On the Topology of Collective Integrable Systems

by

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Abstract

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This thesis studies the topological properties of momentum maps of a large family of completely integrable systems called collective completely integrable systems.

The first result concerns the topological monodromy of a collective completely integrable system on a product of three two-spheres. This system is called a “Heisenberg spin-chain” for its connection with a quantum integrable system of the same name.

The remainder of the thesis is concerned with collective systems that are “constructed by Thimm’s trick” with the “action coordinates of Guillemin and Sternberg.” We observe that the open dense subset of a symplectic manifold where such systems define a Hamiltonian torus action is connected. This observation was absent from the literature until this point. We also prove that if a system is constructed in this manner from a chain of Lie subalgebras then the image is given by an explicit set of inequalities called branching inequalities.

When the symplectic manifold in question is a multiplicity free Hamiltonian $U(n)$-manifold, the construction of Thimm’s trick with the action coordinates of Guillemin and Sternberg yields a completely integrable Hamiltonian torus action on a connected open dense subset. If the momentum map is proper, then we are able to prove lower bounds for the Gromov width of the symplectic manifold from the classification of the connected open dense subset as a non-compact symplectic toric manifold. Convex multiplicity free manifolds of compact Lie groups have been classified by [51] and, accordingly, our lower bounds are given in terms of the combinatorics of the classifying data: the momentum set and a lattice. This result is the first estimate for the Gromov width of general multiplicity free manifolds of a nonabelian group.

This result relies crucially on connectedness of the open dense subset and the explicit description of the momentum map image. The proof is a generalization of methods used to prove lower bounds for the Gromov width of $U(n+1)$ coadjoint orbits [71] which are an example of multiplicity free $U(n)$-manifolds.

Dedication

In memory of Blake Dickens.

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Contents

Chapter 1

Introduction

A *symplectic manifold* is a smooth manifold M, necessarily of even dimension, equipped with a closed, nondegenerate 2-form ω. A smooth function f on M defines a *Hamiltonian vector field* $X_f \in \mathfrak{X}(M)$ via Hamilton's equation

$$\iota_{X_f}\omega = df.$$

The symplectic form also equips $C^\infty(M)$ with a Poisson bracket, defined by

$$\{f, g\} = \iota_{X_g}\iota_{X_f}\omega.$$

A *completely integrable system* or *Liouville integrable system* on a symplectic manifold (M, ω) of dimension $2n$ can be defined as n real valued functions[1] $f_1, \ldots, f_n \in C^\infty(M)$ that Poisson commute and whose differentials are independent on an open dense subset of M [3]. Given a completely integrable system, the Hamiltonian vector fields X_{f_i} commute and, provided they are complete, their flows define an action of $\mathbb{R}^n$ on M. Completely integrable systems originated in the study of Hamiltonian mechanics where a Hamiltonian function H is called completely integrable if there is a completely integrable system consisting of functions that are invariants of the Hamiltonian vector field X_H [3].

The *momentum map*[2] of a completely integrable system is the map $F = (f_1, \ldots, f_n) : M \to \mathbb{R}^n$ and the decomposition of M into connected components of the fibers of F is called the *Liouville foliation*[3] of the completely integrable system [11]. The momentum map factors through the space B of connected

[1] In fact, the important data of a completely integrable system is not the functions $f_1, \ldots, f_n$ but rather the subspace of $C^\infty(M)$ that they span. Thus more than n functions may also define a completely integrable system. For now we are content to fix a basis of this subspace.

[2] There is a minor disagreement in the literature over the use of "moment" vs. "momentum." Several interesting comments on the history of this terminology can be found in the mathoverflow discussion [1].

[3] Contrary to what this terminology suggests, the Liouville foliation is often not a foliation.

components of the fibers of F, which is called the *base space* of the system [88].

$$\begin{tikzcd} M \arrow[d, "\pi"'] \arrow[dr, "F"] & \\ B \arrow[r, "\widetilde{F}"'] & \mathbb{R}^n \end{tikzcd}$$

Suppose M is connected and F is the momentum map of a completely integrable system on M. The symplectic manifold (M, ω) together with the integrable system F is classified up to isomorphism[4] by the image of F provided that the following conditions are true.

(i) The $\mathbb{R}^n$ action on M defined by the flows of the Hamiltonian vector fields X_{f_i} induces an effective action of a compact torus T on M. In this case the action of T is a *completely integrable Hamiltonian T-action.*

(ii) The fibers of F are connected.

(iii) F is open as a map to it's image.

(iv) The image of F is convex.

If (i) is satisfied then B is a manifold with corners and (M, ω, F) is classified up to isomorphism by a cohomology class on B, the map $\widetilde{F}$, and a lattice in $\mathbb{R}^n$ determined by T [46]. If in addition (ii) and (iii) are satisfied, then the map $\widetilde{F}$ is a homeomorphism onto it's image. If (iv) is also satisfied, then $F(M)$ is contractible so the cohomology class on B vanishes and (M, ω, F) is determined up to isomorphism by $F(M)$ and a lattice in $\mathbb{R}^n$ determined by T.

For example, if M is compact and F satisfies condition (i), then conditions (ii)-(iv) follow by the Atiyah-Guillemin-Sternberg convexity theorem[5] [4, 27]. By the Delzant theorem, (M, ω, F) is classified up to isomorphism by $F(M)$ [17].

This thesis studies the topological properties of momentum maps of a large family of completely integrable systems called collective completely integrable systems, with an emphasis on collective systems that are "constructed by Thimm's trick". We are able to show that for these systems conditions (i)-(iv) hold on a connected open dense subset. Applying the classification of [46] to this open dense subset we are able to give lower bounds for the Gromov widths of a large family of symplectic manifolds that admit collective completely integrable systems constructed by Thimm's trick.

[4] Two symplectic manifolds M, M' with completely integrable systems F, F' are isomorphic if there is a symplectomorphism $\Psi : M \to M'$ and an integral affine transformation g such that $F' \circ \Psi = g \circ F$. There are other interesting definitions of 'equivalence' of completely integrable systems [11] – isomorphism is the strongest.

[5] Openness of the momentum map of a Hamiltonian torus action was not observed to be an important property in the literature until some time later, see for example [50].

Global topology of completely integrable systems

In general, the momentum map of a completely integrable system satisfies none of the conditions (i)-(iv). In order to motivate the study in this thesis, we now give an overview of some of the pathologies that can occur. For a more detailed study of the global topology of completely integrable systems, the reader may consult [18, 16, 86, 88].

As before, suppose M is a connected symplectic manifold and F is the momentum map of a completely integrable system on M. If F is proper, then in a neighbourhood of a connected component $\pi^{-1}(b)$ of a regular fiber[6] of F, the $\mathbb{R}^n$ action induces an action of a compact torus T. This is the content of the *Liouville-Arnold theorem*: there exists a neighbourhood U of $\pi^{-1}(b)$, an open set $V \subseteq \mathbb{R}^n$, a diffeomorphism $\Psi : U \to V \times T$, and a diffeomorphism $\underline{a} : F(U) \to V$, such that the following diagram

$$\begin{tikzcd} U \arrow[r, "\Psi"] \arrow[d, "F"'] & V \times T \arrow[d, "\mathrm{pr}_1"] \\ F(U) \arrow[r, "\underline{a}"'] & V \end{tikzcd}$$

commutes and Ψ is a symplectomorphism with respect to ω and the standard symplectic form on $V \times T$ [3]. The map $\underline{a}$ is called *local action coordinates* for the integrable system [3]. Since the diagram commutes and Ψ is a symplectomorphism the Hamiltonian vector fields of functions $a_1 \circ F, \ldots, a_n \circ F$ induce a completely integrable Hamiltonian action of the torus T on U.

A completely integrable system can be modified to satisfy condition (i) if there is a smooth map $\underline{a} : F(M) \to \mathbb{R}^n$ such that $\underline{a} \circ F$ is a momentum map for a completely integrable Hamiltonian T-action on M. The map $\underline{a}$, if it exists, is called *global action coordinates* for F.

If $M_{\text{reg}} \subseteq M$ denotes the union of fibers $\pi^{-1}(b)$ that do not contain critical points of F, then by the Liouville-Arnold theorem $B_{\text{reg}} = \pi(M_{\text{reg}})$ is a smooth manifold of dimension n and the map $\pi : M_{\text{reg}} \to B_{\text{reg}}$ is a smooth torus bundle. Local action coordinates for F define functions on patches of B_{reg} and one can try to extend local action coordinates to global action coordinates by continuation. This may fail to define action coordinates on M_{reg} for several reasons.

First, it may not be possible to extend local action coordinates to all of B_{reg} by continuation; local action coordinates on B_{reg} may be multivalued. The possibly multivalued nature of action coordinates was first described by [18] who related it to the topological monodromy of a covering space of B_{reg} – a lattice $L \subseteq T^*B_{\text{reg}}$ called the *period lattice* (see Section 2.2). Fixing a point $b \in B_{\text{reg}}$, the topological monodromy of the period lattice is a homomorphism

[6] Note that it is sufficient to assume F has no critical points in the connected component of the fiber, $\pi^{-1}(b)$.

$$\pi_1(B_{\text{reg}}, b) \longrightarrow GL(L_b),$$

where L_b is the fiber of the lattice over b, and the topological monodromy is trivial if and only if action coordinates are not multi-valued (and thus can be extended to B_{reg} by continuation). In particular, if B_{reg} is simply connected then there exist global action coordinates on B_{reg}. However, conditions (ii), (iii), and (iv) are not sufficient to guarantee that B_{reg} is simply connected; it is possible that there are critical values in the interior of $F(M)$ that make $F(M_{\text{reg}})$ ($\cong B_{\text{reg}}$ in this case) non-simply connected[7]. Examples where the topological monodromy of a completely integrable system is non-trivial have been given in dimension 4 by various authors [18, 7]. A main result of Chapter 2 is the computation of the topological monodromy of a completely integrable system on $S^2 \times S^2 \times S^2$.

Second, even if action coordinates exist on B_{reg} – there exists a map $g : B_{\text{reg}} \to \mathbb{R}^n$ such that $g \circ \pi$ defines *local* action coordinates on M_{reg} – it is possible that g does not induce a function on $F(M_{\text{reg}})$ for the simple reason that the fibers of F restricted to M_{reg} may not be connected (see for example [80], Chapter 5, figure 29). Thus the failure of condition (ii) can obstruct condition (i).

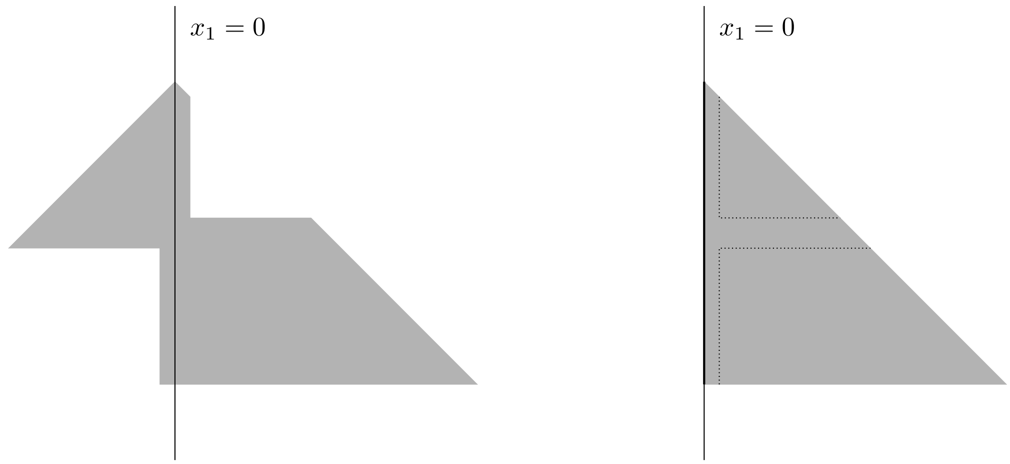
If condition (i) is satisfied and F generates a completely integrable Hamiltonian T-action on M then it is still possible that F fails to satisfy conditions (ii), (iii), and (iv).

Example 1.1. Let $\mathcal{A} \subseteq \mathbb{R}^2$ be an annulus and let $\widetilde{\mathcal{A}}$ be it's universal cover. Let $T = \mathbb{R}^2/\mathbb{Z}^2$ and consider the manifold $M = \widetilde{\mathcal{A}} \times T$ with symplectic form $dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ where (x_1, x_2) are coordinates on $\mathbb{R}^2$ and (y_1, y_2) are coordinates on $\mathbb{R}^2/\mathbb{Z}^2$. The projection $F(x_1, x_2, y_1, y_2) = (x_1, x_2)$ generates a completely integrable Hamiltonian T action on M. However, the fibers of F are not connected and $F(M) = \mathcal{A}$ is not convex. This example can also be modified by symplectic cutting so that the momentum map is not open as a map to $\mathcal{A}$.

On the other hand, even if one can define action coordinates on M_{reg}, it may happen that they do not extend smoothly from $F(M_{\text{reg}})$ to $F(M)$ as the following example demonstrates.

Example 1.2 (geodesic flow on the circle). Consider the cylinder $M = S^1 \times \mathbb{R}$ equipped with coordinates (q, p) and the symplectic form $dq \wedge dp$. The function $f_1(q, p) = p^2$ is a completely integrable system with momentum map $F = f_1$. The set $M_{\text{reg}} = \{(q, p) \in M : p \neq 0\}$ is not connected. The flow of the Hamiltonian vector field of the function $|p| = \sqrt{F}$ induces a S^1 action on M_{reg}, so the system admits action coordinates on M_{reg}. However, the function $a(x) = \sqrt{x}$ does not extend to a smooth function on $F(M) = \mathbb{R}_{\geq 0}$. The image $F(M)$ is convex but the fibers of F are not all connected.

[7] In fact, this is precisely what happens in the classic examples of integrable systems that do not admit global action coordinates [18, 7] as well as the example studied in Chapter 2.



(a) The open set $S \subseteq \mathbb{R}^2$.

(b) Image of M under action coordinates.

Figure 1.1: Example 1.3.

If F admits smooth action coordinates $\underline{a}$ on an open dense subset $\mathcal{U} \subseteq M$, then the following are sufficient conditions for the map $\underline{a} \circ F|_{\mathcal{U}}$ to satisfy (ii), (iii), and (iv).

(1) $\mathcal{U}$ is connected, and

(2) $F|_{\mathcal{U}} \colon \mathcal{U} \to \mathbb{R}^n$ is proper as a map to a convex subset of $\mathbb{R}^n$,

If (1) and (2) hold, then conditions (ii), (iii), and (iv) hold for the map $\underline{a} \circ F|_{\mathcal{U}}$ by the convexity theorem for proper Hamiltonian torus actions [10, 9]. We end this section with an example of what can happen if (1) and (2) are not true.

Example 1.3 (a false simplex). Let S be the open subset of $\mathbb{R}^2$ drawn in Figure 1.1 (a) and let $M = S \times \mathbb{R}^2/\mathbb{Z}^2$ equipped with the symplectic form $dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ where (x_1, x_2) are coordinates on $\mathbb{R}^2$ and (y_1, y_2) are coordinates on $\mathbb{R}^2/\mathbb{Z}^2$. The map $F(x_1, x_2, y_1, y_2) = (x_1^2, x_2)$ is the momentum map of a completely integrable system on M that admits action coordinates on the open dense submanifold $\mathcal{U}$ that is the complement of the subset where $x_1 = 0$. Note that condition (1) fails: $\mathcal{U}$ is not connected.

The image of $\mathcal{U}$ under the action coordinates $\underline{a} \circ F(x_1, x_2, y_1, y_2) = (|x_1|, x_2)$ is the open simplex $\Delta^\circ \subseteq \mathbb{R}^2$ drawn in Figure 1.1 (b), but $\mathcal{U}$ is not isomorphic to the symplectic manifold $(\Delta^\circ \times \mathbb{R}^2/\mathbb{Z}^2, dx_1 \wedge dy_1 + dx_2 \wedge dy_2)$, nor does it contain connected components that are. The restriction of $\underline{a} \circ F$ to $\mathcal{U}$ satisfies conditions (i), (iii), and (iv), but not (ii). Furthermore, condition (2) fails: the map $\underline{a} \circ F|_{\mathcal{U}}$ is not proper as a map to any convex set.

Collective completely integrable systems and Thimm's trick

The key observation of Chapter 3 is that conditions (1) and (2) hold for the momentum maps of a large family of collective completely integrable systems that admit action coordinates on an open dense set. We now introduce collective functions and collective completely integrable systems.

An action of a connected Lie group G with Lie algebra $\mathfrak{g}$ on a connected symplectic manifold (M, ω) is *Hamiltonian* if there is an equivariant[8] map $\mu : M \to \mathfrak{g}^*$ such that for all $X \in \mathfrak{g}$ the fundamental vector field $\underline{X} \in \mathfrak{X}(M)$ is the Hamiltonian vector field of the function $\langle \mu, X\rangle$, i.e. it satisfies Hamilton's equation

$$\iota_{\underline{X}}\omega = d\langle \mu, X\rangle.$$

The map μ is equivariant with respect to the action of G on M and the coadjoint action of G on $\mathfrak{g}^*$. The tuple (M, ω, μ) is called a *Hamiltonian G-manifold.* Given a Hamiltonian G-manifold, a *collective function* is the pullback $\mu^* f$ of a smooth function $f \in C^\infty(\mathfrak{g}^*)$. Collective functions appear naturally in the study of systems with Hamiltonian symmetries [26].

A *collective completely integrable system* on a Hamiltonian G-manifold is a completely integrable system consisting of collective functions $\mu^* f_1, \ldots, \mu^* f_n$. For a Hamiltonian G-manifold to admit a collective completely integrable system, it is necessary that the G-action is completely integrable in an appropriate non-commutative sense. This motivates the following definition [30, 66].

Definition 1.4. A Hamiltonian G-manifold is *multiplicity free* if the Poisson subalgebra of invariant functions $C^\infty(M)^G$ is abelian.

If a Hamiltonian G-manifold admits a collective completely integrable system, then it must be multiplicity free [29]. Following the definition of multiplicity free G-manifolds, their local and global structure was studied extensively, culminating with the classification of convex[9] multiplicity free G-manifolds for any compact connected Lie group G [51, 57].

From this point forward, we assume that G is compact. A sufficient[10] condition for there to exist a collective completely integrable system on any multiplicity free G-manifold M is the existence of n Poisson commuting functions $f_1, \ldots, f_n \in C^\infty(\mathfrak{g}^*)$ such that

- the functions $f_1, \ldots, f_r$ generate $C^\infty(\mathfrak{g}^*)^G$, and
- the restriction of the functions $f_{r+1}, \ldots, f_n$ to any symplectic leaf of $\mathfrak{g}^*$ is a completely integrable system [29].

[8] With respect to the action of G on M and the coadjoint action of G on $\mathfrak{g}^*$.

[9] A Hamiltonian G-manifold is *convex* if μ satisfies conditions (ii), (iii), and (iv) [50]. In particular, if $\mu\colon M \to \mathfrak{g}^*$ is proper, then (M, ω, μ) is convex [14, 39, 55], but a convex Hamiltonian G-manifold need not have μ proper.

[10] Since the image $\mu(M)$ may not equal $\mathfrak{g}^*$, this condition may not be necessary.

Casimirs give r independent functions that generate $C^\infty(\mathfrak{g}^*)^G$ so the problem of constructing collective completely integrable systems on multiplicity free G-manifolds reduces to finding additional Poisson commuting functions $f_{r+1}, \ldots, f_n$ whose restriction to coadjoint orbits are sufficiently independent. Methods for constructing additional functions were explored in the 1970's and 1980's, notably resulting in the method of "argument shifting" also called "translation of invariants" [59, 65] (see also the book [22]). One motivation for this work was the problem of determining whether geodesic flows on symmetric spaces G/H, which can be viewed as Hamiltonian systems on T^*G/H, are completely integrable [30].

In [76], additional commuting functions were constructed on $\mathfrak{g}^*$ by considering collective functions *on* $\mathfrak{g}^*$ corresponding to subalgebras $\mathfrak{h} \subseteq \mathfrak{g}$. The connected subgroup H corresponding to a subalgebra $\mathfrak{h}$ has an induced action on $\mathfrak{g}^*$ and the restriction of this action to a coadjoint orbit is Hamiltonian, generated by the restriction of the linear map $p : \mathfrak{g}^* \to \mathfrak{h}^*$ dual to the inclusion of $\mathfrak{h}$ in $\mathfrak{g}$. Pullbacks p^*f of invariant functions $f \in C^\infty(\mathfrak{h}^*)^H$ are collective functions on $\mathfrak{g}^*$.

If subalgebras used to construct collective functions on $\mathfrak{g}^*$ form a chain $\mathfrak{h}_1 \subseteq \cdots \subseteq \mathfrak{h}_k$ or satisfy the condition

$$[\mathfrak{h}_i, \mathfrak{h}_j] \subseteq \mathfrak{h}_i \text{ or } \mathfrak{h}_j$$

for all i, j, then the collective functions constructed from invariants of each of the subalgebras Poisson commute and one obtains a large number of additional functions on $\mathfrak{g}^*$. This construction of additional collective functions is often referred to as the "method of Thimm" or "Thimm's trick" after the author of [76][11].

Remarkably, collective functions constructed by Thimm's trick on a multiplicity free G-manifold admit natural global action coordinates on an open dense set[12][28]. Given a choice of maximal torus T with positive Weyl chamber $\mathfrak{t}^*_+$, the *sweeping map* is the continuous map $s\colon \mathfrak{g}^* \to \mathfrak{t}^*_+$ that sends an element of $\mathfrak{g}^*$ to the unique element of the intersection of its G-orbit with $\mathfrak{t}^*_+$. The sweeping map is not smooth on $\mathfrak{g}^*$ but if M is a Hamiltonian G-manifold then the pullback $\mu^*s : M \to \mathfrak{t}^*_+$ is smooth on an open dense subset of M and generates a Hamiltonian torus action there. Given subalgebras $\mathfrak{h}_1, \ldots, \mathfrak{h}_k$ that form a chain or otherwise satisfy conditions of Thimm, one obtains continuous maps in this manner whose restriction to open dense subsets of M generate Hamiltonian torus actions. These torus actions commute on the intersection of these open dense subsets, thus one has

- a continuous map $F\colon M \to \mathbb{R}^N$ which we say is "constructed by Thimm's trick" with the "action coordinates of Guillemin and Sternberg," and

[11] The method of obtaining additional commuting functions on $\mathfrak{g}^*$ from subalgebras also appeared in the slightly earlier paper [79].

[12] In [28], Guillemin and Sternberg only address the case when the principal stratum of the Hamiltonian G-manifold is maximal. A full proof is given in [83].

- an open dense subset $\mathcal{U} \subseteq M$ such that the restriction of F to $\mathcal{U}$ generates a Hamiltonian torus action (i.e. $F|_{\mathcal{U}}$ satisfies (i)).

It follows from properties of Hamiltonian G-manifolds that the open dense set $\mathcal{U}$ is connected (Lemma 3.26), so $F|_{\mathcal{U}}$ satisfies condition (1). If in addition the map $F|_{\mathcal{U}}$ satisfies condition (2) – it is proper as a map to a convex set – then $(\mathcal{U}, \omega|_{\mathcal{U}}, F|_{\mathcal{U}})$ is a proper Hamiltonian torus manifold. If moreover the torus action on $\mathcal{U}$ is completely integrable, then it follows by that $(\mathcal{U}, \omega|_{\mathcal{U}}, F|_{\mathcal{U}})$ is classified as a non-compact symplectic toric manifold by the image $F(\mathcal{U})$ and a lattice in $\mathbb{R}^N$ [46]. This is in contrast to Example 1.3, where conditions (1) and (2) fail.

We end this subsection by noting that functions constructed by Thimm's trick with Guillemin and Sternberg's action coordinates are a key player in the recent paper [38] which studies the "symplectic contraction" construction. This is discussed briefly in Section 3.6.

Gelfand-Zeitlin systems on coadjoint orbits

In [28], Guillemin and Sternberg used their action coordinates for Thimm's trick to construct completely integrable torus actions on open dense subsets of unitary coadjoint orbits, which they called *Gelfand-Zeitlin systems.*

A $U(n)$ coadjoint orbit can be identified with the set of $n \times n$ Hermitian matrices $\mathcal{O}_\lambda$ with fixed eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$. Thimm's trick with Guillemin and Sternberg's action coordinates then defines maps

$$\Lambda_j \colon \mathcal{O}_\lambda \to \mathbb{R}^j$$

that send a Hermitian matrix to the ordered eigenvalues of its upper left $j \times j$ submatrix, $\lambda_1^{(j)} \geq \cdots \geq \lambda_j^{(j)}$ (since Hermitian matrices are described up to conjugacy by their eigenvalues, it should be clear that this map describes projection to $\mathfrak{u}(j)^*$ composed with the sweeping map from $\mathfrak{u}(j)^*$ to a Weyl chamber). These functions satisfy *interlacing inequalities*, and it follows that the the image of the map

$$\Lambda = (\Lambda_{n-1}, \ldots, \Lambda_1) \colon \mathcal{O}_\lambda \to \mathbb{R}^{n-1} \oplus \cdots \oplus \mathbb{R}^1$$

is contained in the convex polytope $\Delta_\lambda \subseteq \mathbb{R}^{n-1} \oplus \cdots \oplus \mathbb{R}^1$ defined by the inequalities

$$
\begin{array}{ccccccccccccc}
\lambda_1 & & & & \lambda_2 & & & & \lambda_3 & \cdots & \lambda_{n-1} & & & & \lambda_n \\
& \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\geq$} & & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\geq$} & & & & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\geq$} & \\
& & \lambda_1^{(n-1)} & & & & \lambda_2^{(n-1)} & & & \cdots & & & \lambda_{n-1}^{(n-1)} & & \\
& & & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\geq$} & & \rotatebox{-45}{$\geq$} & & & & \rotatebox{45}{$\geq$} & & & \\
& & & & \lambda_1^{(n-2)} & & & & \ddots & & \iddots & & & & \\
& & & & & \rotatebox{-45}{$\geq$} & & & & & & & & & \\
& & & & & & \ddots & & & \iddots & & & & & \\
& & & & & & & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\geq$} & & & & & \\
& & & & & & & & \lambda_1^{(1)} & & & & & &
\end{array}
\tag{1.5}
$$

which is called the *Gelfand-Zeitlin polytope* corresponding to λ. Guillemin and Sternberg proved[13] that $\Lambda(\mathcal{O}_\lambda) = \Delta_\lambda$; the image of the Gelfand-Zeitlin system on $\mathcal{O}_\lambda$ is the Gelfand-Zeitlin polytope Δ_λ.

Depending on λ, the inequalities (1.5) force some functions $\lambda_i^{(j)}$ to be constant on $\mathcal{O}_\lambda$. The remaining non-constant Gelfand-Zeitlin functions $\lambda_i^{(j)}$ are smooth on the open dense subset $\mathcal{U} \subseteq \mathcal{O}_\lambda$ where both of the neighbouring horizontal inequalities $\lambda_{i-1}^{(j)} \geq \lambda_i^{(j)} \geq \lambda_{i+1}^{(j)}$ are strict, but they may fail to be smooth on the complement of this set (this is not surprising since the functions $\lambda_i^{(j)}$ are defined to be ordered eigenvalues of $j \times j$ submatrices). Guillemin and Sternberg also showed that the non-constant Gelfand-Zeitlin functions generate an effective, completely integrable Hamiltonian torus action on the open dense subset $\mathcal{U}$ where they are smooth[14] [28, 29].

Thus for Gelfand-Zeitlin systems on coadjoint orbits, it was previously known that the Gelfand-Zeitlin functions define an effective, completely integrable Hamiltonian torus action on the open dense set $\mathcal{U}$ (condition (i)) and the image of the Gelfand-Zeitlin systems are the Gelfand-Zeitlin polytopes (condition (iv)), but this is not sufficient to conclude that the open dense subsets $\mathcal{U}$ are classified as non-compact symplectic toric manifolds by their momentum map images (cf. Example 1.3). In Example 3.27 we apply the results of our study of the construction of Thimm's trick with Guillemin and Sternberg's action coordinates to conclude the following.

Theorem 1.6. *Let $\mathcal{O}_\lambda$ be a unitary (or orthogonal[15]) coadjoint orbit. The open dense subset $\mathcal{U} \subseteq \mathcal{O}$ where the Gelfand-Zeitlin functions define an effective, completely integrable torus action is a proper toric manifold. In particular, it is classified up to isomorphism by the subset of the Gelfand-Zeitlin*

[13] Although they describe a method to prove this fact in general, [28] only gives a full proof for regular unitary coadjoint orbits. This fact was proven for arbitrary unitary coadjoint orbits in [70].

[14] Again, the argument was given only for regular coadjoint orbits. Details for arbitrary coadjoint orbits appear in [70].

[15] Gelfand-Zeitlin polytopes for orthogonal coadjoint orbits are defined by a different set of inequalities. See [70, 71].

polytope Δ_λ where no extra horizontal inequalities in (1.5) *are equalities.*

Lower bounds for the Gromov width of multiplicity free $U(n)$ manifolds

Gromov's non-squeezing theorem says that if there exists a symplectic embedding of a ball of radius $r > 0$, $B^{2n}(r)$ equipped with the standard symplectic structure into the cylinder $Z^{2n}(R) = B^2(R) \times \mathbb{R}^{2n-2}$ also equipped with the standard symplectic structure, then $r \leq R$ [25]. Thus there is an obstruction of the existence of symplectic embeddings beyond the requirement that symplectic embeddings preserve Liouville volume. For an arbitrary symplectic manifold (M, ω) of dimension $2n$, the *Gromov width* of M,

$$\mathrm{GWidth}(M, \omega) = \sup_{r>0} \left\{ \pi r^2 \colon \exists \text{ a symplectic embedding } B^{2n}(r) \hookrightarrow M \right\},$$

records the obstruction of symplectic embeddings of balls into M.

In [71], Theorem 1.6 was combined with the classification of proper symplectic toric manifolds [46] to prove lower bounds for the Gromov width of unitary and orthogonal coadjoint orbits by studying the combinatorics of Gelfand-Zeitlin polytopes. These lower bounds agreed with the known upper bounds and this work was an important step towards the eventual computation of the Gromov width for all coadjoint orbits of all simple Lie groups [21, 13].

Inspired by the approach of [71] and our newfound understanding of the topology of completely integrable systems constructed by Thimm's trick with the action coordinates of Guillemin and Sternberg, we make the following observations. Suppose (M, ω, μ) is a connected multiplicity free Hamiltonian $U(n)$-manifold with $\mu : M \to \mathfrak{u}(n)^*$ proper. Then we know that

I) The construction of Thimm's trick with the action coordinates of Guillemin and Sternberg defines a continuous map $F \colon M \to \mathbb{R}^N$ whose restriction to an open dense subset $\mathcal{U} \subseteq M$ generates a completely integrable Hamiltonian torus action.

II) The open dense submanifold $(\mathcal{U}, \omega|_{\mathcal{U}}, F|_{\mathcal{U}})$ is classified as a non-compact symplectic toric manifold by its image $F(\mathcal{U})$ together with a lattice (the torus action generated by F may not be effective).

III) The image $F(\mathcal{U})$ can be described by the momentum set $\square = \mu(M) \cap \mathfrak{t}^*_+$ and the interlacing inequalities for the Gelfand-Zeitlin functions.

Thus we can prove lower bounds for the Gromov width of arbitrary multiplicity free $U(n)$-manifolds by studying the sets $F(\mathcal{U})$ and the corresponding lattices, following work by [78, 73, 58, 71].

Convex multiplicity free Hamiltonian $U(n)$-manifolds are classified up to isomorphism by two pieces of data: the convex locally polyhedral set $\square = \mu(M) \cap \mathfrak{t}^*_+$ and a lattice $\Gamma_M \subseteq \mathfrak{t}^*$ that is determined by the principal isotropy group for the action of $U(n)$ on M [51, 57]. We define a quantity $\mathcal{W}(\Box, \Gamma_M)$ that depends only on $\Box$ and Γ_M and prove the following theorem.

Theorem 4.1: *Let (M, ω, μ) be a connected multiplicity free Hamiltonian $U(n)$-manifold with μ proper. Then*

$$\mathcal{W}(\Box, \Gamma_M) \leq \mathrm{GWidth}(M, \omega).$$

Remark 1.7. The Gromov width of a symplectic manifold is the minimal symplectic capacity [41], i.e. for any symplectic capacity c,

$$\mathrm{GWidth}(M, \omega) \leq c(M, \omega).$$

Thus theorem 4.1 provides lower bounds for all symplectic capacities of multiplicity free $U(n)$-manifolds.

The lower bound obtained in Theorem 4.1 are not optimal – they are strictly worse than the lower bounds of [71] for some non-regular $U(n)$ coadjoint orbits (see the discussion in section 4.4.2). We expect that the lower bound can be improved, and hope to find a combinatorial invariant of $\Box$ and Γ_M that equals the maximal lower bound that can be achieved in this manner.

1.1 Overview

Each chapter was written independently and is more or less self contained (with the exception of the dependence of Chapter 4 on the results of Chapter 3). There is some overlap in the background and minor differences in notation[16]

Chapter 2: Topological monodromy of a completely integrable Heisenberg spin chain This chapter studies the topology of the Liouville foliation of the symplectic manifold $(S^2 \times S^2 \times S^2, \omega_{STD} \oplus \omega_{STD} \oplus \omega_{STD})$ given by the completely integrable system

$$H(X, Y, Z) = \langle X \cdot Y \rangle + \langle Y, Z \rangle + \langle Z, X \rangle,$$
$$I(X, Y, Z) = \langle X + Y + Z, e_3 \rangle,$$
$$J(X, Y, Z) = \det(X, Y, Z)$$

where $(X, Y, Z) \in S^2 \times S^2 \times S^2$ are considered as vectors in $\mathbb{R}^3$, $\langle , \rangle$ is the standard inner product on $\mathbb{R}^3$, and e_3 is the unit vector $(0, 0, 1)$. In particular we

[16] In Chapter 3 momentum maps for nonabelian group actions are denoted by Φ whereas in Chapter 4 they are denoted by μ. The best of both worlds.

- Compute the image of the momentum map (Proposition 2.20).
- Show that the fibers of the momentum map are connected (Theorem 2.24).
- Compute the sets of critical values and critical points of the momentum map (Proposition 2.21).
- Explicitly compute the topological monodromy of the system on M_{reg} (Theorem 2.1).

This integrable system can be viewed as a collective integrable system: if $S^2 \times S^2 \times S^2$ is viewed as a coadjoint orbit in $\mathfrak{g}^* = \mathfrak{so}(3)^* \oplus \mathfrak{so}(3)^* \oplus \mathfrak{so}(3)^*$, then the functions H, I, J are the restriction of Poisson commuting functions on $\mathfrak{g}^*$ to the coadjoint orbit.

Chapter 3: Convexity and Thimm's trick After recalling background and definitions for Hamiltonian group actions, Lemma 3.26 and Theorem 3.1 are proven. In Example 3.27, we apply these results to conclude that the open dense subsets of unitary and orthogonal coadjoint orbits where Gelfand-Zeitlin systems define completely integrable Hamiltonian torus action are proper symplectic toric manifolds (and thus are classified up to isomorphism by the corresponding subsets of Gelfand-Zeitlin polytopes).

In the special case of Thimm's trick where the commuting functions are constructed from a chain of subalgebras of $\mathfrak{g}$, we also prove that the image of the map constructed by Thimm's trick is defined by the inequalities that define the Kirwan set $\Phi(M) \cap \mathfrak{t}_+^*$ and the branching inequalities [6] for the chain of subalgebras (Proposition 3.49).

Chapter 4: Lower bounds for the Gromov width of multiplicity free $U(n)$-manifolds The main result of this chapter is Theorem 4.1. This result relies crucially on the convexity results of Chapter 3.

Completely integrable Hamiltonian T-manifolds with proper momentum maps and the rules for obtaining lower bounds for their Gromov width from their momentum map images are carefully introduced in Section 4.2. In Section 4.3, basic facts about Hamiltonian group actions and multiplicity free spaces are recalled, as well as the relevant results for Gelfand-Zeitlin systems on $U(n)$ coadjoint orbits due to [71]. The proof of Theorem 4.1 and a discussion of the result is contained in Section 4.4.

Appendix A: Fibers of maps constructed by toric degeneration are connected This appendix gives a short proof that the fibers of completely integrable systems constructed via toric degeneration in [34] are connected. The proof is an application of Lemma 3.28.

Appendix B: Completely integrable systems This appendix gives a review of the basic theory behind the definition of completely integrable systems and the details of a theorem about collective completely integrable systems from [29] which we use in Chapter 4.

Appendix C: Thimm's trick This appendix contains proofs of some of the basic facts about torus actions constructed by Thimm's trick with the action coordinates of Guillemin and Sternberg.

The publication statuses of the chapters of this document are as follows.

- Chapter 2, *Topological monodromy of a completely integrable Heisenberg spin chain*, was published in [53].
- Chapter 3 is in the process of peer review. The current version of Chapter 3 is the current iteration of *Convexity and Thimm's trick* in the peer-review process, which does not appear on the arXiv. Previous versions of this chapter appear on the arXiv [52].
- Chapter 4 has not appeared.
- Appendix A appeared in arXiv version 3 of [52], but was removed in the subsequent version.

Chapter 2

Topological Monodromy of an Integrable Heisenberg Spin Chain

We investigate topological properties of a completely integrable system on $S^2 \times S^2 \times S^2$ which was recently shown to have a Lagrangian fiber diffeomorphic to $\mathbb{R}P^3$ not displaceable by a Hamiltonian isotopy [Oakley J., Ph.D. Thesis, University of Georgia, 2014]. This system can be viewed as integrating the determinant, or alternatively, as integrating a classical Heisenberg spin chain. We show that the system has non-trivial topological monodromy and relate this to the geometric interpretation of its integrals.

2.1 Introduction

The Hamiltonian

$$H(X,Y,Z) = \sqrt{3 + \langle X, Y\rangle + \langle Y, Z\rangle + \langle Z, X\rangle}$$

models pairwise interaction of three identical spin vectors $X, Y, Z \in S^2$ which are fixed to the vertices of an equilateral triangle. Systems of this type – called Heisenberg spin chains – are of interest to physicists as they provide a classical model for quantum spin in a fixed lattice. Together with the Hamiltonians

$$I(X,Y,Z) = \langle X + Y + Z, e_3\rangle \qquad \text{and} \qquad J(X,Y,Z) = \det(X,Y,Z),$$

this Heisenberg spin chain becomes a completely integrable system on $(S^2 \times S^2 \times S^2, \omega_{\mathrm{STD}} \oplus \omega_{\mathrm{STD}} \oplus \omega_{\mathrm{STD}})$. Lagrangian fibers of this system were recently studied in [68], and it was shown that the system has a Lagrangian fiber $L \cong \mathbb{R}P^3$ which is not displaceable by Hamiltonian diffeomorphisms.

There is an analogous coupled spin system given by the Hamiltonians

$$H_1(X,Y) = \sqrt{1 - \langle X, Y\rangle} \qquad \text{and} \qquad H_2(X,Y) = \langle X+Y, e_3\rangle,$$

which has provided interesting examples of non-displaceable Lagrangian tori in $(S^2 \times S^2, \omega_{\mathrm{STD}} \oplus \omega_{\mathrm{STD}})$ [20, 23, 69, 84]. In fact, (H_1, H_2) is the moment map of a Hamiltonian T^2-action on the complement of the Lagrangian sphere $\widetilde{\Delta}$ of anti-diagonal elements $(X, -X)$, where H_1 fails to be smooth.

In this note we study the integrable system $\mathcal{H} = (H, I, J)$ from the perspective of Lagrangian torus fibrations and global action-angle coordinates, as introduced by Duistermaat in [18]. Unlike the system on $S^2 \times S^2 \setminus \widetilde{\Delta}$, the system $\mathcal{H}$ is not toric on the set where it is smooth, $S^2 \times S^2 \times S^2 \setminus L$, and cannot be made so: there is a global obstruction to the existence of a diffeomorphism f such that the Hamiltonian flow of $f \circ J$ is periodic. We compute this obstruction, called topological monodromy, explicitly

Theorem 2.1. *The topological monodromy of the system $\mathcal{H}$ is generated by the matrix*

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus the system does not admit global action coordinates (since H and I are already periodic, this implies that there does not exist a diffeomorphism $f\colon \mathrm{Im}(J) \to \mathbb{R}$ such that the Hamiltonian $f \circ J$ is periodic).

Although topological monodromy of a given integrable system may be computed directly using elliptic integrals – as has been done in [7, 18] and many other places – we derive this result from general theorems of Zung and Izosimov about the structure of integrable systems near singularities [43, 86]. Section 2.2 reviews the theory of singularities for integrable systems developed in [86] and the relation of this theory to the topological monodromy obstruction. Section 2.3 establishes basic facts and notation for adjoint orbits in Lie algebras which we use throughout the paper. Section 2.4 introduces the system $\mathcal{H}$ and provides an algebraic proof that the Hamiltonians H, I, J Poisson commute. In order to apply the theorems of Zung and Isozimov to this system, we must describe the topology of the critical fibers of the map $\mathcal{H}$. In Section 2.5 we use the underlying Euclidean geometry of the system and the structure of the Lie algebra $\mathfrak{so}(3)$ to completely describe the critical set, critical values and image of $\mathcal{H}$,

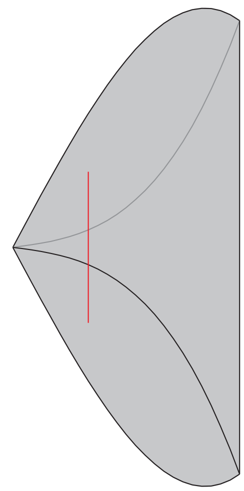


Figure 2.1: The moment map image is a solid with one 'orbifold' corner, four 'toric' faces, and a 'critical line' through the interior (red).

Theorem 2.2. *The image of the moment map $\mathcal{H}$ is the set of points $(r, s, t) \in \mathbb{R}^3$ that satisfy the equations*

$$|t| \leq \sqrt{1 + 2\left(\frac{r^2-3}{6}\right)^3 - 3\left(\frac{r^2-3}{6}\right)^2}, \qquad |s| \leq r, \qquad \textit{and} \qquad 0 \leq r \leq 3 \tag{2.3}$$

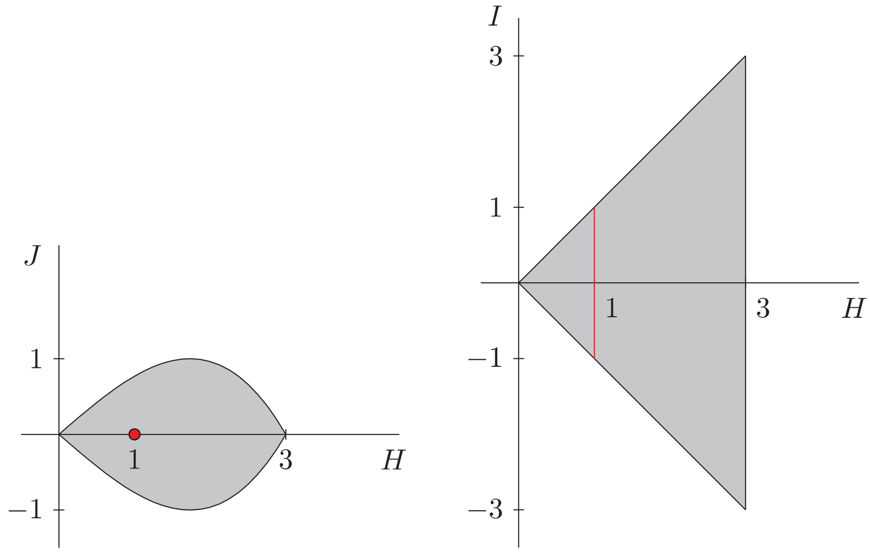
(*see Figs.* 2.1 *and* 2.2). *The set of critical values consists of the boundary of this set, together with the line segment* $\{(1, s, 0)\colon -1 < s < 1\}$.

We show in Section 2.7 that the critical fibers above the line segment $\{(1, s, 0)\colon -1 < s < 1\}$, which we call the 'critical line', are all topologically stable, rank 1, focus-focus (see Section 2.2 for definitions). Theorem 2.1 follows from this by the theorems of Zung and Izosimov.

Although it is not necessary in order to prove Theorem 2.1, we give a hands-on proof that the regular fibers of $\mathcal{H}$ are all connected in Section 2.6. Combining our description of the moment map image, and results of [18], the Lagrangian torus fibration over the set of regular values of $\mathcal{H}$ is determined up to fiber-preserving symplectomorphism by Theorem 2.1 (see Remark 2.29).

2.2 Singularities and topological monodromy of integrable systems

In this section we review the theory of singularities for integrable systems developed in [86] and its relation to topological monodromy. We make several simplifying assumptions (such as fiber connectedness) and the reader should note that Theorem 2.13 is not stated in full generality. Historical background and



(a) (b)

Figure 2.2: Projections of the moment map image.

further details can be found in [11, 86].

Definition 2.4. An *integrable system* is a triple $(M, \omega, \mathcal{H})$ where (M, ω) is a symplectic manifold of dimension $2n$, and $\mathcal{H} = (H_1, \dots, H_n)\colon M \to \mathbb{R}^n$ is a smooth map such that

1) the Poisson bracket $\{H_i, H_j\} = 0$ for all i, j, and

2) there is an open, dense subset of M where the functions $H_1, \dots, H_n$ are all functionally independent (i.e., $dH_1 \wedge \cdots \wedge dH_n \neq 0$).

We denote the image of $\mathcal{H}$ by B, the set of regular values by B_{reg}, and its preimage $\mathcal{H}^{-1}(B_{\text{reg}})$ by M_{reg}.

Definition 2.5. An integrable system $(M, \omega, \mathcal{H})$ admits *global action coordinates* if there is a diffeomorphism $a = (a_1, \dots, a_n)\colon B \to \mathbb{R}^n$ such that the Hamiltonian flow of $a_i \circ \mathcal{H}$ is periodic for each $1 \leq i \leq n$. The map a is often referred to as *global action coordinates* for $\mathcal{H}$.

If an integrable system with compact fibers admits global action coordinates, then $(M, \omega,\, a \circ \mathcal{H})$ is a symplectic toric manifold. Thus, an important question in the study of integrable systems is

Question 2.6. *Does a given integrable system* $(M, \omega, \mathcal{H})$ *admit global action coordinates?*

Suppose the fibers of $\mathcal{H}$ are compact and connected; then the regular fibers are Lagrangian n-tori and the Arnold–Liouville theorem implies that the restricted map $\mathcal{H}\colon M_{\text{reg}} \to B_{\text{reg}}$ is a fiber bundle of compact, connected Lagrangian tori T^n that is locally symplectically equivalent to the trivial system

$$\mathcal{H} = (x_1, \ldots, x_n)\colon\ \big(D^n \times T^n, dx \wedge d\theta\big) \to D^n.$$

Regular integrable systems such as this are called *Lagrangian torus fibrations.* Given a Lagrangian torus fibration, there is a natural fiber-wise action

$$T^*B \times_B M \to M$$

defined by $(\alpha_b, m) \mapsto \phi^1_{\alpha_b}(m)$ where $\phi^1_{\alpha_b}$ is the time-1 flow of the vector-field $\omega^{-1}\mathcal{H}^*\alpha$ for some 1-form $\alpha \in \Omega^1(B)$ such that $\alpha(b) = \alpha_b$. The stabilizer of this action is a smooth submanifold $\Lambda \subset T^*B$ that is a full rank lattice in each fiber, called the *period lattice* of the fibration. Restricting the projection map $\pi\colon T^*B \to B$ to this lattice, we obtain a covering space $\pi\colon \Lambda \to B$ whose topological monodromy is a group homomorphism $m_b\colon \pi_1(B, b) \to GL(n, \mathbb{Z})$ after identifying the fiber Λ_b with $\mathbb{Z}^n$ by a choice of basis (see, e.g., [63] or [18] for full details of this construction). The *topological monodromy* of a Lagrangian torus fibration is the topological monodromy of its period lattice, and its non-triviality is the obstruction to the torus fiber bundles

$$T^*B/\Lambda \to B \qquad \text{and} \qquad M \to B$$

being principal, and thus to $\mathcal{H}$ admitting global action coordinates. In his seminal 1980 paper *On global action-angle coordinates*, Duistermaat proved

Theorem 2.7 ([18]). *A Lagrangian torus fibration $\mathcal{H}\colon M \to B$ admits global action coordinates if and only if the topological monodromy of the period lattice is trivial.*

An integrable system with compact, connected fibers admits global action coordinates only if the corresponding Lagrangian torus fibration $M_{\text{reg}} \to B_{\text{reg}}$ admits global action coordinates, so to answer Question 2.6 one may begin by studying the topological monodromy of this fibration. On the other hand, there is a subtle interplay between the topological monodromy of $M_{\text{reg}} \to B_{\text{reg}}$ and the critical fibers of $\mathcal{H}$. To describe this interplay we need to recall several definitions and theorems from [43, 86].

Let p be a critical point of rank $n-k$ of an integrable system $\mathcal{H} = (H_1, \ldots, H_n)$ on (M, ω). Without loss of generality, we may assume that $dH_1, \ldots, dH_k = 0$. The operators $\omega^{-1}d^2H_1,\ \ldots, \omega^{-1}d^2H_k$ form a commutative subalgebra $\mathfrak{h}$ of $\mathfrak{sp}(L^\perp/L) \cong \mathfrak{sp}(\mathbb{R}, 2k)$ where the subspace $L \subset T_pM$ is the span of the vector fields $X_{H_{k+1}}, \ldots, X_{H_n}$ and $L^\perp$ is its symplectic ortho-complement. A generic critical point of $\mathcal{H}$

will satisfy the following Morse–Bott type condition for integrable systems:

Definition 2.8. A rank $n-k$ critical point p of an integrable system $\mathcal{H}$ is *non-degenerate* if the subalgebra $\mathfrak{h}$ defined above is a Cartan subalgebra. Similarly, a critical fiber of an integrable system is *non-degenerate* if all of its critical points are non-degenerate.

Equivalently, p is non-degenerate if some linear combination of the operators $\omega^{-1}d^2H_1, \ldots, \omega^{-1}d^2H_k$ has $2k$ distinct eigenvalues (and the operators are independent). It was shown in [81] that a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{sp}(\mathbb{R}, 2k)$ decomposes as a direct sum of three types of Lie subalgebra, called *elliptic*, *hyperbolic*, and *focus-focus*, and the conjugacy class of a given Cartan subalgebra $\mathfrak{h}$ can be described by the three positive integers h_e, h_h, and h_f, which denote the number of elliptic, hyperbolic, and focus-focus blocks respectively in this decomposition[1].

Definition 2.9. The *Williamson type* of a non-degenerate critical point p for an integrable system $\mathcal{H}$ is the 3-tuple (h_e, h_h, h_f) corresponding to the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{sp}(\mathbb{R}, 2k)$.

The remainder of this paper is primarily concerned with critical points for which the Williamson type is $(h_e, 0, h_f)$; such critical points are sometimes called *almost toric*. A fiber is called *almost toric* if all the critical points it contains are almost toric.

It was shown in [86] that the Williamson type of a critical point of lowest rank in a non-degenerate critical fiber is an invariant of the fiber, which allows the definition,

Definition 2.10. The *Williamson type* of a non-degenerate critical fiber N in an integrable system $(M, \omega, \mathcal{H})$ with compact and connected fibers is the Williamson type of any critical point of lowest rank in the fiber. The *rank* of N is the rank of these critical points.

Assuming once again that all the fibers of an integrable system are compact and connected, we are interested in the foliation of a neighbourhood $\mathcal{U}(N)$ of a critical fiber N by the fibers of $\mathcal{H}$ (this is the *Liouville foliation*, although it is defined differently in more general settings, see [86]). We consider neighbourhoods of the form $\mathcal{U}(N) = \mathcal{H}^{-1}(D)$ where D is a small disc centred at $\mathcal{H}(N)$. Two foliations are said to be *topologically equivalent* if they are related by a foliation preserving homeomorphism. A *singularity* of an integrable system is an equivalence class of a Liouville foliation in a neighbourhood of N.

Let (H_1, H_2) be an integrable system with compact, connected fibers on a symplectic 4-manifold M. The (topological equivalence class of a) Liouville foliation in a neighbourhood of critical fiber N whose only critical points are non-degenerate, rank 0, focus-focus is called a *stable focus-focus singularity*. The

[1] See Appendix 2.9 for a more explicit description of these subalgebras.

critical fiber N of such a singularity is a disjoint union of $c \geq 1$ critical points $x_1, \ldots, x_c$ and c open annuli $A_1, \ldots A_c$, $A_i \cong \mathbb{R} \times S^1$, such that each annulus A_i has $\{x_i, x_{i+1}\}$ as its boundary.

Theorem 2.11 ([87])**.** *If N is a stable focus-focus singularity of an integrable system which contains $c \geq 1$ critical points then there is a neighbourhood $\mathcal{U}(N) = \mathcal{H}^{-1}(D^2)$ such that $\mathcal{H}(N) = 0$ is the only critical value in D^2, and the topological monodromy of the torus fibration $\mathcal{U}(N) \setminus N \to D^2 \setminus \{0\}$ is generated by*

$$A = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$$

(i.e., up to a choice of basis for $\mathbb{Z}^n$, $m_b(\gamma) = A$ for a representative γ of a generator of $\pi_b(D^2 \setminus \{0\})$). Furthermore, every two such Liouville foliations are topologically equivalent.

Note that since m_b is a group homomorphism,

$$m_b(\gamma^{-1}) = A^{-1} = \begin{pmatrix} 1 & -c \\ 0 & 1 \end{pmatrix},$$

but A^{-1} and A are in the same $GL(n, \mathbb{Z})$ conjugacy class.

In particular, a model Liouville foliation $\mathcal{U}(N_c^f) \to D^2$ can be constructed for any $c \geq 1$ [87]. It has been shown that this prototypical model can be used to understand almost toric Liouville foliations in higher degrees of freedom up to topological equivalence, under an additional assumption:

Definition 2.12 ([86, Definition 6.3])**.** A non-degenerate singularity $\mathcal{H} \colon \mathcal{U}(N) \to D^n$ of an integrable system is called *topologically stable* if the local critical value set of the moment map restricted to $\mathcal{U}(N)$ coincides with the critical value set of the moment map restricted to a small neighbourhood of a singular point of minimal rank in N.

The following theorem was proven for more general singularities, but we only state it for almost toric fibers.

Theorem 2.13 ([86])**.** *Suppose N is a rank $n-k$, topologically stable, almost toric critical fiber of an integrable system $\mathcal{H}$ with compact and connected fibers. Then the Liouville foliation of $\mathcal{U}(N)$ is topologically equivalent to a finite quotient of the product foliation:*

$$\big((D^{n-k} \times T^{n-k}) \times \mathcal{U}(N^e_{h_e}) \times \mathcal{U}(N^f_{c_1}) \times \cdots \times \mathcal{U}(N^f_{c_{h_f}})\big)/G,$$

where G is a finite group which acts freely and component-wise on each factor in a foliation preserving manner[2]*. Moreover, G acts trivially on the elliptic component.*

This result was later strengthened by Izosimov,

Theorem 2.14 ([43])**.** *The Liouville foliation of $\mathcal{U}(N)$ in the preceding theorem is topologically equivalent to the product foliation of*

$$\left(D^{n-k} \times T^{n-k}\right) \times \mathcal{U}\left(N^e_{h_e}\right) \times \left(\mathcal{U}\left(N^f_{c_1}\right) \times \cdots \times \mathcal{U}\left(N^f_{c_{h_f}}\right)\right)/G.$$

Given an almost toric critical fiber, this theorem implies that the associated Lagrangian torus fibration is topologically equivalent to the associated Lagrangian torus fibration of a model product system. In particular, both fibrations will have the same topological monodromy, since it is a topological invariant of torus fibrations. The topological monodromy of a product of Lagrangian torus fibrations $\mathcal{H} \times \mathcal{H}' \colon M \times M' \to B \times B'$ is a the topological monodromy of the lattice $\Lambda_{\mathcal{H}\times\mathcal{H}'} \subset T^*B \times T^*B'$ which decomposes as the product covering space $\Lambda_{\mathcal{H}} \times \Lambda_{\mathcal{H}'} \to B \times B'$, and the topological monodromy of this covering space decomposes as

$$m_b(\gamma) = \begin{pmatrix} m_{b,\mathcal{H}}(\gamma) & 0 \\ 0 & m_{b,\mathcal{H}'}(\gamma) \end{pmatrix}.$$

Remark 2.15. In Section 2.7 we will use Theorem 2.14 to conclude that (for the Heisenberg spin system) the Liouville foliation above the critical line is homeomorphic to a product, and thus the topological monodromy around the critical line decomposes as a product, as in the preceding discussion. In fact, one can prove this more directly by using the description in [86] of the group G appearing in Theorem 2.13, and our explicit identification of the singular fibers with the product $S^1 \times N^f_3$ (Remark 2.27) to show that the group G will vanish, and hence the foliation is homeomorphic to a product.

2.3 Symplectic geometry of coadjoint orbits

Let G be a compact, connected Lie group with Lie algebra $\mathfrak{g}$ endowed with an Ad-invariant inner product $\langle\, , \rangle$. After G-equivariant identification of $\mathfrak{g}$ with $\mathfrak{g}^*$ via the inner product, the Kostant–Kirillov–

[2]Here $\mathcal{U}(N^e_{h_e})$ is the model Liouville foliation $D^{2h_e} \to \mathbb{R}^{h_e}$ given by $(x_1, y_1, \ldots, x_{h_e}, y_{h_e}) \mapsto (x_1^2 + y_1^2, \ldots, x_{h_e}^2 + y_{h_e}^2)$ as proven by [19].

Souriau symplectic structure on an adjoint orbit $\mathcal{O}_Z$ of $Z \in \mathfrak{g}$ is given by

$$\omega_Z([X,Z],[Y,Z]) = \langle Z, [X,Y] \rangle,$$

where $X, Y \in \mathfrak{g}$. Hamilton's equation for a function $H \colon \mathfrak{g} \to \mathbb{R}$ can be written as

$$\frac{dZ}{dt} = X_H(Z) = [\nabla H(Z), Z],$$

where ∇H is the gradient vector field defined by the equation $\langle \nabla H(Z), Y \rangle = dH_Z(Y)$ for all $Y \in T_Z\mathcal{O}$. Hence, the Poisson bracket of two functions H, F can be conveniently written as

$$\{H, F\}_Z = \omega_Z(X_H, X_F) = \omega_Z([\nabla H, Z], [\nabla F, Z]) = \langle Z, [\nabla H, \nabla F] \rangle.$$

A direct sum of semisimple Lie algebras $\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_N$ is endowed with direct sum Lie brackets and Killing forms. An adjoint orbit in $\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_N$ is a product of adjoint orbits $\mathcal{O}_{Z_1} \times \cdots \times \mathcal{O}_{Z_N}$ and the symplectic structure coincides with the direct sum of their respective symplectic structures, $\omega = \omega_1 \oplus \cdots \oplus \omega_N$.

The moment map for the adjoint action of G on an orbit in $\mathfrak{g}$ is inclusion of the orbit into $\mathfrak{g}^*$ via the Ad-equivariant identification. The moment map for the diagonal adjoint action of G on $\mathcal{O}_{Z_1} \times \cdots \times \mathcal{O}_{Z_N}$ is hence the map $(X_1, \ldots, X_N) \mapsto \sum X_i$.

Example 2.16. Let $G = \mathrm{SO}(3)$ be the group of rotations of $\mathbb{R}^3$ equipped with the standard basis and inner product. Its Lie algebra is

$$\mathfrak{so}(3) = \left\{ \begin{pmatrix} 0 & x_1 & x_2 \\ -x_1 & 0 & x_3 \\ -x_2 & -x_3 & 0 \end{pmatrix} : x_1, x_2, x_3 \in \mathbb{R} \right\},$$

which has the Ad-invariant inner product $\langle X, Y \rangle = -\frac{1}{2}\operatorname{tr}(XY)$ and Lie bracket $[X,Y] = XY - YX$. The map $\Psi \colon \mathfrak{so}(3) \to (\mathbb{R}^3, \times)$ given by

$$\begin{pmatrix} 0 & x_1 & x_2 \\ -x_1 & 0 & x_3 \\ -x_2 & -x_3 & 0 \end{pmatrix} \mapsto (x_1, x_2, x_3) \in \mathbb{R}^3$$

is an isomorphism of Lie algebras, where $(\mathbb{R}^3, \times)$ is the cross-product Lie algebra. Under this identification, the adjoint action of $\mathrm{SO}(3)$ on $\mathfrak{so}(3)$ is identified with the standard action on $\mathbb{R}^3$, and the

Ad-invariant inner product is simply the standard Euclidean inner product:

$$\langle X, Y\rangle = x_1y_1 + x_2y_2 + x_3y_3 = -\frac{1}{2}\operatorname{tr}(XY).$$

Thus, the adjoint orbits are identified with concentric spheres in $\mathbb{R}^3$ and the symplectic structure on an adjoint orbit through the point $Z = (z_1, z_2, z_3)$ is

$$\omega_Z([X,Z],[Y,Z]) = \langle Z, [X,Y]\rangle = \det\begin{pmatrix} z_1 & z_2 & z_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix},$$

which is precisely the standard symplectic structure on a sphere with radius $|Z|$. In what follows, we will consider the adjoint orbit in $\mathfrak{so}(3)\oplus\mathfrak{so}(3)\oplus\mathfrak{so}(3)$ that is a product of three spheres with radius 1, $(S^2\times S^2\times S^2, \omega_{\mathrm{STD}}\oplus\omega_{\mathrm{STD}}\oplus\omega_{\mathrm{STD}})$.

2.4 An integrable Heisenberg spin chain

As in Example 2.16, consider a product of three spheres of radius 1 whose elements are triples $(X, Y, Z) \in M = S^2\times S^2\times S^2$. Define the Hamiltonians

$$H(X,Y,Z) = |X+Y+Z| = \sqrt{3 + \langle X,Y\rangle + \langle Y,Z\rangle + \langle Z,X\rangle},$$
$$I(X,Y,Z) = \langle X+Y+Z, e_3\rangle, \qquad \text{and} \qquad J(X,Y,Z) = \langle X, [Y,Z]\rangle = \det(X,Y,Z).$$

By ad-invariance of the inner product, the Hamiltonian vector fields of these functions are

$$X_H = [\nabla H, (X,Y,Z)] = \frac{1}{H(X,Y,Z)}([Y+Z,X],[X+Z,Y],[X+Y,Z]),$$
$$X_I = [\nabla I, (X,Y,Z)] = ([e_3,X],[e_3,Y],[e_3,Z]), \qquad \text{and}$$
$$X_J = [\nabla J, (X,Y,Z)] = ([[Y,Z],X],[[Z,X],Y],[[X,Y],Z]).$$

The flow $\varphi^t_{[v,X]}$ of a vector field $[v,X]$ acts by rotation of the vector X around the axis v with period $2\pi/|v|$, which we denote as R^v_t,

$$\varphi^t_{[v,X]}X = R^v_t X.$$

Thus, the Hamiltonian flow of I acts by rotating each sphere around the e_3-axis with period 2π,

$$\varphi^t_{X_I}(X,Y,Z) = \big(R^{e_3}_t X, R^{e_3}_t Y, R^{e_3}_t Z\big).$$

Where defined, the Hamiltonian flow of H rotates each sphere around the axis $X+Y+Z$ with period 2π,

$$\varphi^t_{X_H}(X,Y,Z) = \big(R^{v}_t X, R^{v}_t Y, R^{v}_t Z\big),$$

where $v = (X+Y+Z)/|X+Y+Z|$. This is perhaps best visualized as rotating the polygon with edges X, Y, Z, $-X-Y-Z$ around the edge $-X-Y-Z$.

Proposition 2.17. $\{H,J\} = \{H,I\} = \{J,I\} = 0$.

Proof. It is a nice exercise to see that this is true based on the geometric description of the Hamiltonians and their flows given above. More algebraically, one can see this using the Lie algebra structure that is present. For example,

$$\begin{aligned}\{J,I\}_{(X,Y,Z)} &= \langle (X,Y,Z), [([Y,Z],[Z,X],[X,Y]), (e_3,e_3,e_3)]\rangle \\ &= \langle [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]], e_3\rangle = 0\end{aligned}$$

by the Jacobi identity and ad-invariance of our inner product. A similar calculation shows that $\{H,I\} = \{H,J\} = 0$. $\square$

Remark 2.18. The Hamiltonian flow of J is less straightforward to describe, but we can say something about how it acts on the submanifold

$$L = \{(X,Y,Z) \in M \colon X+Y+Z = 0\} = H^{-1}(0).$$

There the vectors X, Y, and Z are coplanar and the vector field

$$X_J(X,Y,Z) = ([[X,Y],X],[[Y,Z],Y],[[Z,X],Z]).$$

The flow of this vector field acts on L by rotation of each vector X, Y and Z around the axis $\overline{n} = [X,Y] = [Y,Z] = [Z,X]$ with constant period: since each component of X_J is tangent to the plane spanned by X, Y, and Z, and the flow of X_J preserves L, the vector $[X,Y]$ is preserved by X_J, so

$$X_J = ([\overline{n}, X], [\overline{n}, Y], [\overline{n}, Z]).$$

In [68] it was shown that the fiber L is a non-displaceable Lagrangian submanifold of $(S^2 \times S^2 \times S^2, \omega_{\mathrm{STD}} \oplus \omega_{\mathrm{STD}} \oplus \omega_{\mathrm{STD}})$. To see that it is an embedded Lagrangian $\mathbb{R}P^3$, observe that it is the zero level set for the moment map of the diagonal SO(3)-action, $(X,Y,Z) \mapsto X+Y+Z$, and the diagonal action of SO(3) is free and transitive.

Remark 2.19. The Hamiltonian H has several important symmetries. First, note that H is invariant under the diagonal Hamiltonian SO(3)-action, whose moment map is

$$S^2 \times S^2 \times S^2 \xrightarrow{(X,Y,Z)\mapsto X+Y+Z} \mathfrak{so}(3) \cong \mathfrak{so}(3)^*.$$

Thus H is non-commutatively integrable, or super-integrable. Indeed, there are three independent choices for our second integral

$$I_v(X,Y,Z) = \langle X+Y+Z, v\rangle,$$

which do not pairwise Poisson commute.

In addition, all three Hamiltonians, H, I, J have a symplectic $\mathbb{Z}_3$-symmetry by cyclic permutations $(X,Y,Z) \mapsto (Z,X,Y)$. As we will see, this symmetry of the system by a group of order 3 is also a symmetry of the rank 1 focus-focus critical fibers, and thus the system's topological monodromy contains the number 3 (see Theorem 2.11).

2.5 Image of the moment map

In this section we give a complete description the image, critical set, and critical values for the moment map $\mathcal{H} = (H, I, J)$.

The image of the moment map has two reflective symmetries coming from the fact that $J(X,Y,Z) = -J(X,Z,Y)$ and $I(-X,-Y,-Z) = -I(X,Y,Z)$. It is obvious that $|I| \le H$, with equality when $X+Y+Z \in \mathrm{span}(e_3)$, and that $0 \le H \le 3$ with equality when $X=Y=Z$.

Observe that

$$H = \sqrt{3 + 2(a+b+c)} \qquad \text{and} \qquad |J| = \sqrt{1 + 2abc - \left(a^2+b^2+c^2\right)},$$

where $a = \langle X, Y\rangle$, $b = \langle Y, Z\rangle$ and $c = \langle Z, X\rangle$ (the second formula is the volume of a parallelpiped). If we maximize $|J|$ with the constraint $H = \text{const}$, then we must have $a = b = c$ (the interior angles between the three vectors are the same). Using this we can deduce that

$$|J| \leq \sqrt{1 + 2\left(\frac{H^2-3}{6}\right)^3 - 3\left(\frac{H^2-3}{6}\right)^2},$$

and equality is achieved when $a = b = c$.

Proposition 2.20. *The image of the moment map $\mathcal{H}$ is the set of points $(r, s, t) \in \mathbb{R}^3$ that satisfy the equations*

$$|t| \leq \sqrt{1 + 2\left(\frac{r^2-3}{6}\right)^3 - 3\left(\frac{r^2-3}{6}\right)^2}, \qquad |s| \leq r, \qquad \textit{and} \qquad 0 \leq r \leq 3.$$

Proof. Observe that the level sets of H are all connected (see the proof of Theorem 2.24 in Section 2.6). For a given tuple $(r, s, t) \in \mathbb{R}^3$ that satisfies the inequalities of (2.3), consider the restriction of the map J to the connected set $H^{-1}(r)$. Since $0 \leq r \leq 3$, we can construct a tuple $(X, Y, Z) \in H^1(r)$ such that

$$\langle X, Y\rangle = \langle Y, Z\rangle = \langle Z, X\rangle = \frac{r^2-3}{6}.$$

$J(X, Y, Z)$ is maximal or minimal depending on the orientation of the basis $\{X, Y, Z\}$ (if $r = 0$ or 3 then $\{X, Y, Z\}$ does not form a basis and $J(X, Y, Z) = 0$ is both maximal and minimal). It follows by the Intermediate Value Theorem that there is a tuple $(X, Y, Z) \in S^2 \times S^2 \times S^2$ such that $H(X, Y, Z) = r$ and $J(X, Y, Z) = t$. Using the Intermediate Value Theorem again, it is easy to see that there exists a $\theta \in [0, 2\pi]$ such that $I(R_\theta^{e_3} X, R_\theta^{e_3} Y, R_\theta^{e_3} Z) = s$. Since H and J are invariant under diagonal rotations, we are done. $\square$

Next, we turn our attention to the critical set for the system. The critical set consists of several subsets:

1. The sets where H is critical:

   (a) the embedded $\mathrm{SO}(3) \cong H^{-1}(0)$, which lies over the vertex $(0, 0, 0)$,

   (b) the three embedded spheres $S_1 = \{(-X, X, X)\}$, $S_2 = \{(X, -X, X)\}$, and $S_3 = \{(X, X, -X)\}$, which lie over the critical line ($H = 1$ and $J = 0$), and

   (c) the diagonally embedded sphere $S_4 = \{(X, X, X)\}$, which lies over the edge $H = 3$.

2. The set $C_1 = \{(X, Y, Z) \colon X + Y + Z \in \mathrm{Span}(e_3)\}$ where dI is proportional to dH. This contains the critical set of I and is mapped to two opposite faces of the moment map image.

3. The set $C_2 = \{(X, Y, Z) \colon \langle X, Y\rangle = \langle Y, Z\rangle = \langle Z, X\rangle\}$ where dJ is proportional to dH. Note that $S_4, H^{-1}(0) \subset C_2$. This set maps to the other two opposite faces of the moment map image. Points in $C_1 \cap C_2$ map to edges of the moment map image.

Proposition 2.21. *The critical set for $\mathcal{H}$ is*

$$C = C_1 \cup C_2 \cup S_1 \cup S_2 \cup S_3.$$

The set $\mathcal{H}(C_1 \cup C_2)$ is the boundary of the image $\mathcal{H}(M)$ and the set $\mathcal{H}(S_1) = \mathcal{H}(S_2) = \mathcal{H}(S_3)$ is the line segment $\mathcal{H}(M) \cap \{H = 1, J = 0\}$ (see Fig. 2.1*).*

In the terminology of integrable systems, the set of critical values is the system's 'bifurcation diagram'. This description of the bifurcation diagram will be of use to us in Section 2.7. Note that the three critical spheres S_1, S_2, S_3 are permuted by the system's $\mathbb{Z}_3$-symmetry (cf. Remark 2.19).

Proof. Throughout we use the fact that $df = 0$ if and only if $X_f = 0$ (when f is smooth).

1. Since 0 is the global minimum for H, the set $H^{-1}(0)$ is critical. To find the other critical sets of H, observe that $X_H = 0$ if and only if $[X+Y+Z, X] = 0$, $[X+Y+Z, Y] = 0$, and $[X+Y+Z, Z] = 0$. If $X+Y+Z \neq 0$, this occurs if and only if $X+Y+Z$ is is contained in the lines spanned by X, Y, and Z, so this happens if and only if X, Y, and Z are collinear. This entails cases (1b) and (1c).

2. Observe that $X_I = \alpha X_H$ for some $\alpha \neq 0$ if and only if

$$[e_3, X] = \alpha[X + Y + Z, X], \qquad [e_3, Y] = \alpha[X + Y + Z, Y], \qquad \text{and}$$
$$[e_3, Z] = \alpha[X + Y + Z, Z],$$

which is true if and only if $X + Y + Z \in \mathrm{span}(e_3)$.

3. Observe that $X_J = \alpha X_H$ if and only if

$$[[Y, Z], X] = \alpha[Y + Z, X], \qquad [[Z, X], Y] = \alpha[X + Z, Y], \qquad \text{and}$$
$$[[X, Y], Z] = \alpha[X + Y, Z]$$

for some α. If $\alpha = 0$ then the 3-tuple X, Y, Z forms an oriented or anti-oriented orthonormal frame, or X, Y and Z are collinear. If $\alpha \neq 0$ then

(a) $[Y, Z]$, $Y + Z$, and X are coplanar,

(b) $[Z, X]$, $X + Z$, and Y are coplanar, and

(c) $[X, Y]$, $X + Y$, and Z are coplanar.

Since X, Y and Z all have the same length, the three items listed above are true if and only if X, Y, Z are collinear, or $\langle X, Y\rangle = \langle Y, Z\rangle = \langle Z, X\rangle$ (this fact is a straightforward exercise in Euclidean geometry).

Finally, suppose that $\alpha X_I + \beta X_H + \gamma X_J = 0$ for $\alpha, \beta, \gamma \in \mathbb{R}$ not all zero. Then

$$\alpha[e_3, X] + \beta[Y + Z, X] + \gamma[[Y, Z], X] = 0,$$
$$\alpha[e_3, Y] + \beta[X + Z, Y] + \gamma[[Z, X], Y] = 0, \qquad \text{and}$$
$$\alpha[e_3, Z] + \beta[X + Y, Z] + \gamma[[X, Y], Z] = 0.$$

Adding these equations together we obtain

$$\alpha[e_3, X + Y + Z] + \gamma\left([[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y]\right) = 0,$$

which by the Jacobi identity reduces to $\alpha[e_3, X+Y+Z] = 0$. If $[e_3, X+Y+Z] = 0$, then $(X, Y, Z) \in C_1$. If $\alpha = 0$, then (X, Y, Z) is in the set $S_1 \cup S_2 \cup S_3 \cup C_2$. $\square$

Combining Propositions 2.20 and 2.21, we have Theorem 2.2. Since we now know that our Hamiltonians are independent on an open dense subset of M, we can conclude:

Corollary 2.22. *$\mathcal{H}$ is a completely integrable system. In particular, the regular level sets of $\mathcal{H}$ are homeomorphic to a disjoint union of finitely many* 3*-tori.*

Corollary 2.23. *The set of regular values is homotopy-equivalent to S^1.*

In the next section, we will see that the regular level sets are connected and in Section 2.7 we will describe the structure of the associated Lagrangian torus fibration.

2.6 Connectedness of regular level sets

In this section we prove

Theorem 2.24. *The regular fibers of the map $\mathcal{H} = (H, I, J)$ are all connected.*

Proof. The proof will have two parts. For $H \neq 1$ we can make a general argument and for $H = 1$ we will apply Ehresmann's theorem.

Pick a regular value (r, s, t) with $r \neq 1$. Since H generates a Hamiltonian S^1-action on $M \setminus H^{-1}(0)$, it is a Morse–Bott function such that all critical sets have even index [4]. Thus, the level sets of H are connected. Since the regular level sets of H are compact and connected, the symplectic reductions $M_r \equiv H^{-1}(r)/S^1$ are all compact, connected symplectic manifolds. Since s is a regular value of the reduced Hamiltonian $\tilde{I}$, and $\tilde{I}$ generates a free S^1-action, we can reduce once more to obtain the compact and connected manifold $M_{r,s} = \tilde{I}^{-1}(s)/S^1$.

The image of the twice-reduced Hamiltonian $\tilde{J}$ on $M_{r,s}$ is a line segment with the only critical values being the maximum and minimum. Recall from Proposition 2.21 that in the unreduced manifold, dJ is dependent on dH and dI at a point $(X, Y, Z) \in H^{-1}(r) \cap I^{-1}(s)$ if and only if dJ is proportional to dH, and this occurs if and only if $\langle X, Y \rangle = \langle Y, Z \rangle = \langle Z, X \rangle$. It is easy to see geometrically that the set of all 3-tuples of unit length vectors which satisfy the conditions

1) $\langle X, Y \rangle = \langle Y, Z \rangle = \langle Z, X \rangle$,

2) $|X + Y + Z| = r$, and

3) $\langle X + Y + Z, e_3 \rangle = s$.

is two orbits of the Hamiltonian T^2-action generated by (H, I), which are distinguished by whether the basis $\{X, Y, Z\}$ is positively or negatively oriented, corresponding to being the maximum or minimum set for J on $H^{-1}(r) \cap I^{-1}(s)$. Thus there are two critical points of $\tilde{J}$ on $M_{r,s}$ and the regular fibers $\tilde{J}^{-1}(s)$ are all connected. This implies that the regular fiber $\mathcal{H}^{-1}(r, s, t) \subset M$ is connected.

Now consider a regular value $(1, s, t)$. Since the map $\mathcal{H}$ is proper, Ehresmann's theorem implies that the fiber $\mathcal{H}^{-1}(1, s, t)$ is diffeomorphic to a nearby fiber $\mathcal{H}^{-1}(r, s, t)$ with $r \neq 1$, which we have just shown is connected. $\square$

Remark 2.25. The image of the invariant Lagrangian $L = H^{-1}(0)$ in the reduction at 0 by I is a Lagrangian S^2.

Remark 2.26. In [72] it is shown that if $(M^4, \omega, (H_1, H_2))$ is a non-degenerate completely integrable system (cf. Definition 2.8) with two degrees of freedom, whose set of critical values has no vertical tangent lines, then the system has connected fibers. After rotating the moment map image, and checking that the boundary of $\mathcal{H}(M)$ consists generically (almost all values of H or I) of non-degenerate elliptic critical values one could deduce connectedness of almost all the fibers of $\mathcal{H}$ or I by applying this theorem to the reduced systems, then applying Ehresmann to the remaining fibers. For example, the systems obtained by reducing at $H = 0$ or ± 1 will have degenerate critical points (the Lagrangian S^2 of the previous remark and image of intersections $S_i \cap C_1$, respectively), so the theorem of [72] cannot be applied directly.

2.7 Topological monodromy around the critical line

As we have seen, the regular fibers of the moment map $\mathcal{H} = (H, I, J)$ for the Heisenberg spin chain are connected and, because of the critical line, the set of regular values B_{reg} is homotopy equivalent to S^1. Thus, the associated Lagrangian torus fibration $M_{\mathrm{reg}} \to B_{\mathrm{reg}}$ of the Heisenberg spin chain may have non-trivial topological monodromy. This section uses our understanding of the critical set and critical values of $\mathcal{H}$ from Section 2.5, together with a non-degeneracy computation that is relegated to Appendix 2.9, to deduce the topological monodromy directly from Theorem 2.14. Recall

Theorem 1.1. *The topological monodromy of the Lagrangian torus fibration associated to the integrable system (H, I, J) is generated by the matrix*

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus the system does not admit global action coordinates.

Proof. Consider critical the fibers $N(s) = \mathcal{H}^{-1}(1, s, 0)$ for $-1 < s < 1$. The condition $J = 0$ implies that the unit-length vectors X, Y, Z are coplanar and the condition $H = 1$ implies that they form three sides of the parallelogram X, Y, Z, $-X - Y - Z$. The set of all such vectors is compact and connected (see the remark following the proof). By Proposition 2.21, the set of critical points in the fiber $N(s)$ contains three connected components:

$$N(s) \cap S_1 = \left\{(-X, X, X) \in S^2 \times S^2 \times S^2 \colon \langle X, e_3 \rangle = s\right\},$$
$$N(s) \cap S_2 = \left\{(X, -X, X) \in S^2 \times S^2 \times S^2 \colon \langle X, e_3 \rangle = s\right\},$$
$$N(s) \cap S_3 = \left\{(X, X, -X) \in S^2 \times S^2 \times S^2 \colon \langle X, e_3 \rangle = s\right\},$$

each homeomorphic to S^1, and they are permuted by the system's $\mathbb{Z}_3$ symmetry (cf. Remark 2.19). Each critical fiber is topologically stable (Definition 2.12) since the system's $\mathbb{Z}_3 \times S^1$-symmetry acts transitively on the critical set in each fiber $N(s)$, and preserves the map $\mathcal{H}$.

By Proposition 2.32 the critical points in $N(s)$ are rank 1 non-degenerate, with Williamson type $(0,0,1)$. Thus by Theorem 2.14, the Liouville foliation of $\mathcal{U}(N(s))$ is topologically equivalent to the product foliation of $(D^1 \times S^1) \times \mathcal{U}(N_3^f)$. As noted in Section 2.2, this implies that the topological monodromy of our system is the same as the product foliation of $(D^1 \times S^1) \times \mathcal{U}(N_3^f)$, which decomposes into blocks for each component of the direct sum,

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix}.$$

By Proposition 2.11, the bottom right minor is

$$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}.$$

□

Since the Hamiltonian flows of H and I are periodic, the 1-forms dH and dI are global sections of the period lattice $\Lambda \to B_{\text{reg}}$. Thus if $\{dH(b), dI(b), \alpha_b\}$ is a $\mathbb{Z}$-basis for Λ_b, and γ is a choice of generator for $\pi_1(B_{\text{reg}}, b)$, then we must have

$$m_b(\gamma) = \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & \pm 1 \end{pmatrix}.$$

Theorem 1.1 tells us that

$$m_b(\gamma) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

up to a choice of basis fixing $dH(b)$ and $dI(b)$.

Remark 2.27. It is easy to see how the fiber $N(s)$ is homeomorphic to $S^1 \times N_3^f$, where N_3^f is the focus-focus fiber with three critical points as described in Section 2.2. First note that the set

$$S = \{(X,Y,Z) \in S^2 \times S^2 \times S^2 :\ X + Y + Z = \beta e_1 + s e_3, \beta \in \mathbb{R}^+\}$$

is a slice for the Hamiltonian S^1-action generated by I in a neighbourhood of $N(s)$. Since this S^1-action is free in a neighbourhood of $N(s)$, the fiber $N(s)$ is homeomorphic to $S^1 \times (N(s) \cap S)$. The set $N(s) \cap S$ consists of: the three annuli,

$$\{(X, \beta e_1 + se_3, -X) \in S |\, X \in S^2 \setminus \{\pm(\beta e_1 + se_3)\}\},$$
$$\{(\beta e_1 + se_3, X, -X) \in S |\, X \in S^2 \setminus \{\pm(\beta e_1 + se_3)\}\},$$
$$\{(X, -X, \beta e_1 + se_3) \in S |\, X \in S^2 \setminus \{\pm(\beta e_1 + se_3)\}\}$$

together with the three points,

$$(-\beta e_1 - se_3, \beta e_1 + se_3, \beta e_1 + se_3), \qquad (\beta e_1 + se_3, -\beta e_1 - se_3, \beta e_1 + se_3),$$
$$(\beta e_1 + se_3, \beta e_1 + se_3, -\beta e_1 - se_3).$$

Remark 2.28. The fact that this system has non-trivial monodromy should be unsurprising for the following reason: the topology of the H-level sets changes as you pass through the critical value 1. This can be seen directly with Morse theory, but there is also a natural interpretation in terms of the topology of polygon spaces (as introduced by [44]). There is an obvious diffeomorphism of the level set $H^{-1}(r)$ with the manifold $M(1, 1, 1, r)$ of closed 4-gons in $\mathbb{R}^3$ with side lengths 1, 1, 1, and r. When $r \neq 1$, it has been observed by Knutson, Hausmann [36], and Kapovitch and Millson [44] that the quotient $M(1, 1, 1, r)/\mathrm{SO}(3)$ is homeomorphic to S^2, and that the quotient map $\pi\colon M(1, 1, 1, r) \to S^2$ is a principal SO(3)-bundle. Further, the characteristic classes of this principal SO(3)-bundle were described by Knutson and Hausmann in their paper [35]. Their result says that for $0 < r < 1$ the bundle is trivial, whereas for $1 < r < 3$ the bundle is non-trivial. It is a fun exercise to check that $\mathrm{SO}(3) \times S^2$ and the total space of the non-trivial principal SO(3)-bundle over S^2 are not homeomorphic[3].

It was observed in [15] that such a change in the topology of the level set $H^{-1}(r)$ as r passes through an interior critical value indicates that there *must* be non-trivial monodromy around the associated critical fibres, since this forces the pullback of the torus bundle to any circle around the critical line to be non-trivial.

Remark 2.29. Since B_{reg} is homotopy equivalent to S^1, the Chern class of the torus fibration $M_{\mathrm{reg}} \to B_{\mathrm{reg}}$ is trivial, so there exists a global section $\sigma\colon B_{\mathrm{reg}} \to M_{\mathrm{reg}}$. Since the Lagrangian Chern class [18] also vanishes, σ can be chosen to be Lagrangian, so the map

$$\Psi\colon\ T^*(B_{\mathrm{reg}})/\Lambda \times_{B_{\mathrm{reg}}} M_{\mathrm{reg}} \to M_{\mathrm{reg}}$$

[3]Hint: compute their fundamental groups.

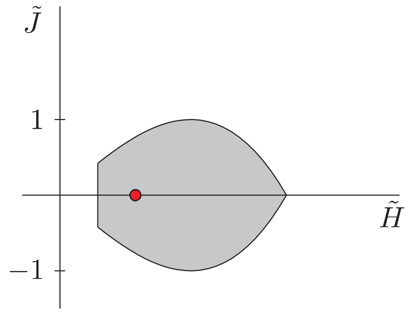


Figure 2.3: Reduced system on M_s for $-1 < s < 1$, $t \neq 0$.

gives a symplectomorphism $\alpha \mapsto \Psi(\alpha, \sigma(\pi(\alpha)))$ which is an isomorphism of the Lagrangian torus fibrations $(T^*(B_{\mathrm{reg}})/\Lambda, d\lambda) \to B_{\mathrm{reg}}$ and $(M_{\mathrm{reg}}, \omega) \to B_{\mathrm{reg}}$ (see [18, 63] for details).

Remark 2.30. For $-1 < s < 1$, the reduced system on $M_s = I^{-1}(s)/S^1$ has a stable focus-focus critical fiber with three critical points. The manifold M_s is the blow-up $Bl_3\mathbb{C}P^2$, and the reduced moment map image has 3 vertices (see Fig. 2.3). A quick comparison with the list of almost toric systems in [56] shows that this checks out.

2.8 Further directions

In quantum mechanics, the spin-S isotropic Heisenberg spin chain on the lattice $L = \mathbb{Z}$ is an operator $\mathcal{K}$ on the tensor product $V = \bigotimes_L \mathbb{C}^{2S+1}$ of irreducible $\mathfrak{su}(2)$ representations, defined as

$$\mathcal{K} = \sum_{i \in L} \langle X_i, X_{i+1} \rangle = \sum_{i \in L} \left(X_i^1 \circ X_{i+1}^1 + X_i^2 \circ X_{i+1}^2 + X_i^3 \circ X_{i+1}^3 \right),$$

where X_i^j are the Pauli matrices

$$X_i^1 = \frac{1}{S} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad X_i^2 = \frac{1}{S} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad X_i^3 = \frac{1}{S} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

acting on V by the irreducible representation of $\mathfrak{su}(2)$ in the ith factor. This operator models 'nearest-neighbour' spin coupling on the lattice. One may consider the infinite chain $L = \mathbb{Z}$ or a chain with boundary conditions, $L = \mathbb{Z}_N$. For a chain with boundary conditions, the large-S limit of this system is the Hamiltonian

$$K = \sum_{i \in L} \langle X_i, X_{i+1} \rangle$$

on the symplectic manifold $M_N = (S^2 \times \cdots \times S^2, \omega_{\text{STD}} \oplus \cdots \oplus \omega_{\text{STD}})$ with elements $(X_1, \ldots, X_N)$. In the case $N = 3$, this is related to the Hamiltonian H studied in this paper by the identity

$$3 - H^2 = K$$

(of course, the analogous identity fails for $N > 3$). In [24], the algebraic structure of the spin-$\frac{1}{2}$ representation was used to find a recursive formula for the operators commuting with $\mathcal{K}$ for any lattice L and $S = \frac{1}{2}$. Inspired by this, one might ask if there is a similar combinatorial pattern generating integrals of the classical Hamiltonians K or H on M_N, where we might define H_N for $N > 3$ as the Hamiltonian describing 'complete graph' coupling

$$H_N = \sum_{i,j \in L,\, i \neq j} \langle X_i, X_j \rangle.$$

Question 2.31. *How is the structure of the Lie algebra $\mathfrak{so}(3) \cong \mathfrak{su}(2)$ reflected in the conservation laws of the Hamiltonians H_N on M_N.*

For example, for arbitrary N one can check using the Jacobi identity and ad-invariance of $\langle\, ,\, \rangle$, as in the proof of Proposition 2.17 that H_N has integrals

$$I_v = \sum_{i \in L} \langle X_i, v \rangle, \quad v \in \mathbb{R}^3, \qquad J = \sum_{i \in L} \langle X_i, [X_{i+1}, X_{i+2}] \rangle$$

with $\{H_N, J\} = \{H_N, I_v\} = \{J, I_v\} = 0$, and one could ask if further independent integrals exist. As in Remark 2.28, the level sets $H_N^{-1}(h)$ are diffeomorphic to the space $M(1, \ldots, 1, h)$ of N-gons in $\mathbb{R}^3$ with corresponding edge lengths, and come with a quotient map to the associated polygon space $M(1, \ldots, 1, h)/\mathrm{SO}(3)$. As h passes through critical values (which are even or odd integers depending on the parity of N) the topology of this bundle will change, as described in [35], so if H_N is completely integrable we anticipate non-trivial topological monodromy, provided these critical values lie in the interior of the moment map image.

2.9 Non-degeneracy computation

In this appendix we prove

Proposition 2.32. *The critical points of $\mathcal{H} = (H, I, J)$ which lie above the critical line $\{(1, s, 0)\colon -1 < s < 1\}$ are non-degenerate and have Williamson type $(0, 0, 1)$ (see Definitions 2.8 and 2.10).*

As we have seen, this set of critical points is precisely

$$\{(X,Y,Z) \in S_1 \cup S_2 \cup S_3 \colon\ I(X,Y,Z) \neq \pm 1\}\,.$$

On this set $dJ = dH = 0$ and $dI \neq 0$, so these points are all rank 1. To show the non-degeneracy of these critical points, we need to show that for each such p, the operators $A_J(p) = \omega^{-1}d^2J(p)$, and $A_H(p) = \omega^{-1}d^2H(p)$ span a Cartan subalgebra (see Section 2.2). Equivalently, we must find a linear combination of the operators $A_J(p)$ and $A_H(p)$ which has 4 distinct eigenvalues. Note that it is sufficient to check non-degeneracy of a single critical point in each fiber $\mathcal{H}^{-1}(1,s,0)$ because each connected component of the critical set is an orbit of the Hamiltonian flow of X_I, and the symplectic $\mathbb{Z}_3$-action generated by the permutation $\sigma(X,Y,Z) = (Z,X,Y)$ preserves $\mathcal{H}$ and permutes these connected components transitively.

According to the classification of [81], the Williamson type of p will then be determined by the form of these eigenvalues: in $\mathfrak{sp}(\mathbb{R},4)$ there are four conjugacy classes of Cartan subalgebras corresponding to four possible combinations of eigenvalues for a generic element:

1) elliptic-elliptic: $\pm iA$, $\pm iB$,

2) elliptic-hyperbolic: $\pm A$, $\pm iB$,

3) hyperbolic-hyperbolic: $\pm A$, $\pm B$, and

4) focus-focus: $A \pm iB$, $-A \pm iB$.

In Darboux coordinates the operator $A_H(p)$ is equal to the linearization of the Hamiltonian vector field X_H at p, since

$$\frac{\partial X^i_H}{\partial x^j} = \frac{\partial}{\partial x^j}\left(\omega^{ik}\frac{\partial f}{\partial x^k}\right) = \omega^{ik}\frac{\partial^2 f}{\partial x^j \partial x^k} = \left(\omega^{-1}d^2H\right)^i_j.$$

Consider the cylindrical coordinates $(\theta, z) \in (-\pi/2, 3\pi/2) \times (-1,1)$ with symplectic form $d\theta_i \wedge dz_i$. The map $\phi\colon (-\pi/2, 3\pi/2) \times (-1,1) \to S^2$ given by

$$\phi(\theta,z) = \left(\left(1-z^2\right)^{1/2}\cos(\theta), \left(1-z^2\right)^{1/2}\sin(\theta), z\right)$$

is a symplectomorphism. In cylindrical coordinates $(\theta_1, z_1, \theta_2, z_2, \theta_3, z_3)$, the Hamiltonians are

$$\hat{H} = \left(\sum_j \left(1 - z_j^2\right)^{1/2} \cos(\theta_j) \right)^2 + \left(\sum_j \left(1 - z_j^2\right)^{1/2} \sin(\theta_j) \right)^2 + \left(\sum_j z_j \right)^2 ,$$

$$\hat{J} = \sum_{j=1,2,3} z_j \left(1 - z_{j+1}^2\right)^{1/2} \left(1 - z_{j-1}^2\right)^{1/2} \sin(\theta_{j-1} - \theta_{j+1}), \qquad \hat{I} = z_1 + z_2 + z_3$$

(where we have pulled back $(H^2 - 3)/2$ instead of H for computational convenience). Hamilton's equations tell us that

$$X_f = \frac{\partial f}{\partial z_i} \frac{\partial}{\partial \theta_i} - \frac{\partial f}{\partial \theta_i} \frac{\partial}{\partial z_i}.$$

The linearization of X_f at a fixed point of the flow of f is then

$$A_f = \begin{pmatrix} \left(\dfrac{-\partial^2 f}{\partial z_k \partial \theta_i} \right)_{ik} & \left(\dfrac{-\partial^2 f}{\partial \theta_k \partial \theta_i} \right)_{ik} \\ \left(\dfrac{\partial^2 f}{\partial z_k \partial z_i} \right)_{ik} & \left(\dfrac{\partial^2 f}{\partial \theta_k \partial z_i} \right)_{ik} \end{pmatrix}.$$

In order to check non-degeneracy, one therefore computes the partial derivatives:

$$\frac{\partial^2 \hat{H}}{\partial z_k \partial z_i} = \begin{cases} -2\big(1-z_i^2\big)^{-3/2}\left(\displaystyle\sum_{j\neq i}\big(1-z_j^2\big)^{1/2}\cos(\theta_i-\theta_j)\right), & k=i,\\ 2z_iz_k\big(1-z_i^2\big)^{-1/2}\big(1-z_k^2\big)^{-1/2}\cos(\theta_i-\theta_k)+2, & k\neq i,\end{cases}$$

$$\frac{\partial^2 \hat{H}}{\partial \theta_k \partial z_i} = \begin{cases} 2z_i\big(1-z_i^2\big)^{-1/2}\left(\displaystyle\sum_{j}\big(1-z_j^2\big)^{1/2}\sin(\theta_i-\theta_j)\right), & k=i,\\ -2z_i\big(1-z_i^2\big)^{-1/2}\big(1-z_k^2\big)^{1/2}\sin(\theta_i-\theta_k), & k\neq i,\end{cases}$$

$$\frac{\partial^2 \hat{H}}{\partial \theta_k \partial \theta_i} = \begin{cases} -2\big(1-z_i^2\big)^{1/2}\left(\displaystyle\sum_{j\neq i}\big(1-z_j^2\big)^{1/2}\cos(\theta_i-\theta_j)\right), & k=i,\\ 2\big(1-z_i^2\big)^{1/2}\big(1-z_k^2\big)^{1/2}\cos(\theta_i-\theta_k), & k\neq i,\end{cases}$$

$$\frac{\partial^2 \hat{J}}{\partial z_k \partial z_i} = \begin{cases} -\big(1-z_i^2\big)^{-3/2}\Big(z_{i-1}\big(1-z_{i+1}^2\big)^{1/2}\sin(\theta_{i+1}-\theta_i) \\ \quad + z_{i+1}\big(1-z_{i-1}^2\big)^{1/2}\sin(\theta_i-\theta_{i-1})\Big), & k=i,\\ \dfrac{-z_i}{\big(1-z_i^2\big)^{1/2}}\left(\big(1-z_{i+1}^2\big)^{1/2}\sin(\theta_{i+1}-\theta_i)-\dfrac{z_{i+1}z_{i-1}}{\big(1-z_{i-1}^2\big)^{1/2}}\sin(\theta_i-\theta_{i-1})\right) \\ \quad -\dfrac{z_{i-1}\big(1-z_{i+1}^2\big)^{1/2}}{\big(1-z_{i-1}^2\big)^{1/2}}\sin(\theta_{i-1}-\theta_{i+1}), & k=i-1,\\ \dfrac{-z_i}{(1-z_i^2)^{1/2}}\left(\dfrac{-z_{i-1}z_{i+1}}{\big(1-z_{i+1}^2\big)^{1/2}}\sin(\theta_{i+1}-\theta_i)+\big(1-z_{i-1}^2\big)^{1/2}\sin(\theta_i-\theta_{i-1})\right) \\ \quad -\dfrac{z_{i+1}\big(1-z_{i-1}^2\big)^{1/2}}{\big(1-z_{i+1}^2\big)^{1/2}}\sin(\theta_{i-1}-\theta_{i+1}), & k=i+1,\end{cases}$$

$$\frac{\partial^2 \hat{J}}{\partial \theta_k \partial z_i} = \begin{cases} -z_i\big(1-z_i^2\big)^{-1/2}\Big(z_{i+1}\big(1-z_{i-1}^2\big)^{1/2}\cos(\theta_i-\theta_{i-1}) \\ \quad - z_{i-1}\big(1-z_{i+1}^2\big)^{1/2}\cos(\theta_{i+1}-\theta_i)\Big), & k=i,\\ z_iz_{i+1}\big(1-z_{i-1}^2\big)^{1/2}\big(1-z_i^2\big)^{-1/2}\cos(\theta_i-\theta_{i-1}) \\ \quad + \big(1-z_{i+1}^2\big)^{1/2}\big(1-z_{i-1}^2\big)^{1/2}\cos(\theta_{i-1}-\theta_{i+1}), & k=i-1,\\ -z_iz_{i-1}\big(1-z_{i+1}^2\big)^{1/2}\big(1-z_i^2\big)^{-1/2}\cos(\theta_{i+1}-\theta_i) \\ \quad - \big(1-z_{i+1}^2\big)^{1/2}\big(1-z_{i-1}^2\big)^{1/2}\cos(\theta_{i-1}-\theta_{i+1}), & k=i+1,\end{cases}$$

$$\frac{\partial^2 \hat{J}}{\partial \theta_k \partial \theta_i} = \begin{cases} (1-z_i)^{1/2}\Big(-z_{i+1}\big(1-z_{i-1}^2\big)^{1/2}\sin(\theta_i-\theta_{i-1}) \\ \quad - z_{i-1}\big(1-z_{i+1}^2\big)^{1/2}\sin(\theta_{i+1}-\theta_i)\Big), & k=i,\\ z_{i+1}(1-z_i)^{1/2}\big(1-z_{i-1}^2\big)^{1/2}\sin(\theta_i-\theta_{i-1}), & k=i-1,\\ z_{i-1}(1-z_i)^{1/2}\big(1-z_{i+1}^2\big)^{1/2}\sin(\theta_{i+1}-\theta_i), & k=i+1.\end{cases}$$

Let $p = (0, s, 0, s, \pi, -s)$ in cylindrical coordinates. The linearization of $X_{\hat{J}}$ at p is

$$A_{\hat{J}}(p) = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 & 0 \end{pmatrix}.$$

The linearization of $X_{\hat{H}}$ at p is

$$A_{\hat{H}}(p) = 2 \begin{pmatrix} 0 & 0 & 0 & 0 & -b^2 & b^2 \\ 0 & 0 & 0 & -b^2 & 0 & b^2 \\ 0 & 0 & 0 & b^2 & b^2 & -2b^2 \\ 0 & b^{-2} & b^{-2} & 0 & 0 & 0 \\ b^{-2} & 0 & b^{-2} & 0 & 0 & 0 \\ b^{-2} & b^{-2} & 2b^{-2} & 0 & 0 & 0 \end{pmatrix},$$

where $b^2 = 1 - s^2$. These operators are independent and a quick computation shows that for any $-1 < s < 1$ the operator on $L^{\perp}/L$ induced by $A_{\hat{J}} + A_{\hat{H}}$ has four distinct complex eigenvalues of the form $A \pm iB$, $-A \pm iB$. Hence the critical point is rank 1 non-degenerate focus-focus.

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Chapter 3

Convexity and Thimm's Trick

In this chapter we study topological properties of maps constructed by Thimm's trick with Guillemin and Sternberg's action coordinates on a connected Hamiltonian G-manifold M. Since these maps only generate a Hamiltonian torus action on an open dense subset of M, convexity and fibre-connectedness of such maps does not follow immediately from Atiyah-Guillemin-Sternberg's convexity theorem, even if M is compact. The core contribution of this chapter is to provide a simple argument circumventing this difficulty.

In the case where the map is constructed from a chain of subalgebras we prove that the image is given by a list of inequalities that can be computed explicitly in many examples. This generalizes the fact that the images of the classical Gelfand-Zeitlin systems on coadjoint orbits are Gelfand-Zeitlin polytopes. Moreover, we prove that if such a map generates a completely integrable torus action on an open dense subset of M, then all its fibres are smooth embedded submanifolds.

3.1 Introduction

A connected symplectic manifold (M, ω) equipped with a Hamiltonian action of a compact torus T generated by a momentum map $\mu\colon M \to \mathfrak{t}^*$ is a *proper Hamiltonian T-manifold*[1] if μ is proper as a map to a convex subset of $\mathfrak{t}^*$. The convexity theorem for proper Hamiltonian torus manifolds says that if (M, ω, μ) is a proper Hamiltonian T-manifold, then $\mu(M)$ is convex, the fibres of μ are connected, and μ is open as a map to it's image (cf. [10, Theorem 30] or [9]). If in addition the action of T on M is effective and completely integrable, then (M, ω, μ) is called a *proper toric T-manifold*. It follows from

[1] This differs slightly from the definition of proper Hamiltonian T-manifolds given in [47], who require that the convex subset of $\mathfrak{t}^*$ is also open. This extra assumption is not necessary in the context of this chapter.

[46, Theorem 1.3] that proper toric T-manifolds are classified up to isomorphism by $\mu(M)$ together with the weight lattice of the torus. If M is compact, then this classification reduces to Delzant's theorem [17].

Let G be a compact connected Lie group. It was observed by Guillemin and Sternberg that a collective integrable system constructed by Thimm's trick on a Hamiltonian G-manifold (M, ω, Φ) admits natural action coordinates on an open dense subset $\mathcal{U} \subseteq M$ [28]. The most important examples of this construction are the classical Gelfand-Zeitlin[2] systems on $U(n)$ and $SO(n)$ coadjoint orbits for which Guillemin and Sternberg's action coordinates generate effective completely integrable Hamiltonian torus actions on open dense subsets[3]. The images of the classical Gelfand-Zeitlin systems were shown in [28] to be convex polytopes defined by the interlacing inequalities for eigenvalues of Hermitian matrices, which are called *Gelfand-Zeitlin polytopes.*

In [71, Proposition 3.1], it is claimed that for $U(n)$ and $SO(n)$ coadjoint orbits the open dense subsets where the classical Gelfand-Zeitlin systems define Hamiltonian torus actions are proper toric manifolds[4] and this claim is combined with the classification of proper toric manifolds to deduce tight lower bounds for $U(n)$ and $SO(n)$ coadjoint orbits' Gromov width from the geometry of their Gelfand-Zeitlin polytopes. Unfortunately, a review of the literature cited in [71] and related work does not yield a direct explanation of why these open dense subsets are proper Hamiltonian torus manifolds.

In this chapter we describe the general construction of a map F by "Thimm's trick with Guillemin and Sternberg's action coordinates" (Equation (3.24)). We observe that if (M, ω, Φ) is a connected Hamiltonian G-manifold, then it follows from properties of Hamiltonian G-manifolds that the open dense subset $\mathcal{U} \subseteq M$ where F generates a Hamiltonian action of a big torus T' is connected (Lemma 3.26). For a classical Gelfand-Zeitlin system on a $U(n)$ or $SO(n)$ coadjoint orbit, it then follows that the open dense subset where F defines an effective, completely integrable torus action is a proper toric manifold (Example 3.27). More generally, we prove the following theorem.

Theorem 3.1. *Suppose that (M, ω, Φ) is a connected Hamiltonian G-manifold and F is a map constructed on M by Thimm's trick with Guillemin and Sternberg's action coordinates. If F is proper as a map to a convex set, then*

1. *$(\mathcal{U}, \omega|_{\mathcal{U}}, F|_{\mathcal{U}})$ is a proper Hamiltonian T'-manifold,*

2. *$F(M)$ is convex, and*

[2] One should note that there are multiple spellings of Zeitlin in the literature, including Tsetlin and Cetlin. Works by Kostant-Wallach, Kogan-Miller, and Guillemin-Sternberg respectively each use a different spelling.

[3] This is proven for regular coadjoint orbits in [28]. A proof for arbitrary coadjoint orbits can be found in [70].

[4] The definition of proper Hamiltonian T-manifold given in [71] is that of [47], but the assumption of openness for the convex subset of $\mathfrak{t}^*$ is actually not needed in the context of [71].

3. The fibres of F are connected.

We also prove a straightforward generalization of the fact that the images of the classical Gefland-Zeitlin systems are Gelfand-Zeitlin polytopes: if (M, ω, Φ) is a connected Hamiltonian G-manifold with Φ proper, then the image of the map F constructed by Thimm's trick with Guillemin and Sternberg's action coordinates from a chain of subalgebras

$$\mathfrak{h}_1 \subseteq \ldots \subseteq \mathfrak{h}_k = \mathfrak{g}$$

is the locally polyhedral set defined by the inequalities of the momentum set $\Phi(M) \cap \mathfrak{t}^*_+$ and the branching inequalities corresponding to the chain of subalgebras (Proposition 3.49). Moreover, if the torus action generated by F on an open dense subset is completely integrable, then we prove that the fibres of F are embedded submanifolds (Proposition 3.52).

If (M, ω, Φ) is a multiplicity free Hamiltonian $U(n)$ or $SO(n)$-manifold, then the map F constructed by Thimm's trick with Guillemin and Sternberg's action coordinates from chains of subalgebras

$$\mathfrak{u}(1) \subseteq \mathfrak{u}(2) \subseteq \cdots \subseteq \mathfrak{u}(n) \text{ or } \mathfrak{so}(2) \subseteq \mathfrak{so}(3) \subseteq \cdots \subseteq \mathfrak{so}(n) \tag{3.2}$$

respectively generates a completely integrable Hamiltonian torus action on an open dense subset of M. If Φ is proper, then it follows by Theorem 3.1 that this open dense subset is a proper, completely integrable Hamiltonian torus manifold (in general, the torus action may not be effective) and by Proposition 3.49 one has an explicit description of the image of F. This can then be used to prove explicit lower bounds for the Gromov widths of a much larger family of symplectic manifolds than was studied in [71] (namely multiplicity free $U(n)$-manifolds with proper momentum maps), which is done in Chapter 4. Unfortunately, applying Thimm's trick to similar chains of subalgebras for groups other than $U(n)$ and $SO(n)$ does not yield completely integrable torus actions [29, p. 225], although one should note that Harada was able to extend the construction of classical Gelfand-Zeitlin systems to construct completely integrable systems on $Sp(n)$ coadjoint orbits in [33].

Nishinou-Nohara-Ueda proved that the classical Gelfand-Zeitlin systems on $U(n)$ coadjoint orbits can be constructed by toric degeneration [67]. This was later generalized by Harada-Kaveh who proved that, under various technical assumptions, a toric degeneration of a smooth projective variety endows an open dense subset with the structure of a proper toric manifold [34, Theorem B]. These results were applied in [48] to prove lower bounds for the Gromov width of smooth projective varieties in terms of their Newton-Okounkov bodies and in [32] and [21] to finish the proof of tight lower bounds for the

Gromov width of coadjoint orbits of compact simple Lie groups (tight upper bounds were proven [13]).

In contrast to toric degeneration, the construction of Hamiltonian torus actions by Thimm's trick with Guillemin and Sternberg's action coordinates

- does not require the manifold to be projective or even Kähler and
- typically does not yield a completely integrable torus action.

Recently, Hilgert-Martens-Manon described a symplectic analogue of toric degeneration called "symplectic contraction" and showed that maps constructed via symplectic contraction are the same as the maps constructed by Thimm's trick [38]. The relation between this chapter and the results of [38] is discussed in Section 3.6.

The contents of this chapter are as follows. In Section 2 we recall basic definitions and results pertaining to Hamiltonian group actions. Section 3 studies the details of Thimm's trick with Guillemin and Sternberg's action coordinates and gives the proof of Theorem 3.1. Section 3 also contains a brief subsection describing how maps constructed by Thimm's trick interact with symplectic reduction. In Section 4 we recall the general branching inequalities, multiplicity free G-manifolds, and prove Proposition 3.49 and Proposition 3.52. In Section 5 we illustrate the Thimm construction and applications in symplectic topology with several low-dimensional examples. In Section 3.6 we discuss the related work by [38].

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3.2 Hamiltonian Group Actions

3.2.1 Basic Definitions

Let G be a compact, connected Lie group with Lie algebra $\mathfrak{g}$. We write

$$\langle -, - \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R} \tag{3.3}$$

for the dual pairing, $Ad_g X$ for the adjoint action of $g \in G$ on $\mathfrak{g}$, and $Ad_g^* \xi$ for the coadjoint action of G on $\mathfrak{g}^*$. Given an action of G on a manifold M, let $\underline{X}$ denote the fundamental vector field of $X \in \mathfrak{g}$. A manifold M is *symplectic* if it is equipped with a closed, non-degenerate 2-form ω. Recall,

Definition 3.4. An action of G on a symplectic manifold (M, ω) is *Hamiltonian* if there is an equivariant map $\Phi : M \to \mathfrak{g}^*$ such that

$$\iota_{\underline{X}}\omega = d\langle \Phi, X\rangle \tag{3.5}$$

for all $X \in \mathfrak{g}$. If this is the case, then Φ is called a *momentum map* for the action and (M, ω, Φ) is called a *Hamiltonian G-manifold.*

Given a function $f \in C^\infty(M)$, the *Hamiltonian vector field X_f* is defined by Hamilton's equation

$$\iota_{X_f}\omega = df. \tag{3.6}$$

Definition 3.7. A *Poisson bracket* on $C^\infty(M)$ is a Lie bracket $\{\cdot,\cdot\}$ such that $\{f,\cdot\}$ is a derivation for all $f \in C^\infty(M)$. A map between manifolds with Poisson brackets, $\Phi : (M, \{\cdot,\cdot\}) \to (M', \{\cdot,\cdot\}')$, is *Poisson* if for all $f, g \in C^\infty(M')$,

$$\{f \circ \Phi, g \circ \Phi\} = \{f, g\}' \circ \Phi. \tag{3.8}$$

Two functions $f, g \in C^\infty(M)$ are said to *Poisson commute* if $\{f, g\} = 0$.

The Poisson bracket on a symplectic manifold is

$$\{f, g\} = \omega\left(X_f, X_g\right). \tag{3.9}$$

The Kostant-Kirillov-Souriou Poisson bracket on $\mathfrak{g}^*$, is defined by

$$\{f, g\}_\xi = \langle \xi, [df_\xi, dg_\xi]\rangle \tag{3.10}$$

where the linear functional $df_\xi : T_\xi\mathfrak{g}^* = \mathfrak{g}^* \to \mathbb{R}$ is identified with an element of $\mathfrak{g}$. The symplectic leaves of this Poisson bracket are the coadjoint orbits which are equipped with the Kostant-Kirillov-Souriau symplectic form. If $\Phi : M \to \mathfrak{g}^*$ is a momentum map for a Hamiltonian G-action, then Φ is Poisson with respect to these brackets [5].

3.2.2 Properties of the Sweeping Map

In this section we recall important facts about the sweeping map. These appear in various places, such as [31, 14, 39] along with a generalization to orbifolds in [55].

Fix a choice of maximal torus $T \subseteq G$ with Lie algebra $\mathfrak{t}$. Let $\mathfrak{t}_+$ be a choice of closed[5] positive

[5] By this we mean that $\mathfrak{t}_+$ is the closed polyhedral cone in $\mathfrak{t}$ defined as an intersection of closed half-spaces corresponding to reflections generating of the Weyl group.

Weyl chamber. Since G is compact there is a non-degenerate, positive definite, bilinear form $(-,-)$ on $\mathfrak{g}$ which is invariant under the adjoint action of G. The map $X \mapsto (X,-)$ is a G-equivariant vector space isomorphism of $\mathfrak{g}$ with $\mathfrak{g}^*$, with respect to the adjoint and coadjoint actions. We also call the image of $\mathfrak{t}_+$ under this map a *positive Weyl chamber* and denote it by $\mathfrak{t}^*_+$. The positive Weyl chamber is a fundamental domain for the coadjoint action of G.

Definition 3.11. Let G be a compact, connected Lie group and let $\mathfrak{t}^*_+$ be a positive Weyl chamber. The *sweeping map* $s : \mathfrak{g}^* \to \mathfrak{t}^*_+$ is defined by letting $s(\xi)$ be the unique element of the set $(G \cdot \xi) \cap \mathfrak{t}^*_+$.

The sweeping map is continuous and induces a homeomorphism $\mathfrak{g}^*/G \cong \mathfrak{t}^*_+$. If σ is a stratum[6] of the polyhedral cone $\mathfrak{t}^*_+$, then $\Sigma_\sigma = G \cdot \sigma$ is a connected component of an orbit-type stratum in $\mathfrak{g}^*$ and the restricted map $s : \Sigma_\sigma \to \sigma$ is smooth. We recall the following detail of the symplectic cross-section theorem (see e.g. [55, Theorem 3.1]).

Theorem 3.12. *Let (M, ω, Φ) be a connected Hamiltonian G-manifold. There exists a unique stratum $\sigma \subseteq \mathfrak{t}^*_+$ with the property that $\Phi(M) \cap \mathfrak{t}^*_+ \subseteq \overline{\sigma}$ and $\Phi(M) \cap \sigma$ is non-empty.*

The unique stratum σ of the preceding theorem is called the *principal stratum* of M. Unpacking the details of the proof of Theorem 3.12, one has the following proposition which will be useful in the proof of Theorem 3.1.

Proposition 3.13. *Let σ be the principal stratum of a connected Hamiltonian G-manifold (M, ω, Φ). The pre-image $\Phi^{-1}(\Sigma_\sigma)$ is a connected, dense open submanifold of M and its complement is contained in a locally finite union of submanifolds of codimension at least 2 in M.*

The stabilizer subgroup G_ξ of a point ξ in a stratum σ of $\mathfrak{t}_+$ is independent of the point ξ, so we refer to it as G_σ. Let $\mathfrak{g}_\sigma$ be the Lie algebra of G_σ and let $\mathfrak{t}_\sigma = \mathfrak{z}(\mathfrak{g}_\sigma) \subseteq \mathfrak{t}$ be its centre. The span of σ in $\mathfrak{g}^*$ is the subspace of points $(\mathfrak{g}^*)^{G_\sigma}$ fixed under the coadjoint action of G_σ, which is identified with $\mathfrak{t}^*_\sigma$ via the inner product on $\mathfrak{g}$. Let $T_\sigma \subseteq T$ be the torus with Lie algebra $\mathfrak{t}_\sigma$.

We may use the action of G to define a new action of T_σ on $(\Phi)^{-1}(\Sigma_\sigma)$ by letting

$$t * m = (g^{-1} t g) \cdot m \tag{3.14}$$

for all $m \in (\Phi)^{-1}(\Sigma_\sigma)$ and $t \in T_\sigma$. Here $g \in G$ is an element such that $Ad^*_g \Phi(m) \in \sigma$, and $(g^{-1}tg)$ is acting on M as an element of G. One checks that this action is independent of the choice of g in the

[6] $\mathfrak{t}^*_+$ has a natural stratification as a polyhedral set. The maximal stratum is the interior of $\mathfrak{t}^*_+$ and the lower dimensional strata are the relative interiors of the intersections of the faces of $\mathfrak{t}^*_+$.

coset $G_\sigma g$ (since T_σ is contained in the centre of G_σ). Note that the new action of T_σ commutes with the action of G.

In [28, Theorem 3.4], Guillemin and Sternberg observed that if σ is maximal (i.e. $\sigma = (\mathfrak{t}^*_+)^{\mathrm{int}}$), then the new action of $T_\sigma = T$ on $\Phi^{-1}(\Sigma_\sigma)$ is Hamiltonian, generated by $s \circ \Phi$. More generally, if $p_\sigma \colon \mathfrak{t}^* \to \mathfrak{t}^*_\sigma$ is the projection dual to the inclusion $\mathfrak{t}_\sigma \subseteq \mathfrak{t}$, then we have the following proposition (see [83, Proposition 3.4] for a proof).

Proposition 3.15 (Guillemin and Sternberg's action coordinates)**.** *The new action of T_σ on $\Phi^{-1}(\Sigma_\sigma)$ defined by equation* (3.14) *is Hamiltonian and*

$$p_\sigma \circ s \circ \Phi : \Phi^{-1}(\Sigma_\sigma) \to \mathfrak{t}^*_\sigma$$

is a momentum map for this action.

In what follows, we identify $\mathfrak{t}^*_\sigma$ with the subspace of $\mathfrak{t}^*$ spanned by σ so that $p_\sigma \circ s \circ \Phi = s \circ \Phi$.

3.3 Thimm's Trick

3.3.1 Thimm's trick

A completely integrable system on a symplectic manifold (M, ω) of dimension $2n$ is a set of n Poisson commuting functions $f_1, \ldots, f_n \in C^\infty(M)$ such that the map

$$(f_1, \ldots, f_n) : M \to \mathbb{R}^n \tag{3.16}$$

is a submersion on an open dense subset of M. A classical problem in the study of integrable systems is the construction of completely integrable systems on a Hamiltonian G-manifold from collective functions, where a function on a Hamiltonian G-manifold (M, ω, Φ) is *collective* if it is of the form $\Phi^* f$, for some $f \in C^\infty(\mathfrak{g}^*)$. Since Φ is Poisson, if two functions $f, g \in C^\infty(\mathfrak{g}^*)$ commute with respect to the Kostant-Kirillov-Souriau Poisson bracket, then their pullbacks $\Phi^* f$, $\Phi^* g$ commute with respect to the Poisson bracket on M.

Thus one may construct a completely integrable system of collective functions on M^{2n} by constructing n independent Poisson commuting functions $f_1, \ldots, f_n \in C^\infty(\mathfrak{g}^*)$. By Chevalley's theorem, one can find $\mathrm{rank}(G)$ independent functions in the ring of Ad^*-invariant functions, $C^\infty(\mathfrak{g}^*)^G$, and these functions Poisson commute, but it is often the case that $\mathrm{rank}(G) < n$. In order to find additional independent Poisson commuting functions on $\mathfrak{g}^*$, Thimm proved the following proposition [76, Proposition 4.1].

Proposition 3.17 (Thimm's Trick). *Let G be a compact connected Lie group and let $\mathfrak{h}_1$, $\mathfrak{h}_2$ be two subalgebras of $\mathfrak{g}$ with dual projection maps $p_i : \mathfrak{g}^* \to \mathfrak{h}_i^*$, $i = 1, 2$. If $h_1 \in C^\infty(\mathfrak{h}_1^*)^{H_1}$ and $h_2 \in C^\infty(\mathfrak{h}_2^*)$, then the Poisson bracket of $p_1^* h_1$ and $p_2^* h_2$ vanishes identically on $\mathfrak{g}^*$ if*

$$[\mathfrak{h}_1, \mathfrak{h}_2] \subseteq \mathfrak{h}_1. \tag{3.18}$$

This holds in particular if $\mathfrak{h}_2 \subseteq \mathfrak{h}_1$ or $[\mathfrak{h}_1, \mathfrak{h}_2] = \{0\}$.

Thus one can find additional Poisson commuting functions on $\mathfrak{g}^*$ by pulling back invariant functions from subalgebras that satisfy (3.18). This construction of additional commuting functions is *Thimm's trick*[7].

Remark 3.19. The statement of Proposition 3.17 differs from the statement of [76, Proposition 4.1] in two ways. First, we note that it is only necessary for the function $h_1 \in C^\infty(\mathfrak{h}_1^*)$ to be H_1-invariant. This is implicit in the proof provided in [76] which does not use the assumption that h_2 is H_2-invariant. Second, it is not necessary to require G to be compact (it is only assumed in [76] that the subalgebras $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are nondegenerate).

Remark 3.20. In the earlier paper [79] – also concerned with the construction of collective completely integrable systems – Trofimov also used chains of subalgebras $\mathfrak{h}_2 \subseteq \mathfrak{h}_1$ to find additional collective integrals. There are other constructions of additional commuting functions on $\mathfrak{g}^*$ that may give collective integrable systems such as the method of argument shifting [59, 65].

If the coordinates of a completely integrable system $f_1, \ldots, f_n$ generate Hamiltonian S^1-actions, then the map $(f_1, \ldots, f_n)$ is said to provide *action coordinates* for the integrable system. In [28, p. 119], Guillemin and Sternberg essentially applied Proposition 3.15 to show that there are natural action coordinates for a collective completely integrable system on a Hamiltonian G-manifold (M, ω, Φ) that has been constructed by Thimm's trick from a chain of subalgebras

$$\mathfrak{h}_1 \subseteq \cdots \subseteq \mathfrak{h}_d$$

(although they make the extra assumption that the principal stratum corresponding to M is $(\mathfrak{t}_+^*)^{\mathrm{int}}$). In general, systems of commuting collective functions constructed by Thimm's trick may not give a completely integrable system on M (i.e. there are fewer than n independent functions) and the principal stratum corresponding to M may not be maximal. Nevertheless, it is still possible to construct a

[7] There is a minor confusion in the literature between Thimm's trick and Guillemin and Sternberg's action coordinates. Thimm's paper [76] is solely concerned with constructing integrable systems whereas natural action coordinates for systems constructed by Thimm's trick originate in [28].

continuous map that generates a Hamiltonian torus action on an open dense subset as we will now see.

Proposition 3.21. *Let (M, ω, Φ) be a connected Hamiltonian G-manifold and suppose that $\mathfrak{h}_1$, $\mathfrak{h}_2$ are subalgebras of $\mathfrak{g}$ with corresponding connected subgroups H_1 and H_2, such that*

$$[\mathfrak{h}_1, \mathfrak{h}_2] \subseteq \mathfrak{h}_1.$$

Let σ_1 be the principal stratum of a Weyl chamber $\mathfrak{t}^*_{1,+} \subseteq \mathfrak{h}^*_1$ corresponding to the induced action of H_1 on M. The action of H_2 on M leaves $(p_1 \circ \Phi)^{-1}(\Sigma_{\sigma_1})$ invariant and commutes with the new action of T_{σ_1} defined there.

Proof. See Proposition C.5. $\square$

Let σ_2 be the principal stratum of $\mathfrak{t}^*_{2,+}$ corresponding to the induced H_2-action on M, which is generated by $p_2 \circ \Phi$. A new action of the torus T_{σ_2} on $(p_2 \circ \Phi)^{-1}(\Sigma_{\sigma_2})$ is defined as in equation (3.14) via the action of elements of H_2. It follows by the preceding proposition that this action leaves $(p_1 \circ \Phi)^{-1}(\Sigma_{\sigma_1})$ invariant and commutes with the new action of T_{σ_1}. Thus we have proven the following corollary.

Corollary 3.22. *If $[\mathfrak{h}_1, \mathfrak{h}_2] \subseteq \mathfrak{h}_1$ or $\mathfrak{h}_2$, then the new T_{σ_1}-action generated by $s \circ p_1 \circ \Phi$ and the new T_{σ_2}-action generated by $s \circ p_2 \circ \Phi$ commute on the open, dense subset where they are both defined,*

$$(p_1 \circ \Phi)^{-1}(\Sigma_{\sigma_1}) \cap (p_2 \circ \Phi)^{-1}(\Sigma_{\sigma_2}).$$

Let $\mathfrak{h}_1, \dots, \mathfrak{h}_d$ be subalgebras of $\mathfrak{g}$ such that for all $1 \leq i, j \leq d$,

$$[\mathfrak{h}_i, \mathfrak{h}_j] \subseteq \mathfrak{h}_i \text{ or } \mathfrak{h}_j \tag{3.23}$$

and let (M, ω, Φ) be a connected Hamiltonian G-manifold. For each $1 \leq k \leq d$, fix maximal tori T_k and Weyl chambers $\mathfrak{t}^*_{k,+} \subseteq \mathfrak{t}^*_k$. The maps $p_k \circ \Phi$ are momentum maps for the induced actions of H_k on M and by Proposition 3.12 there are principal strata $\sigma_k \subseteq \mathfrak{t}^*_{k,+}$ corresponding to each of the Hamiltonian H_k-actions on M. For each k, let U_k denote the connected open dense subset $(p_k \circ \Phi)^{-1}(\Sigma_{\sigma_k})$ where $\Sigma_{\sigma_k} = H_k \cdot \sigma_k$. For each k, let T_{σ_k} be the torus with Lie algebra $\mathfrak{t}_{\sigma_k}$.

Let F be the composition of Φ with the map

$$(s \circ p_1, \dots, s \circ p_d) : \mathfrak{g}^* \to \mathfrak{t}^*_1 \oplus \cdots \oplus \mathfrak{t}^*_d. \tag{3.24}$$

We say that F is *constructed by Thimm's trick with Guillemin and Sternberg's action coordinates* from the subalgebras $\mathfrak{h}_1, \ldots, \mathfrak{h}_d$. The map F is continuous since the sweeping maps are continuous, but in general the map F is not smooth.

By Corollary 3.22, the set

$$\mathcal{U} = F^{-1}(\sigma_1 \times \cdots \times \sigma_d) = \bigcap_{1 \leq k \leq d} U_k \tag{3.25}$$

is invariant under the new actions of the tori T_{σ_k}, and the new actions of the tori T_{σ_k} commute on $\mathcal{U}$. Thus the new actions define an action of a big torus $T' = T_{\sigma_1} \times \cdots \times T_{\sigma_d}$ on $\mathcal{U}$ and this action is Hamiltonian with momentum map $F|_{\mathcal{U}} \colon \mathcal{U} \to \mathfrak{t}_1^* \oplus \cdots \oplus \mathfrak{t}_d^*$.

Lemma 3.26. *The open dense set $\mathcal{U}$ is connected.*

Proof. By equation (3.25), $\mathcal{U}$ is the finite intersection of the sets $U_k \subseteq M$. By Proposition 3.13, the complement of $\mathcal{U}$ is a finite union of closed sets that are each contained in a locally finite union of submanifolds of codimension at least 2 in M, so $\mathcal{U}$ is connected. □

Thus, given a connected Hamiltonian G-manifold (M, ω, Φ) and subalgebras $\mathfrak{h}_1, \ldots, \mathfrak{h}_d$ that pairwise satisfy (3.23),

$$(\mathcal{U}, \omega|_{\mathcal{U}}, F|_{\mathcal{U}})$$

is a connected Hamiltonian T'-manifold, where $T' = T_{\sigma_1} \times \cdots \times T_{\sigma_d}$.

Example 3.27 (The classical Gelfand-Zeitlin systems). Let $(\mathcal{O}, \omega)$ be a $U(n)$ coadjoint orbit equipped with the Kostant-Kirillov-Souriau symplectic form. The induced action of a subgroup $U(n-1) \subseteq U(n)$ is Hamiltonian, and the restriction of the projection $p \colon \mathfrak{u}(n)^* \to \mathfrak{u}(n-1)^*$ to $\mathcal{O}$ is a momentum map for the induced action. The classical Gelfand-Zeitlin system on $\mathcal{O}$ is the torus action constructed by Thimm's trick with Guillemin and Sternberg's action coordinates from a chain of subalgebras

$$\mathfrak{u}(1) \subseteq \cdots \subseteq \mathfrak{u}(n-1).$$

Since $\mathcal{O}$ is compact, the continuous map

$$F \colon \mathcal{O} \to \mathfrak{t}_1^* \oplus \cdots \oplus \mathfrak{t}_{n-1}^*.$$

is proper. By definition, $\mathcal{U} = F^{-1}(\sigma_1 \times \cdots \times \sigma_{n-1})$, so the restriction $F|_{\mathcal{U}}$ is proper as a map to the convex set

$$\sigma_1 \times \cdots \times \sigma_{n-1} \subseteq \mathfrak{t}_1^* \oplus \cdots \oplus \mathfrak{t}_{n-1}^*.$$

Combining this with Lemma 3.26, we have shown that $(\mathcal{U}, \omega|_{\mathcal{U}}, F|_{\mathcal{U}})$ is a proper Hamiltonian T'-manifold. Combining this with the fact that the T' action on $\mathcal{U}$ is effective and completely integrable [28, 70], we have shown that $(\mathcal{U}, \omega|_{\mathcal{U}}, F|_{\mathcal{U}})$ is a proper toric T'-manifold.

The construction of the classical Gelfand Zeitlin systems on $SO(n)$ coadjoint orbits is completely analogous, except that one considers a chain of subalgebras

$$\mathfrak{so}(2) \subseteq \cdots \subseteq \mathfrak{so}(n-1)$$

instead.

More generally, if $(\mathcal{U}, \omega|_{\mathcal{U}}, F|_{\mathcal{U}})$ is a proper Hamiltonian T'-manifold, then it follows by the convexity theorem for proper Hamiltonian torus actions (cf. [10, Theorem 30]), that

(a) $F(\mathcal{U})$ is convex,

(b) the fibres of $F|_{\mathcal{U}}$ are connected, and

(c) the domain and codomain restricted map $F|_{\mathcal{U}} \colon \mathcal{U} \to F(\mathcal{U})$ is open (with respect to the subspace topologies on $\mathcal{U}$ and $F(\mathcal{U})$).

For instance, if $F \colon M \to \mathfrak{t}_1^* \oplus \cdots \oplus \mathfrak{t}_d^*$ is proper (e.g. if M is compact), then $F|_{\mathcal{U}}$ is proper as a map to the convex set $\sigma_1 \times \cdots \times \sigma_n$. It is natural to wonder whether properties (a), (b) and (c) are also true for the map $F \colon M \to \mathfrak{t}_1^* \oplus \cdots \oplus \mathfrak{t}_d^*$ if F is proper. This prompts the following lemma.

Lemma 3.28. *Let X be a Hausdorff topological space and let $f \colon X \to \mathbb{R}^n$ be a continuous proper map. Suppose S is a dense subset of X saturated*[8] *by f such that $f(S)$ is convex, the fibres of $f|_S$ are connected, and $f|_S \colon S \to f(S)$ is open. Then the fibres of f are connected.*

Proof. Since the fibres of $f|_S$ are connected and S is saturated by f, it remains to show that the fibres $f^{-1}(x)$, $x \in f(X \setminus S)$ are connected.

Fix $x \in f(X \setminus S)$ and let $F = f^{-1}(x)$. Suppose that $F = U \cup V$ for open sets $U, V \subseteq F$. Since U and V are open with respect to the subspace topology on F, there are open sets $U', V' \subseteq X$ with $U = U' \cap F$, $V = V' \cap F$. Our aim is to show that $U \cap V \neq \emptyset$.

First, we claim that there is a $\delta > 0$ such that $f^{-1}(B_\delta(x)) \subseteq U' \cup V'$, where $B_\delta(x)$ is the open ball of radius δ centred at x. If not then, for all $n \geq 1$, there is a $y_n \in f^{-1}(B_{1/n}(x))$ such that $y_n \notin U' \cup V'$. Since f is proper, $f^{-1}(\overline{B}_1(x))$ is compact in X. Thus there is a convergent subsequence $y_{n_k} \to y$, and

[8] A subset $S \subseteq X$ is saturated by a map $f \colon X \to Y$ if it is a union of level sets of f.

$y \in F$ by continuity. Since y_n is contained in the closed set $X \setminus (U' \cup V')$, so is the limit y, which contradicts our assumption that $F = U \cup V$.

For every $\gamma > 0$, the set $B_\gamma(x) \cap f(S)$ is convex and non-empty. Further,

$$A_\gamma = B_\gamma(x) \cap f(U' \cap S) \text{ and } C_\gamma = B_\gamma(x) \cap f(V' \cap S)$$

are non-empty, open subsets of $f(S)$, since $f|_S$ is open as a map to $f(S)$ and S is dense. By the previous paragraph, $B_\gamma(x) \cap f(S) \subseteq A_\gamma \cup C_\gamma$ for all positive $\gamma < \delta$. Thus since $B_\gamma(x) \cap f(S)$ is connected and non-empty, $A_\gamma \cap C_\gamma \neq \emptyset$ for all positive $\gamma < \delta$.

Fix $1/N < \delta$. For every $n \geq N$, we have shown there is an element $x_n \in A_{1/n} \cap C_{1/n}$. Thus the intersections $U' \cap f^{-1}(x_n)$ and $V' \cap f^{-1}(x_n)$ are both non-empty. Since $f^{-1}(x_n) \subseteq U' \cup V'$. and $f^{-1}(x_n)$ is connected, there exists a $y_n \in U' \cap V' \cap f^{-1}(x_n)$. Once again, by properness of f, we can find a convergent subsequence y_{n_k} with limit $y \in F$.

It follows that $U \cap V \neq \emptyset$. Suppose $y \notin V$ (and thus, $y \in U$). Since $y_n \in V'$, $y \in \overline{V'}$. Observe that $\overline{V'} \cap F = \overline{V' \cap F} = \overline{V}$ (where the second and third closures are taken in the subspace topology) so $y \in \overline{V}$. This contradicts $U \cap V = \emptyset$ since U is an open neighbourhood of y in F. □

Remark 3.29. Note that Lemma 3.28 is false if various assumptions are dropped. For example,

- Let X be the sphere of radius 1 in $\mathbb{R}^3$ and let $f\colon X \to \mathbb{C}$ be the map $f(x,y,z) = e^{\pi i(z+1)}$. Then

  - f is a proper continuous map.
  - The set $S = X \setminus \{(0,0,1),(0,0,-1)\}$ is dense in X and saturated by f.
  - The restriction $f|_S\colon S \to f(S)$ is open.
  - The fibres of $f|_S$ are connected

  But $f(S)$ not convex and the fibre $f^{-1}(1)$ is not connected.

- Let X be the unit square $[0,1] \times [0,1]$ minus the set $\{1\} \times (0,1)$. Let $f\colon X \to \mathbb{R}$ be projection to the first coordinate. Then

  - f is a continuous map.
  - The set $S = [0,1) \times [0,1]$ is dense in X and saturated by f.
  - The restriction $f|_S\colon S \to f(S)$ is open.
  - The fibres of $f|_S$ are connected.

But f is not proper and the fibre $f^{-1}(1)$ is not connected.

We can now prove Theorem 3.1.

Proof of Theorem 1: (1) By Lemma 3.26, $\mathcal{U}$ is connected. Since F is proper as a map to a convex set, call this set $\mathcal{T}$, $F|_{\mathcal{U}}$ is proper as a map to the convex set $\mathcal{T} \cap (\sigma_1 \times \cdots \times \sigma_d)$. Therefore $(\mathcal{U}, \omega|_{\mathcal{U}}, F|_{\mathcal{U}})$ is a proper Hamiltonian T'-manifold.

(2) By (1) and the convexity theorem for proper Hamiltonian torus actions (cf. [10, Theorem 30]), $F(\mathcal{U})$ is convex. Since $\mathcal{U}$ is dense in M and F is a proper continuous map, $F(M) = F(\overline{\mathcal{U}}) = \overline{F(\mathcal{U})}$. Thus $F(M)$ is convex.

(3) We check the hypotheses of Lemma 3.28. F is continuous and by assumption, F is proper. The set $\mathcal{U}$ is dense in M and by definition it is saturated by F. By (1) and the convexity theorem for proper Hamiltonian torus actions (cf. [10, Theorem 30]), the restriction $F|_{\mathcal{U}}$ satisfies conditions (a), (b), and (c). These are precisely the remaining hypotheses of Lemma 3.28, so the result follows. $\square$

Remark 3.30. The non-abelian convexity theorem for Hamiltonian group actions says that if (M, ω, Φ) is a Hamiltonian G-manifold with Φ proper, then the *momentum set*

$$\square(M) := s \circ \Phi(M) = \Phi(M) \cap \mathfrak{t}^*_+$$

is a convex, locally polyhedral set, the fibres of $s \circ \Phi$ are connected, and the map $s \circ \Phi$ is open as a map to its image[9]. Thus if F is a map constructed by Thimm's trick with Guillemin and Sternberg's action coordinates from one subalgebra $\mathfrak{h}$, Theorem 3.1 reduces to the non-abelian convexity theorem. Conversely, Theorem 3.1 does not follow directly from the non-abelian convexity theorem for Hamiltonian group actions; a product of maps with convex images and connected fibres does not necessarily have a convex image and connected fibres.

Remark 3.31. We were unable to determine in the context of Theorem 3.1 whether F is open as a map to its image (property (c) above). It is known that F is open as a map to its image in the special case when $F = s \circ \Phi$ by the non-abelian convexity theorem for Hamiltonian group actions [55, Theorem 1.2].

[9]The first fact is standard in the literature (see e.g. [55, Theorem 1.1]). The second fact is equivalent to fibre connectedness for the map Φ (see the argument in [55, p. 256]). Openness of $s \circ \Phi$ as a map to its image is described in [55, Theorem 1.2]).

3.3.2 Symplectic Reduction and Thimm's Trick

Suppose that (M, ω, Φ) is a connected Hamiltonian G-manifold and F is a map constructed by Thimm's trick with Guillemin and Sternberg's action coordinates from subalgebras $\mathfrak{h}_1, \ldots, \mathfrak{h}_d$ of $\mathfrak{g}$ as in the preceding section.

Let K be a closed, connected subgroup of G. The induced action of K on M is Hamiltonian with momentum map $\Phi_K = p_K \circ \Phi$. The *reduction* of M at a level $\mu \in \mathfrak{k}^*$ is the quotient space $M_\mu = \Phi_K^{-1}(K \cdot \mu)/K$, where $K \cdot \mu$ is the coadjoint orbit of K through μ. If μ is a regular value of Φ_K, then this quotient space is a smooth manifold or an orbifold with a Marsden-Weinstein symplectic structure. Proposition 3.33 is purely topological and will hold in the general setting where μ may be a critical value. Let $\pi\colon \Phi_K^{-1}(K \cdot \mu) \to M_\mu$ be the quotient projection map.

Lemma 3.32. *If $[\mathfrak{h}_k, \mathfrak{k}] \subseteq \mathfrak{h}_k$ for all $1 \leq k \leq d$, then F induces a map $\widetilde{F}$ on M_μ and the restriction of $\widetilde{F}$ to $\pi(\mathcal{U} \cap \Phi_K^{-1}(K \cdot \mu))$ generates a Hamiltonian torus action.*

Note that the set $\mathcal{U} \cap \Phi_K^{-1}(K \cdot \mu)$ could be empty.

Proof. By Proposition 3.21, the actions of K preserves $\mathcal{U}$ and commutes with the action of the big torus T' defined there. The map F is K-invariant on M by continuity (since $\mathcal{U}$ is dense in M). Thus F induces a map $\widetilde{F}$ on M_μ and $\widetilde{F}$ is a momentum map for the induced T'-action on $\pi(\mathcal{U} \cap \Phi_K^{-1}(K \cdot \mu))$ if it is non-empty (see [75, Example 1.11 and Lemma 3.2]). $\square$

Proposition 3.33. *Suppose that either F or Φ_K is proper and $[\mathfrak{h}_k, \mathfrak{k}] \subseteq \mathfrak{h}_k$ for all $1 \leq k \leq d$. Then the induced map $\widetilde{F}\colon M_\mu \to \mathfrak{t}_1^* \oplus \cdots \oplus \mathfrak{t}_d^*$ has connected fibres and $\widetilde{F}(M_\mu)$ is convex.*

Proof. By Corollary 3.22, the map

$$(F, s \circ \Phi_K)\colon M \to (\mathfrak{t}_1^* \oplus \cdots \oplus \mathfrak{t}_d^*) \oplus \mathfrak{t}_K^* \tag{3.34}$$

is a momentum map for the new action of $T' \times T_{\sigma_K}$ on

$$\mathcal{U} \cap (s \circ \Phi_K)^{-1}(\sigma_K) = (F, s \circ \Phi_K)^{-1}(\sigma_1 \times \cdots \times \sigma_d \times \sigma_K). \tag{3.35}$$

Since one of F or Φ_K is proper, the continuous map $(F, s \circ \Phi_K)$ is proper. Thus by Theorem 3.1, the image of $(F, s \circ \Phi_K)$ is convex and the fibres are connected. The image $\widetilde{F}(M_\mu)$ is the projection to $\mathfrak{t}_1^* \oplus \cdots \oplus \mathfrak{t}_d^* \subseteq (\mathfrak{t}_1^* \oplus \cdots \oplus \mathfrak{t}_d^*) \oplus \mathfrak{t}_K^*$ of the intersection

$$(F, s \circ \Phi_K)(M) \cap (\sigma_1 \times \cdots \times \sigma_d \times \{\mu\}) \tag{3.36}$$

which is convex, so $\widetilde{F}(M_\mu)$ is convex. The map $\widetilde{F}$ has connected fibres since $(F, s\circ\Phi_K)$ has connected fibres that are preserved by the action of K. $\square$

3.4 Branching and Gelfand-Zeitlin systems

3.4.1 The branching cone

We begin by recalling the branching cone, as described in [6].

Let G be a compact, connected Lie group. With respect to the trivialization of the cotangent bundle T^*G by left-invariant vector fields the cotangent lifts of the left and right actions of G on itself are given by

$$\mathcal{L}_{g'}(g,\xi) = (g'g,\xi), \text{ and } \mathcal{R}_{g'}(g,\xi) = (g(g')^{-1}, Ad^*_{g'}\xi). \tag{3.37}$$

and these actions are Hamiltonian with respect to the canonical symplectic form on T^*G, generated by momentum maps

$$\Phi_{\mathcal{L}}(g,\xi) = -Ad^*_g\xi \text{ and } \Phi_{\mathcal{R}}(g,\xi) = \xi. \tag{3.38}$$

Let $K \leq G$ be a closed, connected subgroup, and consider T^*G as a Hamiltonian $K\times G$-manifold, where

$$(k,g')\cdot(g,\xi) = \mathcal{R}_k\mathcal{L}_{g'}(g,\xi). \tag{3.39}$$

This action is generated by the momentum map

$$(g,\xi)\mapsto \left(p^K_G\circ\Phi_{\mathcal{R}}(g,\xi), \Phi_{\mathcal{L}}(g,\xi)\right) = \left(p^K_G(\xi), -Ad^*_g\xi\right) \tag{3.40}$$

where $p^K_G\colon \mathfrak{g}^* \to \mathfrak{k}^*$ is the projection dual to the inclusion $i\colon \mathfrak{k}\to\mathfrak{g}$. Following [6], we consider the modified map

$$\Psi\colon T^*G \to \mathfrak{k}^*\times\mathfrak{g}^*,\ \Psi(g,\xi) = \left(p^K_G(\xi), Ad^*_g\xi\right). \tag{3.41}$$

If $\mathfrak{t}^*_{G,+}$ and $\mathfrak{t}^*_{K,+}$ are positive Weyl chambers then

$$\mathfrak{t}^*_{K,+}\times\mathfrak{t}^*_{G,+}\subseteq\mathfrak{k}^*\times\mathfrak{g}^*$$

is a positive Weyl chamber for $K\times G$. The map Ψ is proper so by the non-abelian convexity theorem for Hamiltonian group actions [55, Theorem 1.1], the set

$$\begin{aligned} C_{K,G} &= \Psi(T^*G) \cap \left(\mathfrak{t}^*_{K,+} \times \mathfrak{t}^*_{G,+}\right) \\ &= \left\{\left(\pi^K_G(\xi), Ad^*_g\xi\right) \in \mathfrak{t}^*_{K,+} \times \mathfrak{t}^*_{G,+} \colon (g,\xi) \in G \times \mathfrak{g}^*\right\} \\ &= \left\{(\eta,\xi) \in \mathfrak{t}^*_{K,+} \times \mathfrak{t}^*_{G,+} \colon \eta \in p^K_G(\mathcal{O}_\xi)\right\}, \end{aligned} \tag{3.42}$$

is a convex polyhedral cone. This set is called the *branching cone* for the subgroup $K \leq G$ [6]. Since it is a polyhedral cone, $C_{K,G}$ is defined as a subset of $\mathfrak{t}^*_K \times \mathfrak{t}^*_G$ by a finite list of inequalities,

$$\langle a_i, \xi\rangle + \langle b_i, \eta\rangle \leq \kappa_i,\ i = 1, \dots, N \tag{3.43}$$

for $a_i \in \mathfrak{t}_G$, $b_i \in \mathfrak{t}_K$, and $\kappa_i \in \mathbb{R}$, which are called *branching inequalities.*

Note that if we fix $\xi \in \mathfrak{t}^*_{G,+}$, then the momentum set of the G coadjoint orbit through ξ, as a Hamiltonian K-manifold, is the set

$$p^K_G(\mathcal{O}_\xi) \cap \mathfrak{t}^*_{K,+} = \left\{\eta \in \mathfrak{t}^*_{K,+} \colon\ \eta \in p^K_G(\mathcal{O}_\xi)\right\} \tag{3.44}$$

which can be identified with the projection to $\mathfrak{t}^*_K$ of the intersection $(\mathfrak{t}^*_K \times \{\xi\}) \cap C_{K,G}$. Accordingly, the momentum set of $\mathcal{O}_\xi$ is equal to the subset of $\mathfrak{t}^*_K$ defined by the inequalities (3.43) with ξ fixed.

Remark 3.45. If $U(n)$ is embedded in $U(n+1)$ as a subgroup of block diagonal matrices $\mathrm{diag}(A, 1)$ then the inequalities defining the branching cone $C_{U(n),U(n+1)}$ can be described as the classical interlacing inequalities for eigenvalues of principal submatrices of Hermitian matrices, as observed in [28]. Similar inequalities describe the branching cone for pairs $SO(n) \leq SO(n+1)$ [71]. Inequalities defining the branching cone of a general pair $K \leq G$ were described in [8].

3.4.2 Branching and Thimm's trick

Suppose we have a connected Hamiltonian G-manifold (M, ω, Φ) along with a chain of subalgebras $\mathfrak{h}_1 \subseteq \cdots \subseteq \mathfrak{h}_d = \mathfrak{g}$ with corresponding connected subgroups H_k, maximal tori T_k, and choices of positive Weyl chambers $\mathfrak{t}^*_{k,+} \subseteq \mathfrak{t}^*_k$. For each k, let $\sigma_k \subseteq \mathfrak{t}^*_{k,+}$ be the principal stratum corresponding to the induced Hamiltonian action of H_k on M and let T_{σ_k} be the corresponding subtorus of T_k. We consider the map F constructed by Thimm's trick from this chain,

$$F = \left(s \circ p^1_d \circ \Phi, \dots, s \circ p^{d-1}_d \circ \Phi, s \circ \Phi\right) : M \longrightarrow \mathfrak{t}^*_1 \oplus \cdots \oplus \mathfrak{t}^*_d, \tag{3.46}$$

which generates a Hamiltonian action of the torus $T' = T_{\sigma_1} \times \cdots \times T_{\sigma_d}$ on the open dense subset $\mathcal{U}$. The projections $p^i_j \colon \mathfrak{h}^*_j \to \mathfrak{h}^*_i$ satisfy the identities $p^i_j = p^i_k \circ p^k_j$ for all $i < k < j$. For each $1 \le k < d$, let

$$\langle a_{i,k+1}, \xi \rangle + \langle b_{i,k},\, \eta \rangle \le \kappa_{i,k},\; i = 1, \ldots, N_k \tag{3.47}$$

be the inequalities defining the branching cone $C_{H_k,H_{k+1}}$ as a subset of $\mathfrak{t}^*_k \times \mathfrak{t}^*_{k+1}$, where $a_{i,k+1} \in \mathfrak{t}_{k+1}$, $b_{i,k} \in \mathfrak{t}_k$ and $\kappa_{i,k} \in \mathbb{R}$. If Φ is proper, then by the non-abelian convexity theorem for Hamiltonian group actions [55, Theorem 1.1] the momentum set $\square\colon = \Phi(M) \cap \mathfrak{t}^*_{d,+}$ is a convex, locally polyhedral set defined by a list of inequalities

$$\langle \alpha_j, \xi \rangle \le \upsilon_j,\; j \in S \tag{3.48}$$

where $\alpha_j \in \mathfrak{t}_d$ and $\upsilon_j \in \mathbb{R}$, and S is a set indexing the inequalities (note that without the assumption that M is compact, S may be infinite).

Proposition 3.49. *Let (M, ω, Φ) be a connected Hamiltonian G-manifold with Φ proper and let $\mathfrak{h}_1 \subseteq \cdots \subseteq \mathfrak{h}_d = \mathfrak{g}$ be a chain of subalgebras. The image of the map F is the convex, locally polyhedral subset of $\mathfrak{t}^*_1 \oplus \cdots \oplus \mathfrak{t}^*_d$ defined by the inequalities (3.47) and (3.48). Furthermore, the fibres of F are connected.*

Proof. Since Φ is proper and $\mathfrak{h}_d = \mathfrak{g}$, the map F is proper, so it follows by Theorem 3.1 that $F(M)$ is convex and the fibres of F are connected. It remains to show that $F(M)$ is equal to the set $\Delta \subseteq \mathfrak{t}^*_1 \oplus \cdots \oplus \mathfrak{t}^*_d$ defined by the inequalities (3.47) and (3.48).

Suppose that $(\xi_1, \ldots, \xi_d) = F(m)$ for some $m \in M$. Since $\xi_d = s \circ \Phi(m)$, the inequalities (3.48) are satisfied. For each $1 \le k < d$, since $p^k_d = p^k_{k+1} \circ p^{k+1}_d$, it is true that $\xi_k \in p^k_{k+1}(H_{k+1} \cdot p^{k+1}_d(\xi_d))$, so the inequalities (3.47) are satisfied. Thus $F(M) \subseteq \Delta$

The proof that $\Delta \subseteq F(M)$ follows from a repeated application of the branching inequalities.

Let $(\xi_1, \ldots, \xi_d) \in \Delta$. Since ξ_1, ξ_2 satisfy the inequalities (3.48), we know that $\xi_1 \in p^1_2(H_2 \cdot \xi_2)$. Thus there exists $h_2 \in H_2$ such that $\xi_1 = p^1_2(Ad^*_{h_2}\xi_2)$. Similarly, for each $2 \le k < d$, since ξ_k, ξ_{k+1} satisfy the inequalities (3.47), for each $h_k \in H_k$, there exists $h_{k+1} \in H_{k+1}$ such that $Ad^*_{h_k}\xi_k = p^k_{k+1}(Ad^*_{h_{k+1}}\xi_{k+1})$.

Continuing in this fashion, we find $h_d \in H_d$ such that $s \circ p^k_d(Ad^*_{h_d}\xi_d) = \xi_k$ for all $1 \le k < d$. Finally, since ξ_d satisfies the inequalities (3.48) there exists $m \in M$ such that $\Phi(m) = Ad^*_{h_d}\xi_d$ and it follows that $(\xi_1, \ldots, \xi_d) = F(m)$. $\square$

3.4.3 Smooth fibres and Gelfand-Zeitlin systems

If μ is the momentum map for a completely integrable Hamiltonian torus action, then the connected components of the fibres of μ are isotropic tori that are generically Lagrangian. In comparison, if a map F constructed by Thimm's trick with Guillemin and Sternberg's action coordinates generates a completely integrable torus action on the open dense subset $\mathcal{U}$ then very little is known about the fibres of F in the complement of $\mathcal{U}$. In this subsection we give a short proof that if the torus action generated by F is completely integrable, then the fibres in the complement of $\mathcal{U}$ are smooth embedded submanifolds.

Recall that a Hamiltonian G-manifold M is *multiplicity free* if the Poisson subalgebra of G-invariant functions, $C^\infty(M)^G$, is abelian.

Proposition 3.50. *[82, Proposition A.1] Let G be a compact, connected Lie group and let (M, ω, Φ) be a connected Hamiltonian G-manifold with Φ proper*[10]*. Then*

1. *M is multiplicity free if and only G acts transitively on $\Phi^{-1}(\mathcal{O}_\xi)$ for every $\xi \in \Phi(M)$.*
2. *If in addition G acts locally freely on a dense set then M is multiplicity free if and only if $\dim(M) = \dim(G) + rank(G)$.*

In particular, if G is a torus acting effectively on M with principal orbit-type (1), then M is multiplicity free if and only if $\dim(M) = 2\dim(G)$.

Proposition 3.51. *Let (M, ω, Φ) be a connected Hamiltonian G-manifold with Φ proper. If M admits a collective completely integrable system then it must be multiplicity free.*

Proof. This theorem is proven in [29, p. 223] under a cleanness assumption: the image $W = \Phi(M)$ is a submanifold of $\mathfrak{g}^*$ and the map $\Phi\colon M \to W$ is a submersion [29, p. 221].

If Φ is proper then by the non-abelian convexity theorem for proper momentum maps [55, Theorem 1.1], $\Box = \Phi(M) \cap \mathfrak{t}^*_+$ is convex. By Proposition 3.12, the $\Box^{\text{rel-int}}$ is contained in the principal stratum σ corresponding to M, so the set $G \cdot \Box^{\text{rel-int}}$ is a submanifold of $\mathfrak{g}^*$. It follows from the symplectic cross-section theorem that $\Phi^{-1}(G \cdot \Box^{\text{rel-int}})$ is an open dense subset of M and the restricted map $\Phi\colon \Phi^{-1}(G \cdot \Box^{\text{rel-int}}) \to G \cdot \Box^{\text{rel-int}}$ is a submersion. Thus $(\Phi^{-1}(G \cdot \Box^{\text{rel-int}}), \omega, \Phi)$ satisfies the cleanness assumption.

Applying [29, p. 223], it follows that $\Phi^{-1}(G \cdot \Box^{\text{rel-int}})$ is multiplicity free so by continuity, M is multiplicity free. ∎

Proposition 3.52. *Let (M, ω, Φ) be a connected Hamiltonian $U(n)$ or $SO(n)$-manifold with Φ proper and let F be a map constructed by Thimm's trick from a chain of subalgebras* (3.2)*. If the torus action generated by F on the open dense subset $\mathcal{U}$ is completely integrable, then the fibres of F are connected, embedded submanifolds.*

[10] In [82] it is assumed that M is compact. This assumption can be removed using the non-abelian convexity theorem for proper momentum maps [55, Theorem 1.1].

Proof. Let $G = U(n)$ or $SO(n)$ and let (M, ω, Φ) be a connected Hamiltonian G-manifold. Since Φ is proper and $\mathfrak{h}_n = \mathfrak{g}$, the map F is proper. Thus by Theorem 3.1, the fibres of F are connected.

Let

$$F = \left(s \circ p_n^1 \circ \Phi, \ldots, s \circ p_n^{n-1} \circ \Phi, s \circ \Phi\right) : M \longrightarrow \mathfrak{t}_1^* \oplus \cdots \oplus \mathfrak{t}_n^*, \tag{3.53}$$

be the map constructed by Thimm's trick with Guillemin and Sternberg's action coordinates from the chain (3.2) and fix $(\xi_1, \ldots, \xi_n) \in F(M)$. Let $\mathcal{O}_{\xi_n}$ be the G coadjoint orbit through ξ_n. The fibre $F^{-1}(\xi_1, \ldots, \xi_n)$ equals the preimage under Φ of the fiber $H^{-1}(\xi_1, \ldots, \xi_{n-1})$ the map

$$H = (s \circ p_n^1, \ldots, s \circ p_n^{n-1})\colon \mathcal{O}_{\xi_n} \to \mathfrak{t}_1^* \oplus \cdots \oplus \mathfrak{t}_{n-1}^* \tag{3.54}$$

(which is a classical Gelfand-Zeitlin system).

By Proposition 3.51, M is multiplicity free. By Proposition 3.50, G acts transitively on $\Phi^{-1}(\mathcal{O}_{\xi_n})$, so it is an embedded submanifold. Since Φ is G-equivariant, the restricted map $\Phi\colon \Phi^{-1}(\mathcal{O}_{\xi_n}) \to \mathcal{O}_{\xi_n}$ is a submersion. Thus if $H^{-1}(\xi_1, \ldots, \xi_{n-1})$ is an embedded submanifold of $\mathcal{O}_{\xi_n}$, it follows that $F^{-1}(\xi_1, \ldots, \xi_n)$ is an embedded submanifold of M.

It remains to show that the preimages $H^{-1}(\xi_1, \ldots, \xi_{n-1}) \subseteq \mathcal{O}_{\xi_n}$ are embedded submanifolds. Since every $U(n)$ coadjoint orbit is a multiplicity free $U(n-1)$-manifold[11] (respectively, every $SO(n)$ coadjoint orbit is a multiplicity free $SO(n-1)$-manifold), this follows inductively by applying the argument above. □

Remark 3.55. The geometry of the classical Gelfand-Zeitlin systems on $U(n)$ coadjoint orbits near these fibres have been studied by Eva Miranda and N.T. Zung [64] although no results have been published. A low dimensional example appears in [2]. The fibres of bending flow systems on moduli spaces of oriented polygons (see Example 3.75) were studied recently by Damien Bouloc, who showed that they are embedded coisotropic submanifolds and gave an explicit description of their geometry [12].

3.5 Examples

In this section we give several examples (mainly in low dimensions) to illustrate the construction and its applications.

[11]This was shown for generic $U(n)$ coadjoint orbits in [29]. For arbitrary $U(n)$ and $SO(n)$ coadjoint orbits, this follows from Proposition 3.51 and the fact that the classical Gelfand-Zeitlin systems are completely integrable (one can also give a direct proof using Proposition 3.50). This in turn follows from the fact that for any coadjoint orbit the T' action of the classical Gelfand-Zeitlin system is effective [70, Proposition 4.2.2.] and the dimension of T' is half the dimension of $\mathcal{O}$ [70, Proposition 4.3.7.].

Example 3.56 (Compact multiplicity free $SU(2)$ 4-manifolds). Let (M^4, ω) be a compact, connected, multiplicity free Hamiltonian $SU(2)$ 4-manifold with momentum map Φ. These spaces are classified by their momentum set and principal isotropy group [42][12].

Fix the maximal torus and positive Weyl chamber

$$\begin{aligned} T &= \left\{ \begin{pmatrix} e^{ix} & 0 \\ 0 & e^{-ix} \end{pmatrix} : x \in \mathbb{R} \right\}, \\ \mathfrak{t} &= \left\{ \begin{pmatrix} ix & 0 \\ 0 & -ix \end{pmatrix} : x \in \mathbb{R} \right\} \cong \mathbb{R},\ \mathfrak{t}_+ = \left\{ \begin{pmatrix} ix & 0 \\ 0 & -ix \end{pmatrix} : x \geq 0 \right\} \cong \mathbb{R}^+ \end{aligned} \tag{3.57}$$

and identify $\mathfrak{t} \cong \mathfrak{t}^*$ and $\mathfrak{t}_+ \cong \mathfrak{t}_+^*$ via the nondegenerate form $(X, Y) = \mathrm{tr}(XY)$. We can construct a Gelfand-Zeitlin system on M from the chain $\mathfrak{t} \subseteq \mathfrak{su}(2)$,

$$F = (p \circ \Phi, s \circ \Phi) : M \to \mathbb{R} \times \mathbb{R}^+ \subseteq \mathbb{R}^2 \tag{3.58}$$

where $p\colon \mathfrak{su}(2)^* \to \mathfrak{t}^*$ is the projection and $s\colon \mathfrak{su}(2)^* \to \mathfrak{t}_+^*$ is the sweeping map. The momentum set of M is an interval $s \circ \Phi(M) = [a, b] \subseteq \mathbb{R}^+$. By Proposition 3.49 and the interlacing inequalities, the image of the Gelfand-Zeitlin system on M is the set

$$F(M) = \left\{ (x, y) \in \mathbb{R}^2 \colon\ a \leq y \leq b \text{ and } -y \leq x \leq y \right\}. \tag{3.59}$$

If $a = 0$ then the principal isotropy group is trivial and M is isomorphic to $\mathbb{C}P^2$ with the action of $SU(2)$ as a subgroup of $SU(3)$ and $\Phi^{-1}(0)$ is an isolated fixed point for the $SU(2)$ action [42]. One can check that the map F is smooth on M and generates an effective, completely integrable torus action whose weight lattice is

$$L^* = \mathbb{Z} \left\langle \left(0, \frac{1}{\pi}\right), \left(\frac{1}{2\pi}, \frac{1}{2\pi}\right) \right\rangle. \tag{3.60}$$

With respect to the lattice structure, $F(M)$ is equivalent to the standard Delzant triangle of $\mathbb{C}P^2$.

If $a > 0$ then M is a Hirzebruch surface, the principal isotropy group is $\mathbb{Z}_m$ for some positive integer m, and M is symplectomorphic to a blow-up of $\mathbb{C}P^2$ if m is odd or $S^2 \times S^2$ if m is even [42]. The map F is smooth on M and one checks that it generates an effective, completely integrable torus action on M whose weight lattice is

$$L_m^* = \mathbb{Z} \left\langle \left(0, \frac{m}{\pi}\right), \left(\frac{1}{2\pi}, \frac{m}{2\pi}\right) \right\rangle \tag{3.61}$$

[12] These results are outlined in Section IV.5 of [5].

if m is odd and

$$L_m^* = \mathbb{Z}\left\langle \left(0, \frac{m}{2\pi}\right), \left(\frac{1}{\pi}, 0\right)\right\rangle. \tag{3.62}$$

if m is even. Thus one sees that for all m, $F(M)$ is equivalent to a standard Delzant polytope of a Hirzebruch surface.

Recall that following [25], the Gromov width of a symplectic manifold of dimension $2n$ is defined as

$$\mathrm{GWidth}(M, \omega) = \sup_{r>0} \left\{\pi r^2 \colon \exists \text{ a symplectic embedding } B^{2n}(r) \to M\right\} \tag{3.63}$$

where $B^{2n}(r)$ is the open ball of radius $r > 0$ in $\mathbb{R}^{2n}$ with the standard symplectic structure.

Example 3.64 (G_2 coadjoint orbits). There are four families of G_2 coadjoint orbits: the trivial orbit, the regular orbits, and the two one-parameter families of non-regular coadjoint orbits corresponding to maximal parabolic subgroups of G_2. Both of the non-regular coadjoint orbits admit Gelfand-Zeitlin systems; one can be viewed as a multiplicity free Hamiltonian $SU(3)$-manifold, for the action of the subgroup $SU(3) \leq G_2$ [82, Example 7.4], and the other is isomorphic to a $SO(8)$ coadjoint orbit. Proposition 3.49 was applied in [54] to prove strict lower bounds for Gromov width of the former coadjoint orbit. The lower bound for the Gromov width of the latter coadjoint orbit follows from [71].

Another classical example from the integrable systems literature is the construction of integrable systems on cotangent bundles of homogeneous spaces.

Example 3.65 (Cotangent bundles). Let G act on a manifold Q and consider the cotangent bundle $M = T^*Q$ with its canonical symplectic structure. The cotangent lift of the action of G is Hamiltonian. If the momentum map for this action is proper then we can apply Theorem 3.1 to a map F constructed by Thimm's trick from a list of subalgebras that contains $\mathfrak{g}$. In particular, if $Q = G/K$ is a homogeneous G-manifold, then Φ is proper and M is a multiplicity free G-manifold if and only if (G, K) is a Gelfand-pair [30]. If G is one of $U(n)$ or $SO(n)$, then one can construct a Gelfand-Zeitlin system on M [29].

One can construct many interesting examples of non-compact multiplicity free $U(n)$ or $SO(n)$-manifolds by applying non-abelian symplectic cutting to cotangent bundles, as the next example shows.

Example 3.66 (A non-compact $U(2)$-manifold with finite Gromov width). Consider the standard action of $U(2)$ on the unit sphere $S^3 \subseteq \mathbb{C}^2$. The cotangent lift of this action is a Hamiltonian action of $U(2)$ on T^*S^3. The momentum map for this action is proper and by Proposition 3.50(2) the action is multiplicity free. Fix the maximal torus

$$T = \left\{ \begin{pmatrix} e^{ix_1} & 0 \\ 0 & e^{ix_2} \end{pmatrix} : x_1, x_2 \in \mathbb{R} \right\} \tag{3.67}$$

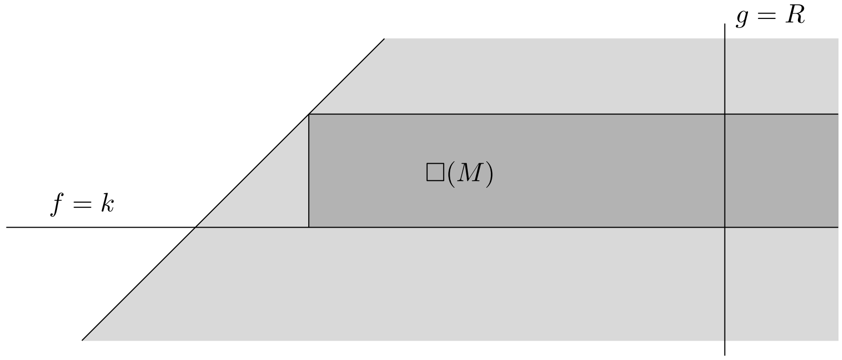

Figure 3.1: Momentum set of a noncompact multiplicity free $U(2)$-manifold.

and identifications $\mathfrak{t}^* \cong \mathfrak{t} \cong \mathbb{R}^2$, $\mathfrak{t}^*_+ \cong \{(x_1, x_2) \in \mathbb{R}^2 \colon\ x_1 \geq x_2\}$. The momentum set of T^*S^3 is

$$\square(T^*S^3) = \Phi(T^*S^3) \cap \mathfrak{t}^*_+ = \left\{(x_1, x_2) \in \mathbb{R}^2 \colon\ 0 \leq x_1 \text{ and } x_2 \leq 0\right\}. \tag{3.68}$$

Consider the functions

$$f, g \colon \mathfrak{t}^*_+ \to \mathbb{R},\ f(x_1, x_2) = -x_2,\ g(x_1, x_2) = x_1. \tag{3.69}$$

We can perform non-abelian symplectic cutting [82] with respect to the collective function $f \circ s$ at some level $k > 0$ to produce a non-compact, multiplicity free $U(2)$-manifold (M, ω) with momentum set

$$\square(M) = \left\{(x_1, x_2) \in \mathbb{R}^2 \colon\ 0 \leq x_1 \text{ and } -k \leq x_2 \leq 0\right\} \tag{3.70}$$

(see Figure 3.1). Identify $U(1)$ with the subgroup of diagonal matrices $\operatorname{diag}(e^{i\theta}, 1)$. The chain of subgroups $U(1) \leq U(2)$ equips M with a Gelfand-Zeitlin system

$$F = (p \circ \Phi, s \circ \Phi) \colon M \to \mathbb{R} \times \mathbb{R}^2. \tag{3.71}$$

By Proposition 3.49, the image is

$$F(M) = \left\{(x_0, x_1, x_2,) \in \mathbb{R}^3 \colon\ 0 \leq x_1,\ -k \leq x_2 \leq 0, \text{ and } x_2 \leq x_0 \leq x_1\right\}. \tag{3.72}$$

Since M is multiplicity free, the map F generates a completely integrable T^3-action on the open dense set $\mathcal{U} = M \setminus F^{-1}(0,0,0)$ (Theorem 3.51) and one can check that this action is effective. Combining this with Theorem 3.1, it follows that $(\mathcal{U}, \omega|_{\mathcal{U}}, F|_{\mathcal{U}})$ is a proper toric T^3-manifold. Note that with the identifications above, the weight lattice for T^3 is $\frac{1}{2\pi}\mathbb{Z}^3 \subseteq \mathbb{R}^3$.

Let $S \subseteq F(\mathcal{U})$ be the interior of the simplex with vertices $(0,0,0)$, $(0,0,-k)$, $(0,k,-k)$, and $(-k,0,k)$. Since $(\mathcal{U}, \omega|_{\mathcal{U}}, F|_{\mathcal{U}})$ is a proper toric T^3-manifold, the preimage $F^{-1}(S)$ inherits the structure of a proper toric T^3-manifold (in particular, by the convexity theorem for proper Hamiltonian torus manifolds [10, Theorem 30], the submanifold $F^{-1}(S)$ is connected). By the classification of proper toric manifolds, $F^{-1}(S)$ is isomorphic to the proper toric T^3-manifold

$$(S \times T^3, \omega_0 = dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + dx_3 \wedge dy_3, \mathrm{pr}_1).$$

By [71, Proposition 2.1], $2\pi k \leq \mathrm{GWidth}(S \times T^3, \omega_0)$ (in fact, this is an equality). It follows that

$$2\pi k \leq \mathrm{GWidth}(F^{-1}(S), \omega|_{F^{-1}(S)}) \leq \mathrm{GWidth}(M, \omega).$$

On the other hand, given a symplectic embedding of a ball $B^6(r)$ into M, for every $\rho < r$ there is a $R > 0$ such that the image of the closed ball $\overline{B}^6(\rho)$ is contained in the sublevel set $g \circ s \circ \Phi(x) < R$ (see Figure 3.1). Performing non-abelian symplectic cutting by the collective function $g \circ s$ at the level R, we obtain a multiplicity free Hamiltonian $U(2)$-manifold M_R with momentum set

$$\square(M_R) = \left\{(x_1, x_2) \in \mathbb{R}^2 \colon\ 0 \leq x_1 \leq R \text{ and } -k \leq x_2 \leq 0\right\}. \tag{3.73}$$

By the classification of convex multiplicity free manifolds [51, Theorem 11.2] (and since the principal isotropy group for the action of $U(2)$ on M_R is trivial), M_R is isomorphic as a multiplicity free Hamiltonian $U(2)$-manifold to the $U(3)$-coadjoint orbit $(\mathcal{O}_\lambda, \omega_\lambda)$ with the same momentum set (this coadjoint orbit can be identified with the set of Hermitian 3×3 matrices whose eigenvalues are $R, 0$, and $-k$). Thus

$$\pi r^2 \leq \mathrm{GWidth}(M_R, \omega) = \mathrm{GWidth}(\mathcal{O}_\lambda, \omega_\lambda) = \min\{2\pi k, 2\pi R\} \leq 2\pi k$$

where the last equality follows from the known upper bound for Gromov width of coadjoint orbits of compact Lie groups [13]. Thus $\mathrm{GWidth}(M, \omega) \leq 2\pi k$.

Therefore $\mathrm{GWidth}(M, \omega) = 2\pi k$. In contrast, the Hofer-Zehnder capacity of M is infinite: one can construct a sequence of admissible collective functions on M with unbounded oscillation (see [40] for definition of the Hofer-Zehnder capacity).

If a compact symplectic manifold admits a completely integrable Hamiltonian torus action, then it also admits an invariant Kähler structure [17]. As the following example demonstrates, this is not true for Gelfand-Zeitlin systems.

Example 3.74 (A G-Z system with no invariant Kähler structure). In [77], Tolman constructed a symplectic 6-manifold with a Hamiltonian T^2-action that has no invariant Kähler structure. Woodward showed that such examples can be obtained as non-abelian symplectic cuttings $M_{\leq a}$ of $U(3)$ coadjoint orbits, considered as Hamiltonian $U(2)$-manifolds; there is no invariant Kähler structure for the action of the maximal torus of $U(2)$ on $M_{\leq a}$ [83, Figure 3]. Being multiplicity free $U(2)$-manifolds, one can use the chain $U(1) \leq U(2)$ (as in the previous example) to construct a Gelfand-Zeitlin system on $M_{\leq a}$. This system generates an effective Hamiltonian T^3-action on an open dense subset of $M_{\leq a}$. The maximal torus of $U(2)$ acts on this open dense set as a subtorus of T^3, therefore there is no Kähler structure on $M_{\leq a}$ that is invariant under the T^3-action.

It was observed in [67, Remark 3.9] that the standard complex structure on a $U(n)$ coadjoint orbit is not invariant under the Gelfand-Zeitlin torus action.

Our last example demonstrates an application of Proposition 3.33 and a construction of a map by Thimm's trick that does not use a chain of subalgebras.

Example 3.75 (Bending flow systems on moduli spaces). Let H be a compact, connected Lie group and suppose that $M_1, \dots, M_n$ are Hamiltonian H-manifolds. The symplectic direct sum $M = M_1 \times \cdots \times M_n$ is a Hamiltonian $G = H \times \cdots \times H$-manifold. Let $\mathcal{P}$ be a convex n-gon with vertices labelled $1, \dots, n$ clockwise. Fix a triangulation $\mathcal{T}$ of $\mathcal{P}$. To the interior edges of $\mathcal{T}$, which we denote (i,j), $i < j$, we can associate subalgebras of the form

$$\mathfrak{h}_{i,j} = \{(0, \dots, 0, X, \dots, X, 0, \dots, 0) \in \mathfrak{g} \colon X \in \mathfrak{h}\} \cong \mathfrak{h} \tag{3.76}$$

where the n-tuple $(0, \dots, 0, X, \dots, X, 0, \dots, 0)$ is zero in the first $i-1$ entries, equal to $X \in \mathfrak{h}$ in entries i to j, and equal to zero in entries $j+1$ to n. If $i_1 \leq i_2 < j_2 \leq j_1$, then

$$[\mathfrak{h}_{i_1,j_1}, \mathfrak{h}_{i_2,j_2}] \subseteq \mathfrak{h}_{i_2,j_2}. \tag{3.77}$$

If $j_1 < i_2$, then

$$[\mathfrak{h}_{i_1,j_1}, \mathfrak{h}_{i_2,j_2}] = \{0\} \subseteq \mathfrak{h}_{i_2,j_2}. \tag{3.78}$$

Thus for any triangulation $\mathcal{T}$, the subalgebras $\mathfrak{h}_{i,j}$, $(i,j) \in \mathcal{T}$ pairwise satisfy condition 3.23. Further, if $\mathfrak{h}_{1,n}$ is the diagonal subalgebra, then $[\mathfrak{h}_{1,n}, \mathfrak{h}_{i,j}] \subseteq \mathfrak{h}_{i,j}$ for all $i < j$. Assuming properness, e.g. if M is compact, we can apply Proposition 3.33 to show that the map $\tilde{F}$ induced on the diagonally reduced space $M /\!\!/_0 H$ has convex image and connected fibres.

For example, if $M_k = S^2_{r_k}$ is the sphere of radius $r_k > 0$ and $H = SO(3)$, then this construction recovers the bending flow system on the moduli space of oriented polygons in $\mathbb{R}^3$, which can also be obtained as a reduction of the standard Gelfand-Zeitlin systems on coadjoint orbits [37, 45]. The fact that the open dense subsets where the bending flows generate Hamiltonian torus actions are proper toric manifolds was used in [60] to prove lower bounds on the Gromov width of these moduli spaces. The fibres of these systems have been studied by [12].

3.6 Thimm's trick and symplectic contraction

A connection between maps constructed by Thimm's trick with Guillemin and Sternberg's action coordinates and maps constructed by degeneration was recently explored in [38]. Given a Hamiltonian G-manifold (M, ω, Φ), Hilgert-Martens-Manon define a *symplectic contraction map* Φ_M which is a continuous, surjective, and proper map from M to a singular space M^{sc} called the *symplectic contraction of* M. The symplectic contraction map mimics the time-1 flow of the gradient-Hamiltonian vector field of a toric degeneration. The space M^{sc} comes equipped with a continuous map $\mu_{\mathbb{T}} \colon M^{sc} \to \mathfrak{t}^*$ such that the following diagram commutes

$$\begin{array}{ccc} M & \xrightarrow{\Phi_M} & M^{sc} \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\mu_{\mathbb{T}}} \\ \mathfrak{g}^* & \xrightarrow{s} & \mathfrak{t}^*. \end{array}$$

By construction, the space M^{sc} is stratified by smooth symplectic manifolds. If $\sigma \subseteq \mathfrak{t}^*_+$ is the principal stratum corresponding to M, then $\mu_{\mathbb{T}}^{-1}(\sigma)$ contains the open dense stratum of M^{sc} and the restriction of Φ_M to the intersection of $\Phi^{-1}(\Sigma_\sigma)$ with the principal orbit-type stratum of the action of G on M is a symplectomorphism onto this stratum [38, Proposition 4.3].

Given a chain of subgroups $H_1 \leq \cdots \leq H_d \leq G$ (or more generally, a chain of homomorphisms), Hilgert-Martens-Manon also define a *branching contraction map* Φ_M and *a branching contraction space* M^{sc} with similar properties as before: the map Φ_M is continuous, surjective, and proper and there is an open dense submanifold on which the restriction of Φ_M is a symplectomorphism [38, Proposition 7.16]. Further, there is a continuous map $\mu_{\mathbb{T}} \colon M^{sc} \to \mathfrak{t}_1^* \oplus \cdots \oplus \mathfrak{t}_d^*$ such that the following diagram commutes

$$\begin{array}{ccc} M & \xrightarrow{\Phi_M} & M^{sc} \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\mu_{\mathbb{T}}} \\ \mathfrak{g}^* & \xrightarrow{(s\circ p_1, \ldots, s\circ p_d)} & \mathfrak{t}_1^* \oplus \cdots \oplus \mathfrak{t}_d^* \end{array} \tag{3.79}$$

where the composition of Φ with $(s \circ p_1, \ldots, s \circ p_d)$ is precisely the map F constructed by Thimm's trick with Guillemin and Sternberg's action coordinates from the chain of subalgebras $\mathfrak{h}_1 \subseteq \cdots \subseteq \mathfrak{h}_d$ (3.24).

In this case, if for each $1 \leq k \leq d$, σ_k is the principal stratum of $\mathfrak{t}^*_{k,+}$ corresponding to the induced action of H_k on M, then $\mu_{\mathbb{T}}^{-1}(\sigma_1 \times \cdots \times \sigma_d)$ contains the open dense stratum of the branching contraction space M^{sc} and the restriction of Φ_M to the intersection of $\mathcal{U} = F^{-1}(\sigma_1 \times \cdots \times \sigma_d)$ with the principal orbit-type stratum for the action of G is a symplectomorphism onto this open dense stratum [38, Proposition 7.16]. Since the diagram (3.79) commutes and the contraction map Φ_M is continuous and surjective, we note that convexity and fibre connectedness for the map F (Theorem 3.1) is equivalent to convexity and fibre connectedness for the map $\mu_{\mathbb{T}}$ if either of these maps are proper (The fact that $F(M)$ is convex if and only if $\mu_{\mathbb{T}}(M^{sc})$ is convex follows immediately by surjectivity of Φ_M. The fact that the fibres of F are connected iff the fibres of $\mu_{\mathbb{T}}$ are connected follows by Lemma 3.28 and the fact that the restriction of Φ_M to the open dense intersection of $\mathcal{U}$ and the principal orbit type stratum is a homeomorphism onto its image).

Chapter 4

Lower bounds for the Gromov width of multiplicity free $U(n)$-manifolds

4.1 Introduction

Let G be a compact connected Lie group. Introduced in the late 1970's and early 1980's, multiplicity free Hamiltonian G-manifolds are the Hamiltonian G-manifolds that may admit collective completely integrable systems [29, 66, 30]. Similar to Delzant's classification of completely integrable Hamiltonian torus actions on compact connected symplectic manifolds, convex[1] multiplicity free G-manifolds (M, ω, μ) are classified by their invariant momentum image $\square = \mu(M) \cap \mathfrak{t}^*_+$ and the principal isotropy group for the action of G on M [50, 57]. For example, $U(n)$ and $SO(n)$ coadjoint orbits, which are multiplicity free with respect to the induced actions of $U(n-1)$ and $SO(n-1)$ subgroups respectively, and these coadjoint orbits have remarkable collective completely integrable systems called Gelfand-Zeitlin systems that admit action coordinates on an open dense subset [28].

Gromov width is an invariant of symplectic manifolds introduced by [25, 40] that is generally difficult to compute or estimate[2]. Remarkably, the Gromov width of coadjoint orbits of compact simple Lie groups was computed in a series of papers [47, 58, 85, 71, 32, 21, 13]. In particular, Pabiniak proved the first tight lower bounds for the Gromov width of non-regular coadjoint orbits of type A, B, and D by applying an embedding construction of [78, 73] to the completely integrable torus actions constructed by Thimm's trick on open dense subsets of $U(n)$ and $SO(n)$ coadjoint orbits [71].

[1]A multiplicity free Hamiltonian G manifold is *convex* if the set $\square$ is convex, and the invariant momentum map is proper as a map to $\square$ [50]. In particular, if the momentum map is proper, then the Hamiltonian G-manifold is convex [55].

[2]The book [74] provides a good survey of some embedding constructions and problems related to Gromov width.

This chapter extends the approach of [71] to prove lower bounds for the Gromov width of multiplicity free Hamiltonian $U(n)$-manifolds with proper momentum maps by studying the collective completely integrable torus actions constructed on open dense subsets of these manifolds by Thimm's trick with the action coordinates of Guillemin and Sternberg. We also call these integrable torus actions Gelfand-Zeitlin systems. For a multiplicity free Hamiltonian $U(n)$-manifold (M, ω, μ) with μ proper, we define a quantity $\mathcal{W}(\square, \Gamma_M)$ that depends only on the classifying data of [51] and prove the following theorem.

Theorem 4.1. *Let (M, ω, μ) be a connected multiplicity free Hamiltonian $U(n)$-manifold with μ proper. Then*

$$\mathcal{W}(\square, \Gamma_M) \leq \mathrm{GWidth}(M, \omega).$$

The proof of this theorem relies crucially on the results of Chapter 3 which show that

1. The open dense subset $\mathcal{U} \subseteq M$ where the collective completely integrable Hamiltonian torus action is defined is connected (Lemma 3.26), and the momentum map that generates the Hamiltonian torus action on $\mathcal{U}$ is proper as a map to a convex set (Theorem **??**). It follows from this that $\mathcal{U}$ is classified as a proper toric manifold by its image and a lattice in the dual of the Lie algebra of the torus.

2. The image of $\mathcal{U}$ is given explicitly by the inequalities that define $\square$ and the interlacing inequalities of the classic Gelfand-Zeitlin systems on coadjoint orbits (Proposition 3.49)

The theorem follows by combining these facts with a careful analysis of the kernel of the Gelfand-Zeitlin torus action on $\mathcal{U}$, the classification of completely integrable Hamiltonian torus actions with proper momentum map (see [46]), and a combinatorial fact about Gelfand-Zeitlin polytopes from [71].

The structure of this chapter is as follows. Section 4.2 reviews Gromov width, completely integrable Hamiltonian torus actions, and lower bounds for Gromov width obtained from momentum map images. Section 4.3 reviews multiplicity free spaces and Gelfand-Zeitlin systems on multiplicity free $U(n)$-manifolds. Section 4.4 contains the definition of the quantity $\mathcal{W}(\square, \Gamma_M)$, proof of Theorem 4.1, several examples, and a discussion of the results.

4.2 Lower bounds for the Gromov width of symplectic toric manifolds

In this section we review symplectic toric manifolds and lower bounds for their Gromov width.

4.2.1 Gromov width of symplectic manifolds

Consider $\mathbb{C}^n$ with coordinates $(z_1, \ldots, z_n)$, $z_k = x_k + iy_k$ and standard symplectic form $\omega_0 = \sum_{k=1}^n dx_k \wedge dy_k$. The *standard open ball of radius* r is

$$B^{2n}(r) = \left\{ z \in \mathbb{C}^n \colon |z_1|^2 + \cdots + |z_n|^2 < r^2 \right\}$$

together with the restriction of ω_0.

Definition 4.2. The *Gromov width* of a symplectic manifold (M, ω) of dimension $2n$ is

$$\mathrm{GWidth}(M, \omega) = \sup_{r>0} \left\{ \pi r^2 \colon \exists \text{ a symplectic embedding } B^{2n}(r) \to M \right\}.$$

Since symplectic embeddings preserve the Liouville volume, there is an upper bound for the Gromov width of a symplectic manifold sometimes referred to as the *volume obstruction*,

$$\mathrm{GWidth}(M, \omega)^n \leq \int_M \omega^n$$

That Gromov width is a non-trivial invariant of symplectic manifolds follows from Gromov's non-squeezing theorem [25].

In order to prove lower bounds for the Gromov width of a given symplectic manifold M, one must construct, or prove the existence of, symplectic embeddings of standard open balls into M. If M is equipped with a completely integrable Hamiltonian torus action then there several methods for constructing symplectic embeddings of $B^{2n}(r)$ into M. In this section we will focus on two: torus equivariant embeddings and Traynor's non-equivariant lower bounds. These two methods were notably employed in [71, 32, 58].

4.2.2 Hamiltonian T-manifolds and symplectic toric manifolds

Throughout this section, T is a compact torus with Lie algebra $\mathfrak{t}$ and $\mathfrak{t}^*$ is the dual of $\mathfrak{t}$.

Definition 4.3. A *Hamiltonian T-manifold* is a tuple (M, ω, μ) where (M, ω) is a connected symplectic manifold equipped with a Hamiltonian action of T, generated by a choice of momentum map

$$\mu \colon M \to \mathfrak{t}^*.$$

The following theorem was first stated and proven for M compact in [4, 27]. A proof of the theorem as stated below appears in [10] (Theorem 30, page 990). Many other authors have studied convexity for

Hamiltonian group actions on connected manifolds with proper momentum maps, including [50, 39, 14, 9, 55].

Theorem 4.4. *Let (M, ω, μ) be a Hamiltonian T-manifold. Suppose that μ is proper as a map to some convex subset of $\mathfrak{t}^*$. Then the image of μ is convex, the level sets are connected, and μ is open as a map to its image.*

For this reason, we refer to a Hamiltonian T-manifold such that the momentum map is proper as a map to a convex subset of $\mathfrak{t}^*$ as a *proper Hamiltonian T-manifold.*

We say that a Hamiltonian T-manifold is *completely integrable* if the components of the momentum map μ form a completely integrable system in the classical sense: they span a maximal abelian subalgebra of $C^\infty(M)$. If (M, ω, μ) is a completely integrable Hamiltonian T-manifold, then $\dim T \geq \frac{1}{2} \dim M$ and T acts transitively on the connected components of the fibers of μ.

In order to simplify the discussion of lower bounds for the Gromov width of completely integrable Hamiltonian T-manifolds, we begin by describing the special case where the action of T on M is effective[3]. Lower bounds for the Gromov width of non-effective completely integrable T-manifolds can then be described by reducing to the case that the action is effective. This reduction is described in section 4.2.4.

Definition 4.5. A *symplectic toric T-manifold* is a completely integrable Hamiltonian T-manifold (M, ω, μ) such that the action of T on M is effective.

If $L \subseteq \mathfrak{t}$ is the lattice $\ker(\exp\colon \mathfrak{t} \to T)$ then the weight lattice of T is identified with the dual lattice

$$L^* = \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z}) \subseteq \mathfrak{t}^*.$$

Example 4.6. Let $T = \mathbb{R}^n/\mathbb{Z}^n$ with Lie algebra $\mathbb{R}^n$. The lattice L is $\mathbb{Z}^n$. With respect to the standard dual identifications, $(\mathbb{R}^n)^* = \mathbb{R}^n$ and $(\mathbb{Z}^n)^* = \mathbb{Z}^n$, the map

$$\mu\colon B^{2n}(r) \to \mathbb{R}^n,\ \mu(z_1, \ldots, z_n) = \left(\pi|z_1|^2, \ldots, \pi|z_n|^2\right)$$

generates the standard representation of T on $\mathbb{C}^n$. The Hamiltonian T-manifold $(B^{2n}(r), \omega_0, \mu)$ is completely integrable and the action of T is effective. The image of this map is the *simplex of size* πr^2,

$$\Delta^n(\pi r^2) = \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n\colon\ x_1 + \cdots + x_n < \pi r^2 \text{ and } 0 \leq x_i \text{ for all } i = 1, \ldots, n\right\}. \tag{4.7}$$

[3] A group action is *effective* if the kernel of the action consists of the identity.

A set $S \subseteq \mathfrak{t}^*$ is *polyhedral* if it is the intersection of finitely many closed half-spaces. A set $S \subseteq \mathfrak{t}^*$ is *locally polyhedral* if for all $x \in S$, there is a neighbourhood U of x in $\mathfrak{t}^*$ and a polyhedral set $S' \subseteq \mathfrak{t}^*$ such that $S \cap U = S' \cap U$. A locally polyhedral set $S \subseteq \mathfrak{t}^*$ is *locally Delzant* if for all $x \in S$, there is an open neighbourhood U of $x \in \mathfrak{t}^*$, vectors $X_1, \ldots, X_k \in L$, and scalars $a_1, \ldots, a_k \in \mathbb{R}$ such that

- The set $\{X_1, \ldots, X_k\}$ extends to a $\mathbb{Z}$-basis for L, and
- $S \cap U = S' \cap U$, where S' is the convex polyhedral cone given as the intersection of closed half-spaces

$$S' = \bigcap_{i=1}^{k} \{y \in \mathfrak{t}^* \colon \langle y, X_i \rangle \geq a_i\}.$$

Let V and V' be vector spaces and suppose $L \subseteq V$ and $L' \subseteq V'$ are full rank lattices. A map $\psi \colon V \to V'$ of the form

$$\psi(\mathbf{v}) = A\mathbf{v} + \mathbf{b}$$

where $A \colon V \to V'$ is a linear map that sends any $\mathbb{Z}$-basis of L to a $\mathbb{Z}$-basis of L' and $\mathbf{b} \in V'$ is called an *integral affine map* from (V, L) to (V', L'). An integral affine map from (V, L) to itself is often called an *integral affine transformation*.

If (M, ω, μ) is a symplectic toric T-manifold and $\psi \colon (\mathfrak{t}^*, L^*) \to (\mathfrak{t}^*, L^*)$ is an integral affine transformation, then the composition $\psi \circ \mu$ is a moment map for the action of T on M given by

$$T \times M \xrightarrow{\overline{\psi} \times \mathrm{id}_M} T \times M \xrightarrow{(t,m) \mapsto t \cdot m} M$$

where $\overline{\psi}$ is the automorphism of T induced by the adjoint $A^* \colon \mathfrak{t} \to \mathfrak{t}$.

Two symplectic toric T-manifolds (M, ω, μ) and (M', ω', μ') are *equivalent* (or *isomorphic*) if there is a pair of maps: a symplectomorphism $\phi \colon M \to M'$ and an integral affine transformation $\psi \colon \mathfrak{t}^* \to \mathfrak{t}^*$ such that the following diagram commutes.

$$\begin{array}{ccc} M & \xrightarrow{\phi} & M' \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu'} \\ \mathfrak{t}^* & \xrightarrow{\psi} & \mathfrak{t}^* \end{array}$$

Note that since this diagram commutes, $\phi\left(\overline{\psi}(t) \cdot m\right) = t \cdot \phi(m)$.

The following classification was proven for M compact in [17]. The more general version for connected symplectic toric T-manifolds with proper momentum maps follows from [46].

Theorem 4.8. *Let T be a compact torus. The map sending a symplectic toric T-manifold to its momentum map image induces a bijection between the set of symplectic toric T-manifolds (M, ω, μ) such that μ is proper as a map to a convex subset of $\mathfrak{t}^*$ up to equivalence, and the set of convex locally Delzant sets $S \subseteq \mathfrak{t}^*$ up to integral affine transformations.*

4.2.3 Lower bounds for Gromov width from momentum map images

If (M, ω, μ) is a symplectic toric T-manifold and $S \subseteq \mu(M)$ is open with respect to the subspace topology on $\mu(M)$, then $\mu^{-1}(S)$ is an open T-invariant submanifold of M. Thus

$$\left(\mu^{-1}(S), \omega|_{\mu^{-1}(S)}, \mu|_{\mu^{-1}(S)}\right)$$

satisfies the definition a symplectic toric manifold, except that $\mu^{-1}(S)$ may not be connected. If μ is proper as a map to a convex subset of $\mathfrak{t}^*$ and S is convex, then by Theorem 4.4, $\mu^{-1}(S)$ is connected so $\left(\mu^{-1}(S), \omega|_{\mu^{-1}(S)}, \mu|_{\mu^{-1}(S)}\right)$ is a symplectic toric T-manifold. Further, in this case $\mu|_{\mu^{-1}(S)}$ is proper as a map to S, so by Theorem 4.8, the symplectic toric T-manifold $\left(\mu^{-1}(S), \omega|_{\mu^{-1}(S)}, \mu|_{\mu^{-1}(S)}\right)$ is classified up to equivalence by S.

Equivariant lower bounds

Suppose (M, ω, μ) is a symplectic toric T-manifold and μ is proper as a map to a convex subset of $\mathfrak{t}^*$. Let $\Delta^n(l)$ be the simplex of size l

$$\Delta^n(l) = \{(x_1, \dots, x_n) \in \mathbb{R}^n \colon x_1 + \dots + x_n < l \text{ and } 0 \leq x_i \text{ for all } i = 1, \dots, n\}.$$

Recall from Example 4.6 that $\Delta^n(l)$ is the momentum map image of the standard toric action on $B^{2n}(\sqrt{l/\pi})$.

If $S \subseteq \mu(M)$ is open with respect to the subspace topology on $\mu(M)$ and S is the image of $\Delta^n(l)$ under an integral affine map $\psi \colon (\mathbb{R}^n, \mathbb{Z}^n) \to (\mathfrak{t}^*, L^*)$, then the submanifold $\mu^{-1}(S)$ is a symplectic toric T-manifold equivalent to $B^{2n}\left(\sqrt{l/\pi}\right)$ with the toric structure described in Example 4.6. It follows that

$$l \leq \mathrm{GWidth}(M, \omega).$$

This lower bound is "equivariant" in the sense that the symplectic embedding of $B^{2n}\left(\sqrt{l/\pi}\right)$ into M intertwines the respective T-actions (up to an automorphism of T).

Suppose v_0 is a vertex of $\mu(M)$ with incident edges $E_1, \ldots, E_n$. Since $\mu(M)$ is locally Delzant, there exists primitive integral vectors $v_i - v_0 \in L^*$ that span the edges E_i and form a $\mathbb{Z}$-basis for L^*. The *integral affine length* of the edge E_i is the scalar l_i equal to the length of E_i divided by the length of the primitive integral vector $v_i - v_0$ (note that this quantity is independent of the inner product used to define length). If $l = \min_{1 \le i \le n}\{l_i\}$ is the minimum integral affine length of the edges E_i then, since $\mu(M)$ is convex, the convex hull

$$S = \operatorname{conv}\left(\{v_0, v_0 + \alpha(v_1 - v_0), \ldots, v_0 + \alpha(v_n - v_0) \colon 0 \le \alpha < l\}\right)$$

is an open subset of $\mu(M)$ equal to the image of $\Delta^n(l)$ under the integral affine map $\psi(\mathbf{x}) = A\mathbf{x} + v_0$, where A is a linear transformation sending the standard basis for $\mathbb{R}^n$ to the $\mathbb{Z}$-basis $\{v_1 - v_0, \ldots, v_n - v_0\}$. Note that conversely, any open subset of $\mu(M)$ equal to the image of $\Delta^n(l)$ under an integral affine map must be of this form.

Proposition 4.9. *Suppose that (M, ω, μ) is a symplectic toric T-manifold and μ is proper as a map to a convex subset of $\mathfrak{t}^*$. Then*

$$l \le \operatorname{GWidth}(M, \omega)$$

where

$$l = \sup_{v \in \mathcal{V}} \min_{1 \le i \le n} \{l_i(v)\}\,. \tag{4.10}$$

Here $\mathcal{V}$ is the set of vertices[4] *of $\mu(M)$ and for a given vertex v, $l_i(v)$ are the integral affine lengths of the edges of $\mu(M)$ incident to v.*

Example 4.11. Let $T = \mathbb{R}^2/\mathbb{Z}^2$ and let (M, ω, μ) be the symplectic toric T-manifold obtained from $(\mathbf{C}P^2, \frac{3}{\pi}\omega_{\mathrm{FS}})$ with the standard toric structure by three torus equivariant blow-ups of size 1. The image $\mu(M)$ is drawn in Figure 4.1. The lower bound obtained from Proposition 4.9 is

$$1 \le \operatorname{GWidth}(M, \omega).$$

By [62], $\operatorname{GWidth}(M, \omega) = 2$. Thus even for compact symplectic toric manifolds equivariant lower bounds for Gromov width may not be tight.

[4]Note that it is possible that $\mathcal{V}$ is empty. If M is compact then $\mu(M)$ is a polytope and $\mathcal{V}$ is nonempty.

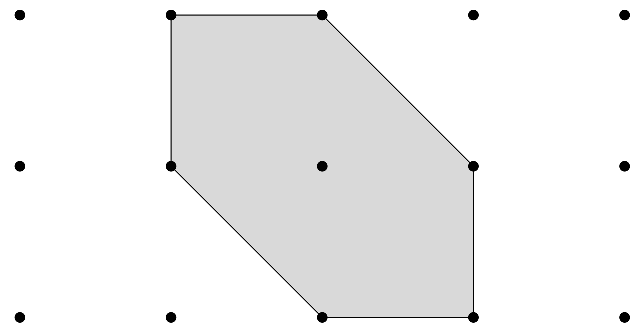

Figure 4.1: Delzant polytope of a three times T-equivariant blow-up of $\mathbb{C}P^2$.

Non-equivariant lower bounds

Suppose that T is a compact torus of dimension n, (M, ω, μ) is a symplectic toric T-manifold and μ is proper as a map to a convex subset of $\mathfrak{t}^*$. Let $\Delta^{n,\circ}(l)$ be the *open simplex of size* l,

$$\Delta^{n,\circ}(l) = \{(x_1, \dots, x_n) \in \mathbb{R}^n \colon x_1 + \cdots + x_n < l \text{ and } 0 < x_i \text{ for all } i = 1, \dots, n\}. \qquad (4.12)$$

If S is a subset of $\mu(M)$ equal to the image of $\Delta^{n,\circ}(l)$ under an integral affine map ψ, then S is open in $\mu(M)$ and by Theorems 4.4 and 4.8, $\mu^{-1}(S)$ is a symplectic toric T-manifold equivalent to

$$(\Delta^{n,\circ}(l) \times T, dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n, \mathrm{pr}_1).$$

Here x_i are standard coordinates on $\mathbb{R}^n$. T is identified with $\mathbb{R}^n/\mathbb{Z}^n$ and y_i are standard coordinates on $\mathbb{R}^n/\mathbb{Z}^n$. The action of T on $\Delta^{n,\circ}(l) \times T$ by translation in the second factor is generated by projection onto the first factor.

It follows from the volume obstruction that

$$\mathrm{GWidth}(\Delta^{n,\circ}(l) \times T, dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n) \leq l.$$

(this is easy to see since the volume of the simplex of size l is $l^n/n!$). In [78] it was proven for $n = 2$ that

$$l \leq \mathrm{GWidth}(\Delta^{n,\circ}(l) \times T, dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n).$$

A proof of this fact for general n appears in [73, 71]. As a consequence, one has the following Proposition, which was applied in [71, 58].

Proposition 4.13. *Let T be a compact torus of dimension n, let (M, ω, μ) be a symplectic toric T-*

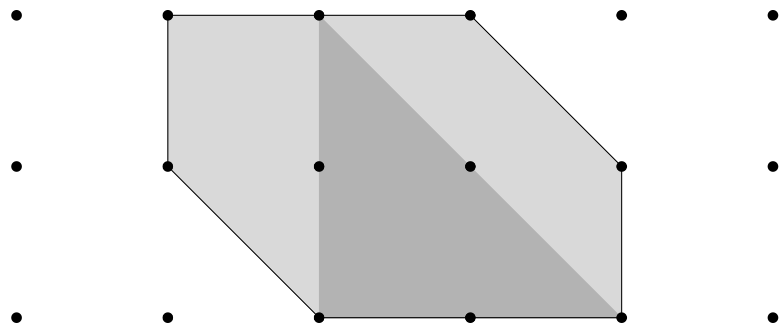

Figure 4.2

manifold, and suppose that μ is proper as a map to a convex subset of $\mathfrak{t}^*$. If there is a subset $S \subseteq \mu(M)$ equal to the image of the open simplex $\Delta^{n,\circ}(l)$ under an integral affine map, then

$$l \leq \mathrm{GWidth}(M, \omega).$$

Example 4.14. Let (M, ω, μ) be the symplectic toric manifold obtained from $(\mathbf{C}P^2, \frac{4}{\pi}\omega_{\mathrm{FS}})$ by performing three torus equivariant blow-ups: one of size 2 and two of size 1. The polytope $\mu(M)$ is illustrated in Figure 4.2. By [62], the Gromov width of M is 2. A tight lower bound is obtained by applying Proposition 4.13 to the shaded open simplex of size 2 in Figure 4.2. On the other hand, the lower bound obtained from Proposition 4.9 is 1. Thus we see that non-equivariant lower bounds may exceed equivariant lower bounds.

Example 4.15. The maximum size l such that $\Delta^{2,\circ}(l)$ can be integral affinely embedded in the polytope in Figure 4.1 is 3/2. On the other hand, the Gromov width of the symplectic toric manifold is 2 [62].

Remark 4.16. Lower and upper bounds for the Gromov width of symplectic toric manifolds were studied in [58], which gives equivalent descriptions of the lower bounds of Proposition 4.9 and 4.13, as well as combinatorial formulas for upper bounds. As the previous example demonstrates, these lower bounds may not be tight.

4.2.4 Completely integrable Hamiltonian T-manifolds

One can use the momentum map image of a completely integrable T-manifold to study its Gromov width as in the previous sections, provided one is careful to work with the correct lattice.

Let (M, ω, μ) be a completely integrable Hamiltonian T-manifold. Let K be the kernel of the action of T on M and let $\mathfrak{k}$ be its Lie algebra. We have the diagram

$$\begin{array}{ccc} \mathfrak{t} & \xrightarrow{p} & \mathfrak{t}/\mathfrak{k} \\ \big\downarrow{\scriptstyle\exp} & & \big\downarrow{\scriptstyle\exp} \\ T & \xrightarrow{p} & T/K \end{array}$$

Since the diagram commutes,

$$p\,(\ker(\exp\colon \mathfrak{t} \to T)) \subseteq \ker(\exp\colon \mathfrak{t}/\mathfrak{k} \to T/K)$$

with equality if and only if K is connected. Denote the lattice $\ker(\exp\colon \mathfrak{t}/\mathfrak{k} \to T/K)$ by $L_{T/K}$.

The action of T induces an effective action of T/K on M. Let $\iota\colon (\mathfrak{t}/\mathfrak{k})^* \to \mathfrak{t}^*$ be the linear map adjoint to p, which is an isomorphism onto its image. The image of ι is the annihilator of $\mathfrak{k}$,

$$\mathfrak{k}^0 = \{\xi \in \mathfrak{t}^*\colon \langle \xi, X\rangle = 0 \text{ for all } X \in \mathfrak{k}\}\,.$$

The image $\mu(M)$ spans an affine subspace of $\mathfrak{t}^*$ which is a translate of $\mathfrak{k}^0$. If $f\colon \mathfrak{t}^* \to \mathfrak{t}^*$ is a translation such that $\psi(\mu(M)) \subseteq \mathfrak{k}^0$, then the composition

$$M \xrightarrow{\mu} \mathfrak{t}^* \xrightarrow{f} \mathfrak{k}^0 \xrightarrow{\iota^{-1}} (\mathfrak{t}/\mathfrak{k})^* \tag{4.17}$$

is a momentum map for the action of T/K induced on M by the action of T. This action is effective, so $(M, \omega, \iota^{-1} \circ f \circ \mu)$ is a symplectic toric T/K-manifold. Observe that μ is proper as a map to a convex subset of $\mathfrak{t}^*$ if and only if $\iota^{-1} \circ f \circ \mu$ is proper as a map to a convex subset of $(\mathfrak{t}/\mathfrak{k})^*$.

Provided that μ is proper as a map to a convex subset of $\mathfrak{t}^*$, we can apply Proposition 4.13 to $(M, \omega, \iota^{-1} \circ f \circ \mu)$. Thus, one obtains lower bounds for $\mathrm{GWidth}(M, \omega)$ from subsets of $(\iota^{-1} \circ f \circ \mu)(M)$ that are the image of the simplexes $\Delta^{n,\circ}(l)$ under integral affine maps $\psi\colon (\mathbb{R}^n, \mathbb{Z}^n) \to ((\mathfrak{t}/\mathfrak{k})^*, L^*_{T/K})$ where $n = \dim T/K$.

Equivalently, one obtains lower bounds for $\mathrm{GWidth}(M, \omega)$ from subsets of $\mu(M)$ that are the image of simplexes $\Delta^{n,\circ}(l)$ under affine maps $\psi\colon \mathbb{R}^n \to \mathfrak{t}^*$ such that the vectors $\psi(e_i) - \psi(0)$ form a basis for the lattice $\iota(L^*_{T/K}) \subseteq \mathfrak{k}^0$. We record this as the following proposition.

Proposition 4.18. *Let (M, ω, μ) be a completely integrable Hamiltonian T-manifold, let K be the kernel of the action of T, and suppose that μ is proper as a map to a convex subset of $\mathfrak{t}^*$. If $\psi\colon \mathbb{R}^n \to \mathfrak{t}^*$ is an affine map such that*

1. The vectors $\psi(e_i) - \psi(0)$, $1 \leq i \leq n$, form a $\mathbb{Z}$-basis for the lattice $\iota(L^*_{T/K})$, and

2. The image of $\Delta^{n,\circ}(l)$ is an open subset of $\mu(M)$,

then

$$l \leq \mathrm{GWidth}(M, \omega).$$

Note that if K is connected then $\iota(L^*_{T/K}) = L^*_T \cap \mathfrak{k}^0$, so the condition that the vectors $\psi(e_i) - \psi(0)$, $1 \leq i \leq n$, form a $\mathbb{Z}$-basis for the lattice $\iota(L^*_{T/K})$ is equivalent to the condition that $\psi(e_i) - \psi(0)$, $1 \leq i \leq n$, can be extended to a $\mathbb{Z}$-basis for L^*_T.

4.3 Gelfand-Zeitlin systems

4.3.1 Multiplicity free Hamiltonian G-manifolds

Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$. Fix a maximum torus T and a Weyl chamber $\mathfrak{t}_+ \subseteq \mathfrak{t}$. Fix an invariant, nondegenerate inner product on $\mathfrak{g}$ and an isomorphism of $\mathfrak{g}$ with $\mathfrak{g}^*$. Identify the image of $\mathfrak{t}$ under this isomorphism with $\mathfrak{t}^*$, and denote the image of $\mathfrak{t}_+$ by $\mathfrak{t}^*_+$. The set $\mathfrak{t}^*_+$ is also called a *Weyl chamber*.

Let (M, ω) be a connected symplectic manifold and let G act on M in a Hamiltonian fashion, generated by a momentum map $\mu\colon M \to \mathfrak{g}^*$. The tuple (M, ω, μ) is a *Hamiltonian G-manifold*.

The Weyl chamber $\mathfrak{t}^*_+$ is a polyhedral cone and has a natural stratification by the relative interiors of the intersections of the faces of $\mathfrak{t}^*_+$. The stabilizer subgroups for the coadjoint action of G of any two points in a stratum σ of $\mathfrak{t}^*_+$ are equal and we denote this group by G_σ. The following is the "symplectic cross-section theorem," versions of which appear in [14, 39, 55].

Theorem 4.19. *Let (M, ω, μ) be a Hamiltonian G-manifold.*

1) There exists a unique stratum σ of the Weyl chamber $\mathfrak{t}^*_+$ with the property that $\mu(M) \cap \sigma$ is dense in $\mu(M) \cap \mathfrak{t}^*_+$.

2) The preimage $\mu^{-1}(\sigma)$ is a connected symplectic G_σ-invariant submanifold of M and the restriction of μ to $\mu^{-1}(\sigma)$ is a momentum map for the action of G_σ.

3) The set $G \cdot \mu^{-1}(\sigma)$ is open and dense in M.

4) All points in the intersection of $\mu^{-1}(\sigma)$ with the principal stratum $M_{prin} \subseteq M$ for the action of G have the same stabilizer subgroup, $G_0 \subseteq G$.

The unique stratum of $\mathfrak{t}^*_+$ defined in the cross-section theorem is called the *principal stratum*[5] corresponding to M and the pre-image $\mu^{-1}(\sigma)$ is called the *symplectic cross-section.* Note that σ may not be the maximal[6] stratum of $\mathfrak{t}^*_+$.

The following theorem was first proven for compact Hamiltonian G-manifolds in [49]. This version of the theorem was proven in [14, 39, 55].

Theorem 4.20. *Let (M, ω, μ) be a Hamiltonian G-manifold. If $\mu\colon M \to \mathfrak{g}^*$ is proper, then $\mu(M) \cap \mathfrak{t}^*_+$ is a convex, locally polyhedral set, and the fibres of μ are connected.*

Remark 4.21. Connectedness of the fibres of μ is equivalent to connectedness of preimages of orbits in $\mathfrak{g}^*$ (see [55], second paragraph on page 256).

The following definition originated with the study of collective integrable systems [29, 65].

Definition 4.22. A Hamiltonian G-manifold (M, ω, μ) is *multiplicity free* if the Poisson subalgebra $C^\infty(M)^G$ of smooth G-invariant functions is abelian.

The geometric meaning of this definition is given by the following proposition (a proof for M compact can be found in [82], Appendix A. One can weaken this assumption to M connected and μ proper using Theorem 4.20).

Proposition 4.23. *[82] Let (M, ω, μ) be a Hamiltonian G-manifold and suppose that μ is proper as a map to $\mathfrak{g}^*$.*

1) (M, ω, μ) is multiplicity free if and only if the stabilizer subgroup G_x acts transitively on the fiber $\mu^{-1}(x)$ for all $x \in \mu(M)$.

2) If in addition G acts locally freely on a dense set then M is multiplicity free if and only if $\dim(M) = \dim(G) + \operatorname{rank}(G)$.

Example 4.24. Coadjoint orbits of the Lie groups $U(n)$ and $SO(n)$ are multiplicity free manifolds for the induced actions of $U(n-1)$ and $SO(n-1)$ subgroups respectively, but the same is not true for $Sp(n)$ coadjoint orbits [29]. The cotangent bundle of a homogeneous space G/K is multiplicity free with respect to the action of G by the cotangent lift of the left or right action of G on G/K if and only if (G, K) is a "Gelfand pair" [30]. For example, if G/K is a compact symmetric space, then (G, K) is a Gelfand pair [29]. Given a multiplicity free G-manifold, new multiplicity free G-manifolds can be constructed using the equivariant symplectic surgery described in [83]. If G is a torus, then the definition of multiplicity free G-manifold coincides with the definition of a completely integrable Hamiltonian G-manifold.

[5]Some authors, such as [55], choose to call it the *principal open face.*
[6]With respect to the partial ordering $\tau \leq \sigma$ if $\tau \subseteq \overline{\sigma}$, the maximal stratum of $\mathfrak{t}^*_+$ is the relative interior of $\mathfrak{t}^*_+$.

By Proposition 4.23, if (M, ω, μ) is a multiplicity free Hamiltonian G-manifold with proper momentum map and principal stratum $\sigma \subseteq \mathfrak{t}^*_+$, then the group G_σ acts transitively on the fibers of μ restricted to the symplectic cross-section $\mu^{-1}(\sigma)$. Let Z_σ be the centre of G_σ and let Z^0_σ be the connected component of the identity of Z_σ. If G_0 is the principal isotropy group of Proposition 4.19, then G_σ/G_0 is abelian[7] ([29], page 222) and it follows that the homomorphism

$$Z_\sigma \to G_\sigma \to G_\sigma/G_0$$

is surjective onto the connected component of the identity of G_σ/G_0 and induces an isomorphism of the torus $Z^0_\sigma/(Z^0_\sigma \cap G_0)$ with the connected component of the identity of G_σ/G_0. Thus $Z^0_\sigma/(Z^0_\sigma \cap G_0)$ is a torus whose action on the symplectic cross-section $\mu^{-1}(\sigma)$ is effective and completely integrable.

Identify $\mathfrak{z}^*_\sigma$ with the subspace of $\mathfrak{t}^*$ spanned by σ (this is the image of $\mathfrak{z}_\sigma$ under the isomorphism if $\mathfrak{g}$ with $\mathfrak{g}^*$). The image under the map

$$\iota : (\mathfrak{z}_\sigma/(\mathfrak{z}_\sigma \cap \mathfrak{g}_0))^* \to \mathfrak{z}^*_\sigma \subseteq \mathfrak{t}^* \tag{4.25}$$

(dual to the projection $\mathfrak{z}_\sigma \to \mathfrak{z}_\sigma/(\mathfrak{z}_\sigma \cap \mathfrak{g}_0)$) of the weight lattice of the torus $Z^0_\sigma/(Z^0_\sigma \cap G_0)$ is a subgroup of L^*_T that we denote by Γ_M (this lattice is denoted by Λ_M on page 571 of [51]). This lattice is determined by the principal isotropy group of the action of G on M and plays a crucial role in the classification of Hamiltonian G-manifolds.

Theorem 4.26. *[51] Suppose (M, ω, μ) and (M', ω', μ') are two convex multiplicity free Hamiltonian G-manifolds. If $\mu(M) \cap \mathfrak{t}^*_+ = \mu'(M') \cap \mathfrak{t}^*_+$ and $\Gamma_M = \Gamma_{M'}$, then there is a symplectomorphism $\phi : M \to M'$ such that $\mu = \mu' \circ \phi$ (in other words, the Hamiltonian G-manifolds are isomorphic).*

Remark 4.27. A Hamiltonian G-manifold is convex if $s \circ \mu(M)$ is convex, the fibers of μ are connected, and $s \circ \mu$ is open as a map to its image [50]. In particular, if a Hamiltonian G-manifold has a proper momentum map, then it is convex.

Remark 4.28. The full classification theorem ([51], Theorem 11.2, page 597) includes an existence statement for multiplicity free Hamiltonian G-manifolds with proper momentum maps corresponding to pairs $(\Box, \Gamma)$ of locally polyhedral sets $\Box \subseteq \mathfrak{t}^*_+$ and subgroups $\Gamma \leq L^*$ that satisfy some local conditions.

The action of the torus Z^0_σ on the symplectic cross-section extends by G-equivariance to a new

[7] In [29], it is effectively shown that if $\dim \mathcal{O}_{\mu(m)}$ is maximal among coadjoint orbits contained in $\mu(M)$, then $G_{\mu(m)}/G_m$ is abelian, which is equivalent.

Hamiltonian action of Z_σ^0 on the open dense subset $G \cdot \mu^{-1}(\sigma)$ generated by the restriction of the map

$$s \circ \mu \colon M \xrightarrow{\mu} \mathfrak{g}^* \xrightarrow{s} \overline{\sigma} \subseteq \mathfrak{z}_\sigma^* \subseteq \mathfrak{t}^*$$

to the connected open dense subset $G \cdot \mu^{-1}(\sigma) \subseteq M$, where s is the *sweeping map* that sends an element of $\mathfrak{g}^*$ to the unique element in the intersection of its G-orbit with $\mathfrak{t}_+^*$ (see Equation (3.14) and Proposition 3.15). Since the kernel of the action of Z_σ^0 on the symplectic cross-section is $Z_\sigma^0 \cap G_0$, the kernel of the new action on $G \cdot \mu^{-1}(\sigma)$ defined by G-equivariance is also $Z_\sigma^0 \cap G_0$.

4.3.2 Gelfand-Zeitlin systems on multiplicity free $U(n)$-manifolds

In this subsection we introduce Gelfand-Zeitlin systems on multiplicity free $U(n)$-manifolds. Gelfand-Zeitlin systems on $U(n)$ and $SO(n)$ coadjoint orbits were introduced in [28] and studied extensively in [70, 71, 67]. Collective integrable systems on multiplicity free spaces were described in [29]. The topology of Gelfand-Zeitlin systems on general multiplicity free spaces was studied in Chapter 3.

Fix the following identifications (we follow most conventions of [71] with some minor differences).

- For $j \geq 1$, identify the Lie algebra $\mathfrak{u}(j)$ of skew-Hermitian matrices with the space $\mathcal{H}_j = \sqrt{-1}\mathfrak{u}(j)$ of Hermitian $j \times j$ matrices. Equip $\mathcal{H}_j$ with the non-degenerate bilinear form $(X, Y) = \operatorname{tr}(XY)$. Using this bilinear form, identify $\mathfrak{u}(j)^*$ with $\mathcal{H}_j$. Note that these identifications are equivariant with respect to the adjoint and coadjoint actions of $U(j)$ and the action of $U(j)$ on $\mathcal{H}_j$ by conjugation.

- Let T^j be the maximal torus of diagonal matrices in $U(j)$ with Lie algebra $\mathfrak{t}_j$. Both $\mathfrak{t}_j$ and $\mathfrak{t}_j^*$ are identified with the set of diagonal matrices in $\mathcal{H}_j$.

- With our identifications, the lattice

$$L_j = \ker\left(\exp \colon \mathfrak{t}_j \to T^j\right) = \{\operatorname{diag}(2\pi k_1, \dots, 2\pi k_j) : \, k_1, \dots, k_j \in \mathbb{Z}\} \subseteq \mathcal{H}_j$$

  and the dual lattice

$$L_j^* = \operatorname{Hom}_{\mathbb{Z}}(L_j, \mathbb{Z}) = \left\{ \operatorname{diag}\left(\frac{k_1}{2\pi}, \dots, \frac{k_j}{2\pi}\right) : \, k_1, \dots, k_j \in \mathbb{Z} \right\} \subseteq \mathcal{H}_j.$$

- Fix the identification of $\mathbb{R}^j/\mathbb{Z}^j$ and T^j induced by the isomorphism of $(\mathbb{R}^j, \mathbb{Z}^j)$ and $(\mathfrak{t}_j, L_j)$ given by

$$(y_1^j, \dots, y_j^j) \mapsto \operatorname{diag}\left(2\pi y_1^j, \dots, 2\pi y_j^j\right)$$

where $(y_1^j, \ldots, y_j^j)$ are coordinates on $\mathbb{R}^j$. This identification is not necessary, but it removes a pesky factor of 2π from the lattices L_j and L_j^* (it is also the convention made implicitly in [71]). This induces an identification of $(\mathbb{R}^j, \mathbb{Z}^j)$ and $(\mathfrak{t}_j^*, L_j^*)$ (after making the standard identification of $\mathbb{R}^j$ with its dual).

- Fix the Weyl chamber

$$\mathfrak{t}_{j,+}^* = \left\{ (y_1^j, \ldots, y_j^j) \in \mathbb{R}^j : y_1^j \geq \ldots \geq y_j^j \right\} \subseteq \mathbb{R}^j.$$

The sweeping map $s : \mathcal{H}_j \to \mathbb{R}^j$ sends a $j \times j$ Hermitian matrix X to the point $(y_1^j, \ldots, y_j^j)$ whose coordinates are the ordered eigenvalues of X times a factor of 2π. Coadjoint orbits of $U(j)$ are parameterized by points in $\mathfrak{t}_{j,+}^*$, but one must be careful: the orbit corresponding to the point $(y_1^j, \ldots, y_j^j) \in \mathfrak{t}_{j,+}^*$ is the set of matrices in $\mathcal{H}_j$ with eigenvalues

$$\frac{y_1^j}{2\pi}, \ldots, \frac{y_j^j}{2\pi}.$$

Thus the integral coadjoint orbits are parameterized by points in $\mathbb{Z}^j \cap \mathfrak{t}_{j,+}^*$.

- The strata of $\mathfrak{t}_{j,+}^*$ are the subsets σ_I (where I is a partition of j, $n_0 = 0, n_1, \ldots, n_{m-1}, n_m = j$) of points $(y_1^j, \ldots, y_j^j) \in \mathfrak{t}_{j,+}^*$ such that

$$y_1^j = \cdots = y_{n_1}^j > y_{n_1+1}^j = \cdots = y_{n_2}^j > \cdots > y_{n_{m-1}+1}^j = \cdots = y_j^j.$$

Let $k_i = n_i - n_{i-1}$. The stabilizer $G_{\sigma_I} \subseteq U(j)$ of the stratum σ_I is the subgroup of block diagonal unitary matrices

$$G_{\sigma_I} = U(k_1) \times \cdots \times U(k_m).$$

It follows that the dimension of a $U(j)$ coadjoint orbit $\mathcal{O}_\lambda$ parameterized by a point $\lambda = (\lambda_1, \ldots, \lambda_j) \in \sigma_I$ is

$$\dim(\mathcal{O}_\lambda) = j^2 - \sum_{i=1}^m k_i^2.$$

The centre of G_{σ_I} is the subgroup of block diagonal matrices

$$Z_{\sigma_I} = T^{k_1} \times \cdots \times T^{k_m}$$

where $T^{k_i} \subseteq U(k_i)$ is the maximal torus of diagonal matrices.

Let (M, ω, μ) be a multiplicity free Hamiltonian $U(n)$-manifold with principal stratum σ_n and let $\Sigma_n = U(n) \cdot \sigma_n$. For each $1 \leq j < n$, consider $U(j)$ as the subgroup of $U(n)$ consisting of block diagonal matrices of the form

$$\begin{pmatrix} g & 0 \\ 0 & I_{n-j} \end{pmatrix}, \, g \in U(j).$$

With this identification, $U(j)$ has induced Hamiltonian actions on M and $\mathcal{H}_n$ generated by the maps

$$M \xrightarrow{\mu} \mathcal{H}_n \xrightarrow{p_n^j} \mathcal{H}_j$$

and

$$\mathcal{H}_n \xrightarrow{p_n^j} \mathcal{H}_j$$

respectively. With respect to the identifications above, p_n^j is the map that sends a $n \times n$ Hermitian matrix X to its upper-left $j \times j$ submatrix, $X^{(j)}$. Note that the map μ is equivariant with respect to these $U(j)$ actions. From the interlacing inequalities it can be observed that

- The principal stratum of $\mathfrak{t}_{j,+}^*$ for the action of $U(j)$ on any $\mathcal{O}_\lambda \subseteq \Sigma_n$ depends only on σ_n, and we denote this stratum by σ_j. Let $I_j = \left\{ n_0^j = 0, n_1^j, \ldots, n_{m_j}^j = j \right\}$ be the partition of j corresponding to the stratum σ_j and let $k_i^j = n_i^j - n_{i-1}^j$.
- σ_j equals the principal stratum of $\mathfrak{t}_{j,+}^*$ corresponding to the action of $U(j)$ on M.

We introduce some new notation.

- Let $\Sigma_j = U(j) \cdot \sigma_j$ and let $V_j = (p_n^j)^{-1}(\Sigma_j) \cap \Sigma_n$.
- For each $1 \leq j < n$, let $Z_j = Z_{\sigma_j}$ and let $H^j = H_1^j \times \cdots \times H_{k_j}^j$ be the subtorus of Z_j defined by

$$H_i^j = \begin{cases} 1 & \text{if } k_i^j > 1 \\ T^{k_i^j} & \text{if } k_i^j = 1 \end{cases}$$

  The Lie algebra of H^j is the coordinate subspace

$$\mathfrak{h}^j = \left\{ (y_1^j, \ldots, y_j^j) \in \mathbb{R}^j : y_i^j = 0 \text{ if } i \notin I_{j+1} \right\} \subseteq \mathbb{R}^j.$$

  Identify the dual $(\mathfrak{h}^j)^*$ with $\mathfrak{h}^j$. Let $P^j : \mathbb{R}^j \to \mathfrak{h}^j$ be projection onto the coordinate subspace $\mathfrak{h}^j$.

Define the map

$$\Lambda = \left(s \circ p_n^{n-1}, \ldots, s \circ p_n^1 \right) : \mathcal{H}_n \to \mathbb{R}^{n-1} \oplus \cdots \oplus \mathbb{R}^1.$$

Let

$$P = \left(P^{n-1}, \ldots, P^1\right) : \mathbb{R}^{n-1} \oplus \cdots \oplus \mathbb{R}^1 \to \mathfrak{h}^{n-1} \oplus \cdots \oplus \mathfrak{h}^1.$$

Define

$$\mathcal{V} = \left(\bigcap_{j=1}^{n-1} V_j\right) \cap \Sigma_n = \Lambda^{-1}(\sigma_{n-1} \times \cdots \times \sigma_1) \cap \Sigma_n.$$

Let $T' = H^{n-1} \times \cdots \times H^1$. With our identifications, the weight lattice of T' is $L' = \mathfrak{t}' \cap \left(\mathbb{Z}^{n-1} \times \cdots \times \mathbb{Z}^1\right)$.

The construction of Thimm's trick with the action coordinates of Guillemin and Sternberg says that the restriction of $P \circ \Lambda$ to $\mathcal{O} \cap \mathcal{V} \subseteq \Sigma_n$ induces a Hamiltonian action of the torus T' (see Chapter 3, Proposition 3.22). Note that since $\sigma_j \subseteq \mathfrak{t}^*_{j,+}$ are the principal strata corresponding to the orbits $\mathcal{O} \subseteq \Sigma_n$ as Hamiltonian $U(j)$-manifolds, the intersection $\mathcal{V} \cap \mathcal{O}$ is an open dense subset of $\mathcal{O}$.

Proposition 4.29. *[70] For any orbit $\mathcal{O} \subseteq \Sigma_n$, the action of T' on $\mathcal{O} \cap \mathcal{V}$ generated by $P \circ \Lambda$ is effective and completely integrable.*

Proof. The fact that the T'-action is effective follows from Proposition 4.2.2. in [70]. By Proposition 4.3.7. of [70], the dimension of T' is half the dimension of $\mathcal{O}_\lambda$. Since the action is effective, it follows that it is completely integrable. □

A point $\lambda \in \mathfrak{t}_{n,+}$ defines a Gelfand-Zeitlin polytope in $\mathbb{R}^{n-1} \oplus \cdots \oplus \mathbb{R}^1$, which we denote by Δ_λ, defined by the interlacing inequalities:

$$\begin{array}{ccccccccccccc}
\lambda_1 & & & & \lambda_2 & & & & \lambda_3 & \cdots & \lambda_{n-1} & & \lambda_n \\
& \geq & & \leq & & \geq & & \leq & & & \geq & \leq & \\
& & \lambda_1^{(n-1)} & & & & \lambda_2^{(n-1)} & & & \cdots & & \lambda_{n-1}^{(n-1)} & \\
& & & \geq & & \leq & & \geq & & & \leq & & \\
& & & & \lambda_1^{(n-2)} & & & & \ddots & & \iddots & & \\
& & & & & \geq & & & & & & & \\
& & & & & & \ddots & & \iddots & & & & \\
& & & & & & \geq & \leq & & & & & \\
& & & & & & & \lambda_1^{(1)} & & & & &
\end{array} \tag{4.30}$$

Proposition 4.31. *[70, 28] For all $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathfrak{t}^*_{n,+}$ the image $\Lambda(\mathcal{O}_\lambda)$ equals the Gelfand-Zeitlin polytope Δ_λ.*

For $\lambda \in \sigma_n$, the polytope $\Lambda(\mathcal{O}_\lambda)$ spans an affine subspace of $\mathbb{R}^{n-1} \times \cdots \times \mathbb{R}^1$ parallel to $\mathfrak{t}'$. In what follows we will consider the projection of $\Lambda(\mathcal{O}_\lambda)$ to $\mathfrak{t}'$.

Define the map

$$F = (s \circ \mu, P \circ \Lambda \circ \mu) : M \to \mathbb{R}^n \times \mathfrak{t}'$$

and let

$$\mathcal{U} = F^{-1}(\sigma_n \times \cdots \times \sigma_1) = \mu^{-1}(\mathcal{V}).$$

The map F is continuous and $F|_{\mathcal{U}}$ generates a Hamiltonian action of the torus $Z_n \times T'$ on the open dense set $\mathcal{U}$, where Z_n is the centre of the stabilizer subgroup of σ_n (see Chapter 3, Proposition 3.22). We denote the image $F(M)$ by Δ.

Proposition 4.32. *Let (M, ω, μ) be a multiplicity free Hamiltonian $U(n)$-manifold such that $\mu\colon M \to \mathfrak{u}(n)^*$ is proper. Then*

1) The action of $Z_n \times T'$ on $\mathcal{U}$ is completely integrable.

2) The kernel of the action of $Z_n \times T'$ on $\mathcal{U}$ is $(Z_n \cap G_0) \times \{1\}$.

3) $(\mathcal{U}, \omega|_{\mathcal{U}}, F|_{\mathcal{U}})$ is a proper Hamiltonian $Z_n \times T'$-manifold.

4) The image $\Delta = F(M)$ is the subset of $\mathbb{R}^n \times \mathfrak{t}'$ defined by the inequalities that define the momentum set $\square \subseteq \mathbb{R}^n$ and the interlacing inequalities. Moreover, $\Delta^{\text{rel-int}} \subseteq F(\mathcal{U})$.

Proof. 1) By Proposition 4.29, for any coadjoint orbit $\mathcal{O} \subseteq \Sigma_n$, the restricted map $\Lambda|_{\mathcal{O} \cap \mathcal{V}}$ defines a completely integrable torus action on the open dense set $\mathcal{O} \cap \mathcal{V} \subseteq \mathcal{O}$. The map $s\colon \Sigma_n \to \sigma_n$ is a submersion and the level sets are the coadjoint orbits $\mathcal{O} \subseteq \Sigma_n$. Thus the coordinates of the maps s and Λ satisfy the assumptions of Proposition B.5 on $\mathcal{V}$. Since M is a multiplicity free $U(n)$ manifold with proper momentum map, it follows that the coordinates of the map $F|_{\mathcal{U}} = (s \circ \mu, \Lambda \circ \mu)|_{\mathcal{U}}$ define a completely integrable system on the open dense set $\mathcal{U} = \mu^{-1}(\mathcal{V})$. Thus the action of $Z_n \times T'$ on $\mathcal{U}$ is completely integrable (see the discussion in [29] beginning on pages 224-225 and Appendix B).

2) The map $\mu : \mathcal{U} \to \mathcal{V}$ is equivariant with respect to the Z_n and T' actions defined above. Thus if $(t, t') \in Z_n \times T'$ such that $(t, t') \cdot m = m$, then $(t, t') \cdot \mu(m) = \mu(m)$. The action of Z_n on $\mathcal{V}$ generated by s is trivial (since s is constant on coadjoint orbits), so this implies that $t' \cdot \mu(m) = \mu(m)$. By Proposition 4.29 the action of T' on $\mathcal{V}$ is effective, so $t' = 1$ and t is contained in stabilizer of m for the action of Z_n on $\mathcal{U}$, which is $Z_n \cap G_0$.

3) By Lemma 3.26, $\mathcal{U}$ is connected. Since μ is proper, s is proper, and $P \circ \Lambda$ is continuous, the map $F = (s \circ \mu, P \circ \Lambda \circ \mu)$ is proper. The set $\mathcal{U} = F^{-1}(\sigma_n \times \cdots \times \sigma_1)$, so the restriction of $F|_{\mathcal{U}}$ is proper as a map to the convex set $\sigma_n \times \cdots \times \sigma_1$.

4) This follows by Proposition 3.49. In particular, $\Delta^{\text{rel-int}} \subseteq \sigma_1 \times \cdots \times \sigma_n$, so $\Delta^{\text{rel-int}} \subseteq F(\mathcal{U})$.

$\square$

The integral affine geometry of the polytopes $P(\Delta_\lambda))$ was studied by [71] who proved the following (see Lemmas 4.1 and 4.2 in [71]).

Lemma 4.33. *[71] Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \sigma_n$ and let $N = \dim(\mathfrak{t}')$. There exists an integral affine map*

$$\psi' \colon (\mathbb{R}^N, \mathbb{Z}^N) \to (\mathfrak{t}', L')$$

such that

1) $\psi'(\Delta^{N,\circ}(l)) \subseteq P(\Lambda(\mathcal{O}_\lambda))^{\text{int}}$,

2) The nonzero coordinates of $\psi'(0)$ in the coordinate subspace $\mathfrak{t}'$ are $y_i^j = \lambda_{n+i-j}$.

4.4 Lower bounds for the Gromov width of multiplicity free $U(n)$-manifolds

Let (M, ω, μ) be a multiplicity free Hamiltonian $U(n)$-manifold with proper momentum map. Recall the following notation and identifications from section 4.3:

- $\mathbb{R}^n$ is identified with the Lie algebra of the maximal torus of diagonal matrices in $U(n)$ and $\mathfrak{t}^*_{n,+}$ is the Weyl chamber
$$\mathfrak{t}^*_{n,+} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \colon x_1 \geq \cdots \geq x_n\}.$$
- σ_n is the principal stratum of $\mathfrak{t}^*_{n,+}$ corresponding to (M, ω, μ) as a Hamiltonian $U(n)$-manifold and $\Sigma_n = U(n) \cdot \sigma_n$.
- $\square = \mu(M) \cap \mathfrak{t}^*_+$ and k is the dimension of the affine subspace spanned by $\square$. This is the dimension of the torus $Z_n/(Z_n \cap G_0)$ that acts effectively on the symplectic cross-section $\mu^{-1}(\sigma_n)$.
- Γ_M is the subgroup of $\mathbb{Z}^n$ determined by the principal isotropy group of the action of $U(n)$ on M.
- $\check{R}$ is the set of coroots of $U(n)$.

In order to bound the Gromov width of (M, ω) we introduce the following definition. Suppose $\square \subseteq \mathfrak{t}^*_{n,+}$ is a convex, locally polyhedral set of dimension k whose relative interior is contained in a stratum $\sigma \subseteq \mathfrak{t}^*_{n,+}$. Let Γ be a subgroup of L^*_T that spans the subspace of $\mathfrak{t}^*_n$ parallel to the affine subspace spanned by $\square$.

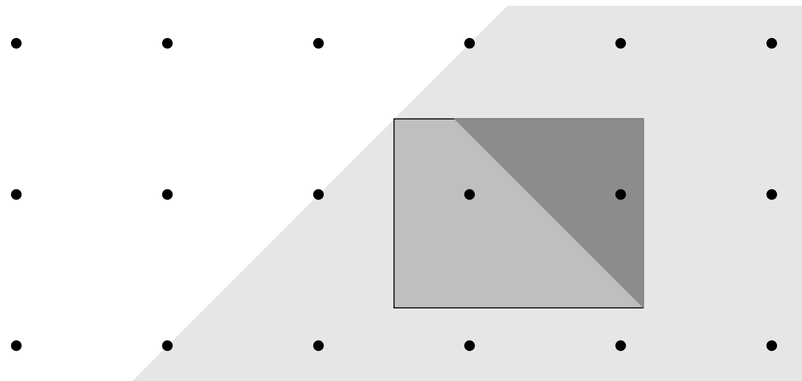

Figure 4.3: The polytope $\square$ of a regular $U(3)$ coadjoint orbit considered as a Hamiltonian $U(2)$-manifold, superimposed on the weight lattice of $U(2)$. The lightly shaded half-space is the principal stratum σ.

Definition 4.34. Define the quantity $\mathcal{W}(\square, \Gamma)$ as the supremum of $l > 0$ such that:

1. There an affine map
$$\psi\colon \mathbb{R}^k \to \mathbb{R}^n$$
such that the vectors $\psi(e_i) - \psi(0)$, $1 \leq i \leq k$, form a basis for Γ (where $e_1, \ldots, e_k$ are the standard basis vectors in $\mathbb{R}^k$) and the image of $\Delta^{k,\circ}(l)$ is contained in $\square \cap \sigma$.

2. $\psi(0) \in \sigma$ and l satisfies the inequality
$$l \leq \min_{\check{\alpha} \in \check{R}} \left\{ \langle \psi(0), \check{\alpha} \rangle \colon \langle \psi(0), \check{\alpha} \rangle > 0 \right\}.$$

Remark 4.35. The quantity $\mathcal{W}(\square, \Gamma)$ is determined by three factors: the shape of $\square$, the lattice Γ, and the position of $\square$ in $\mathfrak{t}^*_{n,+}$ relative to the walls[8] of $\mathfrak{t}^*_{n,+}$ that don't contain σ. For $x \in \sigma$, the quantity
$$\min_{\check{\alpha} \in \check{R}} \left\{ \langle x, \check{\alpha} \rangle \colon \langle x, \check{\alpha} \rangle > 0 \right\}$$
can be interpreted as the smallest of the affine distances (with respect to the the lattice $\mathbb{Z}^n$) from x to the walls of $\mathfrak{t}^*_{n,+}$ that do not contain σ (for a definition of the affine distance from a point to an affine hyperplane, see [61], page 6). If (M, ω, μ) is a multiplicity free Hamiltonian $U(n)$-manifold with proper momentum map and $\square = \mu(M) \cap \mathfrak{t}^*_+$, then the quantity $\mathcal{W}(\square, \Gamma_M)$ depends on $\square$ and Γ_M.

In this section we prove the main theorem.

Theorem 4.36. *Let (M, ω, μ) be a multiplicity free Hamiltonian $U(n)$-manifold with proper momentum*

[8]Faces; codimension one strata of the polyhedral set $\mathfrak{t}^*_{n,+}$.

map. Then

$$\mathcal{W}(\Box, \Gamma_M) \leq \mathrm{GWidth}(M, \omega).$$

The outline of the proof is as follows. Given an affine map ψ embedding a simplex $\Delta^{k,\circ}(l)$ into $\Box \cap \sigma$ as in Definition 4.34, one can construct a new affine map $\overline{\psi}$ embedding $\Delta^{N+k,\circ}(l)$ into $\Delta = F(M)$ and prove that $\overline{\psi}$ respects the lattice $\Gamma_M \times L'$ (see Figure 4.4). This is the content of the following proposition.

Proposition 4.37. *If $\psi \colon \mathbb{R}^k \to \mathbb{R}^n$ is an affine map such that*

1. *The vectors $\psi(e_i) - \psi(0)$, $1 \leq i \leq k$, form a $\mathbb{Z}$-basis for Γ_M,*

2. *$\psi(\Delta^{k,\circ}(l))$ is contained in $\Box \cap \sigma$, and*

3. *$\psi(0) \in \sigma$ and*

$$l \leq \min_{\check{\alpha} \in \check{R}} \{\langle \psi(0), \check{\alpha} \rangle \colon \langle \psi(0), \check{\alpha} \rangle > 0\}$$

then, there exists an affine map $\overline{\psi} \colon \mathbb{R}^k \times \mathbb{R}^N \to \mathbb{R}^n \times \mathfrak{t}'$ such that

1. *The vectors $\overline{\psi}(e_i) - \overline{\psi}(0)$, $1 \leq i \leq N + k$, form a $\mathbb{Z}$-basis for the lattice*

$$\Gamma_M \times L'$$

2. *$\overline{\psi}(\Delta^{N+k,\circ}(l))$ is an open subset of $\Delta^{\text{rel-int}}$.*

Remark 4.38. The definition of $\mathcal{W}(\Box, \Gamma)$ does not depend in any special way on the group $U(n)$. One can write an analogous definition of $\mathcal{W}(\Box, \Gamma_M)$ for $\Box = \mu(M) \cap \mathfrak{t}^*_+$ where (M, ω, μ) is a multiplicity free Hamiltonian G-manifold for any compact connected Lie group G. An obvious question to ask is whether Theorem 4.36 has a corresponding generalization. If G is a torus, then a multiplicity free G-manifold is a completely integrable Hamiltonian G-manifold and $\Box = \mu(M)$. In this case Proposition 4.18 can be interpreted as the statement $\mathcal{W}(\Box, \Gamma_M) \leq \mathrm{GWidth}(M, \omega)$ (in this case condition (2) in the definition of $\mathcal{W}(\Box, \Gamma_M)$ is trivial).

The definition of $\mathcal{W}(\Box, \Gamma_M)$ is illustrated by the following examples.

Example 4.39. Let $\mathcal{O}_\lambda$ be the regular $U(3)$ coadjoint orbit parameterized by $(\lambda_1, \lambda_2, \lambda_3) \in \mathfrak{t}^*_{3,+}$, $\lambda_1 > \lambda_2 > \lambda_3$, and consider $\mathcal{O}_\lambda$ as a multiplicity free Hamiltonian $U(2)$-manifold. The set

$$\Box = \left\{(x_1, x_2) \in \mathbb{R}^2 \colon \lambda_1 \geq x_1 \geq \lambda_2 \geq x_2 \geq \lambda_3\right\} \subseteq \mathbb{R}^2$$

is determined by the interlacing inequalities (4.30) (see Figure 4.3). The principal stratum

$$\sigma = \left\{(x_1, x_2) \in \mathbb{R}^2 : \ x_1 > x_2\right\} \subseteq \mathfrak{t}^*_{2,+}.$$

By Proposition 4.29, the lattice $\Gamma_M = \mathbb{Z}^2$. It is clear from Figure 4.3 that

$$\mathcal{W}(\square, \Gamma_M) = \min\{\lambda_1 - \lambda_2, \lambda_2 - \lambda_3\}$$

and this equals the known Gromov width of $(\mathcal{O}_\lambda, \omega_\lambda)$ [13, 71].

By definition, $\mathcal{W}(\square, \Gamma_M)$ is a lower bound for the Gromov width of the symplectic cross-section $\mu^{-1}(\sigma)$ (cf. Proposition 4.18). As the next example demonstrates, the Gromov width of the symplectic cross-section may be much larger than the Gromov width of M. This example also demonstrates the importance of condition (2) in Definition 4.34.

Example 4.40. Let $\chi\colon U(2) \to U(1)$ be the character $\chi(g) = \det(g)$ and let $U(2)$ act on $\mathbb{C}$ via χ. Let $\mathcal{O}_\lambda$ be the $U(2)$ coadjoint orbit parameterized by $(\lambda_1, \lambda_2) \in \mathfrak{t}^*_{2,+}$, $\lambda_1 > \lambda_2$.

Let M be the symplectic manifold $(\mathcal{O}_\lambda \times \mathbb{C}, \omega_\lambda + dx \wedge dy)$ and let $U(2)$ act on M by the coadjoint action on $\mathcal{O}_\lambda$ and the character χ on $\mathbb{C}$. This action is Hamiltonian, generated by the momentum map

$$\mu(X, z) = X + \frac{|z|^2}{2}.$$

It follows from Proposition 4.23 that M is a multiplicity free Hamiltonian $U(2)$-manifold. It is also easy to see that

$$\square = \left\{(\lambda_2 + x, \lambda_1 + x) \in \mathbb{R}^2 : \ x \geq 0\right\}$$

(see Figure 4.5). The action of the maximal torus of diagonal matrices $T^2 \subseteq U(2)$ on M generated by $s \circ \mu$ has kernel $K = \left\{\mathrm{diag}(e^{i\theta}, e^{-i\theta})\right\}$ and

$$\Gamma_M = \{(k, k) : \ k \in \mathbb{Z}\} = \mathbb{Z}^2 \cap \mathfrak{k}^0.$$

The quantity

$$\mathcal{W}(\square, \Gamma_M) = \lambda_2 - \lambda_1$$

which coincides with the Gromov width of $(\mathcal{O}_\lambda \times \mathbb{C}, \omega_\lambda + dx \wedge dy)$ (note that this quantity equals the affine distance from any point in $\square$ to the wall of $\mathfrak{t}^*_{2,+}$, cf. Remark 4.35). In contrast to this, the symplectic cross-section $\mu^{-1}(\sigma)$ is isomorphic as a symplectic manifold to $(\mathbb{C}, dx \wedge dy)$ which has infinite Gromov

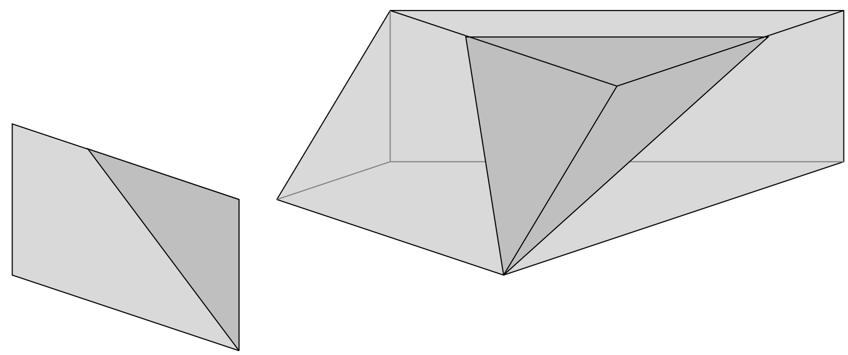

Figure 4.4: The projection $\mathrm{pr}_1 \colon \Delta \to \square$. A simplex embedded in $\square$ can be lifted to a larger dimensional simplex in Δ.

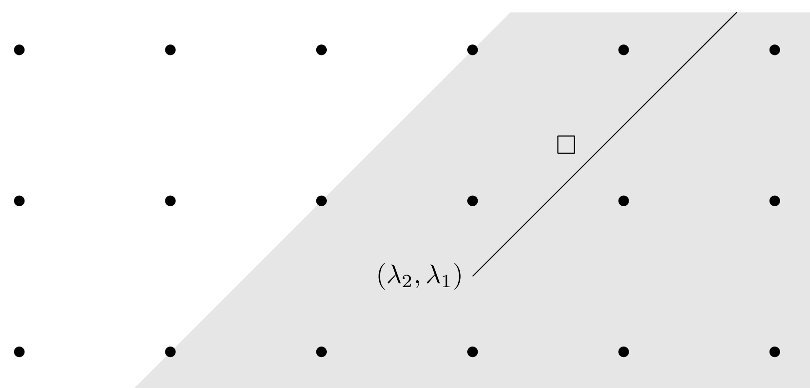

Figure 4.5: The polytope $\square$ of the $U(2)$ action on $\mathcal{O}_\lambda \times \mathbb{C}$ in Example 4.40.

width.

4.4.1 Proof of Proposition 4.37 and Theorem 4.36

Fix a multiplicity free $U(n)$-manifold (M, ω, μ) with μ proper. As before, fix coordinates $(x_1, \ldots, x_n) \in \mathbb{R}^n$ and $(y_1^j, \ldots y_j^j) \in \mathbb{R}^j$ for $1 \leq j < n$.

Suppose there exists an affine map $\psi \colon \mathbb{R}^k \to \mathbb{R}^n$ such that

1. The vectors $\psi(e_i) - \psi(0)$, $1 \leq i \leq k$, form a $\mathbb{Z}$-basis for Γ_M,

2. $\psi(\Delta^{k,\circ}(l))$ is contained in $\square \cap \sigma_n$, and

3. $\psi(0) \in \sigma_n$ and

$$l \leq \min_{\check{\alpha} \in \check{R}} \left\{ \langle \psi(0), \check{\alpha} \rangle \colon \langle \psi(0), \check{\alpha} \rangle > 0 \right\}.$$

Let $N = \dim \Lambda(\mathcal{O}_{\psi(0)}) = \dim(\mathfrak{t}')$. By Lemma 4.33, there exists an integral affine map $\psi' \colon (\mathbb{R}^N, \mathbb{Z}^N) \to (\mathfrak{t}', L')$ such that

(a) $\psi'(\Delta^{N,\circ}(l)) \subseteq P\left(\Lambda(\mathcal{O}_{\psi(0)})\right)^{\mathrm{int}}$

(b) If $\psi(0) = (x_1, \ldots, x_n)$, then

$$\psi'(0) = P\left(\sum_{j=1}^{n-1}\sum_{i=2}^{j} x_{n+i-j} e_i^j\right)$$

Define the linear map

$$q\colon \mathbb{R}^n \to \mathfrak{t}'$$

$$q(\mathbf{x}) = P\left(\sum_{j=1}^{n-1}\sum_{i=2}^{j} x_{n+i-j} e_i^j\right)$$

where e_i^j are the standard basis vectors for $\mathbb{R}^j$, and P is the orthogonal projection

$$P = (P^{n-1}, \ldots, P^1)\colon \mathbb{R}^{n-1} \times \cdots \times \mathbb{R}^1 \to \mathfrak{h}^{n-1} \oplus \cdots \oplus \mathfrak{h}^1 = \mathfrak{t}'.$$

Note that by Proposition 4.31, $q(\mathbf{x}) \in P(\Lambda(\mathcal{O}_{\mathbf{x}}))$. Define the affine map

$$\overline{\psi}\colon \mathbb{R}^k \times \mathbb{R}^N \to \mathbb{R}^n \times \mathfrak{t}'$$

$$\overline{\psi}(\mathbf{z}, \mathbf{w}) = \left(\psi(\mathbf{z}), q(\psi(\mathbf{z})) + \psi'(\mathbf{w}) - \psi'(0)\right).$$

Note that by (b), $q(\psi(0)) = \psi'(0)$.

Lemma 4.41. *The vectors $\overline{\psi}(e_i) - \overline{\psi}(0)$ form a $\mathbb{Z}$-basis for the lattice*

$$\Gamma_M \times L'.$$

Proof.

- For $1 \leq i \leq k$,

$$\begin{aligned}\overline{\psi}(e_i) - \overline{\psi}(0) &= (\psi(e_i), q(\psi(e_i)) + \psi'(0) - \psi'(0)) - (\psi(0), q(\psi(0)) + \psi'(0) - \psi'(0)) \\ &= (\psi(e_i) - \psi(0), q(\psi(e_i) - \psi(0)))\end{aligned}$$

  By definition of ψ, the vectors $\psi(e_i) - \psi(0)$ form a $\mathbb{Z}$-basis for Γ_M. Since Γ_M is a subgroup of $\mathbb{Z}^n$, the coordinates of the vectors $\psi(e_i) - \psi(0)$ are integral. By definition of the map q the vectors $q(\psi(e_i) - \psi(0))$ are integral and contained in $\mathfrak{t}'$, so they are contained in the lattice $L' = \mathfrak{t}' \cap (\mathbb{Z}^{n-1} \times \cdot \times \mathbb{Z}^1)$.

- For $k < i \leq N + k$,

$$\begin{aligned}\overline{\psi}(e_i) - \overline{\psi}(0) &= (\psi(0), q(\psi(0)) + \psi'(e_i) - \psi'(0)) - (\psi(0), q(\psi(0)) + \psi'(0) - \psi'(0)) \\ &= (0, \psi'(e_i) - \psi'(0)))\,.\end{aligned}$$

  By definition of ψ', the vectors $\psi'(e_i) - \psi'(0)$ form a $\mathbb{Z}$-basis for L'.

It follows that the vectors $\overline{\psi}(e_i) - \overline{\psi}(0)$ form a $\mathbb{Z}$-basis for the lattice $\Gamma_M \times L'$. □

We complete the proof of Proposition 4.37 with the following lemma.

Lemma 4.42. *$\overline{\psi}(\Delta^{N+k,\circ}(l))$ is an open subset of $\Delta^{\text{rel-int}}$.*

Proof. Recall that by Proposition 4.32, the image $\Delta = F(M)$ is the subset of $\mathbb{R}^n \times \mathbb{R}^N$ defined by the inequalities that define $\square$ and the interlacing inequalities (4.30).

- The point

$$\overline{\psi}(0,0) = (\psi(0), \psi'(0))$$

  is contained in the closure of Δ since $\psi(0)$ is contained in the closure of $\square$ and $\psi'(0) \in P(\Lambda(\mathcal{O}_{\psi(0)}))$ (which is defined by the interlacing inequalities).

- For $1 \leq i \leq k$, the points

$$\overline{\psi}(le_i, 0) = (\psi(le_i), q(\psi(le_i)))$$

  are contained in the closure of Δ since $\psi(le_i)$ is contained in the closure of $\square$ and by definition of the map q, $q(\psi(le_i))$ is contained in $P(\Lambda(\mathcal{O}_{\psi(le_i)}))$.

- For $k < i \leq N + k$, the points

$$\overline{\psi}(0, le_i) = (\psi(0), \psi'(le_i))$$

  are contained in the closure of Δ since $\psi(0)$ is contained in the closure of $\square$ and by definition of ψ', $\psi'(le_i) \in P(\Lambda(\mathcal{O}_{\psi(0)}))$.

Thus the convex hull of these points is contained in the closure of Δ. The relative interior of the convex hull equals $\overline{\psi}(\Delta^{N+k,\circ}(l))$ and since these sets both have dimension $N + k$, this is an open subset of $\Delta^{\text{rel-int}}$. □

Proof of Theorem 4.36

Let (M, ω, μ) be a multiplicity free $U(n)$-manifold with $\mu \colon M \to \mathfrak{g}^*$ proper. Let $\square = \mu(M) \cap \mathfrak{t}^*_+$ and let $\Delta = F(M) \subseteq \mathbb{R}^n \times \mathfrak{t}'$.

Let $\psi\colon \mathbb{R}^k \to \mathbb{R}^n$ be an affine map that satisfies the assumptions of Definition 4.34. By Proposition 4.37, there is an affine map $\overline{\psi}\colon \mathbb{R}^k \times \mathbb{R}^N \to \mathbb{R}^n \times \mathfrak{t}'$ such that

1. The vectors $\overline{\psi}(e_i) - \overline{\psi}(0)$, $1 \leq i \leq N+k$, form a $\mathbb{Z}$-basis for the lattice

$$\Gamma_M \times L'$$

2. $\overline{\psi}(\Delta^{N+k,\circ}(l))$ is an open subset of $\Delta^{\text{rel-int}}$ which is contained in $F(\mathcal{U})$.

By Proposition 4.32, $(\mathcal{U}, \omega|_{\mathcal{U}}, F|_{\mathcal{U}})$ is a proper, completely integrable Hamiltonian $Z_n \times T'$-manifold and the kernel of the action of $Z_n \times T'$ on $\mathcal{U}$ is $(Z_n \cap G_0) \times \{1\}$. The image of the weight lattice of the torus $Z_n/(Z_n \cap G_0) \times T'$ under the map

$$\iota\colon\ (\mathfrak{z}_n/(\mathfrak{z}_n \cap \mathfrak{g}_0))^* \oplus \mathfrak{t}' \to \mathfrak{z}_n^* \times \mathfrak{t}' \subseteq \mathbb{R}^n \times \mathfrak{t}'$$

is $\Gamma_M \times L'$. Thus $\overline{\psi}$ and $(\mathcal{U}, \omega|_{\mathcal{U}}, F|_{\mathcal{U}})$ satisfy the assumptions of Proposition 4.18 so

$$l \leq \mathrm{GWidth}(\mathcal{U}, \omega|_{\mathcal{U}}) \leq \mathrm{GWidth}(M, \omega).$$

It follows that

$$\mathcal{W}(\square, \Gamma_M) \leq \mathrm{GWidth}(M, \omega).$$

4.4.2 Remarks on the quantity $\mathcal{W}(\square, \Gamma_M)$

Let (M, ω, μ) be a multiplicity free Hamiltonian $U(n)$-manifold with μ proper and let $\square = \mu(M) \cap \mathfrak{t}^*_{n,+}$. The original hope in defining the quantity $\mathcal{W}(\square, \Gamma_M)$ was that $\mathcal{W}(\square, \Gamma_M)$ might equal the size of the largest simplex $\Delta^{N+k,\circ}(l)$ that can be embedded into $\Delta^{\text{rel-int}}$ as in Proposition 4.18. In other words, the hope was that $\mathcal{W}(\square, \Gamma_M)$ would be the largest lower bound for the Gromov width of (M, ω) that can be obtained from the Gelfand-Zeitlin system $(s \circ \mu, \Lambda \circ \mu)$. This is not the case, as the following example demonstrates.

Example 4.43. Let $\mathcal{O}_\lambda$ be the $U(4)$ coadjoint orbit parameterized by $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathfrak{t}^*_{4,+}$, $\lambda_1 = \lambda_2 > \lambda_3 = \lambda_4$ and consider $\mathcal{O}_\lambda$ as a multiplicity free Hamiltonian $U(3)$-manifold. The set $\square = \mu(\mathcal{O}_\lambda) \cap \mathfrak{t}^*_{3,+}$ is

$$\square = \left\{(\lambda_2, x_2, \lambda_3) \in \mathbb{R}^3\colon\ \lambda_2 \geq x_2 \geq \lambda_3\right\}.$$

The principal stratum of $\mathfrak{t}^*_{3,+}$ corresponding to $\mathcal{O}_\lambda$ is

$$\sigma_3 = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \colon x_1 > x_2 > x_3\right\}$$

and the lattice

$$\Gamma_M = \{(0, k, 0) \colon k \in \mathbb{Z}\}.$$

If $\mathbf{x} = (\lambda_2, x_2, \lambda_3)$ is a point contained in $\sigma_3 \cap \square$, i.e. $\lambda_2 > x_2 > \lambda_3$, then

$$\min_{\check{\alpha} \in \check{R}} \{\langle \mathbf{x}, \check{\alpha} \rangle \colon \langle \mathbf{x}, \check{\alpha} \rangle > 0\} = \min\{\lambda_2 - x_2, x_2 - \lambda_3\} \leq \frac{\lambda_2 - \lambda_3}{2}.$$

Thus

$$\mathcal{W}(\square, \Gamma_{\mathcal{O}_\lambda}) \leq \frac{\lambda_2 - \lambda_3}{2}$$

(in fact this is an equality). On the other hand, by Proposition 4.33, there is an integral affine embedding of $\Delta^{4,\circ}(\lambda_2 - \lambda_3)$ into $\Delta^{\text{rel-int}}$. Thus we also see that the lower bound $\mathcal{W}(\square, \Gamma_M) \leq \mathrm{GWidth}(M, \omega)$ may not be tight, even for $U(n+1)$ coadjoint orbits.

Question 4.44. *Is there a quantity that depends only on $(\square, \Gamma_M)$ that equals the size of the largest open simplex that can be embedded into $\Delta^{\text{rel-int}}$?*

4.4.3 Gromov width of multiplicity free Hamiltonian G-manifolds

As noted in Remark 4.38, the quantity $\mathcal{W}(\square, \Gamma_M)$ can be defined for any multiplicity free Hamiltonian G-manifold for any compact connected group G, and this quantity depends only on the classifying data $(\square, \Gamma_M)$. With regards to proving the inequality

$$\mathcal{W}(\square, \Gamma_M) \leq \mathrm{GWidth}(M, \omega)$$

for more general G, we make the following comments.

- If G is a torus the inequality reduces to Proposition 4.13. As observed in Example 4.15, these lower bounds are not tight even for some compact symplectic toric manifolds.

- With a little bit of work, the argument used in this paper for multiplicity free $U(n)$-manifolds should also apply to multiplicity free $SO(n)$-manifolds, since these spaces also have Gelfand-Zeitlin systems and the integral affine geometry of Gelfand-Zeitlin polytopes for $SO(n)$ coadjoint orbits was also studied in [71] (giving an analogue of Lemma 4.33 for those polytopes that could be used in an identical fashion).

- However, the Gelfand-Zeitlin construction of completely integrable torus actions on $\mathfrak{g}^*$, and thus also multiplicity free G-manifolds, using chains of subgroups does not work in the obvious way for groups other than $U(n)$ and $SO(n)$ (cf. [29], page 225). Harada was able to extend the construction to $\mathfrak{sp}(n)^*$ [33] and it may be interesting to study the topology of these systems.

- Recently, toric degeneration has proven a valuable tool for constructing completely integrable torus actions on open dense subsets of projective varieties [34]. In particular, using toric degenerations integrable torus actions can be constructed on integral coadjoint orbits and these actions were used in [21] to finish the proof of precise values of the Gromov width of coadjoint orbits of simple Lie groups for arbitrary Lie type. However, it is not clear if toric degeneration can be adapted to the setting of multiplicity free G-manifolds: multiplicity free G manifolds need not be projective, and it is even possible that they may not be Kähler [83]. Furthermore, from the perspective of constructing collective integrable systems it is not sufficient to construct integrable systems on individual coadjoint orbits; one needs a completely integrable system on (an open dense subset of) the Poisson manifold $\Sigma = G \cdot \sigma$.

Appendix A

Fibers of maps constructed by toric degeneration are connected

In this appendix we prove that the fibers of a continuous map constructed on a smooth projective variety from toric degeneration as in [34] are connected. Although such a map is not smooth on the entire projective variety, it is smooth and generates a completely integrable torus action on an open dense subset[1]. This allows us to apply Lemma 3.28 to conclude that all the fibers of the map are connected (even fibers where the map is not smooth). The contents of this section originally appeared in version 3 of the arXiv preprint [52] but were removed from the subsequent version.

Let T be a compact torus with real dimension n. A *toric degeneration* of a smooth projective variety X of complex dimension n to a projective toric variety (X_0, ω, μ_0, T) of the same dimension is a flat family of irreducible projective varieties $\pi\colon \mathfrak{X} \to \mathbb{C}$ such that the fiber $\pi^{-1}(0) = X_0$ and the restriction of π over $\mathbb{C}^*$ has a trivialization, i.e. there is an isomorphism

$$\rho\colon X \times \mathbb{C}^* \to \mathfrak{X} \setminus X_0,\ \pi(\rho(x,t)) = t.$$

Given a toric degeneration of X to X_0 that satisfies several conditions, there is a continuous, surjective map $\phi\colon X \to X_0$ [34]. This map is defined as the continuous extension of the time-1 flow of the *gradient-Hamiltonian* vector field

$$\frac{X_{\mathrm{Im}(\pi)}}{||X_{\mathrm{Im}(\pi)}||^2}. \tag{A.1}$$

The map ϕ is smooth on a dense, open submanifold $U \subseteq X$, and the restriction of ϕ to this submanifold

[1] See [34] for the list of assumptions that make this true.

is a symplectomorphism onto its image [34]. Let $\mu_0\colon X_0 \to \mathfrak{t}^*$ be the momentum map that generates the action of T on X_0. The image $\Delta = \mu_0(X_0)$ is a rational polytope. Under further conditions on the toric degeneration, the composition

$$\Lambda\colon X \xrightarrow{\phi} X_0 \xrightarrow{\mu_0} \mathfrak{t}^*$$

defines a completely integrable system on X (i.e. the components of Λ Poisson commute on U and are linearly independent on a dense, open subset of U). Since the map ϕ is surjective, $\Lambda(X) = \Delta$. Further, if π has no critical points on the smooth locus of X_0, then the restriction of the map Λ to the submanifold U generates a Hamiltonian torus action and $\mu_0^{-1}(\Delta^{\text{int}}) \subseteq \phi(U)$ [34]. Applying Lemma 3.28 we can prove the following fact, which did not appear in [34].

Proposition A.2. *If $\Lambda = \mu_0 \circ \phi$ is a momentum map defined on a smooth projective variety X by a toric degeneration, as above, then the fibers of Λ are connected.*

Proof. We want to apply Lemma 3.28 to the continuous map $\Lambda\colon X \to \mathfrak{t}^*$.

- Since X is compact, the map Λ is proper.

- Consider the saturated subset $N = \Lambda^{-1}(\Delta^{\text{int}})$. Since $\mu_0^{-1}(\Delta^{\text{int}})$ is dense in X_0, it is dense in $\phi(U)$ which contains $\mu_0^{-1}(\Delta^{\text{int}})$. Since $\phi|_U$ is a diffeomorphism onto its image, it follows that $N = \phi^{-1}(\mu_0^{-1}(\Delta^{\text{int}}))$ is dense in U which in turn is dense in X. Thus N is dense in X.

- The image $\Lambda(N) = \Delta^{\text{int}}$ is convex as it is the interior of a convex polytope[2].

- The map $\mu_0\colon \mu_0^{-1}(\Delta^{\text{int}}) \to \Delta^{\text{int}}$ is a momentum map for the Hamiltonian action of T on the connected symplectic manifold $\mu_0^{-1}(\Delta^{\text{int}})$ and $\mu_0|_{\mu_0^{-1}(\Delta^{\text{int}})}$ is proper as a map to the convex set Δ^{int}. Thus by the convexity theorem for proper Hamiltonian torus actions [10] the map $\mu_0\colon \mu_0^{-1}(\Delta^{\text{int}}) \to \Delta^{\text{int}}$ is open and has connected fibers.

  It follows that the fibers of $\Lambda|_N$ are connected since the fibers of $\mu_0\colon \mu_0^{-1}(\Delta^{\text{int}}) \to \Delta^{\text{int}}$ are connected and $\phi|_U$ is a diffeomorphism onto its image (which contains $\mu_0^{-1}(\Delta^{\text{int}})$).

- Finally, $\Lambda|_N\colon \Lambda(N) \to \Delta^{\text{int}}$ is open as a map to Δ^{int} with the subspace topology. The map $\phi\colon N \to \phi(N) = \mu_0^{-1}(\Delta^{\text{int}})$ is the restriction of the diffeomorphism $\phi|_U$ to the open subset $N \subseteq U$, so it is open as a map to its image, $\mu_0^{-1}(\Delta^{\text{int}})$. As observed above, $\mu_0\colon \mu_0^{-1}(\Delta^{\text{int}}) \to \Delta^{\text{int}}$ is open. Thus $\Lambda|_N$ is the composition of two open maps, so it is open.

$\square$

[2]Note that $\dim(\Delta) = \dim \mathfrak{t}^*$ so the interior is nonempty. Otherwise we would consider the relative interior.

Appendix B

Completely Integrable Systems

In this appendix we recall basic facts about Hamilton's equation and integrable systems and discuss a basic result about collective complete integrable systems from [29].

Let (M, ω) be a connected symplectic manifold. If $f \in C^\infty(M)$ then the Hamiltonian vector field X_f is defined by the equation

$$\iota_{X_f}\omega = df.$$

The Poisson bracket of $f, g \in C^\infty(M)$ is defined to be

$$\{f, g\} = \omega(X_f, X_g).$$

If $\{f, g\} = 0$ then

$$\mathcal{L}_{X_g} f = df(X_g) = \{f, g\} = 0 \tag{B.1}$$

so the flow of X_g preserves the level sets of f (and vis versa, the flow of X_f preserves the level sets of g).

Functions $f_1, \ldots, f_k \in C^\infty(M)$ define two important (singular) distributions on M:

1. Let $F = (f_1, \ldots, f_k) : M \to \mathbb{R}^k$. The first distribution is

$$\ker(dF)_m \subseteq T_m M.$$

If m is a regular point of F, then

$$\dim \ker(dF)_m = \dim M - k. \tag{B.2}$$

2. The subspace spanned by the Hamiltonian vector fields

$$\operatorname{span}\{X_{f_1}(m),\ldots,X_{f_k}(m)\}\subseteq T_mM.$$

If m is a regular point of F, then the differentials $(df_1)_m,\ldots,(df_k)_m$ are independent. Since ω is nondegenerate, this implies that the vectors $X_{f_1}(m),\ldots,X_{f_k}(m)$ are independent and

$$\dim\operatorname{span}\{X_{f_1}(m),\ldots,X_{f_k}(m)\}=k. \tag{B.3}$$

If $\{f_i,f_j\}=0$ for all $1\le i,j\le k$, then

- By (B.1), Hamiltonian vector fields X_{f_i} preserve the level sets of F, so

$$\operatorname{span}\{X_{f_1}(m),\ldots,X_{f_k}(m)\}\subseteq\ker(dF)_m.$$

- For all $1\le i,j\le k$,

$$\omega(X_{f_i},X_{f_j})=\{f_i,f_j\}=0$$

  so for all $m\in M$ the subspace of T_mM spanned by $X_{f_1}(m),\ldots,X_{f_k}(m)$ is isotropic.

- Since

$$\omega(X_{f_i},Y)=df_i(Y),$$

  $Y\in\ker(dF)_m$ if and only if $\omega(X_{f_i},Y)=0$ for all $1\le i\le k$. Thus the subspace

$$\ker(dF)_m=\operatorname{span}\{X_{f_1}(m),\ldots,X_{f_k}(m)\}^{\perp}$$

  (where $\perp$ denotes the symplectic orthocomplement) and $\ker(dF)_m$ is coisotropic.

If $k=\frac{1}{2}\dim M$, then by (B.2) and (B.3), if m is a regular point of F, then

$$\ker(dF)_m=\operatorname{span}\{X_{f_1}(m),\ldots,X_{f_k}(m)\}$$

and this subspace is Lagrangian (isotropic and coisotropic). Thus the regular fibers of F are Lagrangian submanifolds of M. This prompts the following definition.

Definition B.4. Smooth functions $f_1,\ldots,f_n$ on a connected symplectic manifold (M,ω) defines a *completely integrable system* if

1. $n = \frac{1}{2}\dim M$,

2. $\{f_i, f_j\} = 0$ for all $1 \leq i, j \leq n$, and

3. The differentials $df_1, \ldots, df_n$ are independent (F is a submersion) on an open dense subset of M.

B.0.4 Collective integrable systems

In [29], Guillemin and Sternberg studied collective integrable systems and multiplicity free Hamiltonian G-manifolds. In this section we recall the details of one of the important results from that paper.

Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$ and let (M, ω, Φ) be a Hamiltonian G-manifold (recall that our Hamiltonian G-manifolds are assumed to be connected).

Let u_m be the map $\mathfrak{g} \to T_m M$ that sends X to $\underline{X}(m)$. The tangent space to the orbit $G \cdot m$ is the image $u_m(\mathfrak{g})$. It follows from Hamilton's equation that (see e.g. [29], page 221)

$$\ker(d\Phi)_m = u_m(\mathfrak{g})^{\perp}$$

(where $\perp$ denotes the symplectic orthocomplement in $T_m M$).

Proposition B.5. *[29] Suppose (M, ω, Φ) is a multiplicity free Hamiltonian G-manifold and W is a submanifold of an orbit type stratum[1] of $\mathfrak{g}^*$ such that $\Phi(M) \subseteq W$ and $\Phi \colon M \to W$ is a submersion[2]. If $f_1, \ldots, f_n$ are smooth functions on $\mathfrak{g}^*$ such that*

i) The restriction of the functions $f_1, \ldots, f_k$ to any coadjoint orbit $\mathcal{O} \subseteq W$ defines a completely integrable system (with respect to the Kostant-Kirillov-Souriau symplectic structure on $\mathcal{O}$).

ii) The coadjoint orbits $\mathcal{O} \subseteq W$ are regular level sets of the function

$$(f_{k+1}, \ldots, f_n) \colon W \to \mathbb{R}^{n-k}.$$

Then the collective functions $\Phi^* f_1, \ldots, \Phi^* f_n$ define a completely integrable system on M.

Proof. **The number of functions equals $\frac{1}{2}\dim M$:** Since the restriction of the functions $f_1, \ldots, f_k$ to any coadjoint orbit $\mathcal{O} \subseteq W$ defines a completely integrable system,

$$\dim \mathcal{O} = 2k. \tag{B.6}$$

[1] We make this simplifying assumption so that the coadjoint orbits contained in W all have the same dimension.
[2] In the discussion following the proof of this proposition, we discuss the existence of a manifold W with these properties.

Since the coadjoint orbits in W are regular level sets of $(f_{k+1}, \dots, f_n)$,

$$\dim W = (n-k) + 2k = n+k. \tag{B.7}$$

Since $\Phi\colon M \to W$ is a submersion, for all $x \in M$,

$$\begin{aligned} \dim W &= \dim M - \dim \ker(d\Phi)_x \\ &= \dim M - \dim u_x(\mathfrak{g})^{\perp} \\ &= \dim u_x(\mathfrak{g}) + \dim u_x(\mathfrak{g})^{\perp} - \dim u_x(\mathfrak{g})^{\perp} \\ &= \dim u_x(\mathfrak{g}) \end{aligned} \tag{B.8}$$

Since M is a multiplicity free G-manifold, G acts locally transitively on preimages of coadjoint orbits (cf. Proposition 4.23), so by (B.8),

$$\dim \Phi^{-1}(\mathcal{O}) = \dim u_x(\mathfrak{g}) = \dim W. \tag{B.9}$$

Since the map $\Phi : M \to W$ is a submersion and $\mathcal{O} \subseteq W$ is a submanifold,

$$\begin{aligned} \dim \Phi^{-1}(\mathcal{O}) &= \dim M - \mathrm{codim}_W \mathcal{O} \\ &= \dim M - \dim W + \dim \mathcal{O} \end{aligned} \tag{B.10}$$

so by (B.9),

$$2 \dim W - \dim \mathcal{O} = \dim M.$$

Combining this with (B.6) and (B.7), we have that $\dim M = 2n$.

The functions Poisson commute on M: Since the the functions $f_{k+1}, \dots, f_n$ are constant on the symplectic leaves of W and the restrictions of the functions $f_1, \dots, f_k$ to the coadjoint orbits Poisson commute, the functions $f_1, \dots, f_n$ Poisson commute on W. Since Φ is the momentum map of a Hamiltonian group action, it is Poisson so the collective functions $\Phi^* f_1, \dots, \Phi^* f_n$ Poisson commute.

The functions are independent on an open dense subset of M: Since the restriction of the functions $f_1, \dots, f_k$ to any coadjoint orbit $\mathcal{O} \subseteq W$ is a completely integrable system, the differentials of the functions $f_1, \dots, f_k$ are independent on an open dense subset of any coadjoint orbit $\mathcal{O} \subseteq W$, so the differentials of the functions $f_1, \dots, f_k$ are independent on an open dense subset of W.

Since the coadjoint orbits $\mathcal{O} \subseteq W$ are regular level sets of the function $(f_{k+1}, \dots, f_n)\colon W \to \mathbb{R}^{n-k}$ and the differentials of the restrictions of the functions $f_1, \dots, f_k$ are independent on an open dense subset of $\mathcal{O}$, the differentials of the functions $f_1, \ldots, f_n$ are independent on an open dense subset of W. Since $\Phi\colon M \to W$ is a submersion, the differentials of the functions $\Phi^* f_1, \ldots, \Phi^* f_n$ are independent on an open dense subset of M. □

In general, if (M, ω, Φ) is a Hamiltonian G-manifold, then there there does not exist a submanifold $W \subseteq \mathfrak{g}^*$ contained in an orbit type stratum of $\mathfrak{g}^*$ such that

- $\Phi(M) \subseteq W$ and
- $\Phi\colon M \to W$ is a submersion

as in Proposition B.5. However, if Φ is proper then we can restrict to an open dense subset of M and these assumptions will be satisfied:

By the slice theorem for proper group actions, the principal stratum $M_{\text{prin}} \subseteq M$ for the action of G on M is an open dense subset of M. Let $\sigma \subseteq \mathfrak{t}^*_+$ be the principal stratum corresponding to M and let $\Sigma_\sigma = G \cdot \sigma$. By the cross-section theorem the preimage $\Phi^{-1}(\Sigma_\sigma)$ is an open dense submanifold of M. We restrict our attention to the open dense G-invariant submanifold $M_{\text{prin}} \cap \Phi^{-1}(\Sigma_\sigma)$.

Suppose that the momentum map $\Phi\colon M \to \mathfrak{g}^*$ is proper. Then by [55], the momentum set $\square = \Phi(M) \cap \mathfrak{t}^*_+$ is a convex, locally polyhedral set, the relative interior $\square^{\text{rel-int}} \subseteq \sigma$ and

$$\Phi^{-1}\left(G \cdot \square^{\text{rel-int}}\right) = M_{\text{prin}} \cap \Phi^{-1}(\Sigma_\sigma)$$

(see e.g. [55]). The set $W = G \cdot \square^{\text{rel-int}}$ is a G-invariant submanifold contained in an orbit type stratum of $\mathfrak{g}^*$ (it is the image of the submanifold $G/G_\sigma \times \square^{\text{rel-int}} \subseteq G/G_\sigma \times \sigma$ under the diffeomorphism $G/G_\sigma \times \sigma \to \Sigma_\sigma$) such that

- $\Phi\left(M_{\text{prin}} \cap \Phi^{-1}(\Sigma_\sigma)\right) \subseteq W$, and
- $\Phi\colon M_{\text{prin}} \cap \Phi^{-1}(\Sigma_\sigma) \to W$ is a submersion.

Thus if (M, ω, Φ) is a multiplicity free Hamiltonian G-manifold with proper momentum map, and we have functions $f_1, \ldots, f_n\colon \mathfrak{g}^* \to \mathbb{R}$ as in Proposition B.5 (for the manifold $W = G \cdot \square^{\text{rel-int}}$), we can apply Proposition B.5 to conclude that the collective functions $\Phi^* f_1, \ldots, \Phi^* f_n$ define a completely integrable system on the open dense submanifold $M_{\text{prin}} \cap \Phi^{-1}(\Sigma_\sigma)$, thus they define a completely integrable system on M.

In the situation of Proposition 4.32, one has a multiplicity free Hamiltonian space with proper momentum map and functions $f_1, \ldots, f_n\colon \mathfrak{g}^* \to \mathbb{R}$ that are smooth and satisfy the assumptions of

Proposition B.5 on an open dense subset of the manifold $W = G \cdot \square^{\text{rel-int}}$ that is not G-invariant. Since the proof of Proposition B.5 is essentially local, one can apply it with minor modifications to conclude that the collective functions $\Phi^* f_1, \ldots, \Phi^* f_n$ define a completely integrable system on the open dense submanifold of M where they are smooth.

Appendix C

Thimm's trick

Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$. Fix a Weyl chamber $\mathfrak{t}^*_+ \subseteq \mathfrak{g}^*$. The sweeping map

$$s\colon \mathfrak{g}^* \to \mathfrak{t}^*_+$$

is defined by sending ξ to the unique element of the intersection $G \cdot \xi \cap \mathfrak{t}^*_+$. This map is G-invariant by definition.

Proposition C.1. *The sweeping map is continuous.*

Proof. Consider the diagram

$$\begin{tikzcd}
\mathfrak{t}^*_+ \arrow[r, "\iota"] \arrow[dr, "f"'] & \mathfrak{g}^* \arrow[d, "\pi"] \arrow[dr, "s"] & \\
& \mathfrak{g}^*/G \arrow[r, "\tilde{s}"'] & \mathfrak{t}^*_+
\end{tikzcd}$$

where π is the quotient map for the coadjoint action of G, f is the bijective continuous map defined by the composition $\pi \circ \iota$, and $\tilde{s}$ is the map induced by s that sends $[\xi]$ to $s(\xi)$. Since π is a quotient map for a compact group action, it is closed. Since ι is the inclusion of a closed subspace, it is also closed. Thus the composition $f = \pi \circ \iota$ is a closed bijective continuous map, so it is a homeomorphism. Since $\tilde{s} = f^{-1}$, it follows that $s = \tilde{s} \circ \pi$ is continuous. $\square$

Proposition C.2. *For any stratum $\sigma \subseteq \mathfrak{t}^*_+$, the restricted sweeping map*

$$s|_{\Sigma_\sigma} : \Sigma_\sigma \to \sigma$$

is smooth.

We sketch the proof, which appears in more detail in [70].

Proof. The orbit map $G \times \sigma \to \Sigma_\sigma$ induces a diffeomorphism $\phi\colon G/G_\sigma \times \sigma \to \Sigma_\sigma$ (where G_σ is the stabilizer subgroup of σ and $\Sigma_\sigma = G \cdot \sigma$). The map $s|_{\Sigma_\sigma}$ is equal to the composition

$$\Sigma_\sigma \xrightarrow{\phi^{-1}} G/G_\sigma \times \sigma \xrightarrow{\mathrm{pr}_2} \sigma$$

which is smooth. □

Proposition C.3 (Thimm's Trick). *Let (M, ω, Φ) be a connected Hamiltonian G-manifold and suppose that $\mathfrak{h}_1, \mathfrak{h}_2$ are subalgebras of $\mathfrak{g}$ with corresponding connected subgroups H_1 and H_2.*

If $[\mathfrak{h}_1, \mathfrak{h}_2] \subseteq \mathfrak{h}_1$ then for any $f_1 \in C^\infty(\mathfrak{h}_1^*)^{H_1}$ and $f_2 \in C^\infty(\mathfrak{h}_2^*)$, the collective functions $f_1 \circ p_1 \circ \Phi$ and $f_2 \circ p_2 \circ \Phi$ Poisson commute on M.

The proof given here appears in [76], Proposition 4.1.

Proof. Let $f_1 \in C^\infty(\mathfrak{h}_1^*)^{H_1}$ and $f_2 \in C^\infty(\mathfrak{h}_2^*)$. Let $m \in M$ and let $\xi = \Phi(m)$. Since Φ is Poisson,

$$\begin{aligned} \{f_2 \circ p_2 \circ \Phi, f_1 \circ p_1 \circ \Phi\}_m &= \{f_2 \circ p_2, f_1 \circ p_1\}_\xi \\ &= \langle \xi, [d(f_2 \circ p_2)_\xi, d(f_1 \circ p_1)_\xi] \rangle \\ &= \langle ad^*_{d(f_1 \circ p_1)_\xi} \xi, d(f_2 \circ p_2)_\xi \rangle \end{aligned}$$

where $d(f_1 \circ p_1)_\xi$ is identified with an element of $\mathfrak{h}_1$ and $d(f_2 \circ p_2)_\xi$ is identified with an element of $\mathfrak{h}_2$. We have a decomposition $\mathfrak{g}^* = \mathfrak{h}_1^* \oplus \mathfrak{h}_1^0$ and we can write $\xi = \xi_1 + \xi_2$ with $\xi_1 \in \mathfrak{h}_1^*$ and $\xi_2 \in \mathfrak{h}_1^0$. Since f_1 is H_1-invariant,

$$ad^*_{d(f_1 \circ p_1)_\xi} \xi_1 = 0$$

so

$$\begin{aligned} \langle ad^*_{d(f_1 \circ p_1)_\xi} (\xi_1 + \xi_2), d(f_2 \circ p_2)_\xi \rangle &= \langle ad^*_{d(f_1 \circ p_1)_\xi} \xi_2, d(f_2 \circ p_2)_\xi \rangle \\ &= \langle \xi_2, [d(f_2 \circ p_2)_\xi, d(f_1 \circ p_1)_\xi] \rangle \\ &= 0 \end{aligned}$$

where the last equality follows since $[\mathfrak{h}_1, \mathfrak{h}_2] \subseteq \mathfrak{h}_1$ and $\xi_2 \in \mathfrak{h}_1^0$. □

If σ is the principal stratum of $\mathfrak{t}^*_+$ corresponding to a (connected) Hamiltonian G-manifold (M, ω, Φ), then the symplectic cross-section theorem (Proposition 4.19) says that $\Phi^{-1}(\Sigma_\sigma)$ is an open dense submanifold of M, and we can define a new action of the centre Z_σ of G_σ by extending the action of Z_σ on $\Phi^{-1}(\sigma)$ by G equivariance: if $g \in G$ such that $g \cdot m \in \Phi^{-1}(\sigma)$, then for $h \in Z_\sigma$, define

$$h * m = (g^{-1} h g) \cdot m.$$

This action is well-defined since Z_σ commutes with elements of the stabilizer subgroup G_σ. Let Z^0_σ denote the connected component of the identity of Z_σ and let $\mathfrak{z}_\sigma$ denote the Lie algebra of Z_σ.

Proposition C.4 (Guillemin and Sternberg's action coordinates)**.** *The new action of Z^0_σ on*

$$\left(\Phi^{-1}(\Sigma_\sigma), \omega|_{\Phi^{-1}(\Sigma_\sigma)}\right)$$

is Hamiltonian and the map

$$p_\sigma \circ s \circ \Phi : \Phi^{-1}(\Sigma_\sigma) \to \mathfrak{z}^*_\sigma$$

is a momentum map for this action.

The proof is adapted from [83], Propositions 3.4.

Proof. First, we show that Hamilton's equation holds for points in the symplectic cross-section $\Phi^{-1}(\sigma)$. The tangent space at a point m in the symplectic cross-section is a sum

$$T_m M = T_m \Phi^{-1}(\sigma) + T_m(G \cdot m)$$

so it is sufficient to check Hamilton's equation for $Y \in T_m \Phi^{-1}(\sigma)$ and $Y \in T_m(G \cdot m)$.

For $m \in \Phi^{-1}(\sigma)$ and $X \in \mathfrak{z}_\sigma$, the fundamental vector field for the new action of Z_σ coincides with the fundamental vector field of the induced action, $\underline{X}_{new}(m) = \underline{X}(m)$ (we write $\underline{X} \in \mathfrak{X}(M)$ to denote the fundamental vector field of $X \in \mathfrak{g}^*$ corresponding to the action of G on M and $\underline{X}_{new}$ to denote the fundamental vector field of $X \in \mathfrak{z}_\sigma$ corresponding to the new action).

If $Y \in T_m \Phi^{-1}(\sigma)$, then since the action of G is Hamiltonian,

$$\begin{aligned} \omega_m\left(\underline{X}_{new}(m), Y\right) &= \omega_m\left(\underline{X}(m), Y\right) \\ &= \langle d\Phi_m(Y), X\rangle \\ &= \langle d(p_\sigma \circ s \circ \Phi)_m(Y), X\rangle \end{aligned}$$

since $\Phi|_{\Phi^{-1}(\sigma)} = s \circ \Phi|_{\Phi^{-1}(\sigma)}$ and $X \in \mathfrak{z}_\sigma$.

If $Y \in T_m(G \cdot m)$, then $Y = \underline{V}(m)$ for some $V \in \mathfrak{g}$. It follows that for $X \in \mathfrak{z}_\sigma$

$$\langle d(p_\sigma \circ s \circ \Phi)_m(Y), X\rangle = \langle d(s \circ \Phi)_m(Y), X\rangle = \langle \mathcal{L}_{\underline{V}}(s \circ \Phi)(m), X\rangle = 0$$

since $s \circ \Phi$ is G-invariant. For $X \in \mathfrak{z}_\sigma$, let f^X denote the real valued function $f^X(\xi) = \langle \xi, X\rangle$. Since Φ is Poisson, for any $m \in \Phi^{-1}(\sigma)$, and $y = \underline{V} \in T_m(G \cdot m)$,

$$\begin{aligned}
\omega_m\left(\underline{X}_{new}(m), Y\right) &= \omega_m\left(\underline{X}_{old}(m), \underline{V}_{old}(m)\right) \\
&= \left\{f^X \circ \Phi, f^V \circ \Phi\right\}_m \\
&= \left\{f^X, f^V\right\}_{\Phi(m)} \\
&= \langle \Phi(m), [df^X_{\Phi(m)}, df^V_{\Phi(m)}]\rangle \\
&= \langle \Phi(m), [X, V]\rangle \\
&= 0.
\end{aligned}$$

The last equation holds since $\Phi(m) \in \sigma$ and $X \in \mathfrak{z}_\sigma$, so $ad^*_X \Phi(m) = 0$. Thus for $m \in \Phi^{-1}(\sigma)$ and $Y \in T_m(G \cdot m)$,

$$\omega_m\left(\underline{X}_{new}(m), Y\right) = 0 = \langle d(p_\sigma \circ s \circ \Phi)_m(Y), X\rangle.$$

Finally, we show that Hamilton's equation holds at $m \in \Phi^{-1}(\Sigma_\sigma)$. Let $Y \in T_m M$ and $X \in \mathfrak{z}_\sigma$. Let $g \in G$ such that $Ad^*_g \Phi(m) \in \sigma$ (and by equivariance of Φ, $g \cdot m \in \Phi^{-1}(\sigma)$). The new action of $\mathfrak{z}_\sigma$ at m is given by

$$\begin{aligned}
\underline{X}_{new}(m) &= \frac{d}{dt}\exp(tX) * m\Big|_{t=0} \\
&= \frac{d}{dt} g^{-1}\exp(tX) g \cdot m\Big|_{t=0} \\
&= \frac{d}{dt}\exp(tAd_{g^{-1}}X) \cdot m\Big|_{t=0} \\
&= \underline{Ad_{g^{-1}}X}(m).
\end{aligned}$$

By G-invariance of ω and $p_\sigma \circ s \circ \Phi$,

$$\begin{aligned}\omega_m\left(\underline{X}_{new}(m), Y\right) &= \omega_{g\cdot m}\left(g_* \underline{Ad_{g^{-1}}X}(m), g_* Y\right)\\ &= \omega_{g\cdot m}\left(\underline{X}(g\cdot m), g_* Y\right)\\ &= \omega_{g\cdot m}\left(\underline{X}_{new}(g\cdot m), g_* Y\right)\\ &= \langle d(p_\sigma \circ s \circ \Phi)_{g\cdot m}(g_* Y), X\rangle\\ &= \langle d(p_\sigma \circ s \circ \Phi)_{m}(Y), X\rangle\end{aligned}$$

where in the second last we have used the fact that Hamilton's equation holds at the point $g \cdot m \in \Phi^{-1}(\sigma)$. □

In what follows, we identify $\mathfrak{z}_\sigma^*$ with the subspace of $\mathfrak{t}^*$ spanned by σ. The momentum map generating the new action of Z_σ^0 is then $s \circ \Phi$.

Now we prove the following version of Thimm's trick "with action coordinates."

Proposition C.5 (Thimm's trick with Guillemin and Sternberg's action coordinates)**.** *Let (M, ω, Φ) be a connected Hamiltonian G-manifold and suppose that $\mathfrak{h}_1, \mathfrak{h}_2$ are subalgebras of $\mathfrak{g}$ with corresponding connected subgroups H_1 and H_2, such that*

$$[\mathfrak{h}_1, \mathfrak{h}_2] \subseteq \mathfrak{h}_1.$$

Let σ_1 be the principal stratum of $\mathfrak{t}_{1,+}^*$ corresponding to the induced action of H_1 on M.

Then the action of H_2 on M leaves $(p_1 \circ \Phi)^{-1}(\Sigma_{\sigma_1})$ invariant, and the new action of $Z_{\sigma_1}^0$ on $(p_1 \circ \Phi)^{-1}(\Sigma_{\sigma_1})$ commutes with the action of H_2.

It will be sufficient to show that for all $X_1 \in \mathfrak{z}_{\sigma_1}$ and $X_2 \in \mathfrak{h}_2$, the functions

$$\langle p_2 \circ \Phi, X_2\rangle$$

and

$$\langle s \circ p_1 \circ \Phi, X_1\rangle$$

Poisson commute on the open dense set $(p_1 \circ \Phi)^{-1}(\Sigma_{\sigma_1})$ (with respect to the restriction of $\{\cdot,\cdot\}_M$). While both of these functions are smooth on $(p_1 \circ \Phi)^{-1}(\Sigma_{\sigma_1})$, the function

$$\langle s, X_1\rangle \colon \mathfrak{h}_1^* \to \mathbb{R}$$

is only smooth on the orbit type strata of $\mathfrak{h}_1^*$, so we must be careful in applying Thimm's trick.

Proof. Let σ_1 be the principal stratum of $\mathfrak{t}_{1,+}^*$ corresponding to M. We introduce the notation $\Sigma_{\sigma_1} = H_1 \cdot \sigma_1$ and $\mathcal{U} = (p_1 \circ \Phi)^{-1}(\Sigma_{\sigma_1})$. Recall that $\mathfrak{z}_{\sigma_1}^*$ has been identified with the span of σ_1 in $\mathfrak{t}_1^*$. Fix $X_1 \in \mathfrak{z}_{\sigma_1}$.

For $\varepsilon > 0$, let

$$(\sigma_1)_\varepsilon = \sigma_1 \setminus \bigcup_{\tau < \sigma_1} B_\varepsilon(\tau)$$

where $\tau < \sigma_1$ are the strata of $\mathfrak{t}_{1,+}^*$ contained in $\overline{\sigma_1} \setminus \sigma_1$ and

$$B_\varepsilon(\tau) = \left\{\xi \in \mathfrak{z}_{\sigma_1}^* : d(\xi, \tau) < \varepsilon\right\}$$

is the ε-neighbourhood of τ in the subspace $\mathfrak{z}_{\sigma_1}^*$. The set $(\sigma_1)_\varepsilon$ is a closed subset of σ_1 with relative interior

$$(\sigma_1)_\varepsilon^{\text{rel-int}} = \sigma_1 \setminus \bigcup_{\tau < \sigma_1} \overline{B_\varepsilon}(\tau).$$

Since the function $\langle s, X_1 \rangle \colon \Sigma_{\sigma_1} \to \mathbb{R}$ is smooth and Σ_{σ_1} is submanifold of $\mathfrak{h}_1^*$, the restriction of $\langle s, X_1\rangle$ to the closed set $H_1 \cdot (\sigma_1)_\varepsilon \subseteq \Sigma_{\sigma_1}$ is smooth as a function on a closed subset[1] of $\mathfrak{h}_1^*$. By the Whitney Extension Theorem, there exists a smooth function $F_\varepsilon \in C^\infty(\mathfrak{h}_1^*)$ such that

$$F_\varepsilon|_{H_1 \cdot (\sigma_1)_\varepsilon} = \langle s, X_1 \rangle.$$

Since H_1 is compact and $\langle s, X_1\rangle$ is H_1-invariant, we can take F_ε to be H_1-invariant.

By Thimm's trick, for all $X_2 \in \mathfrak{h}_2$, since $F_\varepsilon \in C^\infty(\mathfrak{h}_1^*)^{H_1}$ and $[\mathfrak{h}_1, \mathfrak{h}_2] \subseteq \mathfrak{h}_1$,

$$\left\{\langle p_2 \circ \Phi, X_2 \rangle, F_\varepsilon \circ p_1 \circ \Phi\right\}_M = 0.$$

Let $\mathcal{U}_\varepsilon = (p_1 \circ \Phi)^{-1}\left(H_1 \cdot (\sigma_1)_\varepsilon^{\text{rel-int}}\right)$. Since $\mathcal{U}$ is an open subset of M, the restricted map $s \circ p_1 \circ \Phi \colon \mathcal{U} \to \sigma_1$ is continuous, and the sets $(\sigma)_\varepsilon^{\text{rel-int}} \subseteq \sigma_1$ are open, the sets $\mathcal{U}_\varepsilon \subseteq M$ are open and

$$\mathcal{U} = \bigcup_{\varepsilon > 0} \mathcal{U}_\varepsilon.$$

[1] A function f defined on a closed subset A of a smooth manifold M is smooth if for all $x \in A$ there is a open neighbourhood U of x in M and a smooth function $F \in C^\infty(U)$ such that $F|_{A\cap U} = f|_{A \cap U}$.

Thus we have shown that for all $\varepsilon > 0$,

$$\{\langle p_2 \circ \Phi, X_2\rangle, \langle s \circ p_1 \circ \Phi, X_1\rangle\}\,|_{\mathcal{U}_\varepsilon} = \{\langle p_2 \circ \Phi, X_2\rangle, F_\varepsilon \circ p_1 \circ \Phi\}\,|_{\mathcal{U}_\varepsilon} = 0.$$

so

$$\{\langle p_2 \circ \Phi, X_2\rangle, \langle s \circ p_1 \circ \Phi, X_1\rangle\}_{\mathcal{U}} = 0.$$

Since H_2 is connected and the action of H_2 is Hamiltonian, generated by $p_2 \circ \Phi$, it follows that the action of H_2 preserves $\mathcal{U}$ and the action of H_2 commutes with the action of $Z^0_{\sigma_1}$ on $\mathcal{U}$. $\square$

Let σ_2 be the principal stratum of the Weyl chamber $\mathfrak{t}^*_{2,+}$ corresponding to the induced action of H_2 on M in the preceding proposition and let Z_{σ_2} be the centre of the stabilizer subgroup H_{2,σ_2} of the stratum σ_2 and let $\mathfrak{z}_{\sigma_2}$ be its Lie algebra. Let $Z^0_{\sigma_2}$ be connected component of the identity of Z_{σ_2}.

By the preceding proposition, the components of the map $s \circ p_1 \circ \Phi$ on the subset $(p_1 \circ \Phi)^{-1}(\Sigma_{\sigma_1})$ are invariant under the new action of $Z^0_{\sigma_2}$, thus we have proven the following corollary.

Corollary C.6. *Let (M, ω, Φ) be a Hamiltonian G-manifold and suppose that $\mathfrak{h}_1, \mathfrak{h}_2$ are subalgebras of $\mathfrak{g}$ such that*

$$[\mathfrak{h}_1, \mathfrak{h}_2] \subseteq \mathfrak{h}_1.$$

The new $Z^0_{\sigma_1}$-action generated by $s \circ p_1 \circ \Phi$ and the new $Z^0_{\sigma_2}$-action generated by $s \circ p_2 \circ \Phi$ commute on the open, dense submanifold where they are both defined,

$$(p_1 \circ \Phi)^{-1}(\Sigma_{\sigma_1}) \cap (p_2 \circ \Phi)^{-1}(\Sigma_{\sigma_2}).$$

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