Recovering a Riemannian Metric from Knowledge of the Areas of Properly-Embedded, Area-Minimizing Surfaces

by

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Abstract

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In this thesis, we prove that if (M, g) is a C^3-smooth, 3-dimensional Riemannian manifold with mean convex boundary ∂M, which is additionally either a) C^2 close to Euclidean or b) ϵ_0-thin, then knowledge of the minimal area circumscribed by any simple closed curve on the boundary uniquely determines the metric. In the case where (M, g) only has strictly mean convex boundary at a point $p \in \partial M$, we prove that knowledge of the minimal area circumscribed by any simple closed curve in a neighbourhood $p \in U \subset \partial M$ uniquely determines the metric near p.

Declaration of Originality

The research in this thesis was conducted at the Department of Mathematics, University of Toronto, in the period between June 2013 and April 2017. The material contained in the thesis is original and a product of both collaborative and independent research. Chapters 3, 4, and 5 contains joint work with Spyros Alexakis and Adrian Nachman. This joint research is in preparation for publication.

To Minnie, Marilyn, and Megan

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Chapter 1

Introduction

It is a classical problem from geometry to ask what geometric information is sufficient to determine the metric on a Riemannian manifold. A reasonable conjecture is that knowledge of the distance between any two points on the boundary identifies the Riemannian metric. However, there is an obstruction: if the manifold has a large region of curvature in the interior, length-minimizing geodesics will never pass through this region. Eliminating such cases, Michel [15] conjectured that it is possible to uniquely determine the metric from the geometric data of distance between boundary points; when such minimal lengths determine the metric, we say the manifold is **boundary ridgid**. Special cases of boundary rigidity have been shown by Michel, Gromov, and Croke. More recent work has been done by Pestov and Uhlmann [17] and Burago and Invanov [4], [5]. In this paper, we consider a higher dimensional version of this problem:

Question. Given any simple closed curve γ on the boundary of a Riemannian 3-manifold (Mg), suppose the area of any least-area surface Y_γ circumscribed by the curve is known. Does this information determine the metric g?

Under modest geometric conditions, we show the answer is yes.

Besides purely geometric motivations, from theories posited by AdS/CFT correspondence there is also motivation to consider the problem of using knowledge of the areas of certain submanifolds to determine the metric. In particular, the AdS/CFT correspondence very loosely states that $(n + 2)$-dimensional supergravity theories modelled on an Anti-de Sitter (AdS) space are equivalent to $(n + 1)$-dimensional conformal field theories on the boundary of the AdS space (see [13]). Maldecena [14] has proposed that given curve on the boundary of the AdS spacetime, the "renormalized" area of the minimal surface bounding the curve contains information about the "Wilson loop" of the curve. More recently, it has been conjectured that given a bounded region in an $(n+2)$-dimensional AdS spacetime, the the "entanglement entropy" of the region as computed along the boundary is equivalent to the renormalized area of the region (see [19]). In either of these cases, one must consider a Riemannian, asymptotically hyperbolic manifold (M, g), for which one knows information on the boundary and a notion of area for minimal surfaces bounded by closed loops on the boundary. Hence it is of physical interest to ask if given any simple closed curve on the "boundary-at-infinity" of (M, g), we know the renormalized area of the minimal surfaces bounded by the curve, can we recover the metric from this information?

As stated, the question we consider is a geometric one. Our proofs, however, rely on reformulating the problem in terms of an inverse problem for a partial differential equation on the manifold. The first act of reformulation is to notice that by linearizing, our area data encodes the Dirichlet-to-Neumann map for the stability operator $\Delta_{g_{Y_\gamma}} + (\mathrm{Ric}_g(\vec{n}, \vec{n}) + ||A||_g)$ on any properly embedded, minimal surface $Y_\gamma \subset M$.

From the knowledge of the Dirichlet-to-Neumann map for the stability operator on such an area-minimizing, properly embedded surface Y_γ, we recover the potential $\mathrm{Ric}_g(\vec{n}, \vec{n}) + ||A||_g$, and hence recover information about g. At the time of writing this paper, it is not known whether or not the Dirichlet-to-Neumann map determines the potential of a Schrödinger operator on an arbitrary Riemannian n-manifold with a non-Euclidean metric. However, when Y_γ is a surface, the Laplace-Beltrami operator $\Delta_{g_{Y_\gamma}}$ is conformally invariant. In this case, the surface Y_γ is conformal to a bounded domain

$\Omega \subset \mathbb{R}^2$, and via an isothermal coordinate map $(x^1, x^2) = \Phi : \mathbb{R}^2 \to \mathbb{R}^2$ we are able to relate the operator $\mathcal{J}$ and its Dirichlet-to-Neumann map to a Schrödinger operator and Dirichlet-to-Neumann map in the Euclidean setting. By well-known arguments, we determine the map Φ on $\mathbb{R}^2 \setminus \Omega$ from the Dirichlet-to-Neumann map. Further, if we require that the stability operator $\mathcal{J}$ is nondegenerate – that is, has no zero eigenvalue – by a consequence of the result by Nachman [16], we are able to determine the potential $\mathrm{Ric}_g(\vec{n}, \vec{n}) + ||A||_g$, and any solution ψ to $\mathcal{J}\psi = 0$ from knowledge of the area of Y_γ and nearby perturbations.

That is we show

Proposition 1.1. *Let (M, g) be a C^3 smooth, compact, 3-dimensional Riemannian manifold with strictly mean convex boundary ∂M, and suppose that (M, g) admits a foliation by properly embedded, area-minimizing surfaces. Let γ be a given simple closed curve on ∂M, and g be given on ∂M. Suppose that for γ and any nearby perturbation of $\gamma(s)$, the area of the least-area surface $Y_{\gamma(s)}$ enclosed be $\gamma(s)$ is known.*

Equip a neighbourhood of Y_γ with coordinates (x^α) such that on Y_γ, $x^3 = 0$ and (x^1, x^2) are isothermal coordinates. Then,

1. *the first and second variations of the area of Y_γ determine the Dirichlet-to-Neumann map associated to to the boundary value problem*

$$\Delta_{g_{\mathbb{E}}}\psi + e^{2\phi}\left(Ric_g(\vec{n}, \vec{n}) + ||A||_g^2\right)\psi = 0,$$
$$\psi|_{\partial Y_\gamma} = g\left(\frac{d}{ds}\gamma(s)\Big|_{s=0}, \vec{n}\right),$$

   on Y_γ, where $e^{2\phi}g_{\mathbb{E}} = e^{2\phi}[(dx^1)^2 + (dx^2)^2]$ is the metric on Y_γ in the coordinates (x^1, x^2).

2. *the first and second variations of the area of Y_γ determine the any solution $\psi(x)$ to the above boundary value problem.*

Identifying a relation between geometric inverse problems (i.e recovering the metric from boundary distances) to inverse problems for a differential operator (e.g. recovering a potential from the Dirichlet-to-Neumann map) is a topic of considerable interest. In 2001 Lassas and Uhlmann [12] demonstrated in dimensions $n \geq 3$ that if M is a connected, real analytic manifold with real analytic boundary, then the Dirichlet-to-Neumann map $(f : \partial M \to \mathbb{R}) \mapsto \Lambda(f) := g(\nabla u, \nu)|_{\partial M}$ associated to

$$\Delta_g u = 0, \quad u|_{\partial\Omega} = f,$$

determines the metric on M. By relating information from geodesic distances $d(x, y)$ to information about the map Λ_g, Pestov and Uhlmann [17] were able to prove Michel's conjecture when M is a simple, 2-dimensional manifold. When M is a domain in $\mathbb{R}^2$, Sun and Uhlmann [22] showed that the Dirichlet-to-Neumann map associated to

$$\operatorname{div}_g (c(x)\nabla u) = 0, \quad u|_{\partial M} = f,$$

determines the metric g on M up to conformal class. Most recently in 2016, Stefanov, Uhlmann, and Vasy have demonstrated that if M is of dimension $n \geq 3$ and admits a foliation by strictly convex hypersurfaces, then the lens data on the boundary uniquely determines the metric [21].

Using Proposition 1.1, we recover the Riemannian metric g on Riemannian manifolds which belong to the following classes:

Let (M, g) is a C^3-smooth, 3-dimensional Riemannian manifold with C^3-smooth, mean convex boundary ∂M. Suppose either

1. g is C^2-close to Euclidean, and (M, g) admits a foliation $Y(t)$ by properly embedded, area-minimizing surfaces such that the stability operator on each $Y(t)$ does not nave a zero Dirichlet eigenvalue; or

2. there is an $\epsilon_0 > 0$ such that (M, g) is ϵ_0-thin.

By C^2 close to Euclidean, we mean the following:

Definition 1.2. For $k \in \mathbb{N}$, we say the metric g is $\mathbf{C^k}$ **close** to the Euclidean metric $g_{\mathbb{E}}$ on $\mathbb{R}^3$ if for all $\alpha, \beta = 1, 2, 3$ there is an $0 < \epsilon << 1$ such that $||g_{\alpha\beta} - (g_{\mathbb{E}})_{\alpha\beta}||_{C^k} < \epsilon$.

Additionally,

Definition 1.3. We say a Riemannian 3-manifold (M, g) is ϵ_0**-thin** if

a. there exists global coordinates (y^α), $\alpha = 1, 2, 3$ on (M, g) such that the surfaces $Y(t) := \{y^3 = \text{constant} = t\}$ are properly embedded and area-minimizing;

b. the stability operator on each $Y(t)$ does not nave a zero Dirichlet eigenvalue

c. the magnitudes of the Riemann curvature tensor Rm_g, the second fundamental form of each $Y(t)$, and the metric coefficients of $g|_{Y(t)}$ are all bounded from above by $\frac{1}{\epsilon_0}$;

d. the diameter of each $Y(t)$ is bounded above by ϵ_0.

The restrictions that we place on (M, g) are of a similar nature to the the simplicity assumption in the work of [17], the convex foliation requirement in [21], and the C^2 close to Euclidean restriction needed in [4] and analogous C^3 close to Hyperbolic condition in [5]. Note the since (M, g) is required to have mean convex boundary ∂M, given any closed curve on ∂M we may solve the least area problem for the given curve [9]. The existence of foliations of (M, g) by properly embedded, area-minimizing surfaces allows us to recover the metric globally. Furthermore, the requirement that the operator $\mathcal{J}$ is non-degenerate on such a foliation ensures that our area data does not skip over "bubbles" – that is, minimal 2-spheres – in the interior of M. Our C^2 close to Euclidean or ϵ_0-thin assumption will allow us to uniquely solve for the metric components.

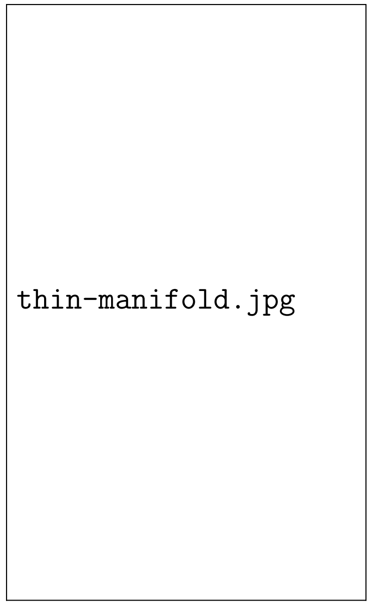

Figure 1.1: Thin manifolds.

While our restrictions are of a similar nature, in contrast to the results in [17], [21], [4], [5] our proofs for the uniqueness of the metric use variational arguments. By considering variations in the curves γ and the resulting variations in the area-minimizing surfaces Y_γ, we prove

Theorem 1.4. *Let (M, g) be a manifold of Class 1 or Class 2 above, and $g|_{\partial M}$ be given. Suppose that for any simple, closed curve γ on ∂M and any nearby perturbation $\gamma(t) \subset \partial M$ with $\gamma(0) = \gamma$, we know the area of the properly embedded surface $Y(t)$ which solves the least-area problem for $\gamma(t)$.*

Then the knowledge of these areas uniquely determines the metric g.

As direct application of Theorem 1.4, we have the local result:

Theorem 1.5. *Let (M, g) be a 3-dimensional Riemannian manifold with boundary ∂M. Assume that ∂M is both C^3-smooth and mean convex at $p \in \partial M$. Let γ be a simple, closed curve on ∂M near p. Suppose that for γ and any nearby perturbation $\gamma(t) \subset \partial M$ with $\gamma(0) = \gamma$, we know the area of the properly embedded surface $Y(t)$ which solves the least-*

area problem fro $\gamma(t)$ Then this information uniquely determines g on a neighbourhood of p.

We will prove Theorem 1.4, and hence Theorem 1.5, as consequences of the the following theorem:

Theorem 1.6. *Let $D(r) \subset \mathbb{R}^2$ be a disk of radius $r > 0$ about the origin, and $T > 0$. Let (M, g_1) be a smooth, 3-dimensional Riemannian manifold which is topologically equivalent to $D(r) \times [0, T]$. Suppose*

1. *(M, g_1) is either*
   - *C^2 close to Euclidean, or*
   - *$r < \epsilon_0 < 1$ and (M, g_1) is ϵ_0-thin.*
2. *for any closed, embedded curve $\gamma : [0, T] \to \partial M$, we know the area of any properly embedded surface in $Y_\gamma \subset M$ which solves the least-area problem for γ.*

Then, if g_2 is a Riemannian metric on M which satisfies the same conditions as (M, g_1), $g_1 = g_2$ on ∂M, and g_2 gives the same area data as g_1, the metric g_2 is isometric to g_1.

We now describe our variational approach to the proof of Theorem 1.6. On a surface Y_γ which minimizes the area circumscribed by γ, Proposition 1.1 allows us to determine all solutions ψ to the problem $\mathcal{J}\psi = 0$ in an isothermal coordinate system on Y_γ. Thus, we are motivated to express the metrics g_1, g_2 on M in a coordinate system which restricts to isothermal coordinates on Y_γ.

We equip the boundary ∂M with a known foliation by simple closed curves, $\gamma(t)$, $t \in [0, 1]$. By solving the least-area problem for each $\gamma(t)$ of the foliation on the boundary, we obtain foliations by properly embedded, area-minimizing surfaces $Y_1(t)$ and $Y_2(t)$ of (M, g_1) and (M, g_2) respectively. By extending (M, g_1) and (M, g_2) to asymptotically flat manifolds, we equip each extended leaf with a unique set isothermal coordinates with a given decay property. In such a coordinate system, the inverse metric component in the direction purely transverse to the leaves is the **lapse function** associated to the foliation:

Definition 1.7. Let $\Omega \subset \mathbb{R}^2$ be a domain with boundary, and $f(\cdot, t) : \Omega \times [0,1] \hookrightarrow M$ a foliation of M by the surfaces $Y(t) : f(\Omega, t)$. Set $\vec{n}_t$ to be a unit normal to $Y(t)$. Then, the normal component of the variational vector

$$g\left(f_*\left(\frac{\partial}{\partial t}\right), \vec{n}_t\right) =: \psi,$$

is called the **lapse function** of the foliation f.

As the lapse function is a Jacobi field on any such area-minimizing surface, by Proposition 1.1 we have determined a component of the metrics g_1 and g_2.

We perturb these lapse functions and compute is first linearization to obtain a system of equations for the metric components of g_1 and g_2 in directions transverse and tangent to the leaves of our chosen area-minimizing foliations. The fact that mean curvature vanishes on each leaf may be expressed as an pseudodifferential equation governing the evolution from leaf to leaf of the metric components of g_1 and g_2 which are purely tangent to the leaves.

We explicitly construct a diffeomorphism which maps our chosen coordinate system on (M, g_1) to the analogous one placed on (M, g_2), and pullback g_2 to (M, g_2) under this diffeomorphism. We then consider the system of equations which arise by taking differences of the equations for the metric components. When the metrics g_1 and g_2 are C^2 close to the Euclidean metric, we show that this system has a unique solution.

Outline of the thesis: In Chapter 2 we provide our notion conventions and breif overviews of area minimizing surfaces and some background information on inverse problems. In chapter 3, we explicitly describe asymptotically flat extensions for (M, g_1) and (M, g_2) and construct the coordinate systems on (M, g_1) and (M, g_2) we work with throughout the rest of the thesis. We also prove Proposition 1.1. In Chapter 4, we collect arguments for determining the lapse function in our chosen coordinates and the first variation of the lapse, as well as a pseudodifferential equation which governs the evolution from leaf to leaf of the metric components of g_1 and g_2 which are purely tangent to the leaves. Finally in Chapter 5, we give the proofs of all theorems. For quick reference, we have included Appendices which collects calculations for the first and section variations of area.

Chapter 2

Preliminaries

2.1 Notation

Throughout this thesis, (M, g) will always denote a 3-dimensional, connected, Riemannian manifold with boundary ∂M, unless stated otherwise. Given such a Riemannian manifold (M, g), we write ∇ for the Levi-Civitá connection associated to g. For a function $u : M \to \mathbb{R}$, we denote the gradient of u as

$$\nabla u := \mathrm{grad}(u) = \sum_{\alpha,\beta} g^{\alpha\beta} \partial_\alpha u \partial_\beta. \tag{2.1}$$

Given smooth vector fields $U, V, W \in TM$, we define the Riemannian curvature tensor of (M, g) by

$$\mathrm{Rm}_g(U, V, W, \cdot) := \nabla_V \nabla_U W - \nabla_U \nabla_V W + \nabla_{[U,V]} W \tag{2.2}$$

(please note the choice of sign convection). The Ricci curvature of (M, g) is then given as $\mathrm{Ric}_g(U, V) := \mathrm{tr}_g \mathrm{Rm}_g(U, \cdot, V, \cdot)$, and the scalar curvature of (M, g) is $R_g := \mathrm{tr}_g \mathrm{Ric}_g(\cdot, \cdot)$.

If $Y \subset M$ is a submanifold of (M, g), $g|_Y$ denotes the first fundamental form of Y. The second fundamental form of Y is written $A(U, V) := g(U, \nabla_V \vec{n})$, where U, V are tangent vectors to Y and $\vec{n}$ is a normal vector field on Y.

We use Einstein summation notation throughout this thesis; that is, an instance of an index in an up and down position indicates a sum over the index; e.g. $T^i S_i := \Sigma_{i=j} T^i S_j$. We further will use the convention that greek indices $\alpha, \beta, \gamma, \ldots$ take values in $\{1, 2, 3\}$, and latin indices $i, j, k, \ldots$ take values in $\{1, 2\}$.

2.2 Area Minimizing Surfaces

The study and theory of area minimizing surfaces is rich and varied. With extreme brevity we provide a few basic properties and facts of area minimizing surfaces with boundary and foliations by such surfaces. We first distinguish the class of maps which define embedded surfaces with boundary in the boundary of (M, g):

Definition 2.1. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with boundary. We say an embedding $f : \Omega \hookrightarrow M$ is a **proper embedding** of Ω in M if $f(\Omega) \cap \partial M = f(\partial\Omega)$. In this setting, we call $Y := f(\Omega)$ **properly embedded** in M.

With this definition in hand,

Definition 2.2. Let $\gamma \subset \partial M$ be a simple closed curve. A properly embedded surface $Y_\gamma \subset M$ bounded by γ is said to be **area minimizing** if

$$Y_\gamma = \text{argmin}\{\text{Area}(Y) \,:\, Y \subset M \text{ is an embedded surface },\, \partial Y = \gamma\}.$$

For an embedded surface $f : \Omega \hookrightarrow Y \subset M$, the **area functional** is defined as

$$\text{Area} : Y \mapsto \text{Area}(Y) := \int_\Omega d\text{Vol}(g|_Y) \tag{2.3}$$

Given a simple closed curve γ, to find such an area minimizer Y_γ we consider a 1-parameter variation of properly embedded surfaces whose boundary is γ. Let f :

$\Omega \times [0,1] \to M$ be such a variation; that is, $f(\cdot, t)$ is a proper embedding for each fixed t, and $f(\Omega, t) =: Y(t)$ is surface bounded by γ. The first variation in area (see appendix ref) is found to be

$$\frac{d}{dt} A(Y(t)) = -\int_{\Omega} g(X_t, H_t \vec{n})\, d\mathrm{Vol}(g_t)$$

where $f_*\left(\frac{\partial}{\partial t}\right) =: X_t$ is the variational vector, $g_t = f^*(g|_{Y(t)})$, $H_t := -\operatorname{div}_g(\vec{n}) = -g_t^{ij} g(\nabla_{X_i} \vec{n}, X_j)$ is the scalar-valued mean curvature of $Y(t)$, and ν is the outward pointing vector normal to $\partial\Omega$ and tangent to Ω.

Definition 2.3. A surface $Y \subset M$ is called **minimal** if it is a critical point of the area functional.

From the first variation in the area formula above, an embedded surface $Y \subset M$ is a critical point of the area functional if the mean curvature of Y vanishes. Thus,

Definition 2.4. A surface $Y \subset M$ is **minimal** if Y has zero mean curvature.

Theorem 2.5. *If $f : \Omega \hookrightarrow Y \subset M$ is an embedded minimal surface, then f is a harmonic map.*

The proof of this result be found in Jost, [11]. We reproduce it below.

Proof. Let $(x^1, x^2) \in \Omega$ be normal coordinates at $p \in \Omega$, and write $f(x^1, x^2) = p = (y^1, y^2, y^3)$ in local coordinates on M. Set

$$X_i := f^* \left(\frac{\partial}{\partial x^i} \right) = \frac{\partial y^k}{\partial x^i} \frac{\partial}{\partial x^k}$$

be a local coordinate frame on Y, and set $\vec{n}$ to be a normal vector field to Y. The mean curvature of Y vanishes means

$$g^{ij} g(\nabla_{X_i} X_j, \vec{n}) = 0; \tag{2.4}$$

that is

$$(g^{ij}\nabla_{X_i}X_j)^{\perp} = 0. \tag{2.5}$$

Write $\bar{\nabla}$ for the connection ∇ restricted to Y. So in these coordinates at p,

$$\begin{aligned}
(\nabla_{X_i}X_i)^{\perp} &= \nabla_{X_i}X_i & (2.6)\\
&= \frac{\partial}{\partial x^i}\left(\frac{\partial y^k}{\partial x^i}\frac{\partial}{\partial x^k}\right)\frac{\partial}{\partial x^k} + \frac{\partial y^k}{\partial x^i}\frac{\partial y^l}{\partial x^i}\Gamma^j_{kl}(f(x))\frac{\partial}{\partial x^j} & (2.7)\\
&= \frac{\partial^2 y^k}{\partial x^i \partial x^i}\frac{\partial}{\partial x^k} + \frac{\partial y^k}{\partial x^i}\frac{\partial x^l}{\partial x^i}\Gamma^j_{kl}(f(x))\frac{\partial}{\partial x^j}. & (2.8)
\end{aligned}$$

where $\Gamma^j_{kl}(f(x)$ are the Christoffel symbols with respect to the metric g on M. So in arbitrary coordinates, the mean curvature of Y is zero implies

$$0 = \Delta_{g|_Y} y^k + (g|_Y)^{ij}\frac{\partial y^k}{\partial x^i}\frac{\partial x^l}{\partial x^i}\Gamma^j_{kl}(f(x)). \tag{2.9}$$

that is, f is a harmonic map.

□

Remark: If $f : \Omega \to \mathbb{R}^3$ is an embedded minimal surface, where $\mathbb{R}^3$ is equipped with the Euclidean metric, then by the previous theorem, the coordinate functions of f are harmonic functions.

Of course, it is not enough that a surface Y_γ is a critical point of the area function to be a surface which minimizes area bounded by γ. To determine surfaces which locally minimize area, we consider the second variation in area.

Definition 2.6. A surface $Y \subset M$ is called **stable** if the second variation in the area of Y is nonnegative.

Thus, surfaces which minimize area are stable. For arbitrary variations $f : \Omega \times [0, 1] \to M$, the expression for the second variation is rather complicated (please see the appendix for the computation). However, the computations simplify greatly in the case where the variation is **normal**, that is, $X_0 := \frac{\partial f_t}{\partial t}\big|_{t=0} = \psi\vec{n}$ where $\vec{n}$ is normal to $f(\Omega, 0) =: Y(0)$ and $\psi : \Omega \to \mathbb{R}$ is called the **lapse function** of the variation $f : \Omega \times [0,1] \to M$. In this case, the second variation in area becomes

$$\frac{d^2}{dt^2} A(Y(t))\bigg|_{t=0} = -\int_\Omega \psi\Delta_{g_0}\psi + \psi^2 \mathrm{Ric}_g(\vec{n},\vec{n}) + \psi^2 ||A||_g^2 \, d\mathrm{Vol}g_0. \tag{2.10}$$

The operator

$$\mathcal{J} := \Delta_{g_0} + \left(\mathrm{Ric}_g(\vec{n},\vec{n}) + ||A||_g^2\right), \tag{2.11}$$

is called the **stability operator** for the surface $Y(0)$. A normal vector field $\psi\vec{n}$, where $\psi : \Omega \to \mathbb{R}$ is a solution to

$$\mathcal{J}(\psi) = \Delta_{g_0}\psi + \left(\mathrm{Ric}_g(\vec{n},\vec{n}) + ||A||_g^2\right)\psi = 0, \tag{2.12}$$

is called a **Jacobi field** on on the surface $Y(0)$. Equation (2.12) is often referred to as the **Jacobi equation**.

Once again, consider a simple closed curve $\gamma : [0,1] \to \partial M$ and let $\Omega \subset \mathbb{R}^2$ be a bounded domain with boundary. Given γ, does there exist an embedded surface Y_γ which minimizes the area enclosed b γ? This question is called **Plateau's problem** or the **least area problem for** γ. In the case where Ω is the unit disk and $M = \mathbb{R}^3$, Plateau's problem was first solved independently by Douglas [**?**] and Rádo [18]. Much work has been done since the seminal results of Douglas and Rádo. For the purposes of this thesis, we only require results which allow us to solve Plateau's problem in the case where $\Omega \subset \mathbb{R}^2$ is a domain and M is a Riemannian 3-manifold with boundary. In particular, if ∂M is **strictly mean convex**, that is, the mean curvature of ∂M is positive define, Meeks and Yau [9] have shown the following:

Theorem 2.7 (Meeks-Yau, 1982). *Let (M, g) be a compact, Riemannian 3-manifold with strictly convex boundary. Let $\gamma \subset \partial M$ be a given simple closed curve which is contractible to a point in M. Then, there exists a proper embedding of the unit disk $f : D \subset \mathbb{R}^2 \hookrightarrow M$ such that $f(\partial D) = \gamma$ and $f(D)$ solves the least area problem for γ.*

In this thesis, we consider Riemannian 3-manifolds (M, g) which admit foliations by properly embedded, area minimizing (and hence stable) surfaces $Y(t)$, for $t \in [0, 1]$.

Definition 2.8. Let (M, g) be a Riemannian m-manifold. A C^k smooth, **codimension** n **foliation** of (M, g) is a disjoint collection of connected subsets $\{L(t) \,:\, t \in I\}$ of M such that the union equals M, and for every t in the indexing set I and every point $p \in L(t) \subset M$, there is an open set $U \subset M$ and a C^k smooth coordinate map $(x^1, x^2, \ldots, x^m) : U \to \mathbb{R}^m$ such that $x^{m-1} = x^{m-2} = \cdots = x^{m-n} = 0$ on $L(t) \cap U$. We call each submanifold $L(t)$ a **leaf** of the foliation.

Some examples of foliations include:

• 1-Parameter family of proper embeddings. If $f : \Omega \times [0, 1] \to M$ is a 1-parameter family of proper embeddings $f(\cdot, t)$ such that $f(\Omega, t_t)$ is transverse to $f(\Omega, t_2)$ for any $t_1, t_2 \in [0, 1]$, and $\mathrm{Im}(f) = M$, then the surfaces $Y(t) := f(\Omega, t)$ define a foliation of M. In this case, normal component of the variational vector $f_* \left(\frac{\partial}{\partial t}\right)$, given by

$$g\left(f_*\left(\frac{\partial}{\partial t}\right), \vec{n}_t\right) =: \psi,$$

is called the **lapse function** of the foliation induced by f.

• Submersions. Let $u : M \to \mathbb{R}$ is a C^k smooth submersion. Then, for each $t \in \mathbb{R}$, we obtain a codimension 1 foliation of (M, g) via the collection of C^k smooth submanifolds $L(t) := u^{-1}(t)$, for $t \in \mathbb{R}$. Choosing any coordinates (x^1, x^2) on $L(t)$ and setting $x^3 = t$ gives the required coordinate chart.

Continuing with this last example, et $u : M \to \mathbb{R}$ be a submersion, and for $t \in \mathbb{R}$, write

$$\{p \in M \,:\, u(p) = t\} =: Y(t)$$

In this setting, $\vec{n} := \frac{\nabla u}{||\nabla u||_g}$ is a normal vector field to $Y(t)$. Set $\{X_1, X_2\}$ to be a tangent frame field on $Y(t)$. If $Y(t)$ is a minimal surface, the mean curvature of $Y(t)$ vanishes, so

$$\begin{aligned} 0 &= g^{ij} g(X_i, \nabla_{X_j} \vec{n}) \\ &= \mathrm{div}_g(\vec{n}) \\ &= \mathrm{div}_g \left(\frac{\nabla u}{||\nabla u||_g} \right). \end{aligned}$$

We call

$$\mathrm{div}_g \left(\frac{\nabla u}{||\nabla u||_g} \right) = 0 \tag{2.13}$$

the **minimal surface equation**. Thus, an alternate form of Plateau's problem is the following: suppose $F : \partial M \to \mathbb{R}$ is a given foliation of ∂M by simple closed curves $\gamma(t)$. Does there exists a foliation defined by $u : M \to \mathbb{R}$ such that

$$\mathrm{div}_g \left(\frac{\nabla u}{||\nabla u||_g} \right) = 0 \tag{2.14}$$

$$u|_{\partial M} = F. \tag{2.15}$$

2.3 A Brief History of Inverse Problems

Motivated by applications to seismic tomography, in 1981 Calderón [6] gave the first rigorous formulation of an inverse problem. **Calderón's problem** was the following: let

$\Omega \subset \mathbb{R}^n$ be a domain, and consider a function $u : \Omega \to \mathbb{R}$ satisfying

$$\nabla \cdot (c(x)\nabla u) = 0, \quad u|_{\partial\Omega} = f,$$

for a known voltage $f : \partial\Omega \to \mathbb{R}$ on the boundary and an unknown conductivity $c : \Omega \to \mathbb{R}$ in the interior. What information is needed to determine the conductivity? Calderón proposed that knowledge of the **Dirichlet-to-Neumann map** Λ_c, defined by

$$(\phi : \partial\Omega \to \mathbb{R}) \mapsto \Lambda_c(\phi) := \int_{\partial\Omega} \phi \cdot c(x) \frac{\partial u}{\partial \nu}$$

(here ν is the outer normal to the boundary) could be used to find the conductivity $c(x)$. Since Calderón's seminal work, there have been several results found for his problem and for other analysis type inverse problems. Sylvester and Uhlmann [23] solved uniqueness for this problem in dimensions $n \geq 3$, for smooth conductivities in smooth domains $\Omega \subset \mathbb{R}^n$. The 2-dimensional isotropic conductivity problem was solved by Nachman [16]. This result also settled the corresponding anisotropic problem in two dimensions, as it could be reduced to the isotropic case using isothermal coordinates.

Using the facts shown in [16], Isakov and Nachman have proved that the Dirichlet-to-Neumann map

$$(\phi : \partial\Omega \to \mathbb{R}) \mapsto \Lambda_q(\phi) := \int_{\partial\Omega} \phi \cdot q(x) \frac{\partial u}{\partial \nu}$$

associated to a linear Schrödinger operator $-\Delta + q(x)$ can be used to reconstruct the potential function $q : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$. We will use their result to reformulate information about minimal areas into information about a Riemannian metric in later chapters. For convenience, we state the version of their result we use below:

Theorem 2.9 (Isakov-Nachman, 1995)**.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded $C^{1,1}$ domain. For some $p > 1$, let $q \in L^p(\Omega)$ be a real-valued potential. Suppose that the first Dirichlet eigenvalue of $-\Delta + q_1$ is positive, and both q and Λ_q are known on $\partial\Omega$. Then, Λ_q can be* *used to recover q in Ω.*

Chapter 3

Asymptotically Flat Extension and Conformal Maps

In this section, we define asymptotically flat extensions of the 3-manifolds we work with. We also define a preferred foliation $\bar{Y}(t)$ of M and a coordinate system (x^α) adapted to the foliation $\bar{Y}(t)$ such that the coordinate system restricted to any leaf is isothermal. We then prove that knowledge of the area of any area-minimizing surface near $\bar{Y}(t)$ determines the Dirichlet-to-Neumann map for the stability operator on $\bar{Y}(t)$ in our preferred coordinates (x^α), and from the information, we also determine the image of $\bar{Y}(t)$ under our chosen isothermal coordinate map.

3.0.1 Extension to an Asymptotically Flat Manifold

Suppose (M, g) is a closed, C^3-smooth, 3-manifold with mean convex boundary ∂M. By a result in [9], we know (M, g) admits properly embedded, codimension 1, area-minimizing foliations. For for any such properly embedded, codimension 1, minimal surface in (M, g), we assume that the area of the surface is known.

In this setting, we utilize the above fact to equip (M, g) with a coordinate system adapted to a chosen background foliation by such surfaces. The preferred coordinate system we construct will be used in several of the proofs of this paper to simplify computations and derive relevant equations.

To derive the desired coordinates, let $\Omega \subset \mathbb{R}^2$ be a bounded $C^{2,\alpha}$ domain for some $\alpha > 0$, and choose $\gamma : \partial\Omega \times [0, T] \to \partial M$ be a given foliation of the boundary by embedded, closed curves. Let $f(\cdot, t) : \Omega \to M$ solve Plateau's problem for $\gamma(\cdot, t)$, for each $t \in [0, T]$. In particular, $f(\cdot, \cdot)$ defines a foliation of M by properly embedded, codimension 1, area-minimizing surfaces such that $f(\partial\Omega \times \{t\}) = \gamma(\cdot, t)$ for each $t \in [0, T]$. We denote the leaves of the foliation f in M by $Y(t) := f(\Omega \times \{t\})$.

Our first choice of coordinate is the parameter identifying each minimal surface $Y(t)$: label the coordinate $x^3 = t$. Now, to obtain two other coordinate functions, one might choose conformal coordinates (x^1, x^2) on each leaf $Y(t)$ of the foliation. Then (x^1, x^2, x^3) is a global coordinate system on (M, g). However, there are many choices for conformal coordinates (x^1, x^2) on a 2-dimensional surface $Y(t)$. To remove this ambiguity, we extend (M, g) to an asymptotically flat manifold and impose decay conditions on the conformal coordinates on the extension of $Y(t)$ which renders the conformal coordinates unique.

To this end, let $(z^1, z^2, t) : M \to \Omega \times [0, T]$ be arbitrary coordinates on M. Via the Whitney extension theorem, we may smoothly extend the metric $g|_{\partial M}$ to a tubular neighbourhood N of ∂M. Let $g_{\mathbb{E}}$ denote the Euclidean metric on $\mathbf{M} : [0, T] \times \mathbb{R}^2$, and let $\chi : \mathbf{M} \to \mathbb{R}$ be a $C^\infty(M)$ cutoff function such that $\chi|_M = 1$ and $\chi = 0$ outside $M \cup N$. In this way we extend (M, g) to an asymptotically flat manifold $(\mathbf{M}, \mathbf{g})$ with metric $\mathbf{g}$ as

$$\mathbf{g} := \chi g + (1 - \chi) g_{\mathbb{E}}.$$

Again via the Whitney extension theorem, we obtain a smooth extension of $\mathbf{f} : \mathbb{R}^2 \times [0, T] \to \mathbf{M}$ of the foliation f. The smooth (but not necessarily minimal with respect to $\mathbf{g}$) extension of $Y(t)$ to $\mathbf{M}$ is then $\mathbf{f}(\mathbb{R}^2, t) =: \mathbf{Y}(t) \cong \mathbb{R}^2$.

Since $\mathbf{Y}(t)$ is a 2-dimensional domain, a well known result of Ahlfors [1] gives the unique existence of **isothermal coordinates** on $\mathbf{Y}(t)$. That is, there exists a conformal map

$$\Phi(t) : \mathbb{R}^2 \to \mathbb{R}^2, \quad z := z^1 + iz^2 \mapsto x^1 + ix^2$$

satisfying

$$\bar{\partial}\Phi(t) = \mu(t)\partial\Phi(t) \tag{3.1}$$

$$\Phi(z,t) - z = L^p(\mathbb{R}^2), \tag{3.2}$$

for $p > 2$ and **dilation** $\mu(t) := \frac{g_{11}+g_{22}+2ig_{12}}{g_{11}+g_{22}+2\sqrt{\mathbf{g}|_{\mathbf{Y}(t)}}}$. In these coordinates, $\Phi(t)$ pushes forward $\mathbf{g}_t := \mathbf{g}|_{\mathbf{Y}(t)}$ to a metric conformal to the Euclidean metric on $\mathbb{R}^2$:

$$\mathbf{g}_t = e^{2\phi(x,t)}[(dx^1)^2 + (dx^2)^2]$$

for some **conformal factor** $\phi(\cdot, t) : \mathbb{R}^2 \to \mathbb{R}$. We call the images (x^1, x^2) **isothermal coordinates** on Ω, and denote the conformal image of Ω as $\tilde{\Omega}(t) := \Phi(\Omega, t)$.

Since $\mathbf{g}$ is C^3-smooth by construction, μ_t is C^3-smooth in t. By a theorem in [2], $\Phi(z,t)$ is a C^3-smooth map in t. Therefore

$$\Phi(z(\cdot), t(\cdot)) : \mathbf{M} \to \tilde{\Omega}(t) \times [0,T]$$

$$\Phi(z(p), t(p)) = (x^1, x^2, x^3)$$

defines a global coordinate chart on $(\mathbf{M}, \mathbf{g})$ In this chart, the metric takes the form

$$\mathbf{g} = \mathbf{g}_{3\alpha} dx^3 dx^\alpha + \mathbf{g}_t,$$

where outside a compact set containing M, the functions $\mathbf{g}_{3k} = 0$ for $k = 1, 2$. In particular, the metric $\mathbf{g}$ restricted to M is written as

$$g = g_{3\alpha}dx^3dx^\alpha + e^{2\phi(t)}[(dx^1)^2 + (dx^2)^2].$$

Next we prove that for such a coordinate system $\Phi(z,t) = (x^1, x^2, x^3)$ on $(\mathbf{M}, \mathbf{g})$ as described above, knowledge of the area of any properly embedded minimal surface in (M, g) determines the conformal map $\Phi(\cdot, t)$ on the complement $\mathbb{R}^2 \setminus \Omega$, for every $t \in [0, T]$. To prove such a statement, we first show that knowledge of a Dirichlet-to-Neumann map for a non-degenerate Schrödinger operator on Ω determines the conformal map Φ satisfying (3.1) and (3.2) on the set $Z := (\mathbb{R}^2 \setminus \Omega) \cup \partial\Omega$ (see [3], [22], for similar results). Then, we prove that the knowledge of the area of any properly embedded minimal surface in (M, g) determines the Dirichlet-to-Neumann map associated to the stability operator on $Y(t)$.

In the proofs below, we will construct solutions to the Dirichlet problem for the Schrödinger operator which have exponential asymptotic behaviour in a weighted L^2 space. The particular weighted space we require has norm

$$||f||_{L^2_{-\delta}(Z)} = \int_Z |f(w)|^2(1 + |w|^2)^{-\delta}\, dw.$$

The following propositions also recall the construction of isothermal coordinates (see above) as well as existence and uniqueness for an exterior problem which will be crucial to the proofs of the theorems in this paper.

Proposition 3.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Set g to be a C^2-smooth Riemannian metric on Ω, and $\mathbf{g}$ to be a C^2-smooth extension of g to $\mathbb{R}^2$ with*

$\mathbf{g} = g_{\mathbb{E}}$ *outside a large compact set containing* Ω,

$\mathbf{g} = g$ *on* Ω,

where $g_{\mathbb{E}}$ is the Euclidean metric on $\mathbb{R}^2$. Write $Z := (\mathbb{R}^2 \setminus \Omega) \cup \partial\Omega$ and ν for the outward pointing unit normal vector field to $\partial\Omega$. Let (z^1, z^2) be coordinates on $\mathbb{R}^2$.

1. *For $p > 2$, there exists a unique conformal map $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying*

$$\Phi(z) - z \in L^p(\mathbb{R}^2) \tag{3.3}$$

$$\Phi_*(\mathbf{g}) = e^{2\phi(x)}[(dx^1)^2 + (dx^2)^2]. \tag{3.4}$$

2. *Let $V \in L^\infty(\Omega)$ and suppose 0 is not a Dirichlet eigenvalue of $\Delta_g + V$. Let $\Lambda : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$ be the Dirichlet-to-Neumann map associated to $\Delta_g\psi + V$.*

   Then there exists and $R_1 > 0$ such that for any $\xi \in \mathbb{C} \setminus \{0\}$ with $|\xi| > R_1$ and $\frac{1}{2} < \delta < 1$, there exists a unique solution $\psi(\cdot, \xi)$ to the exterior problem

$$\psi(\cdot, \xi) \in L^2_{loc}(Z) \text{ and } e^{-iz\xi}\psi(\cdot, \xi) - 1 \in L^2_{-\delta}(Z), \tag{3.5}$$

$$\Delta_g\psi(\cdot, \xi) = 0 \text{ on } Z, \tag{3.6}$$

$$g(\nabla\psi(\cdot, \xi), \nu) = \Lambda(\psi)(\cdot, \xi) \text{ on } \partial\Omega, . \tag{3.7}$$

   Moreover,

$$||e^{-i\Phi(z)\xi}\psi(\cdot, \xi) - 1||_{L^2_{-\delta}(Z)} < \frac{C}{|\xi|}, \tag{3.8}$$

   for some constant $C > 0$.

Proof. 1. This statement is proved in [1].

2. We will prove existence and uniqueness to (3.5), (3.6), (3.7) by transforming the problem into a Euclidean one via the map Φ.

First, note that for $z = z^1 + iz^2 \in \partial\Omega$, in the coordinates $\Phi(z) = x^1 + ix^2 \in \partial\Phi(\Omega)$ the Dirichlet-to-Neumann map Λ is

$$\begin{aligned}\Lambda(\psi)(z) &= g(\nabla\psi(z), \nu(z))|_{\partial\Omega} \\ &= e^{2\phi(x)} g_{\mathbb{E}}(e^{-\phi(x)}\tilde{\nabla}\psi \circ \Phi^{-1}(z), e^{-\phi(x)}\tilde{\nu}(x))|_{\partial\Phi(\Omega)} \\ &= g_{\mathbb{E}}(\tilde{\nabla}\tilde{\psi}(x), \tilde{\nu}(x))|_{\partial\Phi(\Omega)} \\ &=: \tilde{\Lambda}(\tilde{\psi})(x).\end{aligned}$$

Here $\psi \in H^{\frac{1}{2}}(\partial\Omega)$, $\tilde{\psi}(x) := \psi \circ \Phi^{-1}(x^1 + ix^2)$, and $\tilde{\nu}$ is the outward pointing unit normal vector field to $\partial\Phi(\Omega)$, with respect to the metric $g_{\mathbb{E}}$.

The boundary value problem (3.6), (3.7) expressed in the conformal coordinates given by Φ becomes

$$\Delta_{g_{\mathbb{E}}}\tilde{\psi}(\cdot, \xi) = 0 \text{ on } \Phi(Z), \tag{3.9}$$

$$g_{\mathbb{E}}(\tilde{\nabla}\tilde{\psi}(\cdot, \xi), \tilde{\nu}) = \tilde{\Lambda}(\tilde{\psi}(\cdot, \xi)) \text{ on } \Phi(\partial\Omega). \tag{3.10}$$

We claim condition (3.5) becomes

$$\tilde{\psi}(\cdot, \xi) \in L^2_{\text{loc}}(\Phi(Z)) \text{ and } e^{-ix\xi}\tilde{\psi}(\cdot, \xi) - 1 \in L^2_{-\delta}(\Phi(Z)). \tag{3.11}$$

Indeed, if $\psi(\cdot, \xi) \in L^2_{\text{loc}}(Z)$, then $\tilde{\psi}(\cdot, \xi) \in L^2_{\text{loc}}(\Phi(Z))$ follows from change of coordinates. For the second property in (3.5), first notice for $|z| > R_2 > 0$,

$$\begin{aligned}||e^{-ix\xi}\tilde{\psi}(\cdot, \xi) - 1||_{L^2_{-\delta}(\Phi(Z))} &= ||e^{-i[\Phi(z)-z+z]\xi}\psi(\cdot, \xi) - 1||_{L^2_{-\delta}(Z)} \\ &= ||[e^{-i[\Phi(z)-z]\xi} - 1][e^{-iz\xi}\psi(\cdot, \xi)] + e^{-iz\xi}\psi(\cdot, \xi) - 1||_{L^2_{-\delta}(Z)} \\ &\leq ||e^{-i[\Phi(z)-z]\xi} - 1||_{L^\infty}||e^{-iz\xi}\psi(\cdot, \xi) - 1||_{L^2_{-\delta}(Z)} \\ &\quad + ||e^{-i[\Phi(z)-z]\xi} - 1||_{L^2_{-\delta}(Z)} + ||e^{-iz\xi}\psi(\cdot, \xi) - 1||_{L^2_{-\delta}(Z)}\end{aligned}$$

We have $||e^{-iz\xi}\psi(\cdot,\xi)-1||_{L^2_{-\delta}(Z)}<\infty$; to prove $||e^{-ix\xi}\tilde{\psi}(\cdot,\xi)-1||_{L^2_{-\delta}(\Phi(Z))}<\infty$, it remains to show $||e^{-i[\Phi(z)-z]\xi}-1||_{L^2_{-\delta}(Z)}<\infty$.

Now, if $|z|>R_2>0$, for R_2 large, we have $\Delta_{g_{\mathbb{E}}}[\Phi(z)-z]^j=0$ $j=1,2$. So, the Mean Value Theorem applied to $\Phi(z)-z$ over the ball $B_z(|z|-R_2)$ and $\Phi(z)-z\in L^p(\mathbb{R})$ give

$$\begin{aligned}|\Phi(z)-z|^p &\leq \left(\frac{1}{\mathrm{Vol}(B_z(|z|-R_2))}\right)\int_{B_z(|z|-R_2)}|\Phi(z)-z|^p\,dz\\ &\leq C\frac{||\Phi(z)-z||^p_{L^p((B_z(|z|-R_2))}}{|z|-R_2}.\end{aligned}$$

Therefore, for $z>R_2$,

$$\begin{aligned}||e^{-i[\Phi(z)-z]\xi}-1||^p_{L^2_{-\delta}(B_z(|z|-R_2))} &\leq C(|\xi|)|\Phi(z)-z|^p\\ &\leq C(|\xi|)\left[\frac{||\Phi(z)-z||^p_{L^p((B_z(|z|-R_2))}}{|z|-R_2}\right]\\ &\leq C(|\xi|)\int_{R_2}^{\infty}\frac{1}{|z|}(1+|z|^2)^{-\delta}|z|\,d|z|,\end{aligned}$$

which is bounded since we take $\delta>\frac{1}{2}$. Hence

$$\tilde{\psi}(\cdot,\xi)\in L^2_{\mathrm{loc}}(\Phi(Z)) \text{ and } e^{-ix\xi}\tilde{\psi}(\cdot,\xi)-1\in L^2_{-\delta}(\Phi(Z)).$$

Conversely, notive that by the same argument as above, if (3.11) holds, then (3.8) holds.

Now we construct solutions to (3.11), (3.9), and (3.10). Consider $\tilde{\psi}(\cdot,\xi):\mathbb{R}^2\to\mathbb{R}$ defined by the integral equation

$$\tilde{\psi}(x,\xi)=e^{i(x^1+ix^2)\xi}-G_\xi(\tilde{V}\tilde{\psi}_1(\cdot,\xi)), \tag{3.12}$$

where

$$G_\xi(w) = \frac{e^{i\xi(w^1+iw^2)}}{4\pi^2} \int_{\mathbb{R}^2} \frac{e^{iw\cdot\zeta}}{|\zeta|^2 + 2\xi(\zeta^1 + i\zeta^2)}\, d\zeta$$

is Faddeev's Green function (see [20],[16], [8]), and $\tilde{V} := e^{2\phi}(V \circ \Phi^{-1})$ in $\Phi(\Omega)$ and is extended to be zero outside. Equation (3.12) is known to have unique solutions $\tilde{\psi}(x,\xi)$ with $e^{-i(x^1+ix^2)\xi}\tilde{\psi}(x,\xi) - 1 \in L^2_{-\delta}(Z)$, $0 < \delta < 1$, $|\xi| > R_1$ and satisfying (3.9) and (3.10).

Furthermore, the solutions $\tilde{\psi}(\cdot,\xi)$ obey the estimate

$$||e^{-i(x^1+ix^2)\xi}\tilde{\psi}_1(x,\xi) - 1||_{L^2_{-\delta}(\Phi(Z))} \leq \frac{C}{|\xi|}. \tag{3.13}$$

Consider the pullback $\psi(z,\xi) := \tilde{\psi}(\Phi_k(z),\xi)$. By construction the functions $\psi(z,\xi)$ satisfy the exterior problem (3.5), (3.6), (3.7). In addition, from the estimate (3.13) for $\tilde{\psi}(x,\xi)$, the estimate (3.8) holds for $\psi(z,\xi)$. This proves existence. Uniqueness for ψ follows immeadiately from that of $\tilde{\psi}$.

□

Proposition 3.2. *Let (z^1, z^2) be coordinates on $\mathbb{R}^2$, and $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary $\partial\Omega$. Set g_1, g_2 to be two C^2-smooth Riemannian metrics on Ω.*

For $k = 1,2$, let $\mathbf{g}_k$, Φ_k, $V_k \in L^\infty(\Omega)$ be as in Proposition 3.1. Define by Λ_k, $k = 1,2$, the Dirichlet-to-Neumann maps associated to $\Delta_{g_k}\psi_k + V_k\psi_k$ in Ω.

Then, if $\Lambda_1 = \Lambda_2$, the conformal maps $\Phi_1(z) = x^1 + ix^2$, $\Phi_2(z) = y^1 + iy^2$ are equal on the exterior set $Z := (\mathbb{R}^2 \setminus \Omega) \cup \partial\Omega$. In particular, the Dirichlet-to-Neumann maps determine the domain $\Phi_1(\Omega) = \Phi_2(\Omega)$.

Proof. Extend V_k, $k = 1,2$ to all of $\mathbb{R}^2$ such that $V_k = 0$ outside a compact set containing Ω, and $V_1 = V_2$ is known on $\mathbb{R}^2 \setminus \Omega$.

Let $\xi \in \mathbb{C} \setminus \{0\}$ and $\frac{1}{2} < \delta < 1$. For $k = 1, 2$, consider the exterior problems

$$\psi_k(\cdot, \xi) \in L^2_{\text{loc}}(Z) \text{ and } e^{-iz\xi}\psi_k(\cdot, \xi) - 1 \in L^2_{-\delta}(Z), \tag{3.14}$$

$$\Delta_{g_k}\psi_k(\cdot, \xi) + V_k\psi_k(\cdot, \xi) = 0 \text{ on } Z \tag{3.15}$$

$$g_k(\nabla\psi_k(\cdot, \xi), \nu) = \Lambda_k\psi_k(\cdot, \xi) \text{ on } \partial\Omega, \tag{3.16}$$

where ν is the outward pointing unit normal vector field to $\partial\Omega$.

By Proposition 3.1, there exists a unique family of solutions $\psi_k(\cdot, \xi)$ to (3.14), (3.15), (3.16) which additionally satisfy

$$||e^{-i\Phi_k(z)\xi}\psi_k(\cdot, \xi) - 1||_{L^2_{-\delta}(Z)} \leq \frac{C}{|\xi|}. \tag{3.17}$$

Since we imposed $V_1 = V_2$ on Z, and from the assumption that $\Lambda_1 = \Lambda_2$, $\psi_1(z.\xi)$ solves the same problem as $\psi_2(z, \xi)$ on Z. Proposition 3.1 gives uniqueness of the solutions $\psi_k(z, \xi)$ to the exterior problems (3.14), (3.15), (3.16); thus $\psi_1(z, \xi) = \psi_2(z, \xi)$ on Z. we now show that this together with (3.17) implies $\Phi_1 = \Phi_2$ on Z.

Write $\psi(z, \xi) := \psi_1(z, \xi) = \psi_2(z, \xi)$. From the estimates on $\psi(z, \xi)$, we have

$$\frac{2C}{|\xi|} \geq \left|\left|e^{-i\Phi_2(z)\xi}\psi(\cdot, \xi) - e^{-i\Phi_1(z)\xi}\psi(\cdot, \xi)\right|\right|_{L^2_{-\delta}(Z)} \tag{3.18}$$

$$= \left|\left|\left[e^{i(\Phi_1 - \Phi_2)(z)\xi} - 1\right] e^{-i\Phi_1(z)\xi}\psi(\cdot, \xi)\right|\right|_{L^2_{-\delta}(Z)}. \tag{3.19}$$

Using the above estimate and proof by contradiction, we show $\Phi_1(z) = \Phi_2(z)$ for $z \in Z$.

Suppose $|\Phi_1(z_0) - \Phi_2(z_0)| > 0$ for some $z_0 \in Z$. Without loss of genreality, we may assume that $\text{Re}(\Phi_1(z_0) - \Phi_2(z_0)) > 0$. By construction of Φ_j, $j = 1, 2$, there exists $\epsilon > 0$

and $c > 0$ such that

$$\mathrm{Re}(\Phi_1(z) - \Phi_2(z)) > c$$

for $z \in B_{z_0}(\epsilon)$.

Consider $\xi = 0 + i\xi^2$, for $\xi^2 < 0$. We find

$$\begin{aligned} \left|e^{i(\Phi_1-\Phi_2)(z)\xi} - 1\right| &\geq \left|\left|e^{i(\Phi_1-\Phi_2)(z)\xi}\right| - 1\right| \\ &\geq \left|e^{-\mathrm{Re}(\Phi_1(z_0)-\Phi_2(z_0))\xi^2} - 1\right| \\ &\geq \left|e^{-c\xi^2} - 1\right|, \end{aligned}$$

for all $z \in B_{z_0}(\epsilon)$.

Taking $\xi^2 \to -\infty$, we have $\left|e^{i(\Phi_1-\Phi_2)(z)\xi} - 1\right| \to \infty$. This violates (3.19), since the right hand side goes to zero as $|\xi| \to \infty$.

Therefore, $|\Phi_1(z) - \Phi_2(z)| = 0$ on Z. This completes the proof.

□

Recall that for a properly embedded minimal surface $\Omega \hookrightarrow M$, we defined the stability operator as $\Delta_{g_Y} + \left(\mathrm{Ric}_g(\vec{n}, \vec{n}) + ||A||_g^2\right)$ (see (2.11)). Then,

Definition 3.3. The **Dirichlet-to-Neumann map** associated to the stability operator is the map

$$\Lambda_{g_{Y_\gamma}} : H^{\frac{1}{2}}(\partial Y_\gamma) \to H^{-\frac{1}{2}}(\partial Y_\gamma)$$

$$\Lambda_{g_{Y_\gamma}}(\psi) := g_{Y_\gamma}(\nabla\psi, \nu)|_{\partial Y_\gamma},$$

where ν is the outward pointing normal vector field to the boundary ∂Y_γ and g_{Y_γ} is the metric induced on Y_γ by g.

A direct consequence of Proposition 3.4 is thus

Proposition 3.4. *Let g_1, g_2 be two C^3-smooth metrics on M, and $\Omega \subset \mathbb{R}^2$ be a bounded domain. Suppose $g_1 = g_2$ on ∂M. Let $\Omega \hookrightarrow Y_k \subset M$ for $k = 1, 2$ be properly embedded minimal surfaces in (M, g_k) with $\partial Y_1 = \partial Y_2$.*

As defined earlier, let $\mathbf{g}_k$ be the extensions of g_k to an asymptotically flat 3-manifold $\mathbf{M} \supset M$. Set Φ_k to be the unique conformal maps inducing isothermal coordinates on the extensions of Y_k.

Consider the Dirichlet-to-Neumann map Λ_{g_k} associated to the stability operator

$$\Delta_{g_k|_{Y_k}} + Ric_{g_k}(\vec{n}_k, \vec{n}_k) + ||A_{Y_k}||_{g_k},$$

$k = 1, 2$. If $\Lambda_{g_1} = \Lambda_{g_2}$, then $\Phi_1 = \Phi_2$ on $(\mathbb{R}^2 \setminus \Omega) \cup \partial\Omega$.

Key to all subsequent proofs in this paper, we now show that our minimal area data determines the Dirichlet-to-Neumann map associated to the stability operator on an area minimizing surface with boundary.

3.0.2 Areas of Area-Minimizers give Dirichlet-to-Neumann Data

Proposition 3.5. *Let (M, g) be a C^3-smooth, compact, Riemannian 3-manifold with boundary ∂M. Let $g|_{\partial M}$ be given. Suppose that*

1. *M admits properly embedded, area-minimizing foliations*
2. *given a properly embedded, area-minimizing, codimension 1 submanifold $Y \subset M$, the area of Y and any nearby perturbation of Y by area-minimizing surfaces is known.*

Then, the first variations of the area of Y determine the angle at which Y cuts the boundary of M.

Proof. Choose an embedded closed curve γ on ∂M, and set Y to be an area-minimizing hypersurface with boundary γ. Let $\gamma(t, x) : [0, T] \times \gamma \to \partial M$ be a one parameter variation of γ by simple closed curves. Denote by $Y(t)$ the area minimizing surface circumscribed by $\gamma(t)$, and $A(t)$ the area of $Y(t)$. Write $X_0 := \frac{\partial}{\partial t}\big|_{t=0}$. Then, as computed in Appendix, the first variation in the area of Y is

$$\frac{\partial}{\partial t}A(t)\bigg|_{t=0} = \int_Y g(X_0, H)\, d\mathrm{Vol}_{g_Y} + \int_{\partial Y} g(X_0, \nu)\, dS,$$

where g_Y is the metric restricted to Y and ν is the unit outward pointing normal vector to ∂Y and tangent to Y. Since we have assumed knowledge of the area of any minimal perturbation of Y, we know $\frac{\partial}{\partial t}A(t)\big|_{t=0}$; further, Y is minimal implies

$$\frac{\partial}{\partial t}A(t)\bigg|_{t=0} = \int_{\partial Y} g(X_0, \nu)\, dS.$$

By choosing different variations of γ, we thus determine $g(X_0, \nu)$ on $\partial Y := \gamma$.

Note: the above argument holds for arbitrary dimension.

□

Proposition 3.6. *Let (M, g) be a C^3 smooth, compact, 3-dimensional Riemannian manifold with strictly mean convex boundary ∂M, and suppose that (M, g) admits a foliation by properly embedded, area-minimizing surfaces. Let $g|_{\partial M}$ be given.*

Let $\gamma(t)$ be a 1-parameter family of simple closed curves which foliate ∂M, and let $Y(t)$ be the area-minimizing leaves of the foliation induced on M by solving the least-area problem for $\gamma(t)$. Suppose that for each t, the area of $Y(t)$ and any nearby perturbation of $Y(t)$ by area-minimizing surfaces is known.

Then, this data determines the Dirichlet-to-Neumann map associated to the stability operator on $Y(t)$.

Proof. We discover information about the Dirichlet-to-Neumann map associated to the stability operator on each $Y(t) \subset M$ by considering normal variations of $Y(t)$. Such variations need not arise as variations of $\gamma(t)$ on the boundary of M, hence we smoothly extend M and work with this extension. Let N to be a tubular neighbourhood of ∂M; let $\tilde{M} := N \cup M$ and extend g to a $C^3(\tilde{M})$ smooth metric $\tilde{g}$ on $\tilde{M}$ and consider the Riemannian manifold $(\tilde{M}, \tilde{g})$.

For $\epsilon \in [0, 1]$ and fixed $t \in [0, T]$, we may select a 1-parameter family of simple closed curves in $C^3(\tilde{M})$: $\tilde{\gamma}(t, \epsilon) \subset (\tilde{M} \setminus M)$, which satisfy $\tilde{\gamma}(t, 0) = \gamma(t)$, $\tilde{\gamma}(t, 1) \subset (\tilde{M} \setminus M)$, and $\tilde{\gamma}_{t,\epsilon}$ is $C^3(\tilde{M})$ close to $\gamma(t)$. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, and let the maps

$$\tilde{f}_{t,\epsilon} : \Omega \hookrightarrow \tilde{M}$$

$$\tilde{f}_{t,\epsilon}\Big|_{\partial\Omega} = \tilde{\gamma}(t, \epsilon)$$

be embeddings of Ω into $\tilde{M}$ such that $\tilde{Y}(t, \epsilon) := \tilde{f}_{t,\epsilon}(\Omega)$ minimizes the area bounded by $\tilde{\gamma}_{(}t, \epsilon)$. Since $(\tilde{M}, \tilde{g})$ has mean convex boundary, the embeddings $\tilde{f}_{t,\epsilon}$ exist. [9]. Further, $\tilde{f}_{t,\epsilon}$ is $C^3(\tilde{M})$ close to $\tilde{f}_{t,0}$.

Denote by $\vec{n}_{t,\epsilon}$ the unit normal vector field on the surface $\tilde{Y}(t, \epsilon)$. For $s \in [0, 1]$, define $\tilde{\gamma}(s, t, \epsilon)$ to be a normal variation of $\tilde{\gamma}(t, \epsilon)$; that is, $\tilde{\gamma}(0, t, \epsilon) = \tilde{\gamma}(t, \epsilon)$ and $\frac{d}{ds}\big|_{s=0} \tilde{\gamma}(s, t, \epsilon)$ is parallel to $\vec{n}_{t,\epsilon}$.

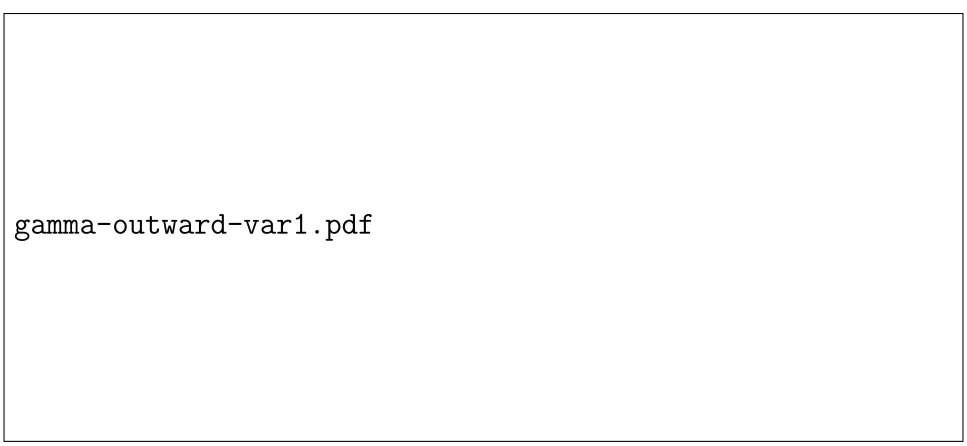

Figure 3.1: Depiction of the curves $\gamma(t)$ and $\tilde{\gamma}(t, \epsilon)$.

For each t, ϵ, let $\tilde{f}_{s,t,\epsilon} : \Omega \hookrightarrow \tilde{M}$ be an embedding of Ω into $\tilde{M}$ with the properties

$$\tilde{f}_{s,t,\epsilon}\Big|_{\partial\Omega} = \tilde{\gamma}(s,t,\epsilon),$$

$$\tilde{f}_{s,t,\epsilon}(\Omega) = \tilde{Y}(s,t,\epsilon) \text{ solves the least area problem for } \tilde{\gamma}(s,t,\epsilon),$$

$$\frac{d}{ds}\tilde{f}_{s,t,\epsilon}\Big|_{s=0} = \psi_{t,\epsilon}\vec{n}_{t\epsilon},$$

where $\psi_{t,\epsilon} : \tilde{Y}(t,\epsilon) \to \mathbb{R}$ is a $C^3(\tilde{M})$ smooth function which solves the boundary value problem

$$\Delta_{\tilde{g}_{t,\epsilon}}\psi_{t,\epsilon} + \left(\mathrm{Ric}_{\tilde{g}}(\vec{n}_{t,\epsilon}, \vec{n}_{t,\epsilon}) + ||A_{t,\epsilon}||^2_{\tilde{g}}\right)\psi_{t,\epsilon} = 0, \qquad \text{on } \tilde{Y}(t,\epsilon) \tag{3.20}$$

$$\psi_{t,\epsilon}|_{\partial\tilde{Y}(t,\epsilon)} = \psi^\sharp_{t,\epsilon}, \qquad \text{on } \partial\tilde{Y}(t,\epsilon), \tag{3.21}$$

for prescribed boundary data $\psi^\sharp_{t,\epsilon} := g\left(V, \vec{n}_{t,\epsilon}\right)$, $V := \frac{d}{ds}\tilde{\gamma}_{s,t,\epsilon}\big|_{s=0}$. Here $\tilde{g}_{t,\epsilon}$ is the metric $\tilde{g}$ restricted to $\tilde{Y}(t,\epsilon)$ and $A_{t,\epsilon}$ is the second fundamental form of $\tilde{Y}_{t,\epsilon}$.

We know the metric on $(\tilde{M} \setminus M) \cup \partial M$. Therefore, by the following lemma (Lemma 3.7), we determine the intersection of $\tilde{Y}(s,t,\epsilon) \cap \partial M$. In particular, the area of $\tilde{Y}(s,t,\epsilon)$, denoted by $\mathrm{Area}(s,t,\epsilon)$, is known. For each t, ϵ, the second variation in area is thus

$$\begin{aligned}
\frac{\partial^2}{\partial s^2}\mathrm{Area}(s,t,\epsilon)\Big|_{s=0} &= \int_{\partial\tilde{Y}(t,\epsilon)} \psi_{t,\epsilon}\tilde{g}(\nabla\psi_{t,\epsilon}, \nu_{t,\epsilon}) + \tilde{g}(\nabla_V V, \nu_{t,\epsilon})\, dS \\
&\quad - \int_{\tilde{Y}(t,\epsilon)} \psi_{t,\epsilon}\Delta_{\tilde{g}_{t,\epsilon}}\psi_{t,\epsilon} + \psi^2_{t,\epsilon}\left(\mathrm{Ric}_{\tilde{g}}(\vec{n},\vec{n}) + ||A||^2_{\tilde{g}}\right)\, d\mathrm{Vol}_{\tilde{g}_{\tilde{Y}(t)}} \\
&= \int_{\partial\tilde{Y}(t,\epsilon)} \psi_{t,\epsilon}\tilde{g}(\nabla\psi_{t,\epsilon}, \nu_{t,\epsilon})\, dS + \int_{\partial\tilde{Y}(t,\epsilon)} \tilde{g}(\nabla_V V, \nu_{t,\epsilon})\, dS,
\end{aligned}$$

where $\nu_{t,\epsilon}$ is the outward pointing normal to $\partial\tilde{Y}(t,\epsilon)$ and tangent to $\tilde{Y}(t,\epsilon)$.

Since the metric $\tilde{g}_{t,\epsilon}$ is given on $\tilde{M} \setminus M$ and V is known on $\tilde{\gamma}(t,\epsilon)$, and since $\psi_{t,\epsilon}$ solves $\Delta_{\tilde{g}_{t,\epsilon}}\psi_{t,\epsilon} + \left(\mathrm{Ric}_{\tilde{g}}(\vec{n},\vec{n}) + ||A_{t,\epsilon}||^2_{\tilde{g}}\right)\psi_{t,\epsilon} = 0$ on $\tilde{Y}(t,\epsilon) \setminus (M \cap \tilde{Y}(t,\epsilon))$, the boundary integral $\int_{\partial\tilde{Y}(t,\epsilon)} \tilde{g}(\nabla_V V, \nu_{t,\epsilon})\, dS$ is determined.

Therefore, we obtain knowledge of the term

$$\int_{\partial\tilde{Y}(t,\epsilon)} \psi_{t,\epsilon}\tilde{g}(\nabla\psi_{t,\epsilon},\nu_{t,\epsilon})\,dS = \frac{\partial^2}{\partial s^2}\text{Area}(s,t,\epsilon)\bigg|_{s=0} + \text{known quantity}. \tag{3.22}$$

Then, by polarizing, our area data has determined the functional

$$L(\phi^\sharp,\psi^\sharp) := \int_{\partial\tilde{Y}(t,\epsilon)} \phi g(\nabla\psi,\nu_{t,\epsilon})\,dS$$

for any functions $\phi^\sharp,\psi^\sharp : \partial\tilde{Y}(t,\epsilon) \to \mathbb{R} \in C^2(\tilde{Y}(t,\epsilon))$.

In particular, we have learned the Dirichlet-to-Neumann map associated to equation (3.21):

$$\Lambda_{t,\epsilon}(\psi^\sharp) := \tilde{g}(\nabla\psi,\nu_{t,\epsilon})|_{\partial\tilde{Y}(t,\epsilon)}.$$

We remark that since the operator $\mathcal{J}_{t,\epsilon} := \Delta_{\tilde{g}_{t,\epsilon}} + \text{Ric}_{\tilde{g}}(\vec{n}_{t,\epsilon},\vec{n}_{t,\epsilon}) + ||A_{t,\epsilon}||^2_{\tilde{g}}$ is non-degenerate for $\epsilon = 0$, it is non-degenerate for $\epsilon > 0$. Indeed, the area of $\tilde{Y}(t,\epsilon)$ increases as ϵ increases, thus since the eigenvalues of $\mathcal{J}_{t,0}$ are bounded below by a positive value, so are the eigenvalues of $\mathcal{J}_{t,\epsilon}$. Hence the Dirichlet-to-Neumann map $\Lambda_{t,\epsilon}$ is injective for each t,ϵ.

Now since the surfaces $\tilde{Y}(t,\epsilon)$ are $C^3(\tilde{M})$ close to $\tilde{Y}(t)$, as $\epsilon \to 0$ the component functions of the metrics $\tilde{g}_{t,\epsilon}$ tend to those of $\tilde{g}_{t,0}$ in the $C^3(\tilde{M})$ norm. Also, for each t, the potentials $\big(\text{Ric}_{\tilde{g}}(\vec{n}_{t,\epsilon},\vec{n}_{t,\epsilon}) + ||A_{t,\epsilon}||^2_{\tilde{g}}\big)$ converge to $\big(\text{Ric}_{\tilde{g}}(\vec{n}_{t,0},\vec{n}_{t,0}) + ||A_{t,\epsilon}||^2_{\tilde{g}}\big)$ in $C^1(\tilde{M})$ as $\epsilon \to 0$. Finally, since each $\psi_{t,\epsilon}$ depends continuously on $g_{t,\epsilon}$ and $\big(\text{Ric}_{\tilde{g}}(\vec{n}_{t,\epsilon},\vec{n}_{t,\epsilon}) + ||A_{t,\epsilon}||^2_{\tilde{g}}\big)$, the functions $\psi_{t,\epsilon}$ converge to $\psi_{t,0}$ in $C^3(\tilde{M})$. Take the limit as $\epsilon \to 0$ of (3.22). Since $Y(t) =: Y(t,0)$, on the original leaf $Y(t)$ we learn

$$\int_{\partial\tilde{Y}(t)} \psi_{t,0}\tilde{g}(\nabla\psi_t,\nu_t)\,dS = \frac{\partial^2}{\partial s^2}\text{Area}(s,t,0)\bigg|_{s=0} + \text{known quantity},$$

and thus determine the Dirichlet-to-Neumann map associated to the stability operator on $Y(t)$:

$$\Lambda_t(\psi^\sharp) := \tilde{g}(\nabla\psi,\nu_t)|_{\partial\tilde{Y}(t)}.$$

Now we prove our assumption about the boundaries and areas of $\tilde{Y}(t,\epsilon)$.

Lemma 3.7. *Let (M,g) be a C^3 smooth, compact, 3-dimensional Riemannian manifold with strictly mean convex boundary ∂M, and suppose that (M,g) admits a foliation by properly embedded, area-minimizing surfaces. Let $(\tilde{M},\tilde{g})$ be a smooth extension of (M,g) such that $\tilde{g}|_M = g$, $\tilde{g}$ is known on $(\tilde{M}\setminus M)\cup\partial M$, and $\partial\tilde{M}$ is strictly convex.*

Let $\gamma(t)$ be a given 1-parameter family of simple closed curves which foliate ∂M, and let $Y(t)$ be the area-minimizing leaves of the foliation induced on M by solving the least-area problem for $\gamma(t)$. Suppose that for each t, the area of $Y(t)$ and any nearby perturbation of $Y(t)$ by area-minimizing surfaces is known.

For each fixed t, choose $\tilde{\gamma}(t,\epsilon)$, $\epsilon\in[0,1]$ to be a family of simple closed curves which lie in $\tilde{M}\setminus(M\cup\partial M)$, and satisfy $\tilde{\gamma}(t,0)=\gamma(t)$ and $\tilde{\gamma}(t,\epsilon)$ is $C^3(\tilde{M})$ close to $\gamma(t)$. Define $Y(t)$ and $\tilde{Y}(t,\epsilon)$ be the surface of least area which bound $\gamma(t)$ and $\tilde{\gamma}(t,\epsilon)$, respectively.

Then,

a. We know the closed curve cut by the intersection $c(t,\epsilon) := \tilde{Y}(t,\epsilon)\cap\partial M$.

b. We know the area of $\tilde{Y}(t,\epsilon)$, with respect to $\tilde{g}$.

Proof. a. Let $c(t,\epsilon)$ the curve cut by $\tilde{Y}(t,\epsilon)\cap\partial M$. Consider the set Σ of all simple closed curves on ∂M which are $C^3(\tilde{M})$ close to $c(t,\epsilon)$. For any curve $\sigma\in\Sigma$, denote by $Y_\sigma\subset M$ the surface which minimizes the area enclosed by σ.

Given any $\sigma\in\Sigma$ the first variations in the area of Y_σ determine the angle at which Y_σ cut the boundary of M (see Proposition 3.5). Thus, we may determine the outward pointing unit normal vector fields ν_σ which are tangent to Y_σ, and normal to the curve σ. Let A_σ to be the area-minimizing annulus which lies between σ and $\tilde{\gamma}(t,\epsilon)$.

The metric $\tilde{g}$ is known on $(\tilde{M}\setminus M)\cup\partial M$, so for any annulus A_σ as above, we can determine the inward pointing unit normal vector field $\tilde{\nu}_\sigma$ tangent to A_σ and normal to the curve σ.

Consider the curve $\sigma^\sharp \in \Sigma$ such that $A_{\sigma^\sharp}$ is the minimal annulus for which $\nu_{\sigma^\sharp}$ and $\tilde{\nu}_{\sigma^\sharp}$ are collinear on ∂M. Note for any $p \in \sigma^\sharp$, the tangent space $T_p A_{\sigma^\sharp}$ coincides with the tangent space $T_p Y_{\sigma^\sharp}$, since they are both spanned by $\nu_{\sigma^\sharp}$ and any vector tangent to $\sigma^\sharp$. Hence, $A_{\sigma^\sharp} \cup Y_{\sigma^\sharp}$ is a $C^1(\tilde{M})$ surface which minimizes area bounded by $\tilde{\gamma}(t,\epsilon)$. We claim that $A_{\sigma^\sharp} \cup Y_{\sigma^\sharp}$ is in fact a smooth minimal surface and further $A_{\sigma^\sharp} \cup Y_{\sigma^\sharp} \equiv \tilde{Y}(t,\epsilon)$.

To prove that $A_{\sigma^\sharp} \cup Y_{\sigma^\sharp}$ is smooth, we express it as a graph of a function z and show that the derivatives of z exist and are continuous. To this end, let $T_{\sigma^\sharp} \subset \tilde{M}$ be the surface obtained by following geodesics $c_\theta(r)$ with $\theta \in \sigma$ and initial direction $\frac{\partial}{\partial r} c_\theta(0) = \nu_{\sigma^\sharp}(\theta)$; that is $T_{\sigma^\sharp} := \{p \in \tilde{M} \,:\, p = c_\theta(r), \text{for some } r \geq 0,\, \theta \in \sigma\}$.

Express $T_{\sigma^\sharp}$ in the natural coordinate system (r,θ). View $A_{\sigma^\sharp}$ as a graph of a function $z = z(r,\theta)$ over $T_{\sigma^\sharp}$. Since $A_{\sigma^\sharp}$ is smooth away from $r = 0$, to show $A_{\sigma^\sharp} \cup Y_{\sigma^\sharp}$ is smooth, we need only show that the derivatives of z at $r = 0$ are continuous. Actually, we need only show that the second order derivatives of z are continuous. This follows from the fact the surface $A_{\sigma^\sharp}$ is minimal, hence $z = z(r,\theta)$ solves the minimal surface equation

$$\operatorname{div}_g \left(\frac{\nabla z}{||\nabla z||_g} \right) = 0;$$

thus by elliptic regularity, it suffices for us to show that $A_{\sigma^\sharp} \cup Y_{\sigma^\sharp}$ is $C^2(\tilde{M})$ at the join σ.

Choose $w : \tilde{M} \to \mathbb{R}$ such that (r,θ,w) form a local coordinate system near $T_{\sigma^\sharp}$. Since $A_{\sigma^\sharp}$ agrees with $T_{\sigma^\sharp}$ on $\sigma^\sharp$ to first order, $z(0,\theta) = 0$ and $\partial_r z(0,\theta) = \partial_\theta z(0,\theta) = 0$. The minimal surface equation written in our chosen coordinates is

$$\begin{aligned} 0 &= \operatorname{div}_g \left(\frac{\nabla z}{||\nabla z||_g} \right) \\ &= \operatorname{div}_g \left(\frac{Dz}{||dw^2 + \partial_r z dr + \partial_\theta z d\theta||_g} \right) \\ &= \frac{||dw^2 + \partial_r z dr + \partial_\theta z d\theta||_g \operatorname{div}_g(Dz) - g(Dz, g(DDz, Dz))}{||dw^2 + \partial_r z dr + \partial_\theta z d\theta||_g^3} \end{aligned}$$

where D denotes the restriction of ∇ to $T_{\sigma^\sharp}$. Substituting $r = 0$ into the above equation and using $z(0,\theta) = 0$, $\partial_r z(0,\theta) = \partial_\theta z(0,\theta) = 0$, we find $D_r D_r z(0,\theta) = 0$. So $z = z(r,\theta)$ is $C^2(\tilde{M})$ on $A_{\sigma^\sharp}$, up to and including $\sigma^\sharp$.

So we have shown that $A_{\sigma^\sharp} \cup Y_{\sigma^\sharp}$ is smooth.

To show that $A_{\sigma^\sharp} \cup Y_{\sigma^\sharp}$ is unique, it is enough to argue that $A_{\sigma^\sharp}$ is unique. Again represent $A_{\sigma^\sharp}$ by $z = z(r,\theta)$. If $z_2 = z_2(r,\theta)$ is the graph of any other minimal annulus with the same properties as $A_{\sigma^\sharp}$, then we must have $z = z(r,\theta)$ and $z_2 = z_2(r,\theta)$ agree to first order. The previous argument for the smoothness of $z = z(r,\theta)$ demonstrates that the second derivatives of $z = z(r,\theta)$ and $z_2 = z_2(r,\theta)$ also agree. Thus $z = z(r,\theta)$ and $z_2 = z_2(r,\theta)$ agree to all order.

Since $A_{\sigma^\sharp}$ is unique, $A_{\sigma^\sharp} \cup Y_{\sigma^\sharp} \equiv \tilde{Y}(t,\epsilon)$. Therefore, $\sigma^\sharp \equiv c(t,\epsilon)$.

b. From part a, for any t we may determine the curves $\gamma(t,\epsilon)$ cut by the intersection $\tilde{Y}(t,\epsilon) \cap \partial M$. In particular, we can find the area of the annulus $\tilde{Y}(t,\epsilon) \setminus (\tilde{Y}(t,\epsilon) \cap M)$.

We have

$$\text{Area}(\tilde{Y}(t,\epsilon)) = \text{Area}(\tilde{Y}(t,\epsilon) \setminus (\tilde{Y}(t,\epsilon) \cap M)) + \text{Area}(\tilde{Y}(t,\epsilon) \cap M).$$

Since the metric $\tilde{g}$ is known on $\tilde{Y}(t,\epsilon) \setminus (\tilde{Y}(t,\epsilon) \cap M)$, we may compute this area. Since we assumed knowledge of any minimal surface M, the area of $\tilde{Y}(t,\epsilon) \cap M$ in known. Therefore, $\text{Area}(\tilde{Y}(t,\epsilon))$ is known.

□

□

Proposition 1.1. *Let (M,g) be a C^3 smooth, compact, 3-dimensional Riemannian manifold with strictly mean convex boundary ∂M, and suppose that (M,g) admits a foliation by properly embedded, area-minimizing surfaces. Let g be given on ∂M. Let $\Omega \subset \mathbb{R}^2$ be a bounded, open set. Suppose that for any properly embedded, area-minimizing, codimension 1 submanifold $f : \Omega \hookrightarrow Y \subset M$, the area of Y and any minimal perturbation of Y is known.*

Equip a neighbourhood of Y with coordinates (x^α) such that on Y, $x^3 = 0$ and $f^{-1}(p) = (x^1, x^2) \in \Omega$ are conformal coordinates. Then,

1. *the first and second variations of the area of Y determine the Dirichlet-to-Neumann map associated to to the boundary value problem*

$$\Delta_{g_{\mathbb{E}}}\psi + e^{2\phi}\left(Ric_g(\vec{n},\vec{n}) + ||A||_g^2\right)\psi = 0,$$
$$\psi|_{\partial\Omega} = \psi_0,$$

   on Ω, where $e^{2\phi}g_{\mathbb{E}}$ is the metric g pulled back to Ω.

2. *the first and second variations of the area of Y determine the any solution $\psi(x)$ to the above boundary value problem.*

Proof. Let g_0 be the metric g restricted to Y. From Lemma 3.6, the minimal area data enables us to find the Dirichlet-to-Neumann map $\Lambda_g(\omega) := g(\nabla\omega, \nu)|_{\partial\Omega}$ associated to the boundary value problem for the stability operator

$$\Delta_{g_0}\omega + \left(\mathrm{Ric}_g(\vec{n},\vec{n}) + ||A||_g^2\right)\omega = 0 \qquad \text{on } \Omega, \tag{3.23}$$
$$\omega|_{\partial\Omega} = \omega_0 \qquad \text{on } \partial\Omega. \tag{3.24}$$

In our chosen coordinate system, the metric g_0 pulled back to Ω takes the form $g_0 = e^{2\phi}g_{\mathbb{E}}$. In these coordinates, the problem (3.23) is transformed to

$$\Delta_{g_{\mathbb{E}}}\psi + e^{2\phi}\left(\mathrm{Ric}_g(\vec{n},\vec{n}) + ||A||_g^2\right)\psi = 0 \qquad \text{on } \Omega, \tag{3.25}$$
$$\psi|_{\partial\Omega} = \psi_0 \qquad \text{on } \partial\Omega, \tag{3.26}$$

and the solutions ω of (3.23) are in one-to-one correspondence with the solutions ψ of (3.25).

The Dirichlet-to-Neumann map transforms as

$$\begin{aligned}\Lambda_g(\omega) &:= g(\nabla\omega, \nu)|_{\partial\Omega} \\ &= e^{2\phi} g_{\mathbb{E}}(e^{-\phi}\tilde{\nabla}\psi, e^{-\phi}\tilde{\nu})|_{\partial\Omega} \\ &= g_{\mathbb{E}}(\tilde{\nabla}\psi, \tilde{\nu})|_{\partial\Omega} \\ &= \Lambda_{g_{\mathbb{E}}}(\psi)\end{aligned}$$

where $\tilde{\nu}$ is the unit outward pointing normal with respect to the Euclidean metric $g_{\mathbb{E}}$, and $\tilde{\nabla}$ denotes the gradient of $\psi := \psi(x^1, x^2)$ with respect to the metric $g_{\mathbb{E}}$.

Thus, knowledge of the area of any minimal surface in M has determined the Dirichlet-to-Neumann map $\Lambda_{g_{\mathbb{E}}}$ associated to the Schrödinger equation in (3.25), with respect to the Euclidean metric.

Employing the result in [10] for linear Schrödinger equations, the Dirichlet-to-Neumann map $\Lambda_{g_{\mathbb{E}}}$ determines the potential

$$e^{2\phi}\left(\mathrm{Ric}_g(\vec{n}, \vec{n}) + ||A||_g^2\right)$$

on Ω. Now that we know this potential in coordinates (x^1, x^2), all solutions $\psi(x^1, x^2)$ to Dirichlet problem (3.25) are known.

□

Chapter 4

Equations for the Components of the Inverse Metric

Proposition 4.1. *Let (M, g) be a C^3 smooth, compact, 3-dimensional Riemannian manifold with strictly mean convex boundary ∂M, and suppose that (M, g) admits a foliation by properly embedded, area-minimizing surfaces. Let g be given on ∂M. Suppose that for any properly embedded, area-minimizing, codimension 1 submanifold $Y \subset M$, the area of Y and of any minimal perturbation of Y is known.*

Let $Y(t)$, $t \in [0, T]$, be a foliation of M by properly embedded, area-minimizing surfaces. As in section 3.0.1, extend (M, g) to an asymptotically flat manifold $(\mathbf{M}, \mathbf{g})$ and extend $Y(t)$ smoothly to $\mathbf{Y}(t)$ in $\mathbf{M}$. Equip $\mathbf{M}$ with coordinates (x^α), $\alpha = 1, 2, 3$ such that $x^3 = t$ and for each t fixed $x^1, x^2 : \mathbf{Y}(t) \to \mathbb{R}$ are the unique conformal coordinates given by Proposition 3.1.

In these coordinates, $g^{33} := ||\nabla x^3||_g^2$, may be recovered on M.

Proof. Recall $\nabla x^3 := \mathrm{grad}(x^3)$. Set $\vec{n} := \frac{\nabla x^3}{||\nabla x^3||_g}$ to be a unit normal vector field on $Y(t)$ for $t \in [0, T]$, and write g_t for the restriction of the metric g to the surface $Y(t)$.

For each fixed t, we may view the nearby leaves of the foliation $Y(t+\delta t)$ as a variation of $Y(t)$ by area-minimizing surfaces. From this viewpoint, the variation is defined by the vector field

$$\frac{\partial}{\partial x^3} := \partial_3.$$

The associated lapse function is

$$\begin{aligned} g(\partial_3, \vec{n}) &= g\left(\partial_3, \frac{\nabla x^3}{||\nabla x^3||_g}\right) \\ &= ||\nabla x^3||_g. \end{aligned}$$

Recall $x^k : Y(t) \to \mathbb{R}$, $k = 1, 2$, are conformal on $Y(t)$. Since the stability operator $\Delta_{g_t} + \left(\mathrm{Ric}_g(\vec{n}, \vec{n}) + ||A||_g^2\right)$ is non-degenerate for each t, $||\nabla x^3||_g : Y(t) \to \mathbb{R}$ is a nontrivial solution of the Jacobi equation

$$\Delta_{g_t}\omega + \left(\mathrm{Ric}_g(\vec{n}, \vec{n}) + ||A||_g^2\right)\omega = 0 \tag{4.1}$$

on $Y(t)$ (see appendix A.3).

Therefore, written in the coordinates (x^α), for $x^3 = t$ fixed, the function $||\nabla x^3||_g$ solves

$$\Delta_{g_{\mathbb{E}}}\psi + e^{2\phi(t)}\left(\mathrm{Ric}_g(\vec{n}, \vec{n}) + ||A||_g^2\right)\psi = 0 \tag{4.2}$$

on $Y(t)$, where the metric on $Y(t)$ is expressed as $g_t = e^{2\phi(t)}[(dx^1)^2 + (dx^2)^2] =: e^{2\phi(t)} g_{\mathbb{E}}$.

Now, we know $g|_{\partial M}$ and the curves $\partial Y(t) = (x^3)^{-1}(t) \cap \partial M$. Thus, the function $||\nabla x^3||_g$ on $\partial Y(t)$ is known. By Proposition 1.1, our minimal area data determines the lapse function $||\nabla x^3||_g$ on $Y(t)$, as it is expressed in the conformal coordinate system given by $(x^1, x^2, x^3 = t)$. Since we now know $||\nabla x^3||_g$ on $Y(t)$ for any $t \in [0, T]$, we have determined $||\nabla x^3||_g$ on M.

Thus, in the coordinates (x^α), we have

$$g^{33} := g(dx^3, dx^3) = ||\nabla x^3||_g^2,$$

hence the metric component g^{33} is known on M.

□

tilt-drawing1-2-3.jpg

Figure 4.1: Depiction of the leaves $Y(s,t)$.

Lemma 4.2. *Let (M, g) be a C^3 smooth, compact, 3-dimensional Riemannian manifold with strictly mean convex boundary ∂M, and suppose that (M, g) admits a foliation by properly embedded, area-minimizing surfaces. Let g be given on ∂M. Suppose that for any properly embedded, area-minimizing, codimension 1 submanifold $Y \subset M$, the area of Y and of any minimal perturbation of Y is known.*

Let $\gamma(t)$, $t \in [0, T]$, be a given foliation of ∂M by simple closed curves, and set $Y(t)$ to be a foliation of M by properly embedded, area-minimizing surfaces such that $\partial Y(t) = \gamma(t)$. Extend M and each $Y(t)$ to asymptotically flat manifolds $\mathbf{M}$ and $\mathbf{Y}(t)$ as defined in section 3.0.1. Further, set $(x^1, x^2) = \Phi(\cdot, t) : \mathbf{Y}(t) \to \mathbb{R}^2$ to be unique isothermal coordinates on $\mathbf{Y}(t)$ given by Proposition 3.1; write $\tilde{\Omega}(t) := \Phi(Y(t))$. Set $x^3 = t$.

Consider a point $p \in Y(t)$. Define $h : [0, S] \times \tilde{\Omega}(t) \to \mathbf{M}$, $h(s, x^1, x^2, t) =: Y(s,t)$ to be a variation of $Y(t) \subset \mathbf{M}$ by properly embedded, area-minimizing surfaces which has the *property that the component of* $h_*\left(\frac{\partial}{\partial s}\big|_{s=0}\right)$ *normal to* $Y(t)$*, denoted by* $\psi_p = \psi_p(x^1, x^2)$*, vanishes at* p*. Let* $(x_s^1, x_s^2) = \Phi(\cdot, s, t) : \mathbf{Y}(s,t) \to \mathbb{R}^2$ *are unique isothermal coordinates on the extended, new foliation* $\mathbf{Y}(s,t)$*, and* $x_s^3 = x^3 + \mathcal{O}(s^2)$*.*

Then, the linearization of $||\nabla x_s^3||_g$ *at the point* p *is*

$$\frac{d}{ds}||\nabla x_s^3||_g(p)\bigg|_{s=0} = g^{3\alpha}(p)\partial_\alpha\psi_p(x^1(p), x^2(p)) + \partial_k||\nabla x^3||_g(p)\dot{x}^k(p),$$

where $\dot{x}^k := \frac{d}{ds}x_s^k\big|_{s=0}$ *is the first variation in the coordinate functions* x_s^k *at* p*, for* $k = 1, 2$*.*

Proof. Without loss of generality, sett $x^3 = t = 0$. Let $\vec{n} := \frac{\nabla x^3}{||\nabla x^3||_g}$ denote the unit normal vector field to $Y(0)$.

Via Taylor expansion, the new coordinate functions (x_s^α), $\alpha = 1, 2, 3$ on $Y(s,t)$ in terms of the "original" coordinate functions (x^α) on $Y(t)$ are expressed as

$$x_s^\alpha = x^\alpha + s\dot{x}^\alpha + \mathcal{O}(s^2). \tag{4.3}$$

Then, linearizing $||\nabla x_s^3||_g^2(p)$ about $s = 0$,

$$\begin{aligned}
\frac{d}{ds}\bigg|_{s=0} ||\nabla x_s^3||_g^2(p) &= \frac{d}{ds}\bigg|_{s=0} \big[\big(||\nabla x^3||_g^2 + 2sg(\nabla x^3, \nabla \dot{x}^3)\big) \circ (x_s^1(p), x_s^2(p), x_s^3(p)) \\
&\quad + s^2||\nabla \dot{x}^3||_g^2 \circ (x_s^1(p), x_s^2(p), x_s^3(p))\big] \\
&= 2g(\nabla x^3, \nabla \dot{x}^3)(x^1(p), x^2(p), x^3(p)) \\
&\quad + \partial_\alpha ||\nabla x^3||_g^2(x^1(p), x^2(p), x^3(p))\dot{x}^\alpha(p) \\
&= 2g^{3\alpha}(p)\partial_\alpha[||\nabla x^3||_g \psi_p](x^1(p), x^2(p)) \\
&\quad + \partial_k ||\nabla x^3||_g^2(p)\dot{x}^k(p) \\
&= 2||\nabla x^3||_g(p) g^{3\alpha}(p)\partial_\alpha \psi_p(x^1(p), x^2(p)) + 0 + \partial_k||\nabla x^3||_g^2(p)\dot{x}^k(p) \\
&= 2||\nabla x^3||_g(p) g^{3\alpha}(p)\partial_\alpha \psi_p(x^1(p), x^2(p)) + \partial_k||\nabla x^3||_g^2(p)\dot{x}^k(p),
\end{aligned}$$

at the chosen point p.

□

It will serve our purposes to find an expression for $\dot{x}^k$ in terms of g^{13}, g^{23} and ϕ. The calculations for such an expression are carried out below.

Lemma 4.3. *Let (M, g), $\tilde{\Omega}(t)$, $Y(t)$, $Y(s,t)$, $p \in Y(t)$, and $\psi_p : \tilde{\Omega}(t) \to \mathbb{R}$ be as defined in Lemma 4.2. For $\alpha = 1, 2, 3$, let $x^\alpha : Y(t) \to \mathbb{R}$ and $x_s^\alpha : Y(s,t) \to \mathbb{R}$ be the isothermal coordinates on $Y(t)$ and $Y(s,t)$, as defined in Lemma 4.2, and write $\dot{x}^k : Y(t) \to \mathbb{R}$, $k = 1, 2$ for the first order change in the isothermal coordinates x_s^k.*

Then on $Y(t)$, the functions $\dot{x}^k$, $k = 1, 2$ are determined via a Poisson equation

$$\Delta_{g_{\mathbb{E}}} \dot{x}^k = \mathcal{F}^k(g^{13}, g^{23}, \phi, \psi_p, d\psi_p, p),$$

where $\phi = \phi(x^1, x^2, t)$ is the conformal factor on $Y(t)$ and $\mathcal{F}^k$ is a second order differential operator acting on g^{13}, g^{23}, and ϕ.

Proof. Without loss of generality, fix $t = 0$ and consider $Y(0)$. Write g_0 for the metric induced by g on $Y(0)$, and $g_{s,0} := g|_{Y(s,0)}$ for the metric induced on the leaves $Y(s,0)$. Recall from Lemma 4.2, we express the foliation $Y(s,0)$ as embeddings $h : [0,S] \times \tilde{\Omega}(0) \to \mathbf{M}$ into the extension $\mathbf{M}$ of M; that is, $h(s, x^1, x^2, 0) = Y(s,0)$.

The equation (4.3) which expresses x_s^k in terms of x^k is

$$x_s^k = x^k + s\dot{x}^k + \mathcal{O}(s^2).$$

Now, to compute how $\dot{x}^k$ depends on the components of the metric g, we linearize x_s^k in s.

The conformal coordinates (x_s^k) on the leaves $Y(s,0)$ are harmonic functions, and thus satisfy

$$\Delta_{g_{s,0}} x_s^1 = 0 = \Delta_{g_{s,0}} x_s^2.$$

Linearizing about $s = 0$ and noting $\partial_j x_s^k = (g_{\mathbb{E}})_j^k$, we derive

$$\begin{aligned}
0 &= \frac{d}{ds}[\Delta_{g_{s,0}} x_s^k] \\
&= \left[\frac{dg_{s,0}^{ij}}{ds}\partial_i\partial_j x_s^k - \frac{d}{ds}[g_{s,0}^{ij}\Gamma_{ij}^k(g_{s,0})] + \Delta_{g_{s,0}}\frac{d}{ds}x_s^k + \mathcal{O}(s)\right]_{s=0} \\
&= 0 - (\dot{g}_0)^{ij}\Gamma_{ij}^k(g_{0,0}) - g_{0,0}^{ij}\dot{\Gamma}_{ij}^k(g_{0,0}) + \Delta_{g_{0,0}}\frac{d}{ds}x_s^k|_{s=0} \\
&= -(\dot{g}_0)^{ij}\Gamma_{ij}^k(g_{0,0}) - \frac{1}{2}g_{0,0}^{ij}g_{0,0}^{kl}[\nabla_j(\dot{g}_0)_{il} + \nabla_i(\dot{g}_0)_{jl} - \nabla_l(\dot{g}_0)_{ij}] + \Delta_{g_{0,0}}\dot{x}^k \\
&= -(\dot{g}_0)^{ij}\Gamma_{ij}^k(g_{0,0}) - \frac{1}{2}[g_{0,0}^{ij}\nabla_j(g_{0,0}^{kl}(\dot{g}_0)_{il}) + g_{0,0}^{ij}\nabla_i(g_{0,0}^{kl}(\dot{g}_0)_{jl}) - g_{0,0}^{kl}\nabla_l(g_{0,0}^{ij}(\dot{g}_0)_{ij})] \\
&\qquad + \Delta_{g_{0,0}}\dot{x}^k \\
&= -\dot{g}^{ij}\Gamma_{ij}^k(g_{0,0}) - g_{0,0}^{ij}\nabla_j(g_{0,0}^{kl}(\dot{g}_0)_{il}) + \frac{1}{2}g_{0,0}^{kl}\nabla_l(g_{0,0}^{ij}(\dot{g}_0)_{ij}) + \Delta_{g_{0,0}}\dot{x}^k,
\end{aligned} \tag{4.4}$$

on $Y(0)$, for $k = 1, 2$.

To perform further analysis, we require the linearization $\dot{g}_0$ of the induced metric on the leaves $Y(s,0)$, as well as the Christoffel symbols associated to the metric $g_{0,0} := g_0$ on $Y(0)$.

In all computations which follow, let i,j,k,l,m,p sum over $1,2$ and α,β,γ sum over $1,2,3$. In the coordinates (x^α), $g_0 = e^{2\phi}g_{\mathbb{E}}$. Hence, the Christoffel symbols of g_0 are

$$\begin{aligned}\Gamma^k_{ij}(g_0) &= \Gamma^k_{ij}(e^{2\phi}g_{\mathbb{E}}) \\ &= \Gamma^k_{ij}(g_{\mathbb{E}}) + {g_{\mathbb{E}}}^k_i\partial_j\phi + {g_{\mathbb{E}}}^k_j\partial_i\phi - (g_{\mathbb{E}})_{ij}{g_{\mathbb{E}}}^{kl}\partial_l\phi \\ &= {g_{\mathbb{E}}}^k_i\partial_j\phi + {g_{\mathbb{E}}}^k_j\partial_i\phi - (g_{\mathbb{E}})_{ij}{g_{\mathbb{E}}}^{kl}\partial_l\phi.\end{aligned}$$

For ease of computation of the linearization $\dot{g}_0$, we employ Gaussian coordinates adapted to $Y(0)$: for $i=1,2$, define the coordinate vector fields $X_i := h(\cdot,s,0)_*\left(\frac{\partial}{\partial x^i}\right)$ and $X_s := h(\cdot,s,0)_*\left(\frac{\partial}{\partial s}\right)$. Then in these coordinates, the components of the metric $g_{s,0}$ induced on the leaves $Y(s,0)$ are given by $(g_{s,0})_{ij} := g(X_i,X_j)$.

Now Taylor expand $g_{s,0}$ in terms of s: $g_{s,0} = g_0 + s\dot{g}_0 + \mathcal{O}(s^2)$. Then,

$$\begin{aligned}(\dot{g}_0)_{ij} := \frac{d}{ds}(g_{s,0})_{ij}\bigg|_{s=0} &= \frac{d}{ds}g(X_i,X_j)\bigg|_{s=0} \\ &= [g(\nabla_{X_s}X_i,X_j) + g(X_i,\nabla_{X_s}X_j)]|_{s=0} \\ &= [g(\nabla_{X_i}X_s,X_j) + g(X_i,\nabla_{X_j}X_s)]|_{s=0} \\ &= g(\nabla_{X_i}(\psi_p\vec{n}),X_j) + g(X_i,\nabla_{X_j}(\psi_p\vec{n})) \\ &= g(X_i(\psi_p)\vec{n},X_j) + \psi_p g(\nabla_{X_i}\vec{n},X_j) \\ &\qquad + g(X_i,X_j(\psi_p)\vec{n}) + \psi_p g(X_i,\nabla_{X_j}\vec{n}) \\ &= \psi_p g(\nabla_{X_i}\vec{n},X_j) + \psi_p g(X_i,\nabla_{X_j}\vec{n}) \\ &= -2\psi_p A_{ij},\end{aligned}$$

where A_{ij} are the components of the second fundamental form of $Y(0)$. Thus, the first order change in $g_{s,0}$ is given by the coordinate free expression

$$\dot{g}_0 = -2\psi_p A. \tag{4.5}$$

Recall $g_{0,0} = g_0$, and substitute $\dot{g}_0 = -2\psi_p A$ into equation (4.4); the resulting PDE describes the first variation in s of the coordinates (x_s^i):

$$\begin{aligned}\Delta_{g_0}\dot{x}^k &= -2\psi_p A^{ij}\Gamma^k_{ij}(g_0) - 2g_0^{ij}\nabla_j(g_0^{kl}\psi_p A_{il}) + g_0^{kl}\nabla_l(g_0^{ij}\psi_p A^{ij}) \\ &= -2\psi_p A^{ij}\Gamma^k_{ij}(g_0) - 2g_0^{ij}\nabla_j(\psi_p A_i^k).\end{aligned} \tag{4.6}$$

Note the term $g_0^{kl}\nabla_l(g_0^{ij}\psi_p A^{ij})$ is zero since the surface $Y(0)$ is minimal.

We now expand each term in equation (4.6) in terms of the components of g and g^{-1} which we aim to to uniquely determine. To this end, a quick calculation gives that in the coordinates (x^α), a normal vector field to the leaves $Y(t)$ is

$$\vec{n} := \frac{\nabla u}{||\nabla x^3||_g} =: \frac{1}{||\nabla x^3||_g} g^{\alpha\beta}\partial_\beta x^3 \partial_\alpha = \frac{1}{||\nabla x^3||_g} g^{\alpha 3}\partial_\alpha.$$

Hence the components of the second fundamental form are

$$\begin{aligned}A_{ij} &:= -g(\nabla_{\partial_i}\partial_j, \vec{n}) \\ &= -\frac{1}{||\nabla x^3||_g}\Gamma^3_{ij}(g) \\ &= -\frac{1}{2||\nabla x^3||_g} g^{3\alpha}(\partial_i g_{\alpha j} + \partial_j g_{i\alpha} - \partial_\alpha g_{ij}).\end{aligned}$$

Raising an index and noting $-g_{\alpha j}\partial_i g^{3\alpha} = \partial_i g_{\alpha j} g^{3\alpha}$ then gives

$$\begin{aligned}A_i^k &= g^{jk}\frac{1}{2||\nabla x^3||_g} g^{3\alpha}(\partial_i g_{\alpha j} + \partial_j g_{i\alpha} - \partial_\alpha g_{ij}) \\ &= -\frac{e^{-2\phi}(g_{\mathbb{E}})^{jk}}{2||\nabla x^3||_g}(g_{\alpha j}\partial_i g^{3\alpha} + g_{i\alpha}\partial_j g^{3\alpha} + g^{3\alpha}\partial_\alpha g_{ij}).\end{aligned} \tag{4.7}$$

So we calculate a factor of the first term in (4.6) to be

$$
\begin{aligned}
-2||\nabla x^3||_g g_0^{im} A_m^j \Gamma_{ij}^k(g_0) &= e^{-2\phi}\cdot (g_{\mathbb{E}})^{im}\cdot e^{-2\phi}(g_{\mathbb{E}})^{jl}\cdot \left(g_{l\alpha}\partial_m g^{3\alpha} + g_{m\alpha}\partial_l g^{3\alpha} + g^{3\alpha}\partial_\alpha g_{ml}\right)\cdot \\
&\quad \left({g_{\mathbb{E}}}_i^k \partial_j\phi + {g_{\mathbb{E}}}_j^k \partial_i\phi - (g_{\mathbb{E}})_{ij}{g_{\mathbb{E}}}^{kp}\partial_p\phi\right) \\
&= e^{-4\phi}\left\{2{g_{\mathbb{E}}}^{im}{g_{\mathbb{E}}}^{jl} g_{m\alpha}\partial_l g^{3\alpha}\left({g_{\mathbb{E}}}_i^k\partial_j\phi + {g_{\mathbb{E}}}_j^k\partial_i\phi - (g_{\mathbb{E}})_{ij}{g_{\mathbb{E}}}^{kp}\partial_p\phi\right)\right. \\
&\quad + g^{3\alpha}{g_{\mathbb{E}}}^{im}{g_{\mathbb{E}}}^{jl}\partial_l(e^{2\phi}(g_{\mathbb{E}})_{ml})\left({g_{\mathbb{E}}}_i^k\partial_j\phi + {g_{\mathbb{E}}}_j^k\partial_i\phi\right) \\
&\quad \left. - g^{3\alpha}{g_{\mathbb{E}}}^{im}{g_{\mathbb{E}}}^{jl}\partial_l(e^{2\phi}(g_{\mathbb{E}})_{ml})(g_{\mathbb{E}})_{ij}{g_{\mathbb{E}}}^{kp}\partial_p\phi\right\} \\
&= 2e^{-4\phi}\left\{g_{\mathbb{E}}^{km} g_{\mathbb{E}}^{jl} g_{l\alpha}\partial_m g^{3\alpha}\partial_j\phi\right. \\
&\quad + g_{\mathbb{E}}^{im} g_{\mathbb{E}}^{kl} g_{l\alpha}\partial_m g^{3\alpha}\partial_i\phi - g_{\mathbb{E}}^{kj} g_{\mathbb{E}}^{ml} g_{l\alpha}\partial_m g^{3\alpha} e^{2\phi}\partial_j\phi \\
&\quad \left. + 2g_{\mathbb{E}}^{kj} g^{3\alpha} e^{2\phi}\partial_\alpha\phi\partial_j\phi - 2g_{\mathbb{E}}^{km} g^{3\alpha} e^{2\phi}\partial_\alpha\phi\partial_m\phi\right\} \\
&= 2e^{-4\phi}\left\{g_{\mathbb{E}}^{km} g_{\mathbb{E}}^{jl} g_{3l}\partial_m g^{33}\partial_j\phi + g_{\mathbb{E}}^{im} g_{\mathbb{E}}^{kl} g_{3l}\partial_m g^{33}\partial_i\phi\right. \\
&\quad - g_{\mathbb{E}}^{kj} g_{\mathbb{E}}^{ml} g_{3l}\partial_m g^{33} e^{2\phi}\partial_j\phi + g_{\mathbb{E}}^{km} e^{2\phi}\partial_m g^{3j}\partial_j\phi \\
&\quad \left. + g_{\mathbb{E}}^{im} e^{2\phi}\partial_m g^{3k}\partial_i\phi - g_{\mathbb{E}}^{kj}\partial_m g^{3m} e^{4\phi}\partial_j\phi + 0\right\}.
\end{aligned}
$$

For the second term in (4.6), observe the partial coordinate derivatives of the components of the second fundamental form are thus

$$
\begin{aligned}
2\partial_j(A_i^k) = -2\partial_j\phi A_i^k - \frac{1}{||\nabla x^3||_g}\partial_j||\nabla x^3||_g A_i^k + \frac{e^{-2\phi}(g_{\mathbb{E}})^{jk}}{||\nabla x^3||_g}&\left\{\partial_j g_{\alpha m}\partial_i g^{3\alpha} + g_{\alpha m}\partial_j\partial_i g^{3\alpha}\right. \\
+ \partial_j g_{\alpha i}\partial_m g^{3\alpha} + g_{\alpha i}\partial_j\partial_m g^{3\alpha} + 2e^{2\phi}(g_{\mathbb{E}})_{im}\partial_j g^{3\alpha}\partial_\alpha\phi& \\
\left. + 2e^{2\phi}(g_{\mathbb{E}})_{im} g^{3\alpha}\partial_j\partial_\alpha\phi - 4e^{2\phi}(g_{\mathbb{E}})_{im} g^{3\alpha}\partial_\alpha\phi\partial_j\phi\right\}.&
\end{aligned}
$$

Substituting the expressions for $\partial_j(A_i^k)$ and $g^{im}A_m^j\Gamma_{ij}^k(g_0)$ above into equation (4.6), the equation for the first order change in conformal coordinates is

$$\begin{aligned}
\Delta_{g_0}\dot{x}^k &= -2\psi_p A^{ij}\Gamma^k_{ij}(g_0) - 2g_0^{ij}\nabla_j(\psi_p A^k_i) \\
&= -2\psi_p g^{im}A^j_m\Gamma^k_{ij}(g_0) - 2g_0^{ij}\nabla_j(\psi_p)A^k_i - 2g_0^{ij}\psi_p[\partial_j A^k_i - \Gamma^m_{ij}(g_0)A^k_m + \Gamma^k_{mj}(g_0)A^m_i] \\
&= -2g_0^{ij}\nabla_j(\psi_p)A^k_i - 2\psi_p g_0^{ij}\partial_j A^k_i - 4\psi_p g^{im}A^j_m\Gamma^k_{ij}(g_0) \\
&= -g_0^{ij}\nabla_j(\psi_p)\frac{e^{-2\phi}(g_{\mathbb{E}})^{jk}}{2||\nabla x^3||_g}(g_{\alpha j}\partial_i g^{3\alpha} + g_{i\alpha}\partial_j g^{3\alpha} + g^{3\alpha}\partial_\alpha g_{ij}) \\
&\quad + 2g_0^{ij}\psi_p\frac{e^{-2\phi}(g_{\mathbb{E}})^{jk}}{2||\nabla x^3||_g}(g_{\alpha j}\partial_i g^{3\alpha} + g_{i\alpha}\partial_j g^{3\alpha} + g^{3\alpha}\partial_\alpha g_{ij}) \\
&\quad + g_0^{ij}\psi_p\frac{1}{||\nabla x^3||_g}\partial_j||\nabla x^3||_g\frac{e^{-2\phi}(g_{\mathbb{E}})^{jk}}{2||\nabla x^3||_g}(g_{\alpha j}\partial_i g^{3\alpha} + g_{i\alpha}\partial_j g^{3\alpha} + g^{3\alpha}\partial_\alpha g_{ij}) \\
&\quad - g_0^{ij}\psi_p\frac{e^{-2\phi}(g_{\mathbb{E}})^{jk}}{||\nabla x^3||_g}\left\{\partial_j g_{\alpha m}\partial_i g^{3\alpha} + g_{\alpha m}\partial_j\partial_i g^{3\alpha} + \partial_j g_{\alpha i}\partial_m g^{3\alpha} + g_{\alpha i}\partial_j\partial_m g^{3\alpha}\right. \\
&\quad \left. + 2e^{2\phi}(g_{\mathbb{E}})_{im}\partial_j g^{3\alpha}\partial_\alpha\phi + 2e^{2\phi}(g_{\mathbb{E}})_{im}g^{3\alpha}\partial_j\partial_\alpha\phi - 4e^{2\phi}(g_{\mathbb{E}})_{im}g^{3\alpha}\partial_\alpha\phi\partial_j\phi\right\} \\
&\quad - 8\psi_p e^{-4\phi}\left\{g_{\mathbb{E}}^{km}g_{\mathbb{E}}^{jl}g_{3l}\partial_m g^{33}\partial_j\phi + g_{\mathbb{E}}^{im}g_{\mathbb{E}}^{kl}g_{3l}\partial_m g^{33}\partial_i\phi - g_{\mathbb{E}}^{kj}g_{\mathbb{E}}^{ml}g_{3l}\partial_m g^{33}e^{2\phi}\partial_j\phi\right. \\
&\quad \left. + g_{\mathbb{E}}^{km}e^{2\phi}\partial_m g^{3j}\partial_j\phi + g_{\mathbb{E}}^{im}e^{2\phi}\partial_m g^{3k}\partial_i\phi - g_{\mathbb{E}}^{kj}\partial_m g^{3m}e^{4\phi}\partial_j\phi\right\}. \qquad (4.8)
\end{aligned}$$

We may express this complicated PDE schematically as

$$\begin{aligned}
\Delta_{g_0}\dot{x}^k &= e^{-2\phi}\psi_p A^{ijk}_l\partial_i\partial_j g^{3l} + e^{-2\phi}\psi_p B^{ik}_\alpha\partial_i\partial_\alpha\phi + e^{-2\phi}\psi_p C^{ijk}_\alpha\partial_i\phi\partial_j g^{3\alpha} \\
&\quad + e^{-2\phi}\psi_p D^{ijk}_{\alpha\beta}\partial_i g^{3\alpha}\partial_j g^{3\beta} + e^{-2\phi}\psi_p F^{ik\alpha}\partial_i\phi\partial_\alpha\phi \\
&\quad + e^{-2\phi}(\psi_p H^{ik}_\alpha + \nabla_j\psi_p I^{ijk}_\alpha)\partial_i g^{3\alpha} + (\psi_p J^{k\alpha}_1 + \nabla_i\psi_p J^{ik\alpha}_2)\partial_\alpha\phi, \\
&=: e^{-2\phi}\mathcal{F}^k(g^{13}, g^{23}, \phi, \psi_p, p) \qquad (4.9)
\end{aligned}$$

where $A^{ijk}_l, B^{ik\alpha}, ..., J^{ik\alpha}_j$ are smooth functions of $\phi, g^{13}, g^{23}, g^{33}$, and the indices range over $i, j, k, l \in \{1, 2\}$, and $\alpha, \beta \in \{1, 2, 3\}$. Now, since $g_0 = e^{2\phi}g_{\mathbb{E}}$, $\Delta_{g_0} = e^{-2\phi}\Delta_{g_{\mathbb{E}}}$; so on

$Y(0)$ we have the equation

$$\Delta_{g_{\mathbb{E}}} \dot{x}^k = \mathcal{F}^k(g^{13}, g^{23}, \phi, \psi_p, p), \tag{4.10}$$

where $\mathcal{F}^k(g^{13}, g^{23}, \phi, \psi_p, p)$ is as in (4.9).

□

In the setting where our manifold (M, g) has been equipped with a foliation by area-minimizing surfaces $Y(t)$ and coordinates (x^α) adapted to the foliation (c.f. section 3.0.1, we next derive a transport-type equation for the conformal factor $\phi = \phi(x^1, x^2, t)$ on the leaf $Y(t)$.

Proposition 4.4. *Let (M, g) be a C^3 smooth, compact, 3-dimensional Riemannian manifold with strictly mean convex boundary ∂M, and suppose that (M, g) admits a foliation by properly embedded, area-minimizing surfaces. Let g be given on ∂M. Suppose that for any properly embedded, area-minimizing, codimension 1 submanifold $Y \subset M$, the area of Y and of any minimal perturbation of Y is known.*

Let $\gamma(t)$, $t \in [0, T]$, be a given foliation of ∂M by simple closed curves, and set $Y(t)$ to be a foliation of M by properly embedded, area-minimizing surfaces such that $\partial Y(t) = \gamma(t)$. Extend M and each $Y(t)$ to asymptotically flat manifolds $\mathbf{M}$ and $\mathbf{Y}(t)$ as defined in section 3.0.1. Further, set $(x^1, x^2) = \Phi(\cdot, t) : \mathbf{Y}(t) \to \mathbb{R}^2$ to be unique isothermal coordinates on $\mathbf{Y}(t)$ given by Proposition 3.1. Set $x^3 = t$.

Then, in the coordinate system (x^α), the metric on the leaf $Y(t)$ is $g_t := e^{2\phi(t)} g_{\mathbb{E}}$, and the evolution from leaf to leaf of the conformal factor $\phi = \phi(x^1, x^2, t)$ is described by the transport-type equation

$$g^{31}\partial_1\phi + g^{32}\partial_2\phi + g^{33}\partial_3\phi + \frac{1}{2}\partial_k g^{3k} - \frac{1}{2}g^{3k}\partial_k \log(g^{33}) = 0. \tag{4.11}$$

Proof. Recall the mean curvature of $Y(t)$ is the trace of the second fundamental form:

$H(Y(t)) := A_i^i$ (we do not average over the dimension).

As demonstrated in the proof of Lemma 4.3, equation (4.7), the second fundamental form may be written as

$$A_i^k = -\frac{e^{-2\phi}(g_{\mathbb{E}})^{jk}}{2||\nabla x^3||_g}(g_{\alpha j}\partial_i g^{3\alpha} + g_{i\alpha}\partial_j g^{3\alpha} + g^{3\alpha}\partial_\alpha g_{ij}).$$

in the coordinates (x^α), $\alpha = 1, 2, 3$.

Therefore the mean curvature of $Y(t)$ is given by

$$\begin{aligned} H := A_i^i &= -\frac{e^{-2\phi}(g_{\mathbb{E}})^{ij}}{2||\nabla x^3||_g}(g_{\alpha j}\partial_i g^{3\alpha} + g_{i\alpha}\partial_j g^{3\alpha} + 2e^{2\phi}(g_{\mathbb{E}})_{ij} g^{3\alpha}\partial_\alpha \phi) \\ &= -\frac{e^{-2\phi}}{2||\nabla x^3||_g}(2g_{\alpha 1}\partial_1 g^{3\alpha} + 2g_{\alpha 2}\partial_2 g^{3\alpha} + 4e^{2\phi} g^{3\alpha}\partial_\alpha\phi) \\ &= -\frac{e^{-2\phi}}{||\nabla x^3||_g}(g_{31}\partial_1 g^{33} + e^{2\phi}\partial_k g^{3k} + g_{32}\partial_2 g^{33} + 2e^{2\phi} g^{3\alpha}\partial_\alpha\phi), \end{aligned}$$

where k sums over $1, 2$.

Since $Y(t)$ is minimal for each $t \in \mathbb{R}$, $H(Y(t)) = 0$ provides the differential equation

$$0 = e^{-2\phi}(g_{31}\partial_1 g^{33} + g_{32}\partial_2 g^{33}) + (\partial_k g^{3k} + 2g^{3\alpha}\partial_\alpha\phi)$$

which we rewrite as

$$g^{31}\partial_1\phi + g^{32}\partial_2\phi + g^{33}\partial_3\phi + \frac{1}{2}\partial_k g^{3k} + \frac{e^{-2\phi}}{2}(g_{31}\partial_1 g^{33} + g_{32}\partial_2 g^{33}) = 0. \tag{4.12}$$

As shown in the appendix, we can express the components g_{31}, g_{32} in terms of the components of the inverse metric as

$$\begin{aligned} g_{31} &= -\frac{g^{31}}{g^{33}} e^{2\phi} \\ g_{32} &= -\frac{g^{32}}{g^{33}} e^{2\phi}. \end{aligned}$$

Substituting the above into equation (4.12), we obtain

$$g^{31}\partial_1\phi + g^{32}\partial_2\phi + g^{33}\partial_3\phi + \frac{1}{2}\partial_k g^{3k} - \frac{1}{2}[g^{31}\partial_1 \log(g^{33}) + g^{32}\partial_2 \log(g^{33})] = 0.$$

□

Remark. Notice that if g^{13}, g^{23}, and g^{33} are known functions on $Y(t)$, then equation (4.11) reduces to a simple differential equation for the conformal factor ϕ, which can be easily solved.

Chapter 5

Proof of the Main Theorems

In this section, we prove the uniqueness theorems given in the introduction. We restate them below for convenience.

Theorem 1.6. *Let $D(r) \subset \mathbb{R}^2$ be a disk of radius $r > 0$ about the origin, and $T > 0$. Let (M, g_1) be a smooth, 3-dimensional Riemannian manifold which is topologically equivalent to $D(r) \times [0, T]$. Suppose*

1. *(M, g_1) is either*
   - *C^2 close to Euclidean, or*
   - *$r < \epsilon_0 < 1$ and (M, g_1) is ϵ_0-thin.*
2. *for any closed, embedded curve $\gamma : [0, T] \to \partial M$, we know the area of any properly embedded surface in $Y_\gamma \subset M$ which solves the least-area problem for γ.*

Then, if g_2 is a Riemannian metric on M which satisfies the same conditions as (M, g_1), $g_1 = g_2$ on ∂M, and g_2 gives the same area data as g_1, the metric g_2 is isometric to g_1.

Proof of Theorem 1.6. To show g_1 is isometric to g_2 on M, we construct coordinate systems on (M, g_1) and (M, g_2), and explicitly construct a diffeomorphism $F : (M, g_1) \to$

(M, g_2) which maps one coordinate system to the other. In this setting, we prove that the components of the inverses of the metrics g_1 and $F^*(g_2)$ satisfy

$$g_1^{\alpha\beta} - F^*(g_2)^{\alpha\beta} = 0.$$

This equation gives our uniqueness result.

Construction of the diffeomorphism $F : (M, g_1) \to (M, g_2)$: As in section 3.0.1, extend (M, g_1) to an asymptotically flat manifold $(\mathbf{M}, \mathbf{g}_1) := (\mathbb{R}^2 \times [0, T], \mathbf{g}_1)$. Smoothly extend each leaf $Y_1(t)$ to an asymptotically flat manifold $\mathbf{Y}_1(t)$ as defined in section 3.0.1. Further, set $(x^1, x^2) = \Phi_1(\cdot, t) : \mathbf{Y}_1(t) \to \mathbb{R}^2$ to be unique isothermal coordinates on $\mathbf{Y}_1(t)$ given by Proposition 3.1; write $\tilde{\Omega}_1(t) := \Phi_1(Y_1(t))$. Set $x^3 = t$.

Let $Y_2(t)$ be a foliation of (M, g_2) by properly embedded, area minimizing surfaces which is found by solving the least-area problem for $\gamma(t)$. As in section 3.0.1, we also extend (M, g_2) to an asymptotically flat manifold $(\mathbf{M}, \mathbf{g}_2) := (\mathbb{R}^2 \times [0, T], \mathbf{g}_2)$ and smoothly extend each leaf $Y_2(t)$ to an asymptotically flat manifold $\mathbf{Y}_2(t)$. As above, we set $(y^1, y^2) = \Phi_2(\cdot, t) : \mathbf{Y}_2(t) \to \mathbb{R}^2$ to be unique isothermal coordinates on $\mathbf{Y}_2(t)$ given by Proposition 3.1; write $\tilde{\Omega}_2(t) := \Phi_1(Y_2(t))$. Set $y^3 = t$.

Then, define

$$F : (\mathbf{M}, \mathbf{g}_1) \to (\mathbf{M}, \mathbf{g}_2)$$

$$F(p) = \Phi_2^{-1} \circ \Phi_1(p).$$

From Proposition 3.2, in section 2, we know $\tilde{\Omega}_1(t) = \tilde{\Omega}_2(t)$ for all $t \in [0, T]$. Thus $F = \mathrm{Id}$ on $\mathbf{M} \setminus M \cup \partial M$. The restriction of F to (M, g_1) is then our desired diffeomorphism.

Notation: Abusing notation, we write g_2 for the pulled-back metric $F^*(g_2)$ on M in all that follows. Note that in the (x^α) coordinates, the metrics g_1 and g_2 take the form

$$g_1 = (g_1)_{3\alpha} dx^3 dx^\alpha + e^{2\phi_1(t)} \left[(dx^1)^2 + (dx^2)^2\right]$$
$$g_2 = (g_2)_{3\alpha} dx^3 dx^\alpha + e^{2\phi_2(t)} \left[(dx^1)^2 + (dx^2)^2\right].$$

Further, since Proposition 3.2 shows our area data determines $\tilde{\Omega}_1(t) = \tilde{\Omega}_2(t)$ for all $t \in [0, T]$, by choosing a new conformal map, we may assume that $\tilde{\Omega}_1(t) = \tilde{\Omega}_2(t) = D(r)$ for some disk of radius $r > 0$. We make this choice for simplicity in the proofs to follow. Again abusing notation, we still write $(x^1, x^2) = \Phi_1$ and $(y^1, y^2) = \Phi_2$ for the resulting maps to the disk $D(r)$.

Lemma 5.1. *In the coordinate system described above, $g_1^{\alpha\beta} - g_2^{\alpha\beta} = 0$ on $\mathbf{M} \setminus M$, and $g_1^{33} - g_2^{33} = 0$ on $\mathbf{M}$.*

Proof. By construction of the asymptotic extension of (M, g_1) and (M, g_2), we have $F = \mathrm{Id}$ on $\mathbf{M} \setminus M \cup \partial M$ and thus $g_1^{\alpha\beta} - g_2^{\alpha\beta} = 0$ on $\mathbf{M} \setminus M$.

By the hypothesis on M, we know the area of $Y_k(t)$ for all t, $k = 1, 2$ as measured by g_1 and g_2 respectively. Proposition 4.1 tells us this area information determines the lapse functions $||\nabla x^3||^2_{g_1}$ and $||\nabla y^3||^2_{g_2}$. Moreover, since we have assumed the area of $Y_1(t)$ as measured by g_1 equals the area of $Y_2(t)$ as measured by g_2 for each $t \in 0, T$, by Proposition 1.1 the Dirichlet-to-Neumann maps associated to the stability operators on the leaves are equal. Hence, in the coordinates (x^α),

$$g_1^{33} = ||\nabla x^3||^2_{g_1} = ||\nabla y^3||^2_{g_2} \circ F = g_2^{33}.$$

□

Next, we prove uniqueness for all the remaining the metric components by showing the differences $g_1^{\alpha\beta} - g_2^{\alpha\beta}$ vanish on M. First, in the coordinates (x^α), we derive a system of equations for the differences

$$\delta g^{3j} := g_1^{3j} - g_2^{3j}, \text{ and } \delta\phi := \phi_1 - \phi_2$$

on $\mathbf{M}$, $j = 1, 2$. Then, using Lemma 4.2 and Lemma 4.3, we will express δg^{3j} as a linear combination of pseudodifferential operators acting on $\delta\phi$ or $\partial_3\delta\phi$. From the expressions for these operators, we will show uniqueness of the components of g_k^{3j}, g_k^{11}, g_k^{11}, and g_k^{12} follows from uniqueness of the conformal factors ϕ_1, ϕ_2.

Finally, to prove uniqueness of the conformal factors ϕ_1, ϕ_2, we use Proposition 4.4 and the pseudodifferential expressions for δg^{3j} to obtain a hyperbolic Cauchy problem for $\delta\phi$. The desired result $\delta\phi = \phi_1 - \phi_2 = 0$ will follow from a very standard energy argument.

Derivation of a system of equations for the metric components: First consider the foliation $Y_1(t)$ of (M, g_1). Notice that in the coordinates (x^α), the gradient of the function $x^3 : M \to \mathbb{R}$ is

$$\nabla x^3 := g_1^{\alpha\beta}\partial_\beta x^3 \partial_\alpha = g_1^{\alpha 3}\partial_\alpha.$$

Geometrically, the components of the inverse metric $g_1^{3\alpha}$, for $\alpha = 1, 2, 3$, describe the normal vector field $\vec{n}_1 := \frac{\nabla x^3}{||\nabla x^3||_{g_1}}$ to a leaf $Y_1(t) := \{x^3 = t\}$. Further, we saw above that if we view $Y(t + \delta t)$ as a variation of $Y_1(t)$ by area-minimizing surfaces, the lapse function for this variation is

$$g_1(\partial_3, \vec{n}_1) := ||\nabla x^3||_{g_1} = g_1^{33} = \sqrt{g_1^{33}}.$$

Below we will consider a variation of the foliation $Y_1(t)$, recover information about the new lapse function for the new foliation, and use this to find equations which describe the differences δg^{3k}.

We construct two variations of $Y_1(t)$ as follows: Consider a point $p \in Y_1(t)$. For $i = 1, 2$, define $h_i : [0, S] \times D(r) \times [0, T] \to \mathbf{M}$, $h_i(s, x^1, x^2, t) =: Y_{i,1}(s, t)$ to be a variation of $Y_1(t) \subset \mathbf{M}$ by properly embedded, area-minimizing surfaces which has the property that the component of $(h_i)_* \left(\frac{\partial}{\partial s}\big|_{s=0}\right)$ normal to $Y(t)$, denoted by $\psi_{p,i,1} = \psi_{p,i,1}(x^1, x^2)$, vanishes at p. We further require that $\nabla\psi_{p,i,1}(x(p)) = \frac{\partial}{\partial x^i}$. We write $(x_s^1, x_s^2) = \Phi_1(\cdot, s, t) : \mathbf{Y}_1(s, t) \to \mathbb{R}^2$ for the unique isothermal coordinates on a smooth, asymptotically flat extension of the new foliation $\mathbf{Y}_1(s, t)$; we impose $x_s^3 = x^3 + \mathcal{O}(s^2)$.

The existence of the desired foliations $Y_{i,1}(s, t)$ is equivalent to the existence of the desired functions $\psi_{p,i,1}$. We claim that by Proposition 1.1, for each $t \in [0, T]$ and for each $p \in Y_1(t) \subset M$, we may obtain two distinct, nontrivial solutions $\psi_{p,i,1} \in C^2(\mathbb{R}^2)$, $i = 1, 2$, of the Jacobi equation

$$\Delta_{g_{\mathbb{E}}}\psi_{p,i,1} + e^{2\phi_1}\left(\mathrm{Ric}_{g_1}(\vec{n}, \vec{n}) + ||A||_{g_1}^2\right)\psi_{p,i,1} = 0$$

on $\mathbb{R}^2$, which additionally satisfy

$$\psi_{p,i,1}(x(p)) = 0,$$
$$\nabla\psi_{p,i,1}(x(p)) = \frac{\partial}{\partial x^i}.$$

In the setting of the first case of Theorem 1.6, we have imposed that all the metric components are C^2 close to Euclidean, and we will show this is enough to obtain the functions $\psi_{p,i,1}$. In the setting of the second case of Theorem 1.6, the condition on the size of the radius of $D(r)$ will play the crucial role in place of the C^2 close assumption, and we will show that we are able to construct the functions $\psi_{p,i,1}$ in this case too. We also seek such solutions $\psi_{p,i,1}$ which live in the following spaces:

Let $\Omega \subset \mathbb{R}^2$. For $k = 0, 1, 2, \ldots$, define

$$C^k_x(\Omega) := \left\{ f(x,p) : \Omega \times \Omega \to \mathbb{R} \,|\, \frac{\partial^{i+j}}{\partial x^i \partial x^j} f \text{ is continuous for all } i + j \leq k \right\}$$
$$C^k_p(\Omega) := \left\{ f(x,p) : \Omega \times \Omega \to \mathbb{R} \,|\, \frac{\partial^{i+j}}{\partial p^i \partial p^j} f \text{ is continuous for all } i + j \leq k \right\}.$$

Lemma 5.2. *Let (M, g) satisfy the conditions of the first or second case of Theorem 1.6 and let $D(r)$ a disk of radius $r > 0$. Let $D(r) \hookrightarrow Y \subset M$ be a properly embedded, area-minimizing surface, $\vec{n}$ the unit normal vector field to Y, and A the second fundamental form of Y. Set (x^1, x^2) to be isothermal coordinates on $D(r)$, so $g|_Y := e^{2\phi} g_{\mathbb{E}}$. Suppose $V := e^{2\phi} \left(Ric_g(\vec{n}, \vec{n}) + ||A||^2_g \right) \in W^{1,q}(D(r))$ with $q > 2$.*

Then, for any $p \in D(r)$, and fixed $i = 1, 2$, there exists a function $\psi_{p,i} \in C^2(D(r))$ which satisfies

1. *$(\Delta_{g_{\mathbb{E}}} + V)\psi_{p,i} = 0$ on $D(r)$,*

2. *$\psi_{p,i}(p) = 0$,*

3. *$\frac{\partial}{\partial x^j} \psi_{p,i}(p) = \delta_{ij}$.*

Moreover,

4. *$||\partial_p \psi_{p,i}||_{C^2_x(D(r))} < K_1$, for $K_1 > 0$ independent of p, and*

5. *$||\psi_{p,i}||_{C^2_x(D(r))} < K_2$, for $K_2 > 0$ independent of p,*

where $\partial_p \psi_{p,i}$ denotes differentiation with respect to p. Lastly, we have

6. *the area data for (M, g) determines $\psi_{p,i}$ on $D(r)$.*

Proof. 1. Let (M, g) and $D(r)$ be as in the first case of Theorem 1.6. Since we assume that the metric g is C^2-close to Euclidean, the norm $||V||_\infty$ is small. Then, by setting $r = 1$ in the proof of part 2 below, we obtain the desired properties.

2. Let (M, g), $0 < r < \epsilon_0$, and $D(r)$ be as in the second case of Theorem 1.6. Without loss of generality, suppose $i = 1$. Write $\psi_p := \psi_{p,i}$ for simplicity.

We will construct ψ_p from linear combinations of solutions which are close to either x^1, x^2, or 1 in $C^2(D(r))$.

To this end, we construct a function χ^1 which solves the boundary value problem

$$(\Delta_{g_{\mathbb{E}}} + V)\chi^1 = 0 \qquad \text{on } D(r),$$
$$\chi^1 = x^1 \qquad \text{on } \partial D(r).$$

Rescale the coordinates (x^i), $i = 1, 2$, to $(\tilde{x}^i)$, $i = 1, 2$, so that we work over the unit disk $D(1)$: define $f : D(1) \to D(r)$ to be the change of coordinates map $f(\tilde{x}) = r\tilde{x} = x$. Set $\tilde{\chi}^1 := \frac{1}{r}\chi^1 \circ f$, $\tilde{V} := V \circ f$, and $\tilde{p} = f^{-1}(p)$. Then

$$f^*(\Delta_{g_{\mathbb{E}}} + V) = r^{-2}\Delta_{g_{\mathbb{E}}} + \tilde{V}, \tag{5.1}$$

and we seek solutions $\tilde{\chi}^1$ to

$$(\Delta_{g_{\mathbb{E}}} + r^2\tilde{V})\tilde{\chi}^1 = 0, \qquad \text{on } D(1) \tag{5.2}$$
$$\tilde{\chi}^1 = \tilde{x}^1 \qquad \text{on } \partial D(1). \tag{5.3}$$

Since $r < \epsilon_0$ and $||V||_{W^{1,q}(D(r))} < \frac{1}{\epsilon_0}$, the operator $(\Delta_{g_{\mathbb{E}}} + r^2\tilde{V}) : W^{1,q}(D(1)) \to C^2(D(1))$ is invertible; from (5.2) and (5.3), we derive the estimate $||r^2\tilde{V}||_{W^{1,q}(D(1))} \leq \epsilon_0$. Thus, there exists a unique solution $\tilde{\chi}^1$ to (5.2), (5.3).

Now consider

$$(\Delta_{g_{\mathbb{E}}} + r^2\tilde{V})\left[\tilde{\chi}^1 - \tilde{x}^1\right] = -(r^2\tilde{V})\tilde{x}^1 \qquad \text{on } D(r),$$
$$\tilde{\chi}^1 - \tilde{x}^1 = 0 \qquad \text{on } \partial D(r).$$

Since r is chosen to be small, for $q > 2$ we have

$$\begin{aligned}
\left|\left|\tilde{\chi}^1 - \tilde{x}^1\right|\right|_{C^2(D(1))} &\leq C\left|\left|\tilde{\chi}^1 - \tilde{x}^1\right|\right|_{W^{3,q}(D(1))} \\
&\leq C\left[||(r^2\tilde{V})\tilde{x}^1||_{W^{1,q}(D(1))}\right] \\
&\leq C\left[r^2||\tilde{V}\tilde{x}^1||_{L^q(D(1))} + r^2||\tilde{\nabla}(\tilde{V}\tilde{x}^1)||_{L^q(D(1))}\right] \\
&\leq Cr^2. \qquad (5.4)
\end{aligned}$$

Therefore, the function $\tilde{\chi}^1$ has the estimate

$$\left|\left|\tilde{\chi}^1 - \tilde{x}^1\right|\right|_{C^2(D(1))} \leq Cr^2. \qquad (5.5)$$

In particular, the estimate (5.5) implies

$$||\partial_{\tilde{x}^1}\tilde{\chi}^1 - 1||_{C^1(D(1))} \leq Cr^2,$$

$$||\partial_{\tilde{x}^2}\tilde{\chi}^1 - 0||_{C^1(D(1))} \leq Cr^2;$$

since r is small, we have

$$\partial_{\tilde{x}^1}\tilde{\chi}^1 \sim 1 \text{ in } C^1(D(1)), \qquad (5.6)$$

$$\partial_{\tilde{x}^2}\tilde{\chi}^1 \sim 0 \text{ in } C^1(D(1)). \qquad (5.7)$$

Similar as above, we construct a function $\tilde{\chi}^2$, which solves the boundary value problem

$$\begin{aligned}
(\Delta_{g_{\mathbb{E}}} + r^2\tilde{V})\tilde{\chi}^2 &= 0, && \text{on } D(1) \\
\tilde{\chi}^2 &= \tilde{x}^2 && \text{on } \partial D(1),
\end{aligned}$$

and satisfies

$$||\tilde{\chi}^2 - \tilde{x}^2||_{C^2(D(1))} \leq Cr^2. \tag{5.8}$$

Further, let ω solve

$$(\Delta_{g_{\mathbb{E}}} + r^2\tilde{V})\omega = 0 \qquad \text{on } D(1),$$

$$\omega = 1 \qquad \text{on } \partial D(1).$$

As argued above, the function ω satisfies

$$||\omega - 1||_{C^2(D(1))} \leq Cr^2. \tag{5.9}$$

Now, for $j = 1, 2$, the function

$$\gamma^j := \omega(\tilde{p})\tilde{\chi}^j - \tilde{\chi}^j(\tilde{p})\omega$$

solves

$$(\Delta_{g_{\mathbb{E}}} + r^2\tilde{V})\gamma^j = 0 \qquad \text{on } D(1),$$

$$\gamma^j = \omega(\tilde{p})\tilde{x}^j - \tilde{\chi}^j(\tilde{p}) \qquad \text{on } \partial D(1),$$

and has the property $\gamma^j(\tilde{p}) = 0$, and obeys

$$||\gamma^j - (\omega(\tilde{p})\tilde{x}^j - \tilde{\chi}^j(\tilde{p}))||_{C^2(D(1))} \leq Cr^2. \tag{5.10}$$

Consider γ^1 and γ^2 as above. From (5.5), (5.8), and (5.9),

$$\partial_{\tilde{x}^1}\gamma^1(\tilde{p})\partial_{\tilde{x}^2}\gamma^2(\tilde{p}) - \partial_{\tilde{x}^2}\gamma^1(\tilde{p})\partial_{\tilde{x}^1}\gamma^2(\tilde{p}) \sim 1 \neq 0.$$

Then, the function

$$\tilde{\psi}_{\tilde{p}}(\tilde{x}) := \frac{\partial_{\tilde{x}^2}\gamma^2(\tilde{p})\gamma^1(\tilde{x}) - \partial_{\tilde{x}^2}\gamma^1(\tilde{p})\gamma^2(\tilde{x})}{\partial_{\tilde{x}^1}\gamma^1(\tilde{p})\partial_{\tilde{x}^2}\gamma^2(\tilde{p}) - \partial_{\tilde{x}^2}\gamma^1(\tilde{p})\partial_{\tilde{x}^1}\gamma^2(\tilde{p})}$$

is well defined and satisfies conditions 2-3. By linearity of the operator $(\Delta_{g_{\mathbb{E}}} + r^2\tilde{V})$, $\tilde{\psi}_{\tilde{p}}$ satisfies condition 1:

$$\begin{aligned} (\Delta_{g_{\mathbb{E}}} + r^2\tilde{V})\tilde{\psi}_{\tilde{p}} &= 0 && \text{on } D(1), \\ \tilde{\psi}_{\tilde{p}} &= \tilde{f}_{\tilde{p}} && \text{on } \partial D(1), \end{aligned}$$

where

$$\tilde{f}_{\tilde{p}}(\tilde{x}) := \frac{\partial_{\tilde{x}^2}\gamma^2(\tilde{p})[\omega(\tilde{p})\tilde{x}^1 - \tilde{\chi}^1(\tilde{p})] - \partial_{\tilde{x}^2}\gamma^1(\tilde{p})[\omega(\tilde{p})\tilde{x}^2 - \tilde{\chi}^2(\tilde{p})]}{\partial_{\tilde{x}^1}\gamma^1(\tilde{p})\partial_{\tilde{x}^2}\gamma^2(\tilde{p}) - \partial_{\tilde{x}^2}\gamma^1(\tilde{p})\partial_{\tilde{x}^1}\gamma^2(\tilde{p})}.$$

Now, we claim that $||\tilde{\psi}_{\tilde{p}}||_{C^2_{\tilde{x}}(D(1))} \leq \tilde{K}_2$, for some constant $K_2 > 0$ independent of $\tilde{p}$. From the estimates 5.5), (5.8), (5.9), and (5.10),

$$\begin{aligned} ||\tilde{\psi}_{\tilde{p}}(\tilde{x})||_{C^0_{\tilde{x}}(D(1))} &\leq \frac{||\partial_{\tilde{x}^2}\gamma^2(\tilde{p})\gamma^1(\tilde{x})||_{C^0_{\tilde{x}}(D(1))} + ||\partial_{\tilde{x}^2}\gamma^1(\tilde{p})\gamma^2(\tilde{x})||_{C^0_{\tilde{x}}(D(1))}}{|\partial_{\tilde{x}^1}\gamma^1(\tilde{p})\partial_{\tilde{x}^2}\gamma^2(\tilde{p}) - \partial_{\tilde{x}^2}\gamma^1(\tilde{p})\partial_{\tilde{x}^2}\gamma^2(\tilde{p})|} \\ &\leq \frac{|\partial_{\tilde{x}^2}\gamma^2(\tilde{p})|||\omega(\tilde{p})\tilde{\chi}^1(\tilde{x}) - \tilde{\chi}^1(\tilde{p})\omega(\tilde{x})||_{C^0_{\tilde{x}}(D(1))}}{1} \\ &\quad + \frac{|\partial_{\tilde{x}^2}\gamma^1(\tilde{p})|||\omega(\tilde{p})\tilde{\chi}^2(\tilde{x}) - \tilde{\chi}^2(\tilde{p})\omega(\tilde{x})||_{C^0_{\tilde{x}}(D(1))}}{1} \\ &\leq (1 + Cr^2)\left[||\tilde{\chi}^1(\tilde{x})||_{C^0_{\tilde{x}}(D(1))} + ||\tilde{\chi}^1(\tilde{x})||_{C^0_{\tilde{x}}(D(1))}||\omega(\tilde{x})||_{C^0_{\tilde{x}}(D(1))}\right] \\ &\quad + (1 + Cr^2)\left[||\tilde{\chi}^2(\tilde{x})||_{C^0_{\tilde{x}}(D(1))} + ||\tilde{\chi}^2(\tilde{x})||_{C^0_{\tilde{x}}(D(1))}||\omega(\tilde{x})||_{C^0_{\tilde{x}}(D(1))}\right] \\ &\leq 1 + Cr^2. \end{aligned} \tag{5.11}$$

For $j = 1, 2$,

$$\begin{aligned}
||\partial_{\tilde{x}^j}\tilde{\psi}_{\tilde{p}}(\tilde{x})||_{C^0_x(D(1))} &\leq \frac{||\partial_{\tilde{x}^2}\gamma^2(\tilde{p})\partial_{\tilde{x}^j}\gamma^1(\tilde{x})||_{C^0_x(D(1))} + ||\partial_{\tilde{x}^2}\gamma^1(\tilde{p})\partial_{\tilde{x}^j}\gamma^2(\tilde{x})||_{C^0_x(D(1))}}{|\partial_{\tilde{x}^1}\gamma^1(\tilde{p})\partial_{\tilde{x}^2}\gamma^2(\tilde{p}) - \partial_{\tilde{x}^2}\gamma^1(\tilde{p})\partial_{\tilde{x}^1}\gamma^2(\tilde{p})|} \\
&\leq \frac{|\partial_{\tilde{x}^2}\gamma^2(\tilde{p})|||\omega(\tilde{p})\partial_{\tilde{x}^j}\tilde{\chi}^1(\tilde{x}) - \tilde{\chi}^1(\tilde{p})\partial_{\tilde{x}^j}\omega(\tilde{x})||_{C^0_x(D(1))}}{1} \\
&\quad + \frac{|\partial_{\tilde{x}^2}\gamma^1(\tilde{p})|||\omega(\tilde{p})\partial_{\tilde{x}^j}\tilde{\chi}^2(\tilde{x}) - \tilde{\chi}^2(\tilde{p})\partial_{\tilde{x}^j}\omega(\tilde{x})||_{C^0_x(D(1))}}{1} \\
&\leq (|\omega(\tilde{p})| + Cr^2)\left[||\partial_{\tilde{x}^j}\tilde{\chi}^1(\tilde{x})||_{C^0_x(D(1))} + ||\tilde{\chi}^1(\tilde{x})||_{C^0_x(D(1))}Cr^2\right] \\
&\quad + Cr^2\left[||\tilde{\chi}^2(\tilde{x})||_{C^0_x(D(1))} + ||\tilde{\chi}^2(\tilde{x})||_{C^0_x(D(1))}Cr^2\right] \\
&\leq 1 + Cr^2.
\end{aligned}$$

Similarly as above for $j, k = 1, 2$,

$$\begin{aligned}
||\partial_{\tilde{x}^k}\partial_{\tilde{x}^j}\tilde{\psi}_{\tilde{p}}(\tilde{x})||_{C^0_x(D(1))} &\leq \frac{||\partial_{\tilde{x}^2}\gamma^2(\tilde{p})\partial_{\tilde{x}^k}\partial_{\tilde{x}^j}\gamma^1(\tilde{x})||_{C^0_x(D(1))} + ||\partial_{\tilde{x}^2}\gamma^1(\tilde{p})\partial_{\tilde{x}^k}\partial_{\tilde{x}^j}\gamma^2(\tilde{x})||_{C^0_x(D(1))}}{|\partial_{\tilde{x}^1}\gamma^1(\tilde{p})\partial_{\tilde{x}^2}\gamma^2(\tilde{p}) - \partial_{\tilde{x}^2}\gamma^1(\tilde{p})\partial_{\tilde{x}^1}\gamma^2(\tilde{p})|} \\
&\leq \frac{|\partial_{\tilde{x}^2}\gamma^2(\tilde{p})|||\omega(\tilde{p})\partial_{\tilde{x}^k}\partial_{\tilde{x}^j}\tilde{\chi}^1(\tilde{x}) - \tilde{\chi}^1(\tilde{p})\partial_{\tilde{x}^k}\partial_{\tilde{x}^j}\omega(\tilde{x})||_{C^0_x(D(1))}}{1} \\
&\quad + \frac{|\partial_{\tilde{x}^2}\gamma^1(\tilde{p})|||\omega(\tilde{p})\partial_{\tilde{x}^k}\partial_{\tilde{x}^j}\tilde{\chi}^2(\tilde{x}) - \tilde{\chi}^2(\tilde{p})\partial_{\tilde{x}^k}\partial_{\tilde{x}^j}\omega(\tilde{x})||_{C^0_x(D(1))}}{1} \\
&\leq (1 + Cr^2)\left[||\partial_{\tilde{x}^k}\partial_{\tilde{x}^j}\tilde{\chi}^1(\tilde{x})||_{C^0_x(D(1))} + ||\tilde{\chi}^1(\tilde{x})||_{C^0_x(D(1))}Cr^2\right] \\
&\quad + (1 + Cr^2)\left[||\tilde{\chi}^2(\tilde{x})||_{C^0_x(D(1))} + ||\tilde{\chi}^2(\tilde{x})||_{C^0_x(D(1))}Cr^2\right] \\
&\leq Cr^2.
\end{aligned}$$

This proves the claim.

Finally we prove $||\tilde{\psi}_{\tilde{p}}||_{C^1_x(D(1))} \leq \tilde{K}_1$ for some $\tilde{K}_1 > 0$ independent of $\tilde{p}$. By the same computation as for estimate (5.11), we have $||\tilde{\psi}_{\tilde{p}}||_{C^0_x(D(1))} \leq 1 + Cr^2$. Next, consider $\partial_{\tilde{p}^1}\tilde{\psi}_{\tilde{p}}(x)$ (the argument for $\partial_{\tilde{p}^2}\tilde{\psi}_{\tilde{p}}$ is analogous). Below we estimate the magnitude of

$||\partial_{\tilde{p}^1}\tilde{\psi}_{\tilde{p}}||_{C^1_{\tilde{x}}(D(1))}$. By construction,

$$\begin{aligned}
\partial_{\tilde{p}^1}\tilde{\psi}_{\tilde{p}}(\tilde{x}) := &\frac{\partial_{\tilde{p}^1}\partial_{\tilde{x}^2}\gamma^2(\tilde{p})[\omega(\tilde{p})\tilde{\chi}^1(\tilde{x}) - \tilde{\chi}^1(\tilde{p})\omega(\tilde{x})]}{\partial_{\tilde{x}^1}\gamma^1(\tilde{p})\partial_{\tilde{x}^2}\gamma^2(\tilde{p}) - \partial_{\tilde{x}^2}\gamma^1(\tilde{p})\partial_{\tilde{x}^1}\gamma^2(\tilde{p})} \\
&+ \frac{\partial_{\tilde{x}^2}\gamma^2(\tilde{p})[\partial_{\tilde{p}^1}\omega(\tilde{p})\tilde{\chi}^1(\tilde{x}) - \partial_{\tilde{p}^1}\tilde{\chi}^1(\tilde{p})\omega(\tilde{x})]}{\partial_{\tilde{x}^1}\gamma^1(\tilde{p})\partial_{\tilde{x}^2}\gamma^2(\tilde{p}) - \partial_{\tilde{x}^2}\gamma^1(\tilde{p})\partial_{\tilde{x}^1}\gamma^2(\tilde{p})} \\
&- \frac{\partial_{\tilde{p}^1}\partial_{\tilde{x}^2}\gamma^1(\tilde{p})[\omega(\tilde{p})\tilde{\chi}^2(\tilde{x}) - \tilde{\chi}^2(\tilde{p})\omega(\tilde{x})]}{\partial_{\tilde{x}^1}\gamma^1(\tilde{p})\partial_{\tilde{x}^2}\gamma^2(\tilde{p}) - \partial_{\tilde{x}^2}\gamma^1(\tilde{p})\partial_{\tilde{x}^1}\gamma^2(\tilde{p})} \\
&- \frac{\partial_{\tilde{x}^2}\gamma^1(\tilde{p})[\partial_{\tilde{p}^1}\omega(\tilde{p})\tilde{\chi}^2(\tilde{x}) - \partial_{\tilde{p}^1}\tilde{\chi}^2(\tilde{p})\omega(\tilde{x})]}{\partial_{\tilde{x}^1}\gamma^1(\tilde{p})\partial_{\tilde{x}^2}\gamma^2(\tilde{p}) - \partial_{\tilde{x}^2}\gamma^1(\tilde{p})\partial_{\tilde{x}^1}\gamma^2(\tilde{p})} \\
&- \frac{\partial_{\tilde{x}^2}\gamma^2(\tilde{p})[\omega(\tilde{p})\tilde{\chi}^1(\tilde{x}) - \tilde{\chi}^1(\tilde{p})\omega(\tilde{x})]}{\partial_{\tilde{p}^1}[\partial_{\tilde{x}^1}\gamma^1(\tilde{p})\partial_{\tilde{x}^2}\gamma^2(\tilde{p})] - \partial_{\tilde{p}^1}[\partial_{\tilde{x}^2}\gamma^1(\tilde{p})\partial_{\tilde{x}^1}\gamma^2(\tilde{p})]} \\
&+ \frac{\tilde{\chi}^1(\tilde{p})\omega(\tilde{x})] - \partial_{\tilde{x}^2}\gamma^1(\tilde{p})[\omega(\tilde{p})\tilde{\chi}^2(\tilde{x}) - \tilde{\chi}^2(\tilde{p})\omega(\tilde{x})]}{\partial_{\tilde{p}^1}[\partial_{\tilde{x}^1}\gamma^1(\tilde{p})\partial_{\tilde{x}^2}\gamma^2(\tilde{p})] - \partial_{\tilde{p}^1}[\partial_{\tilde{x}^2}\gamma^1(\tilde{p})\partial_{\tilde{x}^1}\gamma^2(\tilde{p})]}.
\end{aligned}$$

Using (5.5), (5.8), (5.9), and (5.10), we see

$$
\begin{aligned}
||\partial_{\tilde{p}^1}\tilde{\psi}_{\tilde{p}}(\tilde{x})||_{C^0_x(D(1))} \leq\ & \frac{|\partial_{\tilde{p}^1}\partial_{\tilde{x}^2}\gamma^2(\tilde{p})||\omega(\tilde{p})|||\tilde{\chi}^1(\tilde{x})||_{C^0_x(D(1))}}{|\partial_{\tilde{x}^1}\gamma^1(\tilde{p})\partial_{\tilde{x}^2}\gamma^2(\tilde{p}) - \partial_{\tilde{x}^2}\gamma^1(\tilde{p})\partial_{\tilde{x}^1}\gamma^2(\tilde{p})|} \\
& + \frac{|\partial_{\tilde{p}^1}\partial_{\tilde{x}^2}\gamma^2(\tilde{p})||\tilde{\chi}^1(\tilde{p})|||\omega(\tilde{x})||_{C^0_x(D(1))}}{|\partial_{\tilde{x}^1}\gamma^1(\tilde{p})\partial_{\tilde{x}^2}\gamma^2(\tilde{p}) - \partial_{\tilde{x}^2}\gamma^1(\tilde{p})\partial_{\tilde{x}^1}\gamma^2(\tilde{p})|} \\
& + \frac{|\partial_{\tilde{x}^2}\gamma^2(\tilde{p})||\partial_{\tilde{p}^1}\omega(\tilde{p})|||\tilde{\chi}^1(\tilde{x})||_{C^0_x(D(1))}}{|\partial_{\tilde{x}^1}\gamma^1(\tilde{p})\partial_{\tilde{x}^2}\gamma^2(\tilde{p}) - \partial_{\tilde{x}^2}\gamma^1(\tilde{p})\partial_{\tilde{x}^1}\gamma^2(\tilde{p})|} \\
& + \frac{|\partial_{\tilde{x}^2}\gamma^2(\tilde{p})||\partial_{\tilde{p}^1}\tilde{\chi}^1(\tilde{p})|||\omega(\tilde{x})||_{C^0_x(D(1))}}{|\partial_{\tilde{x}^1}\gamma^1(\tilde{p})\partial_{\tilde{x}^2}\gamma^2(\tilde{p}) - \partial_{\tilde{x}^2}\gamma^1(\tilde{p})\partial_{\tilde{x}^1}\gamma^2(\tilde{p})|} \\
& + \frac{|\partial_{\tilde{p}^1}\partial_{\tilde{x}^2}\gamma^1(\tilde{p})||\omega(\tilde{p})|||\tilde{\chi}^2(\tilde{x})||_{C^0_x(D(1))}}{|\partial_{\tilde{x}^1}\gamma^1(\tilde{p})\partial_{\tilde{x}^2}\gamma^2(\tilde{p}) - \partial_{\tilde{x}^2}\gamma^1(\tilde{p})\partial_{\tilde{x}^1}\gamma^2(\tilde{p})|} \\
& + \frac{|\partial_{\tilde{p}^1}\partial_{\tilde{x}^2}\gamma^1(\tilde{p})||\tilde{\chi}^2(\tilde{p})|||\omega(\tilde{x})||_{C^0_x(D(1))}}{|\partial_{\tilde{x}^1}\gamma^1(\tilde{p})\partial_{\tilde{x}^2}\gamma^2(\tilde{p}) - \partial_{\tilde{x}^2}\gamma^1(\tilde{p})\partial_{\tilde{x}^1}\gamma^2(\tilde{p})|} \\
& + \frac{|\partial_{\tilde{x}^2}\gamma^1(\tilde{p})||\partial_{\tilde{p}^1}\omega(\tilde{p})|||\tilde{\chi}^2(\tilde{x})||_{C^0_x(D(1))}}{|\partial_{\tilde{x}^1}\gamma^1(\tilde{p})\partial_{\tilde{x}^2}\gamma^2(\tilde{p}) - \partial_{\tilde{x}^2}\gamma^1(\tilde{p})\partial_{\tilde{x}^1}\gamma^2(\tilde{p})|} \\
& + \frac{|\partial_{\tilde{x}^2}\gamma^1(\tilde{p})||\partial_{\tilde{p}^1}\tilde{\chi}^2(\tilde{p})|||\omega(\tilde{x})||_{C^0_x(D(1))}}{|\partial_{\tilde{x}^1}\gamma^1(\tilde{p})\partial_{\tilde{x}^2}\gamma^2(\tilde{p}) - \partial_{\tilde{x}^2}\gamma^1(\tilde{p})\partial_{\tilde{x}^1}\gamma^2(\tilde{p})|} \\
& + \frac{|\partial_{\tilde{x}^2}\gamma^2(\tilde{p})||\omega(\tilde{p})|||\tilde{\chi}^1(\tilde{x})||_{C^0_x(D(1))}}{|\partial_{\tilde{p}^1}[\partial_{\tilde{x}^1}\gamma^1(\tilde{p})\partial_{\tilde{x}^2}\gamma^2(\tilde{p})] - \partial_{\tilde{p}^1}[\partial_{\tilde{x}^2}\gamma^1(\tilde{p})\partial_{\tilde{x}^1}\gamma^2(\tilde{p})]|} \\
& + \frac{|\partial_{\tilde{x}^2}\gamma^2(\tilde{p})||\tilde{\chi}^1(\tilde{p})|||\omega(\tilde{x})||_{C^0_x(D(1))}}{|\partial_{\tilde{p}^1}[\partial_{\tilde{x}^1}\gamma^1(\tilde{p})\partial_{\tilde{x}^2}\gamma^2(\tilde{p})] - \partial_{\tilde{p}^1}[\partial_{\tilde{x}^2}\gamma^1(\tilde{p})\partial_{\tilde{x}^1}\gamma^2(\tilde{p})]|} \\
& + \frac{|\partial_{\tilde{x}^2}\gamma^1(\tilde{p})|[|\omega(\tilde{p})|||\tilde{\chi}^2(\tilde{x})||_{C^0_x(D(1))} + |\tilde{\chi}^2(\tilde{p})|||\omega(\tilde{x})||_{C^0_x(D(1))}]}{|\partial_{\tilde{p}^1}[\partial_{\tilde{x}^1}\gamma^1(\tilde{p})\partial_{\tilde{x}^2}\gamma^2(\tilde{p})] - \partial_{\tilde{p}^1}[\partial_{\tilde{x}^2}\gamma^1(\tilde{p})\partial_{\tilde{x}^1}\gamma^2(\tilde{p})]|} \\
\leq\ & C_1r^2 + C_2r^{-2}.
\end{aligned}
$$

As well, for $k = 1, 2$

$$
\begin{aligned}
||\partial_{\tilde{x}^k}\partial_{\tilde{p}^1}\tilde{\psi}_{\tilde{p}}(\tilde{x})||_{C^0_x(D(1))} \leq & \frac{|\partial_{\tilde{p}^1}\partial_{\tilde{x}^2}\gamma^2(\tilde{p})||\omega(\tilde{p})|||\partial_{\tilde{x}^k}\tilde{\chi}^1(\tilde{x})||_{C^0_x(D(1))}}{|\partial_{\tilde{x}^1}\gamma^1(\tilde{p})\partial_{\tilde{x}^2}\gamma^2(\tilde{p}) - \partial_{\tilde{x}^2}\gamma^1(\tilde{p})\partial_{\tilde{x}^1}\gamma^2(\tilde{p})|} \\
& + \frac{|\partial_{\tilde{p}^1}\partial_{\tilde{x}^2}\gamma^2(\tilde{p})||\tilde{\chi}^1(\tilde{p})|||\partial_{\tilde{x}^k}\omega(\tilde{x})||_{C^0_x(D(1))}}{|\partial_{\tilde{x}^1}\gamma^1(\tilde{p})\partial_{\tilde{x}^2}\gamma^2(\tilde{p}) - \partial_{\tilde{x}^2}\gamma^1(\tilde{p})\partial_{\tilde{x}^1}\gamma^2(\tilde{p})|} \\
& + \frac{|\partial_{\tilde{x}^2}\gamma^2(\tilde{p})||\partial_{\tilde{p}^1}\omega(\tilde{p})|||\partial_{\tilde{x}^k}\tilde{\chi}^1(\tilde{x})||_{C^0_x(D(1))}}{|\partial_{\tilde{x}^1}\gamma^1(\tilde{p})\partial_{\tilde{x}^2}\gamma^2(\tilde{p}) - \partial_{\tilde{x}^2}\gamma^1(\tilde{p})\partial_{\tilde{x}^1}\gamma^2(\tilde{p})|} \\
& + \frac{|\partial_{\tilde{x}^2}\gamma^2(\tilde{p})||\partial_{\tilde{p}^1}\tilde{\chi}^1(\tilde{p})|||\partial_{\tilde{x}^k}\omega(\tilde{x})||_{C^0_x(D(1))}}{|\partial_{\tilde{x}^1}\gamma^1(\tilde{p})\partial_{\tilde{x}^2}\gamma^2(\tilde{p}) - \partial_{\tilde{x}^2}\gamma^1(\tilde{p})\partial_{\tilde{x}^1}\gamma^2(\tilde{p})|} \\
& + \frac{|\partial_{\tilde{p}^1}\partial_{\tilde{x}^2}\gamma^1(\tilde{p})||\omega(\tilde{p})|||\partial_{\tilde{x}^k}\tilde{\chi}^2(\tilde{x})||_{C^0_x(D(1))}}{|\partial_{\tilde{x}^1}\gamma^1(\tilde{p})\partial_{\tilde{x}^2}\gamma^2(\tilde{p}) - \partial_{\tilde{x}^2}\gamma^1(\tilde{p})\partial_{\tilde{x}^1}\gamma^2(\tilde{p})|} \\
& + \frac{|\partial_{\tilde{p}^1}\partial_{\tilde{x}^2}\gamma^1(\tilde{p})||\tilde{\chi}^2(\tilde{p})|||\partial_{\tilde{x}^k}\omega(\tilde{x})||_{C^0_x(D(1))}}{|\partial_{\tilde{x}^1}\gamma^1(\tilde{p})\partial_{\tilde{x}^2}\gamma^2(\tilde{p}) - \partial_{\tilde{x}^2}\gamma^1(\tilde{p})\partial_{\tilde{x}^1}\gamma^2(\tilde{p})|} \\
& + \frac{|\partial_{\tilde{x}^2}\gamma^1(\tilde{p})||\partial_{\tilde{p}^1}\omega(\tilde{p})|||\partial_{\tilde{x}^k}\tilde{\chi}^2(\tilde{x})||_{C^0_x(D(1))}}{|\partial_{\tilde{x}^1}\gamma^1(\tilde{p})\partial_{\tilde{x}^2}\gamma^2(\tilde{p}) - \partial_{\tilde{x}^2}\gamma^1(\tilde{p})\partial_{\tilde{x}^1}\gamma^2(\tilde{p})|} \\
& + \frac{|\partial_{\tilde{x}^2}\gamma^1(\tilde{p})||\partial_{\tilde{p}^1}\tilde{\chi}^2(\tilde{p})|||\partial_{\tilde{x}^k}\omega(\tilde{x})||_{C^0_x(D(1))}}{|\partial_{\tilde{x}^1}\gamma^1(\tilde{p})\partial_{\tilde{x}^2}\gamma^2(\tilde{p}) - \partial_{\tilde{x}^2}\gamma^1(\tilde{p})\partial_{\tilde{x}^1}\gamma^2(\tilde{p})|} \\
& + \frac{|\partial_{\tilde{x}^2}\gamma^2(\tilde{p})||\omega(\tilde{p})|||\partial_{\tilde{x}^k}\tilde{\chi}^1(\tilde{x})||_{C^0_x(D(1))}}{|\partial_{\tilde{p}^1}[\partial_{\tilde{x}^1}\gamma^1(\tilde{p})\partial_{\tilde{x}^2}\gamma^2(\tilde{p})] - \partial_{\tilde{p}^1}[\partial_{\tilde{x}^2}\gamma^1(\tilde{p})\partial_{\tilde{x}^1}\gamma^2(\tilde{p})]|} \\
& + \frac{|\partial_{\tilde{x}^2}\gamma^2(\tilde{p})||\tilde{\chi}^1(\tilde{p})|||\partial_{\tilde{x}^k}\omega(\tilde{x})||_{C^0_x(D(1))}}{|\partial_{\tilde{p}^1}[\partial_{\tilde{x}^1}\gamma^1(\tilde{p})\partial_{\tilde{x}^2}\gamma^2(\tilde{p})] - \partial_{\tilde{p}^1}[\partial_{\tilde{x}^2}\gamma^1(\tilde{p})\partial_{\tilde{x}^1}\gamma^2(\tilde{p})]|} \\
& + \frac{|\partial_{\tilde{x}^2}\gamma^1(\tilde{p})||\omega(\tilde{p})|||\partial_{\tilde{x}^k}\tilde{\chi}^2(\tilde{x})||_{C^0_x(D(1))}}{|\partial_{\tilde{p}^1}[\partial_{\tilde{x}^1}\gamma^1(\tilde{p})\partial_{\tilde{x}^2}\gamma^2(\tilde{p})] - \partial_{\tilde{p}^1}[\partial_{\tilde{x}^2}\gamma^1(\tilde{p})\partial_{\tilde{x}^1}\gamma^2(\tilde{p})]|} \\
& + \frac{|\partial_{\tilde{x}^2}\gamma^1(\tilde{p})||\tilde{\chi}^2(\tilde{p})|||\partial_{\tilde{x}^k}\omega(\tilde{x})||_{C^0_x(D(1))}]}{|\partial_{\tilde{p}^1}[\partial_{\tilde{x}^1}\gamma^1(\tilde{p})\partial_{\tilde{x}^2}\gamma^2(\tilde{p})] - \partial_{\tilde{p}^1}[\partial_{\tilde{x}^2}\gamma^1(\tilde{p})\partial_{\tilde{x}^1}\gamma^2(\tilde{p})]|} \\
\leq & C_1 r^2 + C_2 r^{-2}.
\end{aligned}
$$

And for $j, k = 1, 2$

$$\begin{aligned}
||\partial_{\tilde{x}^j}\partial_{\tilde{x}^k}\partial_{\tilde{p}^1}\tilde{\psi}_{\tilde{p}}(\tilde{x})||_{C^0_x(D(1))} \leq& \frac{|\partial_{\tilde{p}^1}\partial_{\tilde{x}^2}\gamma^2(\tilde{p})||\omega(\tilde{p})|||\partial_{\tilde{x}^j}\partial_{\tilde{x}^k}\tilde{\chi}^1(\tilde{x})||_{C^0_x(D(1))}}{|\partial_{\tilde{x}^1}\gamma^1(\tilde{p})\partial_{\tilde{x}^2}\gamma^2(\tilde{p}) - \partial_{\tilde{x}^2}\gamma^1(\tilde{p})\partial_{\tilde{x}^1}\gamma^2(\tilde{p})|} \\
&+ \frac{|\partial_{\tilde{p}^1}\partial_{\tilde{x}^2}\gamma^2(\tilde{p})||\tilde{\chi}^1(\tilde{p})|||\partial_{\tilde{x}^j}\partial_{\tilde{x}^k}\omega(\tilde{x})||_{C^0_x(D(1))]}}{|\partial_{\tilde{x}^1}\gamma^1(\tilde{p})\partial_{\tilde{x}^2}\gamma^2(\tilde{p}) - \partial_{\tilde{x}^2}\gamma^1(\tilde{p})\partial_{\tilde{x}^1}\gamma^2(\tilde{p})|} \\
&+ \frac{|\partial_{\tilde{x}^2}\gamma^2(\tilde{p})||\partial_{\tilde{p}^1}\omega(\tilde{p})|||\partial_{\tilde{x}^j}\partial_{\tilde{x}^k}\tilde{\chi}^1(\tilde{x})||_{C^0_x(D(1))}}{|\partial_{\tilde{x}^1}\gamma^1(\tilde{p})\partial_{\tilde{x}^2}\gamma^2(\tilde{p}) - \partial_{\tilde{x}^2}\gamma^1(\tilde{p})\partial_{\tilde{x}^1}\gamma^2(\tilde{p})|} \\
&+ \frac{|\partial_{\tilde{x}^2}\gamma^2(\tilde{p})||\partial_{\tilde{p}^1}\tilde{\chi}^1(\tilde{p})|||\partial_{\tilde{x}^j}\partial_{\tilde{x}^k}\omega(\tilde{x})||_{C^0_x(D(1))]}}{|\partial_{\tilde{x}^1}\gamma^1(\tilde{p})\partial_{\tilde{x}^2}\gamma^2(\tilde{p}) - \partial_{\tilde{x}^2}\gamma^1(\tilde{p})\partial_{\tilde{x}^1}\gamma^2(\tilde{p})|} \\
&+ \frac{|\partial_{\tilde{p}^1}\partial_{\tilde{x}^2}\gamma^1(\tilde{p})||\omega(\tilde{p})|||\partial_{\tilde{x}^j}\partial_{\tilde{x}^k}\tilde{\chi}^2(\tilde{x})||_{C^0_x(D(1))}}{|\partial_{\tilde{x}^1}\gamma^1(\tilde{p})\partial_{\tilde{x}^2}\gamma^2(\tilde{p}) - \partial_{\tilde{x}^2}\gamma^1(\tilde{p})\partial_{\tilde{x}^1}\gamma^2(\tilde{p})|} \\
&+ \frac{|\partial_{\tilde{p}^1}\partial_{\tilde{x}^2}\gamma^1(\tilde{p})||\tilde{\chi}^2(\tilde{p})|||\partial_{\tilde{x}^j}\partial_{\tilde{x}^k}\omega(\tilde{x})||_{C^0_x(D(1))]}}{|\partial_{\tilde{x}^1}\gamma^1(\tilde{p})\partial_{\tilde{x}^2}\gamma^2(\tilde{p}) - \partial_{\tilde{x}^2}\gamma^1(\tilde{p})\partial_{\tilde{x}^1}\gamma^2(\tilde{p})|} \\
&+ \frac{|\partial_{\tilde{x}^2}\gamma^1(\tilde{p})||\partial_{\tilde{p}^1}\omega(\tilde{p})|||\partial_{\tilde{x}^j}\partial_{\tilde{x}^k}\tilde{\chi}^2(\tilde{x})||_{C^0_x(D(1))}}{|\partial_{\tilde{x}^1}\gamma^1(\tilde{p})\partial_{\tilde{x}^2}\gamma^2(\tilde{p}) - \partial_{\tilde{x}^2}\gamma^1(\tilde{p})\partial_{\tilde{x}^1}\gamma^2(\tilde{p})|} \\
&+ \frac{|\partial_{\tilde{x}^2}\gamma^1(\tilde{p})||\partial_{\tilde{p}^1}\tilde{\chi}^2(\tilde{p})|||\partial_{\tilde{x}^j}\partial_{\tilde{x}^k}\omega(\tilde{x})||_{C^0_x(D(1))]}}{|\partial_{\tilde{x}^1}\gamma^1(\tilde{p})\partial_{\tilde{x}^2}\gamma^2(\tilde{p}) - \partial_{\tilde{x}^2}\gamma^1(\tilde{p})\partial_{\tilde{x}^1}\gamma^2(\tilde{p})|} \\
&+ \frac{|\partial_{\tilde{x}^2}\gamma^2(\tilde{p})||\omega(\tilde{p})|||\partial_{\tilde{x}^j}\partial_{\tilde{x}^k}\tilde{\chi}^1(\tilde{x})||_{C^0_x(D(1))}}{|\partial_{\tilde{p}^1}[\partial_{\tilde{x}^1}\gamma^1(\tilde{p})\partial_{\tilde{x}^2}\gamma^2(\tilde{p})] - \partial_{\tilde{p}^1}[\partial_{\tilde{x}^2}\gamma^1(\tilde{p})\partial_{\tilde{x}^1}\gamma^2(\tilde{p})]|} \\
&+ \frac{|\partial_{\tilde{x}^2}\gamma^2(\tilde{p})||\tilde{\chi}^1(\tilde{p})|||\partial_{\tilde{x}^j}\partial_{\tilde{x}^k}\omega(\tilde{x})||_{C^0_x(D(1))]}}{|\partial_{\tilde{p}^1}[\partial_{\tilde{x}^1}\gamma^1(\tilde{p})\partial_{\tilde{x}^2}\gamma^2(\tilde{p})] - \partial_{\tilde{p}^1}[\partial_{\tilde{x}^2}\gamma^1(\tilde{p})\partial_{\tilde{x}^1}\gamma^2(\tilde{p})]|} \\
&+ \frac{|\partial_{\tilde{x}^2}\gamma^1(\tilde{p})||\omega(\tilde{p})|||\partial_{\tilde{x}^j}\partial_{\tilde{x}^k}\tilde{\chi}^2(\tilde{x})||_{C^0_x(D(1))}}{|\partial_{\tilde{p}^1}[\partial_{\tilde{x}^1}\gamma^1(\tilde{p})\partial_{\tilde{x}^2}\gamma^2(\tilde{p})] - \partial_{\tilde{p}^1}[\partial_{\tilde{x}^2}\gamma^1(\tilde{p})\partial_{\tilde{x}^1}\gamma^2(\tilde{p})]|} \\
&+ \frac{|\partial_{\tilde{x}^2}\gamma^1(\tilde{p})||\tilde{\chi}^2(\tilde{p})|||\partial_{\tilde{x}^j}\partial_{\tilde{x}^k}\omega(\tilde{x})||_{C^0_x(D(1))]}}{|\partial_{\tilde{p}^1}[\partial_{\tilde{x}^1}\gamma^1(\tilde{p})\partial_{\tilde{x}^2}\gamma^2(\tilde{p})] - \partial_{\tilde{p}^1}[\partial_{\tilde{x}^2}\gamma^1(\tilde{p})\partial_{\tilde{x}^1}\gamma^2(\tilde{p})]|} \\
\leq& C_1 r^2 + C_2 r^{-1}.
\end{aligned}$$

Therefore

$$||\partial_{\tilde{p}}\tilde{\psi}_{\tilde{p}}||_{C^2_{\tilde{x}}(D(1))} \leq \tilde{K}_1$$

as desired.

Now we rescale $\tilde{\psi}_{\tilde{p}}$ to achieve the desired function ψ_p on the disk $D(r)$. Set $\psi_p(x) := r\tilde{\psi}_{\tilde{p}}\left(\frac{x}{r}\right)$. We claim that this is the function which satisfies conditions 1-5.

By construction, ψ_p satisfies conditions 1, 2, and 3. It just remains to prove the estimates 4 and 5. Note that the coordinates change as $x = r\tilde{x}$, $p = r\tilde{p}$, $\tilde{\partial} = r\partial$, (where $\tilde{\partial}$ represents $\partial_{\tilde{x}}$ or $\partial_{\tilde{p}}$), and $\tilde{\partial}\tilde{\psi}_{\tilde{p}} = \frac{1}{r}\partial\psi_p$. From the estimates for $\tilde{\psi}_{\tilde{p}}$ and by change of coordinates,

$$\begin{aligned} ||\psi_p||_{C^2(D(r))} &= r\left|\left|\tilde{\psi}_p\right|\right|_{C^2(D(1))} \\ &\leq r(1 + Cr^2) =: K_2 \end{aligned}$$

Similarly,

$$\begin{aligned} ||\partial_p\psi_p||_{C^2_x(D(r))} &= r||\partial_{\tilde{p}}\tilde{\psi}_{\tilde{p}}||_{C^2_{\tilde{p}}(D(1))} \\ &\leq r(C_2 r^{-2}) =: K_1, \end{aligned}$$

By Proposition 1.1, the area data for (M, g) dertmines ψ_p. Thus, ψ_p is the desired function which satisfies conditions 1-6. □

Now, by the above lemmas, for $i = 1, 2$, each fixed $t \in [0, T]$, and $p \in Y_1(t)$, there exists foliations $Y_{i,1}(s, t)$ given by embeddings

$$\begin{aligned} h_{i,1}(\cdot, \cdot, t) &: [0, S] \times D(r) \to M \subset \mathbf{M} \\ (h_{i,1})_* \left(\frac{\partial}{\partial s}\right)\bigg|_{s=0} &= \psi_{p,i,1}\vec{n} \end{aligned}$$

where $\psi_{p,i,1} : \mathbb{R}^2 \to \mathbb{R}$ has the properties defined in Lemma 5.2 on the disk $D(r) \subset \mathbb{R}^2$;

in particular,

$$\Delta_{g_{\mathbb{E}}}\psi_{p,i,1} + e^{2\phi_1}\left(\mathrm{Ric}_{g_1}(\vec{n},\vec{n}) + ||A||^2_{g_1}\right)\psi_{p,i,1} = 0$$

on $D(r)$, and

$$\psi_{p,i,1}(f_1(p)) = 0,$$
$$\nabla\psi_{p,i,1}(f_1(p)) = \frac{\partial}{\partial x^i}.$$

The induced variation of the coordinate x^3 is written in Taylor expanded form as

$$x_i^3(s) = x^3 + s\dot{x}_i^3 + \mathcal{O}(s^2).$$

As shown by Proposition 4.1, knowledge of the areas of $Y_{i,1}(s,t) := h_{i,1}(D(r),s,t)$ determines the functions

$$||\nabla x_i^3(s)||^2_{g_1}(p)$$

on $\mathbb{R}^2$, in the coordinates (x^α), for all $s \in [0,S]$, $i = 1,2$.

Linearizing in s, by Lemma 4.2 we obtain a nonlinear, coupled system of equations for g_1^{k3}, $k = 1,2$ of the form

$$\frac{d}{ds}||\nabla x_1^3(s)||_{g_1}(p)\bigg|_{s=0} = g_1^{3\alpha}\partial_\alpha\psi_{p,1,1}(x^1,x^2) + \partial_k||\nabla x^3||_g(p)\dot{x}_1^k, \tag{5.12}$$
$$\frac{d}{ds}||\nabla x_2^3(s)||_{g_1}(p)\bigg|_{s=0} = g_1^{3\alpha}\partial_\alpha\psi_{p,2,1}(x^1,x^2) + \partial_k||\nabla x^3||_g(p)\dot{x}_2^k, \tag{5.13}$$

where the first order change in conformal coordinates $\dot{x}_i^k$ depends on p, $\psi_{p,i,1}$, δg and the first and second derivatives of δg, as shown in Lemma 4.3.

By the very same argument as above, if we consider instead the foliation $Y_2(t)$ of

(M, g_2), we may obtain a nonlinear system of equations for g_2^{k3}, $k = 1,2$ of the form

$$\frac{d}{ds}||\nabla y_1^3(s)||_{g_2}(p)\Big|_{s=0} = g_2^{3\alpha}\partial_\alpha \psi_{p,1,2}(x^1, x^2) + \partial_k ||\nabla x^3||_g(p)\dot{y}_1^k, \tag{5.14}$$

$$\frac{d}{ds}||\nabla y_2^3(s)||_{g_2}(p)\Big|_{s=0} = g_2^{3\alpha}\partial_\alpha \psi_{p,2,2}(x^1, x^2) + \partial_k ||\nabla x^3||_g(p)\dot{y}_2^k, \tag{5.15}$$

where $\dot{y}_i^k$ depends on p, $\psi_{p,i,2}$, δg and the first and second derivatives of δg, as as shown in Lemma 4.3, and the functions $\psi_{p,i,2} \in H^2(\mathbb{R}^2)$, $i = 1,2$ are solutions of the Jacobi equation

$$\Delta_{g_{\mathbb{E}}}\psi_{p,i,2} + e^{2\phi_2}\left[\left(\mathrm{Ric}_{g_2}(\vec{n}_2, \vec{n}_2) + ||A||_{g_2}^2\right) \circ F \circ \Phi_1\right]\psi_{p,i,2} = 0$$

on $\mathbb{R}^2$, which additionally satisfy the conditions of Lemma 5.2 on $D(r)$.

Lemma 5.3. *In the coordinates* (x^α), $\psi_{p,i,1} = \psi_{p,i,2}$ *for* $i = 1,2$ *on* $\mathbb{R}^2$.

Proof. Fix $x^3 = y^3 = t$, and consider the leaves $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$. Since the foliations $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ agree outside the disk $D(r) \times \{t\} \equiv D(r) \subset \mathbb{R}^2$, so do the functions $\psi_{p,i,1}$ and $\psi_{p,i,2}$. As shown by Proposition 3.6, our area data determines the Dirichlet-to-Neumann maps associated to the operators

$$\mathcal{J}_1 := \Delta_{g_{1,t}} + \left(\mathrm{Ric}_{g_1}(\vec{n}, \vec{n}) + ||A||_{g_1}^2\right)$$

$$\mathcal{J}_2 := \Delta_{g_{2,t}} + \left(\mathrm{Ric}_{g_2}(\vec{n}, \vec{n}) + ||A||_{g_2}^2\right)$$

on the unit disk D, in the coordinates (x^α) and (y^α) respectively. Here $g_{k,t}$ denotes the metric induced on $Y_k(t)$ by g_k, $k = 1,2$. Via the map F, we can express both the operators in the coordinate system (x^α) as

$$\mathcal{J}_1 := \Delta_{g_{\mathbb{E}}} + e^{2\phi_1}\left(\mathrm{Ric}_{g_1}(\vec{n}, \vec{n}) + ||A||_{g_1}^2\right) \tag{5.16}$$

$$\mathcal{J}_2 := \Delta_{g_{\mathbb{E}}} + e^{2\phi_2}\left(\mathrm{Ric}_{g_2}(\vec{n}, \vec{n}) + ||A||_{g_2}^2\right) \circ F \circ \Phi_1. \tag{5.17}$$

By Proposition 1.1, the Dirichlet-to-Neumann maps to these operators is also determined as expressed in the coordinates (x^α). By construction the foliation f_1 agrees the foliation f_2 on the boundary of M, and by hypothesis the area of $Y_1(t)$ as measured by g_1 is equal to the area of $Y_2(t)$ as measured by g_2. Thus, the Dirichlet-to-Neumann maps associated to the operators (5.16) and (5.17) agree as maps $H^{\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$. Then, by the linear result in [10], the potential functions

$$e^{2\phi_1}\left(\mathrm{Ric}_{g_1}(\vec{n},\vec{n}) + ||A||^2_{g_1}\right)$$
$$e^{2\phi_2}\left(\mathrm{Ric}_{g_2}(\vec{n},\vec{n}) + ||A||^2_{g_2}\right) \circ F \circ \Phi_1$$

agree on D, as written in the coordinates (x^α).

So in the coordinates (x^α), $\psi_{p,i,1} = \psi_{p,i,2}$ on D for $i = 1, 2$. □

With the above lemma in hand, we simplify notation a bit and write

$$\psi_{p,i} := \psi_{p,i,1} = \psi_{p,i,2}$$

for $i = 1, 2$.

Now in the above setting, we may obtain equations which relate the differences of the metric components $\delta\phi$, δg^{31}, and δg^{32}:

Lemma 5.4. *For each $p \in M$, the unknown differences $\delta\phi$, δg^{31}, δg^{32} satisfy the following system of equations:*

$$0 = \delta g^{3k}(p)\partial_k\psi_{p,1}(x(p)) + \partial_k||\nabla x^3||_{g_1}(p)\delta\dot{x}^k_1(p) \tag{5.18}$$

$$0 = \delta g^{3k}(p)\partial_k\psi_{p,2}(x(p)) + \partial_k||\nabla x^3||_{g_1}(p)\delta\dot{x}^k_2(p) \tag{5.19}$$

$$0 = g_1^{k3}\partial_k(\delta\phi) + g_1^{33}\partial_3(\delta\phi) + \delta g^{k3}\left(\partial_k\phi_2 - \frac{1}{2}\partial_k\log(g_1^{33})\right) + \frac{1}{2}\partial_k(\delta g^{3k}). \tag{5.20}$$

Furthermore, by construction $\delta\phi = 0$, and $\delta g^{3k} = 0$ for $k = 1, 2$ on $\mathbf{M} \setminus M \cup \partial M$.

Proof. For each fixed $x^3 = y^3 = t$, since we assume the area of $Y_1(t)$ equals the area of $Y_2(t)$, we have

$$\frac{d}{ds}||\nabla x_i^3(s)||_{g_1}^2(p)\bigg|_{s=0} = \frac{d}{ds}||\nabla y_i^3(s)||_{g_2}(p)\bigg|_{s=0}$$

on $\mathbb{R}^2$ in the coordinates (x^α).

Then by the previous lemma and Lemma 4.2, taking the difference of (5.12) and (5.14) we get

$$\begin{aligned}
0 &= \frac{d}{ds}||\nabla x_1^3(s)||_{g_1}(p)\bigg|_{s=0} - \frac{d}{ds}||\nabla y_1^3(s)||_{g_2}(p)\bigg|_{s=0} \\
&= [g_1^{3k} - g_2^{3k}](p)\partial_k\psi_{p,1}(x(p)) \\
&\qquad + \partial_k||\nabla x^3||_{g_1}(p)[\dot{x}^k(p,\phi_1,g_1,\psi_{p,1}) - (\dot{y}_1 \circ (F \circ \Phi_1))^k(p)] \\
&= \delta g^{3k}(p)\partial_k\psi_{p,1}(x(p)) + \partial_k||\nabla x^3||_{g_1}(p)\delta\dot{x}_1^k(p),
\end{aligned}$$

where $k = 1, 2$, $\alpha = 1, 2, 3$. Taking the difference of (5.13) and (5.15) we can derive a similar equation for δg^{3k} in terms of $\psi_{p,2}$ instead of $\psi_{p,1}$:

$$0 = \delta g^{3k}(p)\partial_k\psi_{p,2}(x(p)) + \partial_k||\nabla x^3||_{g_1}(p)\delta\dot{x}_2^k(p)\delta\phi,$$

where $k = 1, 2$, $\alpha = 1, 2, 3$.

Now, we turn our attention to deriving an equation for the differences of the conformal factors. From the minimality of each $Y(t)$, Proposition 4.4 gives the conformal factors ϕ_1, ϕ_2 satisfy

$$0 = g_1^{31}\partial_1\phi_1 + g_1^{32}\partial_2\phi_1 + g_1^{33}\partial_3\phi_1 + \frac{1}{2}\partial_k g_1^{3k} - \frac{1}{2}g_1^{3k}\partial_k\log(g_1^{33}), \tag{5.21}$$

$$0 = g_2^{31}\partial_1\phi_2 + g_2^{32}\partial_2\phi_2 + g_2^{33}\partial_3\phi_2 + \frac{1}{2}\partial_k g_2^{3k} - \frac{1}{2}g_2^{3k}\partial_k\log(g_2^{33}). \tag{5.22}$$

Recall we showed earlier $g_1^{33} = g_2^{33}$. Subtracting (5.22) from (5.21) we find

$$\begin{aligned}
0 &= \left[g_1^{31}\partial_1\phi_1 + g_1^{32}\partial_2\phi_1 + g_1^{33}\partial_3\phi_1 + \frac{1}{2}\partial_k g_1^{3k} - \frac{1}{2}g_1^{3k}\partial_k \log(g_1^{33})\right] \\
&\quad - \left[g_2^{31}\partial_1\phi_2 + g_2^{32}\partial_2\phi_2 + g_2^{33}\partial_3\phi_2 + \frac{1}{2}\partial_k g_2^{3k} - \frac{1}{2}g_2^{3k}\partial_k \log(g_2^{33})\right] \\
&= g_1^{13}\partial_1(\delta\phi) + g_1^{23}\partial_2(\delta\phi) + g_1^{33}\partial_3(\delta\phi) + (\delta g^{13})\partial_1\phi_2 + (\delta g^{23})\partial_2\phi_2 \\
&\quad + \frac{1}{2}\partial_k\delta g^{3k} - \frac{1}{2}\delta g^{3k}\partial_k \log(g_1^{33}) \\
&= g_1^{k3}\partial_k(\delta\phi) + g_1^{33}\partial_3(\delta\phi) + \delta g^{k3}\left(\partial_k\phi_2 - \frac{1}{2}\partial_k \log(g_1^{33})\right) + \frac{1}{2}\partial_k(\delta g^{3k}),
\end{aligned} \tag{5.23}$$

for $k = 1, 2$.

In summary, in the coordinate system (x^α) on M, the following system of three equations governs the behaviour of the three unknown differences $\delta\phi$, δg^{31}, δg^{32}:

$$0 = \delta g^{3k}(p)\partial_k\psi_{p,1}(x(p)) + \partial_k||\nabla x^3||_{g_1}(p)\delta\dot{x}_1^k(p)\delta\phi$$

$$0 = \delta g^{3k}(p)\partial_k\psi_{p,2}(x(p)) + \partial_k||\nabla x^3||_{g_1}(p)e\delta\dot{x}_2^k(p)\delta\phi$$

$$0 = g_1^{k3}\partial_k(\delta\phi) + g_1^{33}\partial_3(\delta\phi) + \delta g^{k3}\left(\partial_k\phi_2 - \frac{1}{2}\partial_k \log(g_1^{33})\right) + \frac{1}{2}\partial_k(\delta g^{3k}).$$

By construction, $F = Id$ on $\mathbf{M} \setminus M \cup \partial M$, so $\delta\phi = 0$, and $\delta g^{3k} = 0$ for $k = 1, 2$, on the set $\mathbf{M} \setminus M \cup \partial M$.

□

Writing δg^{3k} as pseudodifferential operators: Our goal from this point forward is to write δg^{3k}, $k = 1, 2$ as a linear combination of pseudodifferential operators acting on $\delta\phi$ and $\partial_3\delta\phi$. Viewing δg^{3k} is this way, from estimates for these pseudodifferential operators we show equation (5.23) can be written as a hyperbolic pseudodifferential operator acting on $\delta\phi$. A standard energy argument gives us uniqueness for the conformal factors, and in turn, uniqueness for the metrics g_1 and g_2.

Lemma 5.5 (δg^{31} and δg^{31} are ΨDOs)**.** *At each $p \in M$, the differences δg^{31} and δg^{32} can be expressed as*

$$\delta g^{31}(p) =: P^1_0(\delta\phi, p) + Q^1_{-1}(\partial_3\delta\phi, p) \tag{5.24}$$

$$\delta g^{32}(p) =: P^2_0(\delta\phi, p) + Q^2_{-1}(\partial_3\delta\phi, p), \tag{5.25}$$

where $P^k_0 : L^2(D(r)) \to H^1(D(r))$, $Q^k_{-1} : L^2(D(r)) \to H^2(D(r))$, $k = 1, 2$, $s > 0$, are respectively order 0 and -1 pseudodifferential operators in the tangential directions ∂_k, $k = 1, 2$.

Further, we have the estimates

$$||P^k_0(\delta\phi)||_{H^1(D(r))} \leq ||\nabla g^{33}_1||_{L^\infty(D)} C(M, g_1, g_2, p)||\delta\phi||_{L^2(D(r))} \tag{5.26}$$

$$||Q^k_{-1}(\partial_3\delta\phi)||_{H^2(D(r))} \leq ||\nabla g^{33}_1||_{L^\infty(D)} C(M, g_1, g_2, p)||\partial_3\delta\phi||_{L^2(D(r))}, \tag{5.27}$$

$$||\nabla Q^k_{-1}(\partial_3\delta\phi)||_{H^1(D(r))} \leq \epsilon_0 C(M, g_1, g_2, p)||\partial_3\delta\phi||_{L^2(D(r))} \tag{5.28}$$

for $j, k = 1, 2$.

Proof. First we derive the operators P^k_0, Q^k_{-1}. Let $w \in M$, and set $\Delta_{g_{\mathbb{E}}} = \frac{\partial^2}{\partial(w^1)^2} + \frac{\partial^2}{\partial(w^2)^2}$. From Lemma 4.3, for each of our choices $\psi_{p,i}$, $i = 1, 2$, the first order change in the conformal coordinates (x^1, x^2) is given schematically as

$$\begin{aligned}\Delta_{g_{\mathbb{E}}} \dot{x}^k_i = {} & \psi_{p,i} A^{mjk}_l \partial_m\partial_j g^{3l}_1 + \psi_{p,i} B^{mk}_\alpha \partial_m\partial_\alpha\phi_1 + \psi_{p,i} C^{mjk}_\alpha \partial_m\phi\partial_j g^{3\alpha}_1 \\ & + \psi_{p,i} D^{mjk}_{\alpha\beta} \partial_m g^{3\alpha}\partial_j g^{3\beta} + \psi_{p,i} F^{mk\alpha}\partial_m\phi\partial_\alpha\phi_1 \\ & + (\psi_{p,i} H^{mk}_\alpha + \nabla_j\psi_{p,i} I^{mjk}_\alpha)\partial_m g^{3\alpha}_1 + (\psi_{p,i} J^{k\alpha}_1 + \nabla_m\psi_p J^{mk\alpha}_2)\partial_\alpha\phi, \\ =: {} & \mathcal{F}^k(g^{13}, g^{23}, \phi, \psi_p),\end{aligned} \tag{5.29}$$

with an analogous equation for the conformal coordinates (y^1, y^2).

Therefore, the differences in the conformal coordinate functions $\delta\dot{x}^k(w) : x^k - y^k$, $k = 1,2$ satisfy on the the disk $D(r)$ an equation of the form

$$\Delta_{g_{\mathbb{E}}}\delta\dot{x}_i^k(w) = \delta\mathcal{F}^k(w, \delta\phi, \delta g, g_1, g_2, \psi_{p,i})$$

where the differential operator $\delta\mathcal{F}^k$ is written schematically as

$$\begin{aligned}\delta\mathcal{F}^k(w) = \psi_{p,i}\bar{A}_m^{jkl}\partial_l\partial_j\delta g^{3m}(q) + \psi_{p,i}\bar{B}^{jk\alpha}\partial_j\partial_\alpha\delta\phi(w) + (\psi_{p,i}\bar{C}_1^{k\alpha} + \partial_j\psi_{p,i}\bar{C}_2^{jk\alpha})(w)\partial_\alpha\delta\phi(w) \\ + (\psi_{p,i}\bar{C}_3 + \partial_j\psi_{p,i}\bar{C}_4^j)\delta\phi + (\psi_{p,i}\bar{D}_{1m}^{jk} + \partial_l\psi_{p,i}\bar{D}_{2m}^{jkl})(w)\partial_j\delta g^{3m}(w) \\ + (\psi_{p,i}\bar{F}_{1m}^k + \partial_l\psi_{p,i}\bar{F}_{2m}^{kl})(w)\delta g^{3m}(w), \qquad (5.30)\end{aligned}$$

for bounded functions $\bar{A}_m^{jkl}, \ldots, \bar{F}_{2m}^{kl}$, which depend on $w \in D(r)$ and are smooth functions in the unknown metric coefficients g_1^{13}, g_1^{23} and g_2^{13}, g_2^{23} and their first derivatives at q.

At a point $p \in M$, we have the expression

$$\delta\dot{x}_i^k(p) = \int_D G(p,w)\delta\mathcal{F}^k(w)\,dw, \qquad (5.31)$$

where $G(p,w)$ is the Dirichlet Greens function on the disk $D(r)$:

$$\begin{aligned}\Delta_{g_{\mathbb{E}}}G(p,w) &= \delta(p-w) && \text{for } x(p) \in D(r) \\ G(p,w) &= 0 && \text{for } x(p) \in \partial D(r).\end{aligned}$$

As our goal is to write δg^{3k}, $k = 1,2$ as pseudodifferential operators in $\delta\phi, \partial_3\delta\phi$; we thus analyze the form of (5.31) in a bit more detail.

Expanding out (5.31),

$$\int_{\mathbb{R}^2} G(p,w)\delta\mathcal{F}^k(w)\,dw = \int_{D(r)} G(p,w)(\psi_{p,i}\bar{A}^{jkl}_m \partial_l\partial_j \delta g^{3m})(w)\,dw \tag{5.32}$$
$$+ \int_{D(r)} G(p,w)(\psi_{p,i}\bar{B}^{jk\alpha}\partial_j\partial_\alpha\delta\phi)(w)\,dw$$
$$+ \int_{D(r)} G(p,w)([\psi_{p,i}\bar{C}^{k\alpha}_1 + \partial_j\psi_{p,i}\bar{C}^{jk\alpha}_2]\partial_\alpha\delta\phi)(w)\,dw$$
$$+ \int_{D(r)} G(p,w)([(\psi_{p,i}\bar{C}_3 + \partial_j\psi_{p,i}\bar{C}^j_4]\delta\phi)(w)\,dw$$
$$+ \int_{D(r)} G(p,w)[(\psi_{p,i}\bar{D}^{jk}_{1m} + \partial_l\psi_{p,i}\bar{D}^{jkl}_{2m})\partial_j\delta g^{3m}]\,dw$$
$$+ \int_{D(r)} G(p,w)[(\psi_{p,i}\bar{F}^{k}_{1m} + \partial_l\psi_{p,i}\bar{F}^{kl}_{2m})\delta g^{3m}]\,dw.$$

Let $B(q,\epsilon)$ be a ball of radius $1 >> \epsilon > 0$ about $q \in D(r)$, and let ν denote the outward pointing normal to $B(q,\epsilon)$. Write $U(q,\epsilon) := D(r) \setminus B(q,\epsilon)$. Consider the right hand side in line (5.32). Integrating by parts,

$$\int_{U(q,\epsilon)} G(p,w)(\psi_{p,i}\bar{A}^{jkl}_m\partial_l\partial_j\delta g^{3m})(w)\,dw \overset{IBP}{=} \int_{U(q,\epsilon)} \partial_l\partial_j[G(p,w)(\psi_{p,i}\bar{A}^{jkl}_m)(w)]\delta g^{3m}(w)\,dw$$
$$- \int_{\partial B(q,\epsilon)} \partial_j[G(p,w)(\psi_{p,i}\bar{A}^{jkl}_m)(w)]\delta g^{3m}(w)\nu_l\,dw$$
$$+ \int_{\partial B(q,\epsilon)} [G(p,w)(\psi_{p,i}\bar{A}^{jkl}_m)(w)]\partial_l\delta g^{3m}(w)\nu_j\,dw$$

Since we've chosen $\psi_{p,i}$ to vanish at the point $p \in M$ and $\partial_j\psi_{p,i}(p) = \delta_{ij}$ (Kronecker delta), as we take the limit $\epsilon \to 0^+$, the boundary integrals vanish by definition of the Greens function. Thus,

$$\int_{D(r)} G(p,w)(\psi_{p,i}\bar{A}^{jkl}_m\partial_l\partial_j\delta g^{3m})(w)\,dw = \int_D \partial_l\partial_j[G(p,w)(\psi_{p,i}\bar{A}^{jkl}_m)(w)]\delta g^{3m}(w)\,dw.$$

By analogous arguments for the other integral terms, we obtain

$$
\begin{aligned}
\delta\dot{x}_i^k(p) &= \int_{\mathbb{R}^2} G(p,w)\delta\mathcal{F}^k(w)\,dw \\
&\overset{IBP}{=} \int_{D(r)} \partial_l\partial_j[G(p,w)(\psi_{p,i}\bar{A}_m^{jkl})(w)]\delta g^{3m}(w)\,dw \\
&\quad + \int_{D(r)} \partial_j\partial_l[G(p,w)(\psi_{p,i}\bar{B}^{jkl})(w)]\delta\phi(w)\,dw \\
&\quad - \int_{D(r)} \partial_j[G(p,w)(\psi_{p,i}\bar{B}^{jk3})(w)]\partial_3\delta\phi(w)\,dw \\
&\quad - \int_{D(r)} \partial_l[G(p,w)(\psi_{p,i}\bar{C}_1^{kl} + \partial_j\psi_{p,i}\bar{C}_2^{jkl})(w)]\delta\phi(w)\,dw \\
&\quad + \int_{D(r)} G(p,w)(\psi_{p,i}\bar{C}_1^{k3} + \partial_j\psi_{p,i}\bar{C}_2^{jk3})(w)\partial_3\delta\phi(w)\,dw \\
&\quad + \int_{D(r)} G(p,w)([(\psi_{p,i}\bar{C}_3 + \partial_j\psi_{p,i}\bar{C}_4^j]\delta\phi)(w)\,dw \\
&\quad - \int_{D(r)} \partial_j[G(p,w)(\psi_{p,i}\bar{D}_{1m}^{jk} + \partial_l\psi_{p,i}\bar{D}_{2m}^{jkl})(w)]\delta g^{3m}(w)\,dw \\
&\quad + \int_{D(r)} [G(p,w)(\psi_{p,i}\bar{F}_{1m}^{k} + \partial_l\psi_{p,i}\bar{F}_{2m}^{kl})]\delta g^{3m}\,dw.
\end{aligned}
$$

Consider the first integrand: $\partial_l\partial_j[G(p,w)(\psi_{p,i}\bar{A}_m^{jkl})(w)]$. Expanding,

$$
\begin{aligned}
\partial_l\partial_j[G(p,w)(\psi_{p,i}\bar{A}_m^{jkl})(w)] &= \partial_l\partial_j G(p,w)\psi_{p,i}(w)\bar{A}_m^{jkl}(w) + \partial_l G(p,w)\partial_j\psi_{p,i}(w)\bar{A}_m^{jkl}(w) \\
&\quad + \partial_j G(p,w)\partial_l\psi_{p,i}(w)\bar{A}_m^{jkl}(w) + \partial_l G(p,w)\psi_{p,i}(w)\partial_j\bar{A}_m^{jkl}(w) \\
&\quad + \partial_j G(p,w)\psi_{p,i}(w)\partial_l\bar{A}_m^{jkl}(w) + G(p,w)\partial_j\psi_{p,i}(w)\partial_l\bar{A}_m^{jkl}(w) \\
&\quad + G(p,w)\partial_l\psi_{p,i}(w)\partial_j\bar{A}_m^{jkl}(w) + G(p,w)\partial_l\partial_j\psi_{p,i}(w)\bar{A}_m^{jkl}(w) \\
&\quad + G(p,w)\psi_{p,i}(w)\partial_l\partial_j\bar{A}_m^{jkl}(w).
\end{aligned}
$$

Since we have chosen the functions $\psi_{p,i}$ to vanish at the point p, the singularity of the kernel $\partial_l\partial_j[G(p,w)(\psi_{p,i}\bar{A}_m^{jkl})(w)]$ along $w = p$ is of order -1. By similar analysis of the other kernels appearing in $\delta\dot{x}_i^k$, we conclude that $\delta\dot{x}_i^k$ is comprised of order -1 pseudodifferential operators acting on δg^{3k}, $\delta\phi$, and $\partial_3\delta\phi$.

From the expansion above, we denote the respective kernels

$$\begin{aligned}
K_{i,1}^k(p,w) &= \partial_l\partial_j[G(p,w)(\psi_{p,i}\bar{A}_1^{jkl})(w)] - \partial_j[G(p,w)(\psi_{p,i}\bar{D}_{11}^{jk} + \partial_l\psi_{p,i}\bar{D}_{21}^{jkl})(w)] \\
&\quad + G(p,w)(\psi_{p,i}\bar{F}_{11}^k + \partial_l\psi_{p,i}\bar{F}_{21}^{kl}), \\
K_{i,2}^k(p,w) &= \partial_l\partial_j[G(p,w)(\psi_{p,i}\bar{A}_2^{jkl})(w)] - \partial_j[G(p,w)(\psi_{p,i}\bar{D}_{12}^{jk} + \partial_l\psi_{p,i}\bar{D}_{22}^{jkl})(w)] \\
&\quad + G(p,w)(\psi_{p,i}\bar{F}_{12}^k + \partial_l\psi_{p,i}\bar{F}_{22}^{kl}), \\
L_{i,1}^k(p,w) &= \partial_j\partial_l[G(p,w)(\psi_{p,i}\bar{B}^{jkl})(w)] - \partial_l[G(p,w)(\psi_{p,i}\bar{C}_1^{kl} + \partial_j\psi_{p,i}\bar{C}_2^{jkl})(w)], \\
L_{i,2}^k(p,w) &= \partial_j[G(p,w)(\psi_{p,i}\bar{B}^{jk3})(w)] + G(p,w)(\psi_{p,i}\bar{C}_1^{k3} + \partial_j\psi_{p,i}\bar{C}_2^{jk3})(w).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\delta\dot{x}_i^k(p) &= \int_{D(r)} K_{i,1}^k(p,w)\delta g^{31}(w)\,dw + \int_{D(r)} K_{i,2}^k(p,w)\delta g^{32}(w)\,dw \\
&\quad + \int_{D(r)} L_{i,1}^k(p,w)\delta\phi(w)\,dw + \int_{D(r)} L_{i,2}^k(p,w)\partial_3\delta\phi(w)\,dw.
\end{aligned}$$

Denote the pseudodifferential operators

$$\begin{aligned}
\mathcal{K}_{i,j}(f)(p) &:= \partial_k||\nabla x^3||_{g_1}(p)\int_{D(r)} K_{i,j}^k(p,w)f(w)\,dw, \\
\mathcal{L}_{i,j}(f)(p) &:= \partial_k||\nabla x^3||_{g_1}(p)\int_{D(r)} L_{i,j}^k(p,w)f(w)\,dw.
\end{aligned}$$

With the above notations, the equations (5.18), (5.19), which describe δg^{31} and δg^{32}, can be written as the system

$$[1-\mathcal{K}](\delta g^{31},\delta g^{32}) = \mathcal{L}(\delta\phi,\partial\delta\phi) \tag{5.33}$$

where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\mathcal{K} = \begin{pmatrix} \mathcal{K}_{1,1} & \mathcal{K}_{1,2} \\ \mathcal{K}_{2,1} & \mathcal{K}_{2,2} \end{pmatrix},$$

$$\mathcal{L} = \begin{pmatrix} \mathcal{L}_{1,1} & \mathcal{L}_{1,2} \\ \mathcal{L}_{2,1} & \mathcal{L}_{2,2} \end{pmatrix}.$$

To solve this system for δg^{31}, δg^{32} in terms of $\delta\phi$, $\partial\delta\phi$, we need to show that the operators $\mathcal{K}_{i,j}$, $i,j=1,2$, have small norms as operators $L^2(D(r)) \to H^1(D(r))$. First, we prove estimates for $\mathcal{K}_{i,j}$ and $\mathcal{L}_{i,j}$, $i,j=1,2$. Then, we show the necessary smallness requirement.

Consider the operator $\mathcal{K}_{2,2}$:

$$\begin{aligned}
\mathcal{K}_{2,2}\delta g^{32}(p) &:= \partial_k||\nabla x^3||_{g_1}(p)\int_{D(r)} \delta g^{32}(w) K^k_{2,2}(p,w)\,dw \\
&= \partial_k||\nabla x^3||_{g_1}(p)\int_{D(r)} \delta g^{32}(w)\partial_l\partial_j[G(p,w)(\psi_{p,2}\bar{A}_2^{jkl})(w)]\,dw \\
&\quad - \partial_k||\nabla x^3||_{g_1}(p)\int_{D(r)} \partial_j[G(p,w)(\psi_{p,2}\bar{D}^{jk}_{12} + \partial_l\psi_{p,2}\bar{D}^{jkl}_{22})(w)]\,dw \\
&\quad + \partial_k||\nabla x^3||_{g_1}(p)\int_{D(r)} G(p,w)(\psi_{p,2}\bar{F}^k_{12} + \partial_l\psi_{p,2}\bar{F}^{kl}_{22})\,dw.
\end{aligned}$$

Since $\psi_{p,i}(p) = 0$, the above terms containing $\partial_l\partial_j G(p,w)$ vanish at p. Then from the estimates for $||\psi_{p,i}||_{C^2_x}$ proved in Lemma 5.2, we find the estimate

$$\begin{aligned}
\left|\left|\int_{D(r)} \delta g^{32}(w) K^k_{2,2}(\cdot, w)\, dw\right|\right|_{H^1} &\leq \left|\left|\int_{D(r)} \delta g^{32}(w)\partial_l\partial_j[G(p,w)(\psi_{p,2}\bar{A}^{jkl}_2)(w)]\, dw\right|\right|_{H^1} \\
&\quad + \left|\left|\int_{D(r)} \partial_j[G(p,w)(\psi_{p,2}\bar{D}^{jk}_{12} + \partial_l\psi_{p,2}\bar{D}^{jkl}_{22})(w)]\, dw\right|\right|_{H^1} \\
&\quad + \left|\left|\int_{D(r)} G(p,w)(\psi_{p,2}\bar{F}^k_{12} + \partial_l\psi_{p,2}\bar{F}^{kl}_{22})\, dw\right|\right|_{H^1} \\
&\leq C^j(M, g_1, g_2)||\psi_{p,2}||_{C^2_x} \left|\left|\int_{D(r)} \delta g^{32}(w)\partial_j G(p,w)\, dw\right|\right|_{H^1} \\
&\quad + C(M, g_1, g_2)||\psi_{p,2}||_{C^2_x} \left|\left|\int_{D(r)} \delta g^{32}(w) G(p,w)\, dw\right|\right|_{H^1} \\
&\leq C(M, g_1, g_2)||\psi_{p,2}||_{C^2_x}||\delta g^{32}||_{L^2},
\end{aligned}$$

where the norms are over $D(r)$, e.g. $||\cdot||_{H^1} = ||\cdot||_{H^1(D(r))}$. Then,

$$\begin{aligned}
||\mathcal{K}_{2,2}\delta g^{32}||_{H^1} &\leq ||\partial_k||\nabla x^3||_{g_1}||_{L^\infty} \left|\left|\int_{D(r)} \delta g^{32}(w) K^k_{2,2}(\cdot, w)\, dw\right|\right|_{H^1} \\
&\leq C(M, g_1, g_2)||\partial_k||\nabla x^3||_{g_1}||_{L^\infty}||\psi_{p,2}||_{C^2_x}||\delta g^{32}||_{L^2}.
\end{aligned}$$

Note

$$\partial_k||\nabla x^3||_{g_1}(p) = \partial_k\sqrt{g^{33}_1}(p) \leq C(M, g_1)||\nabla g^{33}_1||_{L^\infty(D)},$$

for some constant $C(M, g_1, p)$. Hence,

$$||\mathcal{K}_{2,2}\delta g^{32}||_{H^1(D(r))} \leq ||\nabla g^{33}_1||_{L^\infty(D(r))} C(M, g_1, g_2)||\delta g^{32}||_{L^2(D(r))}.$$

By analogous arguments, we obtain similar bounds for the operators $\mathcal{K}_{i,j}$ and $\mathcal{L}_{i,1}$, $i, j = 1, 2$. For the operators $\mathcal{L}_{i,2}$, $i = 1, 2$ since $\psi_{p,i}(p) = 0$, the term containing $\partial_j G(p, w)$ vanish at p. Hence $\mathcal{L}_{i,2} : L^2(D(r)) \to H^2(D(r))$, and by a similar argument as for the estimate for $||\mathcal{K}_{2,2}\delta g^{32}||_{H^1(D(r))}$, we have

$$||\mathcal{L}_{i,2}\partial_3\delta\phi||_{H^2(D(r))} \leq ||\nabla g_1^{33}||_{L^\infty(D(r))}C(M, g_1, g_2, i)||\partial_3\delta\phi||_{L^2(D(r))}.$$

Recall by hypothesis in Theorem 1.6, $||\nabla g_1^{33}||_{L^\infty(D(r))} \leq \epsilon_0$ is small. So the operators $\mathcal{K}_{i,j}$, $\mathcal{L}_{i,1} : L^2(D(r)) \to H^1(D(r))$ and $\mathcal{L}_{i,2} : L^2(D(r)) \to H^2(D(r))$ have small norm. In particular, $1 - \mathcal{K}$ is invertible.

Therefore the system (5.33) is solvable in terms of $\delta\phi$ and $\partial_3\delta\phi$. In particular,

$$\delta g^{31} =: P_0^1(\delta\phi) + Q_{-1}^1(\partial_3\delta\phi)$$
$$\delta g^{32} =: P^2(\delta\phi) + Q_{-1}^2(\partial_3\delta\phi),$$

where P_0^k, Q_{-1}^k, $k = 1, 2$ are respectively order 0 and -1 pseudodifferential operators in the tangential directions ∂_k, $k = 1, 2$, given by the compositions

$$P_0^1 = \left[1 - \mathcal{K}_{1,1} - \mathcal{K}_{1,2}(1 - \mathcal{K}_{2,2})^{-1}\mathcal{K}_{2,1}\right]^{-1}\left[(1 - \mathcal{K}_{2,2})^{-1}\mathcal{L}_{2,1} + \mathcal{L}_{1,1}\right]$$
$$P_0^2 = (1 - \mathcal{K}_{2,2})^{-1}\mathcal{K}_{2,1}\left[1 - \mathcal{K}_{1,1} - \mathcal{K}_{1,2}(1 - \mathcal{K}_{2,2})^{-1}\mathcal{K}_{2,1}\right]^{-1}\left[(1 - \mathcal{K}_{2,2})^{-1}\mathcal{L}_{2,1} + \mathcal{L}_{1,1}\right]$$
$$+ (1 - \mathcal{K}_{2,2})^{-1}\mathcal{L}_{2,1}$$
$$Q_{-1}^1 = \left[1 - \mathcal{K}_{1,1} - \mathcal{K}_{1,2}(1 - \mathcal{K}_{2,2})^{-1}\mathcal{K}_{2,1}\right]^{-1}\left[\mathcal{L}_{1,2} + (1 - \mathcal{K}_{2,2})^{-1}\mathcal{L}_{2,2}\right]$$
$$Q_{-1}^2 = (1 - \mathcal{K}_{2,2})^{-1}\mathcal{K}_{2,1}\left[1 - \mathcal{K}_{1,1} - \mathcal{K}_{1,2}(1 - \mathcal{K}_{2,2})^{-1}\mathcal{K}_{2,1}\right]^{-1}\left[\mathcal{L}_{1,2}\right.$$
$$\left.+(1 - \mathcal{K}_{2,2})^{-1}\mathcal{L}_{2,2}\right] + (1 - \mathcal{K}_{2,2})^{-1}\mathcal{L}_{2,2}.$$

The bounds on $\mathcal{K}_{i,j}$ and $\mathcal{L}_{i,j}$ for $i, j = 1, 2$ give the claimed inequalities for P_0^k and

Q^k_{-1}, $k = 1, 2$:

$$||P^k_0(\delta\phi)||_{H^1(D(r))} \leq ||\nabla g^{33}_1||_{L^\infty(D(r))} C(M, g_1, g_2) ||\delta\phi||_{L^2(D(r))}$$

$$||Q^k_{-1}(\partial_3\delta\phi)||_{H^2(D(r))} \leq ||\nabla g^{33}_1||_{L^\infty(D(r))} C(M, g_1, g_2) ||\partial_3\delta\phi||_{L^2(D(r))}.$$

So estimates (5.26) and (5.27) have been shown.

Now we seek estimate (5.28); that is, to obtain estimates for the tangential derivatives of the operators Q^k_{-1}.

As was shown in the above expansions, the operators $Q^k_{-1} : L^2(D(r)) \to H^2(D(r))$ are of the form

$$Q^k_{-1}(\partial_3\delta\phi)(p) := \mathcal{S}^k \circ \mathcal{L}_{1,2}(\partial_3\delta\phi)(p) + \mathcal{T}^k \circ \mathcal{L}_{2,2}(\partial_3\delta\phi)(p) \tag{5.34}$$

where

$$\mathcal{S}^1 = \left[1 - \mathcal{K}_{1,1} - \mathcal{K}_{1,2}(1 - \mathcal{K}_{2,2})^{-1}\mathcal{K}_{2,1}\right]^{-1},$$

$$\mathcal{S}^2 = (1 - \mathcal{K}_{2,2})^{-1}\mathcal{K}_{2,1}\left[1 - \mathcal{K}_{1,1} - \mathcal{K}_{1,2}(1 - \mathcal{K}_{2,2})^{-1}\mathcal{K}_{2,1}\right]^{-1},$$

$$\mathcal{T}^1 = \left[1 - \mathcal{K}_{1,1} - \mathcal{K}_{1,2}(1 - \mathcal{K}_{2,2})^{-1}\mathcal{K}_{2,1}\right]^{-1}(1 - \mathcal{K}_{2,2})^{-1},$$

$$\mathcal{T}^2 = (1 - \mathcal{K}_{2,2})^{-1}\mathcal{K}_{2,1}\left[1 - \mathcal{K}_{1,1} - \mathcal{K}_{1,2}(1 - \mathcal{K}_{2,2})^{-1}\mathcal{K}_{2,1}\right]^{-1}(1 - \mathcal{K}_{2,2})^{-1} + (1 - \mathcal{K}_{2,2})^{-1},$$

and once again

$$\mathcal{L}_{i,2}(f)(p) := \partial_k ||\nabla x^3||_{g_1}(p) e^{\phi_1(p)} \int_{D(r)} L^k_{i,2}(p, w) f(w)\, dw, \tag{5.35}$$

$$L^k_{i,2}(p, w) := \partial_j[G(p, w)(\psi_{p,i}\bar{B}^{jk3})(w)] + G(p, w)(\psi_{p,i}\bar{C}^{k3}_1 + \partial_j\psi_{p,i}\bar{C}^{jk3}_2)(w), \tag{5.36}$$

for $i = 1, 2$. Therefore for $j = 1, 2$,

$$\partial_j Q^k_{-1}(\partial_3 \delta\phi)(p) := \partial_j \mathcal{S}^k \circ \partial_j \mathcal{L}_{1,2}(\partial_3 \delta\phi)(p) + \partial_j \mathcal{T}^k \circ \partial_j \mathcal{L}_{2,2}(\partial_3\delta\phi)(p., \tag{5.37}$$

In particular

$$\partial_j \mathcal{L}_{i,2}(\partial_3\delta\phi)(p) := \partial_j\partial_k ||\nabla x^3||_{g_1}(p) e^{\phi_1(p)} \int_{D(r)} L^k_{i,2}(p,w)\partial_3\delta\phi(w)\, dw, \tag{5.38}$$

$$+ \partial_k ||\nabla x^3||_{g_1}(p) e^{\phi_1(p)} \int_{D(r)} \partial_{p^j} L^k_{i,2}(p,w)\partial_3\delta\phi(w)\, dw, \tag{5.39}$$

where

$$\partial_{p^j} L^k_{i,2}(p,w) := \partial_{p^j}\partial_{w^l}[G(p,w)(\psi_{p,i}\bar{B}^{lk3})(w)] + G(p,w)(\psi_{p,i}\bar{C}^{k3}_1 + \partial_{w^l}\psi_{p,i}\bar{C}^{lk3}_2)(w),$$

and ∂_{p^j} denotes differentiation on the variable p^j. Since $\mathcal{K}_{i,j} : L^2(D(r)) \to H^1(D(r))$ are bounded operators, $\mathcal{S}^k$ and $\mathcal{T}^k$ are bounded from $L^2(D(r)) \to H^1(D(r))$; hence the derivatives $\partial_j \mathcal{S}^k$ and $\partial_j \mathcal{T}^k$ are bounded operators from $L^2(D(r)) \to L^2(D(r))$.

Under the hypotheses of Theorem 1.6, the metric g_1 is C^2 close to Euclidean, so we know that ∇g_1^{33} and $\nabla\nabla g_1^{33}$ are bounded above in the $L^\infty(D(r))$ norm by some small

$\epsilon_0 > 0$. By the estimates for $\psi_{p,i}$ in Lemma 5.2 and the expression (5.38), (5.39),

$$\begin{aligned}
||\partial_j Q^k_{-1}(\partial_3\delta\phi)||_{L^2(D(r))} &\leq ||\partial_j \mathcal{S}^k \circ \mathcal{L}_{1,2}||_{L^2\to L^2}||\partial_j\mathcal{L}_{1,2}\partial_3\delta\phi||_{H^1(D(r))} \\
&\quad + ||\partial_j \mathcal{T}^k \circ \mathcal{L}_{2,2}||_{L^2\to L^2}||\partial_j\mathcal{L}_{2,2}\partial_3\delta\phi||_{H^1(D(r))} \\
&\leq ||\nabla\nabla g_1^{33}||_{L^\infty(D(r))}C_1(M,g_1,g_2)||\mathcal{L}_{1,2}\partial_3\delta\phi||_{H^1(D(r))} \\
&\quad + ||\nabla\nabla g_1^{33}||_{L^\infty(D(r))}C_2(M,g_1,g_2)||\mathcal{L}_{2,2}\partial_3\delta\phi||_{H^1(D(r))} \\
&\quad + ||\nabla g_1^{33}||_{L^\infty(D(r))}C_3(M,g_1,g_2)||\partial_j\mathcal{L}_{1,2}\partial_3\delta\phi||_{H^1(D(r))} \\
&\quad + ||\nabla g_1^{33}||_{L^\infty(D(r))}C_4(M,g_1,g_2)||\partial_j\mathcal{L}_{2,2}\partial_3\delta\phi||_{H^1(D(r))} \\
&\leq ||\nabla\nabla g_1^{33}||_{L^\infty(D(r))}C_1(M,g_1,g_2)||\partial_3\delta\phi||_{L^2(D(r))} \\
&\quad + ||\nabla g_1^{33}||_{L^\infty(D(r))}C_2(M,g_1,g_2)||\partial_p\psi_{p,2}||_{C^1_x}||\partial_3\delta\phi||_{L^2(D(r))} \\
&\quad + ||\nabla\nabla g_1^{33}||_{L^\infty(D(r))}C_2(M,g_1,g_2)||\psi_{p,2}||_{C^1_x}||\partial_3\delta\phi||_{L^2(D(r))} \\
&\leq \epsilon_0 C(M,g_1,g_2)||\partial_3\delta\phi||_{L^2(D(r))},
\end{aligned}$$

for $j,k = 1,2$. So we obtain (5.28) in the first case of Theorem 1.6.

In the second case of Theorem 1.6, the metrics g_1 and g_2 are only C^1 close to Euclidean, so the above argument for the bounds on $||\partial_j Q^k_{-1}(\partial_3\delta\phi)||_{L^2(D(r))}$ does not hold.

In this case, recall equation (5.37), and consider the term $\partial_j\mathcal{T}^k \circ \partial_j\mathcal{L}_{2,2}(\partial_3\delta\phi)(p)$. Notice since the Dirichlet Green's function $G(p,w)$ vanishes for p on the boundary, and from the fact we have chosen $\psi_{p,i}(p) = 0$, the kernel $\partial_{p^j}L^k_{i,2}(p,w) = 0$ for $p \in \partial D(r)$. Hence $\partial_j Q^k_{-1}(\partial_3\delta\phi)(p)$ vanishes for p on $\partial D(r)$.

Now, the disk $D(r)$ is of radius $0 < r < \frac{1}{R_{\max}}\epsilon_0$; so invoking the Poincaré Inequality,

$$||\nabla Q^k_{-1}||_{L^2(D(r))} \leq \epsilon_0 C||\nabla\nabla Q^k_{-1}||_{L^2(D(r))}$$

for some constant C. From the expression for Q^k_{-1} as described by equation (5.34), and the expression (5.38), (5.39) for $\partial_j \mathcal{L}_{i,2}$,

$$
\begin{aligned}
||\nabla\nabla Q^k_{-1}||_{L^2(D(r))} \leq{}& ||\nabla\nabla\mathcal{S}^k \circ \mathcal{L}_{1,2}||_{L^2\to H^{-1}}||\partial_j\mathcal{L}_{1,2}\partial_3\delta\phi||^2_{H^1(D(r))} \\
&+ ||\nabla\mathcal{S}^k \circ \mathcal{L}_{1,2}||_{H^1\to L^2}||\nabla\nabla\mathcal{L}_{1,2}\partial_3\delta\phi||_{L^2(D(r))} \\
&+ ||\nabla\nabla\mathcal{T}^k \circ \mathcal{L}_{2,2}||_{L^2\to H^{-1}}||\nabla\mathcal{L}_{2,2}\partial_3\delta\phi||^2_{H^1(D(r))} \\
&+ ||\nabla\mathcal{T}^k \circ \mathcal{L}_{2,2}||_{H^1\to L^2}||\nabla\nabla\mathcal{L}_{2,2}\partial_3\delta\phi||_{L^2(D(r))}.
\end{aligned}
$$

As the operators $\nabla\mathcal{S}^k \circ \mathcal{L}_{1,2}$ and $\nabla\mathcal{T}^k \circ \mathcal{L}_{2,2}$ are bounded operators from $L^2(D(r)) \to L^2(D(r))$, the derivatives $\nabla\nabla\mathcal{S}^k \circ \mathcal{L}_{1,2}$ and $\nabla\nabla\mathcal{T}^k \circ \mathcal{L}_{2,2}$ are bounded operators from $L^2(D(r)) \to H^{-1}(D(r))$. From the estimates for $\psi_{p,i}$ as given in Lemma 5.2, employing a similar argument for the bounds on $||\mathcal{L}_{i,2}(\partial_3\delta\phi)||_{H^2(D(r))}$, we may bound $||\nabla\nabla\mathcal{L}_{i,2}\partial_3\delta\phi||_{L^2(D(r))}$.

Finally, we have the estimate (5.28) in the second case of Theorem 1.6:

$$
||\nabla Q^k_{-1}(\partial_3\delta\phi)||_{L^2(D(r))} \leq \epsilon_0 C(M, g_1, g_2, p)||\partial_3\delta\phi||_{L^2(D(r))},
$$

for $k = 1, 2$.

□

From the fact that δg^{3k} are pseudodifferential operators acting on $\delta\phi$, $\partial_3\delta\phi$, we readily obtain the following uniqueness result for the metric components:

Lemma 5.6. *If* $\delta\phi \equiv 0$ *on* M*, then*

$$g_1^{11} = g_2^{11},$$

$$g_1^{22} = g_2^{22},$$

$$g_1^{31} = g_2^{31},$$

$$g_1^{32} = g_2^{32},$$

$$g_1^{12} = g_2^{12},$$

$$g_1^{21} = g_2^{21},$$

on M*.*

Proof. Since $\delta\phi \equiv 0$, $\phi_1 = \phi_2$ on M.

Above we found that

$$\delta g^{31} = P_0^1(\delta\phi) + Q_{-1}^1(\partial_3\delta\phi)$$

$$\delta g^{32} = P_0^2(\delta\phi) + Q_{-1}^2(\partial_3\delta\phi),$$

Therefore $\delta\phi \equiv 0$ implies

$$\begin{aligned} \delta g^{3k} &= P_0^k(\delta\phi) + Q_{-1}^k(\partial_3\delta\phi) \\ &= P_0^k(0) + Q_{-1}^k(0) \\ &= 0. \end{aligned}$$

Thus $g_1^{3k} = g_2^{3k}$ on M as well.

A little linear algebra (see appendix) gives us the relationships

$$
\begin{aligned}
g_1^{12} &= g_1^{21} = \frac{g_1^{31} g_1^{32}}{g_1^{33}}, \\
g_1^{11} &= -e^{-2\phi_1} - \frac{(g_1^{13})^2 + 2(g_1^{23})^2}{g_1^{33}}, \\
g_1^{22} &= -e^{-2\phi_1} - \frac{2(g_1^{13})^2 + (g_1^{23})^2}{g_1^{33}},
\end{aligned}
$$

and similarly

$$
\begin{aligned}
g_2^{12} &= g_2^{21} = \frac{g_2^{31} g_2^{32}}{g_2^{33}}, \\
g_2^{11} &= -e^{-2\phi_2} - \frac{(g_2^{13})^2 + 2(g_2^{23})^2}{g_2^{33}}, \\
g_1^{22} &= -e^{-2\phi_2} - \frac{2(g_2^{13})^2 + (g_2^{23})^2}{g_2^{33}}.
\end{aligned}
$$

Since we have shown $g_1^{3\alpha} = g_2^{3\alpha}$ for $\alpha = 1, 2, 3$ and $\phi_1 = \phi_2$, the above equations give us the claimed uniqueness on M.

□

In light of the above Lemma, to conclude $g_1 = g_2$ in our chosen coordinates, it only remains to prove $\delta\phi \equiv 0$ on M. Below we show that equation (5.23) for $\delta\phi$ may be expressed as a hyperbolic Cauchy problem for $\delta\phi$ with initial data $\delta\phi = 0$ on $Y(0)$. Then, using a standard energy estimate we prove $\delta\phi = 0$ on M as desired.

A hyperbolic Cauchy problem for $\delta\phi$: Substituting the expressions (5.24), (5.25) into equation (5.23) gives us the following evolution equation for $\delta\phi$ on M:

$$
\begin{aligned}
0 = g_1^{33}\partial_3\delta\phi + g_1^{31}\partial_1(\delta\phi) + g_1^{32}\partial_2(\delta\phi) + \left(\partial_k\phi_2 - \frac{1}{2}\partial_k \log(g_1^{33})\right) P_{-1}^k(\delta\phi) \\
+ \left(\partial_k\phi_2 - \frac{1}{2}\partial_k \log(g_1^{33})\right) Q_{-1}^k(\partial_3\delta\phi) + \frac{1}{2}\partial_k P_{-1}^k(\delta\phi) + \frac{1}{2}\partial_k Q_{-1}^k(\partial_3\delta\phi). \qquad (5.40)
\end{aligned}
$$

Now since P^k_{-1}, Q^k_{-1} are pseudodifferential operators of order -1 respectively, $\partial_k P^k_{-1}(\delta\phi)$, $\partial_k Q^k_{-1}(\delta\phi)$ are respectively order 0 pseudodifferential operators in the tangential directions ∂_k. So, equation (5.40) takes the form

$$(1 - Q_0)\partial_3 \delta\phi + Q_1(\delta\phi) = 0, \tag{5.41}$$

where Q_1 is an order 1 pseudodifferential operator independent of ∂_3, and the operator Q_0 is of order 0 and independent of ∂_3. We now argue that $1 - Q_0$ is invertible.

From Lemma 5.5,

$$||\nabla Q^k_{-1}(\partial_3 \delta\phi)||_{L^2(D(r))} \leq \epsilon_0 C(M, g_1, g_2, p)||\partial_3 \delta\phi||_{L^2(D(r))},$$

for $k = 1, 2$. Therefore,

$$||(1 - Q_0)\partial_3 \delta\phi||_{L^2(D(r))} \geq ||[1 - \epsilon_0 C(M, g_1, g_2, p)]\partial_3 \delta\phi||_{L^2(D(r))}$$

Since $\epsilon_0 > 0$ is small, $(1 - Q_0)$ is invertible.

Inverting $(1 - Q_0)$, we derive a hyperbolic Cauchy problem for $\delta\phi$ of the form

$$\begin{aligned} \partial_3 \delta\phi + \tilde{Q}_1(\delta\phi) &= 0 \qquad \text{on } M \\ \delta\phi(0) &= 0 \qquad \text{on } Y(0) \end{aligned} \tag{5.42}$$

where $\tilde{Q}_1 = (1 - Q_0)^{-1} Q_1$ is an order 1 pseudodifferential operator in the tangential directions.

Lemma 5.7. *The Cauchy problem (5.42) has a unique solution $\delta\phi \equiv 0$.*

Proof. For quick reference, we reproduce a standard energy argument.

Let $(\delta\phi, \delta\phi) := ||\delta\phi||_2^2 := \int_{D(r)} |\partial_3 \delta\phi(x)|^2 \, dx^1 dx^2$. Differentiating with respect to x^3,

$$\begin{aligned}
\partial_3 ||\delta\phi||_2^2 &= (\partial_3 \delta\phi, \delta\phi) + (\delta\phi, \partial_3 \delta\phi) \\
&= (\tilde{Q}_1(\delta\phi), \delta\phi) + (\delta\phi, \tilde{Q}_1(\delta\phi)) \\
&= ([\tilde{Q}_1 + \tilde{Q}_1^*](\delta\phi), \delta\phi) \\
&\leq C(x^3) ||\delta\phi||_2^2
\end{aligned}$$

since $\tilde{Q}_1 + \tilde{Q}_1^*$ is a pseudodifferential operator of order 0. Then, by Gronwall's inequality,

$$||\delta\phi(x)||_2 \leq C ||\delta\phi(x^1, x^2, 0)||_2 = 0,$$

so we obtain $\delta\phi \equiv 0$ for all $x \in D(r) \times [0, T]$. □

With the above lemma in hand, we conclude $g_1 = F^*(g_2)$.

□

Next, we use Theorem 1.6 to extend to the case where M is not topologically a cylinder $D(r) \times [0, T]$, but instead has mean convex boundary.

Theorem 1.4. *Let (M, g) be as in Case 1 or Case 2 above, and $g|_{\partial M}$ be given. Suppose that for any simple, closed curve γ on ∂M and any nearby perturbation $\gamma(t) \subset \partial M$ with $\gamma(0) = \gamma$, we know the area of the properly embedded surface $Y(t)$ which solves the least-area problem fro $\gamma(t)$.*

Then the knowledge of these areas uniquely determines the metric g.

Theorem 1.4 is a consequence of Theorem 1.6 and the following lemma:

Lemma 5.8. *Let (M, g) satisfy the conditions of Theorem 1.4. Let $r > 0$ be sufficiently large so that we may embed M into a cylinder $\tilde{M}(r) := D(r) \times [0, T]$. Equip $\tilde{M}(r)$ with a smooth Riemannian metric $\tilde{g}$ such that $\tilde{g}$ is uniformly C^1 close to Euclidean, $\tilde{g}|_M = g$,* *and* $\tilde{g} = g_{\mathbb{E}}$ *outside a compact set in* $\tilde{M}(r)$ *containing* M*, where* $g_{\mathbb{E}}$ *is the Euclidean metric on* $\tilde{M}(r)$*. The following holds:*

1. *Let* $\tilde{\gamma} : [0,1] \to \tilde{M}(r)$ *be a given an embedded, closed curve which is* C^1 *close to a circle. Set* $Y_{\tilde{\gamma}}$ *to be a properly embedded, area-minimizing surface with boundary given by the image of* $\tilde{\gamma}$*. Then, we know the closed curve cut by the intersection* $\gamma := Y_{\tilde{\gamma}} \cap \partial M$*.*

2. *For any closed, embedded curve* $\tilde{\gamma} : [0,1] \to \partial\tilde{M}(r)$ *which is* C^1 *close to a circle, we know the area of any minimal surface* $Y_{\tilde{\gamma}}$ *in* $(M(r), g)$ *enclosed by* $\tilde{\gamma}$*.*

3. *If* (M, g) *is* ϵ_0*-thin, then we may choose* $r > 0$ *so that* $(\tilde{M}(r), \tilde{g})$ *is* ϵ_0*-thin; likewise if* (M, g) *is* C^2 *close to Euclidean, then we may choose* $r > 0$ *so that* $(\tilde{M}(r), \tilde{g})$ *is* C^2 *close to Euclidean.*

Proof. The proof of parts (1) and (2) are identical to the proof of Lemma 3.7; we repeat the argument here for convenience.

1. Let $\gamma := Y_{\tilde{\gamma}} \cap \partial M$. Consider the set of all simple closed curves on ∂M which are C^3 close to γ. Call this set Σ. For any curve $\sigma \in \Sigma$, denote by Y_σ the minimal surface in M circumscribed by σ.

Given any $\sigma \in \Sigma$ the first variations in the area of Y_σ determine the angle at which Y_σ cut the boundary of M (see Proposition 3.5). Thus, we may determine the outward pointing unit normal vector fields ν_σ which are tangent to Y_σ, and normal to the curve σ. Let A_σ to be the area-minimizing annulus which lies between σ and $\tilde{\gamma}$.

The metric $\tilde{g}$ is known on $(\tilde{M}(r) \setminus M) \cup \partial M$, so for any annulus A_σ as above, we can determine the inward pointing unit normal vector field $\tilde{\nu}_\sigma$ tangent to A_σ and normal to the curve σ.

Consider the curve $\sigma^\sharp \in \Sigma$ such that $A_{\sigma^\sharp}$ is the minimal annulus for which $\nu_{\sigma^\sharp}$ and $\tilde{\nu}_{\sigma^\sharp}$ are collinear on ∂M. Note for any $p \in \sigma^\sharp$, the tangent space $T_p A_{\sigma^\sharp}$ coincides with the tangent space $T_p Y_{\sigma^\sharp}$, since they are both spanned by $\nu_{\sigma^\sharp}$ and any vector tangent to $\sigma^\sharp$.

Hence, $A_{\sigma^\sharp} \cup Y_{\sigma^\sharp}$ is a $C^1(\tilde{M}(r))$ surface which minimizes area bounded by $\tilde{\gamma}$. We claim that $A_{\sigma^\sharp} \cup Y_{\sigma^\sharp}$ is in fact a smooth minimal surface and further $A_{\sigma^\sharp} \cup Y_{\sigma^\sharp} \equiv Y_{\tilde{\gamma}}$.

To prove that $A_{\sigma^\sharp} \cup Y_{\sigma^\sharp}$ is smooth, we express it as a graph of a function z and show that the derivatives of z exist and are continuous. To this end, let $T_{\sigma^\sharp} \subset \tilde{M}$ be the surface obtained by following geodesics $c_\theta(\rho)$ with $\theta \in \sigma$ and initial direction $\frac{\partial}{\partial \rho} c_\theta(0) = \nu_{\sigma^\sharp}(\theta)$; that is $T_{\sigma^\sharp} := \{p \in \tilde{M} \, : \, p = c_\theta(\rho), \text{for some } \rho \geq 0, \, \theta \in \sigma\}$.

Express $T_{\sigma^\sharp}$ in the natural coordinate system (ρ, θ). View $A_{\sigma^\sharp}$ as a graph of a function $z = z(\rho, \theta)$ over $T_{\sigma^\sharp}$. Since $A_{\sigma^\sharp}$ is smooth away from $\rho = 0$, to show $A_{\sigma^\sharp} \cup Y_{\sigma^\sharp}$ is smooth, we need only show that the derivatives of z at $\rho = 0$ are continuous. Actually, we need only show that the second order derivatives of z are continuous. This follows from the fact the surface $A_{\sigma^\sharp}$ is minimal, hence $z = z(\rho, \theta)$ solves the minimal surface equation

$$\operatorname{div}_g \left(\frac{\nabla z}{||\nabla z||_g} \right) = 0;$$

thus by elliptic regularity, it suffices for us to show that $A_{\sigma^\sharp} \cup Y_{\sigma^\sharp}$ is $C^2(\tilde{M}(r))$ at the join σ.

Choose $w : \tilde{M} \to \mathbb{R}$ such that (ρ, θ, w) form a local coordinate system near $T_{\sigma^\sharp}$. Since $A_{\sigma^\sharp}$ agrees with $T_{\sigma^\sharp}$ on $\sigma^\sharp$ to first order, $z(0, \theta) = 0$ and $\partial_\rho z(0, \theta) = \partial_\theta z(0, \theta) = 0$. The minimal surface equation written in our chosen coordinates is

$$0 = \operatorname{div}_g \left(\frac{\nabla z}{||\nabla z||_g} \right) \tag{5.43}$$

$$= \operatorname{div}_g \left(\frac{Dz}{||dw^2 + \partial_\rho z d\rho + \partial_\theta z d\theta||_g} \right) \tag{5.44}$$

$$= \frac{||dw^2 + \partial_\rho z d\rho + \partial_\theta z d\theta||_g \operatorname{div}_g(Dz) - g(Dz, g(DDz, Dz))}{||dw^2 + \partial_\rho z d\rho + \partial_\theta z d\theta||_g^3} \tag{5.45}$$

where D denotes the restriction of ∇ to $T_{\sigma^\sharp}$. Substituting $\rho = 0$ into the above equation and using $z(0, \theta) = 0$, $\partial_\rho z(0, \theta) = \partial_\theta z(0, \theta) = 0$, we find $D_\rho D_r z(0, \theta) = 0$. So $z = z(\rho, \theta)$ is $C^2(\tilde{M}(r))$ on $A_{\sigma^\sharp}$, up to and including $\sigma^\sharp$.

So we have shown that $A_{\sigma^\sharp} \cup Y_{\sigma^\sharp}$ is smooth.

To show that $A_{\sigma^\sharp} \cup Y_{\sigma^\sharp}$ is unique, it is enough to argue that $A_{\sigma^\sharp}$ is unique. Again represent $A_{\sigma^\sharp}$ by $z = z(\rho, \theta)$. If $z_2 = z_2(\rho, \theta)$ is the graph of any other minimal annulus with the same properties as $A_{\sigma^\sharp}$, then we must have $z = z(\rho, \theta)$ and $z_2 = z_2(\rho, \theta)$ agree to first order. The previous argument for the smoothness of $z = z(\rho, \theta)$ demonstrates that the second derivatives of $z = z(\rho, \theta)$ and $z_2 = z_2(\rho, \theta)$ also agree. Thus $z = z(\rho, \theta)$ and $z_2 = z_2(\rho, \theta)$ agree to all order.

Since $A_{\sigma^\sharp}$ is unique, $A_{\sigma^\sharp} \cup Y_{\sigma^\sharp} \equiv Y_\gamma$. Therefore, $\sigma^\sharp \equiv \gamma$.

2. Let $Y_{\tilde{\gamma}} \subset \tilde{M}(r)$ be a minimal surface with boundary given by the image of $\tilde{\gamma}$. As shown in part 1, the surface $Y_{\tilde{\gamma}}$ is determined from our knowledge of the areas minimal surfaces in M. From part 1, we may determine the curve γ cut by $Y_{\tilde{\gamma}} \cap \partial M$. In particular, we can thus determine the surface $Y_{\tilde{\gamma}} \setminus (Y_{\tilde{\gamma}} \cap M)$.

We have

$$\text{Area}(Y_{\tilde{\gamma}}) = \text{Area}(Y_{\tilde{\gamma}} \cap M) + \text{Area}(Y_{\tilde{\gamma}} \setminus (Y_{\tilde{\gamma}} \cap M)). \tag{5.46}$$

Since the metric $\tilde{g}$ is known on $Y_{\tilde{\gamma}} \setminus (Y_{\tilde{\gamma}} \cap M)$, we may compute the area of $Y_{\tilde{\gamma}} \setminus (Y_{\tilde{\gamma}} \cap M)$. Since we assumed knowledge of any minimal surface M, the area of $Y_{\tilde{\gamma}} \cap M$ in known. Therefore, $\text{Area}(Y_{\tilde{\gamma}})$ is known.

3. If (M, g) is thin, then by definition there exists global coordinates (y^α), $\alpha = 1, 2, 3$ on (M, g) such that the surfaces $Y(t) := \{y^3 = \text{constant} = t\}$ are properly embedded and area-minimizing, and further the diameter of each $Y(t)$ is bounded above by ϵ_0. Then, the map coordinate map

$$F : M \to D(\epsilon_0) \qquad p \mapsto \left(y^1, y^2, y^3\right)$$

embeds M into $D(\epsilon_0) \times [0, T]$ for some $T > 0$. Take $r = \epsilon_0$. Then $(\tilde{M}(r), \tilde{g})$ defined above is ϵ_0-thin.

If (M, g) is C^2 close to Euclidean, then by the smoothness of the metric g, we can extend g to a tubular neighbourhood N of M so that g remains C^2 close to Euclidean. By using smooth cutoff functions, we may further extend $(M \cup N, g)$ to asymptotically flat manifold $(\tilde{M}, \mathbf{g})$ as in section 3.0.1. The metric $\mathbf{g}$ is C^2 close to Euclidean by construction. Hence, take any sufficiently large $T, r > 0$ so that $M \subset D(r) \times [0, T] \subset \tilde{M}$ and equip $\tilde{M}(r) := D(r) \times [0, T]$ with the metric $\tilde{g} := \mathbf{g}|_{\tilde{M}(r)}$. Then $(\tilde{M}(r), \tilde{g})$ satisfies the conditions of this lemma and $\tilde{g}$ is C^2 close to Euclidean.

□

Theorem 1.5. *Let (M, g) be a 3-dimensional Riemannian manifold with boundary ∂M. Assume that ∂M is both C^3-smooth and mean convex at $p \in \partial M$. Let γ be a simple, closed curve on ∂M near p. Suppose that for γ and any nearby perturbation $\gamma(t) \subset \partial M$ with $\gamma(0) = \gamma$, we know the area of the properly embedded surface $Y(t)$ which solves the least-area problem for $\gamma(t)$ Then this information uniquely determines g on a neighbourhood of p.*

Proof. Let $U \subset M$ be a neighbourhood near $p \in M$ for which we know the above area information. Further, we may choose U sufficiently small so that $(U, g|_U)$ is a thin manifold. Applying Theorem 1.4 to $(U, g|_U)$, we may recover the metric g near $p \in M$. □

Chapter 6

Appendix

6.1 First and Second Variations of Area

In this section, we include standard calculations for the first and second variations of area for an embedded surface for quick reference. For an excellent introduction to minimal surface theory, as well as computations similar to those presented here, please see [7]. The set up for the computation of the variations is as follows:

Let $(\Omega, g_{\mathbb{E}})$ be a bounded domain in $\mathbb{R}^2$ equipped with the Euclidean metric, and (M, g) be a smooth, Riemannian 3-manifold with boundary ∂M. Let $f_0 : \Omega \hookrightarrow M$ be a proper embedding of the disk into M, and write $Y(0) := f_0(\Omega)$. Define g_0 for the metric on $Y(0)$ induced by g.

Set (x^i) $i = 1, 2$ be conformal coordinates on Y, and define

$$f(x^1, x^2, t) := f_t(x^1, x^2) : \Omega \times [0, T) \to M$$

be a variation of f_0 such that f_t is a proper embedding for each $t \in [0, T)$. Write $Y(t) := f_t(Y) \subset M$, and define a coordinate frame field on the embedded submanifold $Y(t)$ by $X_i := \frac{\partial f_t}{\partial x^i}$, for $i = 1, 2$. In particular, the collection of vector fields $X_t := \frac{\partial f_t}{\partial t}$ and (X_i), $i = 1, 2$, is a coordinate frame field on M. In this frame field, the components of the metric g_t induced on $Y(t)$ by g are $(g_t)_{ij} := g(X_i, X_j)$, $i = 1, 2$.

When $t = 0$, we write $X_0 := \frac{\partial f_t}{\partial t}\Big|_{t=0}$. Since we've chosen conformal coordinates on $Y(0)$, $g_0 := e^{2\phi} g_{\mathbb{E}}$ for some function $\psi : \Omega \to \mathbb{R}$. It will be useful at certain points for express calculations with respect to the orthonormal frame field $\tilde{X}_i := e^{-\phi} X_i$ on $Y(0)$.

In the computations that follow, we do not assume that X_0 is a normal vector field on the surface $Y(0)$, unless specified otherwise.

Finally, the area of $(Y(t), g_t)$ is given by the functional

$$A(Y(t)) := \int_{Y(t)} d\text{Vol}(g) = \int_{\partial\Omega} \sqrt{\det(g_t)}\, dx. \tag{6.1}$$

6.1.1 First Variation of Area

First, we compute that the derivative of the integrand of (6.1) with respect to t is given by

$$\begin{aligned}
\frac{\partial}{\partial t}\sqrt{\det(g_t)} &= \frac{1}{2}(\det(g_t))^{-\frac{1}{2}} \frac{\partial}{\partial t} \det(g_t) \\
&= \frac{1}{2}(\det(g_t))^{-\frac{1}{2}} \cdot \text{tr}\left(\text{Adj}(g_t)\frac{\partial g_t}{\partial t}\right) \\
&= \frac{1}{2}(\det(g_t))^{-\frac{1}{2}} \cdot \text{tr}\left(\det(g_t) g_t^{-1} \frac{\partial g_t}{\partial t}\right) \\
&= \frac{1}{2}\sqrt{\det(g_t)} g_t^{ij} \frac{\partial (g_t)_{ij}}{\partial t}.
\end{aligned}$$

Let ∇ be the Levi-Civita connection associated to the metric g. Then,

$$\begin{aligned}
\frac{\partial (g_t)_{ij}}{\partial t} &= \frac{\partial}{\partial t} g(X_i, X_j) \\
&= g(\nabla_{X_t} X_i, X_j) + g(X_i, \nabla_{X_t} X_j),
\end{aligned}$$

hence

$$\begin{aligned}\frac{\partial}{\partial t}\sqrt{\det(g_t)} &= \frac{1}{2}\sqrt{\det(g_t)}g_t^{ij}[g(\nabla_{X_t}X_i, X_j) + g(X_i, \nabla_{X_t}X_j)] \\ &= g_t^{ij}g(\nabla_{X_t}X_i, X_j)\sqrt{\det(g_t)} \\ &= g_t^{ij}g(\nabla_{X_i}X_t, X_j)\sqrt{\det(g_t)} \\ &= \operatorname{div}_{g_t}(X_t)\sqrt{\det(g_t)},\end{aligned}$$

since $[X_i, X_t] = 0$.

Let $\vec{n}_t$ be a unit normal vector field on Y_t. Now, since $[X_i, X_t] = 0$,

$$\begin{aligned}\frac{d}{dt}A(Y(t)) &= \int_\Omega \operatorname{div}_{g_t}(X_t)\sqrt{\det(g_t)}\,dx \\ &= \int_\Omega \operatorname{div}_{g_t}(X_t^\top + X_t^\perp)\,d\mathrm{Vol}(g_t) \\ &= \int_\Omega \operatorname{div}_{g_t}(X_t^\top)\,d\mathrm{Vol}(g_t) + \int_Y \operatorname{div}_{g_t}(X_t^\perp)\,d\mathrm{Vol}(g_t) \\ &= \int_{\partial\Omega\,\int_\Omega} g(X_t, \nu)\,dS_t - \int_Y g(X_t, H_t\vec{n}_t)\,d\mathrm{Vol}(g_t)\end{aligned}$$

where $H_t := -\operatorname{div}_g(\vec{n}_t) = -g_t^{ij}g(\nabla_{X_i}\vec{n}_t, X_j)$ is the scalar-valued mean curvature of $Y(t)$ and ν is the outward pointing vector normal to ∂D and tangent to D. Thus, $Y(t)$ is extremal for the area functional if $Y(t)$ is a minimal surface.

6.1.2 Second Variation of Area

Write $X_t^\top$ for the component of X_t tangent to $Y(t)$ and $X_t^\perp := \psi_t\vec{n}_t$ for the component of X_t normal to $Y(t)$. From the above calculation for the first variation,

$$\frac{\partial}{\partial t}\sqrt{\det(g_t)} = g_t^{ij}g(\nabla_{X_i}X_t, X_j)\sqrt{\det(g_t)}.$$

So, the second derivative of the integrand in (6.1) is found to be

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\sqrt{\det(g_t)} &= \frac{\partial}{\partial t}\left[g_t^{ij} g(\nabla_{X_i}X_t, X_j)\sqrt{\det(g_t)}\right] \\ &= \frac{\partial g_t^{ij}}{\partial t} g(\nabla_{X_i}X_t, X_j)\sqrt{\det(g_t)} \\ &\quad + g_t^{ij}\left[g(\nabla_{X_t}\nabla_{X_i}X_t, X_j) + g(\nabla_{X_i}X_t, \nabla_{X_t}X_j)\right]\sqrt{\det(g_t)} \\ &\quad + g_t^{ij} g(\nabla_{X_i}X_t, X_j)\frac{\partial}{\partial t}\sqrt{\det(g_t)} \\ &=: (A+B+C)\sqrt{\det(g_t)}.\end{aligned}$$

The partial derivative of the inverse metric components are given by

$$\begin{aligned}&\frac{\partial}{\partial t}((g_t)_{ij}(g_t)^{jl}) = 0 \\ \Rightarrow\quad &\frac{\partial (g_t)^{jl}}{\partial t}(g_t)_{ij} = -\ (g_t)^{jl}\frac{\partial (g_t)_{ij}}{\partial t} \\ \Rightarrow\quad &\frac{\partial (g_t)^{kl}}{\partial t} = -(g_t)^{ik}(g_t)^{jl}\frac{\partial (g_t)_{ij}}{\partial t}.\end{aligned}$$

Below we compute the terms A, B, C:

$$\begin{aligned}A &:= \frac{\partial g_t^{ij}}{\partial t} g(\nabla_{X_i}X_t, X_j) \\ &= -(g_t)^{ik}(g_t)^{jl}\frac{\partial (g_t)_{kl}}{\partial t}\frac{\partial (g_t)_{ij}}{\partial t} g(\nabla_{X_i}X_t, X_j) \\ &= -(g_t)^{ik}(g_t)^{jl}\left[g(\nabla_{X_k}X_t, X_l) + g(X_k, \nabla_{X_l}X_t)\right] g(\nabla_{X_i}X_t, X_j).\end{aligned}$$

When $t = 0$, expressing A with respect to the orthonormal frame $(\tilde{X}_i)$ gives

$$
\begin{aligned}
A|_{t=0} &:= -2(g_t)^{ik}(g_t)^{jl} g(\nabla_{X_k} X_t, X_l) g(\nabla_{X_i} X_t, X_j)|_{t=0} \\
&= -2\sum_{i,j}[g(\nabla_{\tilde{X}_i}(X_0^\top + X_0^\perp), \tilde{X}_j)]^2 \\
&= -\sum_{i,j}[-g((X_0^\top), \nabla_{\tilde{X}_i}\tilde{X}_j) - g(X_0^\perp, \nabla_{\tilde{X}_i}\tilde{X}_j)]^2 \\
&= -2\sum_{i,j}[g((X_0^\top), \nabla_{\tilde{X}_i}\tilde{X}_j) + g(X_0^\perp, \nabla_{\tilde{X}_i}\tilde{X}_j)]^2 \\
&= -2\sum_{i,j}\left[g(X_0^\top, \nabla_{\tilde{X}_i}\tilde{X}_j)^2 + 2g(X_0^\top, \nabla_{\tilde{X}_i}\tilde{X}_j)g(X_0^\perp, \nabla_{\tilde{X}_i}\tilde{X}_j) + g(X_0^\perp, \nabla_{\tilde{X}_i}\tilde{X}_j)^2\right] \\
&=: -2\sum_{i,j}\left[g(X_0^\top, \nabla_{\tilde{X}_i}\tilde{X}_j)^2 + 2g(X_0^\top, \nabla_{\tilde{X}_i}\tilde{X}_j)g(X_0^\perp, \nabla_{\tilde{X}_i}\tilde{X}_j)\right] - 2\psi_0^2||A_0||_g^2,
\end{aligned}
$$

where $A_0(V, W) = g(\vec{n}_0, \nabla_V W)$ the second fundamental form of $Y(0)$, and recall $X_0^\perp := \psi_0\vec{n}_0$.

We expand the second term to find

$$
\begin{aligned}
B &:= g_t^{ij}\left[g(\nabla_{X_t}\nabla_{X_i}X_t, X_j) + g(\nabla_{X_i}X_t, \nabla_{X_t}X_j)\right] \\
&= g_t^{ij}\left[-g(\mathrm{Rm}_g(X_i, X_t)X_t, X_j) + g(\nabla_{X_i}X_t, \nabla_{X_t}X_j) + g(\nabla_{X_j}\nabla_{X_t}X_t, X_i)\right],
\end{aligned}
$$

where $\mathrm{Rm}_g(X_i, X_t)X_t := \nabla_{X_i}\nabla_{X_t}X_t - \nabla_{X_t}\nabla_{X_i}X_t$, and we used the fact

$$
g(\mathrm{Rm}_g(X_i, X_t)X_t, X_j) = g(\nabla_{X_i}\nabla_{X_t}X_t, X_j) - g(\nabla_{X_t}\nabla_{X_i}X_t, X_j).
$$

Then for $t = 0$,

$$
\begin{aligned}
B &= -\mathrm{Ric}_g(X_0, X_0) + g_0^{ij} g(\nabla_{X_i} X_0, \nabla_{X_0} X_j) + g_0^{ij} g(\nabla_{X_j} \nabla_{X_0} X_0, X_i) \\
&= -\mathrm{Ric}_g(X_0, X_0) + g_0^{ij} g((\nabla_{X_i} X_0)^\top, (\nabla_{X_j} X_0)^\top) \\
&\qquad + g_0^{ij} g((\nabla_{X_i} X_0)^\perp, (\nabla_{X_j} X_0)^\perp) + \mathrm{div}_{g_0}(\nabla_{X_0} X_0) \\
&= -\mathrm{Ric}_g(X_0, X_0) + ||(\nabla X_0)^\perp||_{g_0}^2 + \psi_0^2 ||A_0||_g^2 + \mathrm{div}_{g_0}(\nabla_{X_0} X_0) \\
&= -\mathrm{Ric}_g(X_0^\top, X_0^\top) - \psi_0^2 \mathrm{Ric}_g(\vec{n}_0, \vec{n}_0) + ||\nabla \psi_0||_{g_0}^2 + \psi_0^2 ||A_0||_g^2 + \mathrm{div}_{g_0}(\nabla_{X_0} X_0).
\end{aligned}
$$

Finally, the last term is

$$
\begin{aligned}
C &:= \frac{1}{4} (g_t)^{kl} \frac{\partial (g_t)_{kl}}{\partial t} g_t^{ij} \frac{\partial (g_t)_{ij}}{\partial t} \\
&= \frac{1}{4} (g_t)^{ik} [g(\nabla_{X_k} X_t, X_l) + g(X_k, \nabla_{X_l} X_t)] g_t^{ij} [g(\nabla_{X_i} X_t, X_j) + g(X_i, \nabla_{X_j} X_t)] \\
&= \frac{1}{4} (2\,\mathrm{div}_{g_t}(X_t))^2 \\
&= (\mathrm{div}_{g_t}(X_t))^2.
\end{aligned}
$$

So $C|_{t=0} = (\mathrm{div}_{g_0}(X_0))^2$.

Putting together the previous calculations, the second derivative of area is

$$\begin{aligned}
\frac{d^2}{dt^2} A(Y(t))\bigg|_{t=0} &= \int_\Omega \frac{\partial^2}{\partial t^2}\sqrt{\det(g_t)}\bigg|_{t=0} dx \\
&= \int_\Omega (A+B+C)\sqrt{\det(g_0)}\, dx \\
&= -2\int_\Omega \sum_{i,j} \left[g(X_0^\top, \nabla_{\tilde{X}_i}\tilde{X}_j)^2 + 2g(X_0^\top, \nabla_{\tilde{X}_i}\tilde{X}_j) g(X_0^\perp, \nabla_{\tilde{X}_i}\tilde{X}_j)\right] d\mathrm{Vol} g_0 \\
&\quad - \int_\Omega \psi_0^2 ||A_0||_g^2 + \mathrm{Ric}_g(X_0^\top, X_0^\top) + \psi_0^2 \mathrm{Ric}_g(\vec{n}_0, \vec{n}_0)\, d\mathrm{Vol} g_0 \\
&\quad + \int_\Omega ||\nabla \psi_0||_{g_0}^2 + \mathrm{div}_{g_0}(\nabla_{X_0} X_0) + (\mathrm{div}_{g_0}(X_0))^2\, d\mathrm{Vol} g_0 \\
&:= I + J + K.
\end{aligned}$$

Once again we deal with the three terms separately. First, since $Y(0)$ is assumed to be a minimal surface,

$$\begin{aligned}
\int_\Omega (\mathrm{div}_{g_0}(X_0))^2\, d\mathrm{Vol} g_0 &= \int_\Omega (\mathrm{div}_{g_0}(X_0^\top + X_0^\perp))^2\, d\mathrm{Vol}(g_0) \\
&= \int_\Omega (\mathrm{div}_{g_0}(X_0^\top) + g(X_0, H_0\vec{n}_0))^2\, d\mathrm{Vol} g_0 \\
&= \int_\Omega (\mathrm{div}_{g_0}(X_0^\top))^2\, d\mathrm{Vol} g_0.
\end{aligned}$$

Let ν be the outer unit normal to the boundary of $(\Omega, f_0^*(g_0))$. Using the above and integration by parts,

$$\begin{aligned}
K &:= \int_\Omega ||\nabla\psi_0||_{g_0}^2 + \mathrm{div}_{g_0}(X_0^\top) + (\mathrm{div}_{g_0}(X_0^\top))^2\, d\mathrm{Vol} g_0 \\
&= -\int_\Omega \psi_0 \Delta_{g_0}\psi_0\, d\mathrm{Vol} g_0 + \int_{\partial\Omega} g(\nabla\psi_0, \nu) + g(\nabla_{X_0}(X_0^\top), \nu) + \mathrm{div}_{g_0}(X_0^\top) g(X_0, \nu)\, dS.
\end{aligned}$$

Therefore, from all these calculations we have found the expression for the second variation of area at $t = 0$:

$$
\begin{aligned}
\frac{d^2}{dt^2}A(f_t)\Big|_{t=0} &= I + J + K \\
&= -2\int_\Omega \sum_{i,j} \Big[g(X_0^\top, \nabla_{\tilde{X}_i}\tilde{X}_j)^2 + 2g(X_0^\top, \nabla_{\tilde{X}_i}\tilde{X}_j)g(X_0^\perp, \nabla_{\tilde{X}_i}\tilde{X}_j) \Big] \, d\mathrm{Vol}g_0 \\
&\quad - \int_\Omega \psi_0^2 ||A_0||_g^2 + \mathrm{Ric}_g(X_0^\top, X_0^\top) + \psi_0^2 \mathrm{Ric}_g(\vec{n}_0, \vec{n}_0) \, d\mathrm{Vol}g_0 \\
&\quad - \int_\Omega \psi_0 \Delta_{g_0}\psi_0 \, d\mathrm{Vol}g_0 + \int_{\partial\Omega} g(\nabla\psi_0, \nu) \, dS \\
&\quad + \int_{\partial\Omega} g(\nabla_{X_0}(X_0^\top), \nu) + \mathrm{div}_{g_0}(X_0^\top) g(X_0, \nu) \, dS
\end{aligned}
$$

Grouping terms with $X_0^\top$ and terms with $\vec{n}_0$ separately,

$$
\begin{aligned}
\frac{d^2}{dt^2}A(f_t)\Big|_{t=0} &= -2\int_\Omega \sum_{i,j} \Big[g(X_0^\top, \nabla_{\tilde{X}_i}\tilde{X}_j)^2 + 2g(X_0^\top, \nabla_{\tilde{X}_i}\tilde{X}_j)g(X_0^\perp, \nabla_{\tilde{X}_i}\tilde{X}_j) \Big] \, d\mathrm{Vol}g_0 \\
&\quad + \int_\Omega \mathrm{Ric}_g(X_0^\top, X_0^\top) \, d\mathrm{Vol}g_0 \\
&\quad - \int_\Omega \psi_0 \Delta_{g_0}\psi_0 + \psi_0^2 \mathrm{Ric}_g(\vec{n}_0, \vec{n}_0) + \psi_0^2 ||A_0||_g^2 \, d\mathrm{Vol}g_0 \\
&\quad + \int_{\partial\Omega} g(\nabla\psi_0, \nu) \, dS + \int_{\partial\Omega} g(\nabla_{X_0}(X_0^\top), \nu) + \mathrm{div}_{g_0}(X_0^\top) g(X_0, \nu) \, dS.
\end{aligned}
$$

We remark that if $X_0 = X_0^\perp = \psi_0 \vec{n}_0$ is normal to $Y(0)$,

$$
\begin{aligned}
\frac{d^2}{dt^2}A(f_t)\Big|_{t=0} &= -\int_\Omega \psi_0 \Delta_{g_0}\psi_0 + \psi_0^2 \mathrm{Ric}_g(\vec{n}_0, \vec{n}_0) + \psi_0^2 ||A_0||_g^2 \, d\mathrm{Vol}g_0 \\
&\quad + \int_{\partial\Omega} g(\nabla\psi_0, \nu) \, dS + \int_{\partial\Omega} g(\nabla_{X_0}(X_0^\top), \nu) \, dS. \qquad (6.2)
\end{aligned}
$$

Hence we derive the **stability operator** $\mathcal{J}$:

$$
\mathcal{J}(\psi) := \Delta_{g_0}\psi_0 + \left(\mathrm{Ric}_g(\vec{n}_0, \vec{n}_0) + ||A_0||_g^2 \right) \psi_0, \qquad (6.3)
$$

on $Y(0)$.

6.1.3 The Lapse Function Solves the Jacobi Equation

The **lapse function** associated to a variation f_t is defined as the normal component of the vector $X_t := \frac{\partial f_t}{\partial t}$. In keeping with the previous notation, the lapse function is written $\psi_t := g(X_t, \vec{n}_t)$. In a small neighbourhood of $t = 0$, choose t to be the arc length parameter along geodesics with initial velocity $\vec{n}_0$. Then, $\nabla_{\vec{n}_0}\vec{n}_0 = 0$.

Without loss of generality, suppose $X_0^\top = 0$. Then, the first variation in the second fundamental form is

$$\begin{aligned}\frac{d}{dt}A_t(X_i, X_j)\bigg|_{t=0} &= -\frac{d}{dt}\nabla_{X_t} g(\nabla_{X_i}\vec{n}_t, X_j)\bigg|_{t=0} \\ &= -g(\nabla_{X_0}\nabla_{X_i}\vec{n}_0, X_j) - g(\nabla_{X_i}\vec{n}_0, \nabla_{X_0}X_j) \\ &= -g(\nabla_{X_i}\nabla_{X_0}\vec{n}_0, X_j) + g(\mathrm{Rm}_g(X_0, X_i)\vec{n}_0, X_i) \\ &\quad - g(\nabla_{X_i}\vec{n}_0, \nabla_{X_j}X_0).\end{aligned}$$

Now,

$$\begin{aligned}g(\nabla_{X_0}\vec{n}_0, X_i) &= -g(\vec{n}_0, \nabla_{X_0}X_i) \\ &= -g(\vec{n}_0, \nabla_{X_i}X_0) \\ &= -g(\vec{n}_0, \nabla_{X_i}(\psi_0\vec{n}_0)) \\ &= -g(\vec{n}_0, \nabla_{X_i}(\psi_0\vec{n}_0) - g(\vec{n}_0, \psi_0\nabla_{X_i}\vec{n}_0) \\ &= -g(\vec{n}_0, \nabla_{X_i}(\psi_0\vec{n}_0),\end{aligned}$$

since $g(\vec{n}_0, \nabla_{X_i}\vec{n}_0) = 0$. Therefore,

$$\begin{aligned}\frac{d}{dt}A(X_i, X_j)\Big|_{t=0} &= g(\nabla_{X_i}\nabla\psi_0, X_j) + \psi_0 g(\mathrm{Rm}_g(\vec{n}_0, X_i)\vec{n}_0, X_i) \\ &\quad - g(\nabla_{X_i}\vec{n}_0, \nabla_{X_j}(\psi_0)\vec{n}_0 + \psi_0\nabla_{X_j}\vec{n}_0) \\ &= g(\nabla_{X_i}\nabla\psi_0, X_j) + \psi_0 g(\mathrm{Rm}_g(\vec{n}_0, X_i)\vec{n}_0, X_i) \\ &\quad - g(\nabla_{X_i}\vec{n}_0, \psi_0\nabla_{X_j}\vec{n}_0).\end{aligned}$$

Recalling that $A_0(X_i, X_j) = g(\nabla_{X_i}\vec{n}_0, X_j)$ and taking the trace of the above equation, we find

$$\mathrm{tr}_g\left(\frac{d}{dt}A_t(X_i, X_j)\Big|_{t=0}\right) = \Delta_{g_0}\psi_0 + \psi_0\mathrm{Ric}_g(\vec{n}_0.\vec{n}_0) + \psi_0||A_0||^2.$$

Since the mean curvature of $Y(0)$ satisfies $0 = H_0 = \mathrm{tr}_g A_0$, we derive that the lapse function ψ_0 solves

$$0 = \Delta_{g_0}\psi_0 + \psi_0\mathrm{Ric}_g(\vec{n}_0.\vec{n}_0) + \psi_0||A_0||^2.$$

6.2 Algebraic Relationships Between the Components of g and g^{-1}

Let (x^α), $\alpha = 1, 2, 3$ be a local coordinate system on a Riemannian manifold (M, g) such that in these coordinates the metric g takes the form

$$g = \begin{pmatrix} e^{2\phi} & 0 & g_{13} \\ 0 & e^{2\phi} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix},$$

where the functions $g_{13} = g_{31}$ and $g_{23} = g_{32}$. Then, simple cofactor expansion gives

$$g^{-1} := \begin{pmatrix} g^{11} & g^{12} & g^{13} \\ g^{21} & g^{22} & g^{23} \\ g^{31} & g^{32} & g^{33} \end{pmatrix}$$
$$= -\det(g^{-1}) \begin{pmatrix} \frac{e^{2\phi} g_{33} - (g_{32})^2}{e^{2\phi}} & \frac{g_{31} g_{32}}{e^{2\phi}} & -g_{13} \\ \frac{g_{31} g_{32}}{e^{2\phi}} & \frac{e^{2\phi} g_{33} - (g_{31})^2}{e^{2\phi}} & -g_{23} \\ -g_{31} & -g_{32} & e^{2\phi} \end{pmatrix}$$

Thus we have the following relationships:

$$\det(g^{-1}) = -\frac{e^{-2\phi}}{g^{33}}$$
$$g_{31} = \frac{g^{31}}{\det(g^{-1})} = -\frac{g^{31}}{g^{33}} e^{2\phi}$$
$$g_{32} = \frac{g^{32}}{\det(g^{-1})} = -\frac{g^{32}}{g^{33}} e^{2\phi}$$

Now, the determinant of g^{-1} is

$$\det(g^{-1}) = \det(g)^{-1}$$
$$= \left[e^{2\phi} g_{33} + (g_{31})^2 + (g_{32})^2 \right]^{-1}.$$

So

$$g^{33} = -\det(g^{-1}) e^{-2\phi}$$
$$= \frac{e^{-2\phi}}{[e^{2\phi} g_{33} + (g_{31})^2 + (g_{32})^2]},$$

and manipulating the above we obtain an expression for g_{33} in terms of g^{31}, g^{32} and ϕ:

$$g_{33} = -\frac{1}{g^{33}} - \frac{(g_{31})^2 + (g_{32})^2}{(g^{33})^2} e^{2\phi}.$$

Therefore, we derive the following expressions for the components of g^{-1} in the ∂_1, ∂_2 directions in terms of the functions g^{33}, g^{31}, g^{32} and ϕ:

$$\begin{aligned}
g^{12} &= g^{21} = \frac{g^{31}g^{32}}{g^{33}}, \\
g^{11} &= -e^{-2\phi} - \frac{(g^{13})^2 + 2(g^{23})^2}{g^{33}}, \\
g^{22} &= -e^{-2\phi} - \frac{2(g^{13})^2 + (g^{23})^2}{g^{33}}.
\end{aligned}$$

Lastly, we may compute

$$\begin{aligned}
e^{-2\phi}\left[g_{31}\partial_1 g^{33} + g_{32}\partial_2 g^{33}\right] &= e^{-2\phi}\left[-\frac{g^{31}}{g^{33}}e^{2\phi}\partial_1 g^{33} - \frac{g^{32}}{g^{33}}e^{2\phi}\partial_2 g^{33}\right] \\
&= -g^{31}\partial_1 \log(g^{33}) - g^{32}\partial_2 \log(g^{33}).
\end{aligned}$$

Bibliography

[1] Lars Ahlfors. *Lectures on Quasiconformal Mappings*, volume 38 of *University Lecture Series*. American Mathematical Society, second edition, 2006.

[2] Lars Ahlfors and Lipman Bers. Riemann's mapping theorem for variable metrics. *Annals of Mathematics, Second Series*, 72(2):385–404, 1960.

[3] Kari Astala, Matti Lassas, and Lassi Päivärinta. The borderlines of the invisibility and visibility for calderón's inverse problem. *Analysis of PDEs*, 9:43–98, 2016.

[4] Dmitri Burago and Sergei Ivanov. Boundary rigidity and filling volume minimality of metrics close to a flat one. *Annals of Mathematics*, 171(2):1183–1211, 2010.

[5] Dmitri Burago and Sergei Ivanov. Area minimizers and boundary rigidity of almost hyperbolic metrics. *Duke Mathematics Journal*, 162(7):1205–1248, 2013.

[6] Alberto-P. Calderón. On an inverse boundary value problem. In *Seminar on Numerical Analysis and its Applications to Continuum Physics (Rio de Janeiro, 1980)*, pages 65–73. Soc. Brasil. Mat., Rio de Janeiro, 1980.

[7] Tobias Holck Colding and William P. Minicozzi, II. *A course in minimal surfaces*, volume 121 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2011.

[8] Ludvig Dmitrievich Faddeev. Increasing solutions of the schrödinger equation. *Soviet Physics Doklady*, 10:1033–1035, 1966.

[9] William H. Meeks III and Shing-Tung Yau. The classical plateau problem and topology of 3-dimensional manifolds. *Topology*, 21(4):409–442, 1982.

[10] Victor Isakov and Adrian Nachman. Global uniquness for a two-dimensional semi-linear elliptic inverse problem. *Transactions of the American Mathematical Society*, 347(9), 1995.

[11] Jürgen Jost. *Riemannian geometry and geometric analysis*. Universitext. Springer, Heidelberg, sixth edition, 2011.

[12] Matti Lassas and Gunther Uhlmann. On determining a Riemannian manifold from the Dirichlet-to-Neumann map. *Ann. Sci. École Norm. Sup. (4)*, 34(5):771–787, 2001.

[13] Juan Maldacena. The large n limit of superconformal field theories and supergravity. *Advanced Theory of Mathematical Physics*, 2:231–252, 1998.

[14] Juan Maldacena. Wilson loops in large n field theories. *Physical Review Letters*, 80:4859–4862, 1998.

[15] René Michel. Sur la rigidité imposée par la longueur des géodésiques. *Invent. Math.*, 65(1):71–83, 1981/82.

[16] Adrian Nachman. Global uniqueness for a two-dimensional inverse boundary value problem. *Ann. of Math. (2)*, 143(1):71–96, 1996.

[17] Leonid Pestov and Gunther Uhlmann. Two dimensional compact simple Riemannian manifolds are boundary distance rigid. *Ann. of Math. (2)*, 161(2):1093–1110, 2005.

[18] Tibor Radó. The problem of the least area and the problem of Plateau. *Math. Z.*, 32(1):763–796, 1930.

[19] Shinsei Ryu and Tadashi Takayanagi. Holographic derivation of entanglement entropy from ads/cft. *Physical Review Letters*, 96:181602, 2006.

[20] Valery Serov and Lassi Päivärinta. New estimates of the green?faddeev function and recovering of singularities in the two-dimensional schrödinger operator with fixed energy. *Inverse Problems*, 21:1291?1301, 2005.

[21] Plamen Stefanov, Gunther Uhlmann, and Andras Vasy. On the stable recovery of a metric from the hyperbolic dn map with incomplete data. arXiv:1505.02853v1 [math.AP], 2015.

[22] Ziqi Sun and Gunther Uhlmann. Anisotropic inverse problems in two dimensions. *Inverse Problems*, 19:1001–1010, 2003.

[23] John Sylvester and Gunther Uhlmann. A global uniqueness theorem for an inverse boundary value problem. *Ann. of Math. (2)*, 125(1):153–169, 1987.