REPORT ON JACK KLYS THESIS

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This is a report on the Ph.D. thesis of Jack Klys. This is an outstanding thesis. The thesis has three parts, and *any one* of the three parts is sufficient to be a good Ph.D. thesis.

Associated to every number field (finite extension of $\mathbb{Q}$, e.g. $\mathbb{Q}(\sqrt{2})$) there is a class group, which is a finite abelian group that measures the failure of unique factorization of algebraic integers in the number fields. The class group also controls many other things about the field, including its unramified extensions with abelian Galois group, and the structure of modules of the ring of algebraic integers in the fields. Questions about class groups actually go back to Gauss, who studied them in a more concrete form (classes of binary quadratic forms up to linear change of variables). An important question directing research in this area is to understand the distribution of class groups—how often are they what kind of groups. In 1984, Cohen and Lenstra developed conjectures to predict this distribution, and with very few exceptions the conjectures are completely open today.

The very few exceptions are: work of Davenport and Heilbronn (predating the Cohen-Lenstra Heuristics), work of Bhargava on the 2-part of class groups of cubic fields (published in *Annals*), and work of Fouvry and Klüners on the 2-part of the class groups of quadratic fields (published in *Inventiones*). Jack's work on the p-part of the class group (the first part of the thesis) of degree p Galois number fields (for an odd prime p) adds to this very short list of exceptions, each major breakthroughs. For $p = 3$, Jack's work is unconditional. For $p > 3$, Jack assumes a standard extension of the Riemann Hypothesis, but still there is little known about the distribution of class groups even with this assumption.

Jack's method of proof for his work on class groups of degree p fields extends the methods of Fouvry and Klüners, but the method is *not* just an application of the previous methods with numbers changed. First of all, Fouvry and Klüners certainly were aware of this question for $p > 2$ when they wrote their paper and yet did not treat any other cases. Even though they share common ideas, it in fact takes a somewhat creative point of view to see, in the technical details, how Jack's methods are analogous to those of Fouvry and Klüners.

The second part of Jack's thesis, joint with Brandon Alberts, is on counting non-abelian (H_8) unramified extensions of quadratic fields. This is a newer question for mathematicians to be considering, inspired recently due to Bhargava's results and my conjectures on the number of unramified G-extensions of quadratic fields, a non-abelian analog of the Cohen-Lenstra heuristics. The result obtained is an exact asymptotic for the number of extensions. In general, it is extremely difficult to even know where to being counting extensions, but in this case one starts with a formula of Lemmermeyer in terms of Legendre symbols, reducing this to an analytic number theory problem of similar flavor to the first part of the thesis. Again, the main result of the second part of the thesis is a nice enough, sufficient on its own for a PhD thesis. Moreover, this work has opened many new avenues, spelled out in the thesis, that I expect Jack and Brandon to make further progress on.

The third part of the thesis, on reflection principles, proves new results relating class groups of different fields. This is a classical topic, and so again any new results are quite impressive. In particular, Jack proves a conjecture of Lemmermeyer from 2005.

I give my strongest recommendation that this thesis be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

Minor Comments

(1) Into first paragraph, when talking about non-abelian should say "*unramified* Galois extensions"
(2) Notation, for $H^i(\mathcal{F})$ should say what coefficients
(3) p.2, "p-part" is sometimes used to mean p-torsion. Sylow p-subgroup is better
(4) Section 1.2: the quantity $S_X^{\pm}(A)$ depends implicitly on n, and this could be made clearer. Also could clarify what K is ranging over in definition of $S_X^{\pm}(A)$.
(5) After Conjecture 1.2.1: Cohen and Martinet don't really take on the situation where all degree n fields are lumped together (only separated by Galois group), and as far as they do, the set of primes their conjectures avoid is more complicated than $p \mid n$ (it definitely includes more than $p \mid n$, and I'm not sure if it includes all $p \mid n$).
(6) After Conjecture 1.2.1: the cite to Bhargava-Varma is strange. The result is originally due to Bhargava in "Density of discriminants of quartic fields". Bhargava-Varma gives a different proof and proves some generalizations.
(7) Section 1.3: again could clarify what K ranges over
(8) Section 1.3: surjections should be continuous. Maybe add this in notation section.

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