Two results on Asymptotic Behaviour of Random Walks in Random Environment

by

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A thesis submitted in conformity with the requirements
for the degree of Doctor of Philosophy
Graduate Department of Mathematics
University of Toronto

July 11, 2016

Abstract

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In the first chapter of this thesis, we consider a model of directed polymer in $1+1$ dimensions in a product-type random environment $\omega(t,x) = b(t)F(x)$, where the fields F and b are i.i.d., with $F(x)$ continuous, symmetric and bounded, and $b(t) = \pm 1$ with probabilty $1/2$. Thus ω can be viewed as the field F oscillating in time. We consider directed last-passage percolation through this random field; namely, we investigate the behavior of the length n polymer path with maximal action, where the action of a path is simply the sum of the environment variables it moves through.

We prove a law of large numbers for the maximal action of the path from the origin to a fixed endpoint $(n, \lfloor \alpha n \rfloor)$, and investigate the limiting shape function $a(\alpha)$. We prove that this shape function is non-linear, and has a corner at $\alpha = 0$, thus indicating that this model does not belong to the KPZ universality class. We conjecture that this shape function has a linear piece near $\alpha = 0$.

With probability tending to 1, the maximizing path with free endpoint will localize on an edge with F values far from each other. Under an assumption on the arrival time to this localization site, we prove that the path endpoint and the centered action of the path, both rescaled by $n^{-2/3}$, converge jointly to a universal law, given by the maximizer and value of a functional on a Poisson point process.

In the second chapter, we consider a class of multidimensional random walks in random environment, where the environment is of the type $p_0 + \gamma\xi$, with p_0 a deterministic, homogeneous environment with underlying drift, and ξ an i.i.d. random perturbation. Such environments were considered by Sabot in [30], who finds a third-order expansion in the perturbation for the non-null velocity (which is guaranteed to exist by Sznitman and Zerner's LLN [34]). We prove that this velocity is an analytic function of the perturbation, by applying perturbation theory techniques to the Markov operator for a certain chain in the space of environments.

To Padre.

Acknowledgements

To my advisor Kostya Kahnin, I want to thank you for the time and energy you gave me. Ever the optimist, you kept me motivated and helped me work through countless difficulties. You were generous with your time and your ideas. Your intuition was invaluable to my understanding, and you always inspired confidence. It has been a pleasure working with you.

I owe the second chapter of this thesis to Dima Dolgopyat, who I was very fortunate to work with. Visiting you in Maryland was a great experience, and I learned a great deal. You offered so much of your time and guidance, and were always willing to think through things and explain difficult concepts. It was inspiring to have the opportunity to work with you.

The probability graduate course, taught by the members of my thesis committee Jeremy Quastel and Balint Virag, inspired me to study the field. Thank you both for your support and discussion over the years. The probability retreats were great fun, and inspired a sense of camaraderie in the department that I was lucky to be a part of. I first spent time with Almut Burchard at such a retreat, and it was such a pleasure expanding the circle of probability. I am thankful to her for attending my defense.

Many thanks to Jemima Merisca for guiding me through the graduation process with aplomb, taking care of so much behind the scenes. Ever helpful, ever understanding, you made things as smooth as can be.

Ida Bulat was my first introduction to the department, as she was for many graduate students. She was a pure spirit of kindness and compassion, and made coming to U of T a warm and inviting experience. She will be missed.

To my fellow graduate student friends in the department, thank you for the companionship and fun. Special thanks to Jordan Bell and Andrew Stewart for giving your time to help towards the end.

Last year I had to leave my thesis to take care of my mother, and the year after her death was filled with legal and emotional turmoil. Coming back to research was very difficult. I owe so much to the incredible emotional support of my close friends (shumps), and the exceptional characters of Countermeasure. I also want to thank my father and my sister Jennifer for sharing the weight, and being family. Jennifer, I could never have done what you did. Thank you for all the hard work.

I owe special thanks to Tara Park, for pushing me to dive back in, meeting me at the library every day for months, giving me the drive to finish. I couldn't have done this without you.

And most importantly, I owe everything to my wife Amie Everett. I love you.

Contents

Chapter 1

A Zero Temperature Directed Polymer in a Product-Type Environment

1.1 Introduction

A directed polymer in $1+d$ dimensions is a chain of length n in the lattice $\mathbb{N}_0 \times \mathbb{Z}^d$ which is directed in the first coordinate. That is, it can be modeled by a graph $\{(t,\gamma(t))\}_{t=0}^n$, where $\gamma(t)$ is a nearest neighbor path in $\mathbb{Z}^d$ (starting at the origin).

For each point in the lattice (t,x), we assign a random variable $\omega(t,x)$, and a polymer measure on paths which inherits its randomness from the random field. This polymer measure is canonically constructed as a Gibbs measure, by assigning to paths of length n a random action depending on the random field ω,

$$A_n^\omega(\gamma) = \sum_{i=1}^n \omega(i,\gamma(i))$$

and then defining the probability measure

$$\mathbf{P}_{n,\beta}^\omega(\gamma) = \frac{1}{Z_n^\omega(\beta)} \exp\big(\beta A_n^\omega(\gamma)\big)\mathbf{P}(\gamma),$$

where $\mathbf{P}(\cdot)$ is the uniform measure on nearest neighbor paths of length n in $\mathbb{Z}^d$, where $\beta \geq 0$ is the inverse temperature of the system, and where $Z_{n,\beta}^\omega$ is the partition function, the normalizing constant such that $\mathbf{P}^{\omega}_{n,\beta}(\cdot)$ is a probability measure. That is,

$$Z^{\omega}_{n,\beta} = \mathbf{P}\Big(\exp\big(\beta A^{\omega}_n(\gamma)\big)\Big).$$

The random environment ω is distributed by a probability measure $\mathbb{P}$ on the space of environments, Ω. Often this measure is taken to be product measure on the lattice, so that the random variables $\omega(t,x)$ are independent and identically distributed with some distribution, usually taken to be mean zero with all finite exponential moments.

For this system, the real number β controls how strongly the environment influences the behavior of the polymer. When $\beta = 0$ (temperature is infinite), the polymer measure is just the uniform measure and the environment is ignored. When $\beta \to \infty$ (temperature is zero), the polymer measure concentrates on the path with maximum action, $\overline{\gamma}_n$. This model is called the (oriented) last passage percolation model, or the corner growth model (usually rotating the lattice 45 degrees and considering the maximizing up-right path).

In this $\beta \to \infty$ regime (and others), much work is done to characterize the exponents ζ and χ, which describe the order of the maximizing path's endpoint and the order of the fluctuations of the path action, respectively. That is, calling $\overline{\gamma}_n = \operatorname{argmax}_\gamma A^{\omega}_n(\gamma)$,

$$\big|\overline{\gamma}_n(n)\big| \sim n^{\zeta}$$

and

$$\big|A^{\omega}_n(\overline{\gamma}_n) - \mathbb{E}(A^{\omega}_n(\overline{\gamma}_n))\big| \sim n^{\chi}$$

for typical environments ω. (For the case of finite β, one considers the fluctuations of $\log\Big(Z^{\omega}_{n,\beta}\Big)$ instead of $\max_\gamma A^{\omega}_n(\gamma)$.)

It is conjectured that for $\beta \to \infty$, for each dimension $d \geq 1$, these exponents ζ and χ are universal, that is, they do not depend on the distribution of $\omega(t,x)$, as long as they are i.i.d. In dimension 1, values for these exponents have been calculated for some specific distributions of $\omega(t,x)$. Johannson in [19] considers exponentially and geometrically distributed $\omega(t,x)$, and proves that $\chi = 1/3$ with Tracy-Widom law fluctuations. For a Poissonized version of the corner growth model in dimension 1, first studied by Hammersley in [13], Baik, Deift, and Johansson show $\chi = 1/3$ with Tracy-Widom law fluctuations, and $\zeta = 2/3$ is proven by Johansson in [20]. Thus, this model is said to satisfy the so called "KPZ relation",

$$\chi = 2\zeta - 1,$$

a somewhat mysterious but often conjectured relation in many random growth models.

Recently, Chatterjee [5] proved the "KPZ relation" for first passage percolation on the lattice $\mathbb{Z}^d$ for all $d \geq 2$, which considers the (self-avoiding) path which minimizes the Hamiltonian given above for positive $\omega(t, x)$, assuming the exponents χ and ζ exist in a certain sense. In [1], Auffinger and Damron make simplifications and weaken Chatterjee's hypotheses, allowing his results to apply to directed polymers with finite β (see [2]).

In the following, the paths we consider γ are "lazy" in $\mathbb{Z}$ ($d = 1$), so that $\big|\gamma(i) - \gamma(i+1)\big| \leq 1$, and we consider a product-type environment,

$$\omega(t, x) = F(x) \cdot b(t),$$

where $\{F(x)\}$ are i.i.d. random variables with continuous, symmetric distribution supported on an interval $[-c, c]$, and where $\{b(t)\}$ are i.i.d. -1 or 1 with probability 1/2, independent of $\{F(x)\}$. The field $\{b(t)\}$ can be thought of as an oscillating field. Note that in this model, first-passage (minimal action) and last-passage (maximal action) are distributionally equivalent.

We consider paths from the origin to a point (t, x), and call the the maximal action

$$\overline{A}(t, x) = \max_{\gamma:(0,0)\to(t,x)} A_t^\omega(\gamma).$$

Using standard techniques, we can prove a law of large numbers, so that $n^{-1}\overline{A}(n, [\alpha n])$ converges to a shape function $a(\alpha)$ $\mathbb{P}$-a.s, $\alpha \in [-1, 1]$, with the following properties.

Theorem 1.1.1.

1. *The function $a(\alpha)$ is a continuous, concave function on $[-1, 1]$, symmetric across 0,*
2. *$a(0) = c$,*
3. *$a(\alpha)$ has a corner at 0, i.e. $a'(0-) = -a'(0+) > 0$,*
4. *For any $\alpha \in (0, 1)$, $a(\alpha) > c(1 - \alpha)$. Namely, $a(\alpha)$ is non-linear on $[0, 1]$.*

We also conjecture that

Conjecture 1.1.2. *$a(\alpha)$ has a linear piece near the origin.*

Note that the corner at 0 heuristically rules out KPZ behavior, and implies informally that $\chi = \zeta$ (see [2]), which is the behavior we conjecture.

To explore the maximizing path with a free endpoint, $\overline{\gamma}_n$, we note that a path γ can accumulate a large action $A(\gamma) = \sum_{i=1}^{n} \omega(i, \gamma(i))$ by localizing on an edge $\{x, x+1\}$ where $F(x)$ and $F(x+1)$ are far away from each other (near the boundary of the support of F). That is, by localizing on edges with small "discrepancy", $d_x = 2c - \left|F(x+1) - F(x)\right|$. The locations of discrepancies and their values, scaled by $n^{-2/3}$ and $n^{1/3}$ respectively, converge to a Poisson point process.

Theorem 1.1.3. *For each $n \in \mathbb{N}$, we introduce the rescaled point process μ_n as a random measure:*

$$\mu_n = \sum_{k \in \mathbb{Z}} \delta_{(n^{-2/3}k, n^{1/3}d_k)}$$

As $n \to \infty$, the distribution of μ_n converges weakly in the vague topology to the distibution of μ, the Poisson process with driving measure ν absolutely continuous w.r.t. Lebesgue measure with density given by:

$$\frac{d\nu(\mathrm{d}t \times \mathrm{d}y)}{\mathrm{d}t \times \mathrm{d}y} = \begin{cases} \frac{p'(0+)}{2} y, & y > 0 \\ 0, & y \leq 0. \end{cases},$$

where p is the density of d_0.

For the free endpoint maximizing path, we show that the endpoint location $\ell_n = \overline{\gamma}_n(n)$ is order $n^{2/3}$, and that the path localizes on an edge with small discrepancy, a record value, of order $n^{-1/3}$. Calling τ_n the arrival time of $\overline{\gamma}_n$ to this ending edge ℓ_n, we show that with high probability, the path $\overline{\gamma}_n$ does not leave ℓ_n. We conjecture the following, and assume it for our final result.

Assumption 1.1.4 (Bounded Arrival Condition)**.** *There exists a $\kappa > 1$ such that*

$$\mathbb{P}(\tau_n > \kappa|\ell_n|) \longrightarrow 0. \tag{1.1}$$

Note that this assumption implies conjecture 1.1.2, that the shape function is linear near the origin. With this assumption, after normalizing by $n^{2/3}$, we are able to prove that the probability distributions of the centered maximal action $A(\overline{\gamma}_n) - cn$, and the endpoint ℓ_n, converge to a universal limit law.

Theorem 1.1.5. *Under assumption 1.1.4,*

$$\left(\frac{\ell_n}{n^{2/3}}, \frac{A(\overline{\gamma}_n) - cn}{n^{2/3}}\right) \xrightarrow{d} (\overline{X}, g(\overline{X}, \overline{Y})), \tag{1.2}$$

where $(\overline{X},\overline{Y}) = \operatorname{argmax}_{(X,Y)\in\operatorname{supp}(\mu)} g(X,Y)$*, where* μ *is the PPP described in theorem 1.1.3, and where*

$$g(x,y) = -a'(0+)|x| - y/2.$$

In section 1.3 we prove the existence of the shape function and items (i)-(iii) of theorem 1.1.1. The proof of (iv) is given in section 1.6. In section 1.4 we prove that the path localizes on a record discrepancy with probability tending to 1. In section 1.5 we prove theorem 1.1.5, and section 1.7 is the proof of theorem 1.1.3.

1.2 Notation and Preliminaries

We will consider environments $\{\omega(t,x), t\in\mathbb{N}, x\in\mathbb{Z}\}$ of the form

$$\omega(t,x) = F(x)b(t),$$

where $\{F(x) \,|\, x\in\mathbb{Z}\}$ are i.i.d with continuous distribution $f\mathrm{d}m$, m Lebesgue measure, $\operatorname{supp}(f)\subset[-c,c]$ for some $c>0$, and where $\{b(t)\,|\, t\in\mathbb{N}\}$ are i.i.d. with common distribution uniform on $\{-1,+1\}$. The two fields F and b are taken to be statistically independent. We denote by $\mathbb{P}$ the probability distribution of ω just described on the space $\Omega = [-c,c]^{\mathbb{N}\times\mathbb{Z}}$ of environments, endowed with the product σ-algebra.

We require certain conditions on the density f of F, namely, that f is symmetric across 0 with compact support on a bounded interval $[-c,c]$, positive on the whole interval, and $\mathcal{C}^1$, with one sided derivatives at c and $-c$. We let $F\cdot b$ denote the whole field of products, ω.

Since we will be concerned with paths through such environments, let us define the space of directed nearest-neighbor (lazy) paths:

$$\Gamma\big((t_1,x_1),(t_2,x_2)\big) = \Big\{\gamma : [t_1,t_2]\cap\mathbb{N}_0 \to \mathbb{Z} \,\Big|\, \big|\gamma(i)-\gamma(i+1)\big| \le 1 \text{ for } i=t_1,\dots,t_2-1, \gamma(t_1)=x_1, \gamma(t_2)=x_2\Big\}. \tag{1.3}$$

In the special case that the starting point is the origin, we write

$$\Gamma(t,x) = \Gamma\big((0,0),(t,x)\big). \tag{1.4}$$

Leaving any of the parameters as a dot, $\cdot$, means that parameter is free to vary. Given any path γ defined on $[a,b] \cap \mathbb{N}_0$ and a product type environment $\omega = F \cdot b$, we define the action of that path by the random variable

$$A^\omega(\gamma, a, b) = \sum_{i=a+1}^{b} F(\gamma(i)) \cdot b(i). \tag{1.5}$$

If the domain of γ is clear, we simply write $A^\omega(\gamma)$. We are interested in describing the path which maximizes this action on $[0,n]$. We denote the maximizing path of length n with free endpoint (which is unique with probability 1, since $\omega(t,x)$ has continuous distribution) by

$$\overline{\gamma}_n^\omega = \operatorname*{argmax}_{\gamma \in \Gamma(n,\cdot)} A^\omega(\gamma). \tag{1.6}$$

Remark 1.2.1. *A simple observation is that for any $\omega = F \cdot b \in \Omega$, the path which maximizes the above action minimizes the action for $\omega = F \cdot -b$. And since b and $-b$ have the same distribution, the minimum action is equal in distribution to minus the maximum action.*

Denote the maximum action between two points (t_1, x_1) and (t_2, x_2) by

$$\overline{A}^\omega\big((t_1,x_1),(t_2,x_2)\big) = \max\Big\{A^\omega(\gamma) \,\Big|\, \gamma \in \Gamma\big((t_1,x_1),(t_2,x_2)\big)\Big\}.$$

When clear, we will usually omit the ω from the above notation.

Similarly to the path space, in the special case that the starting point is the origin, we write

$$\overline{A}(t,x) = \overline{A}\big((0,0),(t,x)\big). \tag{1.7}$$

We will be considering the action of "pinned paths", whose endpoint we fix as a constant multiple $\alpha \in [-1,1]$ of the length. We obviously have to round values to the nearest integer, and we choose the floor for positive α. For the model to be symmetric across 0, we take the ceiling for negative α. So we denote the symmetric rounding function

$$[x] = \begin{cases} \lfloor x \rfloor & \text{if } x \geq 0 \\ \lceil x \rceil = -\lfloor -x \rfloor & \text{if } x < 0 \end{cases}.$$

1.2.1 The Environment

We note an important observation about the optimal path through an environment $\omega = F \cdot b$.

Remark 1.2.2. *If two adjacent values, $F(x)$ and $F(x+1)$ are such that one of them is close to c and the other is close to $-c$ (the boundaries of the support of f), then a path can do very well in the long run at these two sites, by staying at the c-valued site for all times i where $b(i) = +1$, and then jumping to the $-c$-valued site for all times when $b(i) = -1$.*

It will be important to measure such sites, and denote the optimal path restricted to an edge. For this we define the discrepancy of an edge as the sum of the distances from each point to the boundary points of the support of f, $-c$ and c. The smaller this value, the higher the contribution to the action of a path which stays (optimally) on this edge.

Definition 1.2.3.

- *For each $x \in \mathbb{Z}$, define the discrepancy of the edge $\{x, x+1\}$ to be*

$$d_x = 2c - \big|F(x+1) - F(x)\big| \tag{1.8}$$

- *Define the optimal path restricted to the edge $\{x, x+1\}$ by*

$$\eta_x(t) = \begin{cases} x & \text{if } F(x) \geq F(x+1) \text{ and } b(t) = 1 \\ x & \text{if } F(x) \leq F(x+1) \text{ and } b(t) = -1 \\ x+1 & \text{otherwise} \end{cases} \tag{1.9}$$

Note that if $d_x < c$, then necessarily $F(x)$ and $F(x+1)$ have opposite sign. Since we will focus on edges with small discrepancies, we will usually be looking at sites where this holds. In this case, we have $\big|F(x)\big| + \big|F(x+1)\big| = \big|F(x) - F(x+1)\big| = 2c - d_x$. With this in mind, we obtain the following useful inequalities.

$$2\min\Big(\big|F(x)\big|, \big|F(+1)\big|\Big) \leq 2c - d_x \leq 2\max\Big(\big|F(x)\big|, \big|F(x+1)\big|\Big),$$

(1.10) and

$$c - d_x = \big|F(x)\big| + \big|F(x+1)\big| - c \leq \min\Big(\big|F(x)\big|, \big|F(x+1)\big|\Big).$$

Using these, we also obtain (assuming opposite sign) that

$$
\begin{aligned}
\big|F(x)+F(x+1)\big| &= \max\Big(\big|F(x)\big|,\big|F(x+1)\big|\Big) - \min\Big(\big|F(x)\big|,\big|F(x+1)\big|\Big) \\
&= \big|F(x)\big| + \big|F(x+1)\big| - 2\min\Big(\big|F(x)\big|,\big|F(x+1)\big|\Big) \\
&\leq 2c - d_x - (2c - 2d_x) \\
&= d_x. \qquad (1.11)
\end{aligned}
$$

We will often use the naive bound on the the action of η_x using 1.10,

$$
\begin{aligned}
A(\eta_x, a, b) &\geq (b-a)\min\Big(\big|F(x)\big|,\big|F(x+1)\big|\Big) \qquad (1.12)\\
&\geq (b-a)(c-d_x). \qquad (1.13)
\end{aligned}
$$

For more precise control of the action of η_x, we denote

$$
\begin{aligned}
N^+(a,b) &= \#\Big|\big\{i \in [a+1,b] \,\big|\, b(i) = +1\big\}\Big|, \quad \text{and} \\
N^-(a,b) &= \#\Big|\big\{i \in [a+1,b] \,\big|\, b(i) = -1\big\}\Big|,
\end{aligned} \qquad (1.14)
$$

and then compute the action of η_x between a and b to be

$$
\begin{aligned}
A(\eta_x, a, b) &= \max\big(F(x), F(x+1)\big)\cdot N^+(a,b) - \min\big(F(x),F(x+1)\big)\cdot N^-(a,b) \\
&= (b-a)\Big(c - \frac{1}{2}d_x\Big) + \Big(N^+(a,b) - \frac{b-a}{2}\Big)\big(F(x)+F(x+1)\big). \qquad (1.15)
\end{aligned}
$$

And as we would expect, the contribution on a large time scale T coming from a path staying on the edge $\{x, x+1\}$ will be $T\Big(c - \frac{d_x}{2}\Big)$, plus fluctuations of order $\sqrt{T}$.

The following corollary of theorem 1.1.3 guarantees the existence of good "candidate sites" for the best path, while also guaranteeing that such points are well separated with high probability. Recall from theorem 1.1.3 that $\mu_n = \sum_{k\in\mathbb{Z}} \delta_{(n^{-2/3}k, n^{1/3}d_k)}$.

Corollary 1.2.4. *For any $\epsilon > 0$ and any $a > 0$, there are positive constants N, b, and δ such that for all $n > N$,*

$$
\mathbb{P}\Big(\mu_n(R) \geq 1 \text{ and } \mu_n\big(R_\delta(x,y)\big) \leq 1 \text{ for all } (x,y)\in R\Big) > 1-\epsilon, \qquad (1.16)
$$

where $R = (-a,a)\times(0,b)$ *and* $R_\delta(x,y) = (x-\delta, x+\delta)\times(y-\delta,y+\delta)\cap R$ *.*

Proof. First we prove this fact for the limiting Poisson point process μ with driving measure ν described in theorem 1.1.3. Given a, we can find b such that $\mathbb{P}(\mu(R)\geq 1) > 1-\epsilon$, which is clear from the density of ν. Inside this event, we condition on the number of points in the rectangle, $N_R \geq 1$. Then the points are independent and identically distributed in R with distribution $\nu(\cdot)/\nu(R)$ (see [26] for basics of poisson processes). Thus for any two such points, the probability of both of them being within a given rectangle $R_\delta \subset R$ is less than or equal to $\left(\frac{\nu(R_\delta)}{\nu(R)}\right)^2$. And thus the probability of any two points being within a given rectangle is bounded by $N_R(N_R-1)\nu(R_\delta)^2/\nu(R)^2$.

Now, for each point $(j\frac{\delta}{2}, k\frac{\delta}{2}) \in R$, $j,k\in\mathbb{Z}$, center a square $R_\delta^{j,k} := R_\delta(j\frac{\delta}{2}, k\frac{\delta}{2})$. This covers R such that if all $R_\delta^{j,k}$ contain no more than 1 point, then no two points are within any $R_\delta(x,y)$ for any $(x,y)\in R$. Then using a union bound, we have

$$\begin{aligned}\mathbb{P}\Big(\mu(R_\delta(x,y)) \geq 2 \text{ for some } (x,y)\in R\Big) \\ &= \mathbb{P}\left(\mu(R_\delta^{j,k}) \geq 2 \text{ for some } \left(j\frac{\delta}{2}, k\frac{\delta}{2}\right)\in R\right) \\ &\leq \#\left|\frac{\delta}{2}\mathbb{Z}^2\cap R\right| \mathbb{E}\big[N_R(N_R-1)\big] \max_{(x,y)\in R}\left(\frac{\nu(R_\delta(x,y))}{\nu(R)}\right)^2.\end{aligned}$$

For fixed $a,b>0$, $\#\left|\frac{\delta}{2}\mathbb{Z}^2\cap R\right|$ is a constant times δ^{-2}, and since the density of ν is bounded in R, $\max_{(x,y)\in R}\nu(R_\delta(x,y))$ is a constant times δ^2. Thus we can make the probability arbitrarily small by taking δ small.

Finally, by Theorem 4.2 in [22], vague convergence of μ_n to μ implies that the joint distribution of $\left\{\mu_n\left(R_\delta^{j,k}\right)\right\}_{(j\frac{\delta}{2},k\frac{\delta}{2})\in R}$ converges to the joint distribution of $\left\{\mu\left(R_\delta^{j,k}\right)\right\}_{(j\frac{\delta}{2},k\frac{\delta}{2})\in R}$.

□

Remark 1.2.5. *Note that the driving measure* ν *in theorem 1.1.3 is stationary in the first dimension, so that we can take a more general rectangle* $R = [a_1,a_2]\times[0,b]$ *in the corollary, and* b *only depends on the length* $a_2 - a_1$*.*

1.3 The shape function

In this section we show the basic law of large numbers result for the minimal action along a given direction. Namely, we show the existence of a deterministic function in the plane

$$\Phi(x,y) = \lim_{n\to\infty} n^{-1}\overline{A}\big([nx],[ny]\big).$$

This result, along with several nice properties of the limiting function, follows a classical subadditivity argument. We follow closely the argument in [28] and [31]. We will then investigate the notable properties of the one dimensional shape function mentioned in section 1.2, $a(\alpha) = \Phi(1,\alpha)$.

1.3.1 Existence

Define the cone

$$\mathcal{K} = \Big\{(x,y)\in\mathbb{R}^2 \,\Big|\, 0 \le |y| \le x\Big\}.$$

Theorem 1.3.1. *For the environment distribution $\mathbb{P}$ described in section 1.2, there exists a deterministic function $\Phi : \mathcal{K} \to [0,c]$ such that*

$$\Phi(x,y) = \lim_{n\to\infty} n^{-1}\overline{A}\big([nx],[ny]\big) \ \ \mathbb{P}\text{-almost surely.} \tag{1.17}$$

For (x_1,y_1) and (x_2,y_2) in $\mathcal{K}$, the function Φ is

- *Super-additive:*

$$\Phi(x_1,y_1) + \Phi\big((x_1,y_1),(x_1+x_2,y_1+y_2)\big) \le \Phi(x_1+x_2,y_1+y_2)$$

- *Concave:*

$$t\Phi(x_1,y_1) + (1-t)\Phi(x_2,y_2) \le \Phi(tx_1+(1-t)x_2, ty_1+(1-t)y_2)$$

- *Homogenous:*

$$\Phi(x,y) = k^{-1}\Phi(kx,ky) \text{ for any } k>0.$$

Φ is also continuous on $\mathcal{K}$ and symmetric in y. Quantitatively, $\Phi(x,\pm x) = 0$ and $\Phi(x,0) = c$ for all

$x \in (0, \infty)$.

Proof. The basis of the proof is the superadditivity of the path actions, which is clear from construction, for points $(x_1, y_1), (x_2, y_2) \in \mathcal{K} \cap \mathbb{Z}^2$:

$$\overline{A}(x_1, y_1) + \overline{A}\big((x_1, y_1), (x_1 + x_2, y_1 + y_2)\big) \le \overline{A}(x_1 + x_2, y_1 + y_2). \tag{1.18}$$

From the construction of the model, we have the following bound. For points $(x_1, y_1), (x_2, y_2) \in \mathcal{K} \cap \mathbb{Z}^2$, we have

$$\Big|\overline{A}\big((x_1, y_1), (x_1 + x_2, y_1 + y_2)\big)\Big| \le c x_2, \tag{1.19}$$

and so superadditivity gives the following, which we will use throughout the proof instead of the monotonicity present in the corner growth model. For $(x_1, y_1), (x_2, y_2) \in \mathcal{K} \cap \mathbb{Z}^2$,

$$\overline{A}(x_1, y_1) \le \overline{A}(x_1 + x_2, y_1 + y_2) + c x_2. \tag{1.20}$$

Following Lin Hao's dissertation, [28], and Timo Seppäläinen's lecture notes on the corner growth model, [31], we first consider integer points, then extend to rational points, and finally all real numbers. First, suppose that $(x, y) \in \mathcal{K} \cap \mathbb{Z}^2$. Define $X_{m,n} = \overline{A}\big((mx, my), (nx, ny)\big)$ for $0 \le m < n$. Applying Kingman's subadditive ergodic theorem 1.9.3 to $-X_{m,n}$, we have that

$$n^{-1} X_{0,n} = n^{-1} \overline{A}(xn, yn)$$

tends almost surely to a limit, $\Phi(x, y)$, for any $(x, y) \in \mathcal{K} \cap \mathbb{Z}^2$. For any $k \in \mathbb{N}$, homogeneity is immediate. For superadditivity, we look at

$$n^{-1}\overline{A}(nx_1, ny_1) + n^{-1}\overline{A}\big((nx_1, ny_1), (nx_1 + nx_2, ny_1 + ny_2)\big) \le n^{-1}\overline{A}(nx_1 + nx_2, ny_1 + ny_2). \tag{1.21}$$

The first and last terms converge almost surely to $\Phi(x_1, y_1)$ and $\Phi(x_1 + x_2, y_1 + y_2)$ respectively. The middle term is equal in distribution to $\overline{A}(nx_2, ny_2)$, and thus converges almost surely to $\Phi(x_2, y_2)$ along a subsequence (lemma 1.9.2). Taking the limit of this inequality along this subsequence gives superadditivity of the shape function Φ on integer points in $\mathcal{K}$.

We also have that for any $(x, y) \in \mathcal{K} \cap \mathbb{Z}^2$,

$$\Phi(x, \pm x) = 0, \tag{1.22}$$

$$0 \leq \Phi(x, y) \leq cx, \tag{1.23}$$

and so (1.23) and superadditivity gives a form of monotonicity for $(x_1, y_1), (x_2, y_2) \in \mathcal{K} \cap \mathbb{Z}^2$ for Φ:

$$\Phi(x_1, y_1) \leq \Phi(x_1 + x_2, y_1 + y_2). \tag{1.24}$$

Non-negativity of the shape function in (1.23) follows from the fact that $\Phi(x, y) = \lim_{n\to\infty} n^{-1}\mathbb{E}X_{0,n}$, and by noting that given an arbitrary path from the origin to $(nx, ny) \in \mathcal{K} \cap \mathbb{Z}^2$, the expectation of the action of the path is 0, implying that the max action over all such paths, $X_{0,n}$, has expectation greater than or equal to 0.

Remark 1.3.2. *In fact, we can see that* (1.22) *holds on all of* $\partial\mathcal{K}$*, not just at integer points. The existence of the limit* (1.17) *is immediate for* $(x, y) \in \partial\mathcal{K}$*, since for* $(x, y) \neq (0, 0)$*, the limit on the right hand side is taken along a (non-random) subsequence of either* $n^{-1}\overline{A}(n, n)$ *or* $n^{-1}\overline{A}(n, -n)$*, both of which converge to* 0 *almost surely by the law of large numbers Thus* Φ *is identically* 0 *on* $\partial\mathcal{K}$*.*

Now, suppose $(x, y) \in \mathcal{K} \cap \mathbb{Q}^2$. To extend the proof to rational points, we choose $q \in \mathbb{N}$ such that $(qx, qy) \in \mathbb{Z}^2$, and define

$$\Phi(x, y) = q^{-1}\Phi(qx, qy).$$

This definition is independent of the choice of q by homogeneity for integers. With this definition, we have homogeneity for rational $k > 0$, superadditivity for points in $\mathcal{K} \cap \mathbb{Q}^2$, as well as (1.22), (1.23), and (1.24) for points in $\mathcal{K} \cap \mathbb{Q}^2$.

To show equation (1.17) for rational points $(x, y) \in \mathcal{K} \cap \mathbb{Q}^2$, given any $n \in \mathbb{N}$, write $n = mq + r$, $r \in \{0, \ldots, q-1\}$. Call $R_x = [nx] - mqx = [rx]$ and $R_y = [ny] - mqy = [ry]$. Then $|R_y| \leq R_x$ for $(x, y) \in \mathcal{K} \cap \mathbb{Q}^2$, and thus $(R_x, R_y) \in \mathcal{K} \cap \mathbb{Z}^2$.

Then by (1.20), we have

$$\overline{A}([nx], [ny]) \geq \overline{A}([nx] - R_x, [ny] - R_y) - cR_x, \tag{1.25}$$

and thus, dividing by n and taking $n \to \infty$, we have

$$\liminf_{n\to\infty} n^{-1}\overline{A}([nx],[ny]) \geq \lim_{n\to\infty} \frac{mq}{n}\frac{\overline{A}(mqx,mqy)}{mq} - \frac{cR_x}{n} \tag{1.26}$$

$$= \Phi(x,y). \tag{1.27}$$

For the other direction, we have $nx + qx - rx = (m+1)qx$, and similarly for y, and thus we have $[nx] + qx - R_x = (m+1)qx$. In order to use superadditivity as above, we need to know that $(qx - R_x, qy - R_y) \in \mathcal{K}$. Without loss of generality, we can take $y \geq 0$, since $[ry] = -[-ry]$. So, we need to check that $[rx] - [ry] \leq qx - qy$. If $x - y$ is an integer, then so is $rx - ry$. In this case, it is easy to check that $rx - ry = [rx] - [ry]$, and so the inequality follows, since $r < q$.

Otherwise, note that

$$[rx] - [ry] \leq [rx - ry] + 1 < r(x-y) + 1 \leq (q-1)(x-y) + 1. \tag{1.28}$$

If $q(x-y) \geq (q-1)(x-y)+1$, then $[rx]-[ry] \leq qx - qy$ as needed. Otherwise, since $q(x-y)$ is an integer and $(q-1)(x-y)$ is not by the assumption that $x - y$ is not an integer, $q(x-y)$ must be the unique integer between $(q-1)(x-y)$ and $(q-1)(x-y)+1$. And since $[rx]-[ry]$ is also an integer less than $(q-1)(x-y)+1$ by (1.28), we have the inequality.

So, (1.20) gives

$$\overline{A}([nx],[ny]) \leq \overline{A}([nx] + qx - R_x, [ny] + qx - R_y) + c(qx - R_x), \tag{1.29}$$

and so similarly to above, dividing by n and taking a limit, we obtain the other sided inequality

$$\limsup_{n\to\infty} n^{-1}\overline{A}([nx],[ny]) \leq \Phi(x,y). \tag{1.30}$$

This finishes the existence of the limit for rational points $(x,y) \in \mathcal{K} \cap \mathbb{Q}^2$.

Finally, as in [28], for all $(x,y) \in \mathcal{K}$, extend

$$\Phi(x,y) = \sup\left\{\Phi(u,v) \,\middle|\, (u,v) \in \mathcal{K} \cap \mathbb{Q}^2, (x-u, y-v) \in \mathcal{K}\right\}. \tag{1.31}$$

This agrees with the case of rational (x,y) by (1.24). Superadditivity is straightforward to show from the definition, and (1.22), (1.23), and thus (1.24) are immediate.

For arbitrary $(x, y) \in \mathcal{K}$, we have homogeneity for rational $k > 0$, since

$$k\Phi(x,y) = \sup\Big\{k\Phi(u,v)\,\Big|\,(u,v)\in\mathcal{K}\cap\mathbb{Q}^2, (x-u,y-v)\in\mathcal{K}\Big\} \tag{1.32}$$

$$= \sup\Big\{\Phi(ku,kv)\,\Big|\,(u,v)\in\mathcal{K}\cap\mathbb{Q}^2, (x-u,y-v)\in\mathcal{K}\Big\} \tag{1.33}$$

$$= \sup\Big\{\Phi(u,v)\,\Big|\,(u,v)\in\mathcal{K}\cap\mathbb{Q}^2, (kx-u,ky-v)\in\mathcal{K}\Big\} \tag{1.34}$$

$$= \Phi(kx,ky). \tag{1.35}$$

For general $k > 0$, simply choose $k_1 < k < k_2$ rational, and use (1.24):

$$\Phi(k_1x, k_1y) \leq \Phi(kx, ky) \leq \Phi(k_2x, k_2y).$$

Use homogoneity for k_1 and k_2 and then take the limits $k_1, k_2 \to k$.

Now, let $0 \leq t \leq 1$ and let $(x_1, y_1), (x_2, y_2) \in \mathcal{K}$. By superadditivity and homogeneity, we have concavity:

$$t\Phi(x_1,y_1) + (1-t)\Phi(x_2,y_2) = \Phi(tx_1,ty_1) + \Phi((1-t)x_2,(1-t)y_2) \tag{1.36}$$

$$\leq \Phi(tx_1 + (1-t)x_2, ty_1 + (1-t)y_2), \tag{1.37}$$

and a finite concave function on an open set is continuous, so that we have continuity in $\mathcal{K}^\circ$, the interior of $\mathcal{K}$. Though we know the value along the boundary, $\Phi(x, \pm x) = 0$, we will obtain continuity on the diagonals later.

To compute the limit (1.17) for general $(x, y) \in \mathcal{K}^\circ$, we approximate by rationals. Let $0 < x_1 < x < x_2$ and $y_1 < y < y_2$ be rational such that $(x_1, y_1), (x_2, y_2) \in \mathcal{K}$. To be able to use subadditivity to compare $\overline{A}([nx], [ny])$ and $\overline{A}([nx_1], [ny_1])$, we will need to have that $([nx] - [nx_1], [ny] - [ny_1]) \in \mathcal{K}$ and $([nx_2] - [nx], [ny_2] - [ny]) \in \mathcal{K}$.

Since $(x, y) \in \mathcal{K}^\circ$, then we can choose the rational points such that $2(y - y_1) \leq (x - x_1)$ and $2(y_2 - y) \leq (x_2 - x)$. Taking n large enough so that $n(y - y_1)$ and $n(y_2 - y)$ are greater than 1, we then have that $n(x-x_1)-n(y-y_1)>1$ and $n(x_2-x)-n(y_2-y)>1$, so that

$$n(x-x_1)-n(y-y_1)>1 \tag{1.38}$$

$$\implies \lceil (nx-nx_1)-(ny-ny_1) \rceil \geq 1 \tag{1.39}$$

$$\implies \lceil nx-nx_1 \rceil - \lceil ny-ny_1 \rceil \geq 1 \tag{1.40}$$

$$\implies \lceil nx \rceil - \lceil nx_1 \rceil \geq \lceil ny \rceil - \lceil ny_1 \rceil, \tag{1.41}$$

and the similar argument with $n(x_2-x)-n(y_2-y)>1$.

Then subadditivity gives

$$\overline{A}(\lceil nx \rceil, \lceil ny \rceil) \geq \overline{A}(\lceil nx_1 \rceil, \lceil ny_1 \rceil) - c(\lceil nx \rceil - \lceil nx_1 \rceil) \tag{1.42}$$

and

$$\overline{A}(\lceil nx \rceil, \lceil ny \rceil) \leq \overline{A}(\lceil nx_2 \rceil, \lceil ny_2 \rceil) + c(\lceil nx_2 \rceil - \lceil nx \rceil). \tag{1.43}$$

Dividing by n and taking the limit, we obtain

$$\liminf_{n\to\infty} \overline{A}(\lceil nx \rceil, \lceil ny \rceil) \geq \Phi(x_1,y_1) - c(x-x_1) \tag{1.44}$$

and

$$\limsup_{n\to\infty} \overline{A}(\lceil nx \rceil, \lceil ny \rceil) \leq \Phi(x_2,y_2) + c(x_2-x). \tag{1.45}$$

Continuity of Φ on $\mathcal{K}^\circ$ implies that we can take (x_1,y_1) and (x_2,y_2) to (x,y) to obtain (1.17) for general $(x,y)\in\mathcal{K}^\circ$. And by remark (1.3.2), we see have (1.17) on all of $\mathcal{K}$.

□

We also prove a simple bound for Φ:

Lemma 1.3.3.

$$\Phi(x,y) \leq yD + c(x-|y|), \tag{1.46}$$

where $D=\mathbb{E}\big|F(0)\big|$.

Proof. For any $(x, y) \in \mathcal{K} \cap \mathbb{Z}^2$, taking $y > 0$, the action-maximizing path must move through every $F(\cdot)$-site between 0 and y, and the maximum action that can be achieved through these sites is $\sum_{k=1}^{y} \big|F(k)\big|$. And we can bound the remaining steps by $c(x - y)$. Therefore, for points $(x, y) \in \mathcal{K} \cap \mathbb{Z}^2$ with $y > 0$, we have

$$\overline{A}(x, y) \leq \sum_{k=1}^{y} \big|F(k)\big| + c(x - y), \tag{1.47}$$

and so we have for any $(x, y) \in \mathcal{K}$, $y > 0$,

$$n^{-1}\overline{A}\big([nx], [ny]\big) \leq \sum_{k=1}^{[ny]} \big|F(k)\big| + c([nx] - [ny]). \tag{1.48}$$

Taking limits as $n \to \infty$ and using the distributional symmetry with respect to y of the model, we obtain, for $(x, y) \in \mathcal{K}$,

$$\Phi(x, y) \leq |y|D + c(x - |y|). \tag{1.49}$$

□

1.3.2 Continuity of Φ on $\partial\mathcal{K}$

In this section, we prove that

Lemma 1.3.4. *Φ is continuous on $\partial\mathcal{K}$.*

Proof. Fix $\epsilon > 0$. We first bound $\#\Big|\Gamma\Big(n, \big\lfloor(1-\epsilon)n\big\rfloor\Big)\Big|$, the number of paths from the origin to $\Big(n, \big\lfloor(1-\epsilon)n\big\rfloor\Big)$. We can represent each step in a path by a choice of $+1$, -1, or 0 depending on whether the path moves up, down, or remains where it is respectively. Thus we can represent $\Gamma\Big(n, \big\lfloor(1-\epsilon)n\big\rfloor\Big)$ by the set of sequences

$$\left\{\overline{s} = (s_1, s_2, \ldots, s_n) \in \{-1, 0, 1\}^n \,\middle|\, \sum_{i=1}^{n} s_i = \big\lfloor(1-\epsilon)n\big\rfloor\right\}.$$

Since $\big\lfloor(1-\epsilon)n\big\rfloor = n - \lceil\epsilon n\rceil$, calling v the number of down steps in a sequence (of which there can be at most $\big\lfloor\lceil\epsilon n\rceil/2\big\rfloor$), then there must be $\lceil\epsilon n\rceil - 2v$ steps which remain at a site, and the rest of the steps must be up steps. We can bound the size of the set of paths by summing over the number of down steps, of which there are $\binom{n}{v}$ ways of placing them, and then $\binom{n-v}{\lceil\epsilon n\rceil - 2v}$ ways of placing the steps which remain on a site. So, the size of the set of paths is equal to

$$\sum_{v=0}^{\lfloor \lceil \epsilon n \rceil /2 \rfloor} \binom{n}{v} \binom{n-v}{\lceil \epsilon n \rceil - 2v}.$$

Using the fact that for $1 \leq k \leq n$,

$$\binom{n}{k} \leq \left(\frac{ne}{k}\right)^k,$$

we can bound the number of paths very naively by

$$\begin{aligned}
&\Big(\lfloor \lceil \epsilon n \rceil /2 \rfloor + 1\Big)\left(\max_{0 \leq v \leq \lfloor \lceil \epsilon n \rceil /2 \rfloor} \binom{n}{v}\right)\left(\max_{0 \leq v \leq \lfloor \lceil \epsilon n \rceil /2 \rfloor} \binom{n-v}{\lceil \epsilon n \rceil - 2v}\right) \\
&\leq \Big(\lfloor \lceil \epsilon n \rceil /2 \rfloor + 1\Big)\binom{n}{\lceil \epsilon n \rceil}^2 \\
&\leq \epsilon n \left(\frac{ne}{\epsilon n}\right)^{2\lceil \epsilon n \rceil}.
\end{aligned}$$

We can take this less than or equal to $e^{nI(\epsilon)}$ for all n, where $I(\epsilon) = C\epsilon \log(1/\epsilon)$ with some constant $C > 2$.

Now, fix any path γ in $\Gamma\Big(n, \lfloor (1-\epsilon)n \rfloor\Big)$. Its action $A(\gamma)$ is the sum of $\lfloor (1-\epsilon)n \rfloor$ i.i.d. random variables $X_1, X_2, \ldots, X_{\lfloor (1-\epsilon)n \rfloor}$ corresponding to the up steps of γ, and then $\lceil \epsilon n \rceil$ non-independent random variables corresponding to the other steps, whose sum we call Z. We can bound this second group naively by $c\lceil \epsilon n \rceil$. Then for $\lambda > 0$,

$$\begin{aligned}
\mathbb{P}\Big(A(\gamma) > \lambda n(1-\epsilon) + c\lceil \epsilon n \rceil\Big) &\leq \mathbb{P}\left(\sum_{i=1}^{\lfloor (1-\epsilon)n \rfloor} X_i > \lambda n(1-\epsilon)\right) \\
&\leq \exp\Big\{-C' n \lambda^2 (1-\epsilon)\Big\} \qquad (1.50)
\end{aligned}$$

by Hoeffding's inequality [16].

Finally, we can control the maximum action of paths in $\Gamma\Big(n, \lfloor (1-\epsilon)n \rfloor\Big)$ using a union bound,

$$\begin{aligned}
&\mathbb{P}\Big(\overline{A}\Big(n, \lfloor (1-\epsilon)n \rfloor\Big) > \lambda n(1-\epsilon) + c\lceil \epsilon n \rceil\Big) \\
&\quad \leq \#\Big|\Gamma\Big(n, \lfloor (1-\epsilon)n \rfloor\Big)\Big| \cdot \exp\Big\{-C' n \lambda^2 (1-\epsilon)\Big\} \\
&\quad \leq \exp\Big\{n\Big[I(\epsilon) - C'\lambda^2(1-\epsilon)\Big]\Big\}.
\end{aligned}$$

Since $I(\epsilon) \to 0$ as $\epsilon \to 0$, we can take $\lambda = \lambda(\epsilon)$ such that $I(\epsilon) - C'\lambda^2(1-\epsilon) < 0$ and such that $\lambda \to 0$ as $\epsilon \to 0$. For this choice of λ, we then have that

$$\lim_{n\to\infty} \frac{1}{n}\overline{A}\Big(n, \lfloor (1-\epsilon)n \rfloor\Big) = \Phi(1, 1-\epsilon) \leq \lambda(1-\epsilon) + c\epsilon,$$

and this tends to 0 as ϵ tends to 0. We can extend continuity to all points on $\partial\mathcal{K}$ by homogeneity. □

1.3.3 Properties of the Shape Function

Due to the homogeneity condition in Theorem 1.3.1, if suffices to simply look at the one-dimensional shape function,

$$a(\alpha) := \Phi(1, \alpha). \tag{1.51}$$

We know that $a : [-1, 1] \to [0, c]$ is continuous on $[-1, 1]$, concave on $[-1, 1]$, satisfies $a(1) = 0$ (from remark 1.3.2), and is an even function. We can also prove one more quantitative result:

Lemma 1.3.5. *$a(0) = c$.*

Proof. By corollary 1.2.4, for any $\epsilon > 0$, we can take a, b large enough such that the event

$$A_n = \Big\{\exists\, x \in \mathbb{N} \text{ s.t. } 0 < x < an^{2/3} \text{ and } 0 < d_x < bn^{-1/3}\Big\} \tag{1.52}$$

has probability greater than $1-\epsilon$ for all n. Taking n large enough so that $n^{1/3} >> 2a$, we can guarantee $x << n/2$, so that the path described below can be constructed.

On the event A_n, let $\gamma_{n,x}$ move balistically up to site x, optimally stay on the edge $\{x, x+1\}$ (following η_x as defined in 1.9) as long as possible, and then move balistically back down to 0. That is,

$$\gamma_{n,x}(i) = \begin{cases} i \text{ for } 0 \leq i \leq x \\ \eta_x(i) \text{ for } x < i < n-x \\ n-i \text{ for } n-x \leq i \leq n \end{cases}.$$

Then on this event we can bound the scaled optimal action between c and the scaled action of this path,

$$n^{-1}A(\gamma_{n,x}) \leq n^{-1}\overline{A}(n, 0) \leq c. \tag{1.53}$$

Since the actions of the beginning and ending pieces of $\gamma_{n,x}$ are bounded below by $-2c(x+1)$, and since $A(\eta_x, x, n-x-1) \geq (n-2x-1)(c-d_x)$ by 1.12, it is easy to see that on A_n for n large enough, $n^{-1}A(\gamma_{n,x})$ can be taken as close as we wish to c. And so, since A_n has probability as close to 1 as we wish, 1.53 gives $n^{-1}\overline{A}(n,0) \xrightarrow{p} c$.

□

Remark 1.3.6. *Since $a(0) = c$ is a global maximum for a, concavity tells us that a is non-decreasing on $[-1,0]$ and non-increasing on $[0,1]$.*

An important question when looking at the shape function is its behaviour at 0. For a concave function, right and left sided derivatives exist in the interior of the domain. Applying the bound in (1.46), one immediately obtains

Lemma 1.3.7. *The right sided derivative of the shape function a at 0, $a'(0+) = \lim_{h\to 0^+} \frac{a(h)-c}{h}$, is negative. Thus the shape function a has a corner at 0.*

A possible candidate for the shape function would be simply linear on $[0,1]$, namely $a(\alpha) = c(1-\alpha)$. Indeed, this limit is easily achieved by considering the naive path γ_n^{naive} which moves ballistically up to the best discrepancy in $[0, \lfloor \alpha n \rfloor]$, stays there optimally as long as it can, and then moves ballistically to $\lfloor \alpha n \rfloor$ at time n. Formally, calling the optimal site $r_{n,\alpha} = \operatorname{argmin}_{x\in[0,\lfloor \alpha n\rfloor)} d_x$, set

$$\gamma_n^{naive}(i) = \begin{cases} i & \text{if } \ i \leq r_{n,\alpha} \\ \eta_{r_{n,\alpha}}(i) & \text{if } \ r_{n,\alpha} < i \leq n - \lfloor \alpha n \rfloor + r_{n,\alpha} \\ i - \big(n - \lfloor \alpha n \rfloor\big) & \text{if } \ n - \lfloor \alpha n \rfloor + r_{n,\alpha} < i \leq n \end{cases}.$$

Using equation 1.12, the law of large numbers, and the fact that $d_{r_{n,\alpha}} \to 0$ a.s., it is easy to see that

$$\frac{A(\gamma_n^{naive})}{n} \to c(1-\alpha) \text{ almost surely.}$$

However, it turns out that this is not the behavior of the shape function, which in fact lies above the straight line. We relegate the proof of the following important theorem to section 1.6.

Theorem 1.3.8. *For any $\alpha \in (0,1)$, $a(\alpha) > c(1-\alpha)$. Namely, the shape function a is non-linear on $[0,1]$.*

After numerical analysis, as mentioned in the introduction, we conjecture that there is a linear piece of the shape function near 0

../images/plot.pdf

1.4 The maximizing path

1.4.1 Basic properties

We will often bound the action of a path in terms of the discrepancies it moves through. We would like to say that whenever there is a sign change of $b(\cdot)$, we can bound the action on such two sites by $2c - \min_{x \in \text{Range}(\gamma)} d_x$. But to avoid over-counting such actions (since there can be two sign changes in only three steps), we index pairs only by the even numbers. We then have the following simple lemma.

Lemma 1.4.1. *For any path γ in $\Gamma(a, b)$,*

$$A(\gamma, a, b) \leq c(b-a) - \sum_{\substack{i \in (a,b) \\ i \text{ even}}} \mathbb{1}\big\{b(i) \neq b(i+1)\big\} \cdot \min_{x \in \text{Range}(\gamma)} d_x.$$

We now prove some basic properties of the maximizing path $\overline{\gamma}_n$.

Lemma 1.4.2. *For any $\epsilon > 0$, there exist constants N, k_1, k_2, and B such that for all $n > N$,*

1. $\mathbb{P}\Big(\max_{x \in \text{Range}(\overline{\gamma}_n)} |x| \in \Big(k_1 n^{2/3}, k_2 n^{2/3}\Big)\Big) > 1 - \epsilon$
2. $\mathbb{P}\Big(\exists\, x \in \text{Range}(\overline{\gamma}_n)$ *such that* $d_x < B n^{-1/3}\Big) > 1 - \epsilon.$

Proof. From corollary 1.2.4, for n large enough, there exists a point $\overline{x}$ such that $|\overline{x}| < a n^{2/3}$ and $d_{\overline{x}} < b n^{-1/3}$ with probability greater than $1 - \epsilon$. Inside this event, we consider the path which moves ballistically up to $\overline{x}$ and then remains optimally on the edge $\{\overline{x}, \overline{x}+1\}$. We can lower bound its action by

$$-c|\overline{x}| + A(\eta_{\overline{x}}, |\overline{x}|, n) > -c(an^{2/3}) + \big(n - |\overline{x}|\big)(c - d_x) \tag{1.54}$$

$$> -c(an^{2/3}) + (n - an^{2/3})(c - bn^{-1/3}) \tag{1.55}$$

$$> cn - (2ca + b)n^{2/3}. \tag{1.56}$$

Thus, the maximal path $\overline{\gamma}_n$ must have action greater than or equal to this quantity. Using this fact and lemma 1.4.1, we can give an upper bound on the smallest discrepancy the path must reach.

$$cn - (2ca + b)n^{2/3} \leq cn - \sum_{\substack{i \in (0,n) \\ i \text{ even}}} \mathbb{1}\big\{b(i) \neq b(i+1)\big\} \cdot \min_{x \in \text{Range}(\overline{\gamma}_n)} d_x. \tag{1.57}$$

By the law of large numbers, for any $0 < \lambda < 1/4$, we can take this to be less than

$$cn - (\lambda n) \min_{x \in \text{Range}(\overline{\gamma}_n)} d_x$$

with probability as close to 1 as we wish, for all n large enough. Then inside this event, we have that

$$\min_{x \in \text{Range}(\overline{\gamma}_n)} d_x < (2ac + b)\lambda^{-1} n^{-1/3}.$$

This gives the second statement of the lemma. We will now show the first statement, the bounds on the path.

From theorem 1.1.3, there exists k_1 small enough such that, for all n large enough, the probability of the event

$$\Big\{\mu_n\Big([-k_1, k_1] \times [0, (2ac + b)\lambda^{-1}]\Big) = 0\Big\}$$

is as close to 1 as we wish (Obviously $k_1 < a$ chosen above). So on this event, $\overline{\gamma}_n$ must reach a minimum distance of $k_1 n^{2/3}$ from the origin, by the bound above on $\min_{x \in \text{Range}(\overline{\gamma}_n)} d_x$.

Finally, we upper bound the range of $\overline{\gamma}_n$. Any path with range outside of $(-k_2 n^{2/3}, k_2 n^{2/3})$ moves through all sites in either $[1, k_2 n^{2/3}]$ or $[-k_2 n^{2/3}, -1]$. Using the bounds in 1.10, we can bound the action of any such path from above by

$$\max\left(\sum_{i=1}^{k_2 n^{2/3}} \big|F(i)\big|, \sum_{i=-1}^{-k_2 n^{2/3}} \big|F(i)\big|\right) + (n - k_2 n^{2/3})c. \tag{1.58}$$

The law of large numbers gives that with probability close to 1, for all n large enough, we can take this less than

$$cn - \Big(c - \mathbb{E}\big|F(0)\big| - \epsilon\Big)k_2 n^{2/3}.$$

Thus, choosing k_2 large enough such that

$$k_2\Big(c - \mathbb{E}\big|F(0)\big| - \epsilon\Big) > 2ca + b,$$

a path outside of $(-k_2 n^{2/3}, k_2 n^{2/3})$ has smaller action than the ballistic path to $\overline{x}$ inside the events we have been working, and thus cannot have maximal action.

□

1.4.2 The ending edge

We now define the ending edge of the maximizing path $\overline{\gamma}_n$. To do this, we consider the time that this ending edge is reached, which we call τ_n. That is

Definition 1.4.3.

$$\tau_n = \min\Big\{t \in [0,n] \,\Big|\, \#\Big|\overline{\gamma}_n\big([t,n]\big)\Big| \le 2\Big\}.$$

Note that this is well defined for any path, and that unless $\overline{\gamma}_n$ is identically 0 on $[0,n]$, there are two neighbors in $\overline{\gamma}_n\big([\tau_n, n]\big)$. Call the smaller neighbor of this edge ℓ_n. (Recall that we index discrepancies by the smaller vertex.) Namely,

Definition 1.4.4.

$$\ell_n = \min \overline{\gamma}_n\big([\tau_n, n]\big).$$

Note that, by maximality of the action, $\overline{\gamma}_n(t) = \eta_{\ell_n}(t)$ for all $t \in [\tau_n, n]$. As noted in remark 1.2.2, the maximizing path $\overline{\gamma}_n$ will tend to localize on sites with small discrepancy. In fact, the path will tend to end on edges with record discrepancies, as we show with the lemma 1.4.6. But first, a lemma controlling the probability of neighbors being close to the edge of their support.

Lemma 1.4.5. *For any $m \in \mathbb{N}$, there exists a constant $\lambda = \lambda(m) > 0$ such that for any $\epsilon > 0$ and for any integers a, b with $b - a > 2m$,*

$$\mathbb{P}\left(\prod_{k=-m}^{m}\Big(c - \big|F(x+k)\big|\Big) > \frac{\lambda\epsilon}{b-a} \quad \forall x \in [a+m, b-m]\right) > 1 - \epsilon. \tag{1.59}$$

Proof.

$$\mathbb{P}\left(\exists\, x \in [a+m, b-m] \text{ such that } \prod_{k=-m}^{m}\Big(c - \big|F(x+k)\big|\Big) \le \frac{\lambda\epsilon}{b-a}\right) \tag{1.60}$$

$$\le \mathbb{P}\left(\bigcup_{x\in[a+m,b-m]}\left\{\prod_{k=-m}^{m}\Big(c - \big|F(x+k)\big|\Big) \le \frac{\lambda\epsilon}{b-a}\right\}\right) \tag{1.61}$$

$$\le (b-a-2m+1)\,\mathbb{P}\left(\prod_{k=-m}^{m}\Big(c - \big|F(x+k)\big|\Big) \le \frac{\lambda\epsilon}{b-a}\right) \tag{1.62}$$

$$= (b-a-2m+1)\iiint_{\{x_1\cdots x_{2m+1} \le \frac{\lambda\epsilon}{b-a}\}} g(x_1)\cdots g(x_{2m+1})\,\mathrm{d}x_1\cdots \mathrm{d}x_{2m+1}, \tag{1.63}$$

where here g is the pdf of $c - \big|F(0)\big|$, and by assumption is supported on $[0, c]$, continuously differentiable on $[0, c]$ with finite derivatives at the endpoints, and in particular bounded. And since the Lebesgue measure of $\{x_1 \cdots x_{2m+1} \le \frac{\lambda\epsilon}{b-a}\}$ inside the cube $[0, c]^{2m+1}$ is bounded by a constant times $\frac{\lambda\epsilon}{b-a}$, by choosing λ small enough we can take the probability in (1.60) less than ϵ. ☐

Lemma 1.4.6.

$$\mathbb{P}\left(\ell_n = \operatorname*{argmin}_{x\in\mathrm{Range}(\overline{\gamma}_n)} d_x\right) \to 1. \tag{1.64}$$

Additionally, $\overline{\gamma}_n$ does not leave this site with the best discrepancy once there. That is,

$$\mathbb{P}\Big(\tau_n = \min\big\{t \in [0, n]\,\big|\,\overline{\gamma}_n(t) \in \{\ell_n, \ell_n + 1\}\big\}\Big) \to 1. \tag{1.65}$$

Proof. First, we consider the event such that $\mathrm{Range}\big(\overline{\gamma}_n\big) \subset [-kn^{2/3}, kn^{2/3}]$, which, by lemma 1.4.2, we can take to have probability arbitrarily close to 1 for all n large enough (k as prescribed by the lemma). Call $r = \mathrm{argmin}_{x\in\mathrm{Range}(\overline{\gamma}_n)} d_x$. By the second part of the lemma, we can assume that $d_r < Bn^{-1/3}$ for some $B > 0$. And by corollary 1.2.4, we can also assume that for all $x \in \mathrm{Range}\big(\overline{\gamma}_n\big)$, $d_x - d_r > \delta n^{-1/3}$ for some $\delta > 0$. Again, the intersection of these events can be taken with probability arbitrarily close to 1 for n large enough.

Now, consider the set of pairs with a given small discrepancy

$$S = \Big\{x, x+1 \in [-kn^{2/3}, kn^{2/3}) \,\Big|\, d_x <= B' n^{-1/3}\Big\},$$

where B' is a constant larger than $4B$. Lemma 1.4.5 (with $m = 3$) allows us to work inside the event that S consists only of disjoint pairs. The lemma also gives, calling the neighbors of this set

$$\partial S = \Big\{x \in [-kn^{2/3}, kn^{2/3}] \setminus S \,\Big|\, \{x-1, x+1\} \cap S \neq \emptyset\Big\},$$

that for all $x \in \partial S$,

$$\big|F(x)\big| < c - \lambda,$$

where λ is some positive constant depending on ϵ, B, and k and independent of n.

And finally, call the complement of S in the given range (which includes ∂S)

$$S' = [-kn^{2/3}, kn^{2/3}] \setminus S.$$

Now, consider the event where $\overline{\gamma}_n$ leaves the edge $\{r, r+1\}$ for some interval of time. Let $(t_1, t_2]$ be such an interval, where $\overline{\gamma}_n(t_1) \in \{r, r+1\}$ but $\overline{\gamma}_n\big((t_1, t_2]\big) \cap \{r, r+1\} = \emptyset$. For the path to benefit by moving away, the action of $\overline{\gamma}_n$ on $(t_1, t_2]$ must be larger than the action of the path η_r which remains on the edge $\{r, r+1\}$ optimally. Now, since $r, r+1 \in S$, $\overline{\gamma}_n(t_1+1)$ must be in ∂S. Thus, we have

$$(c - d_r)(t_2 - t_1) \leq A(\eta_r, t_1, t_2) < A(\overline{\gamma}_n, t_1, t_2) \leq c - \lambda + c(t_2 - t_1 - 1), \tag{1.66}$$

which gives that

$$t_2 - t_1 > \lambda d_r^{-1} > \lambda B^{-1} n^{1/3}.$$

So for the path to leave $\{r, r+1\}$, it must leave for an interval of order $n^{1/3}$.

Now, we decompose the path on $(t_1, t_2]$ into two types of intervals. First, we can take intervals $[s_1, s_2]$ such that s_1 is the step before $\overline{\gamma}_n$ moves into S, and s_2 is the last step before it moves back into S' (or t_2). Precisely, $\overline{\gamma}_n(s_1)$ and $\overline{\gamma}_n(s_2+1)$ are in S' and $\overline{\gamma}_n\big([s_1+1, s_2]\big) \subset S$ (And most importantly, $\overline{\gamma}_n(s_1) \in \partial S$). Denote the set of such intervals of this type on $(t_1, t_2]$ by $\mathcal{I}$.

The intervals which remain after considering intervals of this type are intervals of steps for which $\overline{\gamma}_n$ is contained entirely in S', whose leftmost step is an entry of $\overline{\gamma}_n$ into S' from S (hence is a step in ∂S). The last point in the interval will be two steps before $\overline{\gamma}_n$ moves into S, by construction of the intervals in $\mathcal{I}$ (or t_2). Call the set of these intervals $\mathcal{J}$. The intervals in $\mathcal{I}$ and $\mathcal{J}$ form a partition of the points in $(t_1, t_2]$.

We now argue that for any interval in $\mathcal{I}$ or $\mathcal{J}$, $\overline{\gamma}_n$ cannot have action better than the path η_r which remains optimally on the record discrepancy site r, thus concluding that inside the intersection of events we are working on, with measure arbitrarily close to 1 for large enough n, the maximizing path $\overline{\gamma}_n$ will not leave $\{r, r+1\}$ once reaching it. Note that, since both sets in $\mathcal{I}$ and $\mathcal{J}$ begin with a step in ∂S, the same argument above applies, so that for any such interval to have action better than that of η_r, the length of the interval must be at least $\lambda B^{-1} n^{1/3}$.

First we consider an interval $[s_1, s_2]$ in $\mathcal{J}$. We use the simple bound from lemma 1.4.1 to say that the action on this interval of $\overline{\gamma}_n$ is bounded by

$$c(s_2 - s_1 + 1) - \sum_{\substack{i \in [s_2, s_1) \\ i \text{ even}}} \mathbb{1}\big\{b(i) \neq b(i+1)\big\} \cdot \min_{x \in S'} d_x \tag{1.67}$$

$$< c(s_2 - s_1 + 1) - \left(\frac{B}{B'}(s_2 - s_1 + 1)\right)\left(B' n^{-1/3}\right) \tag{1.68}$$

$$< (s_2 - s_1 + 1)(c - d_r), \tag{1.69}$$

since for any $\epsilon > 0$, for n large enough, we can easily lower bound the sum of indicators $\sum_{\substack{i \in [a,b) \\ i \text{ even}}} \mathbb{1}\big\{b(i) \neq b(i+1)\big\}$ by $(1-\epsilon)\frac{b-a+1}{4}$ on any interval $[a, b] \subset [0, n]$ which satisfies, say, $b - a > \lambda B^{-1} n^{1/3}$. The proof of this probabilistic fact can be realized using Hoeffding's Lemma [16] and a union bound over all such intervals. And so, we see that the action on $[s_1, s_2]$ must be smaller than the action of η_r on this interval.

Now we consider an interval $[s_1, s_2]$ in $\mathcal{I}$. To bound the path in such an interval, we only need to say that for long enough time periods ($s_2 - s_1 > \lambda B^{-1} n^{1/3}$), the action of a path restricted to a single edge is primarily given by the length of the interval times c minus half the discrepancy of the edge. By construction, for some $x, x+1 \in S$, $\overline{\gamma}_n$ simply equals η_x (the optimal path on the edge $\{x, x+1\}$) on $(s_1, s_2]$. We compare this action to the action of η_r using equation 1.15, equation 1.11, and the assumption that $d_x - d_r > \delta n^{-1/3}$ for any $x \in \text{Range}(\overline{\gamma}_n)$.

$$
\begin{aligned}
(1.70)\quad & A(\eta_x, s_1, s_2) - A(\eta_r, s_1, s_2) \\
(1.71)\quad & = \frac{1}{2}(s_2 - s_1)(d_r - d_x) + \left(N^+(s_1, s_2) - \frac{s_2 - s_1}{2}\right)\Big(F(x) + F(x+1) - \big(F(r) + F(r+1)\big)\Big) \\
(1.72)\quad & < \frac{1}{2}(s_2 - s_1)(d_r - d_x) + \left|N^+(s_1, s_2) - \frac{s_2 - s_1}{2}\right|(d_x + d_r) \\
(1.73)\quad & < \left|N^+(s_1, s_2) - \frac{s_2 - s_1}{2}\right|\Big((B + B')n^{-1/3}\Big) - \frac{1}{2}(s_2 - s_1)(\delta n^{-1/3}).
\end{aligned}
$$

Similarly to before, using a union bound and Hoeffding's inequality, for any $\epsilon > 0$, for n large enough, we can work inside the event (with probability as close to 1 as we wish) that $\left|N^+(a, b) - \frac{b-a}{2}\right| < \epsilon(b-a)$ for any interval $[a, b] \subset [0, n]$ with $b - a > \lambda B^{-1} n^{-1/3}$. So, taking this ϵ less than $\frac{\delta}{2(B+B')}$, we have that for n large enough, $A(\eta_x, s_1, s_2) - A(\eta_r, s_1, s_2) < 0$. And so, for any interval in $\mathcal{I}$, the action of the path which remains at $\{r, r+1\}$ must be larger than the path which has left.

Thus we have shown that the event in which the action-maximizing path $\overline{\gamma}_n$ leaves $\{r, r+1\}$ is disjoint from the finite intersection of events which have probability arbitrarily close to 1 for n large enough. And so this event has probability which tends to 0.

□

With probability tending to 1, the previous lemma gives that the maximizing path $\overline{\gamma}_n$ hits a record discrepancy edge and then stays there until time n. Thus, with probability tending to 1, the point farthest from the origin that $\overline{\gamma}_n$ reaches is ℓ_n or $\ell_n + 1$. Then, combining the above lemma with lemma 1.4.2, we have

Corollary 1.4.7. *For any $\epsilon > 0$, there exist constants N, k_1, and k_2 such that for all $n > N$,*

$$
\mathbb{P}\Big(|\ell_n| \in \Big(k_1 n^{2/3}, k_2 n^{2/3}\Big)\Big) > 1 - \epsilon. \tag{1.74}
$$

1.4.3 The arrival time

Recall the arrival time τ_n of the maximizing path to the ending edge $\{\ell_n, \ell_n + 1\}$ from definition 1.4.3. By lemma 1.4.6, with probability tending to 1, τ_n is the first hitting time of the edge $\{\ell_n, \ell_n + 1\}$,

$$
\tau_n = \min\big\{t \in [0, n] \,\big|\, \overline{\gamma}_n(t) \in \{\ell_n, \ell_n + 1\}\big\}.
$$

This time τ_n is an important quantity to understand in terms of the location of the final edge.

Numerics suggest that the ratio between them, τ_n/ℓ_n, does not grow to infinity (and in fact, appears to be centered around a value less than or equal to 2). However, proving this has remained elusive. In order to prove the main theorem of this paper, we assume this behavior. Namely, we assume

assumption[Bounded Arrival Condition] There exists a $\kappa > 1$ such that

$$\mathbb{P}(\tau_n > \kappa|\ell_n|) \longrightarrow 0. \tag{1.75}$$

assumption

This behavior implies linear behavior of the shape function near the origin, which we show.

Proof that assumption 1.1.4 $\Rightarrow$ linearity of the shape function near 0. Inside the event $\{\tau_n \leq \kappa|\ell_n|\}$, the maximizing path to the point $\Big(\lfloor 2\kappa|\ell_n|\rfloor, \ell_n\Big)$ is exactly the maximizing path to the point $\Big(\lfloor \kappa|\ell_n|\rfloor, \ell_n\Big)$ which then remains on the edge $\{\ell_n, \ell_n + 1\}$ optimally until time $\lfloor 2\kappa|\ell_n|\rfloor$. Thus on this event, the difference of the maximizing actions to these two points is given by

$$\overline{A}\Big(\lfloor 2\kappa|\ell_n|\rfloor, \ell_n\Big) - \overline{A}\Big(\lfloor \kappa|\ell_n|\rfloor, \ell_n\Big) = A\Big(\eta_{\ell_n}, \lfloor \kappa|\ell_n|\rfloor, \lfloor 2\kappa|\ell_n|\rfloor\Big).$$

Since

$$\frac{1}{\kappa|\ell_n|} A\Big(\eta_{\ell_n}, \lfloor \kappa|\ell_n|\rfloor, \lfloor 2\kappa|\ell_n|\rfloor\Big) \xrightarrow{p} c$$

and since we are supposing that $\mathbb{P}\{\tau_n \leq \kappa|\ell_n|\} \xrightarrow{p} 1$, we have that

$$\frac{1}{\kappa|\ell_n|}\left[\overline{A}\Big(\lfloor 2\kappa|\ell_n|\rfloor, \ell_n\Big) - \overline{A}\Big(\lfloor \kappa|\ell_n|\rfloor, \ell_n\Big)\right] \xrightarrow{p} c.$$

But the left hand side converges to

$$2a\left(\frac{1}{2\kappa}\right) - a\left(\frac{1}{\kappa}\right)$$

in probability, by theorem 1.3.1, and by the fact that $\ell_n \xrightarrow{p} \infty$ from So then by uniqueness of the limit, we must have linearity of a on $[0, 1/\kappa]$,

$$2a\left(\frac{1}{2\kappa}\right) - a\left(\frac{1}{\kappa}\right) = c = a(0).$$

$\square$

Linearity of the shape function near 0 also appears to be supported by numerics, specifically, linearity appears to hold on the interval between 0 and approximately 0.6.

1.5 Distributional Convergence

Define the maximizing path to a point (t, x) by

$$\overline{\xi}_{(t,x)} = \operatorname*{argmax}_{\xi \in \Gamma(t,x)} A(\xi).$$

We consider such a path to the "better" choice between x and $x+1$ at time t,

$$\overline{\xi}_{t,\{x\}} := \overline{\xi}_{(t,x)} \mathbb{1}\{\eta_x(t) = x\} + \overline{\xi}_{(t,x+1)} \mathbb{1}\{\eta_x(t) = x+1\}.$$

Define the path of length n which is this until time $\lfloor \kappa|x| \rfloor < n$, and then is η_x until time n:

$$\gamma_{n,x}(t) := \begin{cases} \overline{\xi}_{\lfloor \kappa|x| \rfloor, \{x\}}(t) & \text{for } 0 \le t \le \lfloor \kappa|x| \rfloor \\ \eta_x(t) & \text{for } \lfloor \kappa|x| \rfloor \le t \le n \end{cases}.$$

Here κ is the arrival ratio given in assumption 1.1.4. Lemma 1.4.6 and assumption 1.1.4 give that $\overline{\gamma}_n$ is a path of this type with probability tending to 1. Thus, we can say that with probability tending to 1,

$$\begin{aligned} \ell_n &= \operatorname*{argmax}_{\left\{x \,\middle|\, \lfloor \kappa|x| \rfloor < n\right\}} A(\gamma_{n,x}) \\ &= \operatorname*{argmax}_{\left\{x \,\middle|\, \lfloor \kappa|x| \rfloor < n\right\}} B_n(x), \end{aligned} \tag{1.76}$$

where $B_n(x)$ is the centered and rescaled action,

$$B_n(x) := n^{-2/3}\big(A(\gamma_{n,x}) - cn\big).$$

Of course we need not look over all $\left\{x \,\middle|\, \lfloor \kappa|x| \rfloor < n\right\}$, since lemma 1.4.2 tells us that the maximizing path is contained in a ball of order $n^{2/3}$ with high probability.

Now, theorem 1.3.1 gives that

$$\frac{1}{|m|} A\left(\overline{\xi}_{(\lfloor \kappa|m| \rfloor, m)}\right) = \frac{1}{|m|} \overline{A}(\lfloor \kappa|m| \rfloor, m) \to \Phi(\kappa, 1)$$

in probability as $|m| \to \infty$ (in fact almost surely, but we won't need this). Note that we have used distributional symmetry with respect to the second coordinate, as m can be positive or negative. We also claim the same about $\frac{1}{|m|}A\Big(\overline{\xi}_{(\lfloor\kappa|m|\rfloor,m+1)}\Big)$. Choosing $0<\epsilon<\kappa-1$, subadditivity gives for $|m|$ large enough that

$$\begin{aligned}\overline{A}\Big(\lfloor(\kappa-\epsilon)|m|\rfloor,m\Big)+\overline{A}\Big((\lfloor(\kappa-\epsilon)|m|\rfloor,m),(\lfloor\kappa|m|\rfloor,m+1)\Big)&\\ \leq \overline{A}\Big(\lfloor\kappa|m|\rfloor,m+1\Big)&\\ \leq \overline{A}\Big(\lfloor(\kappa+\epsilon)|m|\rfloor,m\Big)-\overline{A}\Big((\lfloor\kappa|m|\rfloor,m+1),(\lfloor(\kappa+\epsilon)|m|\rfloor,m)\Big)&\end{aligned}$$

. Dividing by $|m|$, taking a limit, and then using continuity of the shape function Φ taking $\epsilon\to 0$, we obtain that

$$\frac{1}{|m|}A\Big(\overline{\xi}_{(\lfloor\kappa|m|\rfloor,m+1)}\Big)\xrightarrow{p}\Phi(\kappa,1).$$

Then since both random sequences converge in probability to the same limit, we have

$$\frac{1}{|m|}A\Big(\overline{\xi}_{\lfloor\kappa|m|\rfloor,\{m\}}\Big)\xrightarrow{p}\Phi(\kappa,1)=\kappa a(\kappa^{-1}).$$

Now, define the random function

$$g_n(x)=-a'(0+)\frac{|x|}{n^{2/3}}-\frac{d_x}{2}n^{1/3},$$

and call its maximizer

$$\overline{x}_n=\operatorname*{argmax}_{x\in\mathbb{Z}} g_n(x).$$

Now we compare the random functionals $B_n(x)$ and $g_n(x)$. Using equation 1.15 for the second equality, we have

$$\begin{aligned}&B_n(x)-g_n(x)\\ &=\frac{1}{n^{2/3}}\left[A\Big(\overline{\xi}_{\lfloor\kappa|x|\rfloor,\{x\}}\Big)+A\big(\eta_x,\kappa|x|,n\big)-cn-a'(0+)|x|+\frac{d_x}{2}n\right]\\ &=\frac{|x|}{n^{2/3}}\left[\left(\frac{1}{|x|}A\Big(\overline{\xi}_{\lfloor\kappa|x|\rfloor,\{x\}}\Big)-(a'(0+)+\kappa c)\right)+\frac{\kappa d_x}{2}\right.\\ &\qquad\left.+\frac{1}{|x|}\left(N^+(\kappa|x|,n)-\frac{n-\kappa|x|}{2}\right)\big(F(x)+F(x+1)\big)\right].\end{aligned}\tag{1.77}$$

For $x=\ell_n$, each of these terms converge to 0 in probability as $n\to\infty$. The first because $\frac{1}{|x|}A\Big(\overline{\xi}_{\lfloor\kappa|x|\rfloor,\{x\}}\Big)\xrightarrow{p}$

$\kappa a(\kappa^{-1})$ as $|x| \to \infty$, because $a'(0+) + \kappa c = \kappa a(\kappa^{-1})$ by the linearity of the shape function on $[0, \kappa^{-1}]$, and by lemma 1.9.1, because $|\ell_n| \xrightarrow{p} \infty$. The second because $d_{\ell_n} \xrightarrow{p} 0$ by lemma 1.4.6. And the third by corollary 1.4.7, which gives that $\ell_n > kn^{2/3}$ for some k with high probability, together with the central limit theorem. And so we have

$$B_n(\ell_n) - g_n(\ell_n) \xrightarrow{p} 0.$$

Similary we have

$$B_n(\overline{x}_n) - g_n(\overline{x}_n) \xrightarrow{p} 0.$$

To see this, note that the above argument follows for any point of order $n^{2/3}$ (in the way of corollary 1.4.7) whose discrepancies tend to 0 in probability. This is immediately true of $\overline{x}_n$ by theorem 1.1.3. (And thus $\lfloor \kappa |\overline{x}_n| \rfloor < n$ as required to consider $B_n(\overline{x}_n)$ for n large enough.)

We now claim that maximizer of $g_n(x)$ over $x \in \mathbb{Z}$, $\overline{x}_n$, is equal to the maximizer of $B_n(x)$, ℓ_n, with probability tending to 1. To see this, define the difference in the maximum value of the functional g_n and the next maximum value by

$$\zeta_n = \left| g_n(\overline{x}_n) - \operatorname*{argmax}_{x \neq \overline{x}_n} g_n(x) \right|.$$

An immediate consequence of corollary 1.2.4 is that this random variable is bounded away from 0 with high probability.

Lemma 1.5.1. *For every $\epsilon > 0$, there is a $\delta > 0$ uniform in n large enough such that*

$$\mathbb{P}\{\zeta_n > \delta\} > 1 - \epsilon. \tag{1.78}$$

If $\overline{x}_n$ is in the domain of B_n (which has probability tending to 1), and if $\overline{x}_n \neq \ell_n$, then

$$\begin{aligned} \zeta_n &\le g_n(\overline{x}_n) - g_n(\ell_n) \\ &\le \big(g_n(\overline{x}_n) - B_n(\overline{x}_n)\big) + \big(B_n(\ell_n) - g_n(\ell_n)\big), \end{aligned}$$

where we have used the fact that $B_n(\overline{x}_n) \le B_n(\ell_n)$. By the above lemma, and since both of these terms tend to 0 in probability as noted above, we have

Proposition 1.5.2.

$$\mathbb{P}\{\ell_n = \overline{x}_n\} \to 1.$$

Now, g_n can be viwed as the continuous functional $g(x,y) = -a'(0+)|x| - y/2$ on the point process μ_n defined in theorem 1.1.3, and $(\overline{x}_n n^{-2/3}, d_{\overline{x}_n} n^{1/3})$ as the g-maximizing point in μ_n. We argue that

Lemma 1.5.3. *$(\overline{x}_n n^{-2/3}, d_{\overline{x}_n} n^{1/3})$ converges in distribution to $\left(\overline{X}, \overline{Y}\right)$, where*

$$\left(\overline{X}, \overline{Y}\right) = \underset{(X,Y)\in \operatorname{supp}(\mu)}{\operatorname{argmax}} g(X,Y).$$

Here μ is the Poisson Point Process on $\mathbb{R} \times \mathbb{R}^+$ with driving measure ν as described in theorem 1.1.3.

Proof. Call the level set $T_r = g^{-1}(r)$, $r < 0$, which is a one dimensional triangle in $\mathbb{R} \times \mathbb{R}^+$. Inside the (Polish, see [22]) space $\mathfrak{N}$ of point processes in $\mathbb{R}^2$ endowed with the vague topology (see 1.7.1), consider the subset

$$S = \left\{\mu \neq 0 \,\middle|\, \mu(T_r) \leq 1 \text{ for all } r < 0\right\}.$$

The poisson process μ described in theorem 1.1.3, which has continuous driving measure, is inside S with probability 1. This is because the distribution of the points μ in any compact rectangle R is given by N uniformly distributed points, where N is a poisson random variable with parameter $\nu(R)$ (See [26]). It is easy to check that the probability that two such points lie on the same triangle T_r in R is 0. Let $\{R_n\}$ be an increasing sequence of rectangles, $R_n \nearrow \mathbb{R} \times \mathbb{R}^+$, and call the event $A_n = \{\mu(T_r) \leq 1 \; \forall \; T_r \subset R_n\}$. Then $A_1 \subset A_2 \subset A_3 \subset \cdots$, and by continuity of the measure from above,

$$\mathbb{P}(\mu \in S) = \mathbb{P}(\cap A_n) = \lim \mathbb{P}(A_n) = 1.$$

Now, given a point process μ in S, the argmax $M(\mu) = \operatorname{argmax}_{(x,y)\in \operatorname{supp}(\mu)} g(x,y)$ is a well defined function from S to $\mathbb{R} \times \mathbb{R}^+$. In fact, it is a continuous function, where S inherits the topology of vague convergence as described in 1.7.1. To see this, fix an open set $\mathcal{O}$ in $R \times R^+$, and let μ be a point process in $M^{-1}(\mathcal{O})$. Then, calling $(\overline{x}, \overline{y}) = M(\mu) \in \mathcal{O}$, we show that there is an open set around μ in $M^{-1}(\mathcal{O})$. Consider the next maximizing point of μ, $\left(\overline{x}', \overline{y}'\right) = \operatorname{argmax}_{(x,y)\in \operatorname{supp}(\mu)\setminus\{(\overline{x},\overline{y})\}} g(x,y)$. Let B be an open ball around $(\overline{x}, \overline{y})$ in $\mathcal{O}$ which does not contain this next best point $\left(\overline{x}', \overline{y}'\right)$. Define a continuous function h_1 which is positive on the ball B and zero outside of it. Also define another continuous function h_2 which is positive on the open region between the x-axis and the triangle through $\left(\overline{x}', \overline{y}'\right)$, $T_{g\left(\overline{x}', \overline{y}'\right)}$, and zero elsewhere (note that B is contained in the support of h_2 by construction). Then the open set in the vague topology (see 1.7.1)

$$\mathcal{U} = \left\{\nu \in S \,\middle|\, \int h_1 \mathrm{d}\nu > 0\right\} \bigcap \left\{\nu \in S \,\middle|\, \int h_2 \mathrm{d}\nu < 2\right\}$$

contains μ, since it describes elements of S with one point in the ball B around $(\overline{x},\overline{y})$ and no other points until the next best maximizer $\left(\overline{x}',\overline{y}'\right)$. Any such point process has its maximizer in $B\subset\mathcal{O}$, and so $\mathcal{U}$ is contained in $M^{-1}(\mathcal{O})$. Thus, $\mathcal{U}$ is an open set with respect to the inherited vague topology in S, and M is continuous. Since $\mu_n\in S$ converges weakly to μ in the vague topology, the continuous mapping theorem (see [3], theorem 2.7, p. 21) gives that $M(\mu_n)$ converges in distribution to $M(\mu)$.

□

Convergence of $(\overline{x}_n n^{-2/3}, d_{\overline{x}_n} n^{1/3})$ gives convergence of

$$\left(\overline{x}_n n^{-2/3}, g_n(\overline{x}_n)\right)=\left(\overline{x}_n n^{-2/3}, g(\overline{x}_n n^{-2/3}, d_{\overline{x}_n} n^{1/3})\right),$$

and so in light of the fact that

$$\left|\left(\ell_n n^{-2/3}, B_n(\ell_n)\right)-\left(\overline{x}_n n^{-2/3}, g_n(\overline{x}_n)\right)\right|\xrightarrow{p}0,$$

we have our main theorem:

Theorem 1.5.4.

$$\left(\frac{\ell_n}{n^{2/3}},\frac{A(\overline{\gamma}_n)-cn}{n^{2/3}}\right)\xrightarrow{d}(\overline{X},g(\overline{X},\overline{Y})),\tag{1.79}$$

where $(\overline{X},\overline{Y})=\operatorname{argmax}_{(X,Y)\in\operatorname{supp}(\mu)} g(X,Y)$, *with* μ *the Poisson process with driving measure* ν *absolutely continuous w.r.t. Lebesgue measure withdensity given by*

$$\frac{d\nu(\mathrm{d}t\times\mathrm{d}y)}{\mathrm{d}t\times\mathrm{d}y}=\begin{cases}\frac{p'(0+)}{2}y, & y>0\\ 0, & y\le 0\end{cases}$$

and p *the density of* d_0.

1.6 Non-linearity of the Shape Function

In this section, we prove theorem 1.3.8:

Theorem 1.6.1. *For any* $\alpha\in(0,1)$, $a(\alpha)>c(1-\alpha)$. *Namely, the shape function* a *is non-linear on* $[0,1]$.

Proof. Because the shape function is concave on $[-1,1]$ it suffices to simply check non-linearity at a single point, say $\alpha = \frac{1}{2}$. Fix $0 < \delta < \frac{c}{2}$. Call $\ell_n = \operatorname{argmin}_{x \in [0, an^{2/3})} d_x$ the discrepancy minimizer of the points in $[0, an^{2/3})$, where $a > 0$ is an arbitrary positive constant. We will now look at points with small discrepancy above ℓ_n. For simplicity, we take these discrepancies to be independent by only looking at odd points, $I_n = \left\{ i \in [an^{2/3}, \lfloor \alpha n \rfloor) \,\middle|\, i \text{ is odd} \right\}$. Define a sequence in the $\mathbb{Z} \cup \{-\infty\}$

$$x_1 = \sup\{x \in I_n \,|\, d_x < \delta\}, \text{ and } x_{j+1} = \sup\left\{x < x_j \,\middle|\, x \in I_n, d_x < \delta\right\}.$$

This is a non-increasing sequence whose positive elements (if any) are not neighbors. Call $k_n = \operatorname{argmax}_{k \in \mathbb{N}} x_k > 0$ the number of points x in I_n which satisfy $d_x < \delta$. If $k_n > 0$, then by definition, x_{k_n} should be the smallest point in I_n with discrepancy less than δ (if there are any such points in I_n, otherwise $x_{k_n} = -\infty$), and x_1 the largest. Now, consider the event that there are at least order n of such points. Namely, call $p_\delta = \mathbb{P}\{d_0 < \delta\}$, and fix a positive constant $\lambda < \alpha p_\delta / 2$. Call the event $S_n = \{k_n \geq \lambda n\}$. A simple consequence of the law of large numbers is that $\mathbb{P}(S_n) \to 1$.

We now recursively define a sequence of paths whose expected actions inside this event are strictly increasing. First, define the path

$$\gamma_0(t) = \begin{cases} t, & 0 \leq t \leq \ell_n \\ \eta_{\ell_n}(t), & \ell_n < t \leq \ell_n + n - \lfloor an \rfloor \\ t - (n - \lfloor an \rfloor), & \ell_n + n - \lfloor an \rfloor < t \leq n, \end{cases} \tag{1.80}$$

namely, the path which moves balistically to ℓ_n, remains optimally on the edge $\{\ell_n, \ell_n + 1\}$, and then moves balistically to $\lfloor \alpha n \rfloor$. (Recall definition 1.9 of $\eta_x(\cdot)$, the path constrained to $\{x, x+1\}$ which maximizes action). Similarly to the discussion of the naive path in section 1.3.3, it is easy to calculate that the action of this path, which we call a_0, divided by n converges to the line $c(1-\alpha)$ almost surely. Since $A(a_0)/n$ is bounded for all n,

$$\mathbb{E}\left[\frac{a_0}{n}\right] \to c(1-\alpha).$$

For simplicity, we call $m = \lfloor \lambda n \rfloor$. We now define a recursive "borrowing" argument in the event S_n, where we take steps from the portion of the path constrained to $\{\ell_n, \ell_n + 1\}$ to spend more time at the sites $x_1 > x_2 > \cdots > x_m > an^{2/3}$, in order to improve the final ballistic portion of the path. We can do this in a way such that there is an expected positive gain in the action.

Given the path γ_{i-1}, $i = 1, \dots, m$, define

$$\tau_i = \inf\Big\{ t \in (an^{2/3}, n] \,\Big|\, \gamma_{i-1}(t) = x_i + 1 \Big\} \cdot \mathbb{1}_{S_n},$$

the first (and only) time γ_{i-1} hits the site $x_i + 1$ (which will occur inside the event S_n), and define the random variables

$$\sigma_i = \mathbb{1}_{S_n} \cdot \begin{cases} \inf\big\{ t > 0 \,\big|\, \eta_{x_i}(\tau_i - t) \neq x_i + 1 \big\} - 1 & \text{if } \eta_{x_i}(\tau_i) = x_i + 1 \\ 0 & \text{if } \eta_{x_i}(\tau_i) = x_i \end{cases}. \tag{1.81}$$

The non-negative random variable σ_i describes the number of steps we will borrow to remain at $x_i + 1$. The key observation is that if σ_i is positive, then a path which moves through x_i at time $\tau_i - \sigma_i - 1$ and then remains at $x_i + 1$ on $[\tau_i - \sigma_i, \tau_i]$ visits these sites with the correct signs, guaranteeing a positive contribution to the action at each of the $\sigma_i + 2$ steps. We are repeating this process $m = \lfloor \lambda n \rfloor$ times, and we want to guarantee that we have enough steps to borrow each time. Since γ_0 spends $n - \lfloor \alpha n \rfloor$ extra steps on the edge $\{\ell_n, \ell_n + 1\}$, we will be safe if we require $\sigma_i \le \frac{n - \lfloor \alpha n \rfloor}{m}$. Otherwise, we do no borrowing at that step. For n large enough, this is satisfied if we take $\sigma_i \le \frac{1-\alpha}{\lambda}$. And note that $\frac{1-\alpha}{\lambda} > 2$, by our choice of $\alpha = 1/2$ and our restriction on λ from above. So, for simplicity, we will only allow $\sigma_i \le 2$. Otherwise, if $\sigma_i > 2$, then set $\sigma_i = 0$. That is, redefine σ_i to be $\sigma_i \mathbb{1}\{\sigma_i \le 2\}$.

To denote the borrowing, we define the operator

$$\Phi_{j,k,l}(\gamma)(t) = \begin{cases} \gamma(k) & \text{for } k - l \le t \le k \\ \gamma(t + l) & \text{for } j - l \le t < k - l \\ \gamma(t) & \text{otherwise} \end{cases}$$

that shifts the portion of γ on $[j, k]$ left l units, and sets γ equal its value at k for the l units to the right of the shift, up to k.

Let $\tilde{\tau}_i = \max\big\{ t \in [0, n] \,\big|\, \gamma_{i-1}(t) = \ell_n + 1 \big\} \cdot \mathbb{1}_{S_n}$, the last time γ_{i-1} is on the edge $\{\ell_n, \ell_n + 1\}$ before moving ballistically towards x_i. Then for $i = 1, \dots, m$, we recursively define the random paths

$$\gamma_i = \Phi_{\tilde{\tau}_i, \tau_i, \sigma_i}(\gamma_{i-1}) \cdot \mathbb{1}_{S_n} + \gamma_{i-1} \mathbb{1}_{S_n^c}. \tag{1.82}$$

With this definition, in S_n, γ_i borrows σ_i steps from the portion of γ_{i-1} that remains on the edge

$\{\ell_n, \ell_n+1\}$, allowing γ_i to reach x_i+1 σ_i steps earlier and remain there for σ_i steps, which are beneficial by the definition of σ_i. We can compute the difference of actions for subsequent γ_i's in S_n.

$$\begin{aligned} a_i - a_{i-1} = \Bigg(& -A\Big(\eta_{\ell_n}\big|_{[\tilde{\tau}_i-\sigma_i,\tilde{\tau}_i)}\Big) - \sum_{k=\tilde{\tau}_i}^{\tau_i-1} F(\gamma_{i-1}(k))b(k) \\ & + \sum_{k=\tilde{\tau}_i-\sigma_i}^{\tau_i-\sigma_i-2} F(\gamma_{i-1}(k+\sigma_i))b(k) \\ & + F(x_i)b(\tau_i-\sigma_i-1) + \sum_{k=\tau_i-\sigma_i}^{\tau_i} F(x_i+1)b(k) \Bigg) \cdot \mathbb{1}_{S_n} \end{aligned} \tag{1.83}$$

Consider the $\mathbb{P}$-preserving function on environments which flips the sign of each time variable, $h : F \cdot b \to F \cdot -b$, and consider the function on S_n which switches the values of $F(x_i)$ and $F(x_i+1)$ for the environment $\omega = F \cdot b$

$$g_i : F \cdot b \longmapsto F^{x_i \leftrightarrow x_i+1} \cdot b,$$

where

$$F^{a\leftrightarrow b}(x) = \begin{cases} F(a) & \text{if } x = b \\ F(b) & \text{if } x = a \\ F(x) & \text{otherwise .} \end{cases}$$

The function g_i is a measure-preserving bijection from S_n to itself, as it maintains the value of the discrepancy d_{x_i}. And, since the sites $x_1, \dots, x_m$ can not be neighbors, the function $G_k = g_k \circ g_{k-1} \circ \cdots \circ g_1$ is also a measure preserving bijection on S_n for $k = 1, \dots, m$, as is $H_k := G_k \circ h$.

Calling $\tilde{\omega}_i = H_i(\omega)$, note that the path γ_i on ω is exactly equal to γ_i on $\tilde{\omega}_i$ on the interval $[\tilde{\tau}_{i+1}, n]$ for any $\omega \in S_n$ and for each $i = 1, \dots, m$, since $\sigma_j(\omega) = \sigma_j(\tilde{\omega}_i)$ for all $j \le i$. In fact, γ_i will be the same path in ω and $\tilde{\omega}_j$ for all $j = i, \dots, m$ on $[\tilde{\tau}_{i+1}, n]$. Note that the paths will not be equal on $(\ell_n, \tilde{\tau}_{i+1})$, as f switches the signs of b but not of $F(\ell_n)$ and $F(\ell_n+1)$, so that the optimal path η_{ℓ_n} on the edge $\{\ell_n, \ell_n+1\}$ will disagree in ω and $\tilde{\omega}_i$. We can use this to calculate the expectation of terms from equation

(1.83).

$$
\begin{aligned}
\mathbb{E}\left[\left(\sum_{k=\tilde{\tau}_i}^{\tau_i-1} F(\gamma_{i-1}(k))b(k)\right)\mathbb{1}_{S_n}\right] &= \int_{S_n}\sum_{k=\tilde{\tau}_i}^{\tau_i-1}\omega(k,\gamma_{i-1}(k))\mathrm{d}\omega \\
&= \frac{1}{2}\int_{S_n}\sum_{k=\tilde{\tau}_i}^{\tau_i-1}\big(\omega(k,\gamma_{i-1}(k)) + H_{i-1}(\omega)(k,\gamma_{i-1}(k))\big)\mathrm{d}\omega \\
&= \frac{1}{2}\int_{S_n}\sum_{k=\tilde{\tau}_i}^{\tau_i-1} F(\gamma_{i-1}(k))(b(k) + -b(k))\mathrm{d}\omega \\
&= 0, \qquad (1.84)
\end{aligned}
$$

since $H_{i-1}(\omega)$ fixes $\{F\}$ on $\gamma_{i-1}\big([\tilde{\tau}_i,\tau_i)\big) = [\ell_n+1, x_i]$ (γ_{i-1} is simply ballistic on this piece). Similarly, $H_i(\omega)$ fixes σ_i, γ_{i-1} on $[\tilde{\tau}_i,\tau_i)$ (which is ballistic), $\{F\}$ on $[\ell_n+1, x_i)$, and switches signs of $\{b\}$, and so we have

$$
\mathbb{E}\left[\left(\sum_{k=\tilde{\tau}_i-\sigma_i}^{\tau_i-\sigma_i-2} F(\gamma_{i-1}(k+\sigma_i))b(k)\right)\mathbb{1}_{S_n}\right] = 0.
$$

The key observation now is in the final terms of equation (1.83). By construction of σ_i, on the event $S_n(i) = \{\sigma_i > 0\}$, all terms below are positive, so that

$$
\begin{aligned}
\Big[F(x_i)b(\tau_i-\sigma_i-1) + \sum_{k=\tau_i-\sigma_i}^{\tau_i} F(x_i+1)b(k)\Big]\mathbb{1}_{S_n(i)} \\
&= \Big[\big|F(x_i)\big| + \big|F(x_i+1)\big|(\sigma_i+1)\Big]\mathbb{1}_{S_n(i)} \\
&\geq (c-d_{x_i})(\sigma_i+2)\mathbb{1}_{S_n(i)} \qquad (1.85)
\end{aligned}
$$

where we have used equation 1.10. Since we can bound the first term in equation (1.83) very simply by $-c\sigma_i$, we are able to bound $\mathbb{E}[(a_i - a_{i-1})\mathbb{1}_{S_n}]$.

$$
\begin{aligned}
\mathbb{E}[(a_i - a_{i-1})\mathbb{1}_{S_n}] &\geq \mathbb{E}\Big[(-c\sigma_i)\mathbb{1}_{S_n} + (c-d_{x_i})(\sigma_i+2)\mathbb{1}_{S_n(i)}\Big] \\
&\geq \mathbb{E}\Big[\big(2c - \delta(\sigma_i+2)\big)\mathbb{1}_{S_n(i)}\Big] \\
&\geq (2c-4\delta)\mathbb{P}(S_n(i)), \qquad (1.86)
\end{aligned}
$$

where we are using the fact that $\sigma_i \leq 2$ by definition. From the definition of σ_i and the fact that the event S_n is in the sigma field generated by $\{F(i)\}_{i=an^{2/3}}^n$, it is simple to calculate that $\mathbb{P}(S_n(i)) = \frac{3}{16}\mathbb{P}(S_n)$.

Finally, we have

$$\begin{aligned}
\frac{1}{n}\mathbb{E}[\overline{A}(n, \lfloor \alpha n \rfloor)] &\geq \frac{1}{n}\mathbb{E}[a_m \mathbb{1}_{S_n}] \\
&= \frac{1}{n}\mathbb{E}[a_0 \mathbb{1}_{S_n}] + \frac{1}{n}\sum_{i=1}^{m} \mathbb{E}[(a_i - a_{i-1})\mathbb{1}_{S_n}] \\
&\geq \frac{1}{n}\mathbb{E}[a_0 \mathbb{1}_{S_n}] + \frac{m}{n}\left(\frac{3}{8}(c - 2\delta)\right)\mathbb{P}(S_n) \\
&\longrightarrow c(1-\alpha) + \lambda\frac{3}{8}(c - 2\delta), \qquad (1.87)
\end{aligned}$$

since $\mathbb{P}(S_n) \to 1$. And since $\frac{1}{n}\mathbb{E}[\overline{A}(n, \lfloor \alpha n \rfloor)] \longrightarrow a(\alpha)$ and since δ was chosen less than $c/2$, we have that $a(\alpha)$ is strictly greater than $c(1-\alpha)$, proving non-linearity. □

1.7 Proof of Theorem 1.1.3

We assume that $(X_k)_{k\in\mathbb{Z}}$ is a collection of i.i.d. random variables with density f. We assume that f is symmetric with compact support $[-c, c]$, and that it is continuously differentiable on its support (with one sided limits at the end points). For every $k \in \mathbb{Z}$, we define $Y_k = 2c - |X_k - X_{k+1}|$. Let p denote the density of Y_k (which we will express later in terms of f). For each $n \in \mathbb{N}$, we introduce the rescaled point process μ_n as a random measure:

$$\mu_n = \sum_{k\in\mathbb{Z}} \delta_{(n^{-2/3}k, n^{1/3}Y_k)}$$

Theorem 1.7.1. *As $n \to \infty$, the distribution of μ_n converges weakly in the vague topology to the distibution of μ, the Poisson process with driving measure ν absolutely continuous w.r.t. Lebesgue measure with density given by:*

$$\frac{d\nu(\mathrm{d}t \times \mathrm{d}y)}{\mathrm{d}t \times \mathrm{d}y} = \begin{cases} \frac{p'(0+)}{2}y, & y > 0 \\ 0, & y \leq 0. \end{cases}$$

1.7.1 Basics of Poisson Approximation

To prove Theorem 1.1.3 we need some terminology and theory on random point processes that we proceed to describe. This description is based on [22]. We restrict our attention to point processes in $\mathbb{R}^2$ since that is the class of point processes we deal with in this section. The space $\mathfrak{N}$ consists of all integer-valued (nonnegative locally bounded) measures defined on the set $\mathcal{B}$ of bounded Borel sets in $\mathbb{R}^2$. This set is equipped with σ-algebra $\mathcal{N}$ generated by maps $\mu \mapsto \mu(B)$, $B \in \mathcal{B}$. A point process μ is a measurable map from a probability space $(\Omega, \mathcal{F}, \mathsf{P})$ to $(\mathfrak{N}, \mathcal{N})$. Its distribution is, of course, the pushforward of P under μ. A point process μ is called a.s.-simple if with probability 1, all the atoms of μ have weight 1, which corresponds to the situation where no two points of the point process coincide.

The natural topology on $\mathfrak{N}$ is vague topology. Its base is given by finite intersections of $\mathfrak{N}$-sets of the form

$$\{\mu : s < \int f \mathrm{d}\mu < t\},$$

where $s, t \in \mathbb{R}$, and f is any continuous function with compact support.

For any random point process μ we denote

$$\mathcal{B}_\mu = \{B \in \mathcal{B} : \ \mu(\partial B) = 0 \text{ a.s.}\},$$

where ∂B denotes the boundary of B.

A collection $\mathcal{U}$ of bounded Borel sets in $\mathbb{R}^2$ is called a DC(dissecting and covering)-ring if it is a ring such that for any $B \in \mathcal{B}$ and any $\varepsilon > 0$, there is a finite cover of B by $\mathcal{U}$-sets of diameter less than ε. A DC-semiring is a semiring with the same property. An example of a DC-semiring is the collection of all rectangles $[a_1, a_2) \times [b_1, b_2)$. The collection of all finite unions of rectangles is a DC-ring.

It turns out that to check the weak convergence to an a.s.-simple point process in vague topology it is essentially sufficient to check the convergence of the avoidance function that computes the probability that there is no points inside a given set. The following theorem is a specific case of Theorem 4.7 in [22].

Theorem 1.7.2. *Let $(\mu_n)_{n\in\mathbb{N}}$ and μ be point processes in $\mathbb{R}^2$ and assume that μ is a.s.-simple. Suppose that $\mathcal{U} \subset \mathcal{B}_\mu$ is a DC-ring and $\mathcal{I} \subset \mathcal{B}_\mu$ is a DC-semiring. Suppose further that*

$$\lim_{n\to\infty} \mathsf{P}\{\mu_n(U) = 0\} = \mathsf{P}\{\mu(U) = 0\}, \quad U \in \mathcal{U}, \tag{1.88}$$

and

$$\limsup_{n\to\infty} \mathsf{E}\mu_n(I) \le \mathsf{E}\mu(I) < \infty, \quad I \in \mathcal{I}. \tag{1.89}$$

Then μ_n converges to μ weakly in vague topology.

1.7.2 Proof of Theorem 1.1.3

Let us take $\mathcal{I}$ to be the semiring of rectangles, $\mathcal{U}$ to be the ring of finite unions thereof, and check conditions (1.88) and (1.89) of Theorem 1.7.2.

Take a rectangle $I = [a_1, a_2) \times [b_1, b_2)$ and write

$$\mathsf{E}\mu_n(I) = \sum_{n^{2/3}a_1 \le k < n^{2/3}a_2} \mathsf{P}\{Y_k \in [b_1 n^{-1/3}, b_2 n^{-1/3})\} \tag{1.90}$$

Lemma 1.7.3. *For any k, the distribution of Y_k has density w.r.t. Lebesgue measure given by*

$$p(y) = \begin{cases} 0 & y < 0 \\ 2f \star f(2c - y) & 0 \le y \le 2c \\ 0 & 2c < y. \end{cases}$$

In fact, $X_0 - X_1$ is distributed as $X_0 + X_1$, which has density $f \star f$, the convolution of f with itself, with support $[-2c, 2c]$. Since this is symmetric, $|X_0 + X_1|$ is concentrated on $[0, 2c]$ with density at $y \in [0, 2c]$ given by $2f \star f(y)$. The Lemma follows by composing this density with a reflection and then a shift to the right by $2c$.

Now, call F_X and F_Y the CDF's of X_k and Y_k, respectively. F_X has support $[-c, c]$, F_Y has support $[0, 2c]$, and since f is continuously differentiable on $[-c, c]$, p is continuously differentiable on $[0, 2c]$ (convolution maintains the same level of smoothness). Then both F_X and F_Y are twice continuously differentiable, and so by Taylor's theorem, for x greater than (but near) $-c$,

$$F_X(x) = f(-c)(x + c) + \frac{f'(-c+)}{2}(x + c)^2 + o\Big((x + c)^2\Big),$$

and for $x \in [0, 2c]$ (near 0)

$$F_Y(x) = p(0)x + \frac{p'(0+)}{2}x^2 + o(x^2) = \frac{p'(0+)}{2}x^2 + o(x^2),$$

and

since $p(0) = 0$ as the following computation shows:

$$\begin{aligned}
p(0) = 2f \star f(2c) &= \int_{-\infty}^{\infty} f(2c-\tau) f(\tau) \mathrm{d}\tau \\
&= \int_{-\infty}^{\infty} \big(f(2c-\tau)\mathbb{1}\{2c-\tau \in [-c,c]\}\big)\big(f(\tau)\mathbb{1}\{\tau \in [-c,c]\}\big)\mathrm{d}\tau \\
&= \int_{-\infty}^{\infty} \mathbb{1}\{\tau \in [c,3c] \cap [-c,c]\} f(2c-\tau) f(\tau) \mathrm{d}\tau = 0
\end{aligned}$$

Now for large n, we can use the Lemma to rewrite (1.90) as

$$\begin{aligned}
\mathsf{E}\mu_n(I) &= ([n^{2/3}a_2] - [n^{2/3}a_1])\Big(F_Y(b_1 n^{-1/3}) - F_Y(b_2 n^{-1/3})\Big) \\
&= ([n^{2/3}a_2] - [n^{2/3}a_1])\left(\frac{p'(0+)}{2} n^{-2/3}(b_2-b_1)^2 + o(n^{-2/3})\right),
\end{aligned}$$

so that

$$\lim_{n\to\infty} \mathsf{E}\mu_n(I) = \frac{p'(0+)}{2}(a_2-a_1)(b_2^2-b_1^2) = p'(0+)\int_{a_1}^{a_2}\int_{b_1}^{b_2} y \,\mathrm{d}y\mathrm{d}t = \mathsf{E}\mu(I),$$

and(1.89) holds true.

To prove (1.88), we take arbitrary disjoint rectangles

$$U_i = [a_1^{(i)}, a_2^{(i)}) \times [b_1^{(i)}, b_2^{(i)}), \quad i = 1, \ldots, m,$$

define

$$U = \bigcup_{i=1}^m U_i,$$

and compute

$$\lim_{n\to\infty} \mathsf{P}\big\{\mu_n(U) = 0\big\}.$$

We quote a Theorem 4.3 from [6]

Theorem 1.7.4. . *If $X_1, \ldots, X_n$ are m-dependent r.v.'s taking values 0 and 1, then for any function $h : \mathbb{Z}_+ \to [-1,1]$,*

$$\left|\mathsf{E}h\left(\sum_{i=1}^n X_i\right) - \sum_{k=0}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!}h(k)\right| \le 6\min\{\lambda^{-1/2}, 1\}\left[\sum_{i,j:i\neq j} \mathsf{cov}(X_i,X_j) + (4m+1)\sum_{i=1}^n p_i^2\right],$$

where $p_i = \mathsf{P}\{X_i = 1\}$, and $\lambda = \sum_{i=1}^n p_i$.

We shall apply this theorem to our situation. Notice that

$$\mu_n(U) = \sum_k Z_{n,k}$$

is a finite sum of 1-dependent r.v.'s

$$Z_{n,k} = \mathbb{1}_{\{(n^{-2/3}k, n^{1/3}Y_k)\in U\}}, \quad k \in \mathbb{Z}$$

since only $k \in \bigcup_i [n^{2/3}a_1^{(i)}, n^{2/3}a_2^{(i)})$ contribute to this sum. Therefore we can take $m = 1$ in Theorem 1.7.4. Next,

$$\mathsf{P}\{Z_{n,k} = 1\} = \mathsf{P}\{Y_k \in n^{-1/3}U(n^{-2/3}k)\},$$

where $U(x) = \{y : \ (x, y) \in U\}$. Therefore, for large n,

$$p_{n,k} = \mathsf{P}\{Z_{n,k} = 1\} = \int_{n^{-1/3}U(n^{-2/3}k)} \frac{p'(0+)y}{2}\mathrm{d}y = n^{-2/3}\int_{U(n^{-2/3}k)} \frac{p'(0+)y}{2}\mathrm{d}y,$$

so that

$$\lambda_n = \sum_{k\in\mathbb{Z}} \mathsf{P}\{Z_{n,k} = 1\} = \sum_k n^{-2/3}\int_{U(n^{-2/3}k)} \frac{p'(0+)y}{2}\mathrm{d}y \to \lambda, \quad n \to \infty,$$

where

$$\lambda = \int_{\mathbb{R}} \mathrm{d}x \int_{U(x)} \frac{p'(0+)y}{2}\mathrm{d}y = \int_U \frac{p'(0+)y}{2}\mathrm{d}x\mathrm{d}y.$$

Since $\lim_{n\to\infty}\sup_k p_{n,k} = 0$ as $n \to \infty$, we also have

$$\lim_{n\to\infty} \sum_{k\in\mathbb{Z}} p_{n,k}^2 = 0.$$

Most covariances in the estimate provided by Theorem 1.7.4 are equal to zero in our case. The only nontrivial contribution comes from

$$\begin{aligned}
\mathsf{cov}(Z_{n,k}, Z_{n,k+1}) &\le \mathsf{P}\{Z_{n,k} = 1,\ Z_{n,k+1} = 1\} \\
&\le \mathsf{P}\{Y_k \in n^{-1/3}U(n^{-2/3}k),\ Y_{k+1} \in n^{-1/3}U(n^{-2/3}(k+1))\} \\
&\le \mathsf{P}\{Y_k \le n^{-1/3}b_*,\ Y_{k+1} \le n^{-1/3}b_*\},
\end{aligned}$$

where $b_* = \max_i b_2^{(i)}$. For large n, the r.h.s. is bounded by

$$2\mathsf{P}\{X_{k+1} - X_k \geq 2c - n^{-1/3}b_*,\ X_{k+1} - X_{k+2} \geq 2c - n^{-1/3}b_*\},$$

where the factor of 2 appears due to the symmetry in $|X_{k+1} - X_k|$. Since X_k, X_{k+1}, and X_{k+2} are i.i.d., the latter probability equals

$$\begin{aligned}
&\int_{c-n^{-1/3}b_*}^{c} \mathsf{P}\{-c \leq X_k \leq x_{k+1} - 2c + n^{-1/3}b_*\}\mathsf{P}\{-c \leq X_{k+2} \leq x_{k+1} - 2c + n^{-1/3}b_*\}f(x_{k+1})\mathrm{d}x_{k+1} \\
&= \int_{c-n^{-1/3}b_*}^{c} F_X(x_{k+1} - 2c + n^{-1/3}b_*)^2 f(x_{k+1})\mathrm{d}x_{k+1} \\
&= \int_{-c}^{-c+n^{-1/3}b_*} F_X(x)^2 f(x + 2c - n^{-1/3}b_*)\mathrm{d}x,
\end{aligned}$$

where in the last line we used a change of variable. Note that for large n, $F_X(x)$ in the integral can be bounded by a constant C times $x + c$, in light of the Taylor expansion above. So, writing $M_f = \sup_{t\in[-c,c]} f(t)$, we can bound the integral by

$$\begin{aligned}
&M_f \int_{-c}^{-c+n^{-1/3}b_*} \big(C(x+c)\big)^2 \mathrm{d}x \\
&= M_f \int_0^{n^{-1/3}b_*} (Cx)^2 \mathrm{d}x = O(n^{-1})
\end{aligned}$$

.

Since there are $O(n^{2/3})$ indices k contributing nonzero covariances we conclude that the total contribution from the covariance term is $O(n^{-1/3})$. This concludes the proof that the estimate provided by Theorem 1.7.4 converges to 0 as $n \to \infty$. We now take $h(k) = \mathbb{1}_{k=0}$, so that $\mathsf{E}h(\sum Z_{n,k})$ is exactly the avoidance function for measure μ_n, which completes the proof of Theorem 1.1.3. □

1.8 Questions / Future Work

The polymer measure is considered on the set of length n directed polymers, but we can make sense of the optimal path of infinite length, up to a given record discrepancy. Given a realization of space variables $\{F(x)\}$, call the locations of record discrepancies $\{Y_i\}$ in increasing distance from the origin. That is, $Y_0 = 0$, and

$$Y_{i+1} = \operatorname*{argmin}_{\{y\in\mathbb{Z}\,|\, d_y < d_{Y_i}\}} |y|.$$

Consider the set Π_k of infinite paths which "end" at the edge $\{Y_k, Y_k+1\}$, that is, paths $\gamma : \mathbb{N}_0 \to \mathbb{Z}$ such that $\gamma(t) = \eta_{Y_k}(t)$ for all t greater than some $\tau_\gamma > 0$. Of course, the action of these paths are infinite, but even so, we can find the "optimal" such path. We can do this because we can compare any two paths in Π_k as follows.

If $\gamma_1(t) = \gamma_2(t) = \eta_{Y_k}(t)$ for all $t > \tau$, then say $\gamma_1 \le \gamma_2$ if $A(\gamma_1, 0, \tau) \le A(\gamma_2, 0, \tau)$. This defines a total ordering on Π_k. For almost every realization of ω we can choose a unique greatest element, $\overline{\gamma}_k$.

Conjecture 1.8.1. *Define* $\tau_k = \min\big\{\tau \in \mathbb{N} \,\big|\, \overline{\gamma}_k(t) = \eta_{Y_k}(t)\ \forall t \ge \tau_k\big\}$. *For* $\mathbb{P}_1$*-a.e.* $\{F\}$,

$$\frac{Y_k}{\tau_k} \longrightarrow \alpha_{crit}$$

for $\mathbb{P}_2$*-a.e.* $\{b\}$.

An interesting question is what percentage of time this best path spends at the previous discrepancy before reaching Y_k.

Question 1.8.2. *Call* $L_k = \big\{t \in \mathbb{N} \,\big|\, \gamma_k(t) \in \{Y_{k-1}, Y_{k-1}+1\}\big\}$, *and define* $\delta_k = \max L_k - \min L_k$. *For a.e.* ω, *does*

$$\frac{\delta_k}{\tau_k} \longrightarrow 0?$$

We also conjecture that this best path stabilizes as $k \to \infty$. More precisely, we conjecture the following.

Conjecture 1.8.3. *For all* $\tau > 0$, *for all* $\epsilon > 0$, *there exists* $K = K(\tau, \epsilon)$ *such that for any* $k_1, k_2 > K$,

$$\mathbb{P}\Big(\overline{\gamma}_{k_1}(t) = \overline{\gamma}_{k_2}(t) \text{ for all } t = 0, \dots, \tau\Big) > 1 - \epsilon.$$

1.9 Appendix

1.9.1 Probability

Here two simple probabilistic lemmas:

Lemma 1.9.1. *If* $X_n \xrightarrow{p} X$ *and* $Y_n \xrightarrow{p} \infty$, *then* $X_{Y_n} \xrightarrow{p} X$.

Proof. For $\epsilon > 0$ and $\delta > 0$, there exists an N such that for all $n > N$,

$$\mathbb{P}\big(|X_n - X| > \epsilon\big) < \delta.$$

Let M be such that for all $m > M$,

$$\mathbb{P}(Y_m \leq N) < \delta.$$

Then we can write $\mathbb{P}\Big(\big|X_{Y_m} - X\big| > \epsilon\Big)$ as

$$\mathbb{P}\Big(\big|X_{Y_m} - X\big| > \epsilon \,\Big|\, Y_m > N\Big)\mathbb{P}(Y_m > N) \tag{1.91}$$

$$+\mathbb{P}\Big(\big|X_{Y_m} - X\big| > \epsilon \,\Big|\, Y_m \leq N\Big)\mathbb{P}(Y_m \leq N), \tag{1.92}$$

which is less than 2δ. □

Lemma 1.9.2. *Suppose $\{X_n\}$ converges almost surely to a constant L, and let $\{Y_n\}$ be a sequence such that $X_n \stackrel{d}{=} Y_n$ for each n. Then Y_n converges to L along a subsequence.*

Proof. Let c_j be any sequence increasing to L. Then by the convergence of X_n, we can choose an increasing subsequence n_j such that for each j,

$$\mathbb{P}(Y_{n_j} \leq c_j) = \mathbb{P}(X_{n_j} \leq c_j) \leq 2^{-j}.$$

The Borel-Cantelli lemma gives that $\liminf Y_{n_j} \geq L$ a.s. Performing the same argument with a sequence decreasing towards L gives that $\lim Y_{n_j} = L$ almost surely. □

This is Kingman's subadditive ergodic theorem [24, 25]:

Theorem 1.9.3. *Suppose $(X_{m,n})$ is a sequence of random variables indexed by integers $0 \leq m < n$, such that*

1. $$X_{l,n} \leq X_{l,m} + X_{m,n} \text{ whenever } 0 \leq l < m < n.$$

2. *The joint distributions of $\big(X_{m+1,n+1}, 0 \leq m < n\big)$ are the same as those of $\big(X_{m,n}, 0 \leq m < n\big)$.*

3. *For each n, $\mathbb{E}\big|X_{0,n}\big| < \infty$ and $\mathbb{E}X_{0,n} \geq -cn$ for some constant c.*

Then the limit $X = \lim_{n\to\infty} \frac{1}{n} X_{0,n}$ exists almost surely and in L^1, and

$$\mathbb{E}X = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}X_{0,n} = \inf_n \frac{1}{n}\mathbb{E}X_{0,n}$$

Furthermore, if for each $k \in \mathbb{N}$, $\{X_{nk,(n+1)k}\}$ is stationary and ergodic, $X = \mathbb{E}X$ a.s.

Chapter 2

Analyticity of the Effective Velocity for Ballistic RWRE

2.1 Introduction

For $d \geq 1$, the random walk in a random environment (RWRE) on $\mathbb{Z}^d$ is defined as follows. Let $\mathcal{V}$ denote the set of unit vectors in $\mathbb{Z}^d$, $\left\{e \in \mathbb{Z}^d \,\middle|\, |e| = 1\right\} = \{\pm e_1, \dots, \pm e_d\}$, and call $\mathcal{P}^d_\kappa$ the set of uniformly elliptic probability vectors $(p(e))_{e\in\mathcal{V}}$, $\sum p(e) = 1$, where $p(e) > \kappa$ for all $e \in \mathcal{V}$ for some $\kappa > 0$. An environment $\omega = \big(\omega(z,\cdot)\big)_{z\in\mathbb{Z}^d}$ is an element of $\Omega = (\mathcal{P}^d_\kappa)^{\mathbb{Z}^d}$, upon which we put an i.i.d. (product) measure $\mathbb{P} = \nu^{\otimes\mathbb{Z}^d}$, where ν is a probability measure on $\mathcal{P}^d_\kappa$.

For a fixed environment $\omega \in \Omega$ and $z_0 \in \mathbb{Z}^d$, the "quenched" random walk in ω starting at z_0 is a time homogeneous Markov chain with the following transition probabilities:

$$P^\omega_{z_0}\big(X_{n+1} = z + e \,|\, X_n = z\big) = \omega(z, e) \;\; \forall z \in \mathbb{Z}^d,\ e \in \mathcal{V}$$

and $P^\omega_{z_0}(X_0 = z_0) = 1$.

The "annealed" law is the semi-direct product $P_z = \mathbb{P} \times P^\omega_z$ defined by

$$P_z(\cdot) = \int P^\omega_z(\cdot)\mathbb{P}(\mathrm{d}\omega).$$

Under the so called Kalikow's condition [21], a law of large numbers for multidimensional random walk in random environment was established by Sznitman and Zerner [34]. Kalikow's condition is a drift condition on $\mathbb{P}$, and under it, X_n/n converges P_0-almost surely to a non-zero deterministic velocity v.

To prove this, Sznitman and Zerner constructed renewal times $\{\tau_n\}$ in the direction l of drift, where τ_1 is essentially the first time such that after τ_1, $X_n \cdot l$ does not backtrack past $X_{\tau_1} \cdot l$.

The environments we consider exhibit strong drift, the so-called "non-nestling" case, where $\mathbb{P}$-a.s. for all $z \in \mathbb{Z}^d$,

$$\sum_{e \in \mathcal{V}} \omega(z, e) \cdot l \geq \epsilon$$

for some $\epsilon > 0$. Specifically, in this paper we investigate an environment considered by Sabot in [30], where the random environment is a small i.i.d. perturbation of a fixed homogeneous environment p_0, i.e.

$$\omega^\gamma(z, e) = p_0(e) + \gamma \xi(z, e),$$

with γ small (controlling the strength of the perturbation) and ξ random. With a non-zero mean drift, for γ small enough, this walk satisfies Kalikow's condition, and Sabot [30] calculates a quantitative expansion of the limiting velocity to second order for $d \geq 2$,

$$v^\gamma = d_0 + \gamma d_1 + \gamma^2 d_2 + O\Big(\gamma^{3-\epsilon}\Big),$$

where $d_0 = \sum_{e \in \mathcal{V}} e \cdot p(e)$ and $d_1 = \sum_{e \in \mathcal{V}} e \cdot \mathbb{E}[\xi(z, e)]$. In dimension $d = 1$, when $d_0 = 0$, the second order term d_2 has a discontinuity at $\gamma = 0$, though in the one dimensional case, the velocity is explicitly known [32]. Under the stronger assumption that $d_0 \neq 0$, Sabot calculates an expansion to third order. Sabot produces this expansion via Kalikow's random walk, which is a walk in a non-random environment, constructed by taking an average of the random environment, weighted by the Green's function of the RWRE stopped upon exiting some bounded set (or killed with rate $\delta < 1$).

Under the assumption $d_0 \cdot e > 0$ for $e \in \mathcal{V}$, we prove

Theorem 2.1.1. *The map $\gamma \to v^\gamma \cdot e$ is analytic in an interval around 0.*

The method employed is to consider the often useful "environment viewed from the particle" Markov chain, but for the space of ξ's (restricted to a half space), and at the stopping times where the walk advances in a component of the direction of drift e, under the assumption that $d_0 \cdot e \neq 0$. For small γ, we analyze the Markov operator $\mathcal{L}(\gamma)$ of this chain as a linear operator on the Banach space of Lipschitz functions, with a metric chosen to give certain contractive properties. We can extend the operator to include complex values of γ, and we prove that in the space of linear operators, the map $\gamma \mapsto \mathcal{L}(\gamma)$ is holomorphic in a neighborhood of zero. Perturbation theory allows us to conclude that the operator has a spectral gap (quasi-compactness), and that the unique invariant measure for this configuration Markov chain is a holomorphic function of γ. This technique is inspired by Dolgopyat and Liverani in the study of random walks in a Markovian environment [9].

Recently, Guo [12] recovered Sabot's results to first order, in the case where the underlying environment p_0 is random (and not homogeneous). He considers two cases. The first case is when the underlying environment is balanced, and the perturbed environment satisfies a Kalikow-type condition. The second case is when the original environment satisfies Sznitman's (T') condition [33] (conjectured to be equivalent to ballisticity). In both cases, Guo proves law of large numbers, and differentiabilty of $\gamma \mapsto v^\gamma$ at 0. In the balanced case, the results of Lawler [27] give the so-called Einstein relation. It would be nice to extend analyticity results to the random underlying environment cases considered by Guo, particularly in the balanced case, when $d_0 = 0$.

2.2 Background and Notation

Denote the canonical basis vectors for the lattice $\mathbb{Z}^d$, $d \geq 1$, by $(e_1, \ldots, e_d)$, and denote by $\mathcal{V}$ the set of allowable jumps for the nearest neighbor random walk, $\mathcal{V} = \left\{ e \in \mathbb{Z}^d \,\middle|\, |e| = 1 \right\} = \{\pm e_1, \ldots, \pm e_d\}$.

Call the set of uniformly elliptic jump probabilities

$$\mathcal{P}^d_\kappa = \left\{ \big(p(e)\big)_{e\in\mathcal{V}} \,\middle|\, \sum_e p(e) = 1,\ p(e) \in (\kappa, 1]\ \forall e \in \mathcal{V} \right\},$$

with ellipticity constant $\kappa > 0$. We fix a $p_0 \in \mathcal{P}^d_\kappa$, and we assume the following. Calling the drift of p_0

$$d_0 = \sum_{e\in\mathcal{V}} e \cdot p_0(e),$$

we assume that this drift has a positive component in the e_1 direction,

$$d_0 \cdot e_1 > 0.$$

Now, call the set of allowed perturbations at a point

$$\Xi = \left\{ \big((\xi_e)_{e\in\mathcal{V}}\big) \in [-1,1]^{\mathcal{V}} \,\middle|\, \sum_e \xi_e = 0 \right\},$$

and we endow this with the subspace Borel σ-algebra inherited from $[-1,1]^{\mathcal{V}}$. Let μ be a (non-trivial) probability measure on this space.

Denote the space of configurations $\Omega = \Xi^{\mathbb{Z}^d}$ with product measure $\mathbb{P} = \mu^{\otimes \mathbb{Z}^d}$, and consider elements of $\mathcal{P}_\kappa^d$ formed by perturbing p_0 by $\gamma\xi$,

$$p_{\gamma,\xi}(z,e) = p_0(e) + \gamma\xi(z,e).$$

Note that, taking $\gamma_0 > 0$ small enough so that for all $e \in \mathcal{V}$, $p_0(e) - \gamma_0 > \kappa$, we can ensure that for any $|\gamma| < \gamma_0$, we have $p_{\gamma,\xi}(z,\cdot) \in \mathcal{P}_\kappa^d$ for all z. We also take γ_0 small enough so that $p_{\gamma,\xi}$ has positive drift for any ξ, i.e., take γ_0 small enough so that $p_0(e_1) - \gamma_0 > p_0(-e_1) + \gamma_0$.

We consider the nearest-neighbor random walk on $\mathbb{Z}^d$ in the environment $p_{\gamma,\xi}$, starting from z_0, with law $P_{z_0}^{p_{\gamma,\xi}}$ and transition probabilities

$$P_{z_0}^{p_{\gamma,\xi}}\big(X_{n+1} = z + e \,|\, X_n = z\big) = p_{\gamma,\xi}(z,e) \;\; \forall z \in \mathbb{Z}^d,\; e \in \mathcal{V}.$$

We denote the annealed measure of the walk by

$$P_{z_0}^{\gamma}(\cdot) = \mathbb{E}\Big(P_{z_0}^{p_{\gamma,\xi}}(\cdot)\Big). \tag{2.1}$$

The law of large numbers proven by Sznitman and Zerner in [34] holds for random environments satisfying the so called Kalikow condition, as $p_{\gamma,\xi}$ does for all $|\gamma| < \gamma_0$ and all $\xi \in \Omega$, as noted in [30]. Thus there exists a nonzero $v^\gamma \in \mathbb{R}^d$, $v^\gamma \cdot e_1 > 0$, such that

$$\lim_{n\to\infty} \frac{X_n}{n} = v^\gamma \qquad P_\cdot^\gamma - a.s. \tag{2.2}$$

Instead of the standard environment viewed from the particle, we will consider a Markov chain on a half space of Ω, along stopping times. But we first need require some notation.

2.2.1 The Configuration Space

We denote slabs of $\mathbb{Z}^d$ perpendicular to e_1 by

$$\mathcal{A}_{i,j} = \Big\{ z \in \mathbb{Z}^d \,\Big|\, i < z \cdot e_1 \le j \Big\},$$

where $i, j \in [-\infty, +\infty]$. We will also consider slabs of configurations, $\Omega_{i,j} = \Xi^{\mathcal{A}_{i,j}}$. We denote product measure on slabs by $\mathbb{P}_{i,j} = \mu^{\otimes \mathcal{A}_{i,j}}$.

For shorthand, one subscript $\mathcal{A}_j$ denotes the left half up to j, $\mathcal{A}_{-\infty,j}$, and similarly for Ω_j and $\mathbb{P}_j$.

Omitting subscripts altogether denotes the entire space $\mathbb{Z}^d$, so that $\Omega = \Xi^{\mathbb{Z}^d}$, endowed with product measure $\mathbb{P}$.

Given configurations $\xi \in \Omega_{i,j}$ and $\eta \in \Omega_{j,k}$, we denote concatenation by

$$(\xi \star \eta)(z, \cdot) = \begin{cases} \xi(z, \cdot) & \text{for } z \in \mathcal{A}_{i,j} \\ \eta(z, \cdot) & \text{for } z \in \mathcal{A}_{j,k} \end{cases}.$$

For $x \in \mathbb{Z}^d$ with $x \cdot e_1 = k$, denote the shift operator from $\mathcal{A}_{i,j}$ to $\mathcal{A}_{i-k,j-k}$ taking x to the origin by

$$\big(\sigma^x \xi\big)(z, \cdot) = \xi(z + x, \cdot).$$

Note the following properties of the shift, $\sigma^{x+y} = \sigma^x \sigma^y$ and $\sigma^x(\xi_1 \star \xi_2) = \sigma^x \xi_1 \star \sigma^x \xi_2$. Product measure is invariant with respect to this shift.

We will mostly be considering the half space of configurations on the 'left-half" of $\mathbb{Z}^d$, $\Omega_0 = \Xi^{\mathcal{A}_0}$, which is a compact product space endowed with the standard product σ-algebra, which we denote by $\mathcal{F}_0$ (Denote the slab product sigma-algebras similarly to above). We will be considering a complex Banach space of functions on Ω_0, and study the spectrum of a quasi-compact Markov operator on this space.

2.2.2 The Configuration Markov Chain

We now define a Markov chain on Ω_0. The chain is analogous to the environment viewed from the particle Markov chain, but we are shifting the configurations ξ, as opposed to the whole environment. We shift only in one direction along hitting times of the next hyperplane, and restrict to the half plane, so that we are "gluing" a new independent slice of environment at each step.

Precisely, we fix $\xi \in \Omega$, and consider the random walk X_n in $p_{\gamma,\xi}$, started at the origin. Call the hitting time of the hyperplane $\{z \cdot e_1 = k\}$

$$T_k = \inf\{n \geq 0 \,|\, X_n \cdot e_1 = k\}.$$

With our assumptions on p_0 and γ_0 (namely, that $p_0(e_1) - \gamma_0 > p_0(-e_1) + \gamma_0$), for $\mathbb{P}$-a.e. $\xi \in \Omega$, the one dimensional (lazy) random walk $X_n \cdot e_1$ satisfies $\lim_{n\to\infty} X_n \cdot e_1 = \infty$ (see [32]). And thus for $\mathbb{P}$-a.e. $\xi \in \Omega$, $T_k < \infty$ for all $k \in \mathbb{N}_0$. Note that the T_k and $X_1, \ldots, X_{T_k}$ only depends on the environment restricted to $\mathcal{A}_{k-1}$, so in what follows, given ξ in Ω_{k-1} defined only on a half-space, $P_z^{p_{\gamma,\xi}}(X_{T_k} \in A)$ is well defined.

Now, call π the projection from Ω to Ω_0, $\pi(\xi) = \xi|_{\mathcal{A}_0}$, and for $\xi \in \Omega$ and X_n the random walk in $p_{\gamma,\xi}$, call

$$\overline{\xi}_n = \pi(\sigma^{X_{T_n}}\xi).$$

Lemma 2.2.1. *$(\overline{\xi}_n)_{n\in\mathbb{N}_0}$ is a Markov chain on Ω_0 under P_0^γ with initial distribution $\mathbb{P}_0$ and Markov operator*

$$E_0^\gamma\Big[f(\overline{\xi}_{n+1})\ \Big|\ \overline{\xi}_n\Big] = \int \sum_{z:z\cdot e_1=1} f\Big(\sigma^{\mathbf{e}(\pi)}(\xi\star\eta)\Big)P_0^{p_{\gamma,\overline{\xi}_n}}\big(X_{T_1}=z\big)\mathbb{P}_{0,1}(\mathrm{d}\eta). \tag{2.3}$$

Proof. Let $f_0, f_1, \ldots, f_{n+1}$ be bounded measurable functions on Ω_0, and let $\xi\in\Omega$. Using the strong Markov property for X_n under $P_0^{p_{\gamma,\xi}}$, and following [4], we have

$$E_0^{p_{\gamma,\xi}}\Big[f_{n+1}(\overline{\xi}_{n+1})f_n(\overline{\xi}_n)\cdots f_0(\overline{\xi}_0)\Big] = E_0^{p_{\gamma,\xi}}\Big[\ f_{n+1}(\pi(\sigma^{X_{T_{n+1}}}\xi))f_n(\pi(\sigma^{X_n}\xi))\cdots f_0(\pi(\xi))\Big] \tag{2.4}$$

$$= E_0^{p_{\gamma,\xi}}\Big[E_{X_{T_n}}^{p_{\gamma,\xi}}\Big[f_{n+1}(\pi(\sigma^{X_{T_1}}\xi))\Big]f_n(\overline{\xi}_n)\cdots f_0(\overline{\xi}_0)\Big] \tag{2.5}$$

$$= E_0^{p_{\gamma,\xi}}\Big[R^\xi f_{n+1}(X_{T_n})f_n(\overline{\xi}_n)\cdots f_0(\overline{\xi}_0)\Big], \tag{2.6}$$

where

$$R^\xi f(z) = E_z^{p_{\gamma,\xi}}\Big[f\Big(\pi(\sigma^{X_{T_1}\xi})\Big)\Big].$$

We obtain the above statement for E_0^γ as well by performing $\mathbb{E}$-integration. We condition on $X_{T_1},\cdots,X_{T_n}$ and $\xi|_{\mathcal{A}_n}$ (with respect to which $\overline{\xi}_0,\cdots,\overline{\xi}_n$ are measurable) to obtain

$$E_0^\gamma\Big[f_{n+1}(\overline{\xi}_{n+1})f_n(\overline{\xi}_n)\cdots f_0(\overline{\xi}_0)\Big] = \tag{2.7}$$

$$E_0^\gamma\Big[E_0^\gamma\Big[R^\xi f_{n+1}(X_{T_n})\ \Big|\ X_{T_1},\cdots,X_{T_n},\xi|_{\mathcal{A}_n}\Big]f_n(\overline{\xi}_n)\cdots f_0(\overline{\xi}_0)\Big]. \tag{2.8}$$

Now, we consider

$$E_0^\gamma\Big[R^\xi f_{n+1}(X_{T_n})\ \Big|\ X_{T_1},\cdots,X_{T_n},\xi|_{\mathcal{A}_n}\Big] \tag{2.9}$$

$$= E_0^\gamma\left[\sum_{z:z\cdot e_1=1} P_{X_{T_n}}^{p_{\gamma,\xi}}\big(X_{T_1}=z\big)f_{n+1}\Big(\pi(\sigma^{X_{T_n}+z}\xi)\Big)\ \Big|\ X_{T_1},\cdots,X_{T_n},\xi|_{\mathcal{A}_n}\right]. \tag{2.10}$$

First, we note that $P_{X_{T_n}}^{p_{\gamma,\xi}}\big(X_{T_1}=z\big)$ is measurable with respect to $\xi|_{\mathcal{A}_n}$ and X_{T_n}, and in fact equals

$P_0^{p_{\gamma,\bar{\xi}_n}}\big(X_{T_1}=z\big)$. Thus we can pull it out of the expectation to obtain

$$\sum_{z:z\cdot e_1=1} P_0^{p_{\gamma,\bar{\xi}_n}}\big(X_{T_1}=z\big)E_0^{\gamma}\Big[f_{n+1}\Big(\pi(\sigma^{X_{T_n}+z}\xi)\Big)\;\Big|\;X_{T_1},\cdots,X_{T_n},\xi|_{\mathcal{A}_n}\Big].$$

Now, $\pi(\sigma^{X_{T_n}+z}\xi)=\sigma^z\bar{\xi}_n\star\sigma^{X_{T_n}+z}\Big(\xi|_{\mathcal{A}_{n,n+1}}\Big)$, and $\xi|_{\mathcal{A}_{n,n+1}}$ is independent of $\sigma(X_{T_1},\cdots,X_{T_n},\xi|_{\mathcal{A}_n})$. And so

$$E_0^{\gamma}\Big[f_{n+1}\Big(\pi(\sigma^{X_{T_n}+z}\xi)\Big)\;\Big|\;X_{T_1},\cdots,X_{T_n},\xi|_{\mathcal{A}_n}\Big]= \tag{2.11}$$

$$\int_{\Omega_{n,n+1}} f_{n+1}\Big(\sigma^z\bar{\xi}_n\star\sigma^{X_{T_n}+z}\eta\Big)\mathbb{P}_{n,n+1}(\mathrm{d}\eta)=\int_{\Omega_{0,1}} f_{n+1}\Big(\sigma^z\Big(\bar{\xi}_n\star\eta\Big)\Big)\mathbb{P}_{0,1}(\mathrm{d}\eta), \tag{2.12}$$

by translation invariance of product measure. Thus in light of (2.7) we have the Markov property.

□

Remark 2.2.2. *Note that we could define this Markov Chain on this space in the following way. Let $\eta_1,\eta_2,\ldots\in\Omega_{0,1}$ be an i.i.d. sequence of $\mathbb{P}_{0,1}$-distributed slabs. Given a configuration $\bar{\xi}_k\in\Omega_0$, consider the random walk X_n^k in the environment $p_{\gamma,\bar{\xi}_k}$, stopped at the hitting time T_1^k of the hyperplane $\{z\cdot e_1=1\}$, and define*

$$\bar{\xi}_{k+1}=\sigma^{X^k_{T_1^k}}\Big(\bar{\xi}_k\star\eta_k\Big).$$

In the next section, we will extend the Markov operator for this chain to a Banach space of functions with a contractive metric, and for the contraction, we need that the hitting times have exponential tails, a simple consequence of our assumptions on γ and d_0.

Lemma 2.2.3. *For any $\xi\in\Omega_j$, $|\gamma|<\gamma_0$, the hitting time T_k, $k\le j+1$ of the walk X_n with transition probabilities given by $p_{\gamma,\xi}$ satisfies*

$$P_0^{p_{\gamma,\xi}}(T_k>n)\le Ck\lambda^{n/k},$$

where $C=\left(\frac{p}{q}\right)^{1/2}$, and where $\lambda=1-\left(\sqrt{p}-\sqrt{q}\right)^2$, with $p=p_0(e_1)-\gamma_0$ and $q=p_0(-e_1)+\gamma_0$.

Proof. For fixed γ and ξ, call X_n the walk in the environment $p_{\gamma,\xi}$, started at the origin and call $Y_n=X_n\cdot e_1$, the projection to the one dimensional lazy walk along $\{ne_1\,|\,n\in\mathbb{Z}\}$. Via a simple coupling argument, the hitting time of ke_1 for Y_n is stochastically dominated by the hitting time of ke_1 for the following stationary simple random walk, Z_n. Let $P_0(Z_n+1=z+e_1\,|\,Z_n=z)=p$, $P_0(Z_n+1=z-e_1\,|\,Z_n=z)=q$, and $P_0(Z_n+1=z\,|\,Z_n=z)=1-p-q$. Stochastic domination follows from the fact that $0<p<p_{\gamma,\xi}(z,e_1)$ and $p_{\gamma,\xi}(z,-e_1)<q<1$ for any $z\in\mathbb{Z}^d$, $\xi\in\Omega_0$, and $|\gamma|<\gamma_0$. Note that $p>q$ by our assumption on γ_0.

Notice that for the one dimensional homogenous walk $\{Z_n\}$, T_k is equal in distribution to the sum of k independent copies of T_1, the first hitting time of 1. So it suffices to analyze T_1. The standard technique is via martingales, namely, for $\lambda > 0$ set

$$M_n^\lambda = E_0\Big\{-\lambda Z_n - n\log\big(\varphi(\lambda)\big)\Big\}, \text{ where } \varphi(\lambda) = E_0\Big(e^{-\lambda Z_1}\Big). \tag{2.13}$$

This is indeed a non-negative martingale, and by the Optional Stopping Theorem (theorem 5.7.5, [10]),

$$E_0\Big(M_{T_1}^\lambda\Big) \le E_0\Big(M_0^\lambda\Big) \Longrightarrow E_0\Big(\exp\Big\{-\lambda Z_{T_1} - T_1\log\big(\varphi(\lambda)\big)\Big\}\Big) \le 1 \tag{2.14}$$

$$\Longrightarrow E_0\Big(e^{-T_1\log(\varphi(\lambda))}\Big) \le e^\lambda. \tag{2.15}$$

Thus, for $\varphi(\lambda) < 1$,

$$P_0(T_1 > N) = P_0\Big(e^{-T_1\log(\varphi(\lambda))} > e^{-N\log(\varphi(\lambda))}\Big) \tag{2.16}$$

$$\le E_0\Big(e^{-T_1\log(\varphi(\lambda)) + N\log(\varphi(\lambda))}\Big) \le e^{\lambda + N\log(\varphi(\lambda))}, \tag{2.17}$$

and to minimize this tail probability for large N, we can minimize $\varphi(\lambda)$. Using calculus on $\varphi(\lambda) = e^{-\lambda}p + e^\lambda q + 1 - (p+q)$, it is easy to check that $\lambda_0 = \log\Big(\sqrt{p/q}\Big) > 0$ minimizes φ with $\varphi(\lambda_0) = 1 - (\sqrt{p} - \sqrt{q})^2$, as required.

Again, since T_k is distributed as the sum of k independent copies of T_1, the lemma follows from the simple union bound $P_0(T_k > N) \le kP_0\Big(T_1 > \frac{N}{k}\Big)$.

□

2.3 The Operator on a Complex Banach Space

This section will lay the foundation for applying perturbation theory to the spectrum of the Markov operator for the Markov chain of configurations defined above. The Doeblin-Fortet theorem 2.5.4 provides the framework. Namely, if we consider the Banach space of bounded, Lipschitz functions on Ω_0, with a metric that allows us to prove a certain contractive bound, we can conclude quasi-compactness of the operator (spectral gap property).

For the following, we consider complex values of γ. Call B_r the ball of radius $r > 0$ around the origin in $\mathbb{C}$. We now define a metric on Ω_0 that will be critical to our procedure.

2.3.1 The metric

Definition 2.3.1. *Given two configurations $\xi, \xi' \in \Omega_0$, for $\lambda < \theta < 1$ (λ given in lemma 2.2.3) and $0 < \beta$, for $\gamma \in B_{\gamma_0}$ define the metric by*

$$d_\theta\Big(\xi, \xi'\Big) = \sup_{z:\, z\cdot e_1 \le 0} \theta^{-z\cdot e_1}\big(1 + \|z \perp e_1\|\big)^{-\beta} \sup_{\gamma\in B_{\gamma_0}} \left(\max_{e\in\mathcal{V}} \Big|\mathrm{Log}\big(p_{\gamma,\xi}(z,e)\big) - \mathrm{Log}\big(p_{\gamma,\xi'}(z,e)\big)\Big|\right), \tag{2.18}$$

where here $\|\cdot\|$ denotes the taxicab metric in $\mathbb{Z}^d$, $z \perp e_1$ denotes the vector $(z_2, \dots, z_d)$, and Log *denotes the principal branch of the complex logarithm.*

Instead of the difference of logarithms, we could have simplified the metric and used the difference of configurations instead. Note that for $\gamma \in B_{\gamma_0}$, where γ_0 is small enough so that $p_0(e) \pm \gamma_0 \in (\kappa, 1-\kappa)$ for all $e \in \mathcal{V}$, we have that $p_0(e) + \gamma\xi(z,e) \in D$ for any $\xi \in \Omega_0$, for all $z \in \mathbb{Z}^d$ and $e \in \mathcal{V}$, where

$$D = \big\{x + iy \,\big|\, x \in (\kappa, 1-\kappa), |y| < \gamma_0\big\}.$$

By noting that the derivative of $\mathrm{Log}(z)$ is $1/z$, which is bounded in D, and that Log is one-to-one in D, we can find $m, M > 0$ such that

$$m < \frac{\big|\mathrm{Log}(z) - \mathrm{Log}(z')\big|}{|z - z'|} < M \text{ for all } z, z' \in D. \tag{2.19}$$

For $\gamma \in B_{\gamma_0}$ and taking z and z' as $p_{\gamma,\xi}(z,e)$ and $p_{\gamma,\xi'}(z,e)$, we have that

$$\Big|\mathrm{Log}\big(p_{\gamma,\xi}(z,e)\big) - \mathrm{Log}\big(p_{\gamma,\xi'}(z,e)\big)\Big| > m|\gamma|\Big|\xi(z,e) - \xi'(z,e)\Big|$$

and

$$\Big|\mathrm{Log}\big(p_{\gamma,\xi}(z,e)\big) - \mathrm{Log}\big(p_{\gamma,\xi'}(z,e)\big)\Big| < M|\gamma|\Big|\xi(z,e) - \xi'(z,e)\Big|.$$

Taking the supremum over $\gamma \in B_{\gamma_0}$, and calling

$$\overline{d_\theta}(\xi, \xi') = \sup_{z:\, z\cdot e_1 \le 0} \theta^{-z\cdot e_1}\big(1 + \|z \perp e_1\|\big)^{-\beta}\left(\max_{e\in\mathcal{V}}\Big|\xi(z,e) - \xi'(z,e)\Big|\right), \tag{2.20}$$

we see that

$$m|\gamma_0|\overline{d_\theta}(\xi, \xi') \le d_\theta(\xi, \xi') \le M|\gamma_0|\overline{d_\theta}(\xi, \xi'). \tag{2.21}$$

Thus the two metrics are equivalent, and it is easy to see that the metric $\overline{d_\theta}$ generates the Borel product topology on Ω_0 (see [29], Theorem 20.5). So we have the first property in the following lemma.

Lemma 2.3.2.

1. *The metric d_θ generates the product topology on Ω_0.*

2. *Given configurations ξ, ξ' in Ω_{-k}, and a strip $\eta \in \Omega_{-k,0}$,*

$$d_\theta\Big(\xi \star \eta, \xi' \star \eta\Big) = \theta^k d_\theta\Big(\xi, \xi'\Big).$$

3. *Given $x \in \mathbb{Z}^d$, $x \cdot e_1 = 0$, we have*

$$d_\theta\Big(\sigma^x \xi, \sigma^x \xi'\Big) \leq \big(1 + \|x \perp e_1\|\big)^\beta d_\theta\Big(\xi, \xi'\Big).$$

Proof. The formula in (2) follows immediately from the notion of concatenation defined above.

For (3), we note that, for $x \cdot e_1 = 0$, we have $d_\theta^\gamma\big(\sigma^x \xi, \sigma^x \xi'\big)$ equals

$$\sup_{z:\, z\cdot e_1 \leq 0} \theta^{-z\cdot e_1}\big(1 + \|z \perp e_1\|\big)^{-\beta}\Big(\max_{e\in\mathcal{V}}\Big|\mathrm{Log}\big(p_{\gamma,\xi}(x+z,e)\big) - \mathrm{Log}\big(p_{\gamma,\xi'}(x+z,e)\big)\Big|\Big) \tag{2.22}$$

$$= \sup_{z:\, z\cdot e_1 \leq 0} \theta^{(x-z)\cdot e_1}\Big(1 + \big\|(z-x) \perp e_1\big\|\Big)^{-\beta}\Big(\max_{e\in\mathcal{V}}\Big|\mathrm{Log}\big(p_{\gamma,\xi}(z,e)\big) - \mathrm{Log}\big(p_{\gamma,\xi'}(z,e)\big)\Big|\Big) \tag{2.23}$$

$$\leq \sup_{z:\, z\cdot e_1 \leq 0}\left(\frac{\big(1 + \|z \perp e_1\|\big)^\beta}{\Big(1 + \big\|(z-x) \perp e_1\big\|\Big)^\beta}\right) d_\theta^\gamma\Big(\xi, \xi'\Big). \tag{2.24}$$

The supremum term on the left, using the triangle inequality, is less than or equal to

$$\sup_{z:\, z\cdot e_1 \leq 0}\left(1 + \frac{\|x \perp e_1\|}{1 + \big\|(z-x) \perp e_1\big\|}\right)^\beta \leq (1 + \|x \perp e_1\|)^\beta.$$

□

Given a complex-valued function f on Ω_0, denote the supremum norm as

$$|f| = \sup_{\xi\in\Omega_0}\big|f(\xi)\big|,$$

and the Lipschitz norm with respect to d_θ,

$$s(f) = \sup_{\substack{\xi_1,\xi_2\in\Omega_0 \\ \xi_1\neq\xi_2}} \frac{\left|f(\xi_1) - f(\xi_2)\right|}{d_\theta(\xi_1,\xi_2)}.$$

Finally, define the norm

$$\|f\| = |f| + s(f).$$

Define the space of complex valued functions on Ω_0 with finite norm

$$\mathcal{B} = \left\{ f : \Omega_0 \to \mathbb{C} \,\middle|\, f \in \mathcal{F},\ \|f\| < \infty \right\}.$$

It is well known that this is a complex Banach space, and is the Banach space considered for the Doeblin-Fortet theorem, 2.5.4.

2.3.2 The Operator

Denote by Π_R the space of nearest neighbor paths from the origin to the hyperplane $\{z \in \mathbb{Z} \,|\, z \cdot e_1 = R\}$, with elements $\pi = (\pi_i)_{i=0}^{\ell(\pi)}$, where $\ell(\pi)$ denotes the length of the path, $\pi_0 = 0$, and where for $i = 1, \ldots, \ell(\pi) - 1$, $\pi_i \in \mathcal{A}_{R-1}$, $\left|\pi_{i+1} - \pi_i\right| = 1$, and $\pi_{\ell(\pi)} \cdot e_1 = R$. Denote the endpoint $\pi_{\ell(\pi)}$ of the path π by $\mathbf{e}(\pi)$. Denote the steps by $\Delta\pi_i = \pi_{i+1} - \pi_i$.

For a configuration ξ in Ω_{R-1} and $|\gamma| < \gamma_0$, we denote the probability of a path $\pi \in \Pi_R$ with jump probabilities $p_{\gamma,\xi}$

$$P_{\gamma,\xi}[\pi] = P_0^{p_{\gamma,\xi}}\left(X_k = \pi_k \text{ for } k = 0, \ldots, \ell(\pi)\right) \tag{2.25}$$

$$= \prod_{i=0}^{\ell(\pi)-1} p_{\gamma,\xi}(\pi_i, \Delta\pi_i) \tag{2.26}$$

$$= \prod_{i=0}^{\ell(\pi)-1} \left(p_0(\Delta\pi_i) + \gamma\xi(\pi_i, \Delta\pi_i)\right). \tag{2.27}$$

This definition can be extended to all $\gamma \in \mathbb{C}$, though obviously this no longer represents the probability of a random walk taking the path π in the environment $p_{\gamma,\xi}$.

For $\gamma \in B_{\gamma_0}$, $f \in \mathcal{B}$, $\xi \in \Omega_0$, we define a linear operator $\mathcal{L}(\gamma)$ on B:

$$\mathcal{L}(\gamma)f(\xi) = \int \sum_{\pi\in\Pi_1} f\left(\sigma^{\mathbf{e}(\pi)}(\xi \star \eta)\right) P_{\gamma,\xi}[\pi] \mathbb{P}_{0,1}(\mathrm{d}\eta).$$

For $\gamma \in B_{\gamma_0} \cap \mathbb{R}$, this coincides with the Markov operator for the Markov chain $\left(\bar{\xi}_n\right)$ in Ω_0 defined in lemma 2.2.1. For such real γ, and for every $\xi \in \Omega_0$, as noted in section 2.2.2,

$$\sum_{\pi \in \Pi_1} P_{\gamma,\xi}[\pi] = P_0^{p_{\gamma,\xi}}(T_1 < \infty) = 1.$$

This gives that

$$\left|\mathcal{L}(\gamma) f\right| \le |f| \text{ for } \gamma \in B_{\gamma_0} \cap \mathbb{R}.$$

For complex γ,

$$\lim_{\gamma \to 0} \sup_{\substack{\xi \in \Omega_0 \\ z \in \mathcal{A}_0, e \in \mathcal{V}}} \frac{\left|p_0(e) + \gamma \xi(z,e)\right|}{p_0(e) + |\gamma| \xi(z,e)} = 1,$$

so, for $1 < \rho < \theta\lambda^{-1}$, (where λ is given in lemma 2.2.3 and θ in 2.3.1) we can choose γ_0 small enough so that for all $\gamma \in B_{\gamma_0}$, for any $\pi \in \Pi_1$ and $\xi \in \Omega_0$ we have

$$\left|P_{\gamma,\xi}[\pi]\right| \le \rho^{\ell(\pi)} P_{|\gamma|,\xi}[\pi]. \tag{2.28}$$

We can write the operator $\mathcal{L}(\gamma)$ as a series of operators $\sum_{n=1}^{\infty} U_n(\gamma)$, where

$$U_n(\gamma) f(\xi) = \sum_{\substack{\pi \in \Pi_1 \\ \ell(\pi) = n}} \int f\left(\sigma^{\mathbf{e}(\pi)}(\xi \star \eta)\right) P_{\gamma,\xi}[\pi] \mathbb{P}_{0,1}(\mathrm{d}\eta) \tag{2.29}$$

We use the 2.28 for $U_n(\gamma)$:

$$\left|U_n(\gamma) f(\xi)\right| \le |f| \sum_{\substack{\pi \in \Pi_1 \\ \ell(\pi) = n}} \left|P_{\gamma,\xi}[\pi]\right| \tag{2.30}$$

$$\le |f| \rho^n \sum_{\substack{\pi \in \Pi_1 \\ \ell(\pi) = n}} P_{|\gamma|,\xi}[\pi] \tag{2.31}$$

$$= |f| \rho^n P_0^{p_{|\gamma|,\xi}}\left(T_1 = n\right). \tag{2.32}$$

Lemma 2.2.3 then gives that

$$\left|U_n(\gamma) f\right| \le C |f| (\rho\lambda)^n, \tag{2.33}$$

which is summable, since $\rho\lambda < 1$. This implies that the series for $\mathcal{L}(\gamma) f(\xi)$ is absolutely convergent for all $f \in \mathcal{B}$, uniformly for all $\xi \in \Omega_0$ and $\gamma \in B_{\gamma_0}$.

We now show $s\big(\mathcal{L}(\gamma)f\big)$ is bounded for all $f \in \mathcal{B}$. We look at $s\big(U_n(\gamma)f\big)$, and add and subtract the same term to obtain

$$\begin{aligned}\big|U_n(\gamma)f(\xi_1) - U_n(\gamma)f(\xi_2)\big| \leq \int \sum_{\substack{\pi\in\Pi_1\\ \ell(\pi)=n}} \Big|f(\sigma^{\mathbf{e}(\pi)}\xi_1 \star \eta) - f(\sigma^{\mathbf{e}(\pi)}\xi_2 \star \eta)\Big|\big|P_{\gamma,\xi_1}[\pi]\big|\mathbb{P}_{0,1}(\mathrm{d}\eta) \\ + \int \sum_{\substack{\pi\in\Pi_1\\ \ell(\pi)=n}} \Big|f(\sigma^{\mathbf{e}(\pi)}\xi_2 \star \eta)\Big|\big|P_{\gamma,\xi_1}[\pi] - P_{\gamma,\xi_2}[\pi]\big|\mathbb{P}_{0,1}(\mathrm{d}\eta).\end{aligned} \tag{2.34}$$

To bound the left term of the expression, we note that

$$\Big|f(\sigma^{\mathbf{e}(\pi)}\xi_1 \star \eta) - f(\sigma^{\mathbf{e}(\pi)}\xi_2 \star \eta)\Big| \leq \theta\Big(1 + \big\|\mathbf{e}(\pi) \perp e_1\big\|\Big)^{\beta} d_\theta(\xi_1, \xi_2)\, s(f)$$

using the properties of the metric, lemma 2.3.2. Thus the left hand term of (2.34) is less than or equal to

$$\theta d_\theta(\xi_1, \xi_2)\, s(f) \sum_{\substack{\pi\in\Pi_1\\ \ell(\pi)=n}} (1+n)^{\beta}\big|P_{\gamma,\xi_1}[\pi]\big|, \tag{2.35}$$

and in light of (2.33), the supremum over pairs $\xi_1 \neq \xi_2$ of this term divided by $d_\theta(\xi_1, \xi_2)$ is summable.

For the right hand term of (2.34), we need the following lemma.

Lemma 2.3.3. *For $\gamma \in B_{\gamma_0}$,*

$$d_\theta(\xi_1, \xi_2)^{-1} \sum_{\substack{\pi\in\Pi_1\\ \ell(\pi)=n}} \big|P_{\gamma,\xi_1}[\pi] - P_{\gamma,\xi_2}[\pi]\big| \leq C\left(\frac{\lambda\rho}{\theta}\right)^{n}(n+1)^{\beta+1}$$

for any $\xi_1, \xi_2 \in \Omega_0$.

Proof. We first note that $\mathrm{Log}(z)/(z-1)$ is holomorphic on $\mathbb{C}\backslash\{z \in \mathbb{R} \mid z \leq 0\}$, with a removable singularity at 1 with value 1. So $\big|\mathrm{Log}(z)/(z-1)\big|$ is bounded away from zero on the unit disk minus $\{0\}$. Thus, there exists a positive constant C such that for $B_1 \setminus \{0\}$,

$$|1 - z| \leq C|\mathrm{Log}\, z|.$$

Given the pair $P_{\gamma,\xi_1}[\pi]$ and $P_{\gamma,\xi_2}[\pi]$, call $M(\pi)$ the term with larger modulus, and $m(\pi)$ the term with smaller modulus. Then noting that $m(\pi)/M(\pi) \neq 0$, we can bound

$$\big|P_{\gamma,\xi_1}[\pi] - P_{\gamma,\xi_2}[\pi]\big| \leq C\big|M(\pi)\big|\left|\mathrm{Log}\left(\frac{m(\pi)}{M(\pi)}\right)\right|. \tag{2.36}$$

Recall that $\mathrm{Log}(z) = \log|z| + i\,\mathrm{Arg}(z)$, where $\mathrm{Arg}(z) \in (-\pi, \pi]$. Since $\mathrm{Arg}(\prod z_i) = \sum \mathrm{Arg}(z_i) - 2\pi k$, where k is the integer which minimizes the absolute value of $\sum \mathrm{Arg}(z_i)$, it is clear that $\big|\mathrm{Arg}(\prod z_i)\big| \leq \big|\sum \mathrm{Arg}(z_i)\big|$. Then, from the formula for the logarithm above, for $z_i \neq 0$ we have

$$\left|\mathrm{Log}\Big(\prod z_i\Big)\right| \leq \sum \big|\mathrm{Log}(z_i)\big|,$$

since

$$\Re\Big(\mathrm{Log}\Big(\prod z_i\Big)\Big) = \sum \Re\big(\mathrm{Log}(z_i)\big)$$

and by the inequality for Arg (which is the imaginary part of Log) given above.

With this, and noting that $\mathrm{Arg}\big(p_{\gamma,\xi}(z,e)\big) \in (-\pi/2, \pi/2)$ by the choice of γ_0, we have

$$\begin{aligned}
&\sum_{\substack{\pi\in\Pi_1\\ \ell(\pi)=n}} \big|P_{\gamma,\xi_1}[\pi] - P_{\gamma,\xi_2}[\pi]\big| \\
\leq\, &C \sum_{\substack{\pi\in\Pi_1\\ \ell(\pi)=n}} \big|M(\pi)\big| \sum_{i=0}^{n-1} \Big|\mathrm{Log}\big(p_{\gamma,\xi_1}(\pi_i, \Delta\pi_i)\big) - \mathrm{Log}\big(p_{\gamma,\xi_2}(\pi_i, \Delta\pi_i)\big)\Big| \\
\leq\, &Cn \sum_{\substack{\pi\in\Pi_1\\ \ell(\pi)=n}} \big|M(\pi)\big| \sup_{z:\|z\|\leq n, e\in\mathcal{V}} \Big|\mathrm{Log}\big(p_{\gamma,\xi_1}(\pi_i, \Delta\pi_i)\big) - \mathrm{Log}\big(p_{\gamma,\xi_2}(\pi_i, \Delta\pi_i)\big)\Big|
\end{aligned} \tag{2.37}$$

Note that we can bound

$$\sup_{z:\|z\|\leq n, e\in\mathcal{V}} \Big|\mathrm{Log}\big(p_{\gamma,\xi_1}(\pi_i, \Delta\pi_i)\big) - \mathrm{Log}\big(p_{\gamma,\xi_2}(\pi_i, \Delta\pi_i)\big)\Big| \leq (n+1)^{\beta}\theta^{-n} d_\theta(\xi_1, \xi_2) \tag{2.38}$$

from the definition of the metric, 2.3.1. And $\big|M(\pi)\big|$ is less than or equal to the sum $\big|P_{\gamma,\xi_1}[\pi]\big| + \big|P_{\gamma,\xi_2}[\pi]\big|$, each of which is less than or equal to $\rho^{\ell(\pi)}$ times the path probability with real parameter $|\gamma|$, as in (2.28). So we can use lemma 2.2.3 to get that (2.37) is less than or equal to

$$C' d_\theta(\xi_1, \xi_2) \left(\frac{\rho\lambda}{\theta}\right)^n (1+n)^{\beta+1},$$

as required.

□

The bound given by lemma 2.3.3 is summable, since $\lambda\rho < \theta$, and so together with (2.35), we obtain that for all all $f \in \mathcal{B}$, $s\big(\mathcal{L}_n(\gamma)f\big)/\|f\|$ is bounded and summable in n. This with 2.33 gives that $\big\|U_n(\gamma)\big\|$ is bounded by a summable sequence, uniformly for $\gamma \in B_{\gamma_0}$. This implies convergence in norm of $\sum U_n(\gamma)$ to $\mathcal{L}(\gamma)$, uniformly for $\gamma \in B_{\gamma_0}$.

It also gives that $\big\|\mathcal{L}(\gamma)\big\|$ is bounded, so we have that $\mathcal{L}(\gamma)(\mathcal{B}) \subset \mathcal{B}$ for all $\gamma \in B_{\gamma_0}$.

2.4 Proof of Main Result

2.4.1 Quasi-compactness

Now, the form of $\mathcal{L}(\gamma)^n$ is clear when $\mathcal{L}(\gamma)$ is a Markov operator, i.e., when γ is real. We prove that this form holds for complex γ as well.

Lemma 2.4.1. *For all $n \in \mathbb{N}$, $\gamma \in B_{\gamma_0}$, $f \in \mathcal{B}$, $\xi \in \Omega_0$,*

$$\mathcal{L}(\gamma)^n f(\xi) = \int_{\Omega_{0,n}} \sum_{\pi \in \Pi_n} f\Big(\sigma^{\mathbf{e}(\pi)}(\xi \star \eta)\Big) P_{\gamma,\xi\star\eta}[\pi]\mathbb{P}_{0,n}(\mathrm{d}\eta)$$

Proof. Given $\pi \in \Pi_{i+1}$, $\pi' \in \Pi_{j+1}$, denote $\pi \star \pi'$ as the concatenation of the two paths, shifting the start of π' to the end of π:

$$\Big(\pi \star \pi'\Big)_j = \begin{cases} \pi_j & \text{for } j = 1, \dots, \ell(\pi) \\ \pi'_{j-\ell(\pi)} + \mathbf{e}(\pi) & \text{for } j = \ell(\pi)+1, \dots, \ell(\pi)+\ell(\pi'). \end{cases}$$

Then for $\xi \in \Omega_i$, $\eta \in \Omega_{i,j}$, we have

$$P_{\gamma,\xi}[\pi] \cdot P_{\gamma,\sigma^{\mathbf{e}(\pi)}(\xi\star\eta)}[\pi'] = P_{\gamma,\xi\star\eta}[\pi \star \pi']. \tag{2.39}$$

Inductively, assuming the assertion is true for n, then we can write $\mathcal{L}(\gamma)^{n+1} f(\xi) = \mathcal{L}(\gamma)\big(\mathcal{L}(\gamma)^n f(\xi)\big)$

as

$$\int_{\Omega_{0,1}} \sum_{\pi \in \Pi_1} \mathcal{L}(\gamma)^n f\Big(\sigma^{\mathbf{e}(\pi)} \xi \star \eta\Big) P_{\gamma,\xi}[\pi] \mathbb{P}_{0,1}(\mathrm{d}\eta) \tag{2.40}$$

$$= \int_{\Omega_{0,1}} \sum_{\pi \in \Pi_1} \int_{\Omega_{0,n}} \sum_{\pi' \in \Pi_n} f\Big(\sigma^{\mathbf{e}(\pi')}(\sigma^{\mathbf{e}(\pi)}(\xi \star \eta) \star \eta')\Big) P_{\gamma, \sigma^{\mathbf{e}(\pi)}(\xi \star \eta) \star \eta'}[\pi'] \mathbb{P}_{0,n}(\mathrm{d}\eta') P_{\gamma,\xi}[\pi] \mathbb{P}_{0,1}(\mathrm{d}\eta) \tag{2.41}$$

(2.42)

We can write $\sigma^{\mathbf{e}(\pi)}(\xi \star \eta) \star \eta'$ as $\sigma^{\mathbf{e}(\pi)}(\xi \star \eta \star \sigma^{-\mathbf{e}(\pi)} \eta')$, where $\sigma^{-\mathbf{e}(\pi)} \eta'$ is now an element of $\Omega_{1,n+1}$, and since the product measure is translation invariant, we can integrate instead over $\eta^\pi = \sigma^{-\mathbf{e}(\pi)} \eta'$ to obtain

$$\int_{\Omega_{0,1}} \sum_{\pi \in \Pi_1} \sum_{\pi' \in \Pi_n} \int_{\Omega_{1,n+1}} f\Big(\sigma^{\mathbf{e}(\pi')}(\sigma^{\mathbf{e}(\pi)}(\xi \star \eta \star \eta^\pi))\Big) P_{\gamma, \sigma^{\mathbf{e}(\pi)}(\xi \star \eta \star \eta^\pi)}[\pi'] \mathbb{P}_{1,n+1}(\mathrm{d}\eta^\pi) P_{\gamma,\xi}[\pi] \mathbb{P}_{0,1}(\mathrm{d}\eta).$$

Now, we note that there is a bijection from pairs of paths in $\Pi_1 \times \Pi_n$ to Π_{n+1} given above by concatenation (and reversed by splitting a path in Π_{n+1} at the first time the path hits the hyperplane $\{z \cdot e_1 = 1\}$). Thus we can sum over $\pi'' = \pi \star \pi' \in \Pi_{n+1}$, noting that $\mathbf{e}(\pi'') = \mathbf{e}(\pi) + \mathbf{e}(\pi')$. And we can write the double integral as the integral over the product space $\Omega_{0,n+1}$ with product measure $\mathbb{P}_{0,1} \times \mathbb{P}_{1,n+1} = \mathbb{P}_{0,n+1}$, integrating with respect to $\eta'' = \eta' \star \eta^\pi$. Finally, since

$$P_{\gamma, \sigma^{\mathbf{e}(\pi)}(\xi \star \eta \star \eta^\pi)}[\pi'] \cdot P_{\gamma,\xi}[\pi] = P_{\gamma, \xi \star \eta \star \eta^\pi}[\pi \star \pi'] = P_{\gamma, \xi \star \eta''}[\pi''],$$

the claim follows.

□

Now we use this formula to prove a bound on the Lipschitz norm of $\mathcal{L}(0)^k f$. Decomposing as in (2.34), and noting that for $\gamma = 0$ the second term is 0, we have

$$\Big|\mathcal{L}(0)^k f(\xi_1) - \mathcal{L}(0)^k f(\xi_1)\Big| \leq \theta^k s(f) d_\theta(\xi_1, \xi_2) \sum_{\pi \in \Pi_k} \Big(1 + \big\|\mathbf{e}(\pi) \perp e_1\big\|\Big)^\beta P_{0,\cdot}[\pi]. \tag{2.43}$$

We can decompose the sum by the length of the paths, and use lemma 2.2.3 to obtain that this is less than

$$\theta^k s(f) d_\theta(\xi_1, \xi_2) \sum_{n=k}^{\infty} C k \lambda^{\frac{n}{k}} (1+n)^\beta \leq s(f) d_\theta(\xi_1, \xi_2) \left(\frac{C' \; k \theta^k \lambda^{\beta/k}}{(1 - \lambda^{1/k})^\beta} \right),$$

where C' does not depend on k. The final term tends to 0 as k grows, and so we can take k large enough such that there is an $0 < \alpha < 1$ with

$$s\Big(\mathcal{L}(0)^k f\Big) \leq \alpha s(f). \tag{2.44}$$

And since $\big|\mathcal{L}(0)f\big| \leq |f|$, we have that

$$\Big\|\mathcal{L}(0)^k f\Big\| \leq \alpha\|f\| + (1-\alpha)|f|. \tag{2.45}$$

As per the discussion following theorem 2.5.2, as (Ω_0, d_θ) is a compact metric space, and $\mathcal{B}$ is as defined, we can conclude that $\mathcal{L}(0)$ is quasi-compact and of diagonal type, provided the spectral radius of $\mathcal{L}(0)$ is 1. To see this, recall that for any real $|\gamma| < \gamma_0$, $\mathcal{L}(\gamma)$ is a Markov operator, and $\mathbb{1}$ is an eigenvector with eigenvalue 1. So $r\big(\mathcal{L}(0)\big) \geq 1$. To see that $r\big(\mathcal{L}(0)\big) \leq 1$, as shown above we have that for any $f \in \mathcal{B}$,

$$s\Big(\mathcal{L}(0)^k f\Big) \leq s(f)\epsilon_k,$$

where $\epsilon_k \to 0$ as $k \to \infty$ and does not depend on f.

This and the fact that $\Big|\mathcal{L}(0)^k f\Big| \leq |f|$ for any $f \in \mathcal{B}$ gives that

$$\Big\|\mathcal{L}(0)^k\Big\| = \sup_{\substack{f\in\mathcal{B}\\ f\neq 0}} \frac{\Big\|\mathcal{L}(0)^k f\Big\|}{\|f\|} \leq 1 + \epsilon_k.$$

Taking the k-th root and the limit as $k \to \infty$ gives $r\big(\mathcal{L}(0)\big) \leq 1$.

Lemma 2.4.2. *For any $f \in \mathcal{B}$, $\lim_{n\to\infty} \mathcal{L}(0)^n f(\xi) = \overline{f}$, where $\overline{f}$ is a constant function.*

Proof. First note that if $f \in \mathcal{B}$ depends measurably only on $\xi(z,\cdot)$ for $z \cdot e_1 > -N$ for some $N \in \mathbb{N}$, then for $n \geq N$, for any $\pi \in \Pi_n$, $f(\sigma^{\mathbf{e}(\pi)}\xi \star \eta)$ is independent of ξ, and by shift invariance of $\mathbb{P}_{0,n}$, from lemma 2.4.1 we then have that $\mathcal{L}(0)^n f(\xi)$ is a constant.

Now fix an arbitrary $f \in \mathcal{B}$, and define $f_m(\xi) = f\Big(x_m \star \xi|_{\mathcal{A}_{-m,0}}\Big)$, where x_m is an arbitrary but fixed element of Ω_m. f_m depends only on $\xi(z,\cdot)$ for $z \cdot e_1 > -m$, and so $\mathcal{L}(0)^n f_m(\xi)$ is a constant for $n \geq m$, namely, it is equal to $\int_{\Omega_{-n,0}} f_m(\eta)\mathbb{P}_{-n,0}(\mathrm{d}\eta)$, which is equal to $\int_{\Omega_0} f_m(\eta)\mathbb{P}_0(\mathrm{d}\eta)$.

And

$$\big|f_m(\xi) - f(\xi)\big| \leq s(f)\ d_\theta\Big(x_m \star \xi|_{\mathcal{A}_{-m,0}}, \xi\Big) \leq C\ s(f)\ \theta^m,$$

so since $\left|\mathcal{L}(0)^n g\right| \le |g|$ for any n, we have that for all $\xi \in \Omega_0$,

$$\lim_{m\to\infty} \mathcal{L}(0)^n f_m(\xi) = \mathcal{L}(0)^n f(\xi)$$

uniformly in n (and ξ). Then we can interchange limits to obtain

$$\lim_{n\to\infty} \mathcal{L}(0)^n f(\xi) = \lim_{m\to\infty}\lim_{n\to\infty} \mathcal{L}(0)^n f_m(\xi) = \lim_{m\to\infty}\int_{\Omega_0} f_m(\eta)\mathbb{P}_0(\mathrm{d}\eta).$$

By the dominated convergence theorem, the right hand side converges to $\int_{\Omega_0} f(\eta)\mathbb{P}_0(\mathrm{d}\eta)$.

□

This fact immediately gives that 1 is the only peripheral eigenvalue for $\mathcal{L}(0)$, and it is simple.

Then in light of remark 2.5.4 and the decomposition in 2.5.3, we can write

$$\mathcal{L}(0)^n = \mathbb{1}\otimes\varphi_1 + N^n, \tag{2.46}$$

where $\mathbb{1} \in \mathcal{B}$ is the constant function 1, where φ_1 is a bounded linear functional such that $\varphi_1(\mathbb{1}) = 1$, $\mathcal{L}(0)^*\varphi_1 = \varphi_1$, and $r(N) < 1$.

Calling $F = \mathrm{vect}\{\mathbb{1}\}$ and $H = \{f \in \mathcal{B} \,|\, \varphi_1(f) = 0\}$, then we see that $\mathcal{B} = F \oplus H$, as $f = \varphi_1(f)\mathbb{1} + \big(f - \varphi_1(f)\mathbb{1}\big)$. $\mathcal{L}(0)$ takes constant functions to themselves, so that $\mathcal{L}(0)(F) \subset F$, and to see that $\mathcal{L}(0)(H) \subset H$, suppose that there is a function $f \in H$ such that $\mathcal{L}(0)f = C\mathbb{1}$ for some some constant C. Since $\mathcal{L}(0)C\mathbb{1} = C\mathbb{1}$, this implies that $C = \mathcal{L}(0)^n f = N^n f$ (since we are assuming $\varphi_1(f) = 0$), which contradicts $r(N) < 1$ unless $f \equiv 0$.

Thus as in definition 2.5.5, 1 is a dominating simple eigenvalue for $\mathcal{L}(0)$.

2.4.2 Differentiability

Lemma 2.4.3. *The map $\gamma \mapsto \mathcal{L}(\gamma)$ is holomorphic in the disc B_{γ_0}.*

Proof. By theorem 2.5.7, it suffices to show that for any $\varphi \in \mathcal{B}^*$ and $f \in \mathcal{B}$, $\gamma \to \varphi\big(\mathcal{L}(\gamma)f\big)$ is holomorphic in B_{γ_0}. Recalling that $\mathcal{L}(\gamma) = \sum U_n(\gamma)$ as in (2.29), we showed in section 2.3.2 that this series converges in norm uniformly for $\gamma \in B_{\gamma_0}$. Convergence in norm implies weak convergence, so that $\varphi(\mathcal{L}(\gamma)f) = \sum \varphi(U_n(\gamma)f)$. And a consequence of Morera's theorem is that a series of holomorphic functions in a domain D is holomorphic on compact subsets of D if the series is uniformly convergent on compact subset of D (7.3 in [11]).

Thus it suffices to show that $\gamma \to U_n(\gamma)$ is holomorphic in norm for $\gamma \in B_{\gamma_0}$, $n \in \mathbb{N}$

From here on, fix $\gamma \in B_{\gamma_0}$ and call

$$D_{\gamma,\xi}[\pi] := \left(P_{\gamma,\xi}[\pi]\right)^{-1} \frac{\mathrm{d}}{\mathrm{d}\gamma} P_{\gamma,\xi}[\pi] = \sum_{i=0}^{\ell(\pi)-1} \frac{\xi(\pi_i, \Delta\pi_i)}{p_0(\Delta\pi_i) + \gamma\xi(\pi_i, \Delta\pi_i)}. \tag{2.47}$$

We now focus on controlling

$$\left| \frac{P_{\gamma+h,\xi}[\pi] - P_{\gamma,\xi}[\pi]}{h} - D_{\gamma,\xi}[\pi] P_{\gamma,\xi}[\pi] \right|,$$

both in terms of the supremum over ξ, and in terms of the difference between two configurations ξ_1 and ξ_2. Note that

$$\frac{P_{\gamma+h,\xi}[\pi]}{P_{\gamma,\xi}[\pi]} = \prod_{i=0}^{\ell(\pi)-1} \frac{p_0(\Delta\pi_i) + (\gamma+h)\xi(\pi_i, \Delta\pi_i)}{p_0(\Delta\pi_i) + \gamma\xi(\pi_i, \Delta\pi_i)} \tag{2.48}$$

$$= \prod_{i=0}^{\ell(\pi)-1} \left(1 + h\frac{\xi(\pi_i, \Delta\pi_i)}{p_0(\Delta\pi_i) + \gamma\xi(\pi_i, \Delta\pi_i)}\right), \tag{2.49}$$

and call $a_i = \frac{\xi(\pi_i, \Delta\pi_i)}{p_0(\Delta\pi_i) + \gamma\xi(\pi_i, \Delta\pi_i)}$. Then factoring out $\left|P_{\gamma,\xi}[\pi]\right|$, we first control

$$\left| \frac{\left(\frac{P_{\gamma+h,\xi}[\pi]}{P_{\gamma,\xi}[\pi]}\right) - 1}{h} - D_{\gamma,\xi}[\pi] \right| = \left| \frac{\prod_{i=0}^{n-1}(1 + ha_i) - 1}{h} - \sum_{i=0}^{n-1} a_i \right|. \tag{2.50}$$

For notational simplicity, we denote this difference term by

$$A_h^{\gamma,\xi}[\pi] = \frac{\prod_{i=0}^{n-1}(1 + ha_i) - 1}{h} - \sum_{i=0}^{n-1} a_i.$$

To control this term, we use the multi-binomial theorem, which gives

$$\prod_{i=1}^{n}(1 + b_i) = 1 + \sum_{1 \le i_1 \le n} b_{i_1} + \sum_{1 \le i_1 < i_2 \le n} b_{i_1} b_{i_2} + \sum_{1 \le i_1 < i_2 < i_3 \le n} b_{i_1} b_{i_2} b_{i_3} + \tag{2.51}$$

$$\cdots + \sum_{1 \le i_1 < i_2 < \cdots < i_n \le n} b_{i_1} b_{i_2} \cdots b_{i_n}. \tag{2.52}$$

Then we have that

$$A_h^{\gamma,\xi}[\pi] = h \sum_{1\le i_1<i_2\le n} a_{i_1}a_{i_2} + h^2 \sum_{1\le i_1<i_2<i_3\le n} a_{i_1}a_{i_2}a_{i_3}+ \tag{2.53}$$

$$\cdots + h^{n-1} \sum_{1\le i_1<i_2<\cdots<i_n\le n} a_{i_1}a_{i_2}\cdots a_{i_n}. \tag{2.54}$$

This tends to 0 as h tends to 0 uniformly for all $\xi \in \Omega_0$ and $\pi \in \Pi_1$ with $\ell(\pi) = n$, since $|a_i| \le \kappa^{-1}$ for any $\xi \in \Omega_0$. So for $f \in \mathcal{B}$, calling the operator

$$V_n(\gamma)f(\xi) = \int \sum_{\substack{\pi\in\Pi_1\\ \ell(\pi)=n}} f\Big(\sigma^{\mathbf{e}(\pi)}(\xi \star \eta)\Big) P_{\gamma,\xi}[\pi] D_{\gamma,\xi}[\pi] \mathbb{P}_{0,1}(\mathrm{d}\eta),$$

we have that, for fixed n,

$$|f|^{-1} \left| \frac{U_n(\gamma+h)f - U_n(\gamma)f}{h} - V_n(\gamma)f \right| \tag{2.55}$$

tends to 0 as h tends to 0 uniformly for $f \in \mathcal{B}$.

For the Lipschitz norm, fix ξ_1 and ξ_2, and call $a_i^1 = \frac{\xi_1(\pi_i,\Delta\pi_i)}{p_0(\Delta\pi_i)+\gamma\xi_1(\pi_i,\Delta\pi_i)}$ and $a_i^2 = \frac{\xi_2(\pi_i,\Delta\pi_i)}{p_0(\Delta\pi_i)+\gamma\xi_2(\pi_i,\Delta\pi_i)}$ (which depend on π).

We need to show that $\sup_{f\ne 0}\|f\|^{-1} s\bigg(\Big(h^{-1}\big(U_n(\gamma+h) - U_n(\gamma)\big) - V_n\Big)f\bigg)$ tends to 0 as h tends to 0, or that

$$\left| \int \sum_{\substack{\pi\in\Pi_1\\ \ell(\pi)=n}} \Big[f(\sigma^{\mathbf{e}(\pi)\xi_1\star\eta}) P_{\gamma,\xi_1}[\pi] A_h^{\gamma,\xi_1}[\pi] - f(\sigma^{\mathbf{e}(\pi)\xi_2\star\eta}) P_{\gamma,\xi_2}[\pi] A_h^{\gamma,\xi_2}[\pi] \Big] \mathbb{P}_{0,1}(\mathrm{d}\eta) \right|$$

is bounded by $\|f\| d_\theta(\xi_1,\xi_2)$ times something which tends to 0 as h tends to 0, independent of f, ξ_1, and ξ_2. To do this, we decompose similarly to (2.34) and bound this by the sum of two terms,

$$\begin{aligned} &\int \sum_{\substack{\pi\in\Pi_1\\ \ell(\pi)=n}} \Big| f(\sigma^{\mathbf{e}(\pi)}\xi_1 \star \eta) - f(\sigma^{\mathbf{e}(\pi)}\xi_2 \star \eta) \Big| \Big| P_{\gamma,\xi_1}[\pi] A_h^{\gamma,\xi_1}[\pi] \Big| \mathbb{P}_{0,1}(\mathrm{d}\eta) \\ &+ \int \sum_{\substack{\pi\in\Pi_1\\ \ell(\pi)=n}} \Big| f(\sigma^{\mathbf{e}(\pi)}\xi_2 \star \eta) \Big| \Big| P_{\gamma,\xi_1}[\pi] A_h^{\gamma,\xi_1}[\pi] - P_{\gamma,\xi_2}[\pi] A_h^{\gamma,\xi_2}[\pi] \Big| \mathbb{P}_{0,1}(\mathrm{d}\eta). \end{aligned} \tag{2.56}$$

Just as in (2.35), we see that we can bound the first term by $s(f) d_\theta(\xi_1,\xi_2)$ times something which tends to 0 independently of ξ_1 and ξ_2 as h tends to 0.

For the second term, we can bound this by $|f|$ times

$$\sum_{\substack{\pi\in\Pi_1\\ \ell(\pi)=n}} \left|P_{\gamma,\xi_1}[\pi]-P_{\gamma,\xi_2}[\pi]\right|\left|A_h^{\gamma,\xi_1}[\pi]\right|+\left|P_{\gamma,\xi_2}[\pi]\right|\left|A_h^{\gamma,\xi_1}[\pi]-A_h^{\gamma,\xi_2}[\pi]\right|$$

As seen by lemma 2.3.3, the difference of path probabilities in the first term is bounded by a constant times $d_\theta(\xi_1,\xi_2)$ as required.

Finally, we control the second term. For a given path π, we can bound $\left|A_h^{\gamma,\xi_1}[\pi]-A_h^{\gamma,\xi_2}[\pi]\right|$ by

$$\begin{aligned}&|h|\sum_{1\le i_1<i_2\le n}\left|a^1_{i_1}a^1_{i_2}-a^2_{i_1}a^2_{i_2}\right|+|h|^2\sum_{1\le i_1<i_2<i_3\le n}\left|a^1_{i_1}a^1_{i_2}a^1_{i_3}-a^2_{i_1}a^2_{i_2}a^2_{i_3}\right|+\\ &\quad\cdots+|h|^{n-1}\sum_{1\le i_1<i_2<\cdots<i_n\le n}\left|a^1_{i_1}a^1_{i_2}\cdots a^1_{i_n}-a^2_{i_1}a^2_{i_2}\cdots a^2_{i_n}\right|.\end{aligned}\tag{2.57}$$

We control each of these terms inductively, by first noting that

$$\sup_{\|z\|\le n,e\in\mathcal{V}}\left|\xi_1(z,e)-\xi_2(z,e)\right|\le c(n)d_\theta(\xi_1,\xi_2)\tag{2.58}$$

by 2.21, where $c(n)$ is a constant that depends on n (and γ_0, θ, and β) but not on ξ_1 and ξ_2. We can then obtain a bound for $\left|a^1_i-a^2_i\right|$, which by multiplying by $\left|\big(p_0(\Delta\pi_i)+\gamma\xi_1(\pi_i,\Delta\pi_i)\big)\big(p_0(\Delta\pi_i)+\gamma\xi_2(\pi_i,\Delta\pi_i)\big)\right|$ (which is greater than or equal to κ^2), we have then that $\kappa^{-2}\left|a^1_i-a^2_i\right|$ is less than or equal to

$$\left|\xi_1(\pi_i,\Delta\pi_i)\big(p_0(\Delta\pi_i)+\gamma\xi_1(\pi_i,\Delta\pi_i)\big)-\xi_2(\pi_i,\Delta\pi_i)\big(p_0(\Delta\pi_i)+\gamma\xi_1(\pi_i,\Delta\pi_i)\big)\right|.\tag{2.59}$$

And by adding and subtracting the mixed term and noting that for $\|z\|\le n$ and $e\in\mathcal{V}$,

$$\left|\big(p_0(e)+\gamma\xi_1(z,e)\big)-\big(p_0(e)+\gamma\xi_2(z,e)\big)\right|\le c(n)|\gamma_0|d_\theta(\xi_1,\xi_2),\tag{2.60}$$

we obtain that $\left|a^1_i-a^2_i\right|$ is less than or equal to $d_\theta(\xi_1,\xi_2)$ times a constant independent of ξ_1 and ξ_2. Now, for $k\ge 2$ we control individual terms

$$\left|a^1_{i_1}a^1_{i_2}\cdots a^1_{i_k}-a^2_{i_1}a^2_{i_2}\cdots a^2_{i_k}\right|$$

by adding and subtracting a common term to obtain that this is less than or equal to

$$\left|a^1_{i_1}\right|\left|a^1_{i_2}\cdots a^1_{i_k} - a^2_{i_2}\cdots a^2_{i_k}\right| + \left|a^1_{i_1} - a^2_{i_1}\right|\left|a^2_{i_2}\cdots a^2_{i_k}\right| \leq \tag{2.61}$$

$$\kappa^{-1}\left|a^1_{i_2}\cdots a^1_{i_k} - a^2_{i_2}\cdots a^2_{i_k}\right| + Cd_\theta(\xi_1,\xi_2)\kappa^{-(k-1)}. \tag{2.62}$$

We have a recurrence relation on the number of terms, and induction gives that for any indices $i_1,\dots,i_k$ and $j_1,\dots,j_k$, we have

$$\left|a^1_{i_1}a^1_{i_2}\cdots a^1_{i_k} - a^2_{i_1}a^2_{i_2}\cdots a^2_{i_k}\right| \leq C\kappa^{-k}d_\theta(\xi_1,\xi_2). \tag{2.63}$$

Finally, from (2.57) we obtain

$$\sum_{\substack{\pi\in\Pi_1\\ \ell(\pi)=n}} \left|P_{\gamma,\xi_2}[\pi]\right|\left|A^{\gamma,\xi_1}_h[\pi] - A^{\gamma,\xi_2}_h[\pi]\right| \leq \tag{2.64}$$

$$d_\theta(\xi_1,\xi_2)\sum_{\substack{\pi\in\Pi_1\\ \ell(\pi)=n}} \left|P_{\gamma,\xi_2}[\pi]\right|\sum_{k=2}^{n}|h|^{k-1}C\kappa^{-k}\binom{n}{k}, \tag{2.65}$$

with the final term tending to 0 in h as required.

So, since $|f|$ and $s(f)$ are bounded by $\|f\|$, along with (2.55) we have shown that

$$\lim_{h\to 0}\left\|h^{-1}\big(U_n(\gamma+h) - U_n(\gamma)\big) - V_n(\gamma)\right\| = 0.$$

Of course, $V_n(\gamma)$ needs to be a bounded linear operator on $\mathcal{B}$. With uniform boundedness in ξ of $D_{\gamma,\xi}[\pi]$, it is straightforward to check that $\left|V_n(\gamma)\right|/\|f\|$ is bounded. To check that $s\big(V_n(\gamma)f\big)\|f\|$ is bounded, we can again decompose as in (2.34), and the details are similar to the above. The only new detail is bounding

$$d_\theta(\xi_1,\xi_2)^{-1}\sum_{\substack{\pi\in\Pi_1\\ \ell(\pi)=n}} \left|P_{\gamma,\xi_2}[\pi]\right|\left|D_{\gamma,\xi_1}[\pi] - D_{\gamma,\xi_2}[\pi]\right| \tag{2.66}$$

$$\leq d_\theta(\xi_1,\xi_2)^{-1}\sum_{\substack{\pi\in\Pi_1\\ \ell(\pi)=n}}\sum_{k=0}^{n-1}\left|a^1_k - a^2_k\right|, \tag{2.67}$$

which, by the bound (2.58), is bounded uniformly in ξ_1 and ξ_2.

□

Thus the perturbation theorem 2.5.6 gives that, for γ in a ball B_{γ_1} around 0 in the complex plane, some $\gamma_1 < \gamma_0$, $\mathcal{L}(\gamma)$ has one dominating simple eigenvalue. That is, there exist $\lambda(\gamma) \in \mathbb{C}$, $v(\gamma) \in \mathcal{B}$, and $\varphi(\gamma) \in \mathcal{B}^*$, all holomorphic functions of γ, such that

$$\mathcal{L}^n(\gamma) = \lambda(\gamma)^n v(\gamma) \otimes \varphi(\gamma) + N(\gamma)^n, \tag{2.68}$$

where $\mathcal{L}(\gamma)v(\gamma) = \lambda(\gamma)v(\gamma)$, $\mathcal{L}(\gamma)^*\varphi(\gamma) = \lambda(\gamma)\varphi(\gamma)$, $\left|\lambda(\gamma)\right| \geq 1-\eta_1$, $<\varphi(\gamma), v(\gamma)>= 1$, and where $N(\gamma)$ is a bounded linear operator on $\mathcal{B}$ which depends holomorphically on γ, such that $\left\|N(\gamma)^n\right\| \leq c(1-\eta_2)^n$ for $\eta_2 > \eta_1 > 0$.

Now, we know that for real γ, 1 is an eigenvalue of $\mathcal{L}(\gamma)$, i.e. $\lambda(\gamma) = 1$ and $v(\gamma) = \mathbb{1}$ for real γ.

Remark 2.4.4. *The bounded linear functional φ can be extended to the space of bounded continuous complex-valued functions on Ω_0. By the Riesz representation theorem, $\varphi(f) = \int f \mathrm{d}\mu^\gamma$, where μ^γ is the unique $\mathcal{L}(\gamma)$-invariant probability measure. It is easy to check that $\mu^\gamma << \mathbb{P}_0$.*

2.4.3 The Velocity

Now, for $\xi \in \Omega_0$, and for $\gamma \in B_{\gamma_1}$, call

$$g_\gamma(\xi) = \sum_{\pi \in \Pi_1} \ell(\pi) P_{\gamma,\xi}[\pi] = E_0^{p_{\gamma,\xi}}(T_1). \tag{2.69}$$

For real γ, $g_\gamma(\xi) = E_0^{p_{\gamma,\xi}}(T_1)$.

From lemma 2.2.3 and the proof of lemma 2.3.3, it is straightforward to see that $g_\gamma \in \mathcal{B}$. Further, we prove the following.

Lemma 2.4.5. *$\gamma \mapsto g_\gamma$ is holomorphic in norm for $\gamma \in B_{\gamma_0}$.*

Proof. Call

$$g_\gamma^n(\xi) = \sum_{\substack{\pi \in \Pi_1 \\ \ell(\pi)=n}} n P_{\gamma,\xi}[\pi],$$

so that $g_\gamma = \sum g_\gamma^n$. Note that $g_\gamma^n = nU_n(\gamma)\mathbb{1}$, and thus it is easy to see that $\left\|g_\gamma^n\right\|$ is summable, with a bound uniform for $\gamma \in B_{\gamma_0}$. So then $\sum g_\gamma^n$ converges uniformly to g_γ in norm. Thus we can apply a similar argument to that at the beginning of the proof of lemma 2.4.3 using theorem 2.5.7, and so it suffices to check that $\gamma \mapsto g_\gamma^n$ is holomorphic for $\gamma \in B_{\gamma_0}$.

Recall the definitions of $V_n(\gamma)$ and $D_{\gamma,\xi}[\pi]$ from the proof of lemma 2.4.3.

Call

$$G_\gamma^n(\xi) = \sum_{\substack{\pi\in\Pi_1\\ \ell(\pi)=n}} \ell(\pi) P_{\gamma,\xi}[\pi] D_{\gamma,\xi}[\pi]. \tag{2.70}$$

Note that $G_\gamma^n = nV_n(\gamma)\mathbb{1}$, and is thus in $\mathcal{B}$. Then we have that

$$\left\| \frac{g_{\gamma+h}^n - g_\gamma^n}{h} - G_\gamma^n \right\| = n \left\| \frac{U_n(\gamma+h)\mathbb{1} - U_n(\gamma)\mathbb{1}}{h} - V_n(\gamma)\mathbb{1} \right\| \tag{2.71}$$

$$\leq n \left\| \frac{U_n(\gamma+h) - U_n(\gamma)}{h} - V_n(\gamma) \right\|, \tag{2.72}$$

which tends to 0 as $h \to 0$ as shown in lemma 2.4.3. □

Lemma 2.4.6. *For $\xi_0 \in \Omega_0$, $n \in \mathbb{N}_0$*

$$\mathcal{L}(\gamma)^n g_\gamma(\xi_0) = \mathbb{E}\left[E_0^{p_{\gamma,\xi}}\left[T_{n+1} - T_n\right] \,\middle|\, \xi(z,\cdot) = \xi_0(z,\cdot) \text{ for } z\cdot e_1 \leq 0 \right]. \tag{2.73}$$

Proof. For $n = 0$, this is trivial. For $n \geq 1$, using lemma 2.4.1, we have

$$\mathcal{L}(\gamma)^n g_\gamma(\xi_0) = \int_{\Omega_{0,n}} \sum_{\pi\in\Pi_n} g_\gamma\left(\sigma^{\mathbf{e}(\pi)}(\xi_0 \star \eta)\right) P_{\gamma,\xi_0\star\eta}[\pi] \mathbb{P}_{0,n}(\mathrm{d}\eta) \tag{2.74}$$

$$= \int_{\Omega_{0,n}} \sum_{\pi\in\Pi_n} \sum_{\pi'\in\Pi_1} \ell(\pi') P_{\gamma,\sigma^{\mathbf{e}(\pi)}(\xi_0\star\eta)}[\pi'] P_{\gamma,\xi_0\star\eta}[\pi] \mathbb{P}_{0,n}(\mathrm{d}\eta), \tag{2.75}$$

and as in (2.39), we have that $P_{\gamma,\sigma^{\mathbf{e}(\pi)}(\xi_0\star\eta)}[\pi'] P_{\gamma,\xi_0\star\eta}[\pi] = P_{\gamma,\xi_0\star\eta}[\pi\star\pi']$. For a path $\pi \in \Pi_k$, call $\tau_k[\pi] = \min\{j \in \mathbb{N} \,|\, \pi_j \cdot e_1 = k\}$. Then for $\pi'' = \pi\star\pi'$ above, we have $\tau_n[\pi''] = \ell(\pi)$ and $\tau_{n+1} = \ell(\pi) + \ell(\pi')$, so that $\tau_{n+1}[\pi''] - \tau_n[\pi''] = \ell(\pi')$. Then we have

$$\int_{\Omega_{0,n}} \sum_{\pi''\in\Pi_{n+1}} \left(\tau_{n+1}[\pi''] - \tau_n[\pi'']\right) P_{\gamma,\xi_0\star\eta}[\pi''] \mathbb{P}_{0,n}(\mathrm{d}\eta) \tag{2.76}$$

$$= \int_{\Omega_{0,n}} \left(E_\gamma^{\xi_0\star\eta}(T_{n+1}) - E_\gamma^{\xi_0\star\eta}(T_n)\right) \mathbb{P}_{0,n}(\mathrm{d}\eta) \tag{2.77}$$

$$= \mathbb{E}\left[E_0^{p_{\gamma,\xi}}\left[T_{n+1} - T_n\right] \,\middle|\, \xi(z,\cdot) = \xi_0(z,\cdot) \text{ for } z\cdot e_1 \leq 0 \right]. \tag{2.78}$$

□

For the random walk X_n in an environment $p_{\gamma,\xi}$, on the P_0^γ-probability 1 event that $\{\lim_{n\to\infty} X_n \cdot e_1 =$

$\infty\} \subset \{T_n < \infty \; \forall n \in \mathbb{N}\}$, we have that X_{T_n}/T_n has the same limit as X_n/n P_0^γ-a.s., and so

$$v^\gamma \cdot e_1 = \lim_{n\to\infty} \frac{X_n \cdot e_1}{n} = \lim_{n\to\infty} \frac{n}{T_n} \quad P_0^\gamma - a.s. \tag{2.79}$$

Then $\frac{1}{v^\gamma \cdot e_1} = \lim_{n\to\infty} \frac{E_0^\gamma(T_n)}{n}$, as uniform integrability of $\{T_n/n\}$ follows from lemma 2.2.3. And we can write

$$\frac{E_0^\gamma(T_n)}{n} = E_0^\gamma\left[\frac{1}{n}\sum_{k=0}^{n-1}(T_{k+1}-T_k)\right] \tag{2.80}$$

$$= \int_{\Omega_0}\int_{\Omega_{0,\infty}} \frac{1}{n}\sum_{k=0}^{n-1} E_0^{p_{\gamma,\xi\star\eta}}\big[T_{k+1}-T_k\big]\mathbb{P}_{0,\infty}(\mathrm{d}\eta)\mathbb{P}_0(\mathrm{d}\xi) \tag{2.81}$$

$$= \int_{\Omega_0}\left(\frac{1}{n}\sum_{k=0}^{n-1}\mathcal{L}(\gamma)^k g_\gamma(\xi)\right)\mathbb{P}_0(\mathrm{d}\xi) \tag{2.82}$$

$$= \int_{\Omega_0}\left(\frac{1}{n}\sum_{k=0}^{n-1} < \varphi(\gamma), g_\gamma > + N(\gamma)^k g_\gamma(\xi)\right)\mathbb{P}_0(\mathrm{d}\xi). \tag{2.83}$$

From the decay on the norm of $N(\gamma)^k$ in (2.68), we have that

$$\left|\frac{1}{n}\sum_{k=0}^{n-1} N(\gamma)^k g_\gamma(\xi)\right| \le \frac{1}{n}\sum_{k=0}^{-1}\left\|N(\gamma)^k g_\gamma\right\| \le \frac{1}{n}\sum_{k=0}^{n-1} c(1-\eta)^k \left\|g_\gamma\right\|, \tag{2.84}$$

where c and η are positive constants. And thus, taking the limit $n \to \infty$, for γ small enough,

$$v^\gamma \cdot e_1 = \frac{1}{< \varphi(\gamma), g_\gamma >}.$$

This is holomorphic by noting that

$$\lim_{h\to 0}\frac{1}{h}\big(< \varphi(\gamma+h), g_{\gamma+h} > - < \varphi(\gamma), g_\gamma >\big) \tag{2.85}$$

$$= \lim_{h\to 0} < \varphi(\gamma+h), \frac{g_{\gamma+h}-g_\gamma}{h} > + < \frac{\varphi(\gamma+h)-\varphi(\gamma)}{h}, g_\gamma > \tag{2.86}$$

and using the joint continuity of the inner product.

2.5 Appendix

2.5.1 Perturbation Theory for Quasi-compact Operators on a Banach Space

We follow [15], [23], and [18].

Let $(\mathcal{B}, \|\cdot\|)$ be a Banach space, $\mathcal{L}_{\mathcal{B}}$ the space of bounded linear operators on $\mathcal{B}$, and $T \in \mathcal{L}_{\mathcal{B}}$. Denote the spectral radius of a bounded lineary operator by $r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}$.

Definition 2.5.1. *T is quasi-compact if there are two closed, invariant subspaces F and H such that*

$$\mathcal{B} = F \oplus H,$$

where $\dim F < \infty$, each eigenvalue of $T|_F$ has modulus $r(T)$, and $r\big(T|_H\big) < r(T)$.

We include a generalization [14] of the Ionescu Tulcea Marinescu theorem [17] theorem for establishing quasi-compactness, Theorem XIV.3 in [15]:

Theorem 2.5.2. *Let $(\mathcal{B}, \|\cdot\|)$ be a Banach space, $|\cdot|$ a continuous semi-norm on $\mathcal{B}$ and T a bounded linear operator on $\mathcal{B}$, such that*

1. *$T(\{f \in \mathcal{B} \,|\, \|f\| \leq 1\})$ is totally bounded in $(\mathcal{B}, |\cdot|)$,*
2. *there exists a constant M such that, for $f \in \mathcal{B}$, $|Tf| \leq M|f|$,*
3. *there exists $k \in \mathbb{N}$ and real positive numbers r and R such that $r < r(T)$ and*

$$\left\|T^k f\right\| \leq r^k \|f\| + R|f|, \quad \text{for } f \in \mathcal{B}.$$

Then T is quasi-compact and $r_e(T) \leq r$.

Here $r_e(\cdot)$ denotes the essential spectral radius.

Let B^* denote the space of bounded linear functionals on B, and for $\varphi \in B^*$, $f \in \mathcal{B}$, call $<\varphi, f>= \varphi(f)$. For $v \in \mathcal{B}$, write $v \otimes \varphi$ to be the bounded linear operator $(v \otimes \varphi)f =<\varphi, f> v$. Denote the adjoint of T by T^*.

Combining propositions III.1 and lemma III.3 from [15], we have

Lemma 2.5.3. *A quasi-compact operator T is of diagonal type if*

$$\sup_n r(T)^{-n} \left\|T^n\right\| < \infty.$$

Then there exist an integer $s \geq 1$, sequences $(\lambda_k)_{k=1}^s \in \mathbb{C}^s$, $(v_k)_{k=1}^s \in \mathcal{B}^s$ and $(\varphi_k)_{k=1}^s \in (\mathcal{B}^*)^s$, such that, for $k, l = 1, \ldots, s$ and $n \geq 1$,

1. $|\lambda_k| = r(T)$
2. $(\varphi_k, v_l) = \delta_{k,l}$
3. $Tv_k = \lambda_k v_k$, $T^* \varphi_k = \lambda_k \varphi_k$,
4. $T^n = \sum_{k=1}^s \lambda_k^n v_k \otimes \varphi_k + N^n$, *where* $r(N) < r(T)$.

Remark 2.5.4. *The preceding is a generalization of the Doeblin-Fortet theorem [8], which is the case when the Banach space $\mathcal{B}$ is assumed to be the space of continuous bounded complex-valued functions with bounded Lipschitz constant on a compact metric space. The bounded linear operator T is assumed to have spectral radius 1, and satisfy hypotheses (2) and (3) of theorem 2.5.2, with $M = 1$ and $r < 1$. Under these assumptions, the theorem holds using Arzela-Ascoli, and additionally the operator T is of diagonal-type. See [8], [7], and [18].*

Following section III.2 of [15], we give the perturbation theorem we use in this paper, along with a definition (to ensure preserving the number of peripheral eigenvalues under a smooth perturbation). This theorem follows from the techniques in [23].

Definition 2.5.5. *T has $s \in \mathbb{N}$ dominating simple eigenvalues if there exist closed subspaces F and H such that*

1. $\mathcal{B} = F \oplus H$,
2. $T(F) \subset F$, $T(H) \subset H$,
3. $\dim F = s$ *and* $T|_F$ *has* s *simple eigenvalues* λ_k, $k = 1, \ldots, s$,
4. $r\big(T|_H\big) < \min\big\{|\lambda_k| \,\big|\, k = 1, \ldots, s\big\}$.

For an open ball of radius r in $\mathbb{C}$, B_r, let $\mathcal{H}(B_r, \mathcal{G})$ denote the space of holomorphic functions from B_r to the Banach space $\mathcal{G}$.

Theorem 2.5.6. *Let $\big(T(\gamma)\big)_{\gamma \in B_{r_0}}$ be a collection of bounded linear operators on B such that*

1. $T(\cdot) \in \mathcal{H}\big(B_{r_0}, \mathcal{B}\big)$,
2. $T(0)$ *has* s *dominating simple eigenvalues and* $r\big(T(0)\big) = 1$.

Then there exists $r_1 < r_0$ *such that for* $\gamma \in B_{r_1}$, $T(\gamma)$ *has* s *dominating simple eigenvalues. More precisely, there exist* $\eta_2 > \eta_1 > 0$ *and distinct functions* $\lambda_k(\cdot) \in \mathcal{H}\big(B_{r_1}, \mathbb{C}\big)$, $v_k(\cdot) \in \mathcal{H}\big(B_{r_1}, \mathcal{B}\big)$, $\varphi(\cdot) \in \mathcal{H}\big(B_{r_1}, \mathcal{B}^*\big)$, $k = 1, \ldots, s$, $N(\cdot) \in \mathcal{H}\big(B_{r_1}, \mathcal{L}_{\mathcal{B}}\big)$ *such that for* $\gamma \in B_{r_1}$ *and* $k, l = 1, \ldots, s$

1. $T(\gamma)v_k(\gamma) = \lambda_k(\gamma)v_k(\gamma)$, $T(\gamma)^*\varphi_k(\gamma) = \lambda_k(\gamma)\varphi_k(\gamma)$,
2. $\min\Big\{\big|\lambda_k(\gamma)\big| \,\Big|\, k = 1, \ldots, s\Big\} \geq 1 - \eta_1$,
3. $< \varphi_k(\gamma), v_l(\gamma) >= \delta_{k,l}$,
4. *for all* $n \geq 1$, $T^n(\gamma) = \sum_{k=1}^{s} \lambda_k(\gamma)^n v_k(\gamma) \otimes \varphi_k(\gamma) + N(\gamma)^n$, *with* $\big\|N(\gamma)^n\big\| \leq (1 - \eta_2)^n$.

We also recall the following from [23], pages 139 and 152.

Theorem 2.5.7.

1. $\gamma \mapsto f(\gamma) \in \mathcal{B}$ *is holomorphic in norm on a domain* D *of the complex plane if* $\gamma \mapsto < \varphi, f(\gamma) >$ *is holomorphic in* D *for each* $\varphi \in \mathcal{B}^*$.
2. $\gamma \mapsto T(\gamma) \in \mathcal{L}_{\mathcal{B}}$ *is holomorphic in norm on a domain* D *of the complex plane if* $\gamma \mapsto < \varphi, T(\gamma)v >$ *is holomorphic in* $\gamma \in D$ *for each* $v \in \mathcal{B}$ *and* $\varphi \in \mathcal{B}^*$.

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