Logarithmic Algebroids and Meromorphic Line Bundles and Gerbes

by

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Abstract

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In this thesis, we first introduce logarithmic Picard algebroids, a natural class of Lie algebroids adapted to a simple normal crossings divisor on a smooth projective variety. We show that such algebroids are classified by a subspace of the de Rham cohomology of the divisor complement determined by its mixed Hodge structure. We then solve the prequantization problem, showing that under an integrality condition, a log Picard algebroid is the Lie algebroid of symmetries of what is called a meromorphic line bundle, a generalization of the usual notion of line bundle in which the fibres degenerate in a certain way along the divisor. We give a geometric description of such bundles and establish a classification theorem for them, showing that they correspond to a subgroup of the holomorphic line bundles on the divisor complement which need not be algebraic. We provide concrete methods for explicitly constructing examples of meromorphic line bundles, such as for a smooth cubic divisor in the projective plane.

We then proceed to introduce and develop the theory of logarithmic Courant algebroids and meromorphic gerbes. We show that under an integrality condition, a log Courant algebroid may be prequantized to a meromorphic gerbe with logarithmic connection. Lastly, we examine the geometry of Deligne and Deligne-Beilinson cohomology groups and demonstrate how this geometry may be exploited to give quantization results of closed holomorphic and logarithmic differential forms.

This thesis is dedicated to the two most inspiring women in my life: my late mother and my grandmother.

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Contents

Chapter 1

Introduction

In algebraic geometry, it is often extremely advantageous to work with projective varieties. For example, the classical GAGA theorem [36] allows us to move between the holomorphic and the algebraic categories. Another feature of projective varieties is that its cohomology is often well-behaved; it enjoys a number of finiteness properties which are not present when we work with affine or quasi-projective varieties. This gives us an abundance of tools which make cohomology computations on projective varieties a much more amenable task than that of non-projective varieties. One of the persistent driving forces of research in algebraic geometry has been to better understand the phenomenons of non-projective varieties. Celebrated examples of this include Hironaka's resolution of singularities [22], Kato's logarithmic structures [24] and Deligne's theory of mixed Hodge structures [14, 15]. In this thesis, we follow this spirit and develop a systematic study of algebroids, line bundles, and gerbes on non-projective varieties.

We begin this thesis in Chapter 2 with a review of pure and mixed Hodge theory. This is essential as many of our classification theorems thereafter will be both formulated and proved using these Hodge-theoretic frameworks. The focus of Chapter 3 is the study of Picard algebroids. Picard algebroids were introduced by Beĭlinson and Bernstein [1] as the Lie algebroid extensions of the tangent sheaf by the structure sheaf. They were originally used in [1] to study the localization theory of $\mathfrak{g}$-modules for a semisimple algebra $\mathfrak{g}$. Picard algebroids have since emerged as an important tool to study various interesting objects lying at the crossroads of algebraic geometry and geometric representation theory. For example, they are used in the quantization of the Hitchin integrable system in the Geometric Langlands program [2], the Fourier transform of non-commutative algebras [32], and conical symplectic resolutions [5].

Our main contribution in Chapter 3 is a solution to the prequantization question of Picard algebroids. We prove that a Picard algebroid is the Atiyah algebroid of infinitesimal symmetries of a line bundle $\mathcal{L}$ if and only if its curvature class satisfies an integrality constrant which relates it to the first Chern class of the line bundle in question.

In the first half of Chapter 4, we develop a generalization of the above theory in which the tangent sheaf $\mathcal{T}_X$ of a smooth projective variety X is replaced by the logarithmic tangent sheaf $\mathcal{T}_X(-\log D)$ of vector fields tangent to a simple normal crossings divisor $D \subset X$. In Section 4.1.1, we introduce and classify logarithmic Picard algebroids, and observe that due to the failure of the Poincaré lemma for the log de Rham complex, flat connections may not exist even locally. This leads us to focus on log Picard algebroids which are locally trivial, for which local flat connections do exist. The classification of these is particularly interesting: we show in Theorem 4.13 that they correspond to a certain intersection of the Hodge and weight filtrations on the cohomology of the complement $U = X\backslash D$. We then proceed to provide interesting non-trivial examples of log Picard algebroids and prove functorial properties of log Picard algebroids under blowup.

Our main problem is then to determine what type of object such logarithmic Picard algebroids may prequantize to, and what the quantization condition is. In Section 4.2, we show that the natural prequantum object is a meromorphic line bundle, a type of line bundle in which the transition functions may have zeros or poles along the divisor D. While meromorphic line bundles have appeared in the context of meromorphic connections and holonomic $\mathcal{D}$-modules in [28, 29, 35], their basic geometric features are ill-understood. We proceed to make a careful study of these bundles, showing in Section 4.2.1 that they may be realized as holomorphic $\mathbb{C}^*$ principal bundles on U whose fibers degenerate over D to an infinite chain of rational curves. We then classify meromorphic line bundles using the fact that the first Chern class lies in a subgroup of $H^2(U,\mathbb{Z})$ determined by a version of Deligne's mixed Hodge theory with integral coefficients which we developed earlier in Section 2.2.2. Finally, we prove in Theorem 4.41 that a log Picard algebroid may be pre-quantized if and only if its class lies in this subgroup of the integral cohomology of U.

In the last part of Chapter 4, we use ramified coverings and blowups to provide new methods for explicitly constructing meromorphic line bundles, ending with a concrete construction for the group $\mathbb{Z}_3 \oplus \mathbb{Z} \oplus \mathbb{Z}$ of meromorphic line bundles on $\mathbb{C}P^2$ with pole along a smooth cubic divisor; these constructions are necessarily non-algebraic in nature, as meromorphic line bundles are holomorphic but not necessarily algebraic upon restriction to the divisor complement.

In Chapter 5, we develop a generalization of exact Courant algebroids where we replace the tangent sheaf $\mathcal{T}_X$ by the logarithmic tangent sheaf $\mathcal{T}_X(-\log D)$ and replace the cotangent sheaf Ω^1_X by the sheaf of logarithmic 1-forms $\Omega^1_X(\log D)$. We name these new geometric objects as log Courant algebroids. We utilize many of the ideas developed in Section 4.1 to establish both fundamental properties and classification results for these algebroids. We end this section by providing several examples of log Courant algebroids and introduce a preliminary framework for log Dirac structures. Usual Courant algebroids play an important role in generalized geometry [20] and Dirac geometry [11] and we hope that the ideas developed in this chapter may be used in the future to better understand these geometries in the logarithmic setting.

As in the case of log Picard algebroids, we would like to determine what type of object such log Courant algebroids may prequantize to. Using the intuition drawn from meromorphic line bundles in Section 4.2 and the fact that usual Courant algebroids prequantize to gerbes with 0-connection under an integrality condition, we show in Section 5.2 that the prequantum object is a meromorphic gerbe with logarithmic 0-connection. The remainder of Chapter 5 is devoted to the study of these meromorphic gerbes. Our study of gerbes is cohomological rather than categorical, following the approach of Hitchin-Chatterjee towards gerbes [23, 10]. We conclude the chapter by proving in Theorem 5.36 that a log Courant algebroid admits a pre-quantization if and only if its class lies in a subgroup of the integral cohomology $H^3(U,\mathbb{Z}(1))$.

The last chapter of this thesis centers on two themes: the geometry of Deligne and Deligne-Beĭlinson cohomology groups and the quantization of closed differential forms. One of the most important properties of Deligne cohomology is that they are the target of regulator maps

$$c_{n,p} : K_n(X) \to H^{2p-n}_{\mathrm{Del}}(X, \mathbb{Z}(p)),$$

where $K_n(X)$ are the algebraic K-groups of X. This may be viewed as a vast generalization of classical Chern classes of holomorphic vector bundles. They are also of importance in arithmetic geometry as they may be related to L-functions through the work of [3]. It would be ideal to have a general geometric description of Deligne cohomology groups; they are especially difficult to describe geometrically in higher degrees. We give a geometric description of the class of Deligne cohomology groups

$$H^k_{\mathrm{Del}}(X, \mathbb{Z}(k)),$$

using the Hitchin-Chatterjee formulation of higher gerbes with connective structure. We then proceed to study the quantization problem of closed k-forms on X. We prove in Theorem 6.4 that a closed k-form is the curvature of a holomorphic $k-2$ gerbe with connective structure if and only if it has integral periods. This generalizes the classical results of Kostant and Weil for 2-forms [25, 38] and Brylinski and Deligne for 3-forms [8].

We conclude the thesis by extending our results for Deligne-Beĭlinson cohomology: a version of Deligne cohomology on quasi-projective varieties with logarithmic growth along the compactifying divisor D. We prove in Theorem 6.7 that a closed logarithmic k-form is the curvature of a meromorphic $k-2$ gerbe with connective structure if and only if it admits integral periods.

Chapters 3 and 4 represent joint work with Marco Gualtieri.

Chapter 2

Hodge Theory

The purpose of this chapter is to provide an overview of all the Hodge-theoretic facts and results that we will need for the duration of this thesis. We begin by reviewing the basic ideas of Hodge theory on smooth projective varieties and then proceed to discuss Deligne's mixed Hodge theory on quasi-projective varieties. Our classification theorems of log Picard algebroids in Section 4.1 will be dependent on these cohomological tools. Following closely the ideas outlined by Deligne in the complex case, we then proceed to introduce an integral version of mixed Hodge theory. We will use these tools extensively in our classification of meromorphic line bundles in Section 4.2.

2.1 Pure Hodge Theory

Let X be a smooth n-dimensional projective variety over the complex numbers $\mathbb{C}$. We have the ***de Rham complex*** $\Omega_X^\bullet$ of X:

$$\Omega_X^\bullet := [\mathcal{O}_X \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \dots \xrightarrow{d} \Omega_X^{n-1} \xrightarrow{d} \Omega_X^n].$$

The ***de Rham cohomology*** of X is defined by taking the hypercohomology of the de Rham complex $\Omega_X^\bullet$:

$$H^k_{\mathrm{dR}}(X) := \mathbb{H}^k(X, \Omega_X^\bullet).$$

In the holomorphic setting, the Poincaré lemma yields the quasi-isomorphism of complexes

$$\mathbb{C}_X \sim \Omega^\bullet_{X,\mathrm{an}},$$

and so the singular cohomology of X with complex coefficients can be computed as the hypercohomology of the holomorphic de Rham complex $\Omega^\bullet_{X,\mathrm{an}}$:

$$H^k(X, \mathbb{C}) \cong \mathbb{H}^k(X, \Omega^\bullet_{X,\mathrm{an}}).$$

Since X is projective, we can apply the GAGA principle [36] to see that the hypercohomology of the algebraic de Rham complex $\Omega^\bullet_{X,\mathrm{alg}}$ and the hypercohomology of the holomorphic de Rham complex $\Omega^\bullet_{X,\mathrm{an}}$ is isomorphic. This means that the singular cohomology of X over $\mathbb{C}$ may be computed using the algebraic de Rham complex $\Omega^\bullet_{X,\mathrm{alg}}$ of X:

Remark 2.1. The fact that the singular cohomology of X over $\mathbb{C}$ can be computed using the algebraic de Rham complex of X is true even when X is not projective. This is precisely the statement of the algebraic de Rham theorem [19] where the only requirement is that X is a smooth complex algebraic variety.

The ***Hodge filtration*** $\{F^p\}$ on the de Rham complex $\Omega_X^\bullet$ is defined as:

$$F^p\Omega_X^\bullet := \Omega_X^{\geq p}.$$

where $\Omega_X^{\geq p}$ is the truncation of $\Omega_X^\bullet$ in degrees $< p$:

$$\Omega_X^{\geq p} := [0 \to \cdots \to 0 \to \Omega_X^p \to \cdots \to \Omega_X^n].$$

This gives the Hodge filtration on $H^k(X, \mathbb{C})$:

$$F^p H^k(X, \mathbb{C}) = \mathrm{im}(\mathbb{H}^k(F^p\Omega_X^\bullet) \to H^k(X, \mathbb{C})). \tag{2.1}$$

The Hodge filtration is a descending filtration on $H^k(X, \mathbb{C})$:

$$H^k(X, \mathbb{C}) = F^0 \supset F^1 \supset \cdots \supset F^{k-1} \supset F^k \supset \{0\}.$$

Consider the spectral sequence associated to the Hodge filtration on $\Omega_X^\bullet$:

$$\begin{aligned} E_1^{p,q} = H^{p+q}(X, \mathrm{Gr}_F^p\Omega_X^\bullet) &\Rightarrow \mathbb{H}^{p+q}(X, \Omega_X^\bullet) \\ &= H^{p+q}(X, \mathbb{C}). \end{aligned} \tag{2.2}$$

By definition, the associated graded object $\mathrm{Gr}_F^p\Omega_X^\bullet$ is given by

$$\mathrm{Gr}_F^p\Omega_X^\bullet = \frac{F^{p+1}}{F^p} = \Omega_X^p[-p],$$

and so we can rewrite the spectral sequence in (2.2) as:

$$E_1^{p,q} = H^q(X, \Omega_X^p) \Rightarrow H^{p+q}(X, \mathbb{C}).$$

This spectral sequence is commonly referred to as the ***Hodge-de Rham spectral sequence***. By [13, Prop 5.3], this spectral sequence degenerates at E_1 for X smooth and proper. We can use the E_1-degeneration to obtain the following ***Hodge decomposition*** on X:

$$H^k(X, \mathbb{C}) = \oplus_{p+q=k} H^{p,q}(X),$$

where $H^{p,q}(X) = H^q(X, \Omega_X^p)$. We can write the Hodge filtration (2.1) with respect to the Hodge decomposition as

$$\begin{aligned} F^p H^k(X, \mathbb{C}) &= \oplus_{\substack{p' \geq p \\ p+q=k}} H^{p',q}(X) \\ &= H^{k,0}(X) \oplus \cdots \oplus H^{p,k-p}(X). \end{aligned}$$

We conclude this section by presenting a lemma which will be useful for cohomology computations later in the thesis.

Lemma 2.2. *Let X be a projective variety of dimension n. There is a short exact sequence of cohomology groups*

$$0 \to H^{l-1}(\Omega_X^{k+1,cl}) \to H^l(\Omega_X^{k,cl}) \to H^l(\Omega_X^k) \to 0,$$

for $1 \leq k, l, \leq n$.

Proof. Consider the short exact sequence of complexes

$$0 \to \Omega_X^{k+1,\mathrm{cl}}[-1] \to [\Omega_X^k \to \Omega_X^{k+1,\mathrm{cl}}] \to \Omega_X^k \to 0$$

inducing the long exact sequence in hypercohomology

$$\cdots \to H^{l-1}(\Omega_X^k) \to H^{l-1}(\Omega_X^{k+1,\mathrm{cl}}) \to \mathbb{H}^l(\Omega_X^k \to \Omega_X^{k+1,\mathrm{cl}}) \to H^l(\Omega_X^k) \to H^l(\Omega_X^{k+1,\mathrm{cl}}) \to \dots$$

The map from $H^{l-1}(\Omega_X^k)$ to $H^{l-1}(\Omega_X^{k+1,\mathrm{cl}})$ is the d_1 of the Hodge-de Rham spectral sequence. Since the Hodge-de Rham spectral sequence degenerates at E_1, this map is necessarily zero. Similarily, the map from $H^l(\Omega_X^k)$ to $H^l(\Omega_X^{k+1,\mathrm{cl}})$ is also zero. Thus, we have

$$0 \to H^{l-1}(\Omega_X^{k+1,\mathrm{cl}}) \to \mathbb{H}^l(\Omega_X^k \to \Omega_X^{k+1,\mathrm{cl}}) \to H^l(\Omega_X^k) \to 0. \tag{2.3}$$

Lastly, as X is projective, there is a quasi-isomorphism of complexes

$$\Omega_X^{k,\mathrm{cl}} \sim [\Omega_X^k \to \Omega_X^{k+1,\mathrm{cl}}],$$

which induces the isomorphism

$$H^l(\Omega_X^{k,\mathrm{cl}}) \cong \mathbb{H}^l(\Omega_X^k \to \Omega_X^{k+1,\mathrm{cl}}).$$

Combining this fact with (2.3) gives the desired result. □

2.2 Mixed Hodge Theory

2.2.1 Complex Case

We will now work with (X, D) where X is a smooth n-dimensional projective variety over the complex numbers $\mathbb{C}$ and D is a simple normal crossings divisor on X. The complement of D in X will be denoted by $U := X \setminus D$. We first introduce the ***logarithmic de Rham complex*** $\Omega_X^\bullet(\log D)$ of (X, D):

$$\Omega_X^\bullet(\log D) := [\mathcal{O}_X \xrightarrow{d} \Omega_X^1(\log D) \xrightarrow{d} \dots \xrightarrow{d} \Omega_X^{n-1}(\log D) \xrightarrow{d} \Omega_X^n(\log D)]$$

where $\Omega_X^1(\log D)$ is the sheaf of differential 1-forms on X with logarithmic poles on D and $\Omega_X^k(\log D) := \Lambda^k\Omega_X^1(\log D)$.

We will now relate the logarithmic de Rham complex of (X, D) to the usual de Rham complex of the complement U. By [15, Proposition 3.1.8], the restriction of logarithmic forms to usual forms on U defines a quasi-isomorphism of complexes

$$\Omega_X^\bullet(\log D) \hookrightarrow Rj_*\Omega_U^\bullet, \tag{2.4}$$

where $j : U \hookrightarrow X$ is the inclusion. The log de Rham complex therefore computes the cohomology of U:

$$H^k(U, \mathbb{C}) = \mathbb{H}^k(X, \Omega_X^\bullet(\log D)).$$

The ***Hodge filtration*** on $\Omega_X^\bullet(\log D)$ is defined as follows:

$$F^p\Omega_X^\bullet(\log D) := \Omega_X^{\geq p}(\log D),$$

where $\Omega_X^{\geq p}(\log D)$ is the truncation of $\Omega_X^\bullet(\log D)$ in degrees $< p$:

$$\Omega_X^{\geq p}(\log D) := \cdots \to 0 \to \Omega_X^p(\log D) \to \cdots \to \Omega_X^n(\log D).$$

This provides a Hodge filtration on $H^k(U, \mathbb{C})$, as follows:

$$F^p H^k(U, \mathbb{C}) := \operatorname{im}(\mathbb{H}^k(X, F^p\Omega_X^\bullet(\log D)) \to H^k(U, \mathbb{C})).$$

This filtration is a descending filtration on $H^k(U, \mathbb{C})$:

$$H^k(U, \mathbb{C}) = F^0 \supset F^1 \supset \cdots \supset F^{k-1} \supset F^k \supset \{0\}$$

Consider the spectral sequence associated to the Hodge filtration defined on $\Omega_X^\bullet(\log D)$:

$$\begin{aligned} E_1^{p,q} = H^{p+q}(X, \operatorname{Gr}_F^p \Omega_X^\bullet(\log D)) &\Rightarrow \mathbb{H}^{p+q}(X, \Omega_X^\bullet(\log D)) \\ &= H^{p+q}(U, \mathbb{C}) \end{aligned} \tag{2.5}$$

The associated graded object Gr_F^p can be written as

$$\operatorname{Gr}_F^p \Omega_X^\bullet(\log D) = \Omega_X^p(\log D)[-p],$$

and so we can rewrite the spectral sequence (2.5) in the following way

$$E_1^{p,q} = H^q(\Omega_X^p(\log D)) \Rightarrow H^{p+q}(U, \mathbb{C}).$$

This spectral sequence is commonly referred to as the ***logarithmic Hodge-de Rham spectral sequence***. Similar to the Hodge-de Rham spectral sequence, the log Hodge-de Rham spectral sequence degenerates at E_1 [15, Corollary 3.2.13].

We now briefly review Deligne's theory of weight filtrations on $H^k(U, \mathbb{C})$, following the treatment in [15]. The ***weight filtration*** on $\Omega_X^\bullet(\log D)$ is defined as follows:

$$W_m\Omega_X^k(\log D) := \begin{cases} 0 & m < 0 \\ \Omega_X^k(\log D) & m \geq k \\ \Omega_X^{k-m} \wedge \Omega_X^m(\log D) & 0 \leq m \leq k. \end{cases}$$

This induces the weight filtration on $H^k(U, \mathbb{C})$, namely,

$$W_m H^k(U, \mathbb{C}) := \operatorname{im}(\mathbb{H}^k(X, W_{m-k}\Omega_X^\bullet(\log D)) \to H^k(U, \mathbb{C})). \tag{2.6}$$

The weight filtration is an increasing filtration on $H^k(U,\mathbb{C})$:

$$\{0\} \subset W_0 \subset W_1 \subset \cdots \subset W_{2k-1} \subset W_{2k} = H^k(U,\mathbb{C}).$$

The associated graded pieces of the weight filtration, $\mathrm{Gr}_m^W(\Omega_X^\bullet(\log D))$, can be described using the Poincaré residue isomorphism [15, 3.1.5.2]:

$$\mathrm{Gr}_m^W(\Omega_X^\bullet(\log D)) \xrightarrow{\cong} j_{m_*}\Omega^\bullet_{D^{(m)}}[-m], \tag{2.7}$$

where $D^{(m)}$ denotes the disjoint union of all m-fold intersections of different irreducible components of the divisor D with $D^0 = X$ and $j_m : D^{(m)} \hookrightarrow X$ is the natural inclusion. Consider the spectral sequence associated to the weight filtration W_m on $\Omega_X^\bullet(\log D)$

$$E_1^{-m,k+m} = \mathbb{H}^k(X, \mathrm{Gr}_m^W(\Omega_X^\bullet(\log D))) \Rightarrow H^k(U,\mathbb{C}),$$

which we call the weight spectral sequence. Using (2.7), we can simplify and rewrite the weight spectral sequence as

$$E_1^{-m,k+m} = H^{k-m}(D^{(m)},\mathbb{C}) \Rightarrow H^k(U,\mathbb{C}). \tag{2.8}$$

By [15, Corollary 3.2.13], this spectral sequence degenerates at E_2. The associated graded objects of the weight filtration are exactly the E_2-terms:

$$E_2^{-m,k+m} = \mathrm{Gr}_{k+m}^W(H^k(U,\mathbb{C})).$$

We now provide an alternative method to define the weight filtration on $H^k(U,\mathbb{C})$. This will be useful when we subsequently generalize to weight filtrations on the integral cohomology of the complement. First, we define the ***canonical filtrations*** $\sigma_{\leq p}$ of a complex $K^\bullet$ by

$$\sigma_{\leq p}K^\bullet = \ldots K^{p-2} \to K^{p-1} \to \ker(d^k) \to 0 \to 0 \to \ldots$$

The associated graded objects $\mathrm{Gr}_p^\sigma K^\bullet$ of the canonical filtration are given as

$$\mathrm{Gr}_p^\sigma K^\bullet = K^p[-p].$$

By [30, Lemma 4.9], there is a filtered quasi-isomorphism of complexes

$$(\Omega_X^\bullet(\log D), \sigma) \to (\Omega_X^\bullet(\log D), W).$$

Furthermore, since the complexes $Rj_*\mathbb{C}_U$ and $\Omega_X^\bullet(\log D)$ are quasi-isomorphic from (2.4), we can alternatively write the weight filtration on $H^k(U,\mathbb{C})$ in (2.6) as

$$W_m H^k(U,\mathbb{C}) := \mathrm{im}(\mathbb{H}^k(\sigma_{\leq m-k} Rj_*\mathbb{C}_U) \to H^k(U,\mathbb{C})).$$

We can also recover the weight spectral sequence in (2.8) using the definition of the canonical filtration. The Leray spectral sequence associated to the canonical filtration σ on the complex $Rj_*\mathbb{C}_U$ is given by

$$E_2^{k-m,m} = H^{k-m}(R^m j_*\mathbb{C}_U) \Rightarrow H^k(U,\mathbb{C}). \tag{2.9}$$

As there is a filtered quasi-isomorphism between the complexes $(Rj_*\mathbb{C}_U, \sigma)$ and $(\Omega_X^\bullet(\log D), W)$, we obtain the following isomorphism between the graded pieces

$$\mathrm{Gr}_m^\sigma Rj_*\mathbb{C}_U \cong \mathrm{Gr}_m^W \Omega_X^\bullet(\log D),$$

which itself yields the following

$$R^m j_*\mathbb{C}_U \cong j_{m_*}\Omega^\bullet_{D^{(m)}}.$$

We can use this to rewrite the E_2-terms of the Leray spectral sequence in (2.9) as

$$E_2^{k-m,m} = H^{k-m}(D^{(m)}, \mathbb{C}) \Rightarrow H^k(U, \mathbb{C}),$$

which is exactly the weight spectral sequence as described in (2.8).

2.2.2 Integral Case

We will now develop a version of Deligne's mixed Hodge theory with integral coefficients. We work with (X, D) as before and $U := X \setminus D$ will once again denote the complement. Let $j : U \hookrightarrow X$ be the inclusion of U into X and consider the integral cohomology group $H^k(U, \mathbb{Z})$. This can be computed using the complex $Rj_*\mathbb{Z}_U$:

$$H^k(U, \mathbb{Z}) = \mathbb{H}^k(Rj_*\mathbb{Z}_U).$$

We now define the integral weight filtration on $H^k(U, \mathbb{Z})$ by using the canonical filtration σ introduced earlier on the complex $Rj_*\mathbb{Z}_U$. We define the ***integral weights*** $W_m H^k(U, \mathbb{Z})$ as:

$$W_m H^k(U, \mathbb{Z}) := \mathrm{im}(\mathbb{H}^k(\sigma_{\leq m-k} Rj_*\mathbb{Z}_U) \to H^k(U, \mathbb{Z})). \tag{2.10}$$

This gives the ***integral weight filtration*** on $H^k(U, \mathbb{Z})$:

$$H^k(U, \mathbb{Z}) = W_{2k} \supset W_{2k-1} \supset \cdots \supset W_k \supset \{0\}.$$

We have the Leray spectral sequence with respect to the canonical filtration σ:

$$E_2^{k-m,m} = H^{k-m}(R^m j_*\mathbb{Z}_U) \Rightarrow H^k(U, \mathbb{Z}) \tag{2.11}$$

By [30, Lemma 4.9], we have:

$$R^m j_*\mathbb{Z}_U = \alpha_{m*}\mathbb{Z}_{D^{(m)}}(-m) \tag{2.12}$$

where $D^{(m)}$ is the disjoint union of all m-fold intersections of the different irreducible components of the divisor D as before, $\alpha_m : D^{(m)} \hookrightarrow X$ is the natural inclusion, and $\mathbb{Z}_{D^{(m)}}(-m)$ on $D^{(m)}$ is the Tate twist of $\mathbb{Z}$ by $(2\pi i)^{-m}$. Hence, this allows us to re-write the Leray spectral sequence in (2.11) as:

$$E_2^{k-m,m} = H^{k-m}(D^{(m)}, \mathbb{Z})(-m) \Rightarrow H^k(U, \mathbb{Z})$$

This may be viewed as the integral analogue of the weight spectral sequence in (2.8).

Chapter 3

Picard Algebroids

In this chapter, we present the definition and fundamental properties of Picard algebroids on a smooth projective variety X. The two main results of this chapter are Theorem 3.8 and Theorem 3.15. The first theorem is a classification result for isomorphism classes of Picard algebroids in terms of Hodge theory. The second theorem gives necessary and sufficient conditions for a Picard algebroid to be pre-quantizable as the Atiyah algebroid of infinitesimal symmetries of a line bundle on X. While we believe that our pre-quantization result for Picard algebroids is perhaps known to experts, we have been unable to find a precise treatment of this in the literature. In addition to filling this literature gap, we prove several cohomological and geometrical properties of this subgroup of Picard algebroids which admit pre-quantization in Section 3.2.

Definition 3.1. A ***Picard algebroid*** $(\mathcal{A}, [\cdot,\cdot], \sigma, e)$ on X is a locally free $\mathcal{O}_X$-module $\mathcal{A}$ equipped with a Lie bracket $[\cdot,\cdot]$, a bracket-preserving morphism of $\mathcal{O}_X$-modules $\sigma : \mathcal{A} \to \mathcal{T}_X$, and a central section e, such that the Leibniz rule

$$[a_1, fa_2] = f[a_1, a_2] + \sigma(a_1)(f)a_2 \tag{3.1}$$

holds for all $f \in \mathcal{O}_X$, $a_1, a_2 \in \mathcal{A}$, and the sequence

$$0 \to \mathcal{O}_X \xrightarrow{e} \mathcal{A} \xrightarrow{\sigma} \mathcal{T}_X \to 0 \tag{3.2}$$

is exact. We usually denote the quadruple $(\mathcal{A}, [\cdot,\cdot], \sigma, e)$ simply by $\mathcal{A}$.

Picard algebroids over a a fixed X form a Picard category, with monoidal structure given as follows. For $(\mathcal{A}_i, [\cdot,\cdot]_i, \sigma_i, e_i)_{i=1,2}$ Picard algebroids, we define the Baer sum

$$\mathcal{A}_1 \boxplus \mathcal{A}_2 = (\mathcal{A}_1 \oplus_{\mathcal{T}_X} \mathcal{A}_2) / \langle (e_1, -e_2) \rangle, \tag{3.3}$$

equipped with the morphism $\sigma_1 + \sigma_2$ to $\mathcal{T}_X$, componentwise Lie bracket, and the central section $[(e_1, e_2)]$. We refer to [1] for more details of the categorical properties of Picard algebroids.

Definition 3.2. A ***connection*** on a Picard algebroid $\mathcal{A}$ is a splitting

$$\nabla : \mathcal{T}_X \to \mathcal{A}$$

of the sequence (3.2). The sheaf of connections therefore forms a sheaf of affine spaces over Ω^1_X. Given connections ∇, ∇' on $\mathcal{A}, \mathcal{A}'$, the sum $\nabla + \nabla'$ defines a natural connection on the Baer sum $\mathcal{A} \boxplus \mathcal{A}'$, so that the abovementioned Picard category structure extends to Picard algebroids with connection.

The ***curvature*** associates to any connection ∇ the 2-form $F_\nabla \in \Omega^2_X$ defined by

$$[\nabla\xi_1, \nabla\xi_2] - \nabla[\xi_1, \xi_2] = F_\nabla(\xi_1, \xi_2)e, \tag{3.4}$$

for $\xi_1, \xi_2 \in \mathcal{T}_X(-\log D)$.

Lemma 3.3. *The curvature F_∇ is a closed 2-form. Modifying the connection by $A \in \Omega^1_X$ gives the curvature*

$$F_{\nabla+A} = F_\nabla + dA, \tag{3.5}$$

and if $\psi : \mathcal{A} \to \mathcal{A}'$ is an isomorphism of Picard algebroids, then $F_{\psi\circ\nabla} = F_\nabla$. Finally, if ∇, ∇' are connections for $\mathcal{A}, \mathcal{A}'$, then the curvature of the Baer sum is $F_{\nabla+\nabla'} = F_\nabla + F_{\nabla'}$.

Proof. The Leibniz rule 3.1 together with the centrality of e imply that $\xi(f) = [\nabla\xi, fe]$ for $\xi \in \mathcal{T}_X$, $f \in \mathcal{O}_X$. Using this to compute the exterior derivative, we obtain

$$\begin{aligned} dF_\nabla(\xi_1, \xi_2, \xi_3) = [\nabla\xi_1, [\nabla\xi_2, \nabla\xi_3] - \nabla[\xi_2, \xi_3]] + \cdots \\ - ([\nabla[\xi_1, \xi_2], \nabla\xi_3] - \nabla[[\xi_1, \xi_2], \xi_3]) + \cdots , \end{aligned}$$

where the omitted terms are cyclic permutations of the preceding terms. When taken with their cyclic permutations, the second and third displayed summands cancel; similarly, the first and fourth summands vanish using the Jacobi identity. For property (3.5), if $\nabla' = \nabla + A$, then by (3.4) we have

$$F_{\nabla'}(\xi_1, \xi_2) = F_\nabla(\xi_1, \xi_2) + [\nabla\xi_1, A\xi_2] + [A\xi_1, \nabla\xi_2] - A[\xi_1, \xi_2] = (F_\nabla + dA)(\xi_1, \xi_2),$$

as required. For the remaining statements, note firstly that an isomorphism of Picard algebroids is a bracket-preserving isomorphism $\psi : \mathcal{A} \to \mathcal{A}'$ of extensions, giving $F_{\psi\circ\nabla} = F_\nabla$ immediately. The final assertion follows immediately from the fact that the Lie bracket on the Baer sum $\mathcal{A} \boxplus \mathcal{A}'$ is defined componentwise. □

Proposition 3.4. *The splitting determined by the connection ∇ takes the Lie bracket on $\mathcal{A}$ to the bracket on $\mathcal{T}_X \oplus \mathcal{O}_X$ given by*

$$[\xi_1 + f_1, \xi_2 + f_2] = [\xi_1, \xi_2] + \xi_1(f_2) - \xi_2(f_1) + F_\nabla(\xi_1, \xi_2),$$

where $\xi_1, \xi_2 \in \mathcal{T}_X$ and $f_1, f_2 \in \mathcal{O}_X$.

Proof. The claim follows immediately from the identity

$$\begin{aligned} [\nabla\xi_1 + f_1e, \nabla\xi_2 + f_2e] &= [\nabla(\xi_1), \nabla(\xi_2)] + [\nabla(\xi_1), f_2e] - [\nabla(\xi_2), f_1e] + [f_1e, f_2e] \\ &= \nabla[\xi_1, \xi_2] + F_\nabla(\xi_1, \xi_2)e + \xi_1(f_2)e - \xi_2(f_1)e. \end{aligned}$$

□

Definition 3.5. Let $B \in \Omega_X^{2,\mathrm{cl}}$ be a closed 2-form and $\mathcal{A}$ be a Picard algebroid on X. We define the ***twist*** of $\mathcal{A}$ to be the Picard algebroid $(\mathcal{A}, [\cdot,\cdot]_B, \sigma, e)$ where $[\cdot,\cdot]_B$ is given as

$$[a_1, a_2]_B = [a_1, a_2] + B(\sigma(a_1), \sigma(a_2))e,$$

for $a_1, a_2 \in \mathcal{A}$. We denote this algebroid by $\mathcal{A}_B$.

Example 3.6 (Split Picard algebroid)**.** Let $B \in \Omega_X^{2,\mathrm{cl}}$ be a closed 2-form. We can twist the trivial extension $\mathcal{T}_X \oplus \mathcal{O}_X$ by B

$$[\xi_1 + f_1, \xi_2 + f_2] = [\xi_1, \xi_2] + \xi_1(f_2) - \xi_2(f_1) + B(\xi_1, \xi_2)$$

Hence, we obtain a Picard algebroid admitting a global connection with curvature B.

Example 3.7 (Atiyah algebroid of a line bundle)**.** Let $\mathcal{L}$ be a holomorphic line bundle on X, the Atiyah algebroid of $\mathcal{L}$, $\mathrm{At}(\mathcal{L})$, is the sheaf of first-order differential operators on sections of $\mathcal{L}$. We show that this admits the structure of a Picard algebroid on X. Let $\gamma \in \mathrm{At}(\mathcal{L})$ and consider the symbol σ_γ of γ which is defined by

$$\sigma_\gamma(f)s = \xi(fs) - f\xi(s)$$

for $f \in \mathcal{O}_X$ and $s \in \mathcal{L}$. The symbol σ_γ is a derivation of $\mathcal{O}_X$ and so we have a map from $\mathrm{At}(\mathcal{L})$ to $\mathcal{T}_X$ given by $\xi \mapsto \sigma_\gamma$. This is a surjective map and the kernel of this map is the 0-th order differential operators on $\mathcal{L}$ which is just $\mathcal{O}_X$. Hence, we have the short exact sequence

$$0 \to \mathcal{O}_X \to \mathrm{At}(\mathcal{L}) \to \mathcal{T}_X \to 0$$

Moreover, for $\gamma_1, \gamma_2 \in \mathrm{At}(\mathcal{L})$, the bracket of γ_1 and γ_2 satisfies the Leibniz rule

$$[\gamma_1, f\gamma_2] = f[\gamma_1, \gamma_2] + \gamma_1(f)\gamma_2$$

In addition, we have $\sigma_{[\gamma_1,\gamma_2]} = [\sigma_{\gamma_1}, \sigma_{\gamma_2}]$ and so the map $\xi \mapsto \sigma_\gamma$ is a bracket-preserving morphism. Thus, we can conclude that $\mathrm{At}(\mathcal{L})$ is a Picard algebroid on X and from now on, we call this the ***Picard algebroid of infinitesimal symmetries*** of $\mathcal{L}$ and denote it as $\mathcal{A}_{\mathcal{L}}$.

3.1 Classification

Theorem 3.8. *Picard algebroids on X are classified by the hypercohomology group:*

$$\mathbb{H}^2(F^1\Omega_X^\bullet).$$

with addition deriving from the Baer sum operation (3.3)*.*

Proof. Let $\mathcal{A}$ be a Picard algebroid on X and take an open affine cover $\{U_i\}$ of X. Then over each U_i we may choose a connection ∇_i, which has curvature $B_i \in \Omega_{U_i}^{2,\mathrm{cl}}$. On the double overlap $U_i \cap U_j$, we have by Lemma 3.3 that

$$B_i - B_j = dA_{ij},$$

where $A_{ij} = \nabla_i - \nabla_j \in \Omega^1_{U_i \cap U_j}$. Since $A_{ij} + A_{jk} + A_{ki} = 0$ on triple overlaps, we have a Čech-de Rham cocycle (A_{ij}, B_i). Modifying our initial choice of local connections shifts this cocycle by a coboundary, so that we have a well-defined map from Picard algebroids to the hypercohomology group $\mathbb{H}^2(F^1\Omega^\bullet_X)$. An isomorphism $\psi : \mathcal{A} \to \mathcal{A}'$ of Picard algebroids induces an isomorphism of sheaves of connections which preserves the curvature (Lemma 3.3), so that $\mathcal{A}$ and $\mathcal{A}'$ give rise to cohomologous cocycles by the above argument.

To show that this classifying map is surjective, note that any such cocycle (A_{ij}, B_i) may be used to construct a Picard algebroid as follows: on each U_i we define local Picard algebroids $\mathcal{A}_i$ to be trivial extensions $\mathcal{A}_i = \mathcal{O}_{U_i} \oplus \mathcal{T}_{U_i}$ equipped with brackets as in Proposition 3.4:

$$[f_1 + \xi_1, f_2 + \xi_2]_i = [\xi_1, \xi_2] + \xi_1(f_2) - \xi_2(f_1) + B_i(\xi_1, \xi_2).$$

We then glue $\mathcal{A}_i$ to $\mathcal{A}_j$ over $U_i \cap U_j$ using the isomorphism of Picard algebroids

$$\Phi_{ij} : f + \xi \mapsto f + \xi + A_{ij}(\xi),$$

which satisfies $\Phi_{ki}\Phi_{jk}\Phi_{ij} = 1$, yielding a global Picard algebroid and so proving surjectivity. To obtain the correspondence between Baer sum and addition in cohomology, note that by choosing local connections ∇_i, ∇'_i on $\mathcal{A}, \mathcal{A}'$, the sum $\nabla_i + \nabla'_i$ defines local connections for the Baer sum $\mathcal{A} \boxplus \mathcal{A}'$ and by Lemma 3.3 the corresponding cocycle for $\nabla_i + \nabla'_i$ is the sum of the cocycles for the summands, as required. □

Example 3.9 (Family of split Picard algebroids)**.** Given two closed 2-forms $B, B' \in \Omega^{2,\mathrm{cl}}_X$, we can twist the trivial extension $\mathcal{T}_X \oplus \mathcal{O}_X$ by B and B' as shown in Example 3.6 to obtain split Picard algebroids $\mathcal{A}_B$ and $\mathcal{A}_{B'}$ respectively. An automorphism of the trivial extension given by $\Phi : \xi + f \mapsto \xi + f + A(\xi)$ for $A \in H^0(\Omega^1_X)$ takes $[\cdot,\cdot]_B$ to $[\cdot,\cdot]_{B'}$ if and only if $B' - B = dA$. But any global closed form on X is exact if and only if it vanishes, hence we see that $\mathcal{A}_B \cong \mathcal{A}_{B'}$ if and only if $B' = B$. Therefore we obtain a universal family of split Picard algebroids over $H^0(\Omega^{2,\mathrm{cl}}_X)$ by this construction.

Theorem 3.10. *Any extension of $\mathcal{T}_X$ by $\mathcal{O}_X$ admits the structure of a Picard algebroid on X and if $\mathcal{A}, \mathcal{A}'$ are two Picard algebroids with isomorphic underlying extensions, then $\mathcal{A}'$ is isomorphic to $\mathcal{A}$ up to a twist by a closed 2-form as defined in Definition 3.5.*

Proof. Given a Picard algebroid described by the cocycle $\{A_{ij}, B_i\}$, the cocycle $\{A_{ij}\}$ represents the underlying extension class of the Picard algebroid. The forgetful map from $\{A_{ij}, B_i\}$ to $\{A_{ij}\}$ induces the map in cohomology

$$\Phi : \mathbb{H}^2(F^1\Omega^\bullet_X) \to H^1(\Omega^1_X).$$

We will now show that this map fits into the following short exact sequence

$$0 \to H^0(\Omega^{2,\mathrm{cl}}_X) \to \mathbb{H}^2(F^1\Omega^\bullet_X) \xrightarrow{\Phi} H^1(\Omega^1_X) \to 0.$$

By the E_1-degeneration of the Hodge-de Rham spectral sequence [13, Prop 5.3] and the fact that the de Rham complex $\Omega^\bullet_X$ with the Hodge filtration $F^\bullet$ is a biregular filtered complex, we have the short exact sequence in hypercohomology

$$0 \to \mathbb{H}^2(F^2\Omega^\bullet_X) \to \mathbb{H}^2(F^1\Omega^\bullet_X) \to H^1(\Omega^1_X) \to 0.$$

But the group $\mathbb{H}^2(F^2\Omega_X^\bullet)$ can be identified with $H^0(\Omega_X^{2,\mathrm{cl}})$ which completes the proof. □

Definition 3.11. Let $\mathcal{A}$ be a Picard algebroid on X. We say that $\mathcal{A}$ is ***locally trivial*** if a flat connection (a splitting of $\mathcal{A}$ with zero curvature) exists in a neighbourhood of every point.

Proposition 3.12. *Locally trivial Picard algebroids on X are classified by the subgroup*

$$H^1(\Omega_X^{1,\mathrm{cl}}) \subset \mathbb{H}^2(F^1\Omega_X^\bullet).$$

Proof. As described in the proof of Theorem 3.8, a Picard algebroid is classified by the cohomological data $\{A_{ij}, B_i\}$ where $B_i - B_j = dA_{ij}$. In the locally trivial case, local flat splittings may be chosen, in which case B_i and B_j are zero, yielding a Čech 1-cocycle A_{ij} for $\Omega_X^{1,\mathrm{cl}}$, as required. The inclusion of complexes

$$\Omega_X^{1,\mathrm{cl}} \hookrightarrow [\Omega_X^1 \to \Omega_X^{2,\mathrm{cl}}] \tag{3.6}$$

induces an inclusion on first cohomology groups. □

Remark 3.13. Since X is assumed to be projective, the inclusion of complexes in (3.6) is a quasi-isomorphism and so we have an isomorphism $H^1(\Omega_X^{1,\mathrm{cl}}) \cong \mathbb{H}^2(F^1\Omega_X^\bullet)$. This means that all Picard algebroids on X are locally trivial. However, if we work in the Zariski topology and suppose that X is not projective, the inclusion in (3.6) is in general not a quasi-isomorphism. As an example of this, consider the Picard algebroid $\mathcal{T}_X \oplus \mathcal{O}_X$ with the bracket

$$[\xi_1 + f_1, \xi_2 + f_2] = [\xi_1, \xi_2] + \xi_1(f_2) - \xi_2(f_1) + B(\xi_1, \xi_2)$$

as given in Example 3.6. The closed 2-form $B \in \Omega_X^{2,\mathrm{cl}}$ is not necessarily locally exact in the Zariski topology and so this Picard algebroid is not locally trivial.

3.2 Pre-quantization of Picard algebroids

In this section, we will study the pre-quantization problem for Picard algebroids on X. Recall from Example 3.7 that given a line bundle $\mathcal{L}$ on X, there is a Picard algebroid $\mathcal{A}_\mathcal{L}$ associated to $\mathcal{L}$. The pre-quantization problem asks the following: given a Picard algebroid $\mathcal{A}$ on X, under what conditions can $\mathcal{A}$ be pre-quantized, i.e., there exists a line bundle $\mathcal{L}$ on X such that $\mathcal{A} = \mathcal{A}_\mathcal{L}$. We begin by providing a proposition which will be important for solving the pre-quantization problem later.

Proposition 3.14. *Let $\mathcal{L}$ be a line bundle on X and $\mathcal{A}_\mathcal{L}$ be the Picard algebroid of infinitesimal symmetries of $\mathcal{L}$. Suppose that $\{g_{ij}\} \in H^1(\mathcal{O}_X^\times)$ are the transition functions of $\mathcal{L}$ and $[\mathcal{A}_\mathcal{L}] \in H^1(\Omega_X^{1,\mathrm{cl}})$ is the class of $\mathcal{A}_\mathcal{L}$, then, we have $[\mathcal{A}_\mathcal{L}] = \mathrm{dlog}(\{g_{ij}\})$ where* dlog *is the map in cohomology:*

$$H^1(\mathcal{O}_X^\times) \overset{\mathrm{dlog}}{\to} H^1(\Omega_X^{1,\mathrm{cl}})$$

Proof. First, we consider an element $\gamma \in \mathcal{A}_\mathcal{L}$. By definition, γ is a first-order differential operator on $\mathcal{L}$ so we have:

$$\gamma(fs) = f\gamma(s) + \sigma_\gamma(f)s$$

where $\sigma_\gamma \in \mathcal{T}_X$, $f \in \mathcal{O}_X$, $s \in \mathcal{L}$. Now, we choose an open affine cover $\{U_i\}$ of X and take two local splittings s_i, s_j of the short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{A} \to \mathcal{T}_X \to 0$$

On this open cover, we can also trivialize the section s as $\{e_i\}$ with $e_i = g_{ij}e_j$ and where g_{ij} are the transition functions of $\mathcal{L}$.

For the open set U_i, we have the following:

$$\gamma_i(fe_i) = f\gamma_i(e_i) + \sigma_{\gamma_i}(f)e_i$$

where γ_i is the restriction of γ to U_i. We can take $\gamma_i(e_i) = 0$ which implies that $\gamma_i(fe_i) = \sigma_{\gamma_i}(f)e_i$. Analogously, on the open set U_j, we have $\gamma_j(fe_j) = \sigma_{\gamma_j}(f)e_j$. Therefore, on the double intersection U_{ij}, we have:

$$\begin{aligned} \gamma_j - \gamma_i(e_i) &= \gamma_j(e_i) \\ &= \gamma_j(g_{ij}e_j) \\ &= \sigma_{\gamma_{ij}}(g_{ij})e_j \\ &= \sigma_{\gamma_{ij}}(g_{ij})g_{ij}^{-1}e_i \\ &= i_{\sigma_{\gamma_{ij}}} \mathrm{dlog}\{g_{ij}\}e_i \end{aligned}$$

Thus, we have shown that $\gamma_j - \gamma_i = i_{\sigma_{\gamma_{ij}}} \mathrm{dlog}\{g_{ij}\}$. □

Theorem 3.15. *Let $\mathcal{A}$ be a Picard algebroid on X. There exists a pre-quantization for $\mathcal{A}$ if and only if the class of $\mathcal{A}$, $[\mathcal{A}] \in H^1(\Omega_X^{1,cl}) \subset H^2(X, \mathbb{C})$, lies in the image of $H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{C})$.*

Proof. From the above proposition, we know that there exist a pre-quantization for the Picard algebroid $\mathcal{A}$ if and only if its class $[\mathcal{A}]$ lies in the image of the map

$$H^1(\mathcal{O}_X^\times) \overset{\mathrm{dlog}}{\to} H^1(\Omega_X^{1,\mathrm{cl}})$$

To prove our result, we will show that $[\mathcal{A}]$ lies in the image of $H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{C})$ if and only if $[\mathcal{A}]$ lies in the image of the above dlog map. First, consider the following commutative diagram of complexes

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \uparrow & & \uparrow & & & & \\ 0 & \longrightarrow & \mathbb{C}_X^\times & \longrightarrow & \mathcal{O}_X^\times & \xrightarrow{\mathrm{dlog}} & \Omega_X^{1,\mathrm{cl}} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \| & & \\ 0 & \longrightarrow & \mathbb{C}_X & \longrightarrow & \mathcal{O}_X & \xrightarrow{d} & \Omega_X^{1,\mathrm{cl}} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & & & \\ & & \mathbb{Z}_X & = & \mathbb{Z}_X & & & & \\ & & \uparrow & & \uparrow & & & & \\ & & 0 & & 0 & & & & \end{array} \tag{3.7}$$

We have the long exact sequence associated to the top horizontal short exact sequence

$$\cdots \to H^1(\mathcal{O}_X^\times) \stackrel{\text{dlog}}{\to} H^1(\Omega_X^{1,\text{cl}}) \stackrel{\delta}{\to} H^2(X, \mathbb{C}^\times) \to \dots \tag{3.8}$$

If $[\mathcal{A}]$ lies in the image of the dlog map, then $\delta([\mathcal{A}]) = 1 \in H^2(X, \mathbb{C}^\times)$. It follows that $[\mathcal{A}]$ must lie in the kernel of the exponential map $H^2(X, \mathbb{C}) \to H^2(X, \mathbb{C}^\times)$ and hence in the image of $H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{C})$.

Conversely, if $[\mathcal{A}]$ lies in the image of $H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{C})$, then $[\mathcal{A}]$ is mapped to the trivial element in $H^2(X, \mathbb{C}^\times)$ under the exponential map. If we use the sequence (3.8) again, this implies that $[\mathcal{A}]$ must lie in the image of $H^1(\mathcal{O}_X^\times) \stackrel{\text{dlog}}{\to} H^1(\Omega_X^{1,\text{cl}})$ which concludes the proof. □

Remark 3.16. Another interpretation of the above theorem is that a closed 1-form on X is an integral cohomology class if and only if it is the de Rham cohomology class of a holomorphic line bundle $\mathcal{L}$ on X. The class $[\mathcal{A}]$ above is the Atiyah class of the holomorphic line bundle $\mathcal{L}$.

Corollary 3.17. *The Picard algebroids on X which admit pre-quantization are classified by the following subgroup of $H^1(\Omega_X^{1,cl})$:*

$$H^{1,1}(X, \mathbb{Z}) := H^1(\Omega_X^1) \cap im(H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{C})) \subset H^1(\Omega_X^{1,cl})$$

Proof. Let $\mathcal{A}$ be a Picard algebroid on X which admits pre-quantization. From the above theorem, we know that the the class of $\mathcal{A}$, $[\mathcal{A}] \in H^1(\Omega_X^{1,\text{cl}})$, lies in the image of the map $H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{C})$. It remains to prove that $[\mathcal{A}]$ must be a class in $H^1(\Omega_X^1)$. From the E_1-degeneration of the Hodge-de Rham spectral sequence, we have the short exact sequence

$$0 \to H^0(\Omega_X^{2,\text{cl}}) \to H^1(\Omega_X^{1,\text{cl}}) \to H^1(\Omega_X^1) \to 0$$

Now, the group $H^0(\Omega_X^{2,\text{cl}})$ is the holomorphic $H^{2,0}$ part of the Hodge decomposition of $H^2(X, \mathbb{C})$ so its intersection with $H^2(X, \mathbb{Z})$ must be zero. This implies that the class $[\mathcal{A}] \in H^1(\Omega_X^{1,\text{cl}})$ must be a class in $H^1(\Omega_X^1)$.

Suppose that we have a Picard algebroid $\mathcal{A}$ whose class $[\mathcal{A}]$ lies in the subgroup $H^{1,1}(X, \mathbb{Z})$. By the definition of $H^{1,1}(X, \mathbb{Z})$, $[\mathcal{A}]$ lies in the image of $H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{C})$ and so by Theorem 3.15, the Picard algebroid $\mathcal{A}$ admits a pre-quantization. □

Remark 3.18. By the Lefschetz theorem on $(1,1)$-classes, the group $H^{1,1}(X, \mathbb{Z})$ of Picard algebroids on X admitting pre-quantization defined above is precisely the Néron-Severi group of X.

Proposition 3.19. *There is an action on the group $H^{1,1}(X, \mathbb{Z})$ of Picard algebroids on X admitting pre-quantization by the group of flat holomorphic line bundles on X.*

Proof. We know that the Picard algebroids $\mathcal{A}$ on X that can be pre-quantized are the ones whose class $[\mathcal{A}]$ can be lifted under the map

$$H^1(\mathcal{O}_X^\times) \stackrel{\text{dlog}}{\to} H^1(\Omega_X^{1,\text{cl}})$$

Consider the short exact sequence

$$0 \to \mathbb{C}_X^\times \to \mathcal{O}_X^\times \stackrel{\text{dlog}}{\to} \Omega_X^{1,\text{cl}} \to 0,$$

and its associated long exact sequence in cohomology

$$\cdots \to H^0(\Omega_X^{1,\mathrm{cl}}) \to H^1(X, \mathbb{C}^\times) \to H^1(\mathcal{O}_X^\times) \overset{\mathrm{dlog}}{\to} H^1(\Omega_X^{1,\mathrm{cl}}) \to \ldots.$$

Two lifts of the class $[\mathcal{A}]$ differ by a class in $H^1(\mathcal{O}_X^\times)$ given in the image of the map

$$H^1(X, \mathbb{C}^\times) \to H^1(\mathcal{O}_X^\times) \tag{3.9}$$

However, this map is not injective as there are $\mathbb{C}^\times$-local systems on X that are holomorphically trivial as holomorphic line bundles which are given by $H^0(\Omega_X^{1,\mathrm{cl}})$, the group of global 1-forms on X. We denote the image of the above map by $\mathcal{K}$ and we will now study this group. From the short exact sequence

$$0 \to \mathbb{Z}_X \to \mathbb{C}_X \to \mathbb{C}_X^\times \to 0,$$

we obtain that:

$$0 \to \frac{H^1(X, \mathbb{C})}{H^1(X, \mathbb{Z})} \to H^1(X, \mathbb{C}^\times) \to H^2(X, \mathbb{Z})_{\mathrm{tors}} \to 0$$

Using the exponential sheaf sequence on X, the image of the map in (3.9) will surject to $H^2(X, \mathbb{Z})_{\mathrm{tors}}$ with kernel $\frac{H^1(\mathcal{O}_X)}{H^1(X,\mathbb{Z})}$, i.e., the Jacobian of X:

$$0 \to \mathrm{Jac}(X) \to \mathcal{K} \to H^2(X, \mathbb{Z})_{\mathrm{tors}} \to 0$$

The group $\mathcal{K}$ is exactly the group of flat holomorphic line bundles on X. $\square$

Chapter 4

Log Picard Algebroids and Meromorphic Line Bundles

In this chapter, we develop the theory of log Picard algebroids which are generalizations of Picard algebroids where the tangent sheaf $\mathcal{T}_X$ is replaced by the logarithmic tangent sheaf $\mathcal{T}_X(-\log D)$. We use mixed Hodge theory introduced earlier in Section 2.2 to formulate and prove classification results for log Picard algebroids. We then provide concrete non-trivial examples of log Picard algebroids on (X, D) in contexts where non-trivial Picard algebroids do not exist. This illuminates that log Picard algebroids are interesting objects of study on its own right.

In order to solve the prequantization problem for log Picard algebroids, we need to figure out what the prequantum object is. To this end, we introduce the notion of a meromorphic line bundle: a locally free rank one $\mathcal{O}_X(*D)$-module whose transition functions lie in $\mathcal{O}_X^\times(*D)$, where $\mathcal{O}_X(*D)$ and $\mathcal{O}_X^\times(*D)$ are the meromorphic modifications of the sheaf of regular functions $\mathcal{O}_X$ on X and the sheaf of invertible regular functions $\mathcal{O}_X^\times$ on X respectively. It turns out that these meromorphic line bundles display a wide range of interesting properties which warrants its own treatment. We describe explicitly the geometry of the total space of these bundles in Section 4.2.1 and prove classification results in Section 4.2.2 using integral mixed Hodge theory as introduced earlier in Section 2.2.2.

We then proceed to prove the prequantization theorem for log Picard algebroids in Theorem 4.41. Finally, we use holomorphic non-algebraic approaches to provide constructions of meromorphic line bundles for a smooth elliptic curve divisor in the projective plane $\mathbb{P}^2$.

4.1 Log Picard Algebroids

Throughout this section, we will work with (X, D) where X is a smooth projective variety over the complex numbers $\mathbb{C}$ and D is a simple normal crossings divisor in X. We will also work in the analytic topology instead of the Zariski topology.

Definition 4.1. A log Picard algebroid $(\mathcal{A}, [\cdot,\cdot], \sigma, e)$ on (X, D) is a locally free $\mathcal{O}_X$-module $\mathcal{A}$ equipped with a Lie bracket $[\cdot,\cdot]$, a bracket-preserving morphism of $\mathcal{O}_X$-modules $\sigma : \mathcal{A} \to \mathcal{T}_X(-\log D)$, and a central section e, such that the Leibniz rule

$$[a_1, fa_2] = f[a_1, a_2] + \sigma(a_1)(f)a_2 \tag{4.1}$$

holds for all $f \in \mathcal{O}_X$, $a_1, a_2 \in \mathcal{A}$, and the sequence

$$0 \to \mathcal{O}_X \xrightarrow{e} \mathcal{A} \xrightarrow{\sigma} \mathcal{T}_X(-\log D) \to 0 \tag{4.2}$$

is exact. We usually denote the quadruple $(\mathcal{A}, [\cdot,\cdot], \sigma, e)$ simply by $\mathcal{A}$.

Remark 4.2. When the divisor is empty, we recover the notion of a Picard algebroid as given in [1].

Log Picard algebroids over a fixed base (X, D) form a Picard category, with monoidal structure given as follows. For $(\mathcal{A}_i, [\cdot,\cdot]_i, \sigma_i, e_i)_{i=1,2}$ log Picard algebroids, we define the Baer sum

$$\mathcal{A}_1 \boxplus \mathcal{A}_2 = (\mathcal{A}_1 \oplus_{\mathcal{T}_X(-\log D)} \mathcal{A}_2)/\left\langle (e_1, -e_2)\right\rangle, \tag{4.3}$$

equipped with the morphism $\sigma_1 + \sigma_2$ to $\mathcal{T}_X(-\log D)$, componentwise Lie bracket, and the central section $[(e_1, e_2)]$.

Definition 4.3. A ***connection*** on a log Picard algebroid $\mathcal{A}$ is a splitting

$$\nabla : \mathcal{T}_X(-\log D) \to \mathcal{A}$$

of the sequence (4.2). The sheaf of connections therefore forms a sheaf of affine spaces over $\Omega^1_X(\log D)$. Given connections ∇, ∇' on $\mathcal{A}, \mathcal{A}'$, the sum $\nabla + \nabla'$ defines a natural connection on the Baer sum $\mathcal{A} \boxplus \mathcal{A}'$, so that the abovementioned Picard category structure extends to log Picard algebroids with connection.

The ***curvature*** associates to any connection ∇ the 2-form $F_\nabla \in \Omega^2_X(\log D)$ defined by

$$[\nabla\xi_1, \nabla\xi_2] - \nabla[\xi_1, \xi_2] = F_\nabla(\xi_1, \xi_2)e, \tag{4.4}$$

for $\xi_1, \xi_2 \in \mathcal{T}_X(-\log D)$.

Lemma 4.4. *The curvature F_∇ is a closed logarithmic 2-form. Modifying the connection by $A \in \Omega^1_X(\log D)$ gives the curvature*

$$F_{\nabla + A} = F_\nabla + dA, \tag{4.5}$$

and if $\psi : \mathcal{A} \to \mathcal{A}'$ is an isomorphism of log Picard algebroids, then $F_{\psi\circ\nabla} = F_\nabla$. Finally, if ∇, ∇' are connections for $\mathcal{A}, \mathcal{A}'$, then the curvature of the Baer sum is $F_{\nabla+\nabla'} = F_\nabla + F_{\nabla'}$.

Proof. The proof of this is exactly the same as the proof of Lemma 3.3. □

Proposition 4.5. *The splitting determined by the connection ∇ takes the Lie bracket on $\mathcal{A}$ to the bracket on $\mathcal{T}_X(-\log D) \oplus \mathcal{O}_X$ given by*

$$[\xi_1 + f_1, \xi_2 + f_2] = [\xi_1, \xi_2] + \xi_1(f_2) - \xi_2(f_1) + F_\nabla(\xi_1, \xi_2), \tag{4.6}$$

where $\xi_1, \xi_2 \in \mathcal{T}_X(-\log D)$ and $f_1, f_2 \in \mathcal{O}_X$.

Proof. The proof is exactly the same the proof of Proposition 3.4. □

Definition 4.6. Let $B \in \Omega^{2,\mathrm{cl}}_X(\log D)$ be a closed logarithmic 2-form and $\mathcal{A}$ be a log Picard algebroid on (X, D). We define the ***twist*** of $\mathcal{A}$ to be the log Picard algebroid $(\mathcal{A}, [\cdot,\cdot]_B, \sigma, e)$ where $[\cdot,\cdot]_B$ is given as

$$[a_1, a_2]_B = [a_1, a_2] + B(\sigma(a_1), \sigma(a_2))e,$$

for $a_1, a_2 \in \mathcal{A}$. We denote this algebroid by $\mathcal{A}_B$.

Example 4.7 (Split log Picard algebroid)**.** Let $B \in \Omega_X^{2,\mathrm{cl}}(\log D)$ be a closed logarithmic 2-form. We can twist the trivial extension $\mathcal{T}_X(-\log D) \oplus \mathcal{O}_X$ by B

$$[\xi_1 + f_1, \xi_2 + f_2]_B = [\xi_1, \xi_2] + \xi_1(f_2) - \xi_2(f_1) + B(\xi_1, \xi_2).$$

Hence, we obtain a log Picard algebroid admitting a global connection with curvature B.

4.1.1 Classification

Theorem 4.8. *Log Picard algebroids on (X, D) are classified by the hypercohomology group:*

$$\mathbb{H}^2(F^1\Omega_X^\bullet(\log D)),$$

with addition deriving from the Baer sum operation in 4.3.

Proof. The proof of this theorem is essentially identical in nature to the proof of Theorem 3.8. Let $\mathcal{A}$ be a log Picard algebroid on (X, D) and take an open affine cover $\{U_i\}$ of X. Then over each U_i we may choose a connection ∇_i, which has curvature $B_i \in \Omega^{2,\mathrm{cl}}(U_i, \log D)$. On the double overlap $U_i \cap U_j$, we have by Lemma 4.4 that

$$B_i - B_j = dA_{ij},$$

where $A_{ij} = \nabla_i - \nabla_j \in \Omega^1(U_i \cap U_j, \log D)$. Since $A_{ij} + A_{jk} + A_{ki} = 0$ on triple overlaps, we have a Čech-de Rham cocycle (A_{ij}, B_i). Modifying our initial choice of local connections shifts this cocycle by a coboundary, so that we have a well-defined map from log Picard algebroids to the hypercohomology group $\mathbb{H}^2(F^1\Omega_X^\bullet(\log D))$. An isomorphism $\psi : \mathcal{A} \to \mathcal{A}'$ of log Picard algebroids induces an isomorphism of sheaves of connections which preserves the curvature (Lemma 4.4), so that $\mathcal{A}$ and $\mathcal{A}'$ give rise to cohomologous cocycles by the above argument.

To show that this classifying map is surjective, note that any such cocycle (A_{ij}, B_i) may be used to construct a log Picard algebroid as follows: on each U_i we define local log Picard algebroids $\mathcal{A}_i$ to be trivial extensions $\mathcal{A}_i = \mathcal{O}_{U_i} \oplus \mathcal{T}_{U_i}(-\log D)$ equipped with brackets as in Proposition 4.5:

$$[f_1 + \xi_1, f_2 + \xi_2]_i = [\xi_1, \xi_2] + \xi_1(f_2) - \xi_2(f_1) + B_i(\xi_1, \xi_2).$$

We then glue $\mathcal{A}_i$ to $\mathcal{A}_j$ over $U_i \cap U_j$ using the isomorphism of log Picard algebroids

$$\Phi_{ij} : f + \xi \mapsto f + \xi + A_{ij}(\xi),$$

which satisfies $\Phi_{ki}\Phi_{jk}\Phi_{ij} = 1$, yielding a global log Picard algebroid and so proving surjectivity. To obtain the correspondence between Baer sum and addition in cohomology, note that by choosing local connections ∇_i, ∇_i' on $\mathcal{A}, \mathcal{A}'$, the sum $\nabla_i + \nabla_i'$ defines local connections for the Baer sum $\mathcal{A} \boxplus \mathcal{A}'$ and by Lemma 4.4 the corresponding cocycle for $\nabla_i + \nabla_i'$ is the sum of the cocycles for the summands, as required. □

Example 4.9 (Family of split log Picard algebroids)**.** Given two closed logarithmic 2-forms $B, B' \in \Omega_X^{2,\mathrm{cl}}(\log D)$, we can twist the trivial extension $\mathcal{T}_X(-\log D) \oplus \mathcal{O}_X$ by B and B' as shown in Example 4.7

to obtain split log Picard algebroids $\mathcal{A}_B$ and $\mathcal{A}_{B'}$ respectively. An automorphism of the trivial extension given by $\Phi : \xi + f \mapsto \xi + f + A(\xi)$ for $A \in H^0(\Omega^1_X(\log D))$ takes $[\cdot,\cdot]_B$ to $[\cdot,\cdot]_{B'}$ if and only if $B' - B = dA$. But by [15, Proposition 1.3.2], any global closed logarithmic form on (X, D) is exact if and only if it vanishes, hence we see that $\mathcal{A}_B \cong \mathcal{A}_{B'}$ if and only if $B' = B$. Therefore we obtain a universal family of split log Picard algebroids over $H^0(\Omega^{2,\mathrm{cl}}_X(\log D))$ by this construction.

Theorem 4.10. *Any extension of $\mathcal{T}_X(-\log D)$ by $\mathcal{O}_X$ admits the structure of a log Picard algebroid, and if $\mathcal{A}, \mathcal{A}'$ are two log Picard algebroids with isomorphic underlying extensions, then $\mathcal{A}'$ is isomorphic to $\mathcal{A}$ up to a twist by a closed logarithmic 2-form as defined in Definition 4.6.*

Proof. We follow the proof of Theorem 3.10 with appropriate modifications for logarithmic cohomology groups. Consider a log Picard algebroid described by the cocycle $\{A_{ij}, B_i\}$, the cocycle $\{A_{ij}\}$ represents the underlying extension class of the log Picard algebroid. The forgetful map from $\{A_{ij}, B_i\}$ to $\{A_{ij}\}$ induces the map in cohomology

$$\Phi : \mathbb{H}^2(F^1\Omega^\bullet_X(\log D)) \to H^1(\Omega^1_X(\log D)).$$

We will now show that this map fits into the following short exact sequence

$$0 \to H^0(\Omega^{2,\mathrm{cl}}_X(\log D)) \to \mathbb{H}^2(F^1\Omega^\bullet_X(\log D)) \xrightarrow{\Phi} H^1(\Omega^1_X(\log D)) \to 0.$$

The spectral sequence associated to the Hodge filtration on $\Omega^\bullet_X(\log D)$ is exactly the logarithmic Hodge-de Rham spectral sequence, and by [15, Corollary 3.2.13], we know that this spectral sequence degenerates at E_1. In addition, since the log de Rham complex $\Omega^\bullet_X(\log D)$ with the Hodge filtration is a biregular filtered complex and so by [15, Proposition 1.3.2], the following sequence in hypercohomology is short exact

$$0 \to \mathbb{H}^2(F^2\Omega^\bullet_X(\log D)) \to \mathbb{H}^2(F^1\Omega^\bullet_X(\log D)) \to H^1(\Omega^1_X(\log D)) \to 0. \tag{4.7}$$

The group $\mathbb{H}^2(F^2\Omega^\bullet_X(\log D))$ can be identified with $H^0(\Omega^{2,\mathrm{cl}}_X(\log D))$ which completes the proof. ☐

Definition 4.11. Let $\mathcal{A}$ be a log Picard algebroid on (X, D). We say that $\mathcal{A}$ is ***locally trivial*** if a flat connection (a splitting of $\mathcal{A}$ with zero curvature) exists in a neighbourhood of every point.

Proposition 4.12. *Locally trivial log Picard algebroids on (X, D) are classified by the subgroup*

$$H^1(\Omega^{1,\mathrm{cl}}_X(\log D)) \subset \mathbb{H}^2(F^1\Omega^\bullet_X(\log D)).$$

If D is smooth, the above inclusion is an equality, so that all log Picard algebroids on (X, D) are locally trivial.

Proof. As described in the proof of Theorem 4.8, a log Picard algebroid is classified by the cohomological data $\{A_{ij}, B_i\}$ where $B_i - B_j = dA_{ij}$. In the locally trivial case, local flat splittings may be chosen, in which case B_i and B_j are zero, yielding a Čech 1-cocycle A_{ij} for $\Omega^{1,\mathrm{cl}}_X(\log D)$, as required. The inclusion of complexes

$$\Omega^{1,\mathrm{cl}}_X(\log D) \hookrightarrow [\Omega^1_X(\log D) \to \Omega^{2,\mathrm{cl}}_X(\log D)]$$

induces an inclusion on first cohomology groups, and since the Poincaré lemma for logarithmic forms holds in degree ≥ 2 when D is smooth, we obtain surjectivity in that case. ☐

A log Picard algebroid on (X, D) restricts to a usual Picard algebroid on $U = X \setminus D$, and so determines a class in $H^1(\Omega_U^{1,\mathrm{cl}})$, which itself defines a class in $H^2(U, \mathbb{C})$. We now give a classification of locally trivial log Picard algebroids in terms of this cohomology class.

Theorem 4.13. *The restriction of a locally trivial log Picard algebroid on (X, D) to the complement $U = X \setminus D$ defines a cohomology class in the subgroup*

$$F^1H^2(U, \mathbb{C}) \cap W_3H^2(U, \mathbb{C}) \subset H^2(U, \mathbb{C}),$$

and this defines a bijection on isomorphism classes.

Proof. By Proposition 4.12, locally trivial log Picard algebroids on (X, D) are classified by the cohomology group $H^1(\Omega_X^{1,\mathrm{cl}}(\log D))$. Our goal is to prove the following

$$H^1(\Omega_X^{1,\mathrm{cl}}(\log D)) \cong F^1H^2(U, \mathbb{C}) \cap W_3H^2(U, \mathbb{C}), \tag{4.8}$$

using the description of the cohomology of U in terms of logarithmic forms given in Section 2.2. First, we observe that the sheaf $\Omega_X^{1,\mathrm{cl}}(\log D)$ can be resolved by the following complex

$$\Omega_X^1(\log D) \to \Omega_X^1 \wedge \Omega_X^1(\log D) \to \Omega_X^2 \wedge \Omega_X^1(\log D) \to \dots$$

Using the definition of the weight filtration on $\Omega_X^\bullet(\log D)$, we see that this complex is $F^1W_1\Omega_X^\bullet(\log D)$, the first truncation of $W_1\Omega_X^\bullet(\log D)$. Hence, we have the isomorphism

$$H^1(\Omega_X^{1,\mathrm{cl}}(\log D)) \cong \mathbb{H}^2(F^1W_1\Omega_X^\bullet(\log D)).$$

Now, consider the commutative diagram of complexes

$$\begin{array}{ccc} F^1W_1\Omega_X^\bullet(\log D) & \longrightarrow & W_1\Omega_X^\bullet(\log D) \\ \downarrow & & \downarrow \\ F^1\Omega_X^\bullet(\log D) & \longrightarrow & \Omega_X^\bullet(\log D) \end{array}$$

We will now show that if we take the second hypercohomology $\mathbb{H}^2$ of all the complexes in the above diagram, then all of the induced maps are injective. This will prove that the map

$$\mathbb{H}^2(F^1W_1\Omega_X^\bullet(\log D)) \to F^1H^2(U, \mathbb{C}) \cap W_3H^2(U, \mathbb{C}), \tag{4.9}$$

is injective. By the E_1-degeneration of the logarithmic Hodge-de Rham spectral sequence and the fact that $\Omega_X^\bullet(\log D)$ with the Hodge filtration is a biregular filtered complex, we obtain that the map

$$\mathbb{H}^2(F^1\Omega_X^\bullet(\log D)) \to \mathbb{H}^2(\Omega_X^\bullet(\log D))$$

is injective [15, Proposition 1.3.2]. We now prove the injectivity of the map

$$\mathbb{H}^2(W_1\Omega_X^\bullet(\log D)) \to \mathbb{H}^2(\Omega_X^\bullet(\log D)). \tag{4.10}$$

To do this, we first consider the short exact sequence of complexes

$$0 \to W_1\Omega_X^\bullet(\log D) \to W_2\Omega_X^\bullet(\log D) \to \mathrm{Gr}_2^W\Omega_X^\bullet(\log D) \to 0$$

with the associated hypercohomology long exact sequence

$$\cdots \to \mathbb{H}^1(\mathrm{Gr}_2^W\Omega_X^\bullet(\log D)) \to \mathbb{H}^2(W_1\Omega_X^\bullet(\log D)) \to \mathbb{H}^2(W_2\Omega_X^\bullet(\log D)) \to \dots, \tag{4.11}$$

Recall from the Poincaré residue isomorphism in (2.7), we have the following description of the associated graded objects of the weight filtration

$$\mathrm{Gr}_2^W\Omega_X^\bullet(\log D) \cong j_{2_*}\Omega^{\cdot}_{D^{(2)}}[-2],$$

and hence $\mathbb{H}^1(\mathrm{Gr}_2^W\Omega_X^\bullet(\log D)) = 0$. Therefore, we have the injectivity of the map

$$\mathbb{H}^2(W_1\Omega_X^\bullet(\log D)) \to \mathbb{H}^2(W_2\Omega_X^\bullet(\log D)).$$

But the inclusion of $W_k\Omega_X^\bullet(\log D)$ in $\Omega_X^\bullet(\log D)$ is a quasi-isomorphism up to and including degree k, and so the map (4.10) must be injective as well. Using the same argument as above, we obtain also that the following map

$$\mathbb{H}^2(F^1W_1\Omega_X^\bullet(\log D)) \to \mathbb{H}^2(F^1\Omega_X^\bullet(\log D))$$

is injective. To conclude the proof of the theorem, we need to show that the map earlier in (4.9) is surjective. The group $\mathbb{H}^2(F^1W_1\Omega_X^\bullet(\log D))$ injects into $\mathbb{H}^2(W_1\Omega_X^\bullet(\log D))$ with cokernel $H^2(\mathcal{O}_X)$,

$$0 \to \mathbb{H}^2(F^1W_1\Omega_X^\bullet(\log D)) \to \mathbb{H}^2(W_1\Omega_X^\bullet(\log D)) \to H^2(\mathcal{O}_X) \to 0, \tag{4.12}$$

and it injects into $\mathbb{H}^2(F^1\Omega_X^\bullet(\log D))$ with cokernel $H^0(D^{(2)}, \mathbb{C})$,

$$0 \to \mathbb{H}^2(F^1W_1\Omega_X^\bullet(\log D)) \to \mathbb{H}^2(F^1\Omega_X^\bullet(\log D)) \to H^0(D^{(2)}, \mathbb{C}) \to 0. \tag{4.13}$$

Suppose that there exists a class in $F^1H^2(U, \mathbb{C}) \cap W_3H^2(U, \mathbb{C})$ such that it is not in the image of $\mathbb{H}^2(F^1W_1\Omega_X^\bullet(\log D))$ under the map (4.9). Then, this class must either map to $H^2(\mathcal{O}_X)$ or map to $H^0(D^{(2)}, \mathbb{C})$. However, if this class maps to $H^2(\mathcal{O}_X)$, then this contradicts that the class lies in $F^1H^2(U, \mathbb{C})$. If this class maps to $H^0(D^{(2)}, \mathbb{C})$, then this contradicts that the class lies in $W_3H^2(U, \mathbb{C})$. Thus, we obtain the surjectivity of the map in (4.9). □

Example 4.14. Consider $(\mathbb{P}^2, D)$ where D is the union of three projective lines in triangle position. The complement $U := \mathbb{C}^* \times \mathbb{C}^*$ and we have $H^2(U, \mathbb{C}) = \mathbb{C}$. If we write out the $k + m = 4$ row of the $E_1^{-m,k+m}$-page of the weight spectral sequence

$$0 \to \underbrace{H^0(D^{(2)}, \mathbb{C})}_{\mathbb{C}^3} \xrightarrow{d_1} \underbrace{H^2(D, \mathbb{C})}_{\mathbb{C}^3} \xrightarrow{d_1} \underbrace{H^4(X, \mathbb{C})}_{\mathbb{C}} \to 0$$

We obtain that the $k + m = 4$ row of the $E_2^{-m,k+m}$-page of the weight spectral sequence is

$$0 \to \mathbb{C} \to 0 \to 0 \to 0$$

The graded piece $\mathrm{Gr}_4^W H^2(U,\mathbb{C})$ is the $E_2^{-2,4}$-term of the weight spectral sequence and so we have:

$$\mathrm{Gr}_4^W H^2(U,\mathbb{C}) = \mathbb{C}$$

This implies that $W_3 H^2(U,\mathbb{C})$ is necessarily zero and so by the previous theorem, the log Picard algebroids on this pair cannot be locally trivial. We now give a geometric description of the log Picard algebroid for this example. At a vertex of the triangle, D can be given locally as $\{xy = 0\}$. If we consider the following closed log 2-form B,

$$B := \frac{1}{(2\pi i)^2}\frac{dx}{x} \wedge \frac{dy}{y}$$

this is not locally exact near the vertex of the triangle. We can twist the Lie bracket by the form B as illustrated in Example 4.7 to obtain a non-trivial log Picard algebroid.

If we restrict this closed log 2-form B to the complement $\mathbb{C}^* \times \mathbb{C}^*$, this is the curvature of the Deligne line bundle [12].

For the remainder of this thesis, all log Picard algebroids will be taken to be locally trivial log Picard algebroids.

4.1.2 Residues and local system

We now introduce the topological aspects of the theory of locally trivial log Picard algebroids. First, we have the residue short exact sequence

$$0 \to \Omega_X^1 \to \Omega_X^1(\log D) \overset{\mathrm{Res}}{\to} \oplus_{i=1}^m \mathcal{O}_{D_i} \to 0$$

where $D_1, \dots, D_m$ are the irreducible components of the simple normal crossings divisor D. We can describe the residue map locally. Since D is a simple normal crossings divisor, it can be expressed locally by the equation $f_1 \dots f_m = 0$ and as a result, the logarithmic 1-form can be written as

$$\sum_{i=1}^k \alpha_i \frac{df_i}{f_i} + \gamma_i$$

where $\alpha_i, \gamma_i \in \mathcal{O}_X$. The residue map is given by

$$\sum_{i=1}^k \alpha_i \frac{df_i}{f_i} + \gamma_i \overset{\mathrm{Res}}{\mapsto} \oplus_{i=1}^k \alpha_i$$

If we consider closed differential forms, we have the following version of the residue short exact sequence

$$0 \to \Omega_X^{1,\mathrm{cl}} \to \Omega_X^{1,\mathrm{cl}}(\log D) \overset{\mathrm{Res}}{\to} \oplus_{i=1}^k \mathbb{C}_{D_i} \to 0$$

with the following long exact sequence in cohomology

$$\cdots \to H^1(\Omega_X^{1,\mathrm{cl}}) \to H^1(\Omega_X^{1,\mathrm{cl}}(\log D)) \overset{\mathrm{Res}_*}{\to} \oplus_{i=1}^k H^1(D_i,\mathbb{C}) \to \dots \tag{4.14}$$

We explain the geometric meaning of this long exact sequence. The map from the group $H^1(\Omega_X^{1,\mathrm{cl}})$ to the group $H^1(\Omega_X^{1,\mathrm{cl}}(\log D))$ corresponds to locally trivial log Picard algebroids on (X, D) which come from usual Picard algebroids on X. In precise terms, let $\widetilde{\mathcal{A}}$ be a Picard algebroid on X. The log Picard algebroid $\mathcal{A}$ is obtained via the pullback of the anchor map on $\widetilde{\mathcal{A}}$, $\widetilde{\sigma} : \mathcal{A} \to \mathcal{T}_X(-\log D)$, and the inclusion map $i : \mathcal{T}_X(-\log D) \to \mathcal{T}_X$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \widetilde{\mathcal{A}} & \xrightarrow{\widetilde{\sigma}} & \mathcal{T}_X & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow{\scriptstyle i} & & \\ 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{A} & \longrightarrow & \mathcal{T}_X(-\log D) & \longrightarrow & 0 \end{array}$$

The residue map from $H^1(\Omega_X^{1,\text{cl}}(\log D))$ to $\oplus_{i=1}^k H^1(D_i, \mathbb{C})$ can be interpreted as follows: if we take the residue of the logarithmic cohomology class corresponding to a locally trivial log Picard algebroid, we obtain a cohomology class in $H^1(D, \mathbb{C})$ which is the group of isomorphism classes of $\mathbb{C}$-local systems on D.

We describe the geometric meaning of this in greater detail for the situation where D is a divisor which is smooth and irreducible. First, we restrict the log Picard algebroid $\mathcal{A}$ to D

$$0 \to \mathcal{O}_D \to \mathcal{A}|_D \overset{\sigma|_D}{\to} \mathcal{T}_X(-\log D)|_D \to 0$$

The sheaf $\mathcal{T}_X(-\log D)|_D$ is the Atiyah algebroid of the normal bundle N_D, so in particular, it is a Picard algebroid on D. Let E be the central section of this Picard algebroid. We make the following definition:

Definition 4.15. We define the ***residue*** of the log Picard algebroid along D, $\mathcal{A}|_D$, to be the $\mathbb{C}$-local system R formed by taking the sections of the affine line bundle $\sigma^{-1}(E)$ which are central with respect to the Lie bracket on $\mathcal{A}|_D$.

Remark 4.16. We note that if we had worked with the Zariski topology rather than the analytic topology, the group $H^1(D, \mathbb{C})$ is zero. Using the long exact sequence in (4.14), this implies that all locally trivial log Picard algebroids on (X, D) must be pullbacks of Picard algebroids on X.

4.1.3 Examples

We will now provide several examples of log Picard algebroids on pairs (X, D) where D is a smooth irreducible divisor in X. First, we have the following proposition:

Proposition 4.17. *Let X be a projective algebraic surface with first Betti number $b_1(X) = 0$ and D a smooth irreducible curve in X of genus g. The group of isomorphism classes of log Picard algebroids on (X, D) has rank equal to $h^{1,1} + h^{2,0} + 2g - 1$. Moreover, this group is an extension of the group of rank $h^{1,1} + h^{2,0} - 1$ of log Picard algebroids arising from pullback of Picard algebroids on X by the group of rank $2g$ of log Picard algebroids which are in one-to-one correspondence with $\mathbb{C}$-local systems on D.*

Proof. We analyze the long exact sequence in (4.14)

$$\cdots \to H^0(D, \mathbb{C}) \to H^1(\Omega_X^{1,\text{cl}}) \to H^1(\Omega_X^{1,\text{cl}}(\log D)) \overset{\text{Res}_*}{\to} H^1(D, \mathbb{C}) \to H^2(\Omega_X^{1,\text{cl}}) \to \dots$$

Since X is projective, we have the quasi-isomorphism of complexes

$$\Omega_X^{1,\text{cl}} \hookrightarrow [\Omega_X^1 \to \Omega_X^2]$$

which implies

$$H^2(\Omega_X^{1,\mathrm{cl}}) \cong \mathbb{H}^2(\Omega_X^1 \to \Omega_X^2)$$

Moreover, the Hodge-de Rham spectral sequence degenerates at E_1 and so we have the short exact sequence

$$0 \to H^1(\Omega_X^2) \to \mathbb{H}^2(\Omega_X^1 \to \Omega_X^2) \to H^2(\Omega_X^1) \to 0$$

From the hypothesis, we have $b_1(X) = 0$. By Poincare duality, this gives $b_3(X) = 0$ and so the Hodge numbers $h^{1,2}$ and $h^{2,1}$ must be zero. Thus, $H^2(\Omega_X^{1,\mathrm{cl}}) = 0$. By the exact same arguments, we obtain

$$0 \to H^0(\Omega_X^2) \to \mathbb{H}^1(\Omega_X^1 \to \Omega_X^2) \to H^1(\Omega_X^1) \to 0$$

which means that the dimension of $\mathbb{H}^1(\Omega_X^1 \to \Omega_X^2)$ is exactly $h^{1,1} + h^{2,0}$. Moreover, as D is assumed to be a smooth irreducible divisor, the map from $H^0(D, \mathbb{C})$ to $H^1(\Omega_X^{1,\mathrm{cl}})$ is simply the degree map which is injective. Thus, we obtain the exact sequence

$$0 \to H^0(D, \mathbb{C}) \to H^1(\Omega_X^{1,\mathrm{cl}}) \to H^1(\Omega_X^{1,\mathrm{cl}}(\log D)) \to H^1(D, \mathbb{C}) \to 0$$

This shows that the group of isomorphism classes of log Picard algebroids on (X, D) has rank $h^{1,1} + h^{2,0} + 2g - 1$. Since $H^1(D, \mathbb{C}) = \mathbb{C}^{2g}$, the above sequence shows that this is an extension of the group of rank $h^{1,1} + h^{2,0} - 1$ of log Picard algebroids arising from pullback of Picard algebroids on X by the group of rank $2g$ of log Picard algebroids which are in one-to-one correspondence with $\mathbb{C}$-local systems on D. □

We have an obvious source of examples of log Picard algebroids which are obtained by the pullback of Picard algebroids on X as described earlier. We will now provide several examples of log Picard algebroids on (X, D) which do not arise in this obvious fashion.

Example 4.18. Consider $(\mathbb{P}^2, D)$ where D is a smooth projective curve of degree $d \geq 3$ in $\mathbb{P}^2$. Using the result of Proposition 4.17, we have

$$H^1(\Omega_{\mathbb{P}^2}^{1,\mathrm{cl}}(\log D)) \cong \mathbb{C}^{2g}.$$

The log Picard algebroids in this example are in one-to-one correspondence with $\mathbb{C}$-local systems on D.

In the above example, any Picard algebroid on $\mathbb{P}^2$ is trivial as a log Picard algebroid on $(\mathbb{P}^2, D)$ and all log Picard algebroids are in one-to-one correspondence with $\mathbb{C}$-local systems on D. The next example will be an example where this is not the case.

Example 4.19. Let X be the smooth projective cubic surface and D an elliptic curve in X. The relevant Hodge numbers of X here are $h^{1,1} = 7$ and $h^{2,0} = 0$. Using the result of Proposition 4.17, we have

$$H^1(\Omega_X^{1,\mathrm{cl}}(\log D)) \cong \mathbb{C}^8.$$

Now, since $H^1(D, \mathbb{C}) = \mathbb{C}^2$, we have that the above group is an extension of the rank 2 group of $\mathbb{C}$-local systems on D by the group of log Picard algebroids arising as pullback of Picard algebroids on X.

4.1.4 Functorial properties and blowup

In this subsection, we introduce functoriality properties of log Picard algebroids following the treatment for usual Picard algebroids in Section 2.2 of [1]. Let $\varphi : Y \to X$ be a morphism of projective varieties and D a simple normal crossings divisor on X such that $\varphi^* D$ is also a simple normal crossings divisor on Y. Suppose that $\mathcal{A}$ is a log Picard algebroid on (X, D), consider the $\mathcal{O}_Y$-module $\varphi^! \mathcal{A} := \varphi^* \mathcal{A} \times_{\varphi^* \mathcal{T}_X(-\log D)} \mathcal{T}_Y(-\log \varphi^* D)$ with the Lie bracket on its sections defined as

$$[(\xi, \sum_i f_i \otimes P_i), (\eta, \sum_j g_j \otimes Q_j)] = ([\xi, \eta], \sum_{i,j} f_i g_j \otimes [P_i, Q_j] + \xi(g_j) \otimes P_i - \eta(f_i) \otimes Q_j)$$

where $\xi, \eta \in \mathcal{T}_Y(-\log \varphi^* D)$, $f_i, g_j \in \mathcal{O}_Y$ and $P_i, Q_j \in \varphi^{-1}\mathcal{A}$. The projection map $\pi_{\mathcal{T}_Y(-\log \varphi^* D)} : \varphi^! \mathcal{A} \to \mathcal{T}_Y(-\log \varphi^* D)$ preserves the Lie bracket and the central section $e_{\varphi^! \mathcal{A}} : \mathcal{O}_Y \to \varphi^! \mathcal{A}$ is defined as $e_{\varphi^! \mathcal{A}} = (\varphi^* e_{\mathcal{A}}, 0)$. We make the following definition:

Definition 4.20. The quadruple $(\varphi^! \mathcal{A}, [\cdot, \cdot], \pi_{\mathcal{T}_Y(-\log \varphi^* D)}, e_{\varphi^! \mathcal{A}})$ is a log Picard algebroid on $(Y, \varphi^* D)$ and we call this the ***pullback log Picard algebroid*** of $\mathcal{A}$. As before, we simply denote this quadruple by $\varphi^! \mathcal{A}$.

Theorem 4.21. *Let X be a smooth projective algebraic surface X with $b_1(X) = 0$, D a smooth irreducible divisor in X, and $\pi : \widetilde{X} \to X$ the blowup of X at a point p where $p \in D \subset X$. The pullback operation for log Picard algebroids defines an equivalence between the locally trivial log Picard algebroids on (X, D) and the locally trivial log Picard algebroids on $(\widetilde{X}, \pi^* D)$ where $\pi^* D := D + E$ and E is the exceptional divisor of the blowup.*

Proof. Let $\mathcal{A}$ be a locally trivial log Picard algebroid on (X, D), the assignment $\mathcal{A} \mapsto \pi^! \mathcal{A}$ induces the map in cohomology

$$H^1(\Omega_X^{1,\mathrm{cl}}(\log D)) \to H^1(\Omega_{\widetilde{X}}^{1,\mathrm{cl}}(\log \pi^* D))$$

and we want to show that this map is an isomorphism. From the proof of Theorem 4.13, we have

$$H^1(\Omega_{\widetilde{X}}^{1,\mathrm{cl}}(\log \pi^* D)) = F^1 H^2(U, \mathbb{C}) \cap W_3 H^2(U, \mathbb{C}) \tag{4.15}$$

where the Hodge filtration and weight filtration on $H^2(U, \mathbb{C})$ is given by the compactification of U into $\widetilde{X}$ by $D + E$. In order to prove our theorem, we now check that this indeed coincides with the Hodge and weight filtration on $H^2(U, \mathbb{C})$ for the usual compactification of U into X by D. First, we have

$$F^1 H^2(U, \mathbb{C}) = \frac{H^2(U, \mathbb{C})}{H^2(\mathcal{O}_{\widetilde{X}})}$$

If we use the fact that the plurigenus is a birational invariant along with Serre duality, we have $H^2(\mathcal{O}_{\widetilde{X}}) = H^2(\mathcal{O}_X)$. This shows that $F^1 H^2(U, \mathbb{C})$ does indeed coincide with the first Hodge filltration on $H^2(U, \mathbb{C})$ for the usual compactification of U into X by D.

Since D is a smooth irreducible divisor, the weight filtration on $H^2(U, \mathbb{C})$ for the usual compactification of U into X by D is a two step filtration. To conclude the proof, we need to show that the group $W_3 H^2(U, \mathbb{C})$ in (4.15) is $H^2(U, \mathbb{C})$. We do this by showing that $Gr_4^W H^2(U, \mathbb{C})$ is zero. As in Example

4.14, we can write out the $k+m=4$ row of the $E_2^{-m,k+m}$-page of the weight spectral sequence

$$0 \to \underbrace{H^0((D+E)^{(2)},\mathbb{C})}_{\mathbb{C}} \xrightarrow{d_1} \underbrace{H^2(D+E,\mathbb{C})}_{\mathbb{C}^2} \xrightarrow{d_1} \underbrace{H^4(\widetilde{X},\mathbb{C})}_{\mathbb{C}} \to 0.$$

Taking the cohomology gives that the $k+m=4$ row of the $E_2^{-m,k+m}$-page is entirely zero and so we have

$$\mathrm{Gr}_4^W H^2(U,\mathbb{C}) = 0.$$

□

4.2 Meromorphic Line Bundles

In this section, we will again work in the analytic topology unless otherwise specified. First, we consider the sheaf of algebras $\mathcal{O}_X(*D)$ of meromorphic functions on X with poles of arbitrary order on D. There is a local description of this sheaf: suppose that the simple normal crossings divisor D is given by $f_1 f_2 \dots f_m = 0$, we can write $\mathcal{O}_X(*D) = \mathcal{O}_X[f_1^{-1},\dots,f_m^{-1}]$. In addition, the sheaf of algebras $\mathcal{O}_X(*D)$ has a $\mathbb{Z}$-filtration F^k along D defined as:

$$F^k\mathcal{O}_X(*D) := \mathcal{O}_X(kD) \tag{4.16}$$

where $\mathcal{O}_X(kD)$ is the sheaf of meromorphic functions on X with poles of bounded by kD.

Consider now the sheaf of invertible elements of $\mathcal{O}_X(*D)$ which we denote by $\mathcal{O}_X^\times(*D)$. Locally, $\mathcal{O}_X^\times(*D)$ may be described as adjoining group generators $f_1,\dots,f_m$ which define the irreducible components $D_1,\dots,D_m$ of D to the sheaf of groups $\mathcal{O}_X^\times$. Therefore, there is a short exact sequence of abelian groups

$$1 \to \mathcal{O}_X^\times \to \mathcal{O}_X^\times(*D) \overset{\mathrm{ord}}{\to} \oplus_{i=1}^m \mathbb{Z}_{D_i} \to 1 \tag{4.17}$$

where the order map ord is defined as

$$h f_1^{k_1} f_2^{k_2} \dots f_m^{k_m} \overset{\mathrm{ord}}{\mapsto} \oplus_{i=1}^m k_i, \tag{4.18}$$

with $h \in \mathcal{O}_X^\times$ and $k_1,\dots,k_m \in \mathbb{Z}$. Moreover, we have the long exact sequence in cohomology associated to the above short exact sequence:

$$\dots \to H^0(D,\mathbb{Z}) \to \mathrm{Pic}(X) \to H^1(\mathcal{O}_X^\times(*D)) \overset{\mathrm{ord}_*}{\to} H^1(D,\mathbb{Z}) \to H^2(\mathcal{O}_X^\times) \to \dots \tag{4.19}$$

Definition 4.22. A meromorphic line bundle $\mathcal{M}$ on (X,D) is a locally free rank one $\mathcal{O}_X(*D)$-module.

Proposition 4.23. *A meromorphic line bundle $\mathcal{M}$ determines a $\mathbb{Z}$-local system on D, $\mathrm{ord}(\mathcal{M})$, which induces the map in cohomology*

$$H^1(\mathcal{O}_X^\times(*D)) \overset{\mathrm{ord}_*}{\to} H^1(D,\mathbb{Z}).$$

Furthermore, the sheaf of sections of $\mathcal{M}$ is filtered with the filtered pieces labelled by the local system $\mathrm{ord}(\mathcal{M})$.

Proof. Let $\mathcal{M}$ be a meromorphic line bundle and consider two local trivializations of $\mathcal{M}$

$$s_i : \mathcal{M}|_{U_i} \overset{\cong}{\to} \mathcal{O}_X(*D)|_{U_i}$$
$$s_j : \mathcal{M}|_{U_j} \overset{\cong}{\to} \mathcal{O}_X(*D)|_{U_j}$$

The sheaf $\mathcal{O}_X(*D)|_{U_i}$ has a $\mathbb{Z}$-filtration along $D_i := D \cap U_i$ as defined in (4.16) and similarly $\mathcal{O}_X(*D)|_{U_j}$ has a $\mathbb{Z}$-filtration along $D_j := D \cap U_j$. Let $\mathbb{Z}_{D_i}$ and $\mathbb{Z}_{D_j}$ be $\mathbb{Z}$-torsors on D_i and D_j respectively which label these filtrations. On the double overlap U_{ij}, we have transition functions $g_{ij} \in \mathcal{O}_X^\times(*D)(U_{ij})$ such that $s_j = g_{ij}s_i$. Consider the image of g_{ij} under the ord map in (4.18), which we denote by k_{ij}. The map k_{ij} glues together the $\mathbb{Z}$-torsors $\mathbb{Z}_{D_i}$ and $\mathbb{Z}_{D_j}$

$$\mathbb{Z}_{D_i} \overset{k_{ij}}{\to} \mathbb{Z}_{D_j}.$$

This defines a $\mathbb{Z}$-local system on D which we call $\mathrm{ord}(\mathcal{M})$. Since this local system was constructed using the gluing map $k_{ij} = \mathrm{ord}(g_{ij})$, we have that $\mathcal{M} \mapsto \mathrm{ord}(\mathcal{M})$ induces the map in cohomology

$$H^1(\mathcal{O}_X^\times(*D)) \overset{\mathrm{ord}_*}{\to} H^1(D, \mathbb{Z}).$$

□

We can summarize much of the above through the following theorem:

Theorem 4.24. *The* ***meromorphic Picard group*** $\mathrm{Pic}(X, D)$ *of isomorphism classes of meromorphic line bundles on* (X, D) *is given by the cohomology group*

$$H^1(\mathcal{O}_X^\times(*D))$$

The map $\mathcal{M} \mapsto \mathrm{ord}(\mathcal{M})$ *which assigns a meromorphic line bundle to the group of* $\mathbb{Z}$*-local systems on* D *is classified by the map in* (4.19)

$$H^1(\mathcal{O}_X^*(D)) \overset{\mathrm{ord}_*}{\to} H^1(D, \mathbb{Z})$$

Remark 4.25. We note that we must use the analytic topology rather than the Zariski topology in order to clearly address and solve the pre-quantization problem later. Moreover, if we work with the Zariski topology, then there are no interesting $\mathbb{Z}$-local systems on D and so all meromorphic line bundles come from holomorphic line bundles on X.

Theorem 4.26. *If the analytic Brauer group* $H^2(\mathcal{O}_X^\times)$ *is trivial, then the meromorphic Picard group* $\mathrm{Pic}(X, D)$ *is an extension of the group of* $\mathbb{Z}$*-local systems on* D *by the algebraic Picard group of* U*.*

Proof. We consider the following short exact sequence in the Zariski topology

$$0 \to \mathcal{O}_{X,\mathrm{alg}}^\times \to j_*\mathcal{O}_{U,\mathrm{alg}}^\times \to i_*\mathbb{Z}_{D,\mathrm{alg}} \to 0$$

where $j : U \hookrightarrow X$ and $i : D \hookrightarrow X$. The associated long exact sequence is:

$$\cdots \to H^0(D, \mathbb{Z}) \to \mathrm{Pic}(X) \to \mathrm{Pic}^{\mathrm{alg}}(U) \to H^1(D, \mathbb{Z}) \to \dots \tag{4.20}$$

The cohomology group $H^1(D, \mathbb{Z})$ is zero since we are working in the Zariski topology here. Therefore, the algebraic Picard group $\mathrm{Pic}^{\mathrm{alg}}(U)$ is the cokernel of the map from $H^0(D,\mathbb{Z})$ to $\mathrm{Pic}(X)$. Since both D and X satisfy the GAGA principle, we can rewrite the long exact sequence in (4.19) as:

$$0 \to \mathrm{Pic}^{\mathrm{alg}}(U) \to \mathrm{Pic}(X,D) \to H^1(D,\mathbb{Z}) \to H^2(\mathcal{O}_X^\times) \tag{4.21}$$

As the analytic Brauer group $H^2(\mathcal{O}_X^\times)$ is assumed to be trivial, we can conclude that:

$$0 \to \mathrm{Pic}^{\mathrm{alg}}(U) \to \mathrm{Pic}(X,D) \to H^1(D,\mathbb{Z}) \to 0$$

□

We now provide several examples of pairs (X,D) which admit a non-trivial meromorphic Picard group $\mathrm{Pic}(X,D)$. The examples will all share the property that D is a smooth irreducible divisor in X and X has trivial analytic Brauer group.

Example 4.27. Consider $(\mathbb{P}^2,D)$ where D is a smooth irreducible projective curve of degree $d \geq 3$ as in Example 4.18. The group of $\mathbb{Z}$-local systems on D is $\mathbb{Z}^{2g}$ where g is the genus of D. Moreover, the algebraic Picard group of the complement U is given by:

$$\mathrm{Pic}^{\mathrm{alg}}(U) = \mathbb{Z}_d$$

From Theorem 4.26, we obtain:

$$\mathrm{Pic}(\mathbb{P}^2,D) \cong \mathbb{Z}_d \oplus \mathbb{Z}^{2g}$$

In the above example, the meromorphic Picard group admits a torsion subgroup which arises from the algebraic Picard group of the complement. The following is an example where torsion does not appear.

Example 4.28. Consider (X,D) where X is a smooth projective cubic surface and D an elliptic curve in X as in Example 4.19. Since D is smooth and irreducible, the algebraic Picard group of the complement U can be computed from the short exact sequence:

$$0 \to \mathbb{Z} \to \mathrm{Pic}(X) \to \mathrm{Pic}^{\mathrm{alg}}(U) \to 0$$

The Picard group of X is $\mathbb{Z}^7$ and so we obtain $\mathrm{Pic}^{\mathrm{alg}}(U) \cong \mathbb{Z}^6$. Since $H^1(D,\mathbb{Z}) \cong \mathbb{Z}^2$, we can use Theorem 4.26 to obtain:

$$\mathrm{Pic}(X,D) \cong \mathbb{Z}^8$$

and so we have an 8-dimensional family of isomorphism classes of meromorphic line bundles on (X,D).

The meromorphic Picard group can be strictly torsion as well as indicated in this next example.

Example 4.29. Consider $(\mathbb{P}^2,C)$ where C is a smooth irreducible conic curve in $\mathbb{P}^2$. The algebraic Picard group of the complement U is given by:

$$\mathrm{Pic}^{\mathrm{alg}}(U) \cong \mathbb{Z}_2$$

As $H^1(C,\mathbb{Z}) = 0$, we obtain:

$$\mathrm{Pic}(\mathbb{P}^2,C) \cong \mathbb{Z}_2$$

4.2.1 Total space of meromorphic line bundles

We now provide a geometric description of the total space of a meromorphic line bundle for the case where D is a smooth divisor on X. First, we describe how this is done for the case where $X = \mathbb{C}$ and $D = \{0\}$.

Consider a family $(\mathbb{C}^2_i)_{i\in\mathbb{Z}}$ of affine planes indexed by the integers and let (x_i, y_i) be affine coordinates on $\mathbb{C}^2_i$. Let ψ be the equivalence relation on $\coprod_{i\in\mathbb{Z}} \mathbb{C}^2_i$ generated by the identification:

$$(x_{i+1}, y_{i+1}) = (y_i^{-1}, y_i^2 x_i), \quad \text{for } y_i \neq 0. \tag{4.22}$$

Quotienting by this relation, we obtain a smooth complex surface M:

$$M := \coprod_{i\in\mathbb{Z}} \mathbb{C}^2_i \Big/ \psi \,. \tag{4.23}$$

Since the map $(x_i, y_i) \mapsto x_i y_i$ is preserved by ψ, it defines a surjective holomorphic map $\pi : M \to \mathbb{C}$. We also have a natural $\mathbb{C}^*$ action on M defined by:

$$\mathbb{C}^* \ni \lambda \cdot (x_i, y_i) = (\lambda x_i, \lambda^{-1} y_i), \tag{4.24}$$

and π is invariant under this action.

The fibre of π over any nonzero point $z \in \mathbb{C}$ may be identified with the hyperbola $\{(x_i, y_i) \in \mathbb{C}^2_i : x_i y_i = z\}$ for any fixed i, so that the restriction of (M, π) to $\mathbb{C} \setminus \{0\}$ defines a (trivial) principal $\mathbb{C}^*$ bundle. On the other hand, the fibre of π over $0 \in \mathbb{C}$ is

$$\pi^{-1}(0) = \coprod_{i\in\mathbb{Z}} \{(x_i, y_i) \ : \ x_i y_i = 0\} \Big/ \psi, \tag{4.25}$$

which is an infinite chain of rational curves with simple normal crossings, depicted below.

The singular points of the above chain coincide with the fixed points of the $\mathbb{C}^*$ action on M, given by the collection $(x_i)_{i\in\mathbb{Z}}$ where $x_i = (0, 0) \in \mathbb{C}^2_i$.

Theorem 4.30. *Let $D = 0 \subset X = \mathbb{C}$. The sheaf of automorphisms of (M, π) respecting the $\mathbb{C}^*$-action on M is given by $\mathcal{O}_X^\times(*D)$.*

Proof. Away from $\pi^{-1}(0)$, an automorphism ϕ is a principal bundle automorphism, and so is given in any of the above local charts by

$$\phi(x_i, y_i) = (g(z)x_i, g(z)^{-1} y_i),$$

where $z = \pi(x_i, y_i) = x_i y_i$ and g is a nonvanishing holomorphic function on $\mathbb{C} \setminus \{0\}$. We must now investigate the behaviour of $g(z)$ as z approaches zero.

The fixed point $(x_0, y_0) = (0, 0)$ must be sent to another one by the automorphism ϕ, say $(x_k, y_k) = (0, 0)$. So the automorphism defines a germ of a holomorphic map sending the origin of $\mathbb{C}^2_0$ to the origin of $\mathbb{C}^2_k$, so that

$$\phi(x_0, y_0) = (\alpha(x_0, y_0), \beta(x_0, y_0))$$

with α, β holomorphic near zero with nonzero Jacobian at zero.

The equivalence relation ψ identifies (x_0, y_0) with $(x_k, y_k) = (y_0^{-k} x_0^{-k+1}, y_0^{k+1} x_0^k) = (z^{-k} x_0, z^k y_0)$, and so takes $(g(z)x_0, g(z)^{-1}y_0)$ to $(z^{-k}g(z)x_0, z^k g(z)^{-1}y_0)$ in the $\mathbb{C}_k^2$ chart. As a result we have

$$(\alpha(x_0, y_0), \beta(x_0, y_0)) = (z^{-k}g(z)x_0, z^k g(z)^{-1}y_0).$$

This means that $z^{-k}g(z)$ extends to a holomorphic nonvanishing function in a neighbourhood of zero, that is, $g(z) = e^f z^k$ for f holomorphic, showing that $g \in \mathcal{O}_X^\times(*D)$ as required. □

Corollary 4.31. *For a smooth divisor $D \subset X$, a meromorphic $\mathbb{C}^*$ principal bundle may be defined as a usual $\mathbb{C}^*$ principal bundle, but with an alternate local normal form near D, namely (M, π) as described above.*

Any usual principal $\mathbb{C}^*$-bundle P gives rise to a meromorphic principal bundle in the following way. Let L be the complex line bundle associated to P. We construct a family of varieties by iterated blow-up: first blow up the intersection of the zero section Z of L with the restriction L_D of L to D, each viewed as a submanifold of the total space of L. By iteratively blowing up the intersections of the proper transforms of Z and L_D with the exceptional divisor, we obtain a sequence $(\widetilde{M}_k)_{k\geq 0}$ of varieties. Deleting the proper transforms of Z and L_D from each $\widetilde{M}_k$, we obtain a sequence $(M_k)_{k\geq 0}$ of varieties equipped with inclusion maps $M_k \hookrightarrow M_{k+1}$. Taking the direct limit of this system of varieties, we obtain a complex manifold (non-finite-type scheme) which is naturally a meromorphic principal bundle as constructed above.

The functor described above, which maps usual principal bundles to meromorphic ones, categorifies the natural map

$$H^1(X, \mathcal{O}_X^\times) \to H^1(X, \mathcal{O}_X^\times(*D))$$

which derives from the natural inclusion of automorphism sheaves.

4.2.2 Classification

We now apply the theory of integral weight filtrations introduced in 2.2.2 to study meromorphic line bundles on (X, D). From [7, 2-6], there is an exact sequence of sheaves

$$0 \to j_*\mathbb{Z}(1)_U \to \mathcal{O}_X \stackrel{\exp}{\to} \mathcal{O}_X^\times(*D) \to R^1 j_*\mathbb{Z}(1)_U \to 0$$

where $\mathbb{Z}(1)$ is the Tate twist of $\mathbb{Z}$ by $2\pi i$. This implies that the sheaf $\mathcal{O}_X^\times(*D)$ is quasi-isomorphic to the mapping cone:

$$\mathcal{O}_X^\times(*D) \sim \mathrm{Cone}(\sigma_{\leq 1}\mathbb{R}j_*\mathbb{Z}(1)_U \to \mathcal{O}_X) \tag{4.26}$$

We can write the long exact sequence associated to this mapping cone:

$$\begin{aligned} \ldots \to \mathbb{H}^1(\sigma_{\leq 1} Rj_*\mathbb{Z}(1)_U) \to H^1(\mathcal{O}_X) \stackrel{\exp}{\to} \mathrm{Pic}(X, D) \\ \to \mathbb{H}^2(\sigma_{\leq 1} Rj_*\mathbb{Z}(1)_U) \to H^2(\mathcal{O}_X) \to \ldots \end{aligned} \tag{4.27}$$

We would like to make a couple of observations here. From the definition of integral weights (2.10), we have:

$$W_3 H^2(U, \mathbb{Z}(1)) := \mathrm{im}(\mathbb{H}^2(\sigma_{\leq 1} Rj_*\mathbb{Z}(1)_U) \to H^2(U, \mathbb{Z}(1)))$$

If we use the following natural short exact sequence for the canonical filtration σ:

$$0 \to \sigma_{\leq 1} Rj_*\mathbb{Z}(1)_U \to Rj_*\mathbb{Z}(1)_U \to \sigma_{>1} Rj_*\mathbb{Z}(1)_U \to 0 \tag{4.28}$$

with its associated long exact sequence in hypercohomology

$$\cdots \to \mathbb{H}^1(\sigma_{>1} Rj_*\mathbb{Z}(1)_U) \to \mathbb{H}^2(\sigma_{\leq 1} Rj_*\mathbb{Z}(1)_U) \to H^2(U, \mathbb{Z}(1)) \to \dots \tag{4.29}$$

Since $\mathbb{H}^1(\sigma_{>1} Rj_*\mathbb{Z}(1)_U) = 0$, this shows that the map

$$\mathbb{H}^2(\sigma_{\leq 1} Rj_*\mathbb{Z}(1)_U) \to H^2(U, \mathbb{Z}(1))$$

is necessarily injective and so the weight 3 piece $W_3H^2(U, \mathbb{Z}(1))$ can be naturally identified with the hypercohomology group $\mathbb{H}^2(\sigma_{\leq 1} Rj_*\mathbb{Z}(1)_U)$

$$W_3H^2(U, \mathbb{Z}(1)) \cong \mathbb{H}^2(\sigma_{\leq 1} Rj_*\mathbb{Z}(1)_U).$$

The second observation is that if we use the long exact sequence in hypercohomology associated to the short exact sequence in (4.28) again, we have:

$$\begin{aligned} \dots \to \mathbb{H}^0(\sigma_{>1} Rj_*\mathbb{Z}(1)_U) \to \mathbb{H}^1(\sigma_{\leq 1} Rj_*\mathbb{Z}(1)_U) \to H^1(U, \mathbb{Z}(1)) \\ \to \mathbb{H}^1(\sigma_{>1} Rj_*\mathbb{Z}(1)_U) \to \dots \end{aligned}$$

But $\mathbb{H}^0(\sigma_{>1} Rj_*\mathbb{Z}(1)_U)$ and $\mathbb{H}^1(\sigma_{>1} Rj_*\mathbb{Z}(1)_U)$ are zero so this implies that we have an isomorphism:

$$\mathbb{H}^1(\sigma_{\leq 1} Rj_*\mathbb{Z}(1)_U) \cong H^1(U, \mathbb{Z}(1))$$

These two observations allows us to rewrite the sequence in (4.27) as:

$$\cdots \to H^1(U, \mathbb{Z}(1)) \to H^1(\mathcal{O}_X) \stackrel{\exp}{\to} \mathrm{Pic}(X, D) \to W_3H^2(U, \mathbb{Z}(1)) \to H^2(\mathcal{O}_X) \to \dots \tag{4.30}$$

We may think of this sequence as a meromorphic analogue of the usual exponential sheaf sequence on X.

Definition 4.32. We define the ***meromorphic Néron-Severi group*** $\mathrm{NS}(X, D)$ of (X, D) as the following subgroup of $H^2(U, \mathbb{Z}(1))$:

$$\mathrm{NS}(X, D) = F^1H^2(U, \mathbb{C}) \times_{H^2(U,\mathbb{C})} W_3H^2(U, \mathbb{Z}(1)) \subset H^2(U, \mathbb{Z}(1)).$$

Definition 4.33. We define the following quotient group

$$\frac{H^1(\mathcal{O}_X)}{\mathrm{im}(H^1(U, \mathbb{Z}(1)) \to H^1(\mathcal{O}_X))}$$

to be the ***meromorphic Jacobian*** $\mathrm{Jac}(X, D)$ of (X, D). The map $H^1(U, \mathbb{Z}(1)) \to H^1(\mathcal{O}_X)$ is the map given in the sequence in (4.30).

Proposition 4.34. *Suppose that $H^1(U, \mathbb{Z}(1))$ is pure of weight 1, i.e.*

$$W_1 H^1(U, \mathbb{Z}(1)) = H^1(U, \mathbb{Z}(1)),$$

then the meromorphic Jacobian $\mathrm{Jac}(X, D)$ *is isomorphic to the usual Jacobian of* X*.*

Proof. It suffices to show that $W_1 H^1(U, \mathbb{Z}(1))$ is equal to $H^1(X, \mathbb{Z}(1))$. By definition:

$$W_1 H^1(U, \mathbb{Z}(1)) := \mathrm{im}(\mathbb{H}^1(\sigma_{\leq 0} Rj_* \mathbb{Z}(1)_U) \to H^1(U, \mathbb{Z}(1)))$$

If we use the natural short exact sequence for the canonical filtration σ:

$$0 \to \sigma_{\leq 0} Rj_* \mathbb{Z}(1)_U \to Rj_* \mathbb{Z}(1)_U \to \sigma_{>0} Rj_* \mathbb{Z}(1)_U \to 0$$

This gives the long exact sequence in hypercohomology:

$$\cdots \to \mathbb{H}^0(\sigma_{>0} Rj_* \mathbb{Z}(1)_U) \to \mathbb{H}^1(\sigma_{\leq 0} Rj_* \mathbb{Z}(1)_U) \to \mathbb{H}^1(U, \mathbb{Z}(1)) \to \dots$$

Since $\mathbb{H}^0(\sigma_{>0} Rj_* \mathbb{Z}(1)_U) = 0$, this implies that $W_1 H^1(U, \mathbb{Z}(1))$ can be naturally identified with the hypercohomology group $\mathbb{H}^1(\sigma_{\leq 0} Rj_* \mathbb{Z}(1)_U)$. Using the definition of the canonical filtration in addition with the statement in (2.12), we have:

$$\sigma_{\leq 0} Rj_* \mathbb{Z}(1)_U = R^0 j_* \mathbb{Z}(1)_U = \mathbb{Z}(1)_X$$

This shows that $W_1 H^1(U, \mathbb{Z}(1))$ is equal to $H^1(X, \mathbb{Z}(1))$. □

We have the following classification theorem for the meromorphic Picard group $\mathrm{Pic}(X, D)$:

Theorem 4.35. *The first Chern class of meromorphic line bundles* $\mathcal{M}$ *on* (X, D) *restricted to the complement* $U = X \setminus D$ *expresses the meromorphic Picard group* $\mathrm{Pic}(X, D)$ *as an extension of the meromorphic Néron-Severi group* $\mathrm{NS}(X, D)$ *by the meromorphic Jacobian* $\mathrm{Jac}(X, D)$*:*

$$0 \to \mathrm{Jac}(X, D) \to \mathrm{Pic}(X, D) \xrightarrow{c_1} \mathrm{NS}(X, D) \to 0$$

Proof. Using the long exact sequence in (4.30), we obtain:

$$0 \to \mathrm{im}(\mathrm{Pic}(X, D) \to W_3 H^2(U, \mathbb{Z}(1))) \to W_3 H^2(U, \mathbb{Z}(1)) \to H^2(\mathcal{O}_X) \to \dots$$

The map $W_3 H^2(U, \mathbb{Z}(1)) \to H^2(\mathcal{O}_X)$ factors through the following composition of maps:

$$W_3 H^2(U, \mathbb{Z}(1)) \to H^2(U, \mathbb{C}) \to H^2(\mathcal{O}_X)$$

where the first map above is the composition of the inclusion of $W_3 H^2(U, \mathbb{Z}(1)) \hookrightarrow H^2(U, \mathbb{Z}(1))$ and the canonical map from $H^2(U, \mathbb{Z}(1)) \to H^2(U, \mathbb{C})$. The second map $H^2(U, \mathbb{C}) \to H^2(\mathcal{O}_X)$ results from taking the quotient of $H^2(U, \mathbb{C})$ by the first Hodge filtration $F^1 H^2(U, \mathbb{C})$. Hence, we have a commutative diagram:

$$\begin{array}{ccccccc} & & \mathrm{im}(\mathrm{Pic}(X,D)\to W_3H^2(U,\mathbb{Z}(1))) & \longrightarrow & W_3H^2(U,\mathbb{Z}(1)) & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F^1H^2(U,\mathbb{C}) & \longrightarrow & H^2(U,\mathbb{C}) & \longrightarrow & H^2(\mathcal{O}_X) \longrightarrow 0 \end{array}$$

Thus, the image of $\mathrm{Pic}(X,D)$ in $W_3H^2(U,\mathbb{Z}(1))$ is exactly the fibre product $F^1H^2(U,\mathbb{C}) \times_{H^2(U,\mathbb{C})} W_3H^2(U,\mathbb{Z}(1))$ which is the definition of the meromorphic Néron-Severi group $\mathrm{NS}(X,D)$. Now, if we use the long exact sequence in (4.30) again, we find that the kernel of $\mathrm{Pic}(X,D) \to W_3H^2(U,\mathbb{Z}(1))$ is the group:

$$\frac{H^1(\mathcal{O}_X)}{\mathrm{im}(H^1(U,\mathbb{Z}(1)) \to H^1(\mathcal{O}_X)}$$

which is exactly the definition of the meromorphic Jacobian $\mathrm{Jac}(X,D)$. □

We conclude this subsection by proving an equivalence between the meromorphic Picard group on (X,D) and the analytic Picard group of the complement U under specific conditions.

Corollary 4.36. *Let (X,D) be such that X has vanishing Hodge numbers $h^{0,1}=0, h^{0,2}=0$ and D is a smooth ample divisor in X. There is an isomorphism between the meromorphic Picard group $\mathrm{Pic}(X,D)$ and the analytic Picard group $\mathrm{Pic}(U^{an})$ of the complement U:*

$$\mathrm{Pic}(X,D) \cong \mathrm{Pic}(U^{an})$$

Proof. The assumption $H^1(\mathcal{O}_X)=0$ implies that the meromorphic Jacobian $\mathrm{Jac}(X,D)=0$. Moreover, the assumption $H^2(\mathcal{O}_X)=0$ implies that $F^1H^2(U,\mathbb{C}) = H^2(U,\mathbb{C})$ and so the meromorphic Néron-Severi group $\mathrm{NS}(X,D) = W_3H^2(U,\mathbb{Z}(1))$. Using the above theorem, we obtain the following isomorphism

$$\mathrm{Pic}(X,D) \cong W_3H^2(U,\mathbb{Z}(1))$$

Now, if we use the exact sequence in (4.29), we find that the cokernel of the map:

$$W_3H^2(U,\mathbb{Z}(1)) \to H^2(U,\mathbb{Z}(1)) \tag{4.31}$$

is precisely the kernel of the following map:

$$\mathbb{H}^2(\sigma_{>1}Rj_*\mathbb{Z}(1)_U) \to \mathbb{H}^3(\sigma_{\leq 1}Rj_*\mathbb{Z}(1))$$

However, the group $\mathbb{H}^2(\sigma_{>1}Rj_*\mathbb{Z}(1)_U) = H^0(R^2j_*\mathbb{Z}(1)_U)$ and since D was assumed to be a smooth divisor, this group is necessarily zero. Hence, the cokernel of the map in (4.31) is zero and thus,

$$\mathrm{Pic}(X,D) \cong H^2(U,\mathbb{Z}(1))$$

To conclude the proof, we need to show that the analytic Picard group $\mathrm{Pic}(U^{an})$ is also isomorphic to $H^2(U,\mathbb{Z}(1))$. Consider the long exact sequence arising from the exponential sheaf sequence on U:

$$\cdots \to H^1(\mathcal{O}_U) \to \mathrm{Pic}(U^{an}) \to H^2(U,\mathbb{Z}(1)) \to H^2(\mathcal{O}_U) \to \ldots$$

Since U is the complement of a smooth irreducible ample divisor in X, U must be an affine variety. Therefore, the higher cohomologies of coherent sheaves necessarily vanish which yields:

$$\mathrm{Pic}(U^{an}) \cong H^2(U, \mathbb{Z}(1))$$

□

4.2.3 Geometric Pre-quantization

We begin by introducing the sheaf of filtered differential operators on (X, D):

Definition 4.37. We define the ***sheaf of filtered differential operators*** on (X, D) to be the differential operators on $\mathcal{O}_X(*D)$ which preserves the $\mathbb{Z}$-filtration on $\mathcal{O}_X(*D)$ given in (4.16). We denote this sheaf of operators by $\mathcal{D}^{\bullet}_{(X,D)}(\mathcal{M}, \mathcal{M})$.

Definition 4.38. Let $\mathcal{M}$ be a meromorphic line bundle on (X, D). A first-order filtered differential operator on $\mathcal{M}$ is a map $\xi : \mathcal{M} \to \mathcal{M}$ which preserves the $\mathbb{Z}$-filtration along D on $\mathcal{M}$ as in Proposition 4.23 and such that, for any $s \in \mathcal{M}$ and $f \in \mathcal{O}_X(*D)$, the symbol σ_ξ, defined by

$$\sigma_\xi(f)s = \xi(fs) - f\xi(s),$$

is a derivation of $\mathcal{O}_X(*D)$. We denote this sheaf of operators by $\mathcal{D}^1_{(X,D)}(\mathcal{M}, \mathcal{M})$.

Proposition 4.39. *The sheaf $\mathcal{D}^1_{(X,D)}(\mathcal{M}, \mathcal{M})$ of first-order filtered differential operators admits the structure of a log Picard algebroid on (X, D).*

Proof. Let $\xi \in \mathcal{D}^1_{(X,D)}(\mathcal{M}, \mathcal{M})$, we first show that $\sigma_\xi \in \mathcal{T}_X(-\log D)$. To suppress extra notation, we will suppose here that the divisor D is smooth and is given locally by $\{z = 0\}$, the general case of normal crossings follows from the same logic. From the defining property of ξ,

$$\begin{aligned}[\xi, z^{-1}]s &= \xi(z^{-1}s) - z^{-1}\xi(s) \\ &= -z^{-2}\sigma_\xi(z)s,\end{aligned}$$

for $s \in \mathcal{L}$. If s is in the degree k part of the filtration defined on $\mathcal{M}$ by $\mathrm{ord}(\mathcal{M})$ in Proposition 4.23, the left-hand side of the above equation is in the degree $k+1$ part which implies that $\sigma_\xi(z) \subset (z)$. Thus, $\sigma_\xi \in \mathcal{T}_X(-\log D)$.

Hence, the assignment $\xi \mapsto \sigma_\xi$ defines a map from $\mathcal{D}^1_{(X,D)}(\mathcal{M}, \mathcal{M})$ to $\mathcal{T}_X(-\log D)$. This is a surjective map and the kernel of this map is the 0-th order filtered differential operators which is just $\mathcal{O}_X$. Therefore, we have the short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{D}^1_{(X,D)}(\mathcal{M}, \mathcal{M}) \to \mathcal{T}_X(-\log D) \to 0$$

For $\xi_1, \xi_2 \in \mathcal{D}^1_{(X,D)}(\mathcal{M}, \mathcal{M})$, the bracket of ξ_1 and ξ_2 satisfies the Leibniz identity. Moreover, $\sigma_{[\xi_1,\xi_2]} = [\sigma_{\xi_1}, \sigma_{\xi_2}]$ and so the map $\xi \mapsto \sigma_\xi$ is a bracket-preserving morphism. Thus, $\mathcal{D}^1_{(X,D)}(\mathcal{M}, \mathcal{M})$ admits the structure of a log Picard algebroid on (X, D). □

From now on, we will call $\mathcal{D}^1_{(X,D)}(\mathcal{M}, \mathcal{M})$ the log Picard algebroid of infinitesimal symmetries of a meromorphic line bundle $\mathcal{M}$ and we will denote it as $\mathcal{A}_{\mathcal{M}}$. We have the analogue of Proposition 3.14 in the context of log Picard algebroids and meromorphic line bundles:

Proposition 4.40. *Let $\mathcal{M}$ be a meromorphic line bundle on (X, D) and $\mathcal{A}_{\mathcal{M}}$ be the Picard algebroid of infinitesimal symmetries of $\mathcal{M}$. Suppose that $\{g_{ij}\} \in H^1(\mathcal{O}_X^\times(*D))$ are the transition functions of $\mathcal{M}$ and $[\mathcal{A}_{\mathcal{M}}] \in H^1(\Omega_X^{1,\mathrm{cl}}(\log D))$ is the class of $\mathcal{A}_{\mathcal{M}}$, then, we have $[\mathcal{A}_{\mathcal{M}}] = \mathrm{dlog}(\{g_{ij}\})$ where* dlog *is the map in cohomology:*

$$H^1(\mathcal{O}_X^\times(*D)) \overset{\mathrm{dlog}}{\to} H^1(\Omega_X^{1,\mathrm{cl}}(\log D))$$

Proof. The proof of this is exactly the same as the proof given in Proposition 3.14. □

We will now solve the pre-quantization problem for log Picard algebroids on (X, D) where D is a smooth divisor in X:

Theorem 4.41. *Let D be a smooth divisor in X and $\mathcal{A}$ be a log Picard algebroid on (X, D). There exists a pre-quantization for $\mathcal{A}$ if and only if the class of $\mathcal{A}$, $[\mathcal{A}] \in H^1(\Omega_X^{1,cl}(\log D))$, lies in the image of the map $H^2(U, \mathbb{Z}(1)) \to H^2(U, \mathbb{C})$.*

Proof. From the above proposition, we know that there exists a pre-quantization for $\mathcal{A}$ if and only if the class $[\mathcal{A}]$ is in the image of the map

$$H^1(\mathcal{O}_X^\times(*D)) \overset{\mathrm{dlog}}{\to} H^1(\Omega_X^{1,\mathrm{cl}}(\log D))$$

We will show that $[\mathcal{A}]$ lies in the image of the above map if and only $[\mathcal{A}]$ lies in the image of $H^2(U, \mathbb{Z}(1)) \to H^2(U, \mathbb{C})$. First, we have the exact sequence of the sheaves of abelian groups:

$$0 \to \mathbb{C}_X \to \mathcal{O}_X \overset{d}{\to} \Omega_X^{1,\mathrm{cl}}(\log D) \to \mathbb{C}_D \to 0$$

which implies that we have the quasi-isomorphism:

$$\Omega_X^{1,\mathrm{cl}}(\log D) \sim \mathrm{Cone}(\sigma_{\leq 1} Rj_* \mathbb{C}_U \to \mathcal{O}_X)$$

But since D is assumed to be a smooth divisor, $\sigma_{\leq 1} Rj_* \mathbb{C}_U = Rj_* \mathbb{C}_U$ and so we have

$$\Omega_X^{1,\mathrm{cl}}(\log D) \sim \mathrm{Cone}(Rj_* \mathbb{C}_U \to \mathcal{O}_X) \tag{4.32}$$

We also have the following exact sequence of sheaves of abelian groups:

$$0 \to \mathbb{C}_X^\times \to \mathcal{O}_X^\times(*D) \overset{\mathrm{dlog}}{\to} \Omega_X^{1,\mathrm{cl}}(\log D) \to \mathbb{C}_D^\times \to 0$$

which implies that we have the quasi-isomorphism:

$$\Omega_X^{1,\mathrm{cl}}(\log D) \sim \mathrm{Cone}(\sigma_{\leq 1} Rj_* \mathbb{C}_U \to \mathcal{O}_X^\times(*D))$$

Again, by the assumption on D, we have

$$\Omega_X^{1,\mathrm{cl}}(\log D) \sim \mathrm{Cone}(Rj_* \mathbb{C}_U^\times \to \mathcal{O}_X^\times(*D)) \tag{4.33}$$

The quasi-isomorphisms (4.32) and (4.33) implies that we have the following short exact sequence of complexes

$$0 \to Rj_*\mathbb{C}_U \to \mathcal{O}_X \xrightarrow{d} \Omega_X^{1,\mathrm{cl}}(\log D) \to 0$$
$$0 \to Rj_*\mathbb{C}_U^\times \to \mathcal{O}_X^\times(*D) \xrightarrow{\mathrm{dlog}} \Omega_X^{1,\mathrm{cl}}(\log D) \to 0$$

The map between these two short exact sequences gives the following commutative diagram

$$\begin{array}{ccccccc}
Rj_*\mathbb{Z}(1)_U[1] & = & Rj_*\mathbb{Z}(1)_U[1] & & & & Rj_*\mathbb{Z}(1)_U[2] \\
\uparrow & & \uparrow & & & & \uparrow \\
Rj_*\mathbb{C}_U^\times & \longrightarrow & \mathcal{O}_X^\times(*D) & \xrightarrow{\mathrm{dlog}} & \Omega_X^{1,\mathrm{cl}}(\log D) & \longrightarrow & Rj_*\mathbb{C}_U^\times[1] \\
\uparrow & & \uparrow & & \| & & \uparrow \\
Rj_*\mathbb{C}_U & \longrightarrow & \mathcal{O}_X & \xrightarrow{d} & \Omega_X^{1,\mathrm{cl}}(\log D) & \longrightarrow & Rj_*\mathbb{C}_U[1] \\
\uparrow & & \uparrow & & & & \uparrow \\
Rj_*\mathbb{Z}(1)_U & = & Rj_*\mathbb{Z}(1)_U & & & & Rj_*\mathbb{Z}(1)_U[1]
\end{array} \tag{4.34}$$

There is the long exact sequence associated to the top short exact sequence

$$\cdots \to H^1(\mathcal{O}_X^\times(*D)) \xrightarrow{\mathrm{dlog}} H^1(\Omega_X^{1,\mathrm{cl}}(\log D)) \to H^2(U, \mathbb{C}^\times) \to \dots \tag{4.35}$$

If the class $[\mathcal{A}]$ lies in the image of $H^1(\mathcal{O}_X^\times(*D)) \xrightarrow{\mathrm{dlog}} H^1(\Omega_X^{1,\mathrm{cl}}(\log D))$, then $[\mathcal{A}]$ is trivial in $H^2(U, \mathbb{C}^\times)$. By the above diagram, this implies that $[\mathcal{A}]$ lies in the kernel of the exponential map $H^2(U, \mathbb{C}) \to H^2(U, \mathbb{C}^\times)$ and hence in the image of $H^2(U, \mathbb{Z}(1)) \to H^2(U, \mathbb{C})$.

Conversely, suppose that $[\mathcal{A}]$ lies in the image of $H^2(U, \mathbb{Z}(1)) \to H^2(U, \mathbb{C})$, this implies that it is trivial in $H^2(U, \mathbb{C}^\times)$ and using the sequence (4.35), we obtain that $[\mathcal{A}]$ must lie in the image of $H^1(\mathcal{O}_X^\times(*D)) \xrightarrow{\mathrm{dlog}} H^1(\Omega_X^{1,\mathrm{cl}}(\log D))$. □

Corollary 4.42. *Let D be a smooth divisor in X. The log Picard algebroids on (X, D) which admit pre-quantization are classified by the following subgroup of $H^1(\Omega_X^{1,cl}(\log D))$:*

$$H^{1,1}(U, \mathbb{Z}(1)) := H^1(\Omega_X^1(\log D)) \cap \mathrm{im}(H^2(U, \mathbb{Z}(1)) \to H^2(U, \mathbb{C})) \subset H^1(\Omega_X^{1,cl}(\log D)).$$

Proof. Let $\mathcal{A}$ be a log Picard algebroid on (X, D) which admit pre-quantization. From the above theorem, we know that the class of $\mathcal{A}$, $[\mathcal{A}] \in H^1(\Omega_X^{1,\mathrm{cl}}(\log D))$, lies in the image of the map $H^2(U, \mathbb{Z}(1)) \to H^2(U, \mathbb{C})$. It remain to prove that $[\mathcal{A}]$ must be a class in $H^1(\Omega_X^1(\log D))$. Since D is assumed to be smooth, we have the short exact sequence

$$0 \to H^0(\Omega_X^{2,\mathrm{cl}}(\log D)) \to H^1(\Omega_X^{1,\mathrm{cl}}(\log D)) \to H^1(\Omega_X^1(\log D)) \to 0$$

The group $H^0(\Omega_X^{2,\mathrm{cl}}(\log D))$ lies in the residue exact sequence:

$$0 \to H^0(\Omega_X^{2,\mathrm{cl}}) \to H^0(\Omega_X^{2,\mathrm{cl}}(\log D)) \to H^0(\Omega_D^{1,\mathrm{cl}}) \to \dots$$

The groups $H^0(\Omega_X^{2,\mathrm{cl}})$ and $H^0(\Omega_D^{1,\mathrm{cl}})$ lies in the holomorphic part of the Hodge decomposition of X and D

respectively so its intersection with the groups $H^2(X, \mathbb{Z}(1))$ and $H^1(D, \mathbb{Z}(1))$ respectively is zero. Using the integral Gysin sequence

$$\cdots \to H^2(X, \mathbb{Z}(1)) \to H^2(U, \mathbb{Z}(1)) \to H^1(D, \mathbb{Z}(1)) \to \ldots,$$

this implies that the intersection of the group $H^0(\Omega_X^{2,\mathrm{cl}}(\log D))$ with $H^2(U, \mathbb{Z}(1))$ must be zero. Thus, the class $[\mathcal{A}] \in H^1(\Omega_X^{1,\mathrm{cl}}(\log D))$ must lie in $H^1(\Omega_X^1(\log D))$.

Suppose that we have a log Picard algebroid $\mathcal{A}$ whose class $[\mathcal{A}]$ lies in the subgroup $H^{1,1}(U, \mathbb{Z}(1))$. By the definition of $H^{1,1}(U, \mathbb{Z}(1))$, $[\mathcal{A}]$ lies in the image of $H^2(U, \mathbb{Z}(1)) \to H^2(U, \mathbb{C})$ and so by Theorem 3.15, the log Picard algebroid $\mathcal{A}$ admits a pre-quantization. □

The group $H^{1,1}(U, \mathbb{Z}(1))$ of log Picard algebroids on (X, D) admitting pre-quantization is a subgroup of the meromorphic Néron-Severi group $\mathrm{NS}(X, D)$ of (X, D). These two groups are not equal in general unlike the non-logarithmic case, consider the example where $X = \mathbb{P}^2$ and C is the smooth irreducible conic curve. The group $H^{1,1}(U, \mathbb{Z}(1)) = 0$ in this case as $H^1(\Omega^1_{\mathbb{P}^2}(\log C)) = 0$. However, the meromorphic Néron-Severi group for this pair is non-trivial: $\mathrm{NS}(\mathbb{P}^2, C) = \mathbb{Z}_2$.

Proposition 4.43. *Let D be a smooth divisor in X. There is an action on the group $H^{1,1}(U, \mathbb{Z}(1))$ of log Picard algebroids on (X, D) admitting pre-quantization by the group of meromorphic line bundles on (X, D) with flat logarithmic connection.*

Proof. We know that log Picard algebroids $\mathcal{A}$ on X that can be pre-quantized are the ones whose class $[\mathcal{A}]$ can be lifted under the dlog map:

$$H^1(\mathcal{O}_X^\times(*D)) \stackrel{\mathrm{dlog}}{\to} H^1(\Omega_X^{1,\mathrm{cl}}(\log D))$$

Recall we have the long exact sequence associated to the mapping cone (4.33):

$$\cdots \to H^0(\Omega_X^{1,\mathrm{cl}}(\log D)) \to H^1(U, \mathbb{C}^\times) \to H^1(\mathcal{O}_X^\times(*D)) \stackrel{\mathrm{dlog}}{\to} H^1(\Omega_X^{1,\mathrm{cl}}(\log D)) \to \ldots$$

Two lifts of the class $[\mathcal{A}]$ differ by a class in $H^1(\mathcal{O}_X^\times(*D))$ given in the image of the map

$$H^1(U, \mathbb{C}^\times) \to H^1(\mathcal{O}_X^\times(*D)) \tag{4.36}$$

However, this map is not injective as there are $\mathbb{C}^\times$-local systems on U that are trivial as holomorphic line bundles which are given by $H^0(\Omega_X^{1,\mathrm{cl}}(\log D))$, the group of global logarithmic 1-forms. We denote the image of the above map by $\mathcal{K}'$ and we will now study this group. From the short exact sequence

$$0 \to \mathbb{Z}(1)_U \to \mathbb{C}_U \to \mathbb{C}_U^\times \to 0,$$

we obtain that:

$$0 \to \frac{H^1(U, \mathbb{C})}{H^1(U, \mathbb{Z}(1))} \to H^1(U, \mathbb{C}^\times) \to H^2(U, \mathbb{Z})_{\mathrm{tors}} \to 0$$

Using the long exact sequence associated to the middle horizontal sequence in (4.34):

$$\cdots \to H^1(U, \mathbb{Z}(1)) \to H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_X^\times(*D)) \to H^2(U, \mathbb{Z}(1)) \to \ldots$$

we find that the image of the map in (4.36) will surject to $H^2(U,\mathbb{Z}(1))_{\mathrm{tors}}$ with kernel

$$\frac{H^1(\mathcal{O}_X)}{\mathrm{im}(H^1(U,\mathbb{Z}(1)) \to H^1(\mathcal{O}_X))}.$$

This is precisely the meromorphic Jacobian $\mathrm{Jac}(X, D)$ of (X, D) and so we have:

$$0 \to \mathrm{Jac}(X, D) \to \mathcal{K} \to H^2(U,\mathbb{Z}(1))_{\mathrm{tors}} \to 0$$

The group $\mathcal{K}'$ is exactly the group of meromorphic line bundles on (X, D) with a flat logarithmic connection. □

4.2.4 Functorial properties and blowup

Let $\varphi : Y \to X$ be a morphism of projective varieties and D a simple normal crossings divisor on X such that φ^*D is also a simple normal crossings divisor on Y. We define the pullback of meromorphic line bundles:

Definition 4.44. Let $\mathcal{L}$ be a meromorphic line bundle on (X, D) given by transition functions $\{g_{ij}\} \in H^1(\mathcal{O}_X^\times(*D))$, the ***pullback meromorphic line bundle*** $\varphi^*\mathcal{L}$ of $\mathcal{L}$ is the locally-free $\mathcal{O}_Y(*\varphi^*D)$-module $\varphi^*\mathcal{L}$ with transition functions $\{\varphi^*g_{ij}\} \in H^1(\mathcal{O}_Y^\times(*\varphi^*D))$.

We now prove an analogue of Theorem 4.21 in the context of meromorphic line bundles. This will be necessary when we give explicit constructions of meromorphic line bundles later. To do so, we begin with the following lemma:

Lemma 4.45. *Let X be a smooth projective algebraic surface with first Betti number $b_1(X) = 0$ and let $\pi : \widetilde{X} \to X$ be the blowup of X at a point $p \in X$. The analytic Brauer group of $\widetilde{X}$ is isomorphic to the analytic Brauer group of X.*

Proof. Since $b_1(X) = 0$, we have $H^1(\mathcal{O}_X) = 0$ and moreover by Poincaré duality, we have $H^3(X,\mathbb{Z}) = 0$. Hence, we can write the long exact sequence associated to the exponential sheaf sequence on X as follows

$$0 \to \mathrm{Pic}(X) \to H^2(X,\mathbb{Z}) \to H^2(\mathcal{O}_X) \to H^2(\mathcal{O}_X^\times) \to 0 \tag{4.37}$$

Recall again that the first Betti number is a birational invariant for smooth projective surfaces which means that we analogously have the exact sequence for $\widetilde{X}$

$$0 \to \mathrm{Pic}(\widetilde{X}) \to H^2(\widetilde{X},\mathbb{Z}) \to H^2(\mathcal{O}_{\widetilde{X}}) \to H^2(\mathcal{O}_{\widetilde{X}}^\times) \to 0$$

We have $\mathrm{Pic}(\widetilde{X}) = \mathrm{Pic}(X) \oplus \mathbb{Z}$ and $H^2(\widetilde{X},\mathbb{Z}) = H^2(X,\mathbb{Z}) \oplus \mathbb{Z}$. Moreover, by the fact that the plurigenus is a birational invariant along with Serre duality, we have $H^2(\mathcal{O}_{\widetilde{X}}) = H^2(\mathcal{O}_X)$. Hence, we can re-write the above sequence as

$$0 \to \mathrm{Pic}(X) \oplus \mathbb{Z} \to H^2(X,\mathbb{Z}) \oplus \mathbb{Z} \to H^2(\mathcal{O}_X) \to H^2(\mathcal{O}_{\widetilde{X}}^\times) \to 0$$

Therefore, we can fit the above sequence and the sequence in (4.37) in the commutative diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathrm{Pic}(X) & \longrightarrow & H^2(X,\mathbb{Z}) & \longrightarrow & H^2(\mathcal{O}_X) & \longrightarrow & H^2(\mathcal{O}_X^\times) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Pic}(X)\oplus\mathbb{Z} & \longrightarrow & H^2(X,\mathbb{Z})\oplus\mathbb{Z} & \longrightarrow & H^2(\mathcal{O}_X) & \longrightarrow & H^2(\mathcal{O}_{\widetilde{X}}^\times) & \longrightarrow & 0 \end{array}$$

where all the vertical arrows are induced by the pullback π^*. This shows that the map

$$H^2(\mathcal{O}_X^\times) \to H^2(\mathcal{O}_{\widetilde{X}}^\times)$$

is an isomorphism which is what we wanted to prove. □

Theorem 4.46. *Suppose that we have a smooth projective algebraic surface X with $b_1(X) = 0$, D a smooth irreducible divisor in X, and let $\pi : \widetilde{X} \to X$ be the blowup of X at a point p where $p \in D \subset X$. The pullback of meromorphic line bundles defines an equivalence between the meromorphic Picard group $\mathrm{Pic}(X, D)$ on (X, D) and the meromorphic Picard group on $\mathrm{Pic}(\widetilde{X}, \pi^*D)$ of $(\widetilde{X}, \pi^*D)$ where $\pi^*D := D + E$ and E is the exceptional divisor of the blowup.*

Proof. Let $\mathcal{L}$ be a meromorphic line bundle on (X, D), the assignment $\mathcal{L} \mapsto \varphi^*\mathcal{L}$ induces the map in cohomology

$$\mathrm{Pic}(X, D) \to \mathrm{Pic}(\widetilde{X}, \pi^*D)$$

We wish to show that this map is an isomorphism. First, we realize that the blowup of the point p on the divisor D does not change the complement U. Using Theorem 4.26, we have the exact sequence

$$0 \to \mathrm{Pic}^{\mathrm{alg}}(U) \to \mathrm{Pic}(\widetilde{X}, \pi^*D) \to H^1(D,\mathbb{Z}) \oplus H^1(E,\mathbb{Z}) \to H^2(\mathcal{O}_{\widetilde{X}}^\times)$$

As $E \cong \mathbb{P}^1$, we have $H^1(E,\mathbb{Z}) = 0$. Furthermore, from the above lemma, we have that $H^2(\mathcal{O}_{\widetilde{X}}^\times)$ is isomorphic to $H^2(\mathcal{O}_X^\times)$ so we can re-write the above sequence as

$$0 \to \mathrm{Pic}^{\mathrm{alg}}(U) \to \mathrm{Pic}(\widetilde{X}, \pi^*D) \to H^1(D,\mathbb{Z}) \to H^2(\mathcal{O}_X^\times) \tag{4.38}$$

Recall from (4.21), we have

$$0 \to \mathrm{Pic}^{\mathrm{alg}}(U) \to \mathrm{Pic}(X, D) \to H^1(D,\mathbb{Z}) \to H^2(\mathcal{O}_X^\times)$$

We can fit the above sequence and the sequence in (4.38) to obtain the commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Pic}^{\mathrm{alg}}(U) & \longrightarrow & \mathrm{Pic}(X,D) & \longrightarrow & H^1(D,\mathbb{Z}) & \longrightarrow & H^2(\mathcal{O}_X^\times) \\ & & \| & & \downarrow & & \| & & \| \\ 0 & \longrightarrow & \mathrm{Pic}^{\mathrm{alg}}(U) & \longrightarrow & \mathrm{Pic}(\widetilde{X},\pi^*D) & \longrightarrow & H^1(D,\mathbb{Z}) & \longrightarrow & H^2(\mathcal{O}_X^\times) \end{array}$$

This proves that the map

$$\mathrm{Pic}(X, D) \to \mathrm{Pic}(\widetilde{X}, \pi^*D)$$

is an isomorphism which concludes the proof. □

4.3 Explicit Constructions

In this section, we provide a method for constructing meromorphic line bundles on certain ruled surfaces equipped with simple normal crossing divisors. We then employ the method, together with the blowup constructions from the above section to construct all meromorphic line bundles on $\mathbb{C}P^2$ with singularities along a smooth cubic curve.

4.3.1 Meromorphic line bundles on Hirzebruch surfaces

Let $\pi : X \to \mathbb{P}^1$ be a projective line bundle, and let $D \subset X$ be a simple normal crossings divisor which is finite over the base. We define the discriminant locus $\Delta \subset \mathbb{P}^1$ to be the union of the critical values of $\pi|_D : D \to \mathbb{P}^1$ and $\pi(D^{\text{sing}})$, the image of the singular points of D.

As described in Figure 4.1, choose an oriented simple closed curve $\gamma \subset \mathbb{P}^1 \backslash \Delta$, and denote by $\{V_0, V_1\}$ an open covering of $\mathbb{C}P^1$ such that V_0 contains the oriented interior of γ, V_1 contains the exterior, and $V_0 \cap V_1$ is an annular neighbourhood A_γ of γ with no intersection with Δ. The inverse images $U_0 = \pi^{-1}(V_0)$ and $U_1 = \pi^{-1}(V_1)$ then form an open covering of X with intersection $U_0 \cap U_1 = \pi^{-1}(A_\gamma)$.

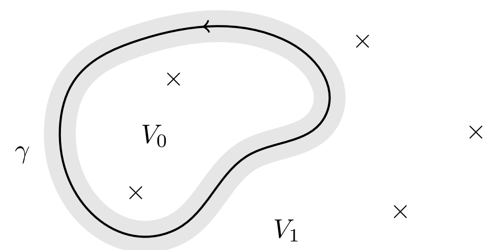

Figure 4.1: A covering of $\mathbb{C}P^1$ by discs V_0, V_1 intersecting in an annular neighbourhood of an oriented simple closed curve γ. The annulus avoids the discriminant locus.

In this way, the choice of the simple closed curve $\gamma \subset \mathbb{P}^1 \backslash \Delta$ defines an analytic open covering of X. We now construct a meromorphic line bundle by specifying a Čech cocycle for this covering. That is, we require a function

$$g_{01} \in \mathcal{O}_X^\times(*D)(U_0 \cap U_1).$$

As the annulus A_γ does not intersect Δ, the restriction $D_{01} = D \cap (U_0 \cap U_1)$ is an unramified covering of A_γ. We now assume that there are two disjoint connected components γ^+, γ^- of $D \cap \pi^{-1}(\gamma)$ which are isomorphic as covering spaces of γ; as a result they lie on linearly equivalent connected components of D_{01} cut out by holomorphic sections s_+, s_- respectively. The resulting meromorphic function

$$g_{01} = \frac{s_+}{s_-}$$

gives the required cocycle. We denote the corresponding meromorphic line bundle by $\mathcal{M}(\gamma^+, \gamma^-)$.

In this situation, the analytic Brauer group of X vanishes, so by Theorem 4.26, the meromorphic Picard group $\text{Pic}(X, D)$ maps surjectively to the moduli of $\mathbb{Z}$-local systems on D via the order map:

$$\text{Pic}(X, D) \stackrel{\text{ord}_*}{\to} H^1(D, \mathbb{Z}) \tag{4.39}$$

The image of $\mathcal{M}(\gamma^+, \gamma^-)$ under the above order map is a $\mathbb{Z}$-local system on D described by the 1-cocycle which is $+1$ on D^+, -1 on D^-, and vanishes on any other components of $D \cap (U_0 \cap U_1)$. Equivalently, for any 1-cycle β in D,

$$\langle \mathrm{ord}_*(\mathcal{M}(\gamma^+, \gamma^-)), [\beta] \rangle = \gamma^+ \cdot \beta - \gamma^- \cdot \beta,$$

where $\gamma^\pm = \pi^{-1}(\gamma) \cap D^\pm$ are equipped with the orientation inherited from γ. We summarize this construction in the following theorem.

Theorem 4.47. *Given an oriented simple closed curve $\gamma \subset \mathbb{P}^1 \backslash \Delta$ having disjoint connected components γ^+, γ^- of $D \cap \pi^{-1}(\gamma)$ of the same degree over γ, the above construction defines a meromorphic line bundle $\mathcal{M}(\gamma^+, \gamma^-)$ whose corresponding local system is Poincaré dual to $[\gamma^+] - [\gamma^-]$, i.e.,*

$$\mathrm{ord}_* \mathcal{M}(\gamma^+, \gamma^-) = \mathrm{PD}([\gamma^+] - [\gamma^-]).$$

Example 4.48. Consider $(\mathbb{P}^2, D)$ where D is the smooth elliptic curve in $\mathbb{P}^2$. If we blow-up a point $p \in D$, we obtain the first Hirzebruch surface $\mathbb{F}_1$ with the divisor $D + E$ where $E \cong \mathbb{P}^1$ is the exceptional divisor of the blow-up. We will now work with $(\mathbb{F}_1, D + E)$. Since the analytic Brauer group $H^2(\mathcal{O}^\times_{\mathbb{F}_1})$ is trivial, by Theorem 4.26, the meromorphic Picard group $\mathrm{Pic}(\mathbb{F}_1, D + E)$ maps surjectively to the moduli of $\mathbb{Z}$-local systems on $D + E$ via the order map

$$\mathrm{Pic}(\mathbb{F}_1, D + E) \overset{\mathrm{ord}_*}{\to} H^1(D, \mathbb{Z}) \oplus H^1(E, \mathbb{Z}).$$

There are no non-trivial $\mathbb{Z}$-local systems on E since $E \cong \mathbb{P}^1$. Hence, we have the surjection

$$\mathrm{Pic}(\mathbb{F}_1, D + E) \overset{\mathrm{ord}_*}{\to} H^1(D, \mathbb{Z}).$$

The first Hirzebruch surface $\mathbb{F}_1$ can be realized as a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ where the divisor $D + E$ on $\mathbb{F}_1$ is a 3-to-1 cover of $\mathbb{P}^1$ ramified at four points on $\mathbb{P}^1$ which we label as p_1, p_2, p_3, p_4. The elliptic curve divisor D is the standard 2-to-1 cover of $\mathbb{P}^1$ ramified at the four points. Consider the oriented simple closed curve γ_{12} containing the points p_1, p_2 in its interior and two disjoint connected components $\gamma_{12}^+, \gamma_{12}^-$ of $(D + E) \cap \pi^{-1}(\gamma_{12})$ where γ_{12}^+ corresponds to one branch of D over γ_{12} and γ_{12}^- corresponds to the branch of E over γ_{12}. By the above theorem, this defines a meromorphic line bundle $\mathcal{M}(\gamma_{12}^+, \gamma_{12}^-)$ and the order of this meromorphic line bundle, $\mathrm{ord}_* \mathcal{M}(\gamma_{12}^+, \gamma_{12}^-)$, is a local system that is Poincaré dual to the cycle $[\gamma_{12}^+]$ in $H_1(D, \mathbb{Z})$.

If we now repeat the same argument as above for a oriented simple closed curve γ_{23} containing the points p_2, p_3 in its interior, we obtain the meromorphic line bundle $\mathcal{M}(\gamma_{23}^+, \gamma_{23}^-)$ where the order of this bundle, $\mathrm{ord}_* \mathcal{M}(\gamma_{23}^+, \gamma_{23}^-)$, is a local system that is Poincaré dual to the cycle $[\gamma_{23}^+]$ in $H_1(D, \mathbb{Z})$.

Each of the cycles γ_{12}^+ and γ_{23}^+ have trivial self-intersection and $\gamma_{12}^+ \cdot \gamma_{23}^+ = 1$. Therefore, the two cycles together form a homology basis for $H_1(D, \mathbb{Z})$. As a result, the local systems $\mathrm{ord}_* \mathcal{M}(\gamma_{12}^+, \gamma_{12}^-)$ and $\mathrm{ord}_* \mathcal{M}(\gamma_{23}^+, \gamma_{23}^-)$ coming from the meromorphic line bundles $\mathrm{ord}_* \mathcal{M}(\gamma_{12}^+, \gamma_{12}^-)$ and $\mathrm{ord}_* \mathcal{M}(\gamma_{23}^+, \gamma_{23}^-)$ respectively generate the cohomology group $H^1(D, \mathbb{Z})$.

4.3.2 Smooth cubic in $\mathbb{P}^2$

We conclude this chapter by providing explicit constructions of meromorphic line bundles on $(\mathbb{P}^2, D)$ where D is the smooth elliptic curve in $\mathbb{P}^2$. By Theorem 4.46, we know that the pullback defines an equivalence between $\mathrm{Pic}(\mathbb{P}^2, D)$ and $\mathrm{Pic}(\mathbb{F}_1, D + E)$ and so we have meromorphic line bundles $\mathcal{M}_1$, $\mathcal{M}_2$

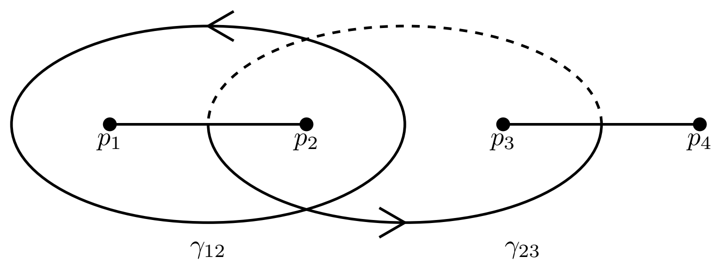

Figure 4.2: The oriented simple closed curves γ_{12}, γ_{23} on $\mathbb{P}^1$ which give rise to cycles $\gamma_{12}^+, \gamma_{23}^+$ on the elliptic curve D.

on $(\mathbb{P}^2, D)$ such that $\pi^*\mathcal{M}_1 = \mathcal{M}(\gamma_{12}^+, \gamma_{12}^-)$, $\pi^*\mathcal{M}_2 = \mathcal{M}(\gamma_{23}^+, \gamma_{23}^-)$ are unique up to isomorphism. From Example 4.27, the meromorphic Picard group of $(\mathbb{P}^2, D)$ is given by

$$\mathrm{Pic}(\mathbb{P}^2, D) = \mathbb{Z}_3 \oplus \mathbb{Z}^2.$$

The following theorem will provide a complete description of the generators of this group.

Theorem 4.49. *Any meromorphic line bundle on* $(\mathbb{P}^2, D)$ *is isomorphic to one of*

$$\mathcal{O}_{\mathbb{P}^2}(1)^k \otimes \mathcal{M}_1^{l_1} \otimes \mathcal{M}_2^{l_2},$$

where $k \in \mathbb{Z}_3$ *and* $l_1, l_2 \in \mathbb{Z}$.

Proof. As the pullbacks of the meromorphic line bundles $\mathcal{M}_1$ and $\mathcal{M}_2$ on $(\mathbb{P}^2, D)$ are the meromorphic line bundles $\mathcal{M}(\gamma_{12}^+, \gamma_{12}^-)$ and $\mathcal{M}(\gamma_{23}^+, \gamma_{23}^-)$ on $(\mathbb{F}_1, D + E)$ respectively, the local systems corresponding to $\mathcal{M}_1$ and $\mathcal{M}_2$ under the order map generates all of $H^1(D, \mathbb{Z})$. If the order is trivial, then it must be in the image of the map

$$\mathrm{Pic}(\mathbb{P}^2) \to \mathrm{Pic}(\mathbb{P}^2, D).$$

However, the image of the above map is the cokernel of the degree map

$$H^0(D, \mathbb{Z}) \xrightarrow{\cdot 3} \mathrm{Pic}(\mathbb{P}^2), \tag{4.40}$$

which is the cyclic group $\mathbb{Z}_3$ generated by the line bundle $\mathcal{O}_{\mathbb{P}^2}(1)$. $\square$

Lastly, we make the following observation. We are working with the smooth elliptic curve D in $\mathbb{P}^2$ and $\mathbb{P}^2$ has vanishing Hodge numbers $h^{0,1}, h^{0,2}$, so by Corollary 4.36, the meromorphic Picard group $\mathrm{Pic}(\mathbb{P}^2, D)$ is isomorphic to the analytic Picard group $\mathrm{Pic}(U^{\mathrm{an}})$ of the complement U.

The algebraic Picard group of U can be characterized as the cokernel of the degree map in (4.40) and so $\mathrm{Pic}^{\mathrm{alg}}(U) \cong \mathbb{Z}_3$. The generator of this cyclic group is $\mathcal{O}_{\mathbb{P}^2}(1)|_U$. Thus, if we restrict the meromorphic line bundles $\mathcal{M}_1$ and $\mathcal{M}_2$ on $(\mathbb{P}^2, D)$ to U, these give holomorphic line bundles on U that are not algebraic.

Chapter 5

Log Courant Algebroids and Meromorphic Gerbes

Our primary aims in this chapter are to replicate the theory of log Picard algebroids and meromorphic line bundles in Chapter 4 in the domain of Courant algebroids and gerbes. We define log Courant algebroids and establish classification properties of these objects through mixed Hodge theory. We proceed to discuss the topological aspects of log Courant algebroids in Section 5.1.3 and how they may be used to give a geometric iinterpretation of the standard residue short exact sequence. Furthermore, we define log Dirac structures on (X, D) which may be viewed as a generalization of Dirac structures. While we do not explore this path in our work, it is our hope that our formulation of log Courant algebroids and log Dirac structures may be used to understand existing geometries on the direct sum bundle $\mathcal{T}_X(-\log D) \oplus \Omega^1_X(\log D)$.

In the second half of this chapter, we introduce the notion of meromorphic gerbes. Our methodology here is inspired by the Hitchin-Chatterjee approach to gerbes - using Čech cocycles rather than sheaves of groupoids to define gerbes. We then proceed to define a logarithmic connective structure on a meromorphic gerbe. In Theorem 5.32, we showed that under certain hypotheses, the meromorphic Brauer group is isomorphic to the analytic Brauer group of the complement. We conclude this chapter by proving the prequantization theorem for log Courant algebroids; demonstrating that under an integrality condition, these algebroids may be prequantized to a meromorphic gerbe with 0-connection.

5.1 Log Courant Algebroids

5.1.1 Basic Setup

Definition 5.1. An ***exact log Courant algebroid*** $(\mathcal{E}, \langle\cdot,\cdot\rangle, [\cdot,\cdot], \pi)$ is a locally free $\mathcal{O}_X$-module $\mathcal{E}$ equipped with a symmetric bilinear nondegenerate pairing $\langle\cdot,\cdot\rangle : \mathcal{E} \times \mathcal{E} \to \mathcal{O}_X$, a bilinear bracket $[\cdot,\cdot]$ on sections of $\mathcal{E}$ which we call the ***log Courant bracket***, a morphism of $\mathcal{O}_X$-modules $\pi : \mathcal{E} \to \mathcal{T}_X(-\log D)$ such that the following hold

- $[e_1, [e_2, e_3]] = [[e_1, e_2], e_3] + [e_2, [e_1, e_3]]2$
- $\pi([e_1, e_2]) = [\pi(e_1), \pi(e_2)]$

- $[e_1, fe_2] = f[e_1, e_2] + (\pi(e_1)f)e_2$
- $\pi(e_1)\langle e_2, e_3\rangle = \langle [e_1, e_2], e_3\rangle + \langle e_2, [e_1, e_3]\rangle$
- $[e_1, e_1] = \mathcal{D}\langle e_1, e_1\rangle$

where $\mathcal{D} := \frac{1}{2}\pi^* \circ d : \mathcal{O}_X \to \mathcal{E},\ e_1, e_2, e_3 \in \mathcal{E},\ f \in \mathcal{O}_X$ and the following sequence is exact

$$0 \to \Omega^1_X(\log D) \xrightarrow{\pi^*} \mathcal{E} \xrightarrow{\pi} \mathcal{T}_X(-\log D) \to 0. \tag{5.1}$$

We will usually denote the quadruple $(\mathcal{E}, \langle\cdot,\cdot\rangle, [\cdot,\cdot], \pi)$ simply by $\mathcal{E}$. For the remainder of the thesis, all log Courant algebroids will be assumed to be exact.

Remark 5.2. When the divisor is empty, we recover the usual definition of exact Courant algebroids on X as introduced in [6]. We refer the reader to [26, 27, 34] for an extensive treatment of Courant algebroids in the smooth category and [21] for a treatment in the holomorphic category.

Similar to log Picard algebroids in Chapter 4, log Courant algebroids over a fixed base (X, D) form a Picard category, with monoidal structure given as follows. For $(\mathcal{E}_i, \langle\cdot,\cdot\rangle, [\cdot,\cdot]_i, \pi_i)_{i=1,2}$ log Courant algebroids, we define the Baer sum

$$\mathcal{E}_1 \boxplus \mathcal{E}_2 = (\mathcal{E}_1 \oplus_{\mathcal{T}_X(-\log D)} \mathcal{E}_2)/\left\langle \pi^*(\alpha), -\pi^*(\alpha)\right\rangle, \tag{5.2}$$

where $\pi^* : \Omega^1_X(\log D) \to \mathcal{E}_1 \oplus_{\mathcal{T}_X(-\log D)} \mathcal{E}_2$ and $\alpha \in \Omega^1_X(\log D)$. The Baer sum is equipped with morphism $\pi_1 + \pi_2$ to $\mathcal{T}_X(-\log D)$, a log Courant bracket defined as

$$[(e_1, e_2), (e_1', e_2')] = ([e_1, e_1'], [e_2, e_2'])$$

and a bilinear pairing defined as

$$\langle (e_1, e_2), (e_1', e_2')\rangle = \langle e_1, e_1'\rangle + \langle e_2, e_2'\rangle$$

where $e_i, e_i' \in \mathcal{E}_i$ for $i = 1, 2$. This gives the Baer sum the structure of an log Courant algebroid.

Definition 5.3. A ***connection*** on a log Courant algebroid $\mathcal{E}$ is a splitting

$$\nabla : \mathcal{T}_X(-\log D) \to \mathcal{E},$$

of the sequence (5.1) which is isotropic, i.e., $\langle \nabla(\xi_1), \nabla(\xi_2)\rangle = 0$. The sheaf of connections therefore forms a sheaf of affine spaces over $\Omega^2_X(\log D)$. Given connections ∇, ∇' on $\mathcal{E}, \mathcal{E}'$, the sum $\nabla + \nabla'$ defines a natural connection on the Baer sum $\mathcal{E} \boxplus \mathcal{E}'$, so that the abovementioned Picard category structure extends to log Courant algebroids with connections.

The ***curvature*** associates to any connection ∇ the 3-form $H_\nabla \in \Omega^2_X(\log D)$ defined by

$$H_\nabla(\xi_1, , \xi_2, \xi_3) = \iota_{\xi_1}([\nabla\xi_2, \nabla\xi_3] - \nabla[\xi_2, \xi_3]) \tag{5.3}$$

for $\xi_1, \xi_2, \xi_3 \in \mathcal{T}_X(-\log D)$.

Lemma 5.4. *The curvature H_∇ is a closed logarithmic 3-form. Modifying the connection by $B \in \Omega^2_X(\log D)$ gives the curvature*

$$H_{\nabla+B} = H_\nabla + dB, \tag{5.4}$$

and if $\psi : \mathcal{E} \to \mathcal{E}'$ is an isomorphism of log Courant algebroids, then $H_{\psi\circ\nabla} = H_\nabla$. Finally, if ∇, ∇' are connections for $\mathcal{E}, \mathcal{E}'$, then the curvature of the Baer sum is $H_{\nabla+\nabla'} = H_\nabla + H_{\nabla'}$.

Proof. The isotropicity of the connection ∇ enables us to write the curvature 3-form as $H_\nabla(\xi_1, \xi_2, \xi_3) = \langle \nabla(\xi_1), [\nabla(\xi_2), \nabla(\xi_3)]\rangle$. The fact that this 3-form is closed follows from the observation that

$$i_{\xi_1} i_{\xi_2} i_{\xi_3} dH_\nabla = \mathrm{Jac}(\nabla(\xi_1), \nabla(\xi_2), \nabla(\xi_3)),$$

where Jac is the Jacobinator defined as:

$$\mathrm{Jac}(\nabla(\xi_1), \nabla(\xi_2), \nabla(\xi_3)) := [\nabla(\xi_1), [\nabla(\xi_2), \nabla(\xi_3)]] - [[\nabla(\xi_1), \nabla(\xi_2)], \nabla(\xi_3)] - [\nabla(\xi_2), [\nabla(\xi_1), \nabla(\xi_3)]].$$

For property 5.4, if $\nabla' = \nabla + B$, we have

$$H_{\nabla'}(\xi_1, \xi_2, \xi_3) = H_\nabla(\xi_1, \xi_2, \xi_3) + i_{\xi_1}([\nabla\xi_2, B\xi_3] + [B\xi_2, \nabla\xi_3] - B[\xi_2, \xi_3]) = (H_\nabla + dB)(\xi_1, \xi_2, \xi_3),$$

as required. For the remaining statements, note that an isomoprhism of log Courant algebroids is an isomorphism of extensions $\psi : \mathcal{E} \to \mathcal{E}'$ which preserves the log Courant bracket $[\cdot, \cdot]$ and the bilinear bracket $\langle \cdot, \cdot \rangle$ which gives $H_{\psi\circ\nabla} = H_\nabla$ immediately. The final assertion follows from the fact that the two brackets on the Baer sum $\mathcal{E} \oplus \mathcal{E}'$ is defined componentwise. ☐

Consider the trivial extension $\mathcal{T}_X(-\log D) \oplus \Omega^1_X(\log D)$. The splitting determined by the connection ∇ takes the log Courant bracket on the log Courant algebroid $\mathcal{E}$ to the bracket on $\mathcal{T}_X(-\log D) \oplus \Omega^1_X(\log D)$ given by

$$[\xi_1 + \tau_1, \xi_2 + \tau_2] = [\xi_1, \xi_2] + \mathcal{L}_{\xi_1}\tau_2 - i_{\xi_2} d\tau_1 + i_{\xi_2} i_{\xi_1} H_\nabla, \tag{5.5}$$

where $\xi_1, \xi_2 \in \mathcal{T}_X(-\log D)$ and $\tau_1, \tau_2 \in \Omega^1_X(\log D)$. Moreover, the isomorphism of locally free $\mathcal{O}_X$-modules given by

$$\nabla + \frac{1}{2}\pi^* : \mathcal{T}_X(-\log D) \oplus \Omega^1_X(\log D) \xrightarrow{\cong} \mathcal{E}$$

yields the following bilinear pairing $\langle \cdot, \cdot \rangle$ on $\mathcal{T}_X(-\log D) \oplus \Omega^1_X(\log D)$:

$$\langle \xi_1 + \tau_1, \xi_2 + \tau_2 \rangle = \frac{1}{2}(\tau_1(\xi_2) + \tau_2(\xi_1)). \tag{5.6}$$

Thus, we obtain an log Courant algebroid structure on $\mathcal{T}_X(-\log D) \oplus \Omega^1_X(\log D)$.

Definition 5.5. Let $H \in \Omega^{3,\mathrm{cl}}_X(\log D)$ be a closed logarithmic 3-form and $\mathcal{E}$ be a log Courant algebroid on (X, D). We define the ***twist*** of $\mathcal{E}$ to be the log Courant algebroid $(\mathcal{E}, \langle \cdot, \cdot \rangle, [\cdot, \cdot]_H, \pi)$ where $[\cdot, \cdot]_H$ is given as

$$[e_1, e_2]_H = [e_1, e_2] + \iota_{\pi(e_2)}\iota_{\pi(e_1)} H,$$

for $e_1, e_2 \in \mathcal{E}$ and $\xi_1, \xi_2 \in \mathcal{T}_X(-\log D)$. We denote this algebroid by $\mathcal{E}_H$.

Example 5.6 (Split log Courant algebroid)**.** Consider a closed logarithmic 3-form $H \in \Omega^{3,\mathrm{cl}}_X(\log D)$.

We can twist the trivial extension $\mathcal{T}_X(-\log D) \oplus \Omega^1_X(\log D)$ by H

$$[\xi_1 + \tau_1, \xi_2 + \tau_2]_H = [\xi_1, \xi_2] + \mathcal{L}_{\xi_1}\tau_2 - i_{\xi_2} d\tau_1 + i_{\xi_2} i_{\xi_1} H.$$

Hence, we obtain a log Courant algebroid admitting a global connection with curvature H.

5.1.2 Classification of log Courant algebroids

Theorem 5.7. *Log Courant algebroids on (X, D) are classified by the cohomology group*

$$\mathbb{H}^3(F^2\Omega^\bullet_X(\log D)),$$

with addition deriving from the Baer sum operation (5.2)*.*

Proof. The structure of this proof is identical to the proof of Theorem 4.8 with appropriate modifications. Let $\mathcal{E}$ be a log Courant algebroid on (X, D) and take an open affine cover $\{U_i\}$ of X. Then over each U_i we may choose a connection ∇_i which has curvature $H_i \in \Omega^{3,\mathrm{cl}}(U_i, \log D)$. On the double overlap $U_i \cap U_j$, by Lemma 5.4, we have

$$H_i - H_j = dB_i j,$$

where $B_{ij} = \nabla_i - \nabla_j \in \Omega^2(U_i \cap U_j, \log D)$. Since we have $B_{ij} + B_{jk} + B_{ki} = 0$ on triple overlaps, this gives a Čech-de Rham cocycle (A_{ij}, B_i). If we modify our choice of local connections, this shifts the cocycle by a Čech coboundary and so we obtain a well-defined map from log Courant algebroids to the hypercohomology group $\mathbb{H}^3(F^2\Omega^\bullet_X(\log D))$. An isomorphism $\psi : \mathcal{E} \to \mathcal{E}'$ of log Courant algebroids induces an isomorphism of sheaves of connections which preserves the curvature (Lemma 5.4), so that $\mathcal{E}$ and $\mathcal{E}'$ give rise to cohomologous cocycles by the above argument.

We proceed now to show that this classifying map is surjective. Any such cocycle (A_{ij}, B_i) may be used to construct a log Courant algebroid as follows: on each U_i we define local log Courant algebroids $\mathcal{E}_i$ to be trivial extensions $\mathcal{E}_i = \mathcal{T}_{U_i}(-\log D) \oplus \Omega^1_{U_i}(\log D)$ equipped with the log Courant bracket in (5.5):

$$[\xi_1 + \tau_1, \xi_2 + \tau_2]_i = [\xi_1, \xi_2] + \mathcal{L}_{\xi_1}\tau_2 + i_{\xi_2} d\tau_1 + i_{\xi_2} i_{\xi_1} H_i,$$

and the bilinear bracket in (5.6):

$$\langle \xi_1 + \tau_1, \xi_2 + \tau_2 \rangle = \frac{1}{2}(\tau_1(\xi_2) + \tau_2(\xi_1)).$$

We then glue $\mathcal{E}_i$ to $\mathcal{E}_j$ using the map

$$\Phi_{ij} : \xi + \tau \mapsto \xi + \tau + B_{ij}(\xi).$$

Note that this map preserves both the log Courant bracket and the bilinear bracket above and so it defines an isomorphism of log Courant algebroids. Moreover, the cocycle condition on B_{ij} implies that $\Phi_{ki}\Phi_{jk}\Phi_{ij} = 1$ and so this yields a global log Courant algebroid which proves surjectivity. For the correspondence between the Baer sum and addition in cohomology, note that by choosing local connections ∇_i, ∇'_i on $\mathcal{E}, \mathcal{E}'$, the sum $\nabla_i + \nabla'_i$ defines local connections for the Baer sum $\mathcal{E} \boxplus \mathcal{E}'$ and by Lemma 5.4 the corresponding cocycle for $\nabla_i + \nabla'_i$ is the sum of the cocycles for the summands, as required. □

Example 5.8 (Family of split log Courant algebroids)**.** Given two closed logarithmic 3-forms $H, H' \in \Omega_X^{3,\mathrm{cl}}(\log D)$, we can twist the trivial extension $\mathcal{T}_X(-\log D) \oplus \Omega^1_X(\log D)$ by H and H' as shown in Example 5.6 to obtain split log Courant algebroids $\mathcal{E}_H$ and $\mathcal{E}_{H'}$ respectively. An automorphism of the trivial extension given by $\Phi : \xi + f \mapsto \xi + f + B(\xi)$ for $B \in H^0(\Omega^2_X(\log D))$ takes $[\cdot,\cdot]_H$ to $[\cdot,\cdot]_{H'}$ if and only if $H' - H = dB$. But by [15, Proposition 1.3.2], any global closed logarithmic form on (X, D) is exact if and only if it vanishes, hence we see that $\mathcal{E}_H \cong \mathcal{E}_{H'}$ if and only if $H' = H$. Therefore we obtain a universal family of split log Courant algebroids over $H^0(\Omega_X^{3,\mathrm{cl}}(\log D))$ by this construction.

Theorem 5.9. *Any extension of $\mathcal{T}_X(-\log D)$ by $\Omega^1_X(\log D)$ admits the structure of a log Courant algebroid on (X, D) and if $\mathcal{E}$, $\mathcal{E}'$ are two log Courant algebroid with isomorphic underlying extensions, then $\mathcal{E}'$ is isomorphic to $\mathcal{E}$ up to a twist by a closed logarithmic 3-form as defined in Definition 5.5.*

Proof. By the above theorem, we know that a log Courant algebroid on (X, D) is given by a cocycle $\{B_{ij}, H_i\}$ where H_i is a closed log 3-form and B_{ij} is a log 2-form such that $H_i - H_j = dB_{ij}$. The forgetful map from $\{B_{ij}, H_i\}$ to $\{B_{ij}\}$ induces the map in cohomology

$$\Phi : \mathbb{H}^3(F^2\Omega^\bullet_X(\log D)) \to H^1(\Omega^2_X(\log D)).$$

We can proceed exactly as in the proof of Theorem 4.10 to obtain that this map fits into the following short exact sequence

$$0 \to H^0(\Omega_X^{3,\mathrm{cl}}(\log D)) \to \mathbb{H}^3(F^2\Omega^\bullet_X(\log D)) \xrightarrow{\Phi} H^1(\Omega^2_X(\log D)) \to 0,$$

which gives the desired result. □

Definition 5.10. Let $\mathcal{E}$ be a log Courant algebroid on (X, D). We say that $\mathcal{E}$ is ***locally trivial*** if a flat connection (a splitting of $\mathcal{E}$ with zero curvature) exists in a neighbourhood of every point.

Proposition 5.11. *Locally trivial log Courant algebroids on (X, D) are classified by the subgroup*

$$H^1(\Omega_X^{2,\mathrm{cl}}(\log D)) \subset \mathbb{H}^3(F^2\Omega^\bullet_X(\log D)).$$

If D is smooth, the above inclusion is an equality, so that all log Courant algebroids on (X, D) are locally trivial.

Proof. As described in Theorem 5.7, a log Courant algebroid is classified by cohomological data $\{B_{ij}, H_{\nabla_i}\}$ where $H_{\nabla_i} - H_{\nabla_j} = dB_{ij}$. In the locally trivial case, local flat splittings may be chosen, in which case H_{∇_i} and H_{∇_j} are zero, yielding a Čech 1-cocycle B_{ij} for $\Omega_X^{2,\mathrm{cl}}(\log D)$, as required. The inclusion of complexes

$$\Omega_X^{2,\mathrm{cl}}(\log D) \hookrightarrow [\Omega^2_X(\log D) \to \Omega_X^{3,\mathrm{cl}}(\log D)],$$

induces an inclusion on first cohomology groups, and since the Poincaré lemma for logarithmic forms holds in degree ≥ 2 when D is smooth, we obtain surjectivity in that case. □

As in the case of log Picard algebroids, a log Courant algebroid on (X, D) restricts to an usual exact Courant algebroid on $U = X \setminus D$, and so determines a class in $H^1(\Omega_U^{2,\mathrm{cl}})$, which itself defines a class in $H^3(U, \mathbb{C})$. We now give the analogue of Theorem 4.13 in the context of locally trivial log Courant algebroids.

Theorem 5.12. *The restriction of a locally trivial log Courant algebroid on (X, D) to the complement $U = X \setminus D$ defines a cohomology class in the subgroup*

$$F^2H^3(U, \mathbb{C}) \cap W_4H^3(U, \mathbb{C}) \subset H^3(U, \mathbb{C}),$$

and this defines a bijection on isomorphism classes.

Proof. The proof of this theorem is similar to the proof of Theorem 4.13 with appropriate modifications. Our goal here is to prove the following

$$H^1(\Omega_X^{2,\mathrm{cl}}(\log D)) \cong F^2H^3(U, \mathbb{C}) \cap W_4H^3(U, \mathbb{C})$$

The sheaf $\Omega_X^{2,\mathrm{cl}}(\log D)$ can be resolved by the following complex

$$\Omega_X^2(\log D) \to \Omega_X^1 \wedge \Omega_X^2(\log D) \to \Omega_X^2 \wedge \Omega_X^2(\log D) \to \dots.$$

This complex is precisely $F^2W_2\Omega_X^\bullet(\log D)$ and so we have the isomorphism

$$H^1(\Omega_X^{2,\mathrm{cl}}(\log D)) \cong \mathbb{H}^3(F^2W_2\Omega_X^\bullet(\log D)). \tag{5.7}$$

There is a commutative diagram of complexes

$$\begin{array}{ccc} F^2W_2\Omega_X^\bullet(\log D) & \longrightarrow & W_2\Omega_X^\bullet(\log D) \\ \downarrow & & \downarrow \\ F^2\Omega_X^\bullet(\log D) & \longrightarrow & \Omega_X^\bullet(\log D) \end{array}$$

Using exactly the same methodology as outlined in the proof of Theorem 4.13, we find that if we take the third hypercohomology $\mathbb{H}^3$ of all the complexes in the above diagram, then all of the induced maps are injective. This proves that the map

$$\mathbb{H}^3(F^2W_2\Omega_X^\bullet(\log D)) \to F^2H^3(U, \mathbb{C}) \cap W_4H^3(U, \mathbb{C}) \tag{5.8}$$

is injective. To show surjectivity, we see that the group $\mathbb{H}^3(F^2W_2\Omega_X^\bullet(\log D))$ injects into $\mathbb{H}^3(W_2\Omega_X^\bullet(\log D))$ with cokernel $\mathbb{H}^3(\mathcal{O}_X \to \Omega_X^1(\log D))$,

$$0 \to \mathbb{H}^3(F^2W_2\Omega_X^\bullet(\log D)) \to \mathbb{H}^3(W_2\Omega_X^\bullet(\log D)) \to \mathbb{H}^3(\mathcal{O}_X \to \Omega_X^1(\log D)) \to 0, \tag{5.9}$$

and it injects into $\mathbb{H}^3(F^2\Omega_X^\bullet(\log D))$ with cokernel $H^0(D^{(3)}, \mathbb{C})$,

$$0 \to \mathbb{H}^3(F^2W_2\Omega_X^\bullet(\log D)) \to \mathbb{H}^3(F_2\Omega_X^\bullet(\log D)) \to H^0(D^{(3)}, \mathbb{C}) \to 0 \tag{5.10}$$

Suppose that there exists a class in $F^2H^3(U, \mathbb{C}) \cap W_4H^3(U, \mathbb{C})$ such that it is not in the image of $\mathbb{H}^3(F^2W_2\Omega_X^\bullet(\log D))$. Then, this class must either map to $\mathbb{H}^3(\mathcal{O}_X \to \Omega_X^1(\log D))$ or map to $H^0(D^{(3)}, \mathbb{C})$. In the former case, this contradicts that this class lies in $F^2H^3(U, \mathbb{C})$ and in the latter case, this contradicts that this class lies in $W_4H^3(U, \mathbb{C})$. Therefore, we obtain surjectivity of the map (5.8). Thus, we have $\mathbb{H}^3(F^2W_2\Omega_X^\bullet(\log D))$ is isomorphic to the group $F^2H^3(U, \mathbb{C}) \cap W_4H^3(U, \mathbb{C})$ and using the isomorphism given in (5.7), we obtain the desired isomorphism

$$H^1(\Omega_X^{2,\mathrm{cl}}(\log D)) \cong F^2H^3(U,\mathbb{C}) \cap W_4H^3(U,\mathbb{C}).$$

□

5.1.3 Residues and relationship to Picard algebroids on D

We now introduce the topological aspects of the theory of locally trivial log Courant algebroids. For this subsection, we restrict our attention to the case where D is a smooth divisor in X. First, we have the residue short exact sequence

$$0 \to \Omega_X^2 \to \Omega_X^2(\log D) \overset{\mathrm{res}}{\to} \Omega_D^1 \to 0.$$

If we restrict ourselves to closed differential forms, we have the following analogue of the residue short exact sequence

$$0 \to \Omega_X^{2,\mathrm{cl}} \to \Omega_X^{2,\mathrm{cl}}(\log D) \overset{\mathrm{res}}{\to} \Omega_D^{1,\mathrm{cl}} \to 0,$$

with the following long exact sequence in cohomology

$$\cdots \to H^1(\Omega_X^{2,\mathrm{cl}}) \to H^1(\Omega_X^{2,\mathrm{cl}}(\log D)) \overset{\mathrm{res}_*}{\to} H^1(\Omega_D^{1,\mathrm{cl}}) \to \dots \tag{5.11}$$

We now describe what this long exact sequence means geometrically. The map from $H^1(\Omega_X^{2,\mathrm{cl}})$ to $H^1(\Omega_X^{2,\mathrm{cl}}(\log D))$ corresponds to locally trivial log Courant algebroids on (X, D) which come from exact Courant algebroids on X. Let $\widetilde{\mathcal{E}}$ be a locally trivial exact Courant algebroid on X

$$0 \to \Omega_X^1 \overset{\widetilde{\pi}^*}{\to} \widetilde{\mathcal{E}} \overset{\widetilde{\pi}}{\to} \mathcal{T}_X \to 0.$$

Consider the pullback square of the map $\widetilde{\pi} : \mathcal{E} \to \mathcal{T}_X$ and the inclusion map $i : \mathcal{T}_X(-\log D) \hookrightarrow \mathcal{T}_X$, we have the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega_X^1 & \xrightarrow{\widetilde{\pi}^*} & \widetilde{\mathcal{E}} & \xrightarrow{\widetilde{\pi}} & \mathcal{T}_X & \longrightarrow & 0\,. \\
 & & \| & & \uparrow & & \uparrow{\scriptstyle i} & & \\
0 & \longrightarrow & \Omega_X^1 & \xrightarrow{\widetilde{\pi}^*} & \overline{\mathcal{E}} & \longrightarrow & \mathcal{T}_X(-\log D) & \longrightarrow & 0
\end{array}$$

Now, if we take the pushout square of the inclusion map $\varphi : \Omega_X^1 \to \Omega_X^1(\log D)$ and the map $\widetilde{\pi}^* : \Omega_X^1 \to \overline{\mathcal{E}}$, we obtain a log Courant algebroid $\mathcal{E}$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega_X^1 & \xrightarrow{\widetilde{\pi}^*} & \overline{\mathcal{E}} & \longrightarrow & \mathcal{T}_X(-\log D) & \longrightarrow & 0\,. \\
 & & \downarrow{\scriptstyle \varphi} & & \downarrow & & \| & & \\
0 & \longrightarrow & \Omega_X^1(\log D) & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{T}_X(-\log D) & \longrightarrow & 0
\end{array}$$

Definition 5.13. Let $\widetilde{\mathcal{E}}$ be an exact Courant algebroid on X, we define the ***induced log Courant algebroid*** to be the log Courant algebroid $\mathcal{E}$ constructed from $\widetilde{\mathcal{E}}$ as above.

The residue map from $H^1(\Omega_X^{2,\mathrm{cl}}(\log D))$ to $H^1(\Omega_D^{1,\mathrm{cl}})$ in (5.11) can be understood in the following way. Let $\mathcal{E}_D$ be the restriction of an exact locally trivial log Courant algebroid $\mathcal{E}$ on D

$$0 \to \Omega^1_X(\log D)|_D \to \mathcal{E}|_D \to \mathcal{T}_X(-\log D)|_D \to 0. \tag{5.12}$$

The sheaf $\mathcal{T}_X(-\log D)|_D$ is the Atiyah algebroid of the normal bundle N_D of D, so in particular, it is a Picard algebroid on D

$$0 \to \mathcal{O}_D \to \mathcal{T}_X(-\log D)|_D \xrightarrow{\sigma} \mathcal{T}_D \to 0. \tag{5.13}$$

The sheaf $\Omega^1_X(\log D)|_D$ lies in the restriction of the residue short exact sequence on D

$$0 \to \Omega^1_D \to \Omega^1_X(\log D)|_D \xrightarrow{\mathrm{res}|_D} \mathcal{O}_D \to 0. \tag{5.14}$$

If we combine the sequences (5.12), (5.13), (5.14) and consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^1_X(\log D)|_D & \longrightarrow & \mathcal{E}|_D & \longrightarrow & \mathcal{T}_X(-\log D)|_D & \longrightarrow & 0\ . \\ & & \downarrow{\scriptstyle \mathrm{res}|_D} & & \downarrow & & \downarrow{\scriptstyle \sigma} & & \\ 0 & \longrightarrow & \mathcal{O}_D & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{T}_D & \longrightarrow & 0 \end{array}$$

There is a Lie bracket $[\cdot,\cdot]$ on the sections of $\mathcal{F}$ which is obtained by descending the bracket on $\mathcal{E}|_D$ and hence, the sheaf $\mathcal{F}$ admits the structure of a Picard algebroid on D.

Definition 5.14. Let $\mathcal{E}$ be a locally trivial log Courant algebroid on (X, D), the ***residue*** of $\mathcal{E}$ along D is the Picard algebroid $\mathcal{F}$ constructed above.

We note that if we take the kernels of all of the vertical maps in the commutative diagram above, we obtain the dual extension $\check{\mathcal{F}}$ of $\mathcal{F}$

$$0 \to \Omega^1_D \to \check{\mathcal{F}} \to \mathcal{O}_D \to 0.$$

This extension is a class in the cohomology group $H^1(\mathcal{T}_D)$ which classifies the deformations of the complex structure on D.

Remark 5.15. For the case of log Picard algebroids, we had to work in the analytic topology in order to obtain interesting examples of log Picard algebroids on (X, D) which do not arise as pullbacks of Picard algebroids on X. In contrast, for log Courant algebroids, we do not need to make this distinction.

5.1.4 Examples

We will now present several examples of locally trivial log Courant algebroids on (X, D).

Example 5.16. Let $(\mathbb{P}^3, Q)$ such that Q is the quadric hypersurface in $\mathbb{P}^3$. We apply the sequence (5.11) to this pair

$$\cdots \to H^1(\Omega^{2,\mathrm{cl}}_{\mathbb{P}^3}) \to H^1(\Omega^{2,\mathrm{cl}}_{\mathbb{P}^3}(\log Q)) \to H^1(\Omega^{1,\mathrm{cl}}_Q) \to H^2(\Omega^{2,\mathrm{cl}}_{\mathbb{P}^3}) \to \dots$$

Using the result of the previous lemma and standard facts about the Hodge numbers of $\mathbb{P}^3$ and Q, we obtain $H^1(\Omega^{2,\mathrm{cl}}_{\mathbb{P}^3}) = 0$, $H^1(\Omega^{1,\mathrm{cl}}_Q) = \mathbb{C}^2$ and $H^2(\Omega^{2,\mathrm{cl}}_{\mathbb{P}^3}) = \mathbb{C}$. Thus, the group $H^1(\Omega^{2,\mathrm{cl}}_{\mathbb{P}^3}(\log Q))$ must be nonzero by the above sequence and we can conclude that there are non-trivial locally trivial log Courant algebroids on this pair.

In the above example, all of the locally trivial log Courant algebroids have non-trivial residues on the divisor. The next example will be a situation where this is not the case and certain classes of the log Courant algebroids will be obtained from exact Courant algebroids.

Example 5.17. Let (X, D) such that X is the cubic hypersurface in $\mathbb{P}^4$ and $D \in |-K_X|$ a K3 surface. If we apply the sequence (5.11), we have

$$\cdots \to H^0(\Omega_D^{1,\mathrm{cl}}) \to H^1(\Omega_X^{2,\mathrm{cl}}) \to H^1(\Omega_X^{2,\mathrm{cl}}(\log D)) \to H^1(\Omega_D^{1,\mathrm{cl}}) \to H^2(\Omega_X^{2,\mathrm{cl}}) \to \dots$$

The divisor D is a K3 surface and so the Hodge numbers are $h_D^{1,0} = h_D^{0,1} = 0$, $h_D^{2,0} = h_D^{0,2} = 1$ and $h_D^{1,1} = 20$. Using Lemma 2.2, we obtain $H^0(\Omega_D^{1,\mathrm{cl}}) = 0$ and $H^1(\Omega_D^{1,\mathrm{cl}}) = \mathbb{C}^{21}$. We can re-write the above sequence

$$0 \to H^1(\Omega_X^{2,\mathrm{cl}}) \to H^1(\Omega_X^{2,\mathrm{cl}}(\log D)) \to \mathbb{C}^{21} \to H^2(\Omega_X^{2,\mathrm{cl}}) \to \dots$$

To determine the groups $H^1(\Omega_X^{2,\mathrm{cl}})$ and $H^1(\Omega_X^{2,\mathrm{cl}})$, we need to find the Hodge numbers $h_X^{3,0}$, $h_X^{2,1}$, $h_X^{3,1}$, $h_X^{2,2}$ of X and then appeal to Lemma 2.2. From the Lefschetz theorem, we have $H^2(X, \mathbb{C}) \cong H^2(\mathbb{P}^4, \mathbb{C}) = \mathbb{C}$ and so $H^4(X, \mathbb{C}) = \mathbb{C}$. This implies that $h_X^{3,1} = 0$ and $h_X^{2,2} = 1$.

We now determine the Hodge numbers $h_X^{3,0}$ and $h_X^{2,1}$. From the adjunction formula, we have $\Omega_X^3 = \mathcal{O}_X(-2)$ and so $h_X^{3,0} = 0$. To determine $h_X^{2,1}$, it suffices to determine $h^{1,2}$ due to the symmetry of the Hodge diamond. From the conormal sequence

$$0 \to \mathcal{O}_X(-3) \to \Omega^1_{\mathbb{P}^4}|_X \to \Omega_X^1 \to 0,$$

we have the long exact sequence in cohomology

$$\cdots \to H^2(\mathcal{O}_X(-3)) \to H^2(\Omega^1_{\mathbb{P}^4}|_X) \to H^2(\Omega^1_X) \to H^3(\mathcal{O}_X(-3)) \to H^3(\Omega^1_{\mathbb{P}^4}|_X) \to \dots$$

By Serre duality, we have $H^2(\mathcal{O}_X(-3)) \cong H^1(\mathcal{O}_X(1)) = 0$ and $H^3(\mathcal{O}_X(-3)) \cong H^0(\mathcal{O}_X(1)) = \mathbb{C}^5$ and so we can re-write the above sequence

$$0 \to H^2(\Omega^1_{\mathbb{P}^4}|_X) \to H^2(\Omega^1_X) \to \mathbb{C}^5 \to H^3(\Omega^1_{\mathbb{P}^4}|_X) \to \dots \tag{5.15}$$

To find the cohomology groups $H^2(\Omega^1_{\mathbb{P}^4}|_X)$ and $H^3(\Omega^1_{\mathbb{P}^4}|_X)$, we can use the Euler sequence

$$0 \to \Omega^1_{\mathbb{P}^4}|_X \to \mathcal{O}_X(-1)^{\oplus 5} \to \mathcal{O}_X \to 0.$$

By the Kodaira vanishing theorem, $H^i(\mathcal{O}_X) = 0$ for $i > 0$. Therefore, the long exact sequence in cohomology for the Euler sequence gives us

$$H^j(\Omega^1_{\mathbb{P}^4}|_X) = H^j(\mathcal{O}_X(-1))^{\oplus 5}.$$

for $j = 2, 3$. Using the Kodaira vanishing theorem again, we get $H^2(\mathcal{O}_X(-1)) = 0$ and moreover,

$H^3(\mathcal{O}_X(-1)) \cong H^0(\mathcal{O}_X(-1)) = 0$. This gives us

$$H^j(\Omega^1_{\mathbb{P}^4}|_X) = 0,$$

for $j = 2, 3$ and if we return now to the sequence (5.15), we have $H^2(\Omega^1_X) \cong \mathbb{C}^5$. Since we have shown $h^{3,0}_X = 0$, $h^{2,1} = 5$, by Lemma 2.2, we have

$$0 \to \mathbb{C}^5 \to H^1(\Omega^{2,\mathrm{cl}}_X(\log D)) \to \mathbb{C}^{21} \to \mathbb{C} \to \dots$$

Thus, we have locally trivial log Courant algebroids on this pair which come from exact Courant algebroids on X and also ones which give non-trivial residues on D.

5.1.5 Log Dirac Geometry

In this subsection, we will work with the log Courant algebroid given by the trivial extension $\mathcal{T}_X(-\log D) \oplus \Omega^1_X(\log D)$ with log Courant bracket given in (5.5)

$$[\xi_1 + \tau_1, \xi_2 + \tau_2] = [\xi_1, \xi_2] + \mathcal{L}_{\xi_1}\tau_2 - i_{\xi_2} d\tau_1,$$

with $\xi_1, \xi_2 \in \mathcal{T}_X(-\log D)$, $\tau_1, \tau_2 \in \Omega^1_X(\log D)$ and the closed 3-form H taken to be trivial. The bilinear bracket $\langle\cdot,\cdot\rangle$ is as given in (5.6):

$$\langle \xi_1 + \tau_1, \xi_2 + \tau_2 \rangle = \frac{1}{2}(\tau_1(\xi_2) + \tau_2(\xi_1)).$$

We will now study a distinguished class of subsheaves of $\mathcal{T}_X(-\log D) \oplus \Omega^1_X(\log D)$ which yield interesting geometric structures.

Definition 5.18. A ***log Dirac structure*** $\mathcal{L} \subset \mathcal{T}_X(-\log D) \oplus \Omega^1_X(\log D)$ is a locally free $\mathcal{O}_X$-module such that:

- $\mathcal{L}$ is Lagrangian: $\mathcal{L} = \mathcal{L}^\perp$ with respect to the pairing $\langle\cdot,\cdot\rangle$
- $[\mathcal{L}, \mathcal{L}] \subset \mathcal{L}$: the sections of $\mathcal{L}$ are involutive under the log Courant bracket

Proposition 5.19. *A log Dirac structure $\mathcal{L}$ admits the structure of a Lie algebroid.*

Proof. The anchor map is obtained from first restricting the projection map

$$\pi : \mathcal{T}_X(-\log D) \oplus \Omega^1_X(\log D) \to \mathcal{T}_X(-\log D),$$

to the subsheaf $\mathcal{L}$ followed by the inclusion $\mathcal{T}_X(-\log D) \hookrightarrow \mathcal{T}_X$. The Lie bracket is obtained from restricting the log Courant bracket on $\mathcal{T}_X(-\log D) \oplus \Omega^1_X(\log D)$ to $\mathcal{L}$. □

Let ω be a logarithmic 2-form which is non-degenerate, $\omega \in H^0(\Omega^2_X(\log D))$. The form ω gives an isomorphism of $\mathcal{O}_X$-modules

$$\omega^\# : \mathcal{T}_X(\log D) \xrightarrow{\cong} \Omega^1_X(\log D),$$

defined by $\xi \mapsto i_\omega \xi$. Consider the graph of ω, $\mathrm{Gr}(\omega)$, which is given by

$$\mathrm{Gr}(\omega) = \{\xi + \omega^\#(\xi) : \xi \in \mathcal{T}_X(-\log D)\}.$$

Proposition 5.20. *The non-degenerate logarithmic 2-form ω is closed if and only if $Gr(\omega)$ is a log Dirac structure. We call such a geometric structure a **log symplectic structure** on (X, D).*

Proof. $\mathrm{Gr}(\omega)$ is clearly Lagrangian and if we take $\xi_1 + \omega^{\#}(\xi_1), \xi_2 + \omega^{\#}(\xi_2) \in \mathrm{Gr}(\omega)$, we have

$$\begin{aligned}[\xi_1 + \omega^{\#}(\xi_1), \xi_2 + \omega^{\#}(\xi_2)] &= [\xi_1 + i_{\xi_1}\omega, \xi_2 + i_{\xi_2}\omega] \\ &= [\xi_1, \xi_2] + \mathcal{L}_{\xi_1} i_{\xi_2}\omega - i_{\xi_2} d i_{\xi_1}\omega \\ &= [\xi_1, \xi_2] + (\mathcal{L}_{\xi_1} i_{\xi_2} - i_{\xi_2}\mathcal{L}_{\xi_1})\omega + (i_{\xi_2}\mathcal{L}_{\xi_1} - i_{\xi_2} d i_{\xi_1})\omega \\ &= [\xi_1, \xi_2] + i_{[\xi_1, \xi_2]}\omega + i_{\xi_2} i_{\xi_1} d\omega.\end{aligned}$$

Hence, we see that involutivity of the log Courant bracket holds if and only if $i_{\xi_2} i_{\xi_1} d\omega = 0$, in other words, ω must be closed. □

Example 5.21. Let X be a Calabi-Yau surface with a reduced anticanonical divisor $D \subset X$. Since $\Omega^3_X(\log D) = 0$, every logarithmic 2-form is necessarily closed. Hence, by the Calabi-Yau condition, $\Omega^2_X(\log D) \cong \mathcal{O}_X(D)$, so every log symplectic structure on (X, D) is given by a non-vanishing section of the line bundle $\mathcal{O}_X(D)$. We refer the reader to [33] for an algebraic study of the theory of log symplectic structures.

5.2 Meromorphic Gerbes

Definition 5.22. A ***meromorphic gerbe*** $\mathcal{G}(\mathcal{M}, \theta)$ on (X, D) with respect to a chosen open covering $\{U_i\}$ of X consists of the following data:

- A family of meromorphic line bundles $\mathcal{M}_{ij} \to U_{ij}$ such that U_{ii} is the trivial meromorphic line bundle, $\mathcal{M}_{ii} \cong \mathcal{O}_{U_i}(*D)$
- For each ordered pair, (i, j), we have $\mathcal{M}_{ij} \cong \mathcal{M}_{ji}^{-1}$
- For i, j, k, a non-vanishing section θ_{ijk} of $\mathcal{M}_{ij} \otimes \mathcal{M}_{jk} \otimes \mathcal{M}_{ki}$ over U_{ijk} satisfying the cocycle condition
$$\theta_{ijkl} \otimes \theta_{jkli} \otimes \theta_{klij} \otimes \theta_{lijk} = 1.$$

We usually denote $\mathcal{G}(\mathcal{M}, \theta)$ simply by $\mathcal{G}$.

Definition 5.23. Fix an open covering $\{U_i\}$ of X. Let $\mathcal{G}$ and $\mathcal{G}'$ be two gerbes on (X, D). We say that $\mathcal{G}$ and $\mathcal{G}'$ are ***gauge-equivalent*** if the following two properties hold:

- There exists $h_{ij} : U_{ij} \to \mathcal{O}^{\times}_{U_{ij}}(*D)$ such that $\theta'_{ijk} = \theta_{ijk} + h_{ij} + h_{jk} + h_{ki}$
- There exists meromorphic line bundles $E_i \to U_i$ such that $\mathcal{M}_{ij} = E_i \otimes \mathcal{M}'_{ij} \otimes E_j^{-1}$. Note that as $\mathcal{M}_{ij} \otimes \mathcal{M}_{jk} \otimes \mathcal{M}_{ki}$ is canonically isomorphic to $\mathcal{M}'_{ij} \otimes \mathcal{M}'_{jk} \otimes \mathcal{M}'_{ki}$, the section θ_{ijk} is a non-vanishing section of $\mathcal{M}'_{ij} \otimes \mathcal{M}'_{jk} \otimes \mathcal{M}'_{ki}$ which satisfies the coherence condition.

We now define the notion of a logarithmic connective structure on meromorphic gerbes.

Definition 5.24. Let $\mathcal{G}$ be a meromorphic gerbe on (X, D). A ***logarithmic connective structure*** (∇, B) on $\mathcal{G}$ consists of the following:

- The ***0-connection*** ∇ which is a family of logarithmic connections ∇_{ij} on the meromorphic line bundles $\mathcal{M}_{ij}$ such that θ_{ijk} is flat with respect to the tensor product connection on $\mathcal{M}_{ij} \otimes \mathcal{M}_{jk} \otimes \mathcal{M}_{ki}$

- The ***1-connection*** B which is a collection of logarithmic 2-forms B_i on U_i such that $B_j - B_i = F_{\nabla_{ij}}$ where $F_{\nabla_{ij}}$ denotes the curvature of the connection ∇_{ij}.

The ***curvature*** of the connective structure is the global closed 3-form H defined as $H_i = dB_i$.

Remark 5.25. If we ignore the 1-connection part in the above definition, we obtain the notion of a ***meromorphic gerbes with 0-connection*** on (X, D).

The notion of gauge-equivalence also extends to meromorphic gerbes with logarithmic connective structure.

Definition 5.26. Let $\mathcal{G}$ and $\mathcal{G}'$ be two meromorphic gerbes with logarithmic connective structure (∇, B) and (∇, B') respectively. We say that these two objects are ***gauge-equivalent*** if the following properties hold:

- $\mathcal{G}$ and $\mathcal{G}'$ are gauge-equivalent gerbes as given in Definition 5.23

- There exists meromorphic line bundles E_i, E_j with logarithmic connections ∇_i, ∇_j such that the logarithmic connections ∇'_{ij} on $\mathcal{M}'_{ij}$ are given by $\nabla_i + \nabla_{ij} - \nabla_j + \mathrm{dlog}(h_{ij})$ where $h_{ij} : U_{ij} \to \mathcal{O}^{\times}_{U_{ij}}(*D)$

- The logarithmic 2-forms B'_i are given by $B_i + F_{\nabla_i}$.

Theorem 5.27. *The **meromorphic Brauer group** $\mathrm{Br}(X, D)$ of gauge-equivalence classes of meromorphic gerbes on (X, D) is given by the cohomology group*

$$H^2(\mathcal{O}^{\times}_X(*D)).$$

Proof. We choose a good open covering of X such that all the meromorphic line bundles $\mathcal{M}_{ij}$ in the datum of the gerbe is trivial. In this case, the sections θ_{ijk} are given by $g_{ijk} : U_{ijk} \to \mathcal{O}^{\times}_{U_{ijk}}(*D)$, a Čech 2-cocycle with values in the sheaf $\mathcal{O}^{\times}_X(*D)$. From Definition 5.23, two gerbes are gauge-equivalent if there exists $h_{ij} : U_{ij} \to \mathcal{O}^{\times}_{U_{ij}}(*D)$ such that $g'_{ijk} = g_{ijk} + h_{ij} + h_{jk} + h_{ki}$. This is a shift of the Čech cocyle $\{g_{ijk}\}$ by a coboundary and so we obtain the desired result. □

Theorem 5.28. *The group of equivalence classes of gauge-equivalence classes of meromorphic gerbes with connective structure on (X, D) is given by the hypercohomology group*

$$\mathbb{H}^2(\mathcal{O}^{\times}_X(*D) \xrightarrow{\mathrm{dlog}} \Omega^1_X(\log D) \xrightarrow{d} \Omega^2_X(\log D)).$$

Proof. As in the proof of the previous theorem, we again take a good open covering of X such that all the meromorphic line bundles $\mathcal{M}_{ij}$ are trivial. The family of 0-connections ∇_{ij} is given by logarithmic 1-forms A_{ij} and these 1-forms satisfy

$$A_{ij} + A_{jk} + A_{ki} = \mathrm{dlog}(g_{ijk}).$$

The logarithmic 2-forms B_i satisfy $B_j - B_i = dA_{ij}$ and so (g_{ijk}, A_{ij}, B_i) is a Čech 2-cocycle. If we use Definition 5.26, we have the following equations for two gauge-equivalent gerbes:

$$\begin{cases} g'_{ijk} &= g_{ijk} + h_{ij} + h_{jk} + h_{ki} \\ A'_{ij} &= A_{ij} + \gamma_i - \gamma_j + \mathrm{dlog}(h_{ij}) \\ B'_i &= B_i + d\gamma_i \end{cases}$$

where γ_i are the logarithmic 1-forms for the connections ∇_i on E_i. This shows that two gauge-equivalent meromorphic gerbes with connective structure correspond to a Čech coboundary and so we obtain the desired result. □

Corollary 5.29. *The group of equivalence classes of meromorphic gerbes with 0-connection on (X, D) as introduced in Remark 5.25 is given by the hypercohomology group*

$$\mathbb{H}^2(\mathcal{O}_X^\times(*D) \to \Omega_X^1(\log D).$$

We now recall the short exact sequence in (4.17)

$$0 \to \mathcal{O}_X^\times \to \mathcal{O}_X^\times(*D) \overset{\mathrm{ord}}{\to} \mathbb{Z}_D \to 0,$$

which induces the long exact sequence in cohomology

$$\cdots \to H^2(\mathcal{O}_X^\times) \to H^2(\mathcal{O}_X^\times(*D)) \overset{\mathrm{ord}_*}{\to} H^2(\mathbb{Z}_D) \to \dots. \tag{5.16}$$

We will now give a geometric description of the map

$$H^2(\mathcal{O}_X^\times(*D)) \overset{\mathrm{ord}_*}{\to} H^2(\mathbb{Z}_D). \tag{5.17}$$

Proposition 5.30. *A meromorphic lgerbe $\mathcal{G}$ determines a flat $\mathbb{Z}$-gerbe on D, $\mathrm{ord}(\mathcal{G})$, which induces the map in cohomology*

$$H^2(\mathcal{O}_X^\times(*D)) \overset{\mathrm{ord}_*}{\to} H^2(D, \mathbb{Z}).$$

Furthermore, the sheaf of sections of $\mathcal{M}$ is filtered with the filtered pieces labelled by the local system $\mathrm{ord}(\mathcal{M})$.

Proof. By Proposition 4.23, the meromorphic line bundle $\mathcal{M}_{ij}$ in the datum of the gerbe $\mathcal{G}$ determines a $\mathbb{Z}$-local system on D with respect to which it is filtered. We denote this local system by Λ_{ij}. Similarly, the meromorphic line bundles $\mathcal{M}_{jk}$ and $\mathcal{M}_{ki}$ determine $\mathbb{Z}$-local systems Λ_{jk} and Λ_{ki} on D with respect to which it is filtered.

Now, consider the class of the gerbe given by $[\theta_{ijk}]$ and its image under the order map (5.17) above which we denote by c_{ijk}. The image c_{ijk} glues together the $\mathbb{Z}$-local systems Λ_{ij}, Λ_{jk} and Λ_{ki} to give rise to a flat $\mathbb{Z}$-gerbe on D. Since this flat $\mathbb{Z}$-gerbe was constructed using the gluing map $c_{ijk} = \mathrm{ord}(g_{ij})$, we have that $\mathcal{G} \mapsto \mathrm{ord}(\mathcal{G})$ induces the map in cohomology

$$H^2(\mathcal{O}_X^\times(*D)) \overset{\mathrm{ord}_*}{\to} H^2(D, \mathbb{Z}).$$

□

Remark 5.31. As in the case of meromorphic line bundles in Section 4.2, we must work in the analytic topology rather than the Zariski topology in order to obtain interesting examples of meromorphic gerbes. If we had work in the Zariski topology, the cohomology group $H^2(D, \mathbb{Z}) = 0$ and using the sequence in (5.16), we find that all meromorphic gerbes on (X, D) arise from holomorphic gerbes on X.

5.2.1 Classification

We now apply the theory of integral weight filtrations in 2.2.2 to provide classification results for meromorphic gerbes on (X, D). Recall from (4.26) that we have the following quasi-isomorphism

$$\mathcal{O}_X^\times(*D) \sim \mathrm{Cone}(\sigma_{\leq 1}\mathbb{R}j_*\mathbb{Z}(1)_U \to \mathcal{O}_X).$$

This induces the long exact sequence in hypercohomology

$$\cdots \to \mathbb{H}^2(\sigma_{\leq 1}\mathbb{R}j_*\mathbb{Z}(1)_U) \to H^2(\mathcal{O}_X) \overset{\exp}{\to} \mathrm{Br}(X, D) \to \mathbb{H}^3(\sigma_{\leq 1}\mathbb{R}j_*\mathbb{Z}(1)_U) \to H^3(\mathcal{O}_X) \to \dots. \tag{5.18}$$

Let us suppose that the divisor D is smooth, this implies that

$$\sigma_{\leq 1}\mathbb{R}j_*\mathbb{Z}(1)_U \cong \mathbb{R}j_*\mathbb{Z}(1)_U,$$

and so we can rewrite the sequence (5.18) in the following way

$$\cdots \to H^2(U, \mathbb{Z}(1)) \to H^2(\mathcal{O}_X) \overset{\exp}{\to} \mathrm{Br}(X, D) \to H^3(U, \mathbb{Z}(1)) \to H^3(\mathcal{O}_X) \to \dots. \tag{5.19}$$

We will now study the image of the meromorphic Brauer group $\mathrm{Br}(X, D)$ in $H^3(U, \mathbb{Z}(1))$. Using the above sequence, we obtain

$$0 \to \mathrm{im}(\mathrm{Br}(X, D) \to H^3(U, \mathbb{Z}(1)) \to H^3(U, \mathbb{Z}(1)) \to H^3(\mathcal{O}_X) \to \dots$$

The map $H^3(U, \mathbb{Z}(1)) \to H^3(\mathcal{O}_X)$ factors through the composition of maps

$$H^3(U, \mathbb{Z}(1)) \to H^3(U, \mathbb{C}) \to H^3(\mathcal{O}_X),$$

where the second map comes from taking the quotient of $H^3(U, \mathbb{C})$ by the Hodge filtration $F^1H^3(U, \mathbb{C})$. Hence, we have the following commutative diagram

$$\begin{array}{ccccccc}
 & & \mathrm{im}(\mathrm{Br}(X, D) \to H^3(U, \mathbb{Z}(1)) & \longrightarrow & H^3(U, \mathbb{Z}(1)) & & \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F^1H^3(U, \mathbb{C}) & \longrightarrow & H^3(U, \mathbb{C}) & \longrightarrow & H^3(\mathcal{O}_X) \longrightarrow 0
\end{array}.$$

Thus, we find that the image of $\mathrm{Br}(X, D)$ in $H^3(U, \mathbb{Z}(1))$ is exactly the fibre product $F^1H^3(U, \mathbb{C}) \times_{H^3(U,\mathbb{C})} H^3(U, \mathbb{Z}(1))$. We can now rewrite the sequence in (5.19) to express the meromorphic Brauer group $\mathrm{Br}(X, D)$ as an extension

$$0 \to \frac{H^2(\mathcal{O}_X)}{\mathrm{im}(H^2(U, \mathbb{Z}(1)) \to H^2(\mathcal{O}_X))} \to \mathrm{Br}(X, D) \to F^1H^3(U, \mathbb{C}) \times_{H^3(U,\mathbb{C})} H^3(U, \mathbb{Z}(1)) \to 0. \tag{5.20}$$

We will now prove that under specific conditions the meromorphic Brauer group $\mathrm{Br}(X, D)$ is isomorphic to the analytic Brauer group $\mathrm{Br}(U^{an})$ of the complement U.

Theorem 5.32. *Let (X, D) such that X has vanishing Hodge numbers $h^{0,2} = 0, h^{0,3} = 0$ and D is a smooth ample divisor in X. There is an isomorphism between the meromorphic Brauer group $\mathrm{Br}(X, D)$ and the analytic Brauer group $\mathrm{Br}(U^{an})$ of the complement U:*

$$\mathrm{Br}(X, D) \cong \mathrm{Br}(U^{an}).$$

Proof. Using the assumptions $h^{0,2} = 0, h^{0,3} = 0$ combined with the sequence in (5.20) shows that

$$\mathrm{Br}(X, D) \cong H^3(U, \mathbb{Z}(1)).$$

To conclude the proof, we need to show that the analytic Brauer group $\mathrm{Br}(U^{an})$ is also isomorphic to $H^3(U, \mathbb{Z}(1))$. Consider the long exact sequence arising from the exponential sheaf sequence on U:

$$\cdots \to H^2(\mathcal{O}_U) \to \mathrm{Br}(U^{an}) \to H^3(U, \mathbb{Z}(1)) \to H^3(\mathcal{O}_U) \to \ldots$$

Since U is the complement of a smooth ample divisor in X, U must be an affine variety. Therefore, the higher cohomologies of coherent sheaves necessarily vanish which yields:

$$\mathrm{Br}(U^{an}) \cong H^3(U, \mathbb{Z}(1)).$$

□

5.2.2 Examples

We will now provide examples of pairs (X, D) which admit non-trivial meromorphic Brauer groups $\mathrm{Br}(X, D)$.

Example 5.33. Consider $(\mathbb{P}^3, Q)$ where Q is the smooth quadric in $\mathbb{P}^3$ obtained by taking a global non-vanishing section of the line bundle $\mathcal{O}_{\mathbb{P}^3}(2)$. Using Theorem 5.32, we know that

$$\mathrm{Br}(\mathbb{P}^3, Q) \cong H^3(U, \mathbb{Z}(1)),$$

where $U := \mathbb{P}^3 \setminus Q$ is the complement. From the integral Gysin sequence, we have

$$\cdots \to H^3(\mathbb{P}^3, \mathbb{Z}(1)) \to H^3(U, \mathbb{Z}(1)) \to H^2(Q, \mathbb{Z}(1)) \to H^4(\mathbb{P}^3, \mathbb{Z}(1)) \to H^4(U, \mathbb{Z}(1)) \to \ldots$$

Since U is an affine variety, we have $H^4(U, \mathbb{Z}(1)) = 0$. The quadric $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ and so $H^2(Q, \mathbb{Z}(1)) = \mathbb{Z}^2$. Moreover, the integral cohomology of $\mathbb{P}^3$ is concentrated in even degrees and so we can rewrite the sequence as

$$0 \to H^3(U, \mathbb{Z}(1)) \to \mathbb{Z}^2 \to \mathbb{Z} \to 0.$$

Thus, $H^3(U, \mathbb{Z}(1)) = \mathbb{Z}$ and hence $\mathrm{Br}(\mathbb{P}^3, Q) = \mathbb{Z}$. The projective 3-space $\mathbb{P}^3$ has a trivial analytic Brauer group and so from (5.16), this one-dimensional family of meromorphic gerbes on $(\mathbb{P}^3, Q)$ must determine non-trivial flat $\mathbb{Z}$-gerbes on Q.

In the previous example, there are no holomorphic gerbes on X and all meromorphic gerbes on (X, D) give rise to flat $\mathbb{Z}$-gerbes on D. The next example will be an example where this is not the case.

Example 5.34. Let (X, D) such that X is the cubic hypersurface in $\mathbb{P}^4$ and $D \in |-K_X|$ a K3 surface as in Example 5.17. We have the exact sequence from (5.16):

$$\cdots \to H^1(D,\mathbb{Z}) \to H^2(\mathcal{O}_X^\times) \to \mathrm{Br}(X,D) \to H^2(D,\mathbb{Z}) \to H^3(\mathcal{O}_X^\times) \to H^3(\mathcal{O}_X^\times(*D)) \to \dots \tag{5.21}$$

Since D is a K3 surface, we have $H^1(D,\mathbb{Z}) = 0$. To prove the vanishing of $H^3(\mathcal{O}_X^\times(*D))$, we use the following long exact sequence associated to the quasi-isomorphism in (4.26)

$$\cdots \to H^3(\mathcal{O}_X) \to H^3(\mathcal{O}_X^\times(*D)) \to H^4(U,\mathbb{Z}(1)) \to H^4(\mathcal{O}_X) \to \dots.$$

The Hodge numbers $h^{k,0}$ of X are all zero for all $k \geq 1$ and moreover, since U is an affine variety, we have $H^4(U,\mathbb{Z}(1)) = 0$. Hence, we obtain the vanishing of $H^3(\mathcal{O}_X^\times(*D))$.

Using the exponential sheaf sequence on X and the vanishing of the Hodge numbers $h^{k,0}$ of X again, we find that $H^2(\mathcal{O}_X^\times) \cong H^3(X,\mathbb{Z}) = \mathbb{Z}^{10}$ and $H^3(\mathcal{O}_X^\times) \cong H^4(X,\mathbb{Z}) = \mathbb{Z}$. Furthermore, we can again use the fact that D is a K3 surface and so $H^2(D,\mathbb{Z}) = \mathbb{Z}^{22}$. Finally, if we return to the sequence in (5.21), we have

$$0 \to \mathbb{Z}^{10} \to \mathrm{Br}(X,D) \to \mathbb{Z}^{22} \to \mathbb{Z} \to 0.$$

Thus, the meromorphic Brauer group $\mathrm{Br}(X,D) = \mathbb{Z}^{31}$. There are 10-dimensions which arise as extensions of holomorphic gerbes on X and 21-dimensions which are in one-to-one correspondence with flat $\mathbb{Z}$-gerbes on D.

5.2.3 Geometric Pre-quantization

In this section, we will discuss the geometric pre-quantization of log Courant algebroids. First, we describe how to canonically associate an log Courant algebroid to a meromorphic gerbe based on ideas given in [21, Theorem 1.10]. Fix an open covering $\{U_i\}$ of X and consider a meromorphic gerbe with 0-connection $(\mathcal{G},\nabla)$ on (X, D) as described in Remark 5.25. Let ∇_{ij} be the logarithmic connection and $F_{\nabla_{ij}}$ be its curvature. This gives a closed logarithmic 2-form on U_{ij}. Consider the trivial extensions $\mathcal{E}_i = \mathcal{T}_{U_i}(-\log D) \oplus \Omega^1_{U_i}(\log D)$ equipped with log Courant bracket:

$$[\xi_1 + \tau_1, \xi_2 + \tau_2] := [\xi_1, \xi_2] + \mathcal{L}_{\xi_1}\tau_2 + i_{\xi_2} d\tau_1$$

and the bilinear bracket:

$$\langle \xi_1 + \tau_1, \xi_2 + \tau_2 \rangle = \frac{1}{2}(\tau_1(\xi_2) + \tau_2(\xi_1)).$$

We can glue $\mathcal{E}_i$ and $\mathcal{E}_j$ using the map

$$\varphi_{ij} : \xi + \tau \mapsto \xi + \tau + F_{\nabla_{ij}}(\xi).$$

As shown in the proof of Theorem 5.7, this construction gives a global log Courant algebroid on (X, D) which we denote as $\mathcal{E}$. It remains to check that this is indeed a well-defined operation. Let $(\mathcal{G}', \nabla')$ be a meromorphic gerbe with 0-connection that is gauge-equivalent to $(\mathcal{G},\nabla)$. The 0-connection for this gerbe is given by $\nabla_i + \nabla_{ij} - \nabla_j$ where ∇_i are logarithmic connections for the auxiliary line bundles E_i

involved in the definition of gauge-equivalence of 0-connections. We can then use the closed logarithmic 2-form $F_{\nabla_i} + F_{\nabla_{ij}} - F_{\nabla_j}$ to construct a log Courant algebroid $\widetilde{\mathcal{E}}$.

There is a map $\varphi : \mathcal{E} \to \mathcal{E}'$ defined as $\varphi|_{U_i} = \varphi_i$ where φ_i is given as

$$\varphi_i : \xi + \tau \mapsto \xi + \tau + F_{\nabla_i}(\xi).$$

The map φ is an isomorphism of log Courant algebroids as it interwines the gluing maps with isomorphisms of the log Courant structure:

$$\begin{array}{ccc} \mathcal{T}_{U_i}(-\log D) \oplus \Omega^1_{U_i}(\log D) & \xrightarrow{\varphi_{ij}} & \mathcal{T}_{U_j}(-\log D) \oplus \Omega^1_{U_j}(\log D) \\ \varphi_i \downarrow & & \downarrow \varphi_j \\ \mathcal{T}_{U_i}(-\log D) \oplus \Omega^1_{U_i}(\log D) & \xrightarrow[\varphi_i + \varphi_{ij} - \varphi_j]{} & \mathcal{T}_{U_j}(-\log D) \oplus \Omega^1_{U_j}(\log D) \end{array}$$

Thus, we obtain a well-defined log Courant algebroid $\mathcal{E}$ associated to a meromorphic gerbe with 0-connection. We denote this algebroid by $\mathcal{E}_{\mathcal{G}}$.

If we trivialize all the meromorphic line bundles $\mathcal{M}_{ij}$, then $(\mathcal{G}, \nabla)$ is given by $\{g_{ijk}\} \in C^2(\mathcal{O}^\times_X(*D))$ and $\{A_{ij}\} \in C^1(\Omega^1_X(\log D))$. The gluing logarithmic 2-forms in the above construction are given by dA_{ij} and so $[\mathcal{E}_{\mathcal{G}}]$ is given by $d[(\mathcal{G}, \nabla)]$. We can summarize this entire construction by the following theorem.

Theorem 5.35. *Let $(\mathcal{G}, \nabla)$ be a meromorphic gerbe with 0-connection on (X, D). There is a canonically associated log Courant algebroid $\mathcal{E}_{\mathcal{G}}$ with isomorphism class given by the map in hypercohomology*

$$\mathbb{H}^2(\mathcal{O}^\times_X(*D) \to \Omega^1_X(\log D)) \xrightarrow{d} H^1(\Omega^{2,\mathrm{cl}}_X(\log D)).$$

We will now address the pre-quantization problem for log Courant algebroids on (X, D). The pre-quantization problem asks the following: given a log Courant algebroid $\mathcal{E}$ on (X, D), under what conditions can $\mathcal{E}$ be pre-quantized, i.e., there exist a meromorphic gerbe with 0-connection $(\mathcal{G}, \nabla)$ such that $\mathcal{E} = \mathcal{E}_{\mathcal{G}}$.

Theorem 5.36. *Let D be a smooth divisor in X and $\mathcal{E}$ be a log Courant algebroid on (X, D). There exists a pre-quantization for $\mathcal{E}$ if and only if the class of $\mathcal{E}$, $[\mathcal{E}] \in H^1(\Omega^{2,\mathrm{cl}}_X(\log D)) \subset H^3(U, \mathbb{C})$, lies in the image of $H^3(U, \mathbb{Z}(1)) \to H^3(U, \mathbb{C})$.*

Proof. From the above theorem, we know there exists a pre-quantization for $\mathcal{E}$ if and only if the class of $[\mathcal{E}]$ is in the image of the map

$$\mathbb{H}^2(\mathcal{O}^\times_X(*D) \to \Omega^1_X(\log D)) \xrightarrow{d} H^1(\Omega^{2,\mathrm{cl}}_X(\log D)).$$

We shall show that $[\mathcal{E}]$ lies in the image of the above map if and only if $[\mathcal{E}]$ lies in the image of the map $H^3(U, \mathbb{Z}(1)) \to H^3(U, \mathbb{C})$. Recall from the proof of Theorem 4.41 that the sheaf $\Omega^{1,\mathrm{cl}}_X(\log D)$ fits into the two short exact sequences

$$0 \to Rj_*\mathbb{C}_U \to \mathcal{O}_X \to \Omega^{1,\mathrm{cl}}_X(\log D) \to 0$$
$$0 \to Rj_*\mathbb{C}^\times_U \to \mathcal{O}^\times_X(*D) \to \Omega^{1,\mathrm{cl}}_X(\log D) \to 0$$

As the Poincaré lemma holds for logarithmic forms for degrees ≥ 2, we obtain the exact sequences

$$0 \to Rj_*\mathbb{C}_U \to [\mathcal{O}_X \to \Omega^1_X(\log D)] \to \Omega^{2,\mathrm{cl}}_X(\log D)[-1] \to 0$$
$$0 \to Rj_*\mathbb{C}^\times_U \to [\mathcal{O}^\times_X(*D) \to \Omega^1_X(\log D)] \to \Omega^{2,\mathrm{cl}}_X(\log D)[-1] \to 0$$

We can map the first complex to the second complex to yield the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & Rj_*\mathbb{C}^\times_U & \longrightarrow & [\mathcal{O}^\times_X(*D) \to \Omega^1_X(\log D)] & \longrightarrow & \Omega^{2,\mathrm{cl}}_X(\log D)[-1] & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & Rj_*\mathbb{C}_U & \longrightarrow & [\mathcal{O}_X \to \Omega^1_X(\log D)] & \longrightarrow & \Omega^{2,\mathrm{cl}}_X(\log D)[-1] & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & Rj_*\mathbb{Z}(1)_U & = & Rj_*\mathbb{Z}(1)_U & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

There is the long exact sequence associated to the top short exact sequence

$$\cdots \to \mathbb{H}^2(\mathcal{O}^\times_X(*D) \to \Omega^1_X(\log D)) \xrightarrow{d} H^1(\Omega^{2,\mathrm{cl}}_X(\log D)) \to H^3(U, \mathbb{C}^\times) \to \ldots$$

Now, we can use the exact same argument as in the proof of Theorem 4.41 to conclude that $[\mathcal{E}]$ lies in the image of $\mathbb{H}^2(\mathcal{O}^\times_X(*D) \to \Omega^1_X(\log D)) \xrightarrow{d} H^1(\Omega^{2,\mathrm{cl}}_X(\log D))$ if and only if $[\mathcal{E}]$ lies in the image of $H^3(U, \mathbb{Z}(1)) \to H^3(U, \mathbb{C})$. ☐

Corollary 5.37. *Let D be a smooth divisor in X. Then log Courant algebroids on (X, D) which admit pre-quantization are classified by the following subgroup of $H^1(\Omega^{2,\mathrm{cl}}_X(\log D))$:*

$$H^{2,1}(U, \mathbb{Z}(1)) = H^1(\Omega^2_X(\log D)) \cap \mathrm{im}(H^3(U, \mathbb{Z}(1)) \to H^3(U, \mathbb{C})) \subset H^1(\Omega^{2,\mathrm{cl}}_X(\log D)).$$

Proof. The proof of this is entirely analogous with the proof of Corollary 4.42. ☐

Chapter 6

Geometry of Deligne and Deligne-Beĭlinson Cohomology

In this chapter, we begin by introducing the basic ideas of Deligne cohomology theory. We proceed to provide a geometric interpretation of a specific class of Deligne cohomology groups using the Hitchin-Chatterjee viewpoint of higher gerbes with connective structure. We then review the classical quantization result of Kostant-Weil which states that a closed 2-form on X is the curvature of some line bundle $\mathcal{L}$ on X if and only if its cohomology class is integral. Using the methodology outlined by Brylinski and Deligne in the degree 3 case, we establish the quantization of closed k-forms on X in Theorem 6.4.

In the latter half of the chapter, we introduce Deligne-Beĭlinson cohomology theory and provide a geometric interpretation of a certain class of these groups as higher meromorphic gerbes with connective structure. We conclude by proving the prequantization of closed logarithmic k-forms in Theorem 6.7.

6.1 Deligne Cohomology

In this section, we will give a brief overview of Deligne cohomology. For a more extensive treatment, we refer to the reader to [37, 18, 17].

Definition 6.1. Let X be a smooth projective variety over the complex numbers $\mathbb{C}$. The ***Deligne complex*** $\mathbb{Z}(k)_{\text{Del}}$ of X is the complex of sheaves

$$\text{Cone}(\mathbb{Z}(k) \oplus F^k\Omega_X^\bullet \to \Omega_X^\bullet)[-1],$$

where $\mathbb{Z}(k) = (2\pi i)^k\mathbb{Z}$ is the Tate twist of $\mathbb{Z}$ by $(2\pi i)^k$. The ***Deligne cohomology*** groups $H^p_{\text{Del}}(X, \mathbb{Z}(k))$ are defined as the hypercohomology groups:

$$H^p_{\text{Del}}(X, \mathbb{Z}(k)) := \mathbb{H}^p(X, \mathbb{Z}(k)_{\text{Del}}).$$

As the Deligne complex $\mathbb{Z}(k)_{\text{Del}}$ is defined as a mapping cone of complexes, we have the long exact sequence in cohomology

$$\cdots \to H^p_{\text{Del}}(X, \mathbb{Z}(k)) \to H^p(X, \mathbb{Z}(k)) \oplus F^k H^p(X, \mathbb{C}) \to H^p(X, \mathbb{C}) \to H^{p+1}_{\text{Del}}(X, \mathbb{Z}(k)) \to \dots$$

Moreover, as $\mathbb{Z}(k)_{\mathrm{Del}}$ is a mapping cone, we can rewrite it as the following complex

$$\mathbb{Z}(k)_{\mathrm{Del}} = [\mathbb{Z}(k) \to \mathcal{O}_X \to \cdots \to \Omega_X^{k-1}].$$

This interpretation of the Deligne complex is useful as we have the following quasi-isomorphism of complexes

$$\begin{array}{ccccccccc} \mathbb{Z}(k) & \longrightarrow & \mathcal{O}_X & \longrightarrow & \Omega_X^1 & \longrightarrow & \dots & \longrightarrow & \Omega_X^{k-1} \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle (2\pi i)^{1-k}} & & & & \downarrow{\scriptstyle (2\pi i)^{1-k}} \\ 0 & \longrightarrow & \mathcal{O}_X^\times & \xrightarrow{\mathrm{dlog}} & \Omega_X^1 & \longrightarrow & \dots & \longrightarrow & \Omega_X^{k-1} \end{array},$$

where $\alpha(f) = \exp((2\pi i)^{1-k} \cdot f)$. Using this quasi-isomorphism of complexes, we have the isomorphism of the Deligne cohomology group with the hypercohomology group

$$H^p_{\mathrm{Del}}(X, \mathbb{Z}(k)) \cong \mathbb{H}^{p-1}(\mathcal{O}_X^\times \to \cdots \to \Omega_X^{k-1}). \tag{6.1}$$

This isomorphism immediately gives a number of geometric interpretations for Deligne cohomology groups in low degrees:

- $H^2_{\mathrm{Del}}(X, \mathbb{Z}(1)) \cong H^1(\mathcal{O}_X^\times)$, the Picard group of holomorphic line bundles on X
- $H^2_{\mathrm{Del}}(X, \mathbb{Z}(2)) \cong \mathbb{H}^1(\mathcal{O}_X^\times \to \Omega_X^1)$, the group of isomorphism classes of holomorphic line bundles on X with connection. We refer to [4] for detailed explanation of this example and how it may be viewed as a geometric construction of a regulator map
- $H^2_{\mathrm{Del}}(X, \mathbb{Z}(k))$ for $k \geq 3$ corresponds to the group of isomorphism classes of holomorphic line bundles on X with flat connection
- $H^3_{\mathrm{Del}}(X, \mathbb{Z}(1)) \cong H^2(\mathcal{O}_X^\times)$, the analytic Brauer group of gerbes on X
- $H^3_{\mathrm{Del}}(X, \mathbb{Z}(3)) \cong \mathbb{H}^2(\mathcal{O}_X^\times \to \Omega_X^1 \to \Omega_X^2)$, the group of equivalence classes of holomorphic gerbes on X with connective structure. This example is studied extensively in Chapter 5 of [9].

It is difficult, in general, to provide a geometric interpretation to Deligne cohomology groups for higher degrees. We will now use the Čech cohomological perspective of higher gerbes with connective structures as introduced in [10, 23, 31] to give a geometric description of the Deligne cohomology group

$$H^k_{\mathrm{Del}}(X, \mathbb{Z}(k)).$$

We will first present the Čech theory of a 1-gerbe with connective structure. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X. A ***1-gerbe with connective structure*** on X is given by the 3-tuple (g, A, B) where $g = \{g_{ijk}\} \in C^2(\mathcal{U}, \mathcal{O}_X^\times)$ is the cochain of transition functions, $A = \{A_{ij}\} \in C^1(\mathcal{U}, \Omega_X^1)$ is the cochain of connection 1-forms, and $B = \{B_i\} \in C^0(\mathcal{U}, \Omega_X^2)$ is the cochain of connection 2-forms. These cochains satisfy the following conditions:

$$\begin{cases} (\delta g)_{ijkl} &= 1 \\ (\delta A)_{ijk} &= d\log(g_{ijk}) \\ (\delta B)_{ij} &= dA_{ij} \end{cases}$$

On the double intersection U_{ij}, we have $B_i - B_j = dA_{ij}$ and so $dB_i = dB_j$. This means that all of the dB_i's glue together to a global 3-form dB. We denote this global 3-form by F and we call this the ***curvature*** of the gerbe.

Given two 1-gerbes with connective structure (g, A, B) and (g', A', B'), we say that these two gerbes are ***gauge equivalent*** if there exists a pair (h, ϕ) such that $h = \{h_{ij}\} \in C^1(\mathcal{U}, \mathcal{O}_X^\times)$ and $\phi = \{\phi_i\} \in C^0(\mathcal{U}, \Omega_X^1)$ which satisfies the following conditions

$$\begin{cases} g'_{ijk} - g_{ijk} &= (\delta f)_{ij} \\ A'_{ij} - A_{ij} &= d\log h_{ij} + (\delta\phi)_{ij}. \end{cases}$$

Gauge equivalences correspond to a Čech coboundary and so 1-gerbes with connective structure are classifed up to gauge equivalence by the hypercohomology group

$$\mathbb{H}^2(\mathcal{O}_X^\times \to \Omega_X^1 \to \Omega_X^2).$$

Recall from (6.1) that we have the isomorphism

$$\mathbb{H}^2(\mathcal{O}_X^\times \to \Omega_X^1 \to \Omega_X^2) \cong H^3_{\mathrm{Del}}(X, \mathbb{Z}(3)),$$

and so we obtain a geometric description for the Deligne cohomology group.

We may now extend this idea to higher gerbes with connective structure. First, consider the Čech-de Rham complex with respect to an open cover $\{U_j\}_{j \in J}$ of X:

$$\begin{array}{ccccccc}
\vdots & & \vdots & & & & \\
\uparrow & & \uparrow & & & & \\
C^0(\Omega_X^2) & \xrightarrow{\delta} & C^1(\Omega_X^2) & \longrightarrow & \cdots & & \\
\uparrow d & & \uparrow d & & \uparrow d & & \\
C^0(\Omega_X^1) & \xrightarrow{\delta} & C^1(\Omega_X^1) & \xrightarrow{\delta} & C^2(\Omega_X^1) & \longrightarrow & \cdots \\
\uparrow \mathrm{dlog} & & \uparrow \mathrm{dlog} & & \uparrow \mathrm{dlog} & & \\
C^0(\mathcal{O}_X^\times) & \xrightarrow{\delta} & C^1(\mathcal{O}_X^\times) & \xrightarrow{\delta} & C^2(\mathcal{O}_X^\times) & \longrightarrow & \cdots
\end{array}$$

The associated total complex $(\mathrm{Tot}^\bullet, \mathcal{D})$ is given by

$$\mathrm{Tot}^k = C^k(\mathcal{O}_X^\times) \oplus C^{k-1}(\Omega_X^1) \oplus \cdots \oplus C^0(\Omega_X^k),$$

where the total differential $\mathcal{D} : \mathrm{Tot}^k \to \mathrm{Tot}^{k+1}$ is defined by

$$D = \delta + (-1)^k d.$$

We define a ***holomorphic k-gerbe with connective structure*** on X to be an element of the kernel of the differential $\mathcal{D} : \mathrm{Tot}^{k+1} \to \mathrm{Tot}^{k+2}$. We can write this as a $k+2$-tuple $(g, A_1, A_2, \ldots, A_k)$ where $g \in C^{k+1}(\mathcal{O}_X^\times)$ and $A_i \in C^{k+1-i}(\Omega_X^i)$ for $i = 1, \ldots, k+1$. The ***curvature*** of the k-gerbe is the global

$k+2$-form F where $F = (dA_k)_j$ for every $j \in J$. We say that two k-gerbes with connective structure $(g, A_1, A_2, \ldots, A_k)$ and $(g', A'_1, A'_2, \ldots, A'_k)$ are gauge-equivalent if there exists a $\mathcal{T} \in \mathrm{Tot}^{k-1}$ such that

$$(g', A'_1, A'_2, \ldots, A'_k) - (g, A_1, A_2, \ldots, A_k) = \mathcal{D}\mathcal{T}.$$

Thus, we can conclude that holomorphic k-gerbes with connective structure are classified up to equivalence by the hypercohomology group

$$\mathbb{H}^k(\mathcal{O}_X^\times \to \cdots \to \Omega_X^k).$$

Using the isomorphism in (6.1), this gives a geometric description of the Deligne cohomology group $H^{k+1}_{\mathrm{Del}}(X, \mathbb{Z}(k+1))$ as the group of gauge-equivalent classes of holomorphic k-gerbes with connective structure.

Remark 6.2. In the above, we have worked with an entirely Čech cohomological viewpoint of k-gerbes. Strictly speaking, this is the gerbe data of a gerbe. We can define k-gerbes in a more geometric fashion as in Section 5.2 using sheaves of groupoids. The reason why we do not pursue this approach is that we wish to abstain ourselves from using higher categories in order to minimize technicalities.

Remark 6.3. In [17], Gajer gives an alternate geometric description of the Deligne cohomology group $H^k_{\mathrm{Del}}(X, \mathbb{Z}(k))$ as the group of equivalence classes of holomorphic principal $B^{k-2}\mathbb{C}^*$-bundles over X with holomorphic i-connections for $i = 1, \ldots, k-1$.

6.1.1 Kostant-Weil theory for k-forms

We now recall the classical quantization result of Kostant and Weil for 2-forms on X [25, 38]. Let $B \in H^0(\Omega_X^{2,\mathrm{cl}})$ be a closed 2-form on X. The result states that B is the curvature of a pair $(\mathcal{L}, \nabla)$ where $\mathcal{L}$ is a holomorphic line bundle on X and ∇ is a connection on $\mathcal{L}$ if and only if B is an integral cohomology class. This was generalized to degree 3 cohomology using the language of gerbes with connections by Brylinski and Deligne in [9]. The following theorem will be a generalization of this result to closed k-forms on X.

Theorem 6.4. *Assume that $k \geq 2$. The curvature of a holomorphic $k-2$-gerbe with connective structure is quantized, i.e., it lies in the image of $H^k(X, \mathbb{Z}(1)) \to H^k(X, \mathbb{C})$. Conversely, if $[F] \in H^0(\Omega_X^{k,\mathrm{cl}})$ is a closed k-form on X such that $[F]$ lies in the image of $H^k(X, \mathbb{Z}(1)) \to H^k(X, \mathbb{C})$, then there exists a holomorphic $k-2$-gerbe with connective structure on X with curvature $[F]$.*

Proof. Let $(g, A_1, \ldots, A_{k-2})$ be the Čech cocyle defining the $k-2$-gerbe with connective structure. Since the curvature of this $k-2$-gerbe F is defined as $F_j = (dA_{k-2})_j$ on every open set U_j, this is obtained by applying the morphism of complexes

$$[\mathcal{O}_X^\times \to \cdots \to \Omega_X^{k-1}] \to \Omega_X^{k,\mathrm{cl}}[-k+1]. \tag{6.2}$$

To prove that the curvature F is quantized, we consider the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & \mathbb{C}_X^\times & \longrightarrow & [\mathcal{O}_X^\times \to \cdots \to \Omega_X^{k-1}] & \longrightarrow & \Omega_X^{k,\mathrm{cl}}[-k+1] & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & \mathbb{C}_X & \longrightarrow & \Omega_X^{\leq k-1} & \longrightarrow & \Omega_X^{k,\mathrm{cl}}[-k+1] & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & \mathbb{Z}(1)_X & = & \mathbb{Z}(1)_X & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

There is the long exact sequence associated to the top horizontal short exact sequence

$$\cdots \to \mathbb{H}^{k-1}(\mathcal{O}_X^\times \to \cdots \to \Omega_X^{k-1}) \to H^0(\Omega_X^{k,\mathrm{cl}}) \to H^k(X, \mathbb{C}^\times) \to \ldots \tag{6.3}$$

It follows that the curvature F belongs to the kernel of the exponential map $H^k(X,\mathbb{C}) \to H^k(X,\mathbb{C}^\times)$ and hence, it lies in the image of $H^k(X,\mathbb{Z}(1)) \to H^k(X,\mathbb{C})$.

Conversely, suppose that $[F] \in H^0(\Omega_X^{k,\mathrm{cl}})$ lies in the image of $H^k(X,\mathbb{Z}(1)) \to H^k(X,\mathbb{C})$. This implies that $[F]$ is mapped to the trivial element in $H^k(X,\mathbb{C}^\times)$. Using the exact sequence (6.3) again, we find that the class $[F]$ has a lift under the map

$$\mathbb{H}^{k-1}(\mathcal{O}_X^\times \to \cdots \to \Omega_X^{k-1}) \to H^0(\Omega_X^{k,\mathrm{cl}}).$$

This lift of the class $[F]$ in $\mathbb{H}^{k-1}(\mathcal{O}_X^\times \to \cdots \to \Omega_X^{k-1})$ is precisely a $k-2$-gerbe with connective structure with $[F]$ as its curvature. □

Corollary 6.5. *Let $[F] \in H^0(\Omega_X^{k,\mathrm{cl}})$ be a closed k-form on X which lies in the image of $H^k(X,\mathbb{Z}(k)) \to H^k(X,\mathbb{C})$. The set of isomorphism classes of k-gerbes with connective structure on X with curvature F is a principal homogeneous space under the group $H^{k-1}(X,\mathbb{C}^\times)$ of isomorphism classes of flat $k-2$ $\mathbb{C}^\times$-gerbes on X.*

Proof. Recall the long exact sequence in (6.3)

$$0 \to H^{k-1}(X,\mathbb{C}^\times) \to \mathbb{H}^{k-1}(\mathcal{O}_X^\times \to \cdots \to \Omega_X^{k-1}) \to H^0(\Omega_X^{k,\mathrm{cl}}) \to \ldots$$

It follows that the set of preimages of F in $\mathbb{H}^{k-1}(\mathcal{O}_X^\times \to \cdots \to \Omega_X^{k-1})$ forms a principal homogeneous space under the group $H^{k-1}(X,\mathbb{C}^\times)$. □

6.2 Deligne-Beĭlinson Cohomology

In this section, we work with (X, D) where X is a smooth projective variety over $\mathbb{C}$ and D is a smooth divisor in X. As usual, we denote the complement by $U := X \setminus D$ and the inclusion by $j : U \hookrightarrow X$. We will follow closely the treatment of [16].

Definition 6.6. The ***Deligne- complex*** $\mathbb{Z}(k)_{\mathrm{DB}}$ of (X, D) is the complex of sheaves

$$\mathrm{Cone}(Rj_*\mathbb{Z}(k)_U \oplus F^k\Omega^\bullet_X(\log D) \to \Omega^\bullet_X(\log D))[-1].$$

The ***Deligne-Beĭlinson cohomology*** groups $H^p_{DB}(U, \mathbb{Z}(k))$ are defined as the hypercohomology groups:

$$H^p_{DB}(U, \mathbb{Z}(k)) := \mathbb{H}^p(X, \mathbb{Z}(k)_{\mathrm{DB}}).$$

As in the case of Deligne cohomology, we have the long exact sequence in cohomology involving Deligne-Beĭlinson cohomology groups

$$\cdots \to H^p_{\mathrm{DB}}(U, \mathbb{Z}(k)) \to H^p(U, \mathbb{Z}(k)) \oplus F^k H^p(U, \mathbb{C}) \to H^p(U, \mathbb{C}) \to H^{p+1}_{\mathrm{DB}}(U, \mathbb{Z}(k)) \to \dots$$

We can also alternatively write the mapping cone in the definition of the Deligne-Beĭlinson complex $\mathbb{Z}(k)_{\mathrm{DB}}$ as

$$\mathrm{Cone}(Rj_*\mathbb{Z}(k)_U \to \Omega^{\leq k-1}_X(\log D))[-1], \tag{6.4}$$

where $\Omega^{\leq k-1}_X(\log D)$ is the truncation of the logarithmic de Rham complex $\Omega^\bullet_X(\log D)$ in degrees $\geq k$. There is an exact sequence of sheaves

$$0 \to \mathbb{Z}(k)_X \to \mathcal{O}_X \xrightarrow{\gamma} \mathcal{O}^\times_X(*D) \to R^1j_*\mathbb{Z}(k)_U \to 0,$$

where $\gamma(f) = \exp((2\pi i)^{1-k} \cdot f)$. This gives a quasi-isomorphism

$$\mathcal{O}^\times_X(*D) \sim \mathrm{Cone}(\sigma_{\leq 1}Rj_*\mathbb{Z}(k)_U \to \mathcal{O}_X).$$

Furthermore, we have the quasi-isomorphism of complexes

$$[\mathcal{O}^\times_X(*D) \to \cdots \to \Omega^{k-1}_X(\log D)] \sim \mathrm{Cone}(\sigma_{\leq 1}Rj_*\mathbb{Z}(k)_U \to \Omega^{\leq k-1}_X(\log D)).$$

However, since D is assumed to be smooth, we have $\sigma_{\leq 1}Rj_*\mathbb{Z}(k)_U = Rj_*\mathbb{Z}(k)_U$. Using the alternate definition of the Deligne-Beĭlinson complex in (6.4), we have the following isomorphism

$$H^p_{\mathrm{DB}}(U, \mathbb{Z}(k)) \cong \mathbb{H}^{p-1}(\mathcal{O}^\times_X(*D) \to \cdots \to \Omega^{k-1}_X(\log D)). \tag{6.5}$$

When $k = 1$, we immediately obtain the following geometric description of the following Deligne-Beĭlinson cohomology groups:

- $H^2_{\mathrm{DB}}(U, \mathbb{Z}(1)) \cong H^1(\mathcal{O}^\times_X(*D))$, the group of isomorphism classes of meromorphic line bundles on (X, D)
- $H^3_{\mathrm{DB}}(U, \mathbb{Z}(1)) \cong H^2(\mathcal{O}^\times_X(*D))$, the group of equivalence classes of meromorphic gerbes on (X, D)
- $H^3_{\mathrm{DB}}(U, \mathbb{Z}(3)) \cong \mathbb{H}^2(\mathcal{O}^\times_X(*D) \to \Omega^1_X(\log D) \to \Omega^2_X(\log D))$, the group of equivalence classes of meromorphic gerbes with connective structure on (X, D)

We now give a geometric description of the Deligne-Beĭlinson cohomology groups $H^k_{\mathrm{DB}}(U, \mathbb{Z}(k))$ using

Čech theory. Consider the Čech-log de Rham complex with respect to an open cover $\{U_j\}_{j\in J}$ of (X, D):

$$\begin{array}{ccccccc}
\vdots & & \vdots & & & & \\
\uparrow & & \uparrow & & & & \\
C^0(\Omega^2_X(\log D)) & \xrightarrow{\delta} & C^1(\Omega^2_X(\log D)) & \longrightarrow & \ldots & & \\
\uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \\
C^0(\Omega^1_X(\log D)) & \xrightarrow{\delta} & C^1(\Omega^1_X(\log D)) & \xrightarrow{\delta} & C^2(\Omega^1_X(\log D)) & \longrightarrow & \ldots \\
\uparrow{\scriptstyle \mathrm{dlog}} & & \uparrow{\scriptstyle \mathrm{dlog}} & & \uparrow{\scriptstyle \mathrm{dlog}} & & \\
C^0(\mathcal{O}^\times_X(*D)) & \xrightarrow{\delta} & C^1(\mathcal{O}^\times_X(*D)) & \xrightarrow{\delta} & C^2(\mathcal{O}^\times_X(*D)) & \longrightarrow & \ldots
\end{array}$$

Again, we have the associated total complex $(\mathrm{Tot}^\bullet, \mathcal{D})$ given by

$$\mathrm{Tot}^k = C^k(\mathcal{O}^\times_X(*D)) \oplus C^{k-1}(\Omega^1_X(\log D)) \oplus \cdots \oplus C^0(\Omega^k_X(\log D)),$$

and where the total differential $\mathcal{D} : \mathrm{Tot}^k \to \mathrm{Tot}^{k+1}$ is defined by

$$\mathcal{D} = \delta + (-1)^k d.$$

We define a ***meromorphic gerbe with*** k***-logarithmic connective structure*** on (X, D) to be an element of the kernel of the differential $\mathcal{D} : \mathrm{Tot}^{k+1} \to \mathrm{Tot}^{k+2}$ which we can write as a $k+2$-tuple $(g, A_1, \ldots, A_k)$ with $g \in C^{k+1}(\mathcal{O}^\times_X(*D))$ and $A_i \in C^{k+1-i}(\Omega^i_X(\log D))$ for $i = 1, \ldots, k+1$. The ***curvature*** of the meromorphic k-gerbe with connective structure is the global logarithmic $k+2$-form F where $F = (dA_k)_j$ for every $j \in J$. As in the case of k-gerbes, we say that two meromorphic k-gerbes with connective structure are gauge-equivalent if they differ up to a Čech k-coboundary. Thus, we see that meromorphic k-gerbes with connective structure are classifed up to equivalence by the hypercohomology group

$$\mathbb{H}^k(\mathcal{O}^\times_X(*D) \to \cdots \to \Omega^k_X(\log D)).$$

Using the isomorphism in (6.5), this gives a geometric description of the Deligne-Beĭlinson cohomology group $H^{k+1}_{\mathrm{DB}}(U, \mathbb{Z}(k+1))$.

6.2.1 Kostant-Weil theory for logarithmic k-forms

We now establish a version of Kostant-Weil quantization for logarithmic k-forms on (X, D).

Theorem 6.7. *Assume that $k \geq 2$. The curvature of a meromorphic $k-2$-gerbe with connective structure is quantized, i.e., it lies in the image of $H^k(U, \mathbb{Z}(1)) \to H^k(U, \mathbb{C})$. Conversely, if $[F] \in H^0(\Omega^{k,\mathrm{cl}}_X(\log D))$ is a closed logarithmic k-form on (X, D) such that $[F]$ lies in the image of $H^k(U, \mathbb{Z}(1)) \to H^k(U, \mathbb{C})$, then there exists a meromorphic $k-2$-gerbe with connective structure on (X, D) with curvature $[F]$.*

Proof. The structure of the proof is similar to the proof of Theorem 6.4. Let $(g, A_1, \ldots, A_k)$ be the Čech cocycle defining the meromorphic $k-2$-gerbe with connective structure. As the curvature of the $k-2$-gerbe F is defined as $F_j = (dA_{k-2})_j$ on every open set U_j, the curvature is then obtained by applying the morphism of complexes

$$[\mathcal{O}_X^\times(*D) \to \dots \to \Omega_X^{k-1}(\log D)] \to \Omega_X^{k,\mathrm{cl}}(\log D)[-k+1]. \tag{6.6}$$

Recall from the proof of Theorem 4.41 that the sheaf $\Omega_X^{1,\mathrm{cl}}(\log D)$ fits into the two short exact sequences

$$0 \to Rj_*\mathbb{C}_U \to \mathcal{O}_X \to \Omega_X^{1,\mathrm{cl}}(\log D) \to 0$$
$$0 \to Rj_*\mathbb{C}_U^\times \to \mathcal{O}_X^\times(*D) \to \Omega_X^{1,\mathrm{cl}}(\log D) \to 0$$

As the Poincaré lemma holds for logarithmic forms for degrees ≥ 2, we obtain the exact sequences

$$0 \to Rj_*\mathbb{C}_U \to \Omega_X^{\leq k-1}(\log D) \to \Omega_X^{k,\mathrm{cl}}(\log D)[-k+1] \to 0$$
$$0 \to Rj_*\mathbb{C}_U^\times \to [\mathcal{O}_X^\times(*D) \to \dots \to \Omega_X^{k-1,\mathrm{cl}}(\log D)] \to \Omega_X^{k,\mathrm{cl}}(\log D)[-k+1] \to 0$$

Now, we can map the first complex to the second complex in the following way which give the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & Rj_*\mathbb{C}_U^\times & \longrightarrow & [\mathcal{O}_X^\times(*D) \to \dots \to \Omega_X^{k-1}(\log D)] & \longrightarrow & \Omega_X^{k,\mathrm{cl}}(\log D)[-k+1] & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & Rj_*\mathbb{C}_U & \longrightarrow & \Omega_X^{\leq k-1}(\log D) & \longrightarrow & \Omega_X^{k,\mathrm{cl}}(\log D)[-k+1] & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & Rj_*\mathbb{Z}(1)_U & = & Rj_*\mathbb{Z}(1)_U & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

We have the long exact sequence associated to the top horizontal short exact sequence

$$\dots \to \mathbb{H}^{k-1}(\mathcal{O}_X^\times(*D) \to \dots \to \Omega_X^{k-1}(\log D)) \to H^0(\Omega_X^{k,\mathrm{cl}}(\log D)) \to H^k(U, \mathbb{C}^\times) \to \dots \tag{6.7}$$

It follows that the curvature F belongs to the kernel of the exponential map $H^k(U,\mathbb{C}) \to H^k(U,\mathbb{C}^\times)$ and hence, it lies in the image of $H^k(U,\mathbb{Z}(k)) \to H^k(U,\mathbb{C})$.

Conversely, suppose that $[F] \in H^0(\Omega_X^{k,\mathrm{cl}}(\log D))$ lies in the image of $H^k(U,\mathbb{Z}(k)) \to H^k(U,\mathbb{C})$. This implies that $[F]$ is mapped to the trivial element in $H^k(U,\mathbb{C}^\times)$. If we use the exact sequence (6.7), we find that the class $[F]$ has a lift under the map

$$\mathbb{H}^{k-1}(\mathcal{O}_X^\times(*D) \to \dots \to \Omega_X^{k-1}(\log D)) \to H^0(\Omega_X^{k,\mathrm{cl}}(\log D)).$$

This lift of $[F]$ in $\mathbb{H}^{k-1}(\mathcal{O}_X^\times(*D) \to \dots \to \Omega_X^{k-1}(\log D))$ is precisely a meromorphic $k-2$-gerbe with connective structure with $[F]$ as its curvature. $\square$

We have the analogue of Corollary 6.5 in the logarithmic case:

Corollary 6.8. *Let* $[F] \in H^0(\Omega_X^{k,\mathrm{cl}}(\log D))$ *be a closed logarithmic* k*-form on* (X, D) *which lies in the image of* $H^k(U, \mathbb{Z}(1)) \to H^k(U, \mathbb{C})$*. The set of isomorphism classes of meromorphic* k*-gerbes with connective structure on* (X, D) *with curvature* F *is a principal homogeneous space under the group* $H^{k-1}(U, \mathbb{C}^\times)$ *of isomorphism classes of flat* $k-2$ $\mathbb{C}^\times$*-gerbes on* U*.*

Proof. Recall the long exact sequence in (6.7)

$$0 \to H^{k-1}(U, \mathbb{C}^\times) \to \mathbb{H}^{k-1}(\mathcal{O}_X^\times(*D) \to \cdots \to \Omega_X^{k-1}(\log D)) \to H^0(\Omega_X^{k,\mathrm{cl}}(\log D)) \to \ldots$$

It follows that the set of preimages of F in $\mathbb{H}^{k-1}(\mathcal{O}_X^\times(*D) \to \cdots \to \Omega_X^{k-1}(\log D))$ forms a principal homogeneous space under the group $H^{k-1}(U, \mathbb{C}^\times)$. □

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