# Tensor product of preprojective algebras and preprojective structure on skew-group algebras

by

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# Abstract

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We investigate properties of finite subgroups $G < SL(n,k)$ for which the skew-group algebra $k[x_1,\ldots,x_n]\#G$ does not have a grading structure of (higher) preprojective algebra. Namely, we prove that if a finite subgroup $G < SL(n,k)$ is conjugate to a finite subgroup of $SL(n_1,k) \times SL(n_2,k)$, for some $n_1, n_2 \geq 1$, $n_1 + n_2 = n$, then the skew-group algebra $R\#G$ is not Morita equivalent to a (higher) preprojective algebra. One of the first examples of such a group $G$ is $\langle \frac{1}{3}(1,2,1,2) \rangle$, the cyclic subgroup of order 3 of $SL(4,k)$ generated by the diagonal matrix

$$\begin{pmatrix} \xi & 0 & 0 & 0 \\ 0 & \xi^2 & 0 & 0 \\ 0 & 0 & \xi & 0 \\ 0 & 0 & 0 & \xi^2 \end{pmatrix},$$

where $\xi$ is a third root of unity. Motivated by this question, we study preprojective algebras over Koszul algebras. We give a quiver construction for the preprojective algebra over a basic Koszul $n$-representation-infinite algebra. Moreover, we show that such algebras are derivation-quotient algebras whose relations are given by a superpotential. The main problem is also related to the preprojective algebra structure on the tensor product $\Pi := \Pi_1 \otimes_k \Pi_2$ of two Koszul preprojective algebras. We prove that a superpotential in $\Pi$ is given by the shuffle product of superpotentials in $\Pi_1$ and $\Pi_2$. Finally, we prove that if $\Pi$ has a grading structure such that it is $n$-Calabi-Yau of Gorentstein parameter 1, then its degree 0 component is the tensor product of a Calabi-Yau algebra and a higher representation-infinite algebra. This implies that it is infinite-dimensional, which means in particular that $\Pi$ is not a preprojective algebra.

# Dedication

À mes parents, Marie-Rose et Claude.
Merci pour votre soutien et vos encouragements.

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# Contents

# Chapter 1

# Introduction

Preprojective algebras were first defined by Gel′fand and Ponomarev in [GP79] to study the representation theory of finite-dimensional hereditary algebras. Baer, Geigle and Lenzing ([BGL87]) later proposed an equivalent definition for the preprojective algebra as a tensor algebra over $\Lambda$ of the inverse Auslander-Reiten translation:

$$\Pi(\Lambda) := T_\Lambda \mathrm{Ext}^1_\Lambda(D\Lambda, \Lambda).$$

In the framework of Iyama's program on higher Auslander-Reiten theory [Iya07], the notion of preprojective algebra was extended ([IO13], [Kel11]) to algebras of higher global dimension. Hereditary algebras were also extended to $n$-hereditary algebras, which include $n$-representation-finite [IO11] and $n$-representation-infinite algebras [HIO14]. These algebras are required to be finite-dimensional. In this thesis, we argue that this restriction is strong in many contexts. The higher $(n+1)$-preprojective algebra $\Pi$ is defined as the direct sum of the inverse of the shifted Serre functor $\mathbb{S}_n := \mathbb{S} \circ [-n] : \mathcal{D}^b(\Lambda) \to \mathcal{D}^b(\Lambda)$ applied to an $n$-hereditary algebra $\Lambda$:

$$\Pi := \bigoplus_{i \geq 0} \mathbb{S}_n^{-i}(\Lambda) = T_\Lambda \mathrm{Ext}^n_\Lambda(D\Lambda, \Lambda).$$

They are characterized by the following homological property: the $(n+1)$-preprojective algebras over $n$-representation-infinite algebras are exactly the bimodule $(n+1)$-Calabi-Yau algebras of Gorenstein parameter 1 ([Kel11], [MM11], [HIO14], [AIR15]). The fact that $n$-hereditary algebras are finite-dimensional is crucial in the extension of Auslander-Reiten theory to algebras of higher global dimensions. For example, the notion of Serre functor does not make sense without this assumption.

Higher preprojective algebras have been studied extensively in the last five years. Numerous algebraic and geometric applications were developed (see, e.g., [IO13], [AIR15], [HIO14]). Buchweitz and Hille [BHon] proved that $n$-representation-infinite algebras arise as the endomorphism algebras of certain tilting sheaves. Moreover, in the setting of higher Auslander-Reiten theory, the authors in [HIMO14] generalized the notion of Geigle-Lenzing spaces [GL87] and canonical algebras, which have had many applications in representation theory. They also appear in Kontsevich's homological mirror symmetry conjecture ([KST07], [KST09]).

In the classical case, preprojective algebras act as bridges between singularity theory and noncommutative algebraic geometry. This was first observed via the McKay correspondence, allowing a whole new understanding of the geometry and algebra of quotient singularities and their resolutions. If $G$ is a finite subgroup of $SL(2,k)$, McKay's observation ([McK83]) was that the dual resolution graph of a Kleinian singularity $k^n/G = \text{spec } R^G$, which is Coxeter-Dynkin of type A-D-E, has a direct interpretation in terms of the finite-dimensional representations of the associated finite subgroups of $SL(2,k)$, through the McKay quiver. This led to many fundamental results whose goals were to understand and interpret this correspondence from the point of view of geometry (see, e.g., [GSV81], [AV85], [EK85]) and algebra (see, e.g., [Aus86], [AR89]). The research exposed in this thesis has roots coming from the algebraic interpretation, known as Auslander's McKay correspondence, which is inscribed into Auslander-Reiten theory. In general, if $R$ is the polynomial ring in $n$ variables and $G < GL(n,k)$ a group without pseudo-reflections acting on it via linear coordinate changes, then the skew-group algebra $R\#G$ is the Auslander algebra over the ring of invariants $R^G$, that is,

$$R\#G \cong \text{End}_{R^G}(R).$$

Coming back to the case where $G < SL(2,k)$, Auslander's McKay correspondence states that the McKay quiver of $G$ can be obtained from the minimal projective resolutions of the simple $k[x,y]\#G$-modules.

Reiten and Van den Bergh ([RVdB89]) established a Morita equivalence between the skew-group algebra $k[x,y]\#G$ and the preprojective algebra $\Pi_G$ of the extended Coxeter-Dynkin quiver associated to $G$ via the McKay correspondence, finding yet another connection between representation theory and the study of quotient singularities. In this case the preprojective algebra $\Pi_G$ is isomorphic to

$$\text{End}_{R^G}(\oplus M_i),$$

where the direct sum is taken over the maximal Cohen-Macaulay $R^G$-modules, which play an important role in the geometry of singularities. This correspondence between singularity theory and representation theory was later extended ([KST07], [LdlP11]) to an equivalence of triangulated categories between $\underline{\text{CM}}(\text{gr } R^G)$, the category of stable graded maximal Cohen-Macaulay modules over $R^G$, and $\mathcal{D}^b(kQ)$, the bounded derived category of the algebra given by the Dynkin quiver $Q$ associated to $G$. This gives rise to an equivalence of categories between $\underline{\text{CM}}(R^G)$ and the so-called cluster category $\mathcal{C}_1(kQ) := \mathcal{D}^b(kQ)/\langle\tau\rangle$, where $\tau$ is the Auslander-Reiten translate on $kQ$.

An interesting problem is to generalize this picture to the setting of higher Auslander-Reiten theory. In [AIR15], the authors proved that if $B$ is a higher $n$-preprojective algebra and $e$ is an idempotent satisfying certain axioms, such that $B$ is the $n$-Auslander algebra of $eBe$, then we get triangle equivalences

$$\underline{\text{CM}}(\text{gr } eBe) \cong \mathcal{D}^b(B_0/\langle e\rangle)$$

and

$$\underline{\text{CM}}(eBe) \cong \mathcal{C}_{n-1}(B_0/\langle e\rangle),$$

generalizing the classical case. If $R$ is the polynomial ring in $n$ variables and $G$ is a finite subgroup of $SL(n,k)$ such that $B := R\#G$ has a grading structure of higher preprojective algebra, then we can apply the previous equivalences to $eBe \cong R^G$, where $e = \frac{1}{|G|}\sum_{g\in G} g$ satisfies the required axioms.

We are thus interested in determining when is $R\#G$ Morita equivalent to a higher preprojective algebra. In [AIR15], the authors showed that if $G$ is cyclic and satisfies an extra condition, then $R\#G$ is isomorphic to a $n$-preprojective algebra. In this thesis, we prove a partial converse to this statement and generalize it to finite subgroups of $SL(n,k)$. Namely, we prove that if $G < SL(n,k)$ is conjugate to a subgroup of $SL(n_1,k)\times SL(n_2,k)$, where $n_1, n_2 \geq 1$, $n_1+n_2 = n$, then $R\#G$ is not Morita equivalent to a higher preprojective algebra. In fact, if we find a grading on $R\#G$, where $G$ is conjugate to a subgroup of $SL(n_1,k)\times SL(n_2,k)$, such that it is bimodule $n$-Calabi-Yau of Gorenstein parameter 1, then we force the degree 0 part to be infinite-dimensional. Except for groups that are conjugate to a subgroup of $SL(m,k)\times\{1\}$, all groups satifying the above property are in $SL(n,k)$, where $n\geq 4$. One of the simplest occurrences of such a group is $G = \langle \frac{1}{3}(1,2,1,2)\rangle$, the cyclic subgroup of order 3 of $SL(4,k)$ generated by the diagonal matrix

$$\begin{pmatrix} \xi & 0 & 0 & 0 \\ 0 & \xi^2 & 0 & 0 \\ 0 & 0 & \xi & 0 \\ 0 & 0 & 0 & \xi^2 \end{pmatrix},$$

where $\xi$ is a third root of unity.

The study of this question naturally leads to the analysis of the preprojective structure of the tensor product of two preprojective algebras. In [HIO14], the authors showed that the tensor product of two higher representation-infinite algebras is again higher representation-infinite. This raises the question whether the same is true for the tensor product of preprojective algebras. In this thesis, we show that such algebras, with an additional Koszulity assumption, cannot be endowed with a grading as required for a preprojective algebra. More precisely, we show that if $\Pi = \Pi_1 \otimes_k \Pi_2$ has a grading structure such that it is Calabi-Yau of Gorenstein parameter 1, then its degree 0 part can be written as $(\Pi)_0 = A_1 \otimes_k A_2$, where $A_1$ is Calabi-Yau and $A_2$ is $n$-representation-infinite, or vice versa. In particular it is infinite-dimensional, therefore $\Pi$ is not a preprojective algebra.

The objects of interest in this thesis are Koszul preprojective algebras. The Koszulity assumption is helpful in our context. In fact, our methods rely on an explicit description of the minimal resolution of the algebras we consider. This is hard to obtain in general, but minimal resolutions of Koszul algebras are known. It makes sense to consider them since skew-group algebras $R\#G$ are Koszul. Moreover, relations in Koszul Calabi-Yau algebras $A$ have a nice description in terms of a superpotential ([BSW10]). They will also take part in the proofs of our results. Superpotentials are bimodule generators of the last term in the minimal resolution of $A$. Relations in $A$ are given by taking "derivatives" of such superpotentials. Thus, they play a dual role: they characterize the projective modules in the minimal resolution of $A$ and they give the relations in $A$.

Classical preprojective algebras over basic hereditary algebras can be described as the path algebra over some quiver modulo relations ([Rin98], [CB99]). In this thesis, we generalize this quiver construction of classical preprojective algebras to higher preprojective algebras over Koszul $n$-representation-infinite algebras. Moreover, we show that the preprojective algebra over a Koszul $n$-representation-infinite algebra has a superpotential and is given by a derivation-quotient algebra, described in [BSW10], generalizing partially a theorem by Keller ([Kel11]) which states that 3-preprojective algebras are Jacobian algebras given by a potential.

The structure of this thesis is as follows. In chapter 2, we give brief background material on homological algebra, representation theory of Artin algebras and higher preprojective algebras. We also define skew-group algebras and the McKay quiver. Finally, we explain Koszul algebras and the notion of a superpotential, described in [BSW10], that plays an important role in the grading structure of the algebras we consider. In chapter 3, we give an explicit construction of the quiver of a basic preprojective algebra, generalizing the quiver construction in the classical case. We then prove that the preprojective algebra $\Pi(\Lambda)$ over a $n$-representation-infinite Koszul algebra $\Lambda$ is a derivation-quotient algebra with quadratic relations. In chapter 4, we first describe the superpotential of a tensor product of two Koszul bimodule Calabi-Yau algebras $\Pi = \Pi_1 \otimes_k \Pi_2$. We show that such superpotential is given as the shuffle product of the superpotentials in $\Pi_1$ and $\Pi_2$. We then use it to show that the tensor product of two Koszul preprojective algebras cannot have a structure of preprojective algebra. In chapter 5, we apply these techniques and results to skew-group algebras, generalizing a partial converse to a theorem in [AIR15]. We explain this theorem and the motivations behind it. We also generalize it by giving properties on abelian groups $G < SL(n,k)$ so that the skew-group algebra is a preprojective algebra. Finally, we show that if a finite subgroup $G < SL(n,k)$ is conjugate to a finite subgroup of $SL(n_1,k) \times SL(n_2,k)$ for some $n_1, n_2 \geq 1$ such that $n_1 + n_2 = n$, then $R\#G$ is not Morita equivalent to a preprojective algebra.

## Notation

We let $k$ be an algebraically closed field of characteristic 0. Every algebra is a connected $k$-algebra, that is, it cannot be written as a product of two algebras. We denote by $D := \mathrm{Hom}_k(-,k)$ the $k$-dual. If $A$ is a $k$-algebra, we denote by $A^e := A \otimes_k A^{op}$ the enveloping algebra, where $A^{op}$ is the opposite ring. Let Mod $A$ denote the category of left $A$-modules, and mod $A$ denote the category of finitely generated left $A$-modules.

# Chapter 2

# Preliminaries

## 2.1 Basic notions in representation theory of Artin algebras

In this section, we explain basic notions in representation theory of Artin algebras. The books [ARS97] and [ASS06] are good references in the subject. Let $A$ be an Artin $k$-algebra.

### 2.1.1 Homological algebra

**Definition 2.1.1.** Let $M$ be a $A$-module. A *projective resolution* of $M$ is a (possibly infinite) exact sequence

$$\cdots \xrightarrow{\delta_{n+1}} P_n \xrightarrow{\delta_n} P_{n-1} \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\epsilon} M \to 0,$$

where each $P_l$ is a projective $A$-module. We add to the term "projective resolution" the adjectives *"left"*, *"right"*, *"bimodule"*, if every module involved in the resolution has this characteristic.

Let $P$ be a projective $A$-module. An epimorphism $f : P \twoheadrightarrow M$ is called *superfluous* if for any submodule $H$ of $P$,

$$H + \ker f = P \Rightarrow H = P.$$

We say that the resolution is *minimal* if $\mathrm{Im}(\delta_i) \subset J(P_{i-1})$ for $i \geq 1$, where $J(-)$ denotes the Jacobson radical, and $P_0 \xrightarrow{\epsilon} M$ is a superfluous epimorphism.

The *length* of a projective resolution is the smallest index $n$ such that $P_l = 0$ for all $l > n$.

**Definition 2.1.2.** The *(left) projective dimension* of a (left) $A$-module $M$, denoted by $\mathrm{pd}_A(M)$, is defined as the minimal length of the (left) projective resolutions of $M$.

The *(left) global dimension* of $A$, denoted by $\mathrm{gl}(A)$, is the supremum of the set of projective dimensions of all finitely generated (left) $A$-modules.

If $\mathrm{gl}(A) = 1$, then we call $A$ a *hereditary algebra*.

**Example 2.1.3.** The polynomial ring $k[x_1, \dots, x_n] = k[V]$ has global dimension $n$. Its minimal bimodule projective resolution is given by

$$0 \to k[V] \otimes_k \bigwedge^n V \otimes_k k[V] \xrightarrow{\delta_n} \cdots \xrightarrow{\delta_3} k[V] \otimes_k \bigwedge^2 V \otimes_k k[V] \xrightarrow{\delta_2}$$
$$k[V] \otimes_k V \otimes_k k[V] \xrightarrow{\delta_1} k[V] \otimes_k k[V] \xrightarrow{\epsilon} k[V] \to 0,$$

where

$$\delta_l(1 \otimes (x_{j_l} \wedge \cdots \wedge x_{j_1}) \otimes 1) = \sum_{i=1}^{l} (-1)^{i-1}((x_{j_i} \otimes (x_{j_l} \wedge \cdots \widehat{x_{j_i}} \cdots \wedge x_{j_1}) \otimes 1) + (-1)^l (1 \otimes (x_{j_l} \wedge \cdots \widehat{x_{j_i}} \cdots \wedge x_{j_1}) \otimes x_{j_i}))$$

and $\epsilon$ is the multiplication map. We use the notation

$$x_{j_l} \wedge \cdots \widehat{x_{j_i}} \cdots \wedge x_{j_1} := x_{j_l} \wedge \cdots \wedge x_{j_{i-1}} \wedge x_{j_{i+1}} \wedge \cdots \wedge x_{j_1}.$$

**Definition 2.1.4.** Let $M$ be a right $A$-module and $N$ be a left $A$-module. The *$n$-Tor-functor*, $\mathrm{Tor}_n^A(M, N)$, is the left derived functor of the tensor product functor over $A$. Its construction is as follows. Consider a projective resolution of $M$

$$P_\bullet = \cdots \to P_2 \to P_1 \to P_0.$$

Now tensor this complex with $- \otimes_A N$:

$$P_\bullet \otimes_A N = \cdots \xrightarrow{\delta_3} P_2 \otimes_A N \xrightarrow{\delta_2} P_1 \otimes_A N \xrightarrow{\delta_1} P_0 \otimes_A N \to 0.$$

Then $\mathrm{Tor}_n^A(M, N) := H_n(P_\bullet \otimes_A N) = \ker(\delta_n)/\mathrm{Im}(\delta_{n+1})$ is the $n$-homology of this complex.

The *$n$-Ext-functor*, $\mathrm{Ext}_A^n(M, N)$, is the right derived functor of the Hom functor over $A$. Consider again a projective resolution $P_\bullet$ of $M$ and apply $\mathrm{Hom}_A(-, N)$ to it:

$$\mathrm{Hom}_A(P_\bullet, N) = 0 \to \mathrm{Hom}_A(P_0, N) \xrightarrow{\delta_1} \mathrm{Hom}_A(P_1, N) \xrightarrow{\delta_2} \mathrm{Hom}_A(P_2, N) \xrightarrow{\delta_3} \cdots$$

Then $Ext_A^n(M, N) := H^n(\mathrm{Hom}_A(P_\bullet, N)) = \ker(\delta_{n+1})/\mathrm{Im}(\delta_n)$.

### 2.1.2 Path algebras and Morita equivalence

Quivers play an important role in representation theory of algebras. Indeed, every finite-dimensional Artin algebra has the same module category as the path algebra over some quiver modulo some relations.

**Definition 2.1.5.** A *quiver* $Q = (Q_0, Q_1, s, t)$ is a directed graph, where $Q_0$ is the set of vertices, $Q_1$ is the set of arrows, and $s, t$ are maps $s : Q_1 \to Q_0,\ s(a) = i$ and $t : Q_1 \to Q_0,\ t(a) = j$, for every arrow $a : i \to j$.

In this thesis, every quiver we consider will have a finite number of vertices and arrows.

**Definition 2.1.6.** A non-trivial *path* in $Q$ is a sequence of arrows $p = a_n a_{n_1} \cdots a_1$ such that $t(a_i) = s(a_{i+1})$ for all $1 \leq i \leq n-1$. The *length* of the path is $n$. For every vertex $i \in Q_0$, we denote the trivial path of length 0 by $e_i$.

**Definition 2.1.7.** Let $Q = (Q_0, Q_1, s, t)$ be a quiver. The *path algebra* $kQ$ of the quiver $Q$ is defined as

$$kQ := \{\sum_{i\in I} a_i p_i \,|\, p_i \text{ is a path in } Q,\, a_i \in k,\; I \text{ finite}\},$$

and the multiplication is given by concatenation, that is, $p_2 \cdot p_1$ is the path given by $p_1$ followed by $p_2$ if $t(p_1) = s(p_2)$, and is 0 otherwise. Then

$$kQ \cong T_{kQ_0} kQ_1 := kQ_0 \oplus kQ_1 \oplus (kQ_1 \otimes_{kQ_0} kQ_1) \oplus kQ_1^{\otimes_{kQ_0} 3} \oplus \dots$$

is a tensor algebra.

We will often abuse terminology and call a path any sum of paths in the sense of Definition 2.1.5.

**Example 2.1.8.** Consider the quiver $Q$

$$1 \xrightarrow{a} 2 \overset{\xrightarrow{b}}{\underset{\xrightarrow{c}}{}} 3$$

The path algebra $kQ$ has basis given by the paths $e_1$, $e_2$, $e_3$, $a$, $b$, $c$, $ba$, $ca$. Note that, as examples, $e_1 a = 0$, $e_2 a = a$ and $ab = 0$.

**Theorem 2.1.1.** *The following properties on $kQ$ are true.*

1. *The $e_i$ form a complete set of primitive orthogonal idempotents in $kQ$, that is, $e_i e_j = 0$ if $i \neq j$, $e_i^2 = e_i$ and $\sum_{i\in Q_0} e_i = 1$, the identity element in $kQ$. Each $kQe_i$ is an indecomposable module. In fact,*
$$kQ = \bigoplus_{i\in Q_0} kQe_i$$
*so each $kQe_i$ is a projective left $kQ$-module. Moreover, the modules $kQe_i$ are pairwise non-isomorphic.*

2. *The $kQ$-module $kQe_i$ (resp. $e_i kQ$) has as $k$-basis elements the distinct paths starting (resp. ending) at vertex $i$.*

3. *The $k$-algebra $kQ$ is hereditary.*

4. *The $k$-algebra $kQ$ is finite-dimensional if and only if $Q$ does not contain oriented cycles.*

5. *The $k$-algebra $kQ_0$, with basis given by the idempotents $e_i$, is semisimple.*

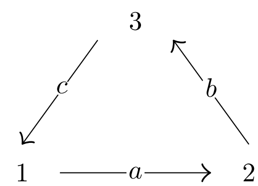
Now let $A$ be a $k$-algebra. Let $\{e_i\}$ be a complete set of primitive idempotents for $A$, that is, $A \cong \oplus A e_i$.

**Definition 2.1.9.** We say that $A$ is *basic* if $A \cong \oplus A e_i$ is the direct sum of pairwise non-isomorphic indecomposable summands.

**Definition 2.1.10.** Let $R_Q^n$ denote the two-sided ideal generated by all paths of length $n$. We say that a two-sided ideal $I \subset kQ$ is *admissible* if there exists $l \in \mathbb{N}$ such that $R_Q^l \subset I \subset R_Q^2$.

The ideal $I$ is called an *ideal of relations*. Admissible ideals contain every combination of paths of length bigger than $l$. Thus, if we quotient $kQ$ by $I$, then we obtain a finite-dimensional algebra, since there is a finite number of residue classes of paths. Note that if $Q$ has no oriented cycle, then any ideal $I \subset R_Q^2$ is admissible, since there exists then $l$ such that $R_Q^l = 0$.

**Example 2.1.11.** Consider the quiver $Q$




bound by the relations $I = \langle ac \rangle$. The ideal is admissible since

$$R_Q^4 \subset I \subset R_Q^2 = \langle ac,\, ba,\, cb \rangle.$$

The path algebra $kQ/I$ has basis $[e_1]$, $[e_2]$, $[e_3]$, $[a]$, $[b]$, $[c]$, $[ba]$, $[cb]$, $[cba]$, and is finite-dimensional.

The following is a famous theorem of Pierre Gabriel, which describes finite-dimensional $k$-algebras as path algebras with relations.

**Theorem 2.1.2** (Gabriel)**.** *Let $A$ be a finite-dimensional basic $k$-algebra. Then there exists a quiver $Q$ and an admissible ideal $I$ in $kQ$ such that*

$$A \cong kQ/I.$$

Note that the ideal of relations $I$ is not unique. However, the quiver $Q$ is uniquely determined by $A$ if we choose $I \subset R_Q^2$.

*Remark* 1. Any finite-dimensional hereditary algebra is Morita equivalent to $kQ$, where $Q$ is a quiver without oriented cycles. Conversely, every path algebra $kQ$ is hereditary.

When studying representation theory of algebras, we are usually interested in the module category over these algebras.

**Definition 2.1.12.** Two rings $R$ and $S$ are said to be *Morita equivalent* if there is an equivalence of categories

$$R - \text{Mod} \cong S - \text{Mod},$$

where $R$-Mod denotes the category of left $R$-modules, and similarly for $S$.

Intuitively, Morita equivalent rings are the same from the point of view of representation theory. They are characterized as follows. Let $M_d(A)$ denote the $d \times d$ matrix-algebra over an algebra $A$, that is, the algebra whose elements are $d \times d$ matrices with entries in $A$. We say that an idempotent $f$ in $A$ is *full* if $AfA = A$.

**Theorem 2.1.3.** *Two algebras $A$ and $B$ are Morita equivalent if and only if there exists a positive integer $d$, and a full idempotent $f \in M_d(B)$ such that*

$$A \cong fM_d(B)f.$$

One of the key facts is that every finite-dimensional $k$-algebra is Morita equivalent to a basic algebra. Therefore, by Gabriel's theorem, we get that every finite-dimensional algebra is Morita equivalent to the path algebra of some quiver modulo admissible relations. Thus, from the point of view of representation theory, studying path algebras of quivers is fundamental.

In this thesis, we will also deal with infinite-dimensional $k$-algebras. However, all $k$-algebras we consider are tensor algebras of the form

$$T_S V / \langle M \rangle,$$

where $S$ is a finite-dimensional semisimple algebra, $V$ a finitely generated $S$-bimodule and $M$ an ideal of relations. In this case, these algebras are also Morita equivalent to the path algebra over some quiver with a finite number of vertices and arrows. The ideal of relations $M$ is not necessarily admissible. In particular, it is not admissible if the algebra is infinite-dimensional.

### 2.1.3 Classical Preprojective algebras

Classical Preprojective algebras appear naturally in the classification of modules over hereditary algebras. In this subsection, we let $\Lambda$ be a finite-dimensional hereditary algebra. All modules are finite-dimensional.

**Definition 2.1.13.** We say that a hereditary algebra $\Lambda$ is *representation-finite* (resp. *representation-infinite*) if there are finitely (resp. infinitely) many isomorphism classes of finite-dimensional indecomposable left $\Lambda$-modules.

**Theorem 2.1.4** (Gabriel)**.** *A hereditary algebra $\Lambda$ is representation-finite if and only if it is Morita equivalent to $kQ$, where $Q$ is a quiver whose underlying graph is a Coxeter-Dynkin diagram of type A-D-E.*

Let $D(-) = \mathrm{Hom}_k(-, k)$ denote the $k$-dual. We consider as well the duality $\mathrm{Hom}_\Lambda(-, \Lambda)$.

**Definition 2.1.14.** The *Nakayama functor* is defined by $\nu(-) := D\mathrm{Hom}_\Lambda(-, \Lambda)$. It gives an equivalence of categories between projective left modules and injective left modules. The inverse functor is given by $\nu^{-1}(-) = \mathrm{Hom}_\Lambda(D\Lambda, -)$.

The *Auslander-Reiten* translate of a left $\Lambda$-module $M$ is given by $\tau(M) = D\mathrm{Ext}^1_\Lambda(M, \Lambda)$. Its inverse is defined as $\tau^{-1}(M) = \mathrm{Ext}^1_\Lambda(D\Lambda, M)$. They give inverse bijections from isomorphism classes of non-projective indecomposable modules to isomorphism classes of non-injective indecomposable modules.

The motivation for defining these objects comes up when trying to classify or understand modules over a certain algebra. If $\Lambda$ is representation-infinite or, more precisely, so called *wild*, this is impossible to do completely. However, the Auslander-Reiten translate allows us, among many other uses, to describe three families of indecomposable modules.

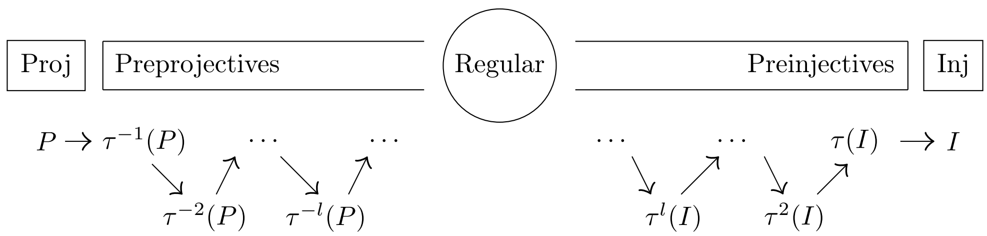
**Definition 2.1.15.** Let $M$ be an indecomposable $\Lambda$-module. We say that

- $M$ is *preprojective* if $M = \tau^{-m}(P)$ for some $m \geq 0$ and indecomposable projective module $P$.
- $M$ is *preinjective* if $M = \tau^{m}(I)$ for some $m \geq 0$ and indecomposable injective module $I$.
- $M$ is *regular* otherwise.

**Lemma 2.1.5.** *Let $M$ and $N$ be indecomposable left modules.*

- *If $N$ is preprojective and $M$ is not, then $Hom_\Lambda(M,N)=0$ and $Ext^1_\Lambda(N,M)=0$.*
- *If $N$ is preinjective and $M$ is not, then $Hom_\Lambda(N,M)=0$ and $Ext^1_\Lambda(M,N)=0$.*

We obtain the following picture for the $\Lambda$-modules.




In this figure, the arrows represent maps between the irreducible modules. By Lemma 2.1.5, there is no map going in the opposite direction.

**Definition 2.1.16** ([GP79], [BGL87])**.** The (classical) preprojective algebra $\Pi(\Lambda)$ over a finite-dimensional hereditary algebra $\Lambda$ is defined as the direct sum over the representatives of the isomorphism classes of preprojective modules:

$$\Pi = \Pi(\Lambda) := T_\Lambda Ext^1_\Lambda(D\Lambda,\Lambda) = \bigoplus_{l\geq 0} \tau^{-l}(\Lambda).$$

The multiplication is induced from the bijective map

$$\mathrm{Hom}_\Lambda(\Lambda,\tau^{-l}(\Lambda)) \otimes_\Lambda \mathrm{Hom}_\Lambda(\Lambda,\tau^{-m}(\Lambda)) \to \mathrm{Hom}_\Lambda(\Lambda,\tau^{-(l+m)}(\Lambda))$$

$$u \otimes v \mapsto \tau^{-m}(u)\circ v,$$

using the isomorphism $\mathrm{Hom}_\Lambda(\Lambda,X)\cong X$ for any left $\Lambda$-module $X$.

The algebra $\Pi(\Lambda)$ is finite-dimensional if $\Lambda$ is representation-finite. Otherwise it is infinite-dimensional.

## 2.2 Higher Preprojective algebras

Higher preprojective algebras appear in Iyama's program of higher Auslander-Reiten theory. The aim is to generalize this theory to algebras of higher global dimension. The notions of higher representation-infinite algebras and higher preprojective algebras are defined in this context. This theory works mainly in the setting of derived categories, which we will explain in a general matter, skipping over some details, since they are not needed for the next chapters of this thesis. The author refers to [Wei94] for a more complete account.

**Definition 2.2.1.** Let $X_\bullet$ and $Y_\bullet$ be chain complexes. A *quasi-isomorphism* is a morphism of chain complexes $X_\bullet \to Y_\bullet$ such that the induced morphisms

$$H^n(X_\bullet) \to H^n(Y_\bullet)$$

are isomorphisms for all $n$.

Let $\Lambda$ be a finite-dimensional $k$-algebra.

**Definition 2.2.2.** Let $\mathcal{A}$ be an abelian category (e.g. the category mod $\Lambda$ of finite-dimensional left $\Lambda$-modules in our context). The *bounded derived category* $\mathcal{D}^b(\mathcal{A})$ has objects given by the bounded chain complexes, that is, complexes with only finitely many non-zero components. The morphisms are obtained by

- identifying chain homotopic morphisms in the homotopy category of bounded chain complexes $K^b(\mathcal{A})$;
- localizing at the set of quasi-isomorphisms. Namely, morphisms $X_\bullet \to Y_\bullet$ are given by

$$X_\bullet \xleftarrow{\alpha} X'_\bullet \xrightarrow{\alpha'} Y,$$

  where $\alpha$ is a quasi-isomorphism and $\alpha'$ a morphism of chain complexes.

In the derived category, quasi-isomorphisms become isomorphisms. Localization induces a functor $Q_{\mathcal{A}} : K^b(\mathcal{A}) \to \mathcal{D}^b(\mathcal{A})$.

We denote the category $\mathcal{D}^b(\text{mod }\Lambda)$ by $\mathcal{D}^b(\Lambda)$.

**Definition 2.2.3.** Let $\Lambda$ be a finite-dimensional $k$-algebras. Let $M \in \text{mod }\Lambda$. Let $\mathcal{I} \subset \text{mod }\Lambda$ be the full subcategory of all injectives. For all objects $A_\bullet \in K^b(\Lambda)$ there exists $I_\bullet \in K^b(\mathcal{I})$, and a quasi-isomorphism $A_\bullet \to I_\bullet$. The composition of the inclusion $\mathcal{I} \subset \text{mod }\Lambda$ with $Q_{\text{mod }\Lambda}$ gives a natural functor $\iota : K^b(\mathcal{I}) \to D^b(\Lambda)$, with quasi-inverse denoted by $\iota^{-1}$. Now consider the functor $F := \text{Hom}_\Lambda(A, -) : \text{mod }\Lambda \to Ab$, the category of abelian groups. This functor induces a functor $K(F) : K^b(\Lambda) \to K^b(Ab)$. The *right derived functor* of $\text{Hom}(A, -)$, denoted by $\mathbf{R}\text{Hom}_\Lambda(A, -)$, is defined as

$$\mathbf{R}\text{Hom}_\Lambda(A, -) = Q_{\text{mod }\Lambda} \circ K(F) \circ \iota^{-1} : D^b(\Lambda) \to D^b(Ab).$$

It has the property that, if $A, B \in \text{mod }\Lambda$ (regarded as complexes concentrated in degree 0), then

$$\text{Ext}^i_\Lambda(A, B) = H^i(\mathbf{R}\text{Hom}_\Lambda(A, B)) \cong \text{Hom}_{\mathcal{D}^b(\Lambda)}(A, B[i]),$$

where $X_\bullet[i]$ denotes the shift in degree $i$, that is, $X_\bullet[i] = X_{\bullet - i}$. We can define dually the notion of *left derived functor* for the tensor product $A \otimes_\Lambda -$, denoted by $A \overset{\mathbf{L}}{\otimes}_\Lambda -$.

Now, let $\Lambda$ be a finite-dimensional algebra of finite global dimension. We can generalize the notions of Nakayama functors and Auslander-Reiten translates.

**Definition 2.2.4.** The *Nakayama functors* $\nu$ and $\nu^{-1}$ are defined as

$$\nu := D\mathbf{R}\text{Hom}_\Lambda(-, \Lambda) : \mathcal{D}^b(\Lambda) \to \mathcal{D}^b(\Lambda),$$
$$\nu^{-1} := \mathbf{R}\text{Hom}_{\Lambda^{op}}(D-, \Lambda) : \mathcal{D}^b(\Lambda) \to \mathcal{D}^b(\Lambda),$$

where $\Lambda^{op}$ denotes the opposite ring. These functors satisfy the isomorphism $\nu \cong (D\Lambda) \overset{\mathbf{L}}{\otimes}_\Lambda -$ and $\nu^{-1} = \mathbf{R}\text{Hom}_\Lambda(D\Lambda, -)$.

The *n-Auslander-Reiten translates* are defined by

$$\tau_n := D\mathrm{Ext}^n_\Lambda(-,\Lambda) : \mathrm{mod}\ \Lambda \to \mathrm{mod}\ \Lambda,$$
$$\tau_n^{-1} := \mathrm{Ext}^n_{\Lambda^{op}}(D-,\Lambda) : \mathrm{mod}\ \Lambda \to \mathrm{mod}\ \Lambda.$$

They are related to the Nakayama functors by $\tau_n = H^0(\nu_n -)$ and $\tau_n^{-1} = H^0(\nu_n^{-1} -)$.

The Nakayama functor $\nu$ is a Serre functor of $\mathcal{D}^b(\Lambda)$, which means that there exists a bifunctorial isomorphism

$$\mathrm{Hom}_{\mathcal{D}^b(\Lambda)}(X,Y) \cong D\mathrm{Hom}_{\mathcal{D}^b(\Lambda)}(Y,\nu(X))).$$

Let $\mathbb{S} := \nu$ denote the Serre functor in $\mathcal{D}^b(\Lambda)$ and denote by $\mathbb{S}_n$ the composition $\mathbb{S}_n := \mathbb{S} \circ [-n]$.

**Definition 2.2.5** ([HIO14])**.** Let $\Lambda$ be a finite-dimensional algebra of global dimension $n$. We say that $\Lambda$ is *n-representation-infinite* if

$$\mathbb{S}_n^{-l}(\Lambda) \in \mathrm{mod}\ \Lambda$$

for any $l \geq 0$.

By definition, $\mathbb{S}_n^{-l}(\Lambda)$ is a complex, so the condition means that it has only cohomology in degree 0. This cohomology is then necessarily equal to

$$\mathrm{Ext}^n_\Lambda(D\Lambda,\Lambda)^{\otimes_\Lambda l} = \tau_n^{-l}(\Lambda).$$

The fact that $\Lambda$ is finite-dimensional is important here. Indeed, the Serre functor is not defined in the context of infinite-dimensional algebras. Moreover, many of the results in (higher)-Auslander-Reiten theory rely on that fact.

**Definition 2.2.6** (See, e.g., [HIO14])**.** Let $\Lambda$ be an $n$-representation-infinite algebra. The $(n+1)$-*preprojective algebra* of $\Lambda$ is defined as the tensor algebra

$$\Pi(\Lambda) := T_\Lambda \mathrm{Ext}^n_\Lambda(D\Lambda,\Lambda) \cong \bigoplus_{l\geq 0} \mathbb{S}_n^{-l}(\Lambda) \cong \bigoplus_{l \geq 0} \tau_n^{-l}(\Lambda).$$

These notions generalize the concepts of representation-infinite algebras and of preprojective algebras for algebras of higher global dimension. In particular, the $(n+1)$-preprojective algebra is always infinite-dimensional and has global dimension equal to $(n+1)$.

**Definition 2.2.7.** Let $B$ be a positively graded $k$-algebra and $a \in \mathbb{Z}_{\geq 0}$. We say that $B$ is *(bimodule) n-Calabi-Yau of Gorenstein parameter a* if $B \in \mathrm{per}\ B^e$, the category of bounded complexes of finitely generated projective $B^e$-modules, and there exists a bounded graded projective $B$-bimodule resolution $P_\bullet$ of $B$ and an isomorphism of graded $B$-bimodules

$$P_\bullet \cong P_\bullet^\vee[n](-a),$$

where $(-)^\vee = \mathrm{Hom}_{B^e}(-,B^e)$.

We call an algebra *(bimodule) n-Calabi-Yau* if it satisfies the previous homological property, forgetting the grading.

**Theorem 2.2.1** ([Kel11], [MM11], [HIO14], [AIR15])**.** *There is a one-to-one correspondence between isomorphism classes of $n$-representation-infinite algebras $\Lambda$ and isomorphism classes of bimodule $(n+1)$-Calabi-Yau algebras $B$ of Gorenstein parameter $1$ such that $B_0$ is finite-dimensional, given by*

$$\Lambda \mapsto \Pi(\Lambda)$$
$$B_0 \leftarrow\!\shortmid B$$

In this paper, we reserve the term *representation-infinite algebra* to finite-dimensional algebras and the term *preprojective algebra* for (higher) preprojective algebras over finite-dimensional algebras. When we use the concept of *bimodule Calabi-Yau of Gorenstein parameter* 1, we do not necessarily require the degree 0 part to be finite-dimensional. In the context of higher Auslander-Reiten theory, the fact that representation-infinite algebras are finite-dimensional is important. The goal of this thesis is to show that this hypothesis may be restrictive in some cases.

## 2.3 Skew-group algebras and the McKay quiver

Let $R$ be the polynomial ring in $n$ variables. One of the main goals of this thesis is to determine for which finite subgroup $G < SL(n,k)$ the skew-group algebra $R\#G$ is Morita equivalent to a higher preprojective algebra. We give in this section the necessary notions to understand this question, which will be treated in detail in Chapter 5.

Throughout this section, let $R = k[x_1, \dots, x_n] = k[V]$ be the polynomial ring in $n$ variables and let $G$ be a finite subgroup of $SL(n,k)$ acting on $R$, via linear change of coordinates:

$$(g \cdot v)(x) := v(g^{-1}(x)),$$

where $v \in k[V]$ and $x \in V$.

**Definition 2.3.1.** The *skew-group algebra* $R\#G$ is defined as

$$R\#G = R \otimes_k kG$$

as vector space, and the multiplication is determined by

$$(v_1, g_1) \cdot (v_2, g_2) = (v_1 g_1(v_2), g_1 g_2),$$

where $(v_i, g_i) = v_i \otimes g_i$. We use this notation to distinguish with other tensor products that will come later.

Note that, as described for example in [BSW10], the skew-group algebra satisfies

$$R\#G \cong T_{kG}(V \otimes_k kG)/\langle M \otimes_k kG \rangle,$$

where $M$ is the vector space of commutativity relations in $R = k[V]$, and the bimodule action of $kG$ on $V \otimes_k kG$ is given by

$$g_1(v \otimes h)g_2 = g_1 v \otimes g_1 h g_2.$$

The algebra structure of $R\#G$ is related to the McKay quiver of $G$.

**Definition 2.3.2** ([McK83]). Let $G$ be a finite subgroup of $GL(n,k)$. The *McKay quiver* $M_G$ of $G$ has a vertex set in which each vertex corresponds to an irreducible representation $\rho$ of $G$. Let $V$ be the given representation of dimension $n$, that is, the representation described by $G \to GL(n,k)$, $g \mapsto g$. Let $S_\rho$ be the simple $kG$-module corresponding to $\rho$. The number of arrows $\rho_1 \to \rho_2$ in $M_G$ is equal to

$$\dim_k(\mathrm{Hom}_{kG}(S_{\rho_2}, V \otimes_k S_{\rho_1})).$$

**Example 2.3.3.** Let $G \cong A_4$ be the alternating group of order 12 as an abstract group, generated by the permutation matrices in $SL(4,k)$. Its character table is given by

| $A_4$ | (1) | (12)(34) | (123) | (132) |
|---|---|---|---|---|
| $\chi_1$ | 1 | 1 | 1 | 1 |
| $\chi_2$ | 1 | 1 | $\omega$ | $\omega^2$ |
| $\chi_3$ | 1 | 1 | $\omega^2$ | $\omega$ |
| $\chi_4$ | 3 | $-1$ | 0 | 0 |

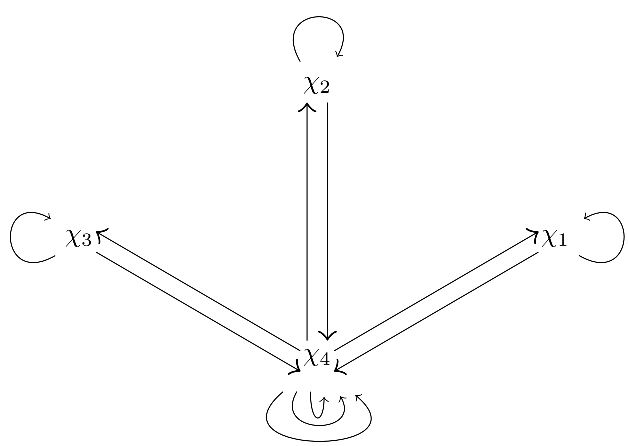
The conjugacy classes are represented by the following matrices:

$$(12)(34) \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \ (123) \mapsto \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ (123) \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The given representation is $V = \chi_1 \oplus \chi_4$. The McKay quiver has four vertices, corresponding to the irreducible representations. We have

$$\begin{aligned}
V \otimes_k \chi_1 &= \chi_1 \oplus \chi_4, && \text{arrows: } \chi_1 \to \chi_1, \quad \chi_1 \to \chi_4 \\
V \otimes_k \chi_2 &= \chi_2 \oplus \chi_4, && \text{arrows: } \chi_2 \to \chi_2, \quad \chi_2 \to \chi_4 \\
V \otimes_k \chi_3 &= \chi_3 \oplus \chi_4 && \text{arrows: } \chi_3 \to \chi_3, \quad \chi_3 \to \chi_4 \\
V \otimes_k \chi_4 &= 3\chi_4 \oplus \chi_1 \oplus \chi_2 \oplus \chi_3 && \text{arrows: } \chi_4 \to \chi_4 \text{ (3 times)}, \quad \chi_4 \to \chi_1, \quad \chi_4 \to \chi_2, \quad \chi_4 \to \chi_3
\end{aligned}$$

The McKay quiver of $G$ is given by



**Theorem 2.3.1** (See, e.g., [GMV02])**.** *Let $G$ be a finite subgroup of $GL(n,k)$, then $R\#G$ is Morita equivalent to $kM_G/\langle M\rangle$, where $M_G$ is the McKay quiver of $G$ and $M$ is the $kG$-bimodule of relations induced from the commutativity relations in $R$.*

*Remark* 2. If two finite subgroups $G_1$ and $G_2$ in $SL(n,k)$ are conjugate, then their skew-group algebras are isomorphic. This implies that their McKay quivers are the same. This is due to the fact that their given representation are isomorphic. Thus, relevant properties on a finite subgroup of $SL(n,k)$ are often stated up to conjugacy.

When $G$ is abelian, the Morita equivalence is an isomorphism, since $R\#G$ is then basic. There is thus an explicit correspondence between the arrows in $M_G$ and the variables in $R$. In this case, we have a description of the McKay quiver of $G$. The group is simultaneously diagonalizable so we can assume that all its elements are diagonal matrices. Let $\rho_i : G \to k^*$ be defined as $\rho_i(g) = \alpha_i$, where $\alpha_i$ is the $i$-th diagonal element of $g$, for $i = 1,\dots,n$. Then the McKay quiver of $G$ is described as follows.

**Theorem 2.3.2** ([CMT07])**.** *Let $G$ be an abelian finite subgroup of $GL(n,k)$. The McKay quiver $M_G$ of $G$ has an arrow $x_i^\rho : \rho \to \rho\rho_i$ for each irreducible representation $\rho$ and all $i = 1,\dots,n$. We say that $x_i^\rho$ is an* arrow of type $x_i$.

The relations are then induced from the relations in $R$. That is, the space $M$ from Theorem 2.3.1 is given by

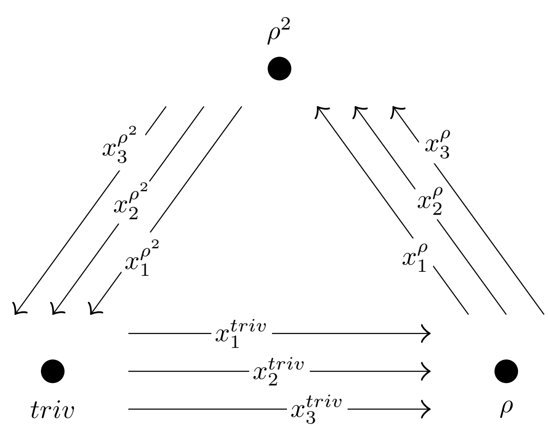
$$\{x_i^{\rho\rho_{i'}} x_{i'}^{\rho} - x_{i'}^{\rho\rho_i} x_i^{\rho} \mid 1 \le i \ne i' \le n \text{ , } \rho \text{ irreducible}\}.$$

If $G$ is a cyclic group of order $r$, then it is generated by a diagonal matrix with entries $\xi^{a_1},\dots,\xi^{a_n}$, where $\xi$ is a primitive $r^{\text{th}}$ root of unity. We denote this group by

$$G = \langle \frac{1}{r}(a_1,\dots,a_n)\rangle.$$

*Remark* 3. If $G$ is non-abelian, then there is no abstract characterization of the McKay quiver of a group $G$. Moreover, it is very hard to describe the relations $M$ induced from the commutativity relations in $R$.

**Example 2.3.4.** Consider the group $G = \langle\frac{1}{3}(1,1,1)\rangle$. Let $R = k[x_1,x_2,x_3]$. The group $G$ has three irreducible representations $triv$, $\rho$ and $\rho^2$, where $triv = \rho^3$ is the trivial representation and $\rho$ is the representation given by the first diagonal elements of the objects in $G$. The skew-group algebra $R\#G$ is isomorphic to $kM_G/\langle M\rangle$, where $M_G$ is the quiver



The relations $\langle M \rangle$ are the ones described above.

## 2.4 Koszul algebras

We describe Koszul algebras, following mostly [BGS96]. One of our main strategies in this thesis is to use the minimal resolution of the algebras we consider to derive a preprojective algebra structure on them, using Theorem 2.2.1. The problem is that constructing minimal resolutions is difficult in general. Fortunately for us, Koszul algebras have an explicit minimal resolution. It also makes sense to consider these algebras, since skew-group algebras are Koszul. Throughout this subsection, we let $A$ be a positively graded $k$-algebra such that $S := A_0$ is a finite-dimensional semisimple algebra. All tensor products are over $S$.

**Definition 2.4.1.** The algebra $A$ is *Koszul* if $S$, considered as a left graded $A$-module, admits a graded projective resolution

$$\cdots \to P^2 \to P^1 \to P^0 \to S \to 0$$

such that $P^l$ is generated in degree $l$.

It is well-known that any Koszul algebra is quadratic, that is, it is given by a tensor algebra

$$T_S V / \langle M \rangle,$$

where $V$ is a $S$-bimodule placed in degree 1 and $M$ is a $S$-bimodule such that $M \subset V \otimes V$.

**Definition 2.4.2.** The *Koszul complex* is described as

$$\cdots \to P^n \to P^{n-1} \to \cdots \to P^0 \to 0,$$

where each $P^l$ is given by

$$A \otimes \bigcap_{\mu} (V^{\otimes \mu} \otimes M \otimes V^{\otimes l-\mu-2}) \subset A \otimes V^{\otimes l}$$

and the differentials are the restrictions of the maps

$$A \otimes V^{\otimes l} \to A \otimes V^{\otimes (l-1)},$$

given by

$$a \otimes v_1 \otimes \cdots \otimes v_l \mapsto av_1 \otimes v_2 \otimes \cdots \otimes v_l.$$

We denote $\bigcap_{\mu} (V^{\otimes \mu} \otimes M \otimes V^{\otimes l-\mu-2})$ by $K_l$.

**Theorem 2.4.1** (see, e.g., [BGS96]). *Let $A = T_S V \langle M \rangle$ be a quadratic algebra. Then $A$ is Koszul if and only if the Koszul complex is a projective left $A$-module resolution of $S$.*

In this case, the Koszul resolution is a subresolution of the *Bar resolution* $\beta(A, S)$ of $S$:

$$\beta(A,S) := \cdots \to A \otimes A_+^{\otimes n} \to A \otimes A_+^{\otimes (n-1)} \to \cdots \to A,$$

where $A_+ := \oplus_{i\geq 1} A_i$, with differentials given by

$$1 \otimes (a_1 \otimes \cdots \otimes a_n) \mapsto a_1 \otimes (a_2 \otimes \cdots \otimes a_n) + \sum_{1\leq l\leq n-1} (-1)^l 1 \otimes (a_1 \otimes \cdots a_{l-1} \otimes a_l a_{l+1} \otimes a_{l+2} \otimes \cdots \otimes a_n).$$

This resolution is often used to compute cohomology, as it always exists. However, it is far from being a minimal resolution. We note that

$$K_l \cong \mathrm{Tor}_l^A(S, S).$$

The Koszul resolution also gives rise to a projective $A$-bimodule resolution of $A$:

$$\cdots \to P^n \to P^{n-1} \to \cdots \to P^0 \to A \to 0,$$

where each $P^l$ is given by

$$A \otimes \bigcap_{\mu} (V^{\otimes \mu} \otimes M \otimes V^{\otimes l-\mu-2}) \otimes A \subset A \otimes V^{\otimes l} \otimes A$$

and the differentials are the restrictions of the maps

$$A \otimes V^{\otimes l} \otimes A \to A \otimes V^{\otimes (l-1)} \otimes A,$$

given by

$$a \otimes v_1 \otimes \cdots \otimes v_l \otimes a' \mapsto av_1 \otimes v_2 \otimes \cdots \otimes v_l \otimes a' + (-1)^l a \otimes v_1 \otimes \cdots \otimes v_{l-1} \otimes v_l a'.$$

*Remark* 4. In this thesis, we generally deal with two gradings. The first one is the natural grading coming from the tensor algebra structure

$$A = T_S V / \langle M \rangle.$$

We consider Koszul algebras with respect to this grading. If $A$ is Calabi-Yau of Gorenstein parameter 1, then we have a second grading on $A$ giving this property. In particular, when $A$ is a preprojective algebra, this grading is given by the natural grading structure on the tensor algebra

$$A = T_\Lambda \mathrm{Ext}^n_\Lambda(D\Lambda, \Lambda).$$

It is in general different from the grading structure coming from $T_S V/\langle M \rangle$.

**Example 2.4.3.** The polynomial ring

$$R = k[x_1, \ldots, x_n] = k[V] = T_k V / \langle M \rangle,$$

where $M$ is the ideal of commutativity relations, is $n$-Calabi-Yau and Koszul. From Example 2.1.3, the complex

$$P_\bullet = R \otimes_k \bigwedge^n V \otimes_k R \to \cdots \to R \otimes_k R \to R$$

is the minimal bimodule resolution of $R$. This is the Koszul resolution of $R$. In fact, there is a vector space isomorphism

$$\bigwedge^l V \cong \bigcap_\mu \left(V^{\otimes\mu} \otimes_k M \otimes_k V^{\otimes l-\mu-2}\right),$$

where the second module is the description of the terms in the Koszul resolution, given by

$$x_{j_l} \wedge x_{j_{l-1}} \wedge \cdots \wedge x_{j_1} \mapsto \sum_{\sigma\in\mathfrak{S}_l} (-1)^{|\sigma|}\sigma(x_{j_l} \otimes x_{j_{l-1}} \otimes \cdots \otimes x_{j_1}),$$

where the elements $\sigma$ of the symmetric group $\mathfrak{S}_l$ act as

$$\sigma(x_{j_l} \otimes x_{j_{l-1}} \otimes \cdots \otimes x_{j_1}) := x_{\sigma^{-1}(j_l)} \otimes v_{\sigma^{-1}(j_{l-1})} \otimes \cdots \otimes v_{\sigma^{-1}(j_1)}$$

and $|\sigma|$ denotes the parity of $\sigma$. The pairing

$$\bigwedge^l V \times \bigwedge^{n-l} V \to k : (v_1, v_2) \mapsto a,$$

where $a$ is such that $v_1 \wedge v_2 = ax_1 \wedge x_2 \wedge \ldots \wedge x_n$, gives a $R$-bimodule isomorphism

$$P_\bullet \cong \mathrm{Hom}_{R^e}(P_\bullet, R^e)[n],$$

which implies the Calabi-Yau property.

Using this example, we obtain that the skew-group algebra $R\#G$ is a Koszul n-Calabi-Yau algebra.

**Theorem 2.4.2** ([BSW10], [HIO14], [AIR15])**.** *Let $G < SL(n,k)$ be finite. Let $V$ be the given representation of $G$. The following complex is a minimal projective $B$-bimodule resolution of $B$:*

$$P_\bullet = B \otimes_{kG} \left(\bigwedge^n V \otimes_k kG\right) \otimes_{kG} B \to B \otimes_{kG} \left(\bigwedge^{n-1} V \otimes_k kG\right) \otimes_{kG} B \to \cdots \to B \otimes_{kG} B.$$

*Moreover, there is a $B$-bimodule isomorphism $P_\bullet \cong P_\bullet^\vee[n]$, where $(-)^\vee = Hom_{B^e}(-, B^e)$.*

*Proof.* Tensoring the Koszul bimodule resolution of $R = k[V]$, given by

$$0 \to R \otimes_k \bigwedge^n V \otimes_k R \to \cdots \to R \otimes_k R \to R \to 0,$$

over $k$ with $kG$ is an exact functor that gives the required resolution. In fact, we have

$$R \otimes_k \bigwedge^l V \otimes_k R \otimes_k kG \cong R \otimes_k kG \otimes_{kG} \bigwedge^l V \otimes_k kG \otimes_{kG} R \otimes_k kG$$

$$\cong R\#G \otimes_{kG} \bigwedge^l V \otimes_k kG \otimes_{kG} R\#G.$$

To prove the second statement, the pairing described in Example 2.4.3

$$\bigwedge^l V \times \bigwedge^{n-l} V \to k : (v_1, v_2) \mapsto a,$$

induces a pairing

$$B \otimes_{kG} \bigwedge^l V \otimes_k kG \otimes_{kG} B \times B \otimes_{kG} \bigwedge^{n-l} V \otimes_k kG \otimes_{kG} B \to B \otimes_{kG} B,$$

which gives the required property. □

The differentials are induced from the differentials in the Koszul resolution of $R$, described in Example 2.1.3. They are defined as

$$\begin{aligned}
&\delta_l((v_1, g_1) \otimes (x_{j_l} \wedge x_{j_{l-1}} \wedge \cdots \wedge x_{j_1} \otimes g) \otimes (v_2, g_2)) \\
&= \delta_l((v_1, 1) \otimes (g_1 x_{j_l} \wedge g_1 x_{j_{l-1}} \wedge \cdots \wedge g_1 x_{j_1} \otimes 1) \otimes (g_1 g v_2, g_1 g g_2)) \\
&:= \sum_{i=1}^{l} (-1)^{i-1} ((v_1 g_1 x_{j_i}, 1) \otimes (g_1 x_{j_l} \wedge \cdots \widehat{g_1 x_{j_i}} \cdots \wedge g_1 x_{j_1} \otimes 1) \otimes (g_1 g v_2, g_1 g g_2) \\
&\qquad + (v_1, 1) \otimes (g_1 x_{j_l} \wedge \cdots \widehat{g_1 x_{j_i}} \cdots \wedge g_1 x_{j_1} \otimes 1) \otimes (g_1 x_{j_i} g_1 g v_2, g_1 g g_2)) \\
&= \sum_{i=1}^{l} (-1)^{i-1} ((v_1 g_1 x_{j_i}, g_1) \otimes (x_{j_l} \wedge \cdots \widehat{x_{j_i}} \cdots \wedge x_{j_1} \otimes g) \otimes (v_2, g_2) \\
&\qquad + (v_1, g_1) \otimes (x_{j_l} \wedge \cdots \widehat{x_{j_i}} \cdots \wedge x_{j_1} \otimes g) \otimes (g^{-1} x_{j_i} v_2, g_2)),
\end{aligned}$$

where

$$x_n \wedge \cdots \widehat{x_l} \cdots \wedge x_1 = x_n \wedge \cdots \wedge x_{l+1} \wedge x_{l-1} \wedge \cdots \wedge x_1.$$

*Remark* 5. Following Example 2.4.3, there is a $kG$-bimodule isomorphism

$$\bigwedge^l V \otimes_k kG \cong \bigcap_{\mu} \left( (V \otimes_k kG)^{\otimes \mu} \otimes_{kG} (M \otimes_k kG) \otimes_{kG} (V \otimes_k kG)^{\otimes l-\mu-2} \right),$$

where the second module is the usual description of the terms in the Koszul resolution, given by

$$x_{j_l} \wedge x_{j_{l-1}} \wedge \cdots \wedge x_{j_1} \otimes 1 \mapsto \sum_{\sigma \in \mathfrak{S}_l} (-1)^{\sigma} \sigma(x_{j_l} \otimes x_{j_{l-1}} \otimes \cdots \otimes x_{j_1}) \otimes 1,$$

where the elements of the symmetric group $\mathfrak{S}_l$ act as in the example. This shows that the skew-group algebra is indeed Koszul.

Another important class of Koszul algebras is given by classical preprojective algebras.

**Theorem 2.4.3** ([EE07]). *Let $\Lambda$ be a representation-infinite hereditary algebra. The 2-preprojective algebra $\Pi(\Lambda)$ over $\Lambda$ is Koszul.*

## 2.5 Superpotentials

Let $A = T_S V / \langle M \rangle$ be a $k$-algebra, where $S$ is a semisimple finite-dimensional $k$-algebra, and $V$ and $M$ are $S$-bimodules. We describe the notion of a superpotential, following [BSW10]. In this section, all tensor products are over $S$.

Let $\mathfrak{S}_l$ be the symmetric group. Let an element $\sigma \in \mathfrak{S}_l$ act on

$$v_l \otimes v_{l-1} \otimes \cdots \otimes v_1 \in V^{\otimes l}$$

by permuting the indices, that is,

$$\sigma(v_l \otimes v_{l-1} \otimes \cdots \otimes v_1) := v_{\sigma^{-1}(l)} \otimes v_{\sigma^{-1}(l-1)} \otimes \cdots \otimes v_{\sigma^{-1}(1)},$$

and extend this action linearly.

**Definition 2.5.1** ([BSW10])**.** A *superpotential* of degree $n$ is an element $\omega$ of degree $n$ in $T_S V$ that commutes with the $S$-action:

$$s\omega = \omega s \quad \forall s \in S,$$

and has the property that $\sigma(\omega) = (-1)^{n-1}\omega$, where $\sigma = (1\ 2\ \cdots\ n-1\ n)$ in $\mathfrak{S}_n$. We say that $\omega$ is *super-cyclically symmetric.*

**Definition 2.5.2** ([BSW10])**.** The *derivation-quotient* algebra of $\omega$ of order $l$ is defined as

$$D(\omega, l) := T_S V / \langle W_{n-l} \rangle,$$

where $W_{n-l}$ is generated by elements of the form $\delta_p \omega$, where $p = v_l \otimes \cdots \otimes v_1$ is an element of order $l$ in $T_S V$ and $\delta_p : V^{\otimes n} \to V^{\otimes n-l}$ is a linear map, called a *l-derivative*, defined as

$$\delta_p v := \begin{cases} r & \text{if } v = p \otimes r \\ 0 & \text{else.} \end{cases}$$

Consider the complex $W_\bullet :=$

$$0 \to A \otimes W_n \otimes A \xrightarrow{d} A \otimes W_{n-1} \otimes A \xrightarrow{d} \cdots \xrightarrow{d} A \otimes W_1 \otimes A \xrightarrow{d} A \otimes W_0 \otimes A \to 0,$$

where the differential $d : A \otimes W_l \otimes A \to A \otimes W_{l-1} \otimes A$ is given by

$$d(a \otimes v_l \otimes \cdots \otimes v_1 \otimes a') = \epsilon_l (a v_l \otimes v_{l-1} \otimes \cdots \otimes v_1 \otimes a' + (-1)^l a \otimes v_l \otimes \cdots \otimes v_2 \otimes v_1 a'),$$

where

$$\epsilon_l := \begin{cases} (-1)^{l(n-l)} & \text{if } l < (n+1)/2 \\ 1 & \text{else.} \end{cases}$$

For any superpotential, this complex is self-dual, leading to a Calabi-Yau property. Moreover, it is a subcomplex of the Koszul complex for $D(\omega, n-2)$.

**Theorem 2.5.1** ([BSW10])**.** *Let $A = T_S V / \langle M \rangle$. Then $A$ is a $n$-Calabi-Yau Koszul algebra if and only if $A \cong D(\omega, n-2)$, for some superpotential $\omega$ of degree $n$, and $W_\bullet$ is exact in positive degree with $H^0(W_\bullet) = A$. Moreover, in this case, $\omega$ is a $S$-bimodule generator of $K_n \cong W_n$.*

By Theorem 2.2.1, preprojective algebras are Calabi-Yau. Since we will consider mainly Koszul algebras, the previous theorem allows us to describe them as derivation-quotient algebras, and their relations can easily be described by superpotentials.

**Example 2.5.3.** We consider, as in Example 2.3.4, the skew-group algebra $A = k[x_1, x_2, x_3] \# G$, where $G = \langle \frac{1}{3}(1, 1, 1) \rangle$. In this case, the semisimple algebra $S = kG$ decomposes as $kG \cong triv \oplus \rho \oplus \rho^2$, where $\rho$ is the representation given by the first diagonal elements of the objects in $G$. The relations are given by

$$M = \{x_{i'}^{\rho^{j+1}} x_i^{\rho^j} - x_i^{\rho^{j+1}} x_{i'}^{\rho^j} \mid 1 \leq i \neq i' \leq 3,\ 0 \leq j \leq 2\}.$$

A $kG$-bimodule generator of $K_3 = V \otimes M \cap M \otimes V$ is given by

$$\omega = \sum_{j=0}^{2} (x_3^{\rho^{j+2}} \otimes x_2^{\rho^{j+1}} \otimes x_1^{\rho^j} - x_3^{\rho^{j+2}} \otimes x_1^{\rho^{j+1}} \otimes x_2^{\rho^j} + x_1^{\rho^{j+2}} \otimes x_3^{\rho^{j+1}} \otimes x_2^{\rho^j}$$

$$-x_1^{\rho^{j+2}} \otimes x_2^{\rho^{j+1}} \otimes x_3^{\rho^j} + x_2^{\rho^{j+2}} \otimes x_1^{\rho^{j+1}} \otimes x_3^{\rho^j} - x_2^{\rho^{j+2}} \otimes x_3^{\rho^{j+1}} \otimes x_1^{\rho^j}).$$

It is a superpotential in the algebra $k[x_1, x_2, x_3]\#G$. In fact, from looking at the McKay quiver in Example 2.3.4, we see that $\omega$ commutes with the action of $kG$, that is, it is a sum of closed paths. Moreover, it is clearly super-cyclically symmetric.

The relations are obtained by applying a 1-derivative on $\omega$. For example,

$$\delta_{x_3^{\rho^2}} \omega = x_2^{\rho^1} x_1^{\rho^0} - x_1^{\rho^1} x_2^{\rho^0}$$

gives a relation in $A$.

Note that, using Remark 5, we can identify $\omega$ with

$$x_3 \wedge x_2 \wedge x_1 \otimes 1 \in \bigwedge^3 V \otimes_k kG.$$

We will use this identification in Chapter 5.

# Chapter 3

# Quiver Construction of Preprojective algebras over Koszul algebras

Let $A = kQ$ be the path algebra over some quiver $Q$. Then $A$ is a basic hereditary algebra. The preprojective algebra

$$\Pi(kQ) = T_{kQ}\mathrm{Ext}^1_{kQ}(DkQ, kQ)$$

is a tensor algebra over $kQ$. It therefore has $kQ$ as a subalgebra in degree 0. To describe $\mathrm{Ext}^1_{kQ}(DkQ, kQ)$, Crawley-Boevey ([CB99]) employs techniques that rely on the minimal bimodule resolution of $kQ$:

$$0 \to \bigoplus_{a:i\to j\in Q_1} kQe_j \otimes_k e_i kQ \xrightarrow{f} \bigoplus_{i\in Q_0} kQe_i \otimes_k e_i kQ \xrightarrow{\text{mult}} kQ \to 0,$$

where $f$ sends $(e_j \otimes e_i)_a$ to $(e_j \otimes a)_j - (a \otimes e_i)_i$, and mult is the multiplication map. The preprojective algebra is itself a basic algebra of global dimension 2. It is thus given as the path algebra of a quiver with relations, as follows:

**Theorem 3.0.1** ([Rin98], [CB99])**.** *Let $Q = (Q_0, Q_1)$ be a quiver with no oriented cycles and consider its path algebra $kQ$. For each arrow $a : i \to j \in Q_1$, define $a^* : j \to i$. Consider $\tilde{Q} = (Q_0, \tilde{Q}_1)$, where $\tilde{Q}_1$ contains all the arrows $a$ of $Q_1$ as well as the new arrows $a^*$. Then, the preprojective algebra of $kQ$ is given by*

$$\Pi(kQ) \cong k\tilde{Q}/(\sum_{a\in Q_1}[a, a^*]),$$

*where $[a, a^*] = aa^* - a^*a$.*

The goal of this chapter is to generalize this quiver construction to higher preprojective algebras over basic Koszul $n$-representation-infinite algebras. Our generalization is inspired by the techniques developed in [CB99]. If $\Lambda$ is a basic finite-dimensional algebra of global dimension $n$, then the maps in the minimal bimodule resolution are unknown in general. However, if we add the assumption that $\Lambda$ is Koszul, then we can describe those maps, as seen in the preliminaries, and we can therefore generalize Theorem 3.0.1 in this case. This will apply in particular to skew-group algebras $R\#G$. We will also show that such algebras are derivation-quotient algebras, as defined in Definition 2.5.2, that is, their relations are given by a superpotential.

In this chapter, we let $\Lambda = T_S V/\langle M\rangle$ be a basic Koszul $n$-representation-infinite algebra, where $S$ is a finite-dimensional semisimple $k$-algebra, $V$ an $S$-bimodule and $M \subset V \otimes_S V$. Let $\Pi$ be a $(n+1)$-preprojective algebra over $\Lambda$. Then $\Pi$ is given by

$$\Pi = T_\Lambda \mathrm{Ext}^n(D\Lambda, \Lambda) \cong T_\Lambda \mathrm{Ext}^n_{\Lambda^e}(\Lambda, \Lambda^e).$$

## 3.1 Quiver construction

From [Hap89], we know that the Koszul bimodule resolution of $\Lambda$ is given by

$$0 \to \bigoplus_{\substack{p:i\to j\\ p\in K_n}} \Lambda e_j \otimes_k e_i\Lambda \xrightarrow{f} \bigoplus_{\substack{p:i\to j\\ p\in K_{n-1}}} \Lambda e_j \otimes_k e_i\Lambda \to \cdots \to \Lambda \otimes_S \Lambda \to \Lambda \to 0,$$

where $K_l$ is the $S$-bimodule defined in section 2.4, for $0 \le l \le n$, and the direct sums are taken over $k$-basis elements $p \in K_l$. If $p = v_n \otimes v_{n-1} \otimes \cdots \otimes v_2 \otimes v_1$ is a path from $i$ to $j$, and $p_1 = v_{n-1} \otimes \cdots \otimes v_1$ is a path from $i$ to $j'$ and $p_2 : v_n \otimes \cdots \otimes v_2$ is a path from $i'$ to $j$, then $f$ is defined as follows:

$$f((e_j \otimes e_i)_p) = (v_n e_{j'} \otimes e_i)_{p_1} + (-1)^n (e_j \otimes e_{i'} v_1)_{p_2}.$$

We extend this definition linearly. It is important to note that this resolution is exactly the Koszul resolution described in the preliminaries, via the isomorphism of $S$-bimodules:

$$\Lambda \otimes_S K_l \otimes_S \Lambda \cong \bigoplus_{\substack{p:i\to j\\ p\in K_l}} \Lambda e_j \otimes_k e_i\Lambda$$
$$1 \otimes p \otimes 1 \mapsto e_j \otimes_k e_i,$$

where $p : i \to j$. Moreover, the map $f$ is then the usual differential in the Koszul resolution.

Applying $\mathrm{Hom}_{\Lambda^e}(-, \Lambda^e)$ to the minimal resolution, we obtain a complex ending as follows:

$$\mathrm{Hom}_{\Lambda^e}(\bigoplus_{\substack{p:i\to j\\ p\in K_{n-1}}} \Lambda e_j \otimes_k e_i\Lambda, \Lambda^e) \xrightarrow{\tilde{f}} \mathrm{Hom}_{\Lambda^e}(\bigoplus_{\substack{p:i\to j\\ p\in K_n}} \Lambda e_j \otimes_k e_i\Lambda, \Lambda^e) \to 0,$$

where $\tilde{f}(\phi) := \phi \circ f$. In general, if $\Omega$ is a $\Lambda$-bimodule, then

$$\mathrm{Hom}_{\Lambda^e}(\Lambda e_j \otimes_k e_i\Lambda, \Omega) \cong e_j \Omega e_i$$
$$\phi \mapsto \phi(e_j \otimes e_i)$$
$$\phi_m : (be_j \otimes e_i b' \mapsto be_i m e_j b') \leftarrow e_j m e_i$$

Thus, taking $\Omega = \Lambda^e$, we get that

$$\mathrm{Hom}_{\Lambda^e}(\bigoplus_{\substack{p:i\to j\\ p\in K_l}} \Lambda e_j \otimes_k e_i\Lambda, \Lambda^e) \cong \bigoplus_{\substack{p:i\to j\\ p\in K_l}} \Lambda e_i \otimes_k e_j\Lambda.$$

Therefore,

$$\mathrm{Ext}^n_{\Lambda^e}(\Lambda,\Lambda^e) \cong \bigoplus_{\substack{p:i\to j\\ p\in K_n}} \Lambda e_i \otimes_k e_j\Lambda/\mathrm{Im}\tilde{f}.$$

We can view the $\Lambda$-bimodule

$$\bigoplus_{\substack{p:i\to j\\ p\in K_n}} \Lambda e_i \otimes_k e_j\Lambda$$

as a space of arrows going from $j \to i$ for every path $p : i \to j$ in $K_n$.

We now describe $\mathrm{Im}\tilde{f}$. We have the following chain of $\Lambda$-bimodule morphisms:

$$\bigoplus_{\substack{p:i\to j\\ p\in K_{n-1}}} \Lambda e_i \otimes_k e_j\Lambda \xrightarrow{\sim} \mathrm{Hom}_{\Lambda^e}(\bigoplus_{\substack{p:i\to j\\ p\in K_{n-1}}} \Lambda e_j \otimes_k e_i\Lambda, \Lambda^e)$$

$$\to \mathrm{Hom}_{\Lambda^e}(\bigoplus_{\substack{q:i\to j\\ q\in K_n}} \Lambda e_j \otimes_k e_i\Lambda, \Lambda^e) \xrightarrow{\sim} \bigoplus_{\substack{q:i\to j\\ q\in K_n}} \Lambda e_i \otimes_k e_j\Lambda,$$

with maps

$$\oplus_p(e_i\otimes e_j)_p \mapsto \oplus_p(\phi_p := (e_j\otimes e_i)_p \mapsto e_j\otimes e_i) \mapsto (\oplus_p\phi_p\circ f) \mapsto (\oplus_p\phi_p\circ f)(\oplus_q(e_j\otimes e_i)_q),$$

where $(-\otimes -)_p$ denotes the $p^{\text{th}}$ component in $\bigoplus_p \Lambda e_i \otimes_k e_j\Lambda$.

If $q : i \to j \in K_n \subset V^{\otimes_S n}$ is expressed as $q = \sum_m k_m v_n^m \otimes \cdots \otimes v_1^m$, for some $k_m \in k$, we get

$$\begin{aligned} f((e_j\otimes e_i)_q) &= \sum_m k_m((v_n^m e_{j'_m}\otimes e_i)_{q'_m} + (-1)^n(e_j\otimes e_{i''_m}v_1^m)_{q''_m}) \\ &= \sum_m k_m((e_j v_n^m\otimes e_i)_{q'_m} + (-1)^n(e_j\otimes v_1^m e_i)_{q''_m}) \\ &= \sum_m k_m((e_i\otimes e_j v_n^m)_{q'_m} + (-1)^n(v_1^m e_i\otimes e_j)_{q''_m}) =: \alpha, \end{aligned}$$

where

$$q'_m : i \to j'_m \in K_{n-1}^{n-1}, \qquad q'_m = k_m v_{n-1}^m \otimes\cdots\otimes v_1^m$$

and

$$q''_m : i''_m \to j \in K_{n-1}^{n-1}, \qquad q''_m = k_m v_n^m \otimes\cdots\otimes v_2^m.$$

Now, if $\rho_1 \neq \rho_2 \in K_{n-1}$, then $\phi_{\rho_1}((e_j\otimes e_i)_{\rho_2}) = 0$, otherwise it acts as the identity. Thus, applying $\phi_{\rho_1}$ to $\alpha$ singles out the elements in $\alpha$ associated to $\rho_1 \in K_{n-1}$.

We conclude that $\mathrm{Im}\tilde{f}$ is given by the following elements in

$$\bigoplus_{\substack{p:i\to j\\ p\in K_n}} \Lambda e_i \otimes_k e_j\Lambda.$$

$$\mathrm{Im}\tilde{f} = \left\{ \sum_{\substack{q'_m \subset q \in K_n \\ q'_m = p}} \sum_m k_m (e_i \otimes e_j v_n^m)_{q'_m} + \sum_{\substack{q''_m \subset q \in K_n \\ q''_m = p}} \sum_m k_m (-1)^n (v_1^m e_i \otimes e_j)_{q''_m} \quad | \quad p \in K_{n-1} \right\},$$

where the paths $q \in K_n^n$ are expressed as $q = \sum_m k_m v_n^m \otimes \cdots \otimes v_1^m$, for some $k_m \in K$ and $q'_m : i \to j'_m \in K_{n-1}^{n-1}$, $q'_m = k_m v_{n-1}^m \otimes \cdots \otimes v_1^m$ and $q''_m : i''_m \to j \in K_{n-1}^{n-1}$, $q''_m = k_m v_n^m \otimes \cdots \otimes v_2^m$.

Define $I := \mathrm{Im}\tilde{f}$. We proved the following theorem.

**Theorem 3.1.1.** *With the setting above, we obtain a $\Lambda$-bimodule isomorphism:*

$$Ext^n_{\Lambda^e}(\Lambda, \Lambda^e) \cong \bigoplus_{\substack{q: i \to j \\ q \in K_n}} \Lambda e_i \otimes_k e_j \Lambda / \langle I \rangle.$$

We can now describe explicitely the quiver of the preprojective algebra over a Koszul $n$-representation-infinite algebra.

**Corollary 3.1.2.** *Assume that $\Pi$ and $\Lambda$ are as above. Then the quiver of $\Pi$ is given by adding an arrow $a_q : j \to i$ for each generator $q : i \to j \in K_n$ in the quiver of $\Lambda$. In addition to the relations in $\Lambda$, $\Pi$ has new quadratic relations given by*

$$\sum_{q \in K_n} \delta_p^{\mathcal{L}} q a_q = 0$$

*and*

$$\sum_{q \in K_n} a_q \delta_p^{\mathcal{R}} q = 0,$$

*for each $p \in K_{n-1}$, where*

$$\delta_p^{\mathcal{L}} q := \begin{cases} a & \text{if } q = p \otimes a \\ 0 & \text{else,} \end{cases}$$

*and*

$$\delta_p^{\mathcal{R}} q := \begin{cases} a & \text{if } q = a \otimes p \\ 0 & \text{else.} \end{cases}$$

*Proof.* Recall that the preprojective algebra is given by

$$\Pi = T_\Lambda Ext^n_\Lambda(\Lambda, \Lambda^e).$$

By Theorem 3.1.1, we thus obtain that

$$\Pi \cong T_\Lambda \left( \bigoplus_{\substack{q: i \to j \\ q \in K_n}} \Lambda e_i \otimes_k e_j \Lambda / \langle I \rangle \right).$$

Since $\Pi$ is a tensor algebra, it is generated in degree 0 and in degree 1. The arrows in the quiver of $\Pi$ are therefore in degree 0 and in degree 1. Thus, the quiver of $\Lambda$, in degree 0, is a subquiver of the quiver of $\Pi$, and the relations in $\Lambda$ are relations in $\Pi$. The quiver of $\Pi$ has new arrows in degree 1, given by

$$\bigoplus_{\substack{q:i\to j\\ q\in K_n}} \Lambda e_i \otimes_k e_j \Lambda.$$

This $\Lambda$-bimodule is generated by elements of the form $(e_i \otimes e_j)_q$, where $q$ is a path from $i$ to $j$ and the bimodule action is given by

$$\alpha(e_i \otimes e_j)_q \alpha' = (\alpha e_i \otimes e_j \alpha')_q,$$

which is non-zero if $\alpha$ starts at vertex $e_i$ and $\alpha'$ ends at vertex $e_j$. Therefore, this generator is given by an arrow $a_q : j \to i$ in the quiver of $\Pi$, and the bimodule action is given by concatenation of paths. The new relations are described by the bimodule $\langle I\rangle$. From its description, we have that

$$I = \left\{ \sum_{\substack{q'_m \subset q\in K_n\\ q'_m = p}} \sum_m k_m a_q v_n^m + \sum_{\substack{q''_m \subset q\in K_n\\ q''_m = p}} \sum_m k_m (-1)^n v_1^m a_q \quad \middle| \quad p \in K_{n-1} \right\},$$

where the paths $q \in K_n^n$ are expressed as $q = \sum_m k_m v_n^m \otimes \cdots \otimes v_1^m$, for some $k_m \in K$ and $q'_m : i \to j'_m \in K_{n-1}^{n-1}$, $q'_m = k_m v_{n-1}^m \otimes \cdots \otimes v_1^m$ and $q''_m : i''_m \to j \in K_{n-1}^{n-1}$, $q''_m = k_m v_n^m \otimes \cdots \otimes v_2^m$.

These are exactly the elements of the form

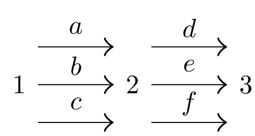
$$\sum_{q\in K_n} \delta_p^{\mathcal{L}} q a_q = 0$$

and

$$\sum_{q\in K_n} a_q \delta_p^{\mathcal{R}} q = 0,$$

for each $p \in K_{n-1}$, as described. □

**Example 3.1.1.** Let $\Lambda$ be the Koszul 2-representation-infinite algebras given by the path algebra of the following quiver and relations.




$$q_1 := db - ea = 0, \quad q_2 := fa - dc = 0, \quad q_3 := ec - fb = 0.$$

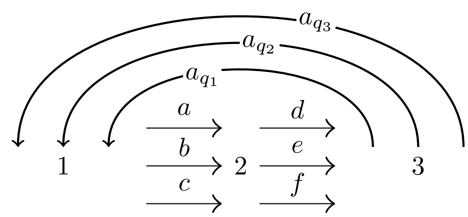
The $k$-module $K_2$ is generated by the relations, thus the preprojective algebra $\Pi(\Lambda)$ has three more arrows $a_{q_i} : 3 \to 1$, $i = 1, 2, 3$. The generators of $K_1$ are just the arrows, so the new relations are given by

$$a_{q_2} f - a_{q_1} e = 0, \quad a_{q_1} d - a_{q_3} f = 0, \quad a_{q_3} e - a_{q_2} d = 0$$

and

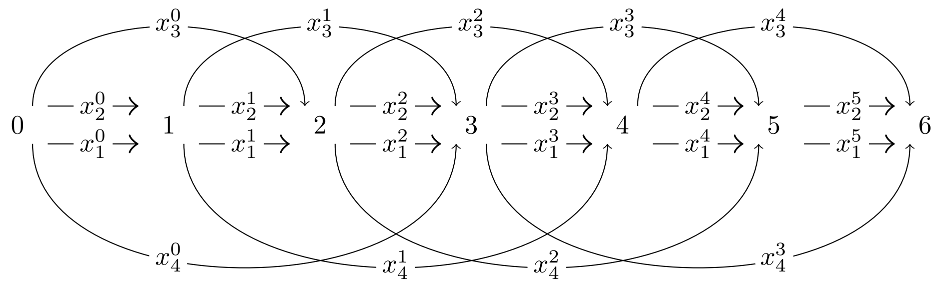
$$b a_{q_1} - f a_{q_3} = 0, \quad c a_{q_3} - a a_{q_1} = 0, \quad a a_{q_2} - b a_{q_3} = 0.$$

The quiver of the preprojective algebra is given by




Note that the path algebra over this quiver with relations is exactly the skew-group algebra $R\#G$, where $R = k[x_1, x_2, x_3]$ and $G = \langle \frac{1}{3}(1,1,1)\rangle$, as seen in Example 2.3.4.

**Example 3.1.2.** Let $\Lambda$ be the Koszul 3-representation-infinite algebra given by




with relations given by $x_{i'}^{j'}x_i^j - x_i^{j''}x_{i'}^j = 0$, whenever the concatenation is not zero. The $k$-bimodule generators of $K_3 = V \otimes_S M \cap M \otimes_S V$, where $M$ is the vector space of relations, are as follows:

$$q_1 = x_4^3x_3^1x_2^0 - x_4^3x_2^2x_3^0 + x_2^5x_4^2x_3^0 - x_2^5x_3^3x_4^0 + x_3^4x_2^3x_4^0 - x_3^4x_4^1x_2^0 : 0 \to 6,$$

$$q_2 = x_4^3x_3^1x_1^0 - x_4^3x_1^2x_3^0 + x_1^5x_4^2x_3^0 - x_1^5x_3^3x_4^0 + x_3^4x_1^3x_4^0 - x_3^4x_4^1x_1^0 : 0 \to 6,$$

$$q_3 = x_4^2x_2^1x_1^0 - x_4^2x_1^1x_2^0 + x_1^4x_4^1x_2^0 - x_1^4x_2^3x_4^0 + x_2^4x_1^3x_4^0 - x_2^4x_4^1x_1^0 : 0 \to 5,$$

$$q_4 = x_3^2x_2^1x_1^0 - x_3^2x_1^1x_2^0 + x_1^3x_3^1x_2^0 - x_1^3x_2^2x_3^0 + x_2^3x_1^2x_3^0 - x_2^3x_3^1x_1^0 : 0 \to 4$$

$$q_5 = x_4^3x_2^2x_1^1 - x_4^3x_1^2x_2^1 + x_1^5x_4^2x_2^1 - x_1^5x_2^4x_4^1 + x_2^5x_1^4x_4^1 - x_2^5x_4^2x_1^1 : 1 \to 6$$

$$q_6 = x_3^3x_2^2x_1^1 - x_3^3x_1^2x_2^1 + x_1^4x_3^2x_2^1 - x_1^4x_2^3x_3^1 + x_2^4x_1^3x_3^1 - x_2^4x_3^2x_1^1 : 1 \to 5$$

$$q_7 = x_3^4x_2^3x_1^2 - x_3^4x_1^3x_2^2 + x_1^5x_3^3x_2^2 - x_1^5x_2^4x_3^2 + x_2^5x_1^4x_3^2 - x_2^5x_3^3x_1^2 : 2 \to 6$$

The preprojective algebra over $\Lambda$ thus have seven more arrows, $a_{q_l}$, $l = 1, \dots, 7$, going in the direction opposite to the associate element $q_l$. The process to obtain these basis elements is as follows. Start with a relation, say $x_3^1x_2^0 - x_2^2x_3^0$. In order to get an element in $V \otimes_S M$ we only need to multiply on the left by an arrow such that the concatenation is non-zero. This gives the element

$$x_4^3x_3^1x_2^0 - x_4^3x_2^2x_3^0.$$

But the elements $q_l$ must be in $M \otimes_S V$ as well. So we need to include the path of length 2 equal to $x_4^3x_3^1$ in $T_SV/\langle M\rangle$, which is $x_3^4x_4^1$, and multiply it by an arrow on the right. And we continue in a similar fashion until the element $q_l$ is closed under cyclic shift of the relations and minimal with this property. It is not easy to do so in general.

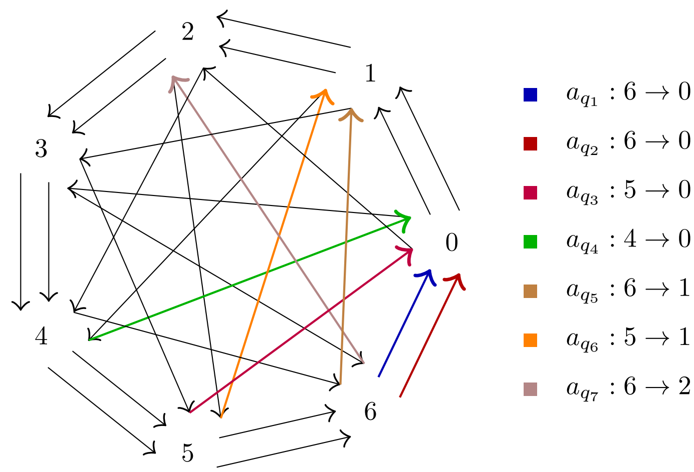
Note that $K_2 = M$ is the vector space of relations in $\Lambda$. We show how to obtain new relations associated to a specific element $p \in K_2$. The idea is similar for the other ones. Define $p \in K_2$ as $p = x_4^3 x_3^1 - x_3^4 x_4^1$. Then we get the relation

$$0 = \sum_{l=1}^{7} \delta_p^{\mathcal{L}} q_l a_{q_l} = x_2^0 a_{q_1} - x_1^0 a_{q_2} : 6 \to 1$$

associated to $p$. The other sum associated to $p$,

$$\sum_{l=1}^{7} a_{q_l} \delta_p^{\mathcal{R}} q_l = 0,$$

does not give any relation. The quiver of the preprojective algebra is given as follows:




We have identified the new arrows in the preprojective algebra with different colors. Note that the path algebra over this quiver, along with the preprojective algebra relations, gives the skew-group algebra $k[x_1, x_2, x_3, x_4]\#G$, with $G = \langle \frac{1}{7}(1,1,2,3) \rangle$. It is an infinite-dimensional algebra of global dimension 4.

## 3.2 Basic preprojective algebras over Koszul algebras are derivation-quotient algebras

We notice that the new relations in the preprojective algebra are quadratic relations. This is a property of Koszul algebras. Thus, since preprojective algebras are Calabi-Yau algebras, a natural question is to ask whether their relations are given by a superpotential.

**Theorem 3.2.1.** *Let $\Lambda = T_S V / \langle M \rangle$ be a basic $n$-representation-infinite algebra and $\Pi$ be the preprojective algebra over $\Lambda$. If $\Pi$ is Koszul, then $\Lambda$ is Koszul. As a partial converse, if $\Lambda$ is Koszul, then $\Pi \cong D(\omega, (n+1) - 2)$ for some superpotential $\omega$.*

*Proof.* Suppose that $\Pi$ is Koszul. Consider its Koszul resolution

$$0 \to \Pi \otimes_S \tilde{K}_{n+1} \otimes_S \Pi \to \Pi \otimes_S \tilde{K}_n \otimes_S \Pi \cdots \to \Pi \otimes_S \Pi \to \Pi \to 0.$$

Taking the preprojective degree 0 part of this resolution, we get a complex

$$0 \to \Lambda \otimes_S K_n \otimes_S \Lambda \to \cdots \to \Lambda \otimes_S \Lambda \to \Lambda \to 0,$$

where $K_l := (\tilde{K}_l)_0$. Since $\tilde{K}_{n+1}$ is generated in degree 1 (see Lemma 4.2.1), we have $K_{n+1} = 0$. The Koszul resolution of $\Pi$ is exact in each degree separately, and thus, in particular in degree 0. Therefore, the latter complex is exact and is the Koszul resolution of $\Lambda$.

Now, if $\Lambda$ is Koszul, then the new relations in $\Pi$ described in Corollary 3.1.2 induce the existence of a superpotential $\omega$ given as follows:

$$\omega := \sum_{q \in K_n} \sum_{0 \leq l \leq n} (-1)^l \sigma^l (q a_q),$$

where $\sigma = (1\ 2\ \cdots\ n-1\ n+1)$ is the element of the symmetric group $\mathfrak{S}_{n+1}$ that acts as

$$\sigma(v_{n+1} \otimes v_n \otimes \cdots \otimes v_1) = v_1 \otimes v_{n+1} \otimes v_n \otimes \cdots \otimes v_2.$$

In fact, by definition of the arrows $a_q$, we get that $\omega$ commutes with the action of $S$. Moreover, since, by definition, every element $q \in K_n$ is in

$$\bigcap_{\mu} (V^\mu \otimes_S M \otimes_S V^{n-\mu-2}),$$

we obtain that $\omega$ is indeed super-cyclically symmetric. By Corollary 3.1.2, the relations in $\Pi$ are given by

$$\delta_p \omega,$$

for any path $p \in \Pi$ of length $((n+1)-2)$, so the preprojective algebra is given by

$$\Pi \cong k\tilde{Q}/\langle \delta_p \omega \rangle = D(\omega, (n+1)-2),$$

where $\tilde{Q}$ is the quiver of $\Pi$ described in the previous corollary.

□

*Remark* 6. This theorem generalizes a result in [Kel11], in which the author proves that a 3-preprojective algebra is a Jacobian algebra given by a potential, which is the same as a derivation-quotient algebra in our context. Moreover, this generalizes Theorem 2.4.3 from [EE07], which states that 2-preprojective algebras over representation-infinite algebras are Koszul, and thus are described by quotient-derivation algebras.

**Question 3.2.1.** If $\Lambda$ is a Koszul $n$-representation-infinite algebra, then $\Pi(\Lambda)$ is a quadratic algebra given by a superpotential. This raises the question as to when it is Koszul. For this we need to show that the associated complex $W_\bullet$, described before Theorem 2.5.1, is a resolution of $\Pi$.

# Chapter 4

# Tensor Product of Preprojective algebras

In this chapter, we study the preprojective grading structure of the tensor product of two preprojective algebras. We use Theorem 2.2.1 to determine whether or not such a structure exists. Preprojective algebras over $n$-representation-infinite algebras are $(n+1)$-Calabi-Yau of Gorenstein parameter 1. While the Calabi-Yau property is independent of the grading, the Gorenstein parameter does depend on it. Therefore, given a Calabi-Yau algebra, it may sometimes be possible to find a grading on it so that it becomes of Gorenstein parameter 1. Yet the degree 0 part need not be finite-dimensional. However, preprojective algebras are tensor algebras over $n$-representation-infinite algebras, which are finite-dimensional, as explained in the introduction and the preliminaries.

When considering the usual tensor product grading on two graded $k$-algebras $A$ and $B$, given by

$$(A \otimes_k B)_i = \bigoplus_{l+m=i} A_l \otimes_k B_m,$$

we get the following proposition.

**Proposition 4.0.1.** *If $A_i$ is a $n_i$-Calabi-Yau algebra of Gorenstein parameter $a_i$, for $i = 1, 2$, then $A_1 \otimes_k A_2$, along with the usual tensor product grading, is $(n_1 + n_2)$-Calabi-Yau of Gorenstein parameter $(a_1 + a_2)$.*

*Proof.* The minimal bimodule resolution $P_\bullet$ of $A_1 \otimes_k A_2$ is the total complex of the tensor product over $k$ of the minimal resolutions $P^1_\bullet$ and $P^2_\bullet$ of $A_1$ and $A_2$, respectively. That is, it is described as

$$P_l = \bigoplus_{i=0}^{l} P^1_i \otimes_k P^2_{l-i}$$

and the differentials are given by

$$\delta(x \otimes y) = (\delta_1 x \otimes y) + (-1)^{\deg(x)}(x \otimes \delta_2 y),$$

where $\delta_i$ is the differential in $P^i_\bullet$ for $i = 1, 2$.

Using the fact that $A_i$ is $n_i$-Calabi-Yau of parameter $a_i$, $i = 1, 2$, we see that

$$\begin{aligned}
P_l &= \bigoplus_{i=0}^{l} P_i^1 \otimes_k P_{l-i}^2 \\
&\cong \bigoplus_{i=0}^{l} \mathrm{Hom}_{A_1^e}(P_{n_1-i}^1, A_1^e)(-a_1) \otimes_k \mathrm{Hom}_{A_2^e}(P_{n_2-l+i}^2, A_2^e)(-a_2) \\
&\cong \bigoplus_{\mu=n_1-l}^{n_1} \mathrm{Hom}_{A_1^e \otimes_k A_2^e}(P_\mu^1 \otimes_k P_{n_2+n_1-l-\mu}^2, A_1^e \otimes_k A_2^e)(-a_1-a_2),
\end{aligned}$$

using the fact that the factors are projective. □

Now let $\Pi_i$ be a basic bimodule $n_i$-Calabi-Yau algebra of Gorenstein parameter 1, for $i = 1, 2$. As shown, the tensor product of two Calabi-Yau algebras is always Calabi-Yau. The main interest of this chapter is to determine whether or not we can put a grading on the tensor product $\Pi_1 \otimes_k \Pi_2$ so that it is an algebra of Gorenstein parameter 1. By the previous theorem, this grading cannot be the standard tensor product grading. There is always a way to do so:

**Proposition 4.0.2.** *Let $\Pi_1$ and $\Pi_2$ be as above. The tensor product $\Pi_1 \otimes_k \Pi_2$ admits a grading giving it Gorenstein parameter* 1*.*

*Proof.* We put $\Pi_1$ in degree 0 and keep the grading on $\Pi_2$. Then the Gorenstein parameter of $\Pi_1 \otimes_k \Pi_2$ is the same as the Gorenstein parameter of $\Pi_2$, which is 1. □

The degree 0 part of the grading that we put on $\Pi_1 \otimes_k \Pi_2$ in Proposition 4.0.2 is of the form $A_1 \otimes_k A_2$, where $A_1 = \Pi_1$ is $n_1$-Calabi-Yau and $A_2$ is $(n_2-1)$-representation-infinite. Note that Calabi-Yau algebras are always infinite-dimensional. Since our main objects of study are preprojective algebras, we want to put a grading in such a way that the degree 0 part is finite-dimensional, as this is a property of higher representation-infinite algebras. We show that this is impossible. More precisely, we show that the only grading giving $\Pi_1 \otimes_k \Pi_2$ Gorenstein parameter 1 is the one described in Proposition 4.0.2, with the grading induced either from the first or the second factor.

**Question 4.0.1.** The algebra $A_1 \otimes_k A_2$, while being infinite-dimensional, has finite global dimension and is the degree 0 part of a bimodule Calabi-Yau algebra of Gorenstein parameter 1. This raises the question whether we can extend the study of $n$-representation-infinite algebras to algebras of infinite dimension, and generalize the notion of preprojective algebras to bimodule Calabi-Yau algebras of Gorenstein parameter 1 such that the degree 0 part is not necessarily finite-dimensional.

**Example 4.0.2.** Let $A$ be a basic Koszul $n$-Calabi-Yau algebra. Then, the Calabi-Yau homological property of $A$ implies that the generators $q \in K_n$ are oriented cycles $q : i \to i$. By Corollary 3.1.2, $\Pi(A)$ has extra arrows $a_q : i \to i$. Thus, taking the preprojective algebra simply adds loops in degree 1 at every vertex, which increases the global dimension by one and the Gorenstein parameter becomes 1.

In [HIO14], the authors study the tensor product of two representation-infinite algebras. They show

**Theorem 4.0.3** ([HIO14])**.** *Let $\Lambda_i$ be a $n_i$-representation-infinite algebra, for $i = 1, 2$. Then*

- *$\Lambda_1 \otimes_k \Lambda_2$ is $(n_1 + n_2)$-representation-infinite;*

- $\mathbb{S}(\Lambda_1 \otimes_k \Lambda_2) = \mathbb{S}(\Lambda_1) \otimes_k \mathbb{S}(\Lambda_2)$;
- $\mathbb{S}_{n_1+n_2}(\Lambda_2 \otimes_k \Lambda_2) = \mathbb{S}_{n_1}(\Lambda_1) \otimes_k \mathbb{S}_{n_2}(\Lambda_2)$.

Combining these results together, we get the following.

**Corollary 4.0.4.** *Let $\Lambda_i$ be as above and let $\Pi_i = \Pi(\Lambda_i)$, for $i = 1, 2$, be the preprojective algebra associated to $\Lambda_i$. Then the Segre product*

$$\Pi_1 * \Pi_2 := \bigoplus_{l \geq 0} ((\Pi_1)_l \otimes_k (\Pi_2)_l)$$

*is a $(n_1 + n_2 + 1)$-preprojective algebra.*

*Proof.* We have that

$$\Pi_1 * \Pi_2 = \bigoplus_{l \geq 0} ((\Pi_1)_l \otimes_k (\Pi_2)_l) = \bigoplus_{l \geq 0} \left(\mathbb{S}_{n_1}^{-l}(\Lambda_1) \otimes_k \mathbb{S}_{n_2}^{-l}(\Lambda_2)\right)$$

$$= \bigoplus_{l \geq 0} \mathbb{S}_{n_1+n_2}^{-l} (\Lambda_1 \otimes_k \Lambda_2) .$$

Since $\Lambda_1 \otimes_k \Lambda_2$ is $(n_1 + n_2)$-representation-infinite, the result follows. □

*Remark* 7. The Segre product naturally appears in geometry. If $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$ are two projective varieties, then the Segre embedding is defined as

$$\sigma : X \times Y \to \mathbb{P}^{(n+1)(m+1)-1}$$

$$\sigma([x_0 : x_1 : \cdots : x_n], [y_0 : y_1 : \cdots : y_m]) = [x_0y_0 : x_0y_1 : \cdots : x_0y_n : x_1y_0 : \cdots : x_iy_j : \cdots : x_ny_m].$$

Under this embedding, the coordinate ring $k[X \times Y]$ of $X \times Y$ is given by the Segre product $k[X] * k[Y]$ of the coordinate rings of these projective varieties.

Corollary 4.0.4 tells us that the Segre product is the right object to consider when taking tensor products. Indeed, it only increases the global dimension of $\Lambda_1 \otimes_k \Lambda_2$ by one, whereas the tensor product increases it by two. Moreover, the degree 1 part of the Segre product lives in degree 2 in the tensor product. This is why the Segre product is of Gorenstein parameter 1, whereas the tensor product is of Gorenstein parameter 2. Thus, the tensor product seems somehow too big to be a preprojective algebra. If we force Gorenstein parameter 1 on it, then we force infinite-dimensionality of its degree 0 part.

## 4.1 Superpotential of a tensor product

Suppose that $A_i = T_{S_i}V_i/\langle M_i \rangle$ is a $n_i$-Calabi-Yau and Koszul algebra, where $S_i$ is a finite-dimensional semisimple $k$-algebra, $V_i$ is a $S_i$-bimodule and $M_i$ are relations, for $i = 1, 2$. Then, $A := A_1 \otimes_k A_2$ is $(n_1 + n_2)$-Calabi-Yau and Koszul. It thus admits a superpotential. We would like to describe this superpotential in terms of $w_1$ and $w_2$, the superpotentials of $A_1$ and $A_2$, respectively. This is needed for the main theorem of this chapter.

The superpotential $\omega$ in $A$ is a $S$-bimodule generator of

$$K_n := \bigcap_{\mu}(V^{\otimes\mu} \otimes_S M \otimes_S V^{\otimes n-\mu-2}) \cong \mathrm{Tor}_n^A(S,S),$$

where $S := S_1 \otimes_k S_2$, $n = n_1 + n_2$, $V = V_1 \oplus V_2$, and $M$ is the $S$-bimodule of relations in $A$. The computation of Tor and Ext is independent of the resolution. Thus, we can use the Koszul resolution of $S$ or the bar resolution described in the preliminaries. Recall that the Bar resolution $\beta(A,S)$ of $S$ is given by

$$\beta(A,S) := \cdots \to A \otimes_S A_+^{\otimes_S n} \to A \otimes_S A_+^{\otimes_S (n-1)} \to \cdots \to A,$$

where $A_+ := \oplus_{i\geq 1} A_i$. It is interesting in our context to compare the resolution of $A_1 \otimes_k A_2$ with the resolutions of $A_1$ and $A_2$. As we have already discussed, the resolution of $A_1 \otimes_k A_2$ is the total complex of the tensor product of the two resolutions. However, we need to understand how $\mathrm{Tor}_{n_i}^{A_i}(S_i, S_i)$ for $i = 1,2$ and $\mathrm{Tor}_n^A(S_1 \otimes_k S_2, S_1 \otimes_k S_2)$ are related. This is done via the shuffle product.

**Definition 4.1.1** (See, e.g., [Wei94])**.** The *shuffle product*

$$\sqcup : \mathrm{Tor}_{n_1}^{A_1}(S_1,S_1) \times \mathrm{Tor}_{n_2}^{A_2}(S_2,S_2) \to \mathrm{Tor}_{n_1}^{A_1}(S_1,S_1) \otimes_k \mathrm{Tor}_{n_2}^{A_2}(S_2,S_2)$$

is defined as

$$v_1^1 \otimes \cdots \otimes v_{n_1}^1 \sqcup v_{n_1+1}^2 \otimes \cdots \otimes v_{n_1+n_2}^2 :=$$
$$\sum_\sigma (-1)^\sigma v^1_{\sigma^{-1}(1)} \otimes v^1_{\sigma^{-1}(2)} \otimes \cdots \otimes v^1_{\sigma^{-1}(n_1)} \otimes v^2_{\sigma^{-1}(n_1+1)} \otimes \cdots \otimes v^2_{\sigma^{-1}(n_1+n_2)},$$

where the sum runs over all $(n_1,n_2)$-shuffles $\sigma$, that is, elements $\sigma$ of the symmetric group $\mathfrak{S}_{n_1+n_2}$ which satisfy $\sigma(1) < \sigma(2) < \cdots < \sigma(n_1)$ and $\sigma(n_1+1) < \sigma(n_1+2) < \cdots < \sigma(n_1+n_2)$.

**Theorem 4.1.1.** *Let $\omega_i$ be a superpotential associated to $A_i$, $i = 1,2$. A $S$-bimodule generator of $Tor_n^A(S,S)$ is given by*

$$\omega := \omega_1 \sqcup \omega_2.$$

*This is a superpotential in the algebra $A$.*

*Proof.* It is known that the shuffle product induces a chain homotopy equivalence of bar resolutions $\beta$ of the $(A_1 \otimes_k A_2)$-modules $S_1 \otimes_k S_2$, see for example ([Wei94], 8.6.13):

$$\mathrm{Tot}(\beta(A_1,S_1) \otimes_k \beta(A_2,S_2)) \xrightarrow{\nabla} \beta(A_1 \otimes_k A_2, S_1 \otimes_k S_2),$$

where Tot denotes the total complex of the tensor product, whose terms are described as the ones in the proof of Proposition 4.0.1. The map sends the tensor of an element in degree $p$ with an element in degree $q$ to an element in degree $p+q$, as follows:

$$\nabla((a \otimes v_1 \otimes \cdots v_p) \otimes_k (a' \otimes v_1' \otimes \cdots v_q')) = \sum_\sigma (-1)^{|\sigma|}(a \otimes a') \otimes \nu_{\sigma(1)} \otimes \cdots \otimes \nu_{\sigma(p+q)},$$

where $\nu_1, \ldots, \nu_{p+q}$ are the elements $v_1 \otimes 1, \ldots, v_p \otimes 1, 1 \otimes v_1', \ldots, 1 \otimes v_q'$, in this order. The sum is over all $(p,q)$-shuffles. We then obtain a $S$-bimodule isomorphism, induced by the shuffle product:

$$\mathrm{Tor}_{n_1}^{A_1}(S_1,S_1) \otimes_k \mathrm{Tor}_{n_2}^{A_2}(S_2,S_2) \to \mathrm{Tor}_{n_1+n_2}^{A}(S_1 \otimes_k S_2, S_1 \otimes_k S_2).$$

Since the algebras are Koszul, the superpotential $\omega_i$ of $A_i$ is a $S_i$-bimodule generator of $\operatorname{Tor}^{A_i}_{n_i}(S_i, S_i)$, for $i = 1, 2$. Moreover, the superpotential $\omega$ of $A_1 \otimes_k A_2$ is a $S$-bimodule generator of

$$\operatorname{Tor}^{A}_{n_1+n_2}(S_1 \otimes_k S_2, S_1 \otimes_k S_2).$$

Thus, it is given by the image of $\omega_1 \otimes_k \omega_2$ via the shuffle map. □

## 4.2 Preprojective structure on the tensor product of Preprojective algebras

The goal of this subsection is to prove that the tensor product of two basic Koszul preprojective algebras cannot have a grading giving it a structure of preprojective algebras. We begin by proving some general statements.

We first describe the quiver of a basic tensor product algebra $A := A_1 \otimes_k A_2$. Following [GMV98] for a complete analysis, let $\Gamma^1$ and $\Gamma^2$ be the quivers of $A_1$ and $A_2$, respectively. Then the quiver $\Gamma$ of $A$ is described as follows:

$$\Gamma_0 = \Gamma^1_0 \times \Gamma^2_0$$

and

$$\Gamma_1 = (\Gamma^1_1 \times \Gamma^2_0) \cup (\Gamma^1_0 \times \Gamma^2_1).$$

Thus for every arrow $a^1 : e^1_i \to e^1_j$ in $\Gamma^1_1$, there are arrows

$$(a^1, e^2_k) : (e^1_i, e^2_k) \to (e^1_j, e^2_k),$$

for every $e^2_k \in \Gamma^2_0$. Similarly, we get the arrows of type $(e^1_k, a^2)$. Moreover, for every path $\sum \mu_i p^1_i \in \Gamma^1$, where $p^1_i = a^1_{i_l} \cdots a^1_{i_1}$, and every vertex $e^2_k \in \Gamma^2_0$, there exists a path

$$\sum \mu_i (a^1_{i_l}, e^2_k) \cdots (a^1_{i_1}, e^2_k).$$

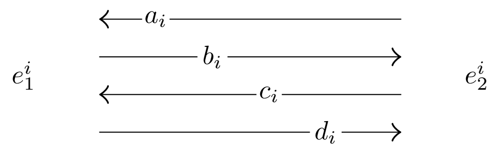
The paths in the second component are obtained in a similar way.

For every arrow $a^1 : e^1_i \to e^1_j \in \Gamma^1$ and $a^2 : e^2_s \to e^2_t \in \Gamma^2$, let $M$ be the ideal generated by the relations

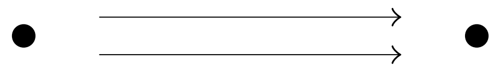
$$(a^1, e^2_t)(e^1_i, a^2) - (e^1_j, a^2)(a^1, e^2_s).$$

Then $A \cong k\Gamma/\langle \tilde{M}_1, \tilde{M}_2, M\rangle$, where $\tilde{M}_1$ is the set consisting of the $(f^1_i, e'')$ for each $f^1_i \in M_1$ and $\tilde{M}_2$ is the set consisting of the $(e', f^2_j)$ for each $f^2_j \in M_2$. When $A$ is Koszul, then the Koszul grading on $A$ is the standard one given by the tensor product.

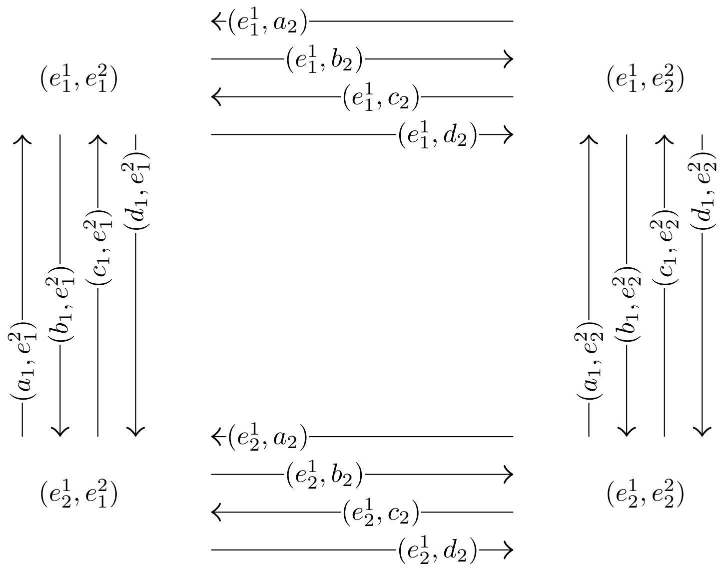
**Example 4.2.1.** Consider the algebras $A_i$ given as the path algebra over the following quiver, for $i = 1, 2$,




with relations $a_ib_i - c_id_i = 0$ and $b_ia_i - d_ic_i = 0$. There is an isomorphism $A_i \cong k[x_1, x_2]\#G$, where $G = \langle \frac{1}{2}(1,1)\rangle$. It is the preprojective algebra over the hereditary algebra given as the path algebra over the quiver



The tensor product $A = A_1 \otimes_k A_2$ is the path algebra over the quiver




with relations induced from the ones in $A_i$, as described, and new commutativity relations induced from the tensor product. An example of such relation is

$$(e_1^1, b_2)(a_1, e_1^2) - (a_1, e_2^2)(e_2^1, b_2) = 0.$$

Now let $\Pi$ be a $n$-Calabi-Yau algebra of Gorenstein parameter 1. Let $P_\bullet$ denote the minimal bimodule resolution of $\Pi$. In the case where we require $\Pi$ to be Koszul, we have that the terms in $P_\bullet$ are given by

$$P_l = \Pi \otimes_S \otimes K_l \otimes_S \Pi,$$

where $K_l \subset V^{\otimes_S l}$ is the usual term in the Koszul resolution defined in the preliminaries and the differentials are given by

$$\delta(a \otimes v_1 \otimes \cdots \otimes v_l \otimes a') = av_1 \otimes \cdots \otimes v_l \otimes a' + (-1)^l a \otimes v_1 \otimes \cdots \otimes v_{l-1} \otimes v_l a'.$$

In the following lemmas we are interested in the preprojective structure of $\Pi$.

**Lemma 4.2.1** ([AIR15]). *If $\Pi$ is $n$-Calabi-Yau of Gorenstein parameter 1, then each term $P_i$ in the minimal bimodule resolution of $\Pi$ is generated in degree 0 or in degree 1. Moreover, $P_0$ is generated in degree 0 and $P_n$ is generated in degree 1.*

*Proof.* Since the resolution of $\Pi$ is minimal and $\Pi$ is positively graded, each $P_i$ is generated in non-negative degrees. Consider the isomorphism

$$P_\bullet \cong P_\bullet^\vee[n](-1).$$

If $P_i$ has a generator in degree $a$, then $P_{n-i}$ has a generator in degree $1-a$. Therefore, $1-a \geq 0$, which implies that $a=0$ or $a=1$. Thus, each $P_i$ is generated in degree 0 or 1. But $\Pi$ is generated in degree 0 as a bimodule over itself, so $P_0$ is generated in degree 0, and thus $P_n$ is generated in degree 1. □

**Lemma 4.2.2.** *Let $\Pi$ be a Koszul $n$-Calabi-Yau algebra of Gorenstein parameter 1. Then*

$$\deg_{P_l}(1 \otimes v_l \otimes \cdots \otimes v_1 \otimes 1) = \sum_i \deg_\Pi(v_i) = \deg_\Pi(v_l \otimes \cdots \otimes v_1),$$

*where $\deg_{P_l}$ and $\deg_\Pi$ denote the degree of an element in $P_l$ and $\Pi$, respectively, coming from the preprojective grading.*

*Proof.* We proceed by induction on $l$ and use the fact that the differentials in the resolution are homogeneous with respect to the preprojective grading. For $l=1$, we have

$$\delta(1 \otimes v_1 \otimes 1) = v_1 \otimes 1 - 1 \otimes v_1 \in \Pi \otimes_S \Pi,$$

so $\deg_{P_1}(1 \otimes v_1 \otimes 1) = \deg_\Pi(v_1)$. In general, we have

$$\delta(1 \otimes (v_1 \otimes \cdots \otimes v_l) \otimes 1) = v_1 \otimes (v_2 \otimes \cdots \otimes v_l) \otimes 1 + (-1)^l 1 \otimes (v_1 \otimes \cdots \otimes v_{l-1}) \otimes v_l,$$

and the statement follows easily by induction. □

Now, let $\Pi_i = T_{S_i}V_i/\langle M_i \rangle$ be a basic bimodule $n_i$-Calabi-Yau connected Koszul algebra of Gorenstein parameter 1 over a semisimple algebra $S_i$, for $i=1,2$, where $n_1, n_2 \geq 1$. Let

$$\Pi := \Pi_1 \otimes_k \Pi_2 \cong T_S V/\langle M \rangle,$$

be a $n$-Calabi-Yau Koszul algebra, where $n = n_1 + n_2$, $S := S_1 \otimes_k S_2$, $V = V_1 \oplus V_2$ and $M$ are induced relations. Let $\omega_i$ be the superpotential of $\Pi_i$, $i=1,2$, and $\omega := \omega_1 \sqcup \omega_2$ be the superpotential of $\Pi$, according to Theorem 4.1.1.

**Theorem 4.2.3.** *Let $\Pi_1$ and $\Pi_2$ be as above and let $\Pi := \Pi_1 \otimes_k \Pi_2$. If $\Pi$ is bimodule $n$-Calabi-Yau of Gorenstein parameter 1, then the degree 0 part is of the form $A_1 \otimes_k A_2$, where $A_i$ is $n_i$-Calabi-Yau and $A_j$ is $(n_j-1)$-representation-infinite, for some $i \neq j \in \{1,2\}$. In particular, $\Pi$ does not admit a structure of preprojective algebra.*

Note that a 0-representation-infinite algebra is isomorphic to $k$.

*Proof.* Let $\Gamma^i$ be the quiver of $\Pi_i$, for $i=1,2$ and $\Gamma$ be the quiver of $\Pi$. We assume that $\Pi$ is $(n_1+n_2)$-Calabi-Yau of Gorenstein parameter 1.

By Lemma 4.2.1, the $S$-bimodule generator $\omega \in K_{n_1+n_2}$ in $P_\bullet$, which is the superpotential of $\Pi$, is homogeneous of degree 1. Moreover, by Lemma 4.2.2 the degree of the elements in $K_l$ is the same as the degree of the associated paths in $\Pi$.

The superpotential $\omega$ of $\Pi$ is of the form $\omega = \omega_1 \sqcup \omega_2$. In particular, all summands of $\omega_1 \otimes_k \omega_2$ are summands of $\omega$. These summands have the property that they are closed paths $(e, e') \to (e, e')$ of length $(n_1 + n_2)$ that are the concatenation of two closed paths $(e, e') \to (e, e')$ of length $n_1$ and $n_2$ respectively, for some vertex $(e, e')$ in $\Gamma$.

Consider one of these summands $(e, q)(p, e')$, where $p$ is a closed path and a summand of $\omega_1$, and $q$ is a closed path and a summand of $\omega_2$. Since $\omega$ is homogeneous of degree 1, the path $(e, q)(p, e')$ is in degree 1. By additivity of the degree, either $(e, q)$ or $(p, e')$ is in degree 1, say $(p, e')$ without loss of generality. Then $(e, q)$ is in degree 0. Denote by $(a, e')$ the arrow in $(p, e')$ which is in degree 1.

Let $\sigma = (1\ 2\ \cdots\ n_1) \in \mathfrak{S}_{n_1}$ be acting as usual on elements in $V_1^{\otimes n_1}$, that is,

$$\sigma^r(v_{n_1} \otimes \cdots \otimes v_1) = v_r \otimes v_{r-1} \otimes \cdots \otimes v_1 \otimes v_{n_1} \otimes v_{n_1 - 1} \otimes \cdots \otimes v_{r+1}.$$

The path

$$\sigma^r(p, e') := (\sigma^r(p), e')$$

is in degree 1 for all $0 \leq r < n_1$, since it contains $(a, e')$. Thus, at any vertex $(e_i, e')$ for which $e_i$ is the start vertex of $\sigma^r(p)$ for some $r$, the path $(e_i, q)(\sigma^r(p), e')$ is in degree 1, therefore $(e_i, q)$ is in degree 0. Now, by the same reasoning, for every vertex $e_j$ in $q$, the path $(p, e_j)$ is in degree 1.

Now consider a vertex $l \in \Gamma^2$ which is not in the path $q$. We claim that the path $(p, l)$ is also in degree 1. We use the symbol $\sharp$ to denote two vertices connected by an arrow. Consider the following non-oriented path

$$(e, f_1)\sharp(e, f_2)\sharp \cdots \sharp(e, l),$$

where $f_1 \in \Gamma^2$ is a vertex in the path $q$. Such a path exists since the quiver is connected. Recall that the path $(p, f_1) : (e, f_1) \to (e, f_1)$ is in degree 1. The arrow between $(e, f_1)$ and $(e, f_2)$ is part of a closed cycle, call it $(e, q')$. This cycle must be in degree 0, as $(p, f_1)$ is in degree 1. Thus, the cycle $(p, f_2)$ must be in degree 1. Continuing in this fashion, we see that the path $(p, l)$ is in degree 1.

Thus, any path of the form $(p, \epsilon')$, where $\epsilon' \in \Gamma_2$, is in degree 1. By the same reasoning, any path of the form $(\epsilon, q)$, where $\epsilon \in \Gamma_1$ is in degree 0.

Now, consider another closed summand $\rho$ of $\omega_1$ starting at a vertex $\mu$. The path $(\mu, q)(\rho, e')$ is in degree 1. Because $(\mu, q)$ is in degree 0, we have that $(\rho, e')$ must be in degree 1. By the previous argument, this implies that any $(\rho, \epsilon')$ is in degree 1 for any vertex $\epsilon' \in \Gamma^2$. By a similar argument, we get that any $(\epsilon, \rho')$ is in degree 0 for any summand $\rho'$ of $\omega_2$ and vertex $\epsilon \in \Gamma^1$.

Therefore, for any $\epsilon \in \Gamma^1$, the superpotential $(\epsilon, \omega_2)$ is in degree 0 and, for every $\epsilon' \in \Gamma^2$, the superpotential $(\omega_1, \epsilon')$ is in degree 1.

It remains to show that if $(a, e)$ is the arrow in $(p, e)$ in degree 1, then $(a, \epsilon')$ is in degree 1 for any

$\epsilon' \in \Gamma^2$. Say $a : i \to j$ in $\Gamma^1$. Consider as before a non-oriented path

$$(i, e)\sharp(i, f_1)\sharp \cdots \sharp(i, \epsilon')$$

and the parallel non-oriented path

$$(j, e)\sharp(j, f_1)\sharp \cdots \sharp(j, \epsilon').$$

Then, consider an arrow $b_1 \in \Pi_2$ between $e$ and $f_1$, say, without loss of generality, that $b : e \to f_1$. Then by the commutativity relations on the tensor product, we have

$$(a, f_1)(i, b) = (j, b)(a, e).$$

Because $b \in \Pi_2$, the arrows $(i, b)$ and $(j, b)$ are in degree 0, using what we just showed. Moreover, $(a, e)$ is in degree 1 by assumption. So $(a, f_1)$ is in degree 1 by homogeneity of the relations. Now, consider $b_2 \in \Pi_2$ between $f_1$ and $f_2$, say $b_2 : f_1 \to f_2$. By the same reasoning,

$$(a, f_2)(i, b) = (j, b)(a, f_1),$$

so $(a, f_2)$ is in degree 1. Continuing in this fashion, we get that $(a, \epsilon')$ is in degree 1. This proves that for any arrow $\alpha \in \Pi_1$, all arrows $(\alpha, \epsilon')$ have the same degree, so their degree is induced from the grading on $\Pi_1$. Therefore, the degree 0 part of $\Pi$ must be of the form $A_1 \otimes_k A_2$, where $A_1$ is $(n_1 - 1)$-representation infinite and $A_2$ is $n_2$-Calabi-Yau. □

**Example 4.2.2.** We consider the algebra $A$ defined in Example 4.2.1 and reproduce the idea of the proof in this particular case. Note that $A$ has global dimension 4, since it is the tensor product of two algebras of global dimension 2. A superpotential $\omega_i$ of $A_i$ is given by the relations

$$\omega_i = a_i b_i - c_i d_i - b_i a_i + d_i c_i,$$

for $i = 1, 2$. A superpotential of $A$ is then given by the shuffle product $\omega = \omega_1 \sqcup\!\sqcup\, \omega_2$. In particular, the closed path

$$(e_1^1, c_2)(e_1^1, d_2)(a_1, e_1^2)(b_1, e_1^2)$$

at vertex $(e_1^1, e_1^2)$ is a summand of $\omega$. Moreover, it is the concatenation of two closed paths at $(e_1^1, e_1^2)$

$$(p, e_1^2) = (a_1, e_1^2)(b_1, e_1^2) \quad \text{and} \quad (e_1^1, q) = (e_1^1, c_2)(e_1^1, d_2).$$

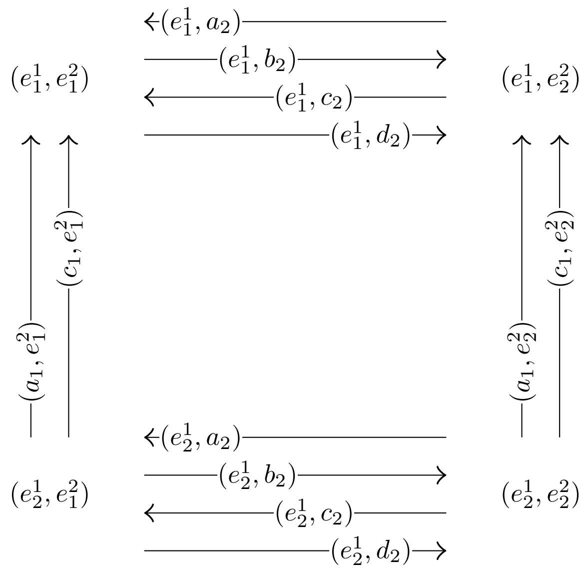
By Lemma 4.2.1 and Lemma 4.2.2, the path $(e_1^1, q)(p, e_1^2)$ has degree 1. Therefore, by additivity of the degree on concatenation, either $(p, e_1^2)$ has degree 1, or $(e_1^1, q)$ has degree 1. Suppose that we are in the former case. Then $(e_1^1, q)$ has degree 0. Now consider, the path $(e_2^1, q)(p, e_1^2)$. Because $(p, e_1^2)$ has degree 1, the path $(e_2^1, q)$ must have degree 0. Finally, considering the path $(e_2^1, q)(p, e_2^2)$, we see that $(p, e_2^2)$ must have degree 1.

By homogeneity of the relations, the paths $(c_1, e_1^2)(d_1, e_1^2)$ and $(c_1, e_2^2)(d_1, e_2^2)$ must be in degree 1 as well, and the paths $(e_1^1, a_2)(e_1^1, b_2)$ and $(e_2^1, a_2)(e_2^1, b_2)$ must be in degree 0.

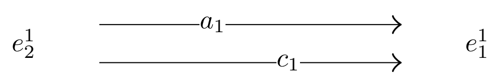
To conclude, suppose that $(a_1, e_1^2)$ and $(c_1, e_1^2)$ are the arrows in degree 1. We want to show that the corresponding arrows in $(-, e_2^2)$ are also in degree 1. We use the relation

$$(e_1^1, d_2)(a_1, e_1^2) - (a_1, e_2^2)(e_2^1, d_2) = 0.$$

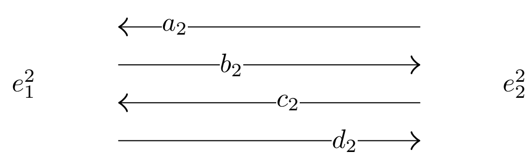
Since $(e_1^1, d_2)$ and $(e_2^1, d_2)$ are in degree 0, we conclude by homogeneity of the relations that $(a_1, e_2^2)$ must be in degree 1. Similarly, $(c_1, e_2^2)$ is also in degree 1. Thus, we showed that making a choice for the degree of $(p, e_1^2)$ and $(e_1^1, q)$ completely characterize the grading of the algebra. In the case where $(p, e_1^2)$ is in degree 1, we then have that the degree 0 part of the algebra is given by the quiver




With the relations induced from the relations on $A$. This algebra is infinite-dimensional. Thus, it is not a 3-representation-infinite algebra. In fact, it is given as the tensor product of




which is a 1-representation-infinite algebra and




with the relations given in $A_2$, which is a 2-preprojective algebra.

# Chapter 5

# Preprojective algebra structure on the Skew-group algebra

The motivation of this chapter is to generalize the following theorem, described in [RVdB89] (see also [CBH98]), to finite subgroups of $SL(n,k)$.

**Theorem 5.0.1** ([RVdB89])**.** *Let $G$ be a finite subgroup of $SL(2,k)$. Then the skew-group algebra $k[x,y]\#G$ is Morita equivalent to $\Pi(Q_G)$, the preprojective algebra over the extended Coxeter-Dynkin quiver associated to $G$ via the McKay correspondence.*

We give a very brief historical account behind this theorem. As these notions are not directly needed for the results of this chapter, we refer to [BD08] and [LW12] for interesting surveys, among many others. The correspondence between finite subgroups $G$ of $SL(2,k)$ and the simply-laced Coxeter-Dynkin diagrams of type A-D-E was known since Patrick DuVal and Michael Artin ([Art66]) via the theory of resolutions of the Klein singularities $k^2/G = \mathrm{Spec}(R^G)$, where $R^G$ denotes the invariant ring over the polynomial ring $R = k[x,y]$.

$$\begin{array}{ccc}
\{\text{Finite subgroups } G \subset SL(2,k)\}/\text{conjugacy} & & \{\text{C-D diagrams of type A-D-E}\} \\
\updownarrow & & \updownarrow \\
\{\text{Klein singularities } X_G = k^2/G\}/\sim & \longleftrightarrow & \{\text{Minimal resolutions } \tilde{X}_G\}/\sim
\end{array}$$

Here, the symbol $\sim$ denotes complex-analytic equivalence. The McKay correspondence ([McK83]) establishes this bijection directly through the representations of the group, without going through quotient singularities, via McKay quivers (recall Definition 2.3.2). In fact, the McKay quivers of the finite subgroups of $SL(2,k)$ are in one-to-one correspondence with the extended Coxeter-Dynkin diagrams of type A-D-E. This correspondence between geometry and representation theory has had many interpretations over the years, one of which is given by preprojective algebras and Auslander-Reiten theory. The Skew-group algebra $R\#G$ is the Auslander algebra over $R^G$, that is,

$$R\#G \cong \mathrm{End}_{R^G}(R).$$

It is Morita equivalent to the preprojective algebra over the extended Coxeter-Dynkin quiver associated to $G$:

$$\Pi_G \cong \mathrm{End}_{R^G}(\oplus M_i),$$

where the direct sum is taken over all indecomposable maximal Cohen-Macaulay modules over $R^G$. Maximal Cohen-Macaulay modules and Auslander algebras have many applications in the theory of singularites. For example, they are related to the so-called Landau-Ginzburg models in string theory. Also, if $X$ is a projective variety, then the stable category of graded maximal Cohen-Macaulay modules over the homogeneous coordinate ring $k[X]$ is equivalent to the derived category of coherent sheaves over $X$ ([Orl09]).

Now recall that, by Theorem 2.3.1, the Skew-group algebra is Morita equivalent the path algebra over the McKay quiver of $G$ modulo some relations. Thus, Theorem 5.0.1 states that, if $G < SL(2, k)$, then the Auslander algebra of the coordinate ring of $k^2/G$, that is $R^G$, is Morita equivalent to a preprojective algebra. Note that this preprojective algebra is basic, since it is given by the path algebra of a quiver with relations. A relevant question is then to ask whether or not this theorem carries in higher Auslander-Reiten theory. We show that, in some cases, the skew-group algebra is not Morita equivalent to a higher preprojective algebra.

Throughout this chapter, let $R = k[x_1, \ldots, x_n] = k[V]$ be the polynomial ring in $n$ variables and let $G$ be a finite subgroup of $SL(n, k)$ acting on $R$ by linear change of coordinates:

$$(g \cdot v)(x) = v(g^{-1}(x)),$$

where $v \in k[V]$ and $x \in V$. The necessary notions on skew-group algebras can be found in Section 2.3.

We want to know for which finite subgroup $G$ of $SL(n, k)$ is the algebra $R\#G$ Morita equivalent to a higher preprojective algebra. We show that if $G$ embeds into a product of special linear groups, then $R\#G$ does not have a structure of preprojective algebra.

Note that $R\#G$ is always $n$-Calabi-Yau, so the difficulty lies in finding a grading that gives $R\#G$ Gorenstein parameter 1. Again, this grading must have the property that the degree 0 part is finite-dimensional, as we want it to be $(n-1)$-representation-infinite. This grading cannot be induced by putting the variables $x_1, \ldots, x_n$ in $R$ in some degree $a_1, \ldots, a_n$, respectively, and $G$ in degree 0, since in this case the Gorenstein parameter would be $\sum_{1 \leq i \leq n} a_i$. If only one $a_i$ is non-zero, then the degree 0 part is clearly infinite-dimensional.

*Remark* 8. In [MM11], the authors prove that $R\#G$, endowed with a grading induced from putting the variables in certain degrees, is graded Morita equivalent to a preprojective algebra, that is, they have the same category of graded modules.

## 5.1 Tensor product of skew-group algebras

We apply Theorem 4.2.3 to the current context. First assume that $G_i$ is a finite subgroup of $SL(n_i, k)$ and $R_i$ is the polynomial ring in $n_i$ variables, for $i = 1, 2$. We have the following $kG$-module isomorphisms:

$$R_1\#G_1 \otimes_k R_2\#G_2 \cong R_1 \otimes_k kG_1 \otimes_k R_2 \otimes_k kG_2 \cong (R_1 \otimes_k R_2) \otimes_k (kG_1 \otimes_k kG_2) \cong R \otimes_k k[G_1 \times G_2],$$

where $R$ is the polynomial ring in $(n := n_1 + n_2)$ variables. The map is given by

$$(f(v), g_1) \otimes_k (g(w), g_2) \mapsto \left( f(v) \otimes_k g(w), \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \right)$$

and it respects the tensor product multiplication, so this is an algebra isomorphism. The block matrix explicitly indicates the given representation. Let $G = G_1 \times G_2$, where each element of $G$ is given by

$$\begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix},$$

for $g_1 \in G_1$ and $g_2 \in G_2$. Then $G < SL(n_1, k) \times SL(n_2, k)$. The algebra $R\#G$ is Morita equivalent to

$$kM_{G_1}/\langle M_1 \rangle \otimes_k kM_{G_2}/\langle M_2 \rangle,$$

where $M_{G_i}$ is the McKay quiver of $G_i$ and $M_i$ are the induced relations, $i = 1, 2$. Since Morita equivalence preserves the bimodule Calabi-Yau property, we obtain the following corollary to Theorem 4.2.3.

**Corollary 5.1.1.** *If $G < SL(n, k)$ is a finite group such that $G$ is conjugate to $G_1 \times G_2$, where $G_i < SL(n_i, k)$, $i = 1, 2$, and $n_1 + n_2 = n$, then $R\#G$ does not have a grading structure making it Morita equivalent to a preprojective algebra.*

This corollary motivates the following definition.

**Definition 5.1.1.** Let $G < SL(n, k)$ be finite. We say that $G$ *embeds* into $SL(n_1, k) \times SL(n_2, k)$, for some $n_1, n_2 \geq 1$ such that $n_1 + n_2 = n$, if $G$ is conjugate to a group such that each of its element is of the form

$$\begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix},$$

where $g_1 \in SL(n_1, k)$ and $g_2 \in SL(n_2, k)$. Equivalently, the given representation $V$ of $G$ can be written as

$$V \cong V_1 \oplus V_2,$$

where $\Lambda^{n_1} V_1$ and $\Lambda^{n_2} V_2$ are the trivial representations.

If $G_1, G_2$ and $H$ are finite groups and $\phi_i : G_i \to H$ is an epimorphism for $i = 1, 2$, define the *fibre product* $G_1 \times_H G_2$ as

$$G_1 \times_H G_2 := \left\{ \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \mid g_1 \in G_1,\, g_2 \in G_2,\, \phi_1(g_1) = \phi_2(g_2) \right\}.$$

By Goursat's lemma, these are exactly the finite subgroups of $G_1 \times G_2$. Therefore, $G$ embeds into

$SL(n_1,k) \times SL(n_2,k)$ if and only if it is conjugate to a fibre product of this form.

*Remark* 9. The group $G$ from Corollary 5.1.1 embeds into $SL(n_1,k)\times SL(n_2,k)$. In the next subsection, we generalize this result to any finite group having this property.

## 5.2 Conditions on subgroups $G < SL(n,k)$

In their paper, C. Amiot, O. Iyama and I. Reiten proved the following:

**Theorem 5.2.1** ([AIR15]). *Let $G$ be a finite cyclic subgroup of $SL(n,k)$ of order $r$. Suppose that there exists a generator $g = \frac{1}{r}(a_1,\dots,a_n)$ of $G$ such that*

1. *$(a_i, r) = 1$ and $0 < a_i < r$ for all $i$;*
2. *$a_1 + \cdots + a_n = r$.*

*Then there exists a grading on $R\#G$ giving it a higher preprojective algebra structure.*

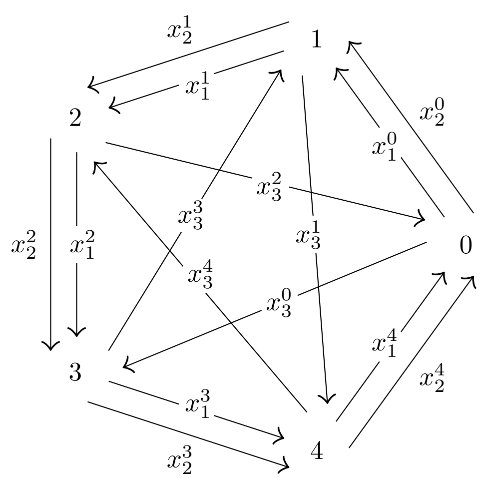
*Remark* 10. The first condition ensures that $R^G$ is an isolated singularity, which is equivalent to the fact that if $1 \neq g \in G$, then $g$ does not have any eigenvalue equal to 1. For our purpose, it is not necessary to assume this condition. Indeed, we will consider products of groups that will contain elements with such eigenvalues. If $0 \le a_i < r$ for all $i$, and $n = 2$ or $n = 3$, then there always exist a generator satisfying the second condition.

Using the description of the McKay quiver of a cyclic group $G$, we can label the vertices from 1 to $r$. There is an arrow of type $x_j$ from $l \to l+a_j$, where the sum is taken modulo $r$, for every vertex $l \in M_G$.

The grading mentioned in Theorem 5.2.1, that defines a preprojective algebra structure on $R\#G$, is given by the following:

$$\deg(x_j^l : l \to l + a_j) = \begin{cases} 1 & \text{if } \overline{l + a_j} < l \text{ in } \mathbb{Z}_r \\ 0 & \text{else.} \end{cases}$$

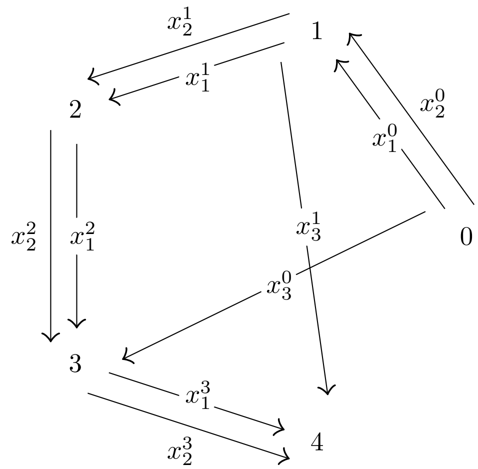
**Example 5.2.1.** Let $G = \langle \frac{1}{5}(1,1,3) \rangle$. The McKay quiver of $G$ is given by:




The algebra $R\#G$ is isomorphic to the path algebra over this quiver with relations given by

$$\{x_{i'}^{j+a_i} \otimes x_i^j - x_i^{j+a_{i'}} \otimes x_{i'}^j\}.$$

Since $G$ satisfies the conditions of Theorem 5.2.1, we obtain that $R\#G$ has a structure of preprojective algebra, where the degree 0 part is given by the following quiver.




### 5.2.1 Abelian case

We extend Theorem 5.2.1 to abelian groups satisfying certain conditions. Let $G \subset SL(n, \mathbb{C})$ be a finite abelian group. Then, $G = \mu_{r_1} \oplus \cdots \oplus \mu_{r_l}$ for some cyclic groups, such that $r_1|r_2|\cdots|r_l$. Thus,

$$G = \langle \frac{1}{r_1}(a_1^1, \ldots, a_n^1)\rangle \oplus \cdots \oplus \langle \frac{1}{r_l}(a_1^l, \ldots, a_n^l)\rangle.$$

However, one has to be very careful with this notation. Indeed, it does not tell you which root of unity is chosen to describe each cyclic group. With only one cyclic group, it often does not matter for our purpose, but with several ones, we need to make sure that it is possible to find roots such that the groups don't intersect themselves. Otherwise the direct sum does not make sense. This is not always possible to do with any choice of roots of unity, and in general it is not clear at first how to determine when a combination is possible. For example, it could be tempting to write

$$G = \langle \frac{1}{2}(1,1)\rangle \oplus \langle \frac{1}{4}(1,3)\rangle$$

as an abelian subgroup of $SL(2,k)$ of order 8. But it is not possible to choose roots of unity in which these two cyclic groups don't intersect. Therefore, this abelian group does not exist, which we already knew since the abelian subgroups of $SL(2,k)$ are cyclic.

Recall from Theorem 2.4.2 that the minimal projective $B$-bimodule resolution of $B := R\#G$ is given by

$$P_\bullet = B \otimes_{kG} \left(\bigwedge^n V \otimes_k kG\right) \otimes_{kG} B \to B \otimes_{kG} \left(\bigwedge^{n-1} V \otimes_k kG\right) \otimes_{kG} B \to \cdots \to B \otimes_{kG} B,$$

and satisfies the property that $P_\bullet \cong P_\bullet^\vee[n]$, where $(-)^\vee = \mathrm{Hom}_{B^e}(-, B^e)$. This property implies that the skew-group algebra is $n$-Calabi-Yau. Moreover, the differentials are given by

$$\delta_l((v_1, g_1) \otimes (x_{j_l} \wedge x_{j_{l-1}} \wedge \cdots \wedge x_{j_1} \otimes g) \otimes (v_2, g_2))$$

$$= \sum_{i=1}^{l} (-1)^{i-1} ((v_1 g_1 x_{j_i}, g_1) \otimes (x_{j_l} \wedge \cdots \widehat{x_{j_i}} \cdots \wedge x_{j_1} \otimes g) \otimes (v_2, g_2)$$
$$+ (v_1, g_1) \otimes (x_{j_l} \wedge \cdots \widehat{x_{j_i}} \cdots \wedge x_{j_1} \otimes g) \otimes (g^{-1} x_{j_i} v_2, g_2)),$$

where we use the notation $(f, g) \in R\#G$ to denote an element $f \otimes_k g \in R \otimes_k kG \cong R\#G$. In the remainder of this chapter, we use the $kG$-bimodule isomorphism from Remark 5

$$\bigwedge^{l} V \otimes_k kG \cong \bigcap_{\mu} \left( (V \otimes_k kG)^{\otimes \mu} \otimes_{kG} (M \otimes_k kG) \otimes_{kG} (V \otimes_k kG)^{\otimes l-\mu-2} \right),$$

given by

$$x_{j_l} \wedge x_{j_{l-1}} \wedge \cdots \wedge x_{j_1} \otimes 1 \mapsto \sum_{\sigma \in \mathfrak{S}_l} (-1)^{\sigma} \sigma(x_{j_l} \otimes x_{j_{l-1}} \otimes \cdots \otimes x_{j_1}) \otimes 1,$$

where the elements of the symmetric group $\mathfrak{S}_l$ act as usual, to describe the superpotential as an element in $\bigwedge^l V \otimes_k kG$.

The $B$-bimodule generators of

$$P_\kappa := B \otimes_{kG} \left( \bigwedge^{\kappa} V \otimes_k kG \right) \otimes_{kG} B$$

are given by the elements $(1,1) \otimes (x_{j_\kappa} \wedge x_{j_{\kappa-1}} \wedge \cdots \wedge x_1 \otimes 1) \otimes (1,1)$, for $\kappa \in \{0, \ldots, n\}$. This implies that a superpotential in $R\#G$ is given by the $S$-bimodule generator

$$\omega \otimes 1 \subset \Lambda^n V \otimes_k kG = K_n,$$

where $\omega := x_n \wedge x_{n-1} \wedge \cdots \wedge x_1$ is the superpotential giving the commutativity relations in $R$.

We would like to generalize Theorem 5.2.1 to abelian groups. We assume that

$$G = \langle \frac{1}{r_1}(a_1^1, \ldots, a_n^1) \rangle \oplus \cdots \oplus \langle \frac{1}{r_l}(a_1^l, \ldots, a_n^l) \rangle$$

exists and that $(a_i^{k_j}, r_j) = 1$ and $0 < a_i^{r_j} < r_j$ for all $i, j$.

**Theorem 5.2.2.** *Let $G \subset SL(n, \mathbb{C})$ be a finite abelian group as above. If there exists an index $1 \leq z \leq l$ such that $\langle \frac{1}{r_z}(a_1^z, \ldots, a_n^z) \rangle$ has a generator $g = \frac{1}{r_z}(\tilde{a}_1^z, \ldots, \tilde{a}_n^z)$ satisfying the equation*

$$\tilde{a}_1^z + \cdots + \tilde{a}_n^z = r_z,$$

*then the skew-group algebra $R\#G$ has a structure of preprojective algebra.*

*Proof.* Let $M_G$ be the McKay graph of $G$. By Theorem 2.3.1, we have that

$$B := R\#G \cong kM_G/\langle M \rangle,$$

where $M$ is the ideal of commutativity relations, as described in the preliminaries.

Let $\{\psi_i^j\}_{i=1}^{r_j}$ be the set of irreducible representations of $\mu_{r_j}$. Then the irreducible representations of

$G$ are given by

$$\psi^1_{i_1} \otimes \cdots \otimes \psi^l_{i_l},$$

for some $i_j \in \{1, \dots, r_j\}$. Consider the bijective map:

$$\chi(G) \to \mathbb{Z}_{r_1} \oplus \cdots \oplus \mathbb{Z}_{r_l},$$

$$\psi^1_{i_1} \otimes \cdots \otimes \psi^l_{i_l} \mapsto (i_1, \dots, i_l),$$

where $\chi(G)$ is the group of irreductible characters of $G$. Then the Mckay graph has a vertex corresponding to every $(i_1, \dots, i_l)$ and there is an arrow

$$a_j^{(i_1,\dots,i_l)} : (i_1, \dots, i_l) \to (i_1 + a^1_j, \dots, i_l + a^l_j)$$

for every $j \in \{1, \dots, n\}$. The relations in $M$ are generated by

$$a_{j'}^{(i_1+a^1_j,\dots,i_l+a^l_j)} a_j^{(i_1,\dots,i_l)} - a_j^{(i_1+a^1_{j'},\dots,i_l+a^l_{j'})} a_{j'}^{(i_1,\dots,i_l)} = 0$$

for $j, j' \in \{1, \dots, n\}$ and every vertex $(i_1, \dots, i_l) \in \mathbb{Z}_{r_1} \oplus \cdots \oplus \mathbb{Z}_{r_l}$.

Choose $z \in \{1, \dots, l\}$ such that $\langle \frac{1}{r_z}(a^z_1, \dots, a^z_n) \rangle$ satisfies the condition of the statement. Without loss of generality, we may assume that $a^z_1 + \cdots + a^z_n = r_z$. The following is analogous to the proof of Theorem 5.2.1 in [AIR15]. Define a grading on the arrows in the algebra $R\#G \cong kM_G/\langle M \rangle$ as follows:

The arrow $a_j^{(i_1,\dots,i_l)}$ has degree

$$\begin{cases} 1 & \text{if } \overline{i_z + a^z_j} < i_z \text{ in } \mathbb{Z}_{r_z} \\ 0 & \text{else.} \end{cases}$$

This grading is compatible with the ideal of relations $M$. In fact, consider a path of length 2

$$a_{j'}^{(i_1+a^1_j,\dots,i_l+a^l_j)} a_j^{(i_1,\dots,i_l)} : (i_1, \dots i_l) \to (\overline{i_1 + a^1_j + a'^1_j}, \dots, \overline{i_l + a^l_j + a'^l_j}).$$

If $\overline{i_z + a^z_j + a'^z_j} < i_z$ in $Z_{r_1}$, then the path is in degree 1. Indeed, since $0 < a^z_\xi < r_j$ for any $1 \leq \xi \leq n$, we get that one arrow must be in degree 0, the other in degree 1. The same is true for the path

$$a_j^{(i_1+a^1_{j'},\dots,i_l+a^l_{j'})} a_{j'}^{(i_1,\dots,i_l)} : (i_1, \dots i_l) \to (\overline{i_1 + a^1_j + a'^1_j}, \dots, \overline{i_l + a^l_j + a'^l_j}).$$

The grading is additive over concatenation. The degree 0 part of $R\#G$ is finite-dimensional. In fact, it does not contain any cycle. This is clear, since if it had one, then in particular this cycle would contain

$$i_z \to i_z + a^z_\alpha \to \cdots \to i_z,$$

in its $z$-th component. But then by definition it must have at least one arrow of degree 1, which is a contradiction.

Consider the following grading on the elements in the $B$-bimodules $P_l$, $l = 0, \dots, n$. We put the element

$$(1,1) \otimes (x_j \otimes e_{(i_1,\dots,i_l)}) \otimes (1,1)$$

in degree

$$\begin{cases} 1 & \text{if } \overline{i_z + a_j^z} < i_z \text{ in } Z_{r_z} \\ 0 & \text{else} \end{cases}$$

Here, $e_{(i_1,\dots,i_l)}$ is the idempotent corresponding to the simple $kG$-module at vertex $(i_1,\dots,i_l)$. This defines a degree on any element

$$1 \otimes (x_{j_\kappa} \wedge x_{j_{\kappa-1}} \wedge \cdots \wedge x_{j_1} \otimes e_{(i_1,\dots,i_l)}) \otimes 1 \in P_\kappa,$$

by putting

$$\deg(1 \otimes (x_{j_\kappa} \wedge x_{j_{\kappa-1}} \wedge \cdots \wedge x_{j_1} \otimes e_{(i_1,\dots,i_l)}) \otimes 1) = \sum_{i=1}^{\kappa} \deg(1 \otimes (x_{j_i} \otimes e_{u_{j_i}}) \otimes 1),$$

for $\kappa \in \{0, \dots, n\}$. We claim that the differentials are homogeneous. This comes from the fact that the degree map is additive. To prove it, we use induction on the degree $\kappa$ in the bimodule resolution of $B$. If $\kappa = 1$, then $\delta_1 : P_1 \to P_0$ is homogeneous since

$$\deg(1 \otimes (x_j \otimes e_{(i_1,\dots,i_l)}) \otimes 1) = \deg((x_j, e_{(i_1,\dots,i_l)}) \otimes 1) = \deg(1 \otimes (x_j, e_{(i_1,\dots,i_l)})),$$

using the fact that the definition of the grading is the same for an element in $B$ and the corresponding element in the $B$-bimodule $V \otimes_k kG$. The induction follows easily from the definition of the differentials.

Also, each bimodule in the resolution is generated in degree 0 and 1. In fact, if there was a path such that

$$\deg(1 \otimes (x_{j_\kappa} \wedge x_{j_{\kappa-1}} \wedge \cdots \wedge x_{j_1} \otimes e_{(i_1,\dots,i_l)}) \otimes 1) > 1,$$

then in particular the associated path in $B$, which, by definition, has the same grading, would cross two vertices having the same $z$-th component, say $i_z$. But this is impossible, as $a_1^z + \cdots a_n^z = r_z$.

This shows that the Gorenstein parameter is 1. In fact, the perfect pairing of Theorem 2.4.2

$$\phi : B \otimes_{\mathbb{C}G} \bigwedge^{\kappa} V \otimes_{\mathbb{C}} \mathbb{C}G \otimes_{\mathbb{C}G} B \times B \otimes_{\mathbb{C}G} \bigwedge^{n-\kappa} V \otimes_{\mathbb{C}} \mathbb{C}G \otimes_{\mathbb{C}G} B \to B \otimes_{\mathbb{C}G} B,$$

gives an isomorphism

$$B \otimes_{\mathbb{C}G} \bigwedge^{\kappa} V \otimes_{\mathbb{C}} \mathbb{C}G \otimes_{\mathbb{C}G} B \cong \operatorname{Hom}_{B^e}\left(B \otimes_{\mathbb{C}G} \bigwedge^{n-\kappa} V \otimes_{\mathbb{C}} \mathbb{C}G \otimes_{\mathbb{C}G} B, B \otimes_{\mathbb{C}G} B\right)$$

which maps

$$\alpha := 1 \otimes (x_{j_\kappa} \wedge x_{j_{\kappa-1}} \wedge \cdots \wedge x_{j_1} \otimes e_{(i_1,\dots,i_l)}) \otimes 1 \mapsto$$

$$(\beta := 1 \otimes (x_{m_{n-\kappa}} \wedge x_{m_{n-\kappa-1}} \wedge \cdots \wedge x_{m_1} \otimes e_{(i'_1,\dots,i'_l)}) \otimes 1 \mapsto \phi(\alpha, \beta)).$$

We see that if $\alpha$ is of degree 1, then $\phi$ must be of degree 0 and vice-versa, since

$$1 \otimes (x_n \wedge x_{n-1} \wedge \cdots \wedge x_1 \otimes e_{(i_1,\ldots,i_l)}) \otimes 1$$

is in degree 1. □

**Example 5.2.2.** Consider the group

$$G = \langle \frac{1}{5}(2,3,1,4)\rangle \oplus \langle \frac{1}{10}(1,3,3,3)\rangle \subset SL(4,\mathbb{C}).$$

The generator $\frac{1}{10}(1,3,3,3)$ is such that $1+3+3+3 = 10$. Therefore, $R\#G$ has a structure of preprojective algebras. It is given by the following grading. There are 50 vertices in the McKay quiver of $G$, which are in correspondence with the elements $(i,j)$ in $\mathbb{Z}_5 \times Z_{10}$. There are 4 arrows going out of each vertex, and their grading is given by

$$\deg((i,j) \to (i+a, j+b)) = \begin{cases} 1 & \text{if } \overline{j+b} < j \text{ in } \mathbb{Z}_{10} \\ 0 & \text{else} \end{cases}$$

where $(a,b) = (2,1)$, $(3,3)$, $(1,3)$ or $(4,3)$.

**Question 5.2.3.** In the next subsection, we will show that if $G$ embeds in $SL(n_1,k) \times SL(n_2,k)$, then $R\#G$ does not have a structure of preprojective algebra. The condition in Theorem 5.2.2 ensures that $G$ does not embed in such product, see Lemma 5.2.3. However, there are abelian groups that do not satisfy the condition, yet do no embed in $SL(n_1,k) \times SL(n_2,k)$ either, for any $n_1$, $n_2$. Such groups are of the form

$$\langle \frac{1}{r_1}(a_1^1,\ldots,a_n^1)\rangle \oplus \cdots \oplus \langle \frac{1}{r_l}(a_1^l,\ldots,a_n^l)\rangle,$$

where the $j$-th component embeds in $SL(n_1^j,k) \times SL(n_2^j,k)$ for all $1 \leq j \leq l$, but the $n_i^j$ are not all the same for each $j$. The group

$$\langle \frac{1}{3}(1,1,1,1,1,1)\rangle \oplus \langle \frac{1}{6}(1,5,1,5,1,5)\rangle < SL(6,k)$$

is such an example, since

$$\langle \frac{1}{3}(1,1,1,1,1,1)\rangle < SL(3,k) \times SL(3,k)$$

and

$$\langle \frac{1}{6}(1,5,1,5,1,5)\rangle < SL(2,k) \times SL(2,k) \times SL(2,k).$$

In this case, it is unknown whether such groups have a structure of preprojective algebra or not.

### 5.2.2 Groups embedding in $SL(n_1,k) \times SL(n_2,k)$

Condition 2 in Theorem 5.2.1 of [AIR15] is specific to cyclic groups. Thus, if we want to generalize this statement or to find a converse for any group (which is the purpose of this section), we need to reinterpret this condition.

With Remark 9 in mind, we see that condition 2 follows from the fact that $G$ does not embed into $SL(n_1, k) \times SL(n_2, k)$.

**Lemma 5.2.3.** *Let $G$ be a finite cyclic subgroup of $SL(n, k)$ of order $r$. If the group $G$ embeds into $SL(n_1, k) \times SL(n_2, k)$ for some $n_1, n_2 \geq 1$ such that $n_1 + n_2 = n$, then either one of the following is true.*

*(a) There exists a generator $g = \frac{1}{r}(a_1, \dots, a_n)$ of $G$ such that $a_i = 0$ for some $i$;*

*(b) Every generator $g = \frac{1}{r}(a_1, \dots, a_n)$ of $G$ is such that $0 < a_i < r$ and the sum $a_1 + \cdots + a_n > r$.*

*The converse is true if $n \leq 4$ or $G$ satisfies condition $(a)$.*

*Proof.* If $G$ embeds into $SL(n_1, k) \times SL(n_2, k)$, then $G$ clearly satisfies condition $(a)$ or $(b)$. Now, if $(a)$ is true, then $G$ embeds into $SL(n - 1, k) \times SL(1, k)$. Finally, if $G < SL(n, k)$, where $n \leq 4$, satisfies condition $(b)$, then the claim follows from [MS84]. □

Let $R_i$ be the polynomial ring in $n_i$ variables, for $i = 1, 2$. If $G$ embeds into $SL(n_1, k) \times SL(n_2, k)$, then the algebra of invariants $R^G$ is an isolated singularity if and only if $G$ is conjugate to

$$G \times_G G < SL(n_1, k) \times SL(n_2, k),$$

where $G \times_G G$ is the diagonal embedding in $G \times G$, and $R_1^G$ and $R_2^G$ are isolated singularities. The McKay quiver of $G \times_G G$ is easy to describe. The vertices are given by the irreducible representations of $G$. Moreover, there is an arrow $\rho_1 \to \rho_2$ if and only if there is an arrow $\rho_1 \to \rho_2$ in $M_G^i$, for some $i \in \{1, 2\}$, where $M_G^i$ is the McKay quiver of the $i^{\text{th}}$ component in $G \times_G G$.

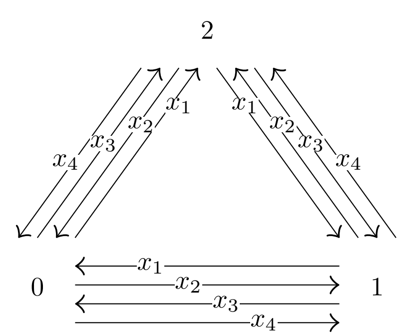
**Example 5.2.4.** Let

$$G = \langle \frac{1}{3}(1, 2, 1, 2) \rangle < SL(4, k).$$

Then

$$G \cong \langle \frac{1}{3}(1, 2) \rangle \times_G \langle \frac{1}{3}(1, 2) \rangle \hookrightarrow SL(2, k) \times SL(2, k).$$

Its McKay quiver is given by

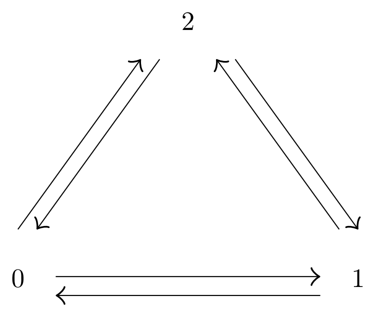



And the relations are given by $x_i \otimes x_j - x_j \otimes x_i = 0$. Note that the superpotential is given by

$$\omega = \sum_{0 \leq i \leq 2} \sum_{\sigma \in \mathfrak{S}_4} (-1)^{|\sigma|} \sigma(x_4 \otimes x_3 \otimes x_2 \otimes x_1^i),$$

where $x_1^i$ is the arrow $x_1$ starting at vertex $i$, and the other arrows in $x_4 \otimes x_3 \otimes x_2 \otimes x_1^i$ start at the vertex where the concatenation is non-zero. The action of the symmetric group is given as usual.

Comparing with the McKay quiver of $\langle \frac{1}{3}(1,2) \rangle$, we see that there is a doubling of the arrows.




In light of Lemma 5.2.3, we prove a partial converse to the theorem in [AIR15], which generalizes to any subgroup of $SL(n,k)$. Let $B := R\#G$. Throughout this section, we suppose that $G$ embeds in $SL(n_1,k) \times SL(n_2,k)$, where $n_1 + n_2 = n$. Equivalently, $G$ is conjugate to a fibre product $G_1 \times_H G_2$, where $G_i$ is a finite subgroup of $SL(n_i,k)$ and $H$ is some finite group such that $G_i$ surjects onto it, for $i = 1, 2$.

**Theorem 5.2.4.** *Let $G$ be a finite subgroup of $SL(n,k)$. If $B := R\#G$ has a grading structure of $n$-preprojective algebra, then $G$ does not embed in $SL(n_1,k) \times SL(n_2,k)$ for any $n_1, n_2 \geq 1$ such that $n_1 + n_2 = n$.*

*Proof.* Suppose that $G = G_1 \times_H G_2$, where $G_1 < SL(n_1,k)$ and $G_2 < SL(n_2,k)$. Then the given representation decomposes as $V \cong V_1 \oplus V_2$, where $V_i$ is a $n_i$-dimensional vector space, $i = 1, 2$. Assume by contradiction that $G$ has a grading structure of preprojective algebra such that the degree 0 part is finite-dimensional.

By Lemma 4.2.1, the superpotential $\omega \otimes 1$ is a $kG$-bimodule generator of $\Lambda^n V \otimes_k kG$ in degree 1. It corresponds to elements of the same degree in $B$, by Lemma 4.2.2. Thus, for any idempotent $e_l$ corresponding to the irreducible representations, the summand $(x_n \otimes x_{n-1} \otimes \cdots \otimes x_1) \otimes e_l$ is in degree 1. Now, using the fact that $G$ embeds in $SL(n_1,k) \times SL(n_2,k)$, we get a decomposition

$$(x_n \otimes x_{n-1} \otimes \cdots \otimes x_1) \otimes e_l = (x_n \otimes x_{n-1} \otimes \cdots \otimes x_{n_2+1}) \otimes e_l \cdot (x_{n_2} \otimes \cdots \otimes x_1) \otimes e_l,$$

and each component commutes with the action of $kG$. By additivity of the degrees, one of these two components must be in degree 0, and the other must be in degree 1. Suppose without loss of generality that

$$(x_n \otimes x_{n-1} \otimes \cdots \otimes x_{n_2+1}) \otimes e_l$$

is in degree 0. Then,

$$\{1 \otimes e_l, (x_n \otimes x_{n-1} \otimes \cdots \otimes x_{n_2+1}) \otimes e_l, (x_n \otimes x_{n-1} \otimes \cdots \otimes x_{n_2+1})^2 \otimes e_l,$$

$$(x_n \otimes x_{n-1} \otimes \cdots \otimes x_{n_2+1})^3 \otimes e_l, \ldots\}$$

is an infinite set of linearly independent elements of degree 0 in $B$. Therefore, $B$ does not have a structure of preprojective algebra. □

*Remark* 11. In the case where $G$ is cyclic, we obtain a partial converse to Theorem 5.2.1, which is a full converse if $n = 4$.

**Example 5.2.5.** Let $G = \langle \frac{1}{3}(1,2,1,2) \rangle < SL(2,k) \times SL(2,k)$ be the group described in Example 5.2.4. The skew-group algebra $R\#G$ does not have a structure of preprojective algebra. As we showed, paths of length 4 in the superpotential go twice through the same vertex. However, since they must be in degree 1, there exists an oriented cycle in degree 0, which means that the degree 0 part is infinite-dimensional.

Now, we want to consider the basic algebra Morita equivalent to $B$. Let $e = \sum e_l$ be the sum of all idempotents in $kG$ associated to the irreducible representations of $G$. If

$$R\#G = T_{kG}(V \otimes_k kG)/\langle M \otimes_k kG \rangle,$$

then $R\#G$ is Morita equivalent to

$$eR\#Ge \cong T_{ekGe}(eV \otimes_k kGe)/\langle eM \otimes_k kGe \rangle,$$

see for example [BSW10] for a proof. In the case where $G$ is abelian, this Morita equivalence is an isomorphism. The superpotential is given by $e(\omega \otimes_k 1)e$.

**Theorem 5.2.5.** *At each vertex $e_j$ in $M_G$, there exists a closed path $p$ of length $n$ which is a summand of $e(\omega \otimes_k 1)e$ and goes through the vertex $e_j$ twice.*

*Proof.* Let $S_j$ be the simple $kG$-module associated to $e_j$. Note that, because $G$ embeds in $SL(n_1,k) \times SL(n_2,k)$, the module

$$\Lambda^{n_i} V \otimes_k S_j$$

contains $S_j$ as a direct summand for $i = 1, 2$. Now, each path $p : t(p) \to h(p)$ of length $l$ corresponds to a map

$$S_{t(p)} \to V^{\otimes_S l} \otimes_k S_{h(p)} \to \Lambda^l V \otimes_k S_{h(p)}.$$

Since there is an obvious map $S_j \to S_j \subset \Lambda^{n_i} V \otimes_k S_j$, we get a closed path $p_i$ of length $n_i$ at $S_j$ for $i = 1, 2$. Concatenating these two paths, we get the required path $p = p_1 \otimes p_2$ of length $n$ that goes twice through the vertex $e_j$. This corresponds to a summand of the superpotential, since it corresponds to an element in $\Lambda^n V \otimes_k kG$. □

Suppose that $eR\#Ge$ has a structure of preprojective algebra. These closed paths are summands of the superpotential in $eR\#Ge$, and thus, by the same reasoning as in the previous subsection, they must be in degree 1. By additivity of the grading on concatenation, we have a closed oriented path in degree 0, the other being in degree 1. However, we cannot directly apply the same idea as in the previous theorem, for some of the powers of the path in degree 0 could be equal to 0 in the algebra $eR\#Ge$.

*Remark* 12. In ([HIO14], Question 5.9), the authors conjecture that the quiver of a $(n-1)$-hereditary algebra must be acyclic. Thus, if we assume this conjecture to be true, the previous discussion already demonstrates that if $G$ embeds in a product of special linear groups, then $B := R\#G$ is not Morita equivalent to a preprojective algebra.

**Lemma 5.2.6.** *If $\Lambda$ is a $(n-1)$-representation-infinite algebra, and $e$ is a full idempotent in $\Lambda$, that is, $\Lambda e \Lambda = \Lambda$, then*

$$eExt^{n-1}_{\Lambda^e}(\Lambda, \Lambda^e)e \cong Ext^{n-1}_{e\Lambda^e e}(e\Lambda e, e\Lambda^e e).$$

*Moreover, if $M_d(-)$ denotes the algebra of $d \times d$ matrices, then*

$$M_d(Ext^{n-1}_{\Lambda^e}(\Lambda, \Lambda^e)) \cong Ext^{n-1}_{M_d(\Lambda^e)}(M_d(\Lambda), M_d(\Lambda^e)).$$

*Proof.* Consider the minimal resolution of $\Lambda$:

$$0 \to \bigoplus_{\substack{p:i\to j \\ p\in K_n}} \Lambda e_j \otimes_k e_i\Lambda \xrightarrow{f} \bigoplus_{\substack{p:i\to j \\ p\in K_{n-1}}} \Lambda e_j \otimes_k e_i\Lambda \to \cdots \to \Lambda \otimes_S \Lambda \to \Lambda \to 0.$$

Then, as we proved in Theorem 3.1.1,

$$\text{Ext}^{n-1}_{\Lambda^e}(\Lambda, \Lambda^e) \cong \bigoplus_{\substack{p:i\to j \\ p\in K_n}} \Lambda e_i \otimes_k e_j\Lambda/\text{Im } \tilde{f},$$

where $\tilde{f} = \text{Hom}_{\Lambda^e}(f, \Lambda^e)$. Therefore

$$\begin{aligned} e\text{Ext}^{n-1}_{\Lambda^e}(\Lambda, \Lambda^e)e &\cong \bigoplus_{\substack{p:i\to j \\ p\in K_n}} e\Lambda e e_i \otimes_k e_j e\Lambda e/e\text{Im } \tilde{f}e \\ &\cong \text{Ext}^{n-1}_{e\Lambda^e e}(e\Lambda e, e\Lambda^e e). \end{aligned}$$

Now idempotents in $M_d(\text{Ext}^{n-1}_{\Lambda^e}(\Lambda, \Lambda^e))$ induce idempotents in $\text{Ext}^{n-1}_{\Lambda^e}(\Lambda, \Lambda^e)$, so we have

$$\text{Ext}^{n-1}_{M_d(\Lambda^e)}(M_d(\Lambda), M_d(\Lambda^e)) \cong M_d(f_{11}\text{Ext}^{n-1}_{M_d(\Lambda^e)}(M_d(\Lambda), M_d(\Lambda^e))f_{11}),$$

where $f_{11}$ is the idempotent corresponding to the entry $(1,1)$ in the matrix algebra. Applying the first part of this proof again, the algebra is isomorphic to

$$M_d(\text{Ext}^{n-1}_{\Lambda^e}(\Lambda, \Lambda^e)).$$

□

We now show that $eR\#Ge$ does not have a structure of preprojective algebra.

**Theorem 5.2.7.** *Let $G$ be a finite subgroup of $SL(n,k)$ such that $G$ embeds in $SL(n_1,k)\times SL(n_2,k)$. Let $e = \sum e_l$ be the sum of all idempotents associated to the irreducible representations of $G$ in $B := R\#G$. Then $eBe$ does not have a grading structure of preprojective algebra.*

*Proof.* Suppose that

$$eBe \cong T_\Lambda \text{Ext}^{n-1}_{\Lambda^e}(\Lambda, \Lambda^e),$$

where $\Lambda$ is a $(n-1)$-representation-infinite algebra. Then

$$B \cong fM_d(T_\Lambda \text{Ext}^{n-1}_{\Lambda^e}(\Lambda, \Lambda^e))f,$$

for some $d$, where $M_d(-)$ denotes the algebra of $d\times d$ matrices and $f$ is a full idempotent in $M_d(T_\Lambda \mathrm{Ext}^{n-1}_{\Lambda^e}(\Lambda,\Lambda^e))$. Let $\mathcal{F}$ be the equivalence of categories

$$\mathcal{F} : \mathrm{Bimod}\ eBe \to \mathrm{Bimod}\ fM_d(eBe)f.$$

Since $M_d(-)$ commute with direct sums and tensor products and $f$ is also a full idempotent in

$$M_d(\Lambda) \subset M_d(T_\Lambda \mathrm{Ext}^{n-1}_{\Lambda^e}(\Lambda,\Lambda^e)),$$

we obtain

$$\mathcal{F}(M \otimes_\Lambda N) \cong \mathcal{F}(M) \otimes_{fM_d(\Lambda)f} \mathcal{F}(N),$$

and the same is true for direct sums. Thus,

$$B \cong T_{fM_d(\Lambda)f} fM_d(\mathrm{Ext}^{n-1}_{\Lambda^e}(\Lambda,\Lambda^e))f.$$

By Lemma 5.2.6, we obtain that

$$B \cong T_{fM_d(\Lambda)f} \mathrm{Ext}^{n-1}_{fM_d(\Lambda^e)f}(fM_d(\Lambda)f, fM_d(\Lambda^e)f),$$

so $B$ is itself $n$-Calabi-Yau of Gorenstein parameter 1. Therefore, by Theorem 5.2.4, we get that $fM_d(\Lambda)f$ is infinite-dimensional. This implies that $\Lambda$ is infinite-dimensional, which is a contradiction. □

**Example 5.2.6.** Let $\mathbb{E}_6$ be the finite subgroup of order 24 of $SL(2,\mathbb{C})$ with generators

$$S = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\ V = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},\ U = \frac{1}{\sqrt{2}} \begin{pmatrix} \gamma & \gamma^3 \\ \gamma & \gamma^7 \end{pmatrix},$$

where $\gamma$ is a primitive eight root of unity, submitted to the relations $U^3 = S^2 = V^2 = -I$. The elements are given by

$$\mathbb{E}_6 = \{\pm U^k, \pm SU^k, \pm VU^k, \pm SVU^k \mid k = 0,1,2\}.$$

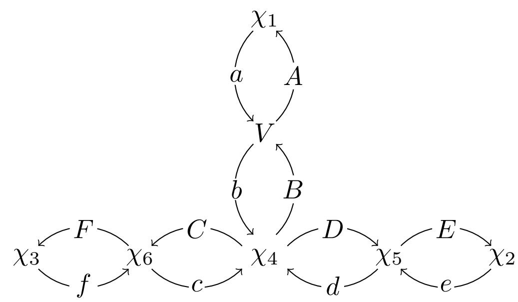
The conjugacy classes of $\mathbb{E}_6$ are the following:

$$\begin{aligned}
C_1 &= \{U^0 = I\},\\
C_2 &= \{U^3 = -I\},\\
C_3 &= \{\pm S, \pm V, \pm SV\},\\
C_4 &= \{U^2, SU^2, VU^2, SVU^2\},\\
C_5 &= \{-U, SU, VU, SVU\},\\
C_6 &= \{-U^2, -SU^2, -VU^2, -SVU^2\},\\
C_7 &= \{U, -SU, -VU, -SVU\},
\end{aligned}$$

and the character table is given by

| | $C_1$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ | $C_6$ | $C_7$ |
|---|---|---|---|---|---|---|---|
| $\chi_1$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\chi_2$ | 1 | 1 | 1 | $\omega$ | $\omega^2$ | $\omega$ | $\omega^2$ |
| $\chi_3$ | 1 | 1 | 1 | $\omega^2$ | $\omega$ | $\omega^2$ | $\omega$ |
| $\chi_4$ | 3 | 3 | $-1$ | 0 | 0 | 0 | 0 |
| $V$ | 2 | $-2$ | 0 | $-1$ | $-1$ | 1 | 1 |
| $\chi_5$ | 2 | $-2$ | 0 | $-\omega$ | $-\omega^2$ | $\omega$ | $\omega^2$ |
| $\chi_6$ | 2 | $-2$ | 0 | $-\omega^2$ | $-\omega$ | $\omega^2$ | $\omega$ |

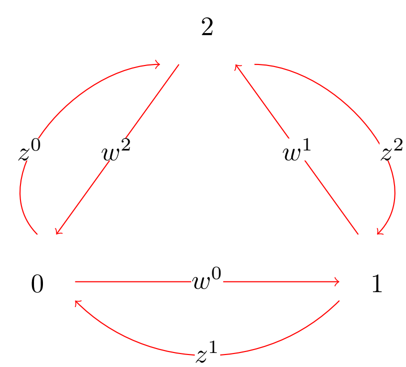
where $\omega$ is a primitive third root of unity and $V$ is the given representation. Its McKay quiver is the extended Coxeter-Dynkin diagram of type $\mathbb{E}_6$, with two arrows of opposite direction instead of edges:




The skew-group algebra $k[x,y]\#\mathbb{E}_6$ is Morita equivalent to the path algebra over this quiver with relations

$$\begin{aligned} aA - Bb &= 0, & eE - Dd &= 0, & fF - Cc &= 0, \\ Aa &= 0, & Ee &= 0, & Ff &= 0, \\ bB + cC + Dd &= 0. \end{aligned}$$

The superpotential $\omega_1$ of $\mathbb{E}_6$ is the sum of those relations. We also consider the cyclic group $\langle \frac{1}{3}(1,2)\rangle$, whose associated skew-group algebra $k[z,w]\#\langle \frac{1}{3}(1,2)\rangle$ is given as the path algebra over the quiver




with superpotential

$$\omega_2 = (z \wedge w) \otimes 1 \sim \sum_{l=1}^{3} (z^{l+1} w^l - w^{l-1} z^l),$$

where $\sim$ represents the identification of Remark 5. Let $\mu_3 = \langle \alpha \rangle$ be the abstract cyclic group of order 3.

Let $G$ be the non-abelian group

$$G = \mathbb{E}_6 \times_{\mu_3} \langle \frac{1}{3}(1,2) \rangle \subset SL(2,\mathbb{C}) \times SL(2,\mathbb{C}).$$

The maps defining the fibre product are given by

$$\phi_1 : \mathbb{E}_6 \to \mu_3, \quad U \mapsto \alpha,\ S \mapsto e,\ V \mapsto e$$

and

$$\phi_2 : \langle \frac{1}{3}(1,2) \rangle \to \mu_3, \quad \frac{1}{3}(1,2) \mapsto \alpha.$$

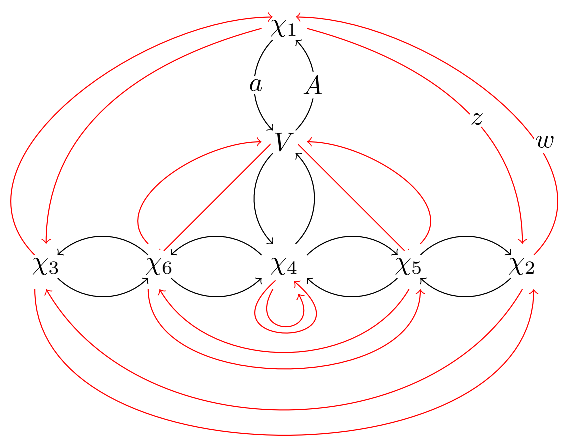
Since $\langle \frac{1}{3}(1,2) \rangle$ is an abelian group, the conjugacy classes are induced from the ones in $\mathbb{E}_6$, and are given by

$$(C_1, I),\ (C_2, I),\ (C_3, I),\ (C_4, \frac{1}{3}(2,1)),\ (C_5, \frac{1}{3}(1,2)),\ (C_6, \frac{1}{3}(2,1)) \text{ and } (C_7, \frac{1}{3}(1,2)),$$

where the elements in the pair $(C_i, A)$ are $4 \times 4$ block matrices, where the first block is in $C_i$ and the second one is $A \in \langle \frac{1}{3}(1,2) \rangle$. We see that $G \cong \mathbb{E}_6$, so their character tables are the same. The given representation $\tilde{V}$ of $G$ is the following:

$$\tilde{V} \cong V \oplus \chi_2 \oplus \chi_3.$$

Therefore the McKay quiver of $G$ has the McKay quiver of $\mathbb{E}_6$ as a subquiver, plus arrows coming from $\chi_2$ and $\chi_3$, that we put in red.




The superpotential $\omega$ of $k[x,y,z,w]\#G$ has summands at every vertex given by concatenation of a black closed path of length 2 followed by a red closed path of length 2. For example,

$$w \otimes z \otimes A \otimes a$$

is such a path at vertex $\chi_1$. Now, this path must have degree 1, so either $w \otimes z$ has degree 1 or $A \otimes a$, the other being in degree 0. The path in degree 0 therefore creates an infinite-dimensional basis of a space in degree 0, which is impossible.

## 5.3 Further Discussion

This section contains interpretations of the results of this chapter. We do not define precisely every term used, as some of the concepts would require too much background for what is needed. The goal is to explore potential further research avenues.

In [AIR15], the authors establish interesting correspondences in higher global dimensions, which are analogous to the classical case $n = 2$. In this case, there are equivalences

$$\underline{\mathrm{CM}}(\mathrm{gr}\ R^G) \cong \mathcal{D}^b(kQ_G)$$

and

$$\underline{\mathrm{CM}}(R^G) \cong \mathcal{C}_1(kQ_G),$$

where $Q_G$ is the (non-extended) Coxeter-Dynkin quiver associated to $G$. Note that

$$\mathcal{C}_1(kQ_G) := \mathcal{D}^b(kQ_G)/\langle\tau\rangle$$

is the cluster category, defined as the orbit category of $\mathcal{D}^b(kQ_G)$ by the action of $\tau$, the Auslander-Reiten translate. We have the following theorem in higher global dimensions:

**Theorem 5.3.1** ([AIR15])**.** *If $B$ is a higher noetherian $n$-preprojective algebra and $e$ is an idempotent such that*

*a) $B/\langle e\rangle$ is a finite-dimensional $k$-algebra;*

*b) $eA(1-e) = 0$,*

*then there are triangle equivalences*

$$\underline{CM}(gr\ eBe) \cong \mathcal{D}^b(B_0/\langle e\rangle)$$

*and*

$$\underline{CM}(eBe) \cong \mathcal{C}_{n-1}(B_0/\langle e\rangle),$$

*where $C_{n-1}$ denotes the $(n-1)$-cluster category.*

The author refers to [Ami11] for a nice survey on generalized cluster categories. If $R$ is the polynomial ring in $n$ variables and $G$ is a finite subgroup of $SL(n,k)$ such that $B := R\#G$ has a grading structure of higher preprojective algebra, then we can apply the previous equivalences to $eBe \cong R^G$, where $e = \frac{1}{|G|}\sum_{g\in G} g$. This then generalizes the classical case and does establish a connection between graded maximal Cohen-Macaulay modules and the algebra $B$.

However, we showed in this chapter that for a certain class of groups $G < SL(n,k)$, with $n \geq 4$ if $R^G$ is isolated, the skew-group algebra cannot be endowed with a grading structure so that it becomes Morita equivalent to a preprojective algebra. We thus have demonstrated that the notion of higher preprojective algebra does not carry the correspondences between all quotient singularities $k^n/G$ and certain types of preprojective algebras coming from McKay quivers, as explained at the beginning of this chapter. Moreover, the equivalences mentioned above do not generalize to all invariant ring $R^G$.

In some sense, the fact that these bijections do not generalize naturally for $n \geq 4$ is not uncommon. In [BKR01], the authors establish a McKay correspondence type theorem for finite subgroups of $SL(n, \mathbb{C})$ satisfying some conditions, related to the existence of the so-called crepant resolutions of $\mathbb{C}^n/G$ in place of minimal resolutions.

**Definition 5.3.1.** Let $G < GL(n, \mathbb{C})$ be finite and $X := \mathbb{C}^n/G$ the quotient space. A *crepant resolution* $f : Y \to X$ is a resolution of singularities such that the canonical divisor $K_Y = f^*(K_Y)$.

In dimension $n = 2$, the minimal resolution of $X$ is crepant. Moreover, crepant resolutions always exist in dimension $n = 3$, but not necessarily for $n \geq 4$.

The McKay correspondence has had many interpretations over the years, both from the algebraic side and the geometric side. In this work we focussed mainly on the former aspect. The following generalization comes from a geometric viewpoint.

The $G$-Hilbert Scheme of $\mathbb{C}^n$ parametrizes $G$-orbits. It is considered a good candidate for a crepant resolution of $X$. Consider $D(G\text{-Hilb } \mathbb{C}^n)$ the derived category of coherent sheaves on $G$-Hilb $\mathbb{C}^n$ and $D^G(\mathbb{C}^n)$ the derived category of coherent $G$-sheaves on $\mathbb{C}^n$.

**Theorem 5.3.2** ([BKR01])**.** *Suppose $n \leq 3$. Then $G$-Hilb $\mathbb{C}^n$ is a crepant resolution of $X$ and there is an equivalence of categories*

$$D(G\text{-}Hilb\ \mathbb{C}^n) \xrightarrow{\sim} D^G(\mathbb{C}^n).$$

This theorem cannot be generalized fully for $n \geq 4$, as crepant resolution of $X$ do not necessarily exist in this case. Thus, from this perspective, the McKay correspondence does not generalize nicely for $n \geq 4$. The existence of crepant resolutions is related to the notion of the *age* of an element in $SL(n, \mathbb{C})$.

**Definition 5.3.2** ([IR96])**.** Let $G < SL(n, \mathbb{C})$ finite. For every $g \in G$, choose a $|g|$-th root of unity, say $\epsilon$. The element $g$ is conjugate to

$$\frac{1}{r}(a_1, \ldots, a_n) := \operatorname{diag}(\epsilon^{a_1}, \ldots, \epsilon^{a_n}),$$

where $r = |g|$ and $0 \leq a_i < r$ for $i = 1, \ldots n$. Define the *age* of $g$ by

$$\operatorname{age}(g) = \frac{1}{r} \sum a_i.$$

Because $G < SL(n, \mathbb{C})$, the age is an integer between 0 and $n - 1$ (only the identity has age 0). We call the elements of age 1 the *junior elements*.

**Theorem 5.3.3** ([IR96])**.** *Suppose that $Y \to X$ is a resolution of the singularity $X = \mathbb{C}^n/G$. There is a one-to-one correspondence between the crepant divisors of $Y$ and the conjugacy classes of the junior elements in $G$.*

This generalizes the McKay correspondence ($n = 2$), which establishes a one-to-one correspondence between the irreducible representations of $G$ (or conjugacy classes of $G$, which are all junior) and the divisors of the minimal resolution (which are all crepant).

In particular, the previous theorem implies that if $G$ does not have any junior element, then $\mathbb{C}^n/G$ does not have any crepant divisor. In this case we say that the singularity is *terminal*. These singularities do not admit crepant resolutions. We have the following links with our embedding condition.

**Proposition 5.3.4.** *If $G$ embeds in $SL(n_1, k) \times SL(n_2, k)$ for some $n_1, n_2 \geq 1$ such that $n_1 + n_2 = n$, then $G$ does not have any junior element. Thus $k^n/G$ is terminal.*

*Proof.* The idea is the same as in Lemma 5.2.3. □

**Proposition 5.3.5.** *Let $G < SL(4, k)$ be a finite cyclic group. The singularity $k^n/G$ is terminal if and only if $R\#G$ does not have a grading structure of preprojective algebra.*

*Proof.* By a result in [MS84], we have that $k^n/G$ is terminal if and only if $G$ embeds in a product $SL(n_1, k) \times SL(n_2, k)$. The result follows from Lemma 5.2.3 (also based on [MS84]), Theorem 5.2.1 and Theorem 5.2.4. □

There thus seems to be a link between the preprojective algebra structure on the skew-group algebra $R\#G$ and the existence of crepant resolutions (or divisors) of $k^n/G$, but insights as to why it is the case are unknown to the author.

**Question 5.3.3.** Is Proposition 5.3.5 true for any finite subgroups of $SL(n, k)$? Similarly, does $R\#G$ admit a preprojective algebra structure if and only if $G$ does not embed in a product $SL(n_1, k) \times SL(n_2, k)$? It is not clear that these two questions are compatible with each other. Namely, is $k^n/G$ terminal if and only if $G$ does not embed in a product $SL(n_1, k) \times SL(n_2, k)$?

The logical next step would then be to completely characterize the groups $G$ such that $R\#G$ admits a grading structure of preprojective algebra. One of the biggest challenge is that it is very hard to describe the McKay quiver of $G$ and the relations in a general abstract matter. Given a group $G$, the best way to compute the McKay quiver is still to find its character table. But doing so can only lead to a case-by-case analysis. Even with a description of the McKay quiver, it is still unknown what are the relations induced from the Morita equivalence between the skew-group algebra $R\#G$ and the path algebra $kM_G$.

Finally, as mentioned in Question 4.0.1, our main results in Chapters 4 and 5 were based on the fact that if we put a grading structure on the Calabi-Yau algebras that we considered so that they have Gorenstein parameter 1, then their degree 0 part must be infinite-dimensional. It may be worth exploring different notions of higher hereditary algebras that include infinite-dimensional algebras. In particular there may exist an interesting notion of higher preprojective algebras such that every skew-group algebra $R\#G$ is Morita equivalent to one of those.

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