This latter quantity is exactly the AH mean dimension mdim_{RSH}. It follows that for a diagonal limit of AH algebras A , $\mathrm{mdim}_{DSH}(A) \leq \mathrm{mdim}_{AH}(A)$.

Example 3.3.3. If X is an infinite compact metric space and $\sigma : X \to X$ is a homeomorphism then we define the dynamical mean dimension of the cross product algebra A to be to be $\mathrm{mdim}_{DYN}(A) = \sup_{\alpha \in \mathcal{C}(A)} \lim_n \mathcal{D}(\sigma^{-1}(\alpha) \vee \cdots \vee \sigma^{-(n+1)}(\alpha))/n$.

It is shown in [8] that if $\mathrm{mdim}_{DYN}(A) = 0$ then for any ϵ and any finite subset $F \subset A$, F can be contained up to ϵ in a ASH subalgebra A_F of A such that $\mathrm{dimRatio}(A_F) < \epsilon$. Therefore by Corr. 3.2.5, it suffices to show that A_F is DSH and if $F \subset G$ then the inclusion of A_F into A_G can be made to be diagonal.

Let Y be the closure of an open set in X. For $y \in Y$, we define $R_Y(y) = \min\{n > 0 : \sigma^n(x) \in Y\}$ to be the first return time for x. By compactness $R_Y(Y)$ is a finite set with maximum value l, and we write $X_i^Y = \overline{R_Y^{-1}(i)}$ for $i = 1, \dots, l$. We have previously given a DSH construction for A_Y, a subalgebra of $\bigoplus_{i=1}^{l} M_i(C(\overline{X_i^Y}))$. In [8], however, a subalgebra is constructed which has a small dimension ratio, and it is this algebra we must show is DSH.

The subalgebra is of the form described in Example 1.0.6. Suppose we have a partition of unity $\{\psi_U : U \in \alpha\}$ for some open cover α for X and let $\Psi : Y \to \mathbb{R}^{|\alpha|}$ be given by $\Psi(x) = \bigoplus_{U\in\alpha} \psi_U$. Furthermore, let $H : X \to [0,1]$ be such that $H^{-1}(0) = Y$. Then we define a map $\xi_i^Y : X_i^Y \to \mathbb{R}^{(|\alpha|+1)i-1}$ given by

$$\xi_i^Y = ((\Psi \circ \sigma^1, \ldots, \Psi \circ \sigma^i), (H \circ \sigma^1, \ldots, H \circ \sigma^{i-1})).$$

It follows that if $\xi_i(x_1) = \xi_i(x_2)$ then $x_1 \in Y \Leftrightarrow x_2 \in Y$ and $\xi_{i'}(y_1) = \xi_{i'}(y_2)$ whenever $\Phi_k(x_m) = y_m \in X_{i'}$ for $m = 1, 2$. Therefore this subalgebra, which we call Ξ_Y, has a DSH decomposition.

Suppose $Y_2 \subseteq Y_1$, hence $A_{Y_1} \subseteq A_{Y_2}$. Then we may choose $H_1, H_2 : X \to [0,1]$ such that $H_m^{-1}(0) = Y_m$ for $m = 1, 2$ and whenever $H_2(x) = H_2(y)$, we have $H_1(x) = H_1(y)$. For fixed $x, y \in X_i^{Y_1} \bigcap X_{i'}^{Y_2}$, we then have that if $\xi_{i'}^{Y_2}(x) = \xi_{i'}^{Y_2}(y)$ then $\xi_i^{Y_1}(x) = \xi_i^{Y_1}(y)$. If $i = i_1 < i_2 < \cdots < i_t = i'$ are the first t values j for which $\sigma^j(x) \in Y_1$, then we find that $\xi_{i_s - i_{s-1}}^{Y_1}(\sigma^{i_{s-1}}(x)) = \xi_{i_s - i_{s-1}}^{Y_1}(\sigma^{i_{s-1}}(y))$ for $s = 2, \ldots, t$. The points $\sigma^{i_{s-1}}(x)$ form exactly the list of points in the diagonal decomposition of the inclusion $A_{Y_1} \subseteq A_{Y_2}$. We can define an analogous map from Ξ_{Y_1} into Ξ_{Y_2}. We do this by way of a diagonal decomposition. If $x \in \xi_i^{Y_2}(X_i)$, then we may choose any value $y \in (\xi_i^{Y_2})^{-1}(x)$ and compute $\sigma^{i_{s-1}}(y)$ for $s = 2, \ldots, t$, and we have shown that $\xi_{i_s - i_{s-1}}^{Y_1}(\sigma^{i_{s-1}}(y))$ is independent of the choice of y. Therefore we can define these as the s points in the diagonal decomposition of the inclusion from Ξ_{Y_1} into Ξ_{Y_2}. Therefore we have a diagonal inclusion, and forming a sequence of these algebras, we obtain the limit A and this sequence has mean dimension 0 by Corr. 3.2.5.

3.4 Internal approximation

For any full spectral layering we can define a subalgebra $A(\alpha)$ of A such that whenever $x \in U \in \alpha(i,j,k)$, $f_i(x)$ has a block point at position i for every $f \in A$. Observe that if $f \in A(\alpha)$ and $\beta \in \vee(\alpha)$ or β refines α then $f \in A(\beta)$. It follows that when α is F, ϵ-compatible we have $|f - f'| < \epsilon$ for some $f' \in A(\beta)$.

We can also define a DSH algebra $A[\alpha]$ with using simplices defined by the covers $\alpha(i,j,k)$. For each j, we define the simplex Δ_j with vertices $[\mathcal{T}]$ indexed by spectral towers $\mathcal{T}$ of height j. Let S_j be the set of spectral towers of height j, and let Δ'_j be the subsimplex of points $\sum_{t=1}^m \lambda_t[\mathcal{T}_t]$ such that there is a non-trivial composition $c \in \mathcal{C}(j)$ such that $c(\mathcal{T}_t)$ refines c for each t.

We will define a DSH algebra $A[\alpha]$ as a subalgebra of $\bigoplus_{j=1}^s C(\Delta_j, M_j)$ such that function values are restricted at each point $w \in \Delta'_j$ given by $\sum_{t=1}^m \lambda_t[\mathcal{T}_t]$. Let $c \in \mathcal{C}(j)$ be the finest composition such that for each t, $c(\mathcal{T}_t)$ refines c. Then we write $c = (c_1, \ldots, c_{|c|})$ and for each $\mathcal{T}_t$ we have a decomposition $\mathcal{T}_t = (\mathcal{T}_{t,1}, \ldots, \mathcal{T}_{t,|c|})$ where $J(\mathcal{T}_{t,s}) = c_s$. Finally we require that $f_j(w) = \operatorname{diag}(f_{c_1}(w_1), \ldots, f_{c_{|c|}}(w_{|c|}))$ where $w_s \in \Delta_{c_s}$ is given by $\sum_{t=1}^m \lambda_t[\mathcal{T}_{t,s}]$.

If α has compatible boundaries, we can then give a diagonal map Ψ from $A[\alpha]$ to A in terms of a partition of unity $\{\psi_T\}$ subordinate to an arbitrary stratification of α. For each point x in each X_i we compute the composition $c(x) = (c_1, \ldots, c_{|c(x)|})$. For each $x \in X_i$ we define a sequence of points given by $\sigma_{i,c_1,a_1+1}(x), \ldots, \sigma_{i,c_{|c(x)|},a_{|c(x)|}+1}(x)$ where $\sigma_{i,j,k} : L(i,j,k) \to \Delta_j$ is given by $\sigma_{i,j,k}(x) = \sum_{\mathcal{T} \in S_j} \psi_{\mathcal{T}(i,k)}(x)[\mathcal{T}]$. For each $f \in A[\alpha]$ and each i we then define a function $\Psi(f)_i : X_i \to M_{n_i}$ given by $\Psi(f)_i(x) = \operatorname{diag}(f_{i_1}(\sigma_{i,c_1,a_1+1}(x)), \ldots, f_{i_{|c(x)|}}(\sigma_{i,c_{|c(x)|},a_{|c(x)|}+1}(x)))$. We construct a piecewise function $\Psi(f)$ on $\bigsqcup X_i$ composed of the functions $\Psi(f)_i$ and we define Ψ to be the map which sends f to $\Psi(f)$.

Lemma 3.4.1. *Ψ is a homomorphism which sends $A[\alpha]$ to A.*

Proof. That Ψ is a homomorphism is immediate, so we check that $\Psi(f)_i$ is continuous. Since $\sigma_{i,j,k}$ is continuous on $L(i,j,k)$, we have that $P[i,j,k]\Psi(f)P[i,j,k]$ is continuous on $L(i,j,k)$. It then suffices to check that $P[i,j,k]\Psi(f)P[i,j,k]$ is continuous on $W(i,j,k)$. Suppose that $c(x) = (c_1, \dots, c_{|c|})$ and $x \in W(i, a_n - a_m, a_m + 1)$ where $1 \leq m < n \leq |c|$. By definition, $P[i, a_n - a_m, a_m + 1]\Psi(f)(x)P[i, a_n - a_m, a_m + 1] = \operatorname{diag}(f_{c_m}(\sigma_{i,c_m,a_m+1}(x)), \dots, f_{c_{n-1}}(\sigma_{i,c_{n-1},a_{n-1}+1}(x)))$.

For every $\mathcal{T} \in S_j$, $\mathcal{T}(i,k) = T$ is a tower in $T(i,j,k)$ and since $x \in L(i, c_t, a_t + 1)$ for each $t \in \{m, \dots, n-1\}$ and $x \in \pi(T)$, $c(T)$ must refine $(c_m, \dots, c_{n-1})$. Then for each t, we have a subtower T_t of T in $T(i, c_t, a_t + 1)$. As y approaches x within $L(i,j,k)$, $\sigma_{i,j,k}(y)$ approaches

$$x' = \sum_{\mathcal{T} \in S_j} \psi_{\mathcal{T}(i,k)}(x)[\mathcal{T}].$$

We can rewrite this as

$$x' = \sum_{T \in T(i,j,k)} \psi_T(x)[\mathcal{T}(T)]$$

where $\mathcal{T}(T)$ refers to the spectral tower $\mathcal{T}$ for which $T = \mathcal{T}_{i,k}$. By the definition of $A[\alpha]$, $f_j(x') = \operatorname{diag}(f_{c_m}(w_m), \dots, f_{c_{n-1}}(w_{n-1}))$ where $w_t \in \Delta_{c_t}$ is given by

$$w_t = \sum_{T \in T(i,j,k)} \psi_T(x)[\mathcal{T}(T_t)].$$

It follows from the definition of a partition of unity that for each $T \in T(i, c_t, a_t + 1)$, $\psi_T(x) = \sum_{T' \in S(T)} \psi_{T'}(x)$ where $S(T)$ is the set of towers in $T(i,j,k)$ which contain T as a subtower. Therefore

$$\begin{aligned} w_t &= \sum_{T \in T(i,j,k)} \psi_T(x)[\mathcal{T}(T_t)] \\ &= \sum_{T \in T(i,c_t,a_t+1)} \sum_{T' \in S(T)} \psi_{T'}(x)[\mathcal{T}(T'_t)] \\ &= \sum_{T \in T(i,c_t,a_t+1)} \sum_{T' \in S(T)} \psi_{T'}(x)[\mathcal{T}(T)] \\ &= \sum_{T \in T(i,c_t,a_t+1)} \psi_T(x)[\mathcal{T}(T)] = \sigma[i, c_t, a_t + 1](x). \end{aligned}$$

Therefore $f_j(x') = P[i,j,k]\Psi(f)_{n_i}(x)P[i,j,k]$, and the former is the limit of

$$P[i,j,k]\Psi(f)_{n_i}(y)P[i,j,k]$$

when $y \in L(i,j,k)$.

We now check the boundary condition on A. If $y \in Y_i$ with corresponding points $x_1, \ldots, x_s$, where $x_t \in X_{i_t}$, then $c(y)$ can be decomposed into s compositions given by $c(x_1), \ldots, c(x_s)$. We write $c(x_t) = (c_{t,1}, \ldots, c_{t,s(t)})$, $a_{t,r} = c_{t,1} + \cdots + c_{t,r-1}$ and $b_{t,r} = n_1 + \cdots + n_{t-1} + a_{t,r} + 1$. If $x_t \in L(i_t, c_{t,r}, a_{t,r} + 1)$, then $y \in L(i, c_{t,r}, b_{t,r})$ and for every $T \in T(i, c_{t,r}, b_{t,r})$ there is a $T' \in T(n_i, c_{t,r}, a_{t,r} + 1)$ such that $\mathcal{T}(T) = \mathcal{T}(T')$ and $\psi_T(y) = \psi_{T'}(x_s)$. It follows that $\sigma_{i,c_{t,r},b_{t,r}}(y) = \sigma_{i_t,c_{t,r},a_{t,r}+1}(x_t)$. Finally we have

$$\Psi(f)_i(y) = \operatorname{diag}(f_{c_{1,1}}(\sigma_{i,c_{1,1},b_{1,1}}(y)), \ldots, f_{c_{1,s(1)}}(\sigma_{i,c_{1,s(1)},b_{1,s(1)}}(y)), f_{c_{2,1}}(\sigma_{i,c_{2,1},b_{2,1}}(y)), \ldots)$$

and

$$\operatorname{diag}(f_{c_{t,1}}(\sigma_{i,c_{t,1},b_{t,1}}(y)),\ldots,f_{c_{t,s(t)}}(\sigma_{i,c_{t,s(1)},b_{t,s(t)}}(y)))=\Psi(f)_{n_t}(x_t),$$

which yields the familiar diagonal decomposition. □

Lemma 3.4.2. dimRatio$(\Psi(A[\alpha]))\leq$ dimRatio(α').

Proof. By Lemma 1.0.5, we can identify $\Psi(A[\alpha])$ with the DSH algebra with spaces given by the closures of the combined images of the maps $\sigma_{i,j,k}$. Each map $\sigma_{i,j,k}$ has a natural extension $\tilde{\sigma}$ to $\overline{L(i,j,k)}$ and we can identify the closure of the image with the image of the extension. By definition, $\tilde{\sigma}_{i,j,k}$ maps $x\in X_i$ to the point $\sum_{T\in T(i,j,k)}\psi_T(x)[\mathcal{T}(T)]$. Then if S_x is the set of towers T in $T(i,j,k)$ such that $\psi_T(x)>0$ then $x\in\alpha'_T$. We conclude that $|S_x|\leq \operatorname{ord}(\alpha'(i,j,k))+1$. Then the image of $\sigma_{i,j,k}$ is always contained within the subcomplex of Δ_j formed by simplices arising from sets of vertices $[\mathcal{T}_1],\ldots,[\mathcal{T}_s]$ such that there exist i,k such that $\alpha'_{\mathcal{T}_1},\ldots,\alpha'_{\mathcal{T}_s}$ have a common intersection. This complex then has dimension at most $\operatorname{ord}(\alpha'(i,j,k))$. The conclusion follows. □

Lemma 3.4.3. *Suppose A is a DSH algebra, α is a full spectral layering with compatible boundaries, F is a finite subset of A, $\epsilon>0$ and assume α is F,ϵ-compatible. Then for any $\alpha'\in\mathcal{A}(\alpha)$ with a corresponding partition of unity we have a diagonal map $\Psi:A[\alpha]\to A(\alpha)$ and for every $f\in F$ there is a $f^*\in\Psi(A[\alpha])$ such that $|f-f^*|<2\epsilon$.*

Proof. Since α is F,ϵ-compatible we may find for each $f\in F$ some f' such that $|f-f'|<\epsilon$, $f'\in A(\alpha)$ and for any $U,V\in S\in\mathcal{S}$, any $x\in U$, $y\in V$, we have

$$\left|P[n_{I(U)},J(U),K(U)](f'_{I(U)}(x))-P[n_{I(V)},J(V),K(V)](f'_{I(V)}(y))\right|<\epsilon.$$

For each $S \in \mathcal{S}$ we choose some $U \in S$ and $x^* \in U$. We then define M_S to be the $J(S)$ by $J(S)$ matrix $P[n_{I(U)}, J(U), K(U)](f'_{I(U)}(x^*))$. We now construct a function $f^* \in A[\alpha]$ by defining f^* first at each vertex. If $\mathcal{T} = (S_1, \ldots, S_t)$ and $J(\mathcal{T}) = j$ then we define $f^*_j([\mathcal{T}]) = \mathrm{diag}(M_{S_1}, \ldots, M_{S_t})$ and we can extend linearly to $\bigcup_j \Delta_j$. For a given $x \in X_i$, we compute $c(x) = (c_1, \ldots, c_{|c|})$ and we have $f'_i(x) = (N_1, \ldots, N_{|c|})$ where N_n is a c_n by c_n matrix. For fixed n, we have towers $T_1, \ldots, T_s \in T(i, c_n, 1+a_n)$ such that $\sum_{t=1}^s \psi_{T_t}(x) = 1$. Each T_t has a corresponding spectral tower $\mathcal{T}_t$, and the corresponding point in the diagonal decomposition of Ψ is $x_n = \sum_{t=1}^s \psi_{T_t}(x)[\mathcal{T}_t]$. If $T_t = (U_1, \ldots, U_m)$, we have $x \in \bigcap_{k=1}^m U_k$ and by F, ϵ-compatibility we have a decomposition of N_n into $\mathrm{diag}(N_{n,1}, \ldots, N_{n,m})$ where $N_{n,k}$ is a $J(U_k)$ dimensional matrix. Again by F, ϵ-compatibility we have that if $U_k \in S$ then $|N_{n,m} - M_S| < \epsilon$. Hence we have $\left|N_n - f^*_{n_s}([\mathcal{T}_t])\right| < \epsilon$. By linearity $f^*_{c_n}(x_n) = \sum_{t=1}^s \psi_{T_t}(x) f^*_{c_n}([\mathcal{T}_t])$, hence

$$\left|N_n - f^*_{c_n}(x_n)\right| \leq \sum_{t=1}^{s} \psi_{T_t}(x) \left|N_n - f^*_{c_n}([\mathcal{T}_t])\right| < \epsilon.$$

Finally, $\Psi(f^*)(x) = \mathrm{diag}(f^*_{c_1}(x_1), \ldots, f^*_{c_{|c|}}(x_{|c|}))$, therefore $|\Psi(f^*)(x) - f'(x)| < \epsilon$. □

Lemma 3.4.4. *Suppose A is a DSH algebra, α is a full spectral layering and $\alpha' \in \mathcal{A}(\alpha)$. Then there exists a refinement β of α with compatible boundaries and $\beta' \in \mathcal{A}(\beta)$ such that* $\mathrm{dimRatio}(\beta') \leq \mathrm{dimRatio}(\alpha')$.

Proof. Suppose that α satisfies the compatible boundary condition for $i' < i$. Then we will construct a β with $\beta|_{i-1} = \alpha|_{i-1}$ that satisfies the compatible boundary condition for i.

By Corollary 3.1.8, we may assume, possibly by passing to a refinement of α', that whenever $x \in \alpha'_T \bigcap W(I(T), j, k)$ where $k \leq K(T) < K(T) + J(T) \leq k + j$ it follows that $x \in \alpha'_{T'}$ for some $T' \in T(i, j, k)$ where $T \leq T'$. We then have that for any $T \in T_\alpha$,

$$\overline{\alpha}'_T = \{x \in X_{I(T)} : x \in \alpha'_{T'} \text{ for some } T' \geq T\}$$

is open.

For each j, k and each $U \in \alpha(i, j, k)$, let $V_U = U \bigcap Y_i$. We then define $U' = \{x \in U : d(x, V_U) < d(x, Y_i \setminus V_U)\}$. For each $T = (U_1, \ldots, U_s) \in T(i, j, k)$, we define $T' = (U'_1, \ldots, U'_s)$. Let $\gamma = \{U' : U \in \alpha\} \bigcup \alpha|_{i-1}$. We may then find an open set $V \subseteq X_i$ containing Y_i such that for every $x \in \overline{V}$ there is a $T \in T_\gamma(i, n_i, 1)$ such that $x \in \pi(T)$. We then define $\tilde{\gamma}$ to be the full layering on a modified version of A_i in which X_i is replaced with $\overline{V}$ given by $\{U' \bigcap \overline{V} : U \in \alpha\} \bigcup \alpha|_{i-1}$. We then find that $\tilde{\gamma}$ has compatible boundaries and that $L_\alpha(i, j, k) \bigcap Y_i = L_{\tilde{\gamma}}(i, j, k) \bigcap Y_i$ for all j, k.

Note that every tower in $T_{\tilde{\gamma}}(i, j, k)$ occurs as T' for some $T \in T_\alpha(i, j, k)$. We can define for each such tower $V_T = \alpha'_T \bigcap Y_i$. We then define for each $T \in T_\alpha(i, j, k)$ a set $\gamma'_T = \{x \in \overline{L_{\tilde{\gamma}}(i, j, k)} \bigcap \overline{\alpha}'_T : d(x, V_T) < d(x, (Y_i \bigcap \overline{L_{\tilde{\gamma}}(i, j, k)}) \setminus V_U)\}$. We find that for every each j, k and $y \in \overline{L_{\tilde{\gamma}}(i, j, k)}$ there is an open subset of $\overline{L_{\tilde{\gamma}}(i, j, k)}$ which contains y. Therefore we may then find an open set $V' \subseteq X_i$ containing Y_i such that for every j, k and $x \in \overline{V'}$ there is a tower $T \in T_\alpha(i, j, k)$ such that $x \in \alpha'_T$ and subtowers $T_1, \ldots, T_s$ such that $(T_1, \ldots, T_s) = T$, T_t has height $c_{\tilde{\gamma}}(j)[k, k + j]_t$ and $x \in \gamma'_{T'_t}$ for $t \in \{1, \ldots, s\}$.

Let $\beta|_{i-1} = \alpha|_{i-1}$. For each $U \in \alpha(i, j, k)$, we define $U'' \in \beta(i, j, k)$ by $U'' = (U' \bigcap \overline{V'}) \bigcup (U \bigcap (X_i \setminus \overline{V'}))$. We can extend this by modifying α on $Y_{i'}$ for $i' > i$ to obtain β as desired. For each j, k and each $T \in T_\alpha(i, j, k)$ we let α''_T is α'_T modified on

∂V to include only those points which lie in γ'_{T_t} for every subtower T_t in $(T_1, \ldots, T_s)$, the decomposition of T corresponding to $c_{\tilde{\gamma}}(j)[k, k+j]$. We can construct β' by defining for each $T \in T_\alpha(i,j,k)$ $\beta'_T = (\gamma'_T \bigcap \overline{V'}) \bigcup (\alpha''_T \bigcap (X_i \setminus V'))$. We can again extend this to a stratification by modifying α' on $Y_{i'}$ for $i' > i$. If $\beta'_{T_1}, \ldots, \beta'_{T_n}$ have a common intersection at x then if $x \notin \overline{V'}$ $\alpha'_{T_1}, \ldots, \alpha'_{T_n}$ all contain x and if $x \notin \overline{V'}$ then $\beta'_{T_1}, \ldots, \beta'_{T_n}$ must intersect at a point in Y_i and this point then lies in each α'_{T_t}. Hence we do not increase the dimRatio. □

Theorem 3.4.5. *Suppose that* $A = \lim(A_n, \phi_n)$ *is an diagonal limit of DSH algebras by diagonal maps and* $\mathrm{mdim}_{DSH}(A) = \gamma$*. Then for any finite subset* $F \subseteq A$ *and* $\epsilon_1, \epsilon_2 > 0$ *there exists a DSH subalgebra* $B \subset A$ *such that* $\mathrm{dist}(F, B) < \epsilon_1$ *and* $\mathrm{dimRatio}(B) < \gamma + \epsilon_2$.

Proof. By Lemma 1.0.5, it suffices to prove this when the maps are injective. Up to $\epsilon_1/3$, we may approximate F by a finite set $F' \subset A_n$ for some n such that $\gamma_n(A) < \gamma + \epsilon_2/2$. By Corollary 3.2.5, there exists a full spectral layering α for A_n that is $F', \epsilon_1/3$-compatible with a stratification α'. We can then find an n' such that $\mathrm{dimRatio}(\phi_{n',n}(\alpha)) < \gamma + \epsilon_2$. Therefore there exists a full spectral layering $\beta \in \vee(\phi_{n',n}(\alpha))$ and a stratification $\beta' \in \mathcal{A}(\beta)$ such that $\mathrm{dimRatio}(\beta') < \gamma + \epsilon_2$. By Lemma 3.4.4, β has a refinement δ that is a full spectral layering with compatible boundaries and a stratification δ' such that $\mathrm{dimRatio}(\delta') < \gamma + \epsilon$. As a refinement of β it is automatic that δ is $\phi_{n',n}(F'), \epsilon_1/3$-compatible. Then by Lemma 3.4.3, we can find a homomorphism $\Psi : A[\delta] \to A_{n'}$ and $F'' \subset \Psi(A[\delta])$ such that $\phi_{n',n}(F')$ is approximated by F'' up to $2\epsilon_1/3$. Furthermore by Lemma 3.4.2, $\mathrm{dimRatio}(\Psi(A[\delta])) \leq \mathrm{dimRatio}(\delta') < \gamma + \epsilon_2$. Therefore the algebra $\Psi(A[\alpha])$ satisfies the lemma. □

Corollary 3.4.6. *If A is a simple, separable, infinite dimensional limit of DSH algebras by diagonal maps and* $\mathrm{mdim}_{DSH}(A) = 0$ *then* $A \otimes \mathcal{Z} \cong A$*, where* $\mathcal{Z}$ *is the Jiang-Su algebra.*

Proof. By Theorem 3.4.5 we can approximate any finite subset of A by elements of a subhomogeneous subalgebra of A of arbitrarily small dimension ratio. It then follows by the same argument as [8] Theorem 4.6 that A is Jiang-Su stable. □

The following is a generalization of [8] Theorem 5.6.

Corollary 3.4.7. *Suppose A_1 and A_2 are simple, separable infinite dimensional limits of DSH algebras by diagonal maps. Then $A_1 \otimes A_2$ is Jiang-Su stable.*

Proof. We can approximate any finite subset F of $A_1 \otimes A_2$ up to $\epsilon/2$ by elements of $F_1 \otimes F_2$ where F_i is a finite subset of a DSH algebra A_i' and we construct full spectral layerings α_1 and α_2 with finite dimension ratios γ_1 and γ_2 respectively such that α_i is $F_i, \epsilon/2$-compatible. Because A_1 and A_2 are infinite dimensional, we can contain each A_i' diagonally in a DSH algebra A_i'' such that the smallest dimension d of any irreducible representation in either algebra is arbitrarily large. In particular we ask that $d \geq 1/\min(\gamma_1, \gamma_2)$. Let $\iota(\alpha_i)$ be the image of α_i in A_i''. By Cor. 3.1.10, this layering has a join β_i such that for every x in every spectral space for A_i'', every entry of $c(x)$ is at least d. By Lemma 3.4.4, β_i has a refinement δ_i such that δ_i has compatible boundaries and $\mathrm{dimRatio}(\delta_i) \leq \gamma_i$. For each δ_i we may construct an embedding $\Psi_i : A_i[\delta_i] \to A_i$ and we have that $F \subseteq_\epsilon \Psi_1(A_1[\delta_1]) \otimes \Psi_2(A_2[\delta_2])$. We define this algebra as B and by [8] Lemma 5.3, we have $\mathrm{dimRatio}(B) \leq (\gamma_1 + \gamma_2)/d$. Since we may choose d to be arbitrarily large, we can make $\mathrm{dimRatio}(B)$ arbitrarily small, so by the same argument as above,

$A_1 \otimes A_2$ is Jiang-Su stable. □

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