On the integrals of the Kudla-Millson theta series

by

Ivan Livinsky

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University of Toronto

Abstract

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The Kudla-Millson theta series θ_{km} of a pseudoeuclidean space V of signature (p,q) and lattice L is a differential form on the symmetric space $\mathcal{D}$ attached to the pseudoorthogonal group $\mathrm{O}(p,q)$ that transforms like a genus n Siegel modular form of weight $(p+q)/2$. Any integral of θ_{km} inherits the modular transformation law and becomes a nonholomorphic Siegel modular form. A special case of such integral is the well-known Zagier Eisenstein series $\mathcal{F}(\tau)$ of weight $3/2$ as showed by Funke.

We show that for $n=1$ and $p=1$ the integral of θ_{km} along a geodesic path coincides with the Zwegers theta function $\widehat{\Theta}_{\mathbf{a},\mathbf{b}}$. We construct a higher-dimensional generalization of Zwegers theta functions as integrals of θ_{km} over geodesic simplices for $n \geqslant 2$.

If Γ is a discrete group of isometries of V that preserve the lattice L and act trivially on the cosets L^*/L, then the fundamental region $\Gamma\backslash\mathcal{D}$ is an arithmetic locally symmetric space. We prove that the integral of θ_{km} over $\Gamma\backslash\mathcal{D}$ converges and compute it in some cases. In particular, we extend the results of Kudla to the cases $p=1$, and q odd.

To my parents

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Contents

Chapter 1

Introduction

In their work [KM1, KM2, KM3, KM4] Kudla and Millson constructed a certain class of Schwartz functions taking values in closed differential forms on the symmetric spaces attached to classical groups $\mathrm{O}(p,q)$, $\mathrm{U}(p,q)$, $\mathrm{Sp}(p,q)$.

Let V be a pseudoeuclidean vector space with a nondegenerate inner product $\langle\cdot,\cdot\rangle$ of signature (p,q). Let G be the identity component of $\mathrm{O}(V)$ and let K be a maximal compact subgroup of G. The corresponding symmetric space $\mathcal{D} = G/K$ is naturally identified with the *negative Grassmannian* $\mathrm{Gr}_q^-(V)$ of V, i.e. with the space of maximum negative-definite subspaces of V.

For $r \geqslant 0$ let $\Omega^r(\mathcal{D})$ be the space of smooth r-forms on $\mathcal{D}$. For $n \geqslant 1$ let $\mathcal{S}(V^n)$ be the space of Schwartz functions on V^n. Recall that the 2-fold metaplectic cover $\mathrm{Mp}(n;\mathbb{R})$ of the symplectic group $\mathrm{Sp}(n;\mathbb{R})$ acts on $\mathcal{S}(V^n)$ via the Weil representation. Let $K' \subset \mathrm{Mp}(n;\mathbb{R})$ be the inverse image of a maximal compact subgroup of $\mathrm{Sp}(n;\mathbb{R})$. Every maximal compact subgroup of $\mathrm{Sp}(n;\mathbb{R})$ is isomorphic to $\mathrm{U}(n)$ and there exists a well-defined character $\det^{\frac{1}{2}} : K' \to \mathbb{C}^*$ whose square descends to $\det : \mathrm{U}(n) \to \mathbb{C}^*$.

For every integer n satisfying $1 \leqslant n \leqslant p$, Kudla and Millson constructed [KM2] an element $\varphi_{km} \in \mathcal{S}(V^n) \otimes \Omega^{nq}(\mathcal{D})$ with the following properties:

(i) The form φ_{km} is G-invariant: $g^*\varphi_{km}(gx) = \varphi_{km}(x)$ for all $g \in G$ and $x \in V^n$, i.e. $\varphi_{km} \in [\mathcal{S}(V^n) \otimes \Omega^{nq}(\mathcal{D})]^G$.

(ii) The form φ_{km} is closed: $d\varphi_{km} = 0$, where $d : \Omega^{nq}(\mathcal{D}) \to \Omega^{nq+1}(\mathcal{D})$ is the exterior derivative.

(iii) The form φ_{km} is an eigenfunction for K' with respect to the Weil representation $\mathbf{R}$ on $\mathcal{S}(V^n)$:

$$\mathbf{R}(k)\varphi_{km} = \det(k)^{\frac{p+q}{2}}\varphi_{km}$$

for all $k \in K'$.

Using φ_{km} we construct a theta series. Let $\mathcal{H}_n = \{\tau = u + \mathrm{i}v | \tau = \tau^\top, v > 0\} \subset \mathrm{M}_n(\mathbb{C})$ be the Siegel upper half-space of genus n. For $\tau = u + \mathrm{i}v \in \mathcal{H}_n$, let

$$g_\tau = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^{\frac{1}{2}} & 0 \\ 0 & v^{-\frac{1}{2}} \end{pmatrix} \in \mathrm{Sp}(n; \mathbb{R}).$$

We slightly abuse the notation and write g_τ also for its pre-image in $\mathrm{Mp}(n; \mathbb{R})$ defined in (2.5), (2.6). Let $L \subset V$ be a maximum rank lattice satisfying $L \subseteq L^*$. Let

$$\theta_{km}(\tau) = \theta_{km}(\mathbf{a}, \mathbf{b}; \tau) = \sum_{x \in L^n + \mathbf{a}} |v|^{-\frac{p+q}{4}} (\mathbf{R}(g_\tau)\varphi_{km})(x) e^{2\pi \mathrm{i} \,\mathrm{tr}\langle x, \mathbf{b}\rangle},$$

where $\mathbf{a}, \mathbf{b} \in V^n$ are *constraints*, and $\langle x, y\rangle$ for $x \in V^n$, $y \in V^m$ denotes the matrix $(\langle x_i, y_j\rangle)_{i,j=1}^{n,m}$.

This is a genus n theta series valued in nq-forms on $\mathcal{D}$. Let $\Gamma_L = \{\gamma \in G | \gamma L = L\}$ be the group of orientation-preserving transformations that also preserve L. Let Γ be a finite index subgroup of Γ_L that fixes the cosets L^*/L, i.e., satisfying $\Gamma|_{L^*/L} = \mathrm{id}_{L^*/L}$.

Theorem 1.1. *The following properties of θ_{km} hold:*

(i) The series θ_{km} is a closed nq-form on $\mathcal{D}$.

(ii) For every $\mathbf{c} \in L^n$ and $\mathbf{d} \in L^{*n}$ the following identities hold:

$$\theta_{km}(\mathbf{a} + \mathbf{c}, \mathbf{b}) = \theta_{km}(\mathbf{a}, \mathbf{b}),$$
$$\theta_{km}(\mathbf{a}, \mathbf{b} + \mathbf{d}) = e^{2\pi \mathrm{i}\,\mathrm{tr}\langle \mathbf{a}, \mathbf{d}\rangle} \theta_{km}(\mathbf{a}, \mathbf{b}),$$
$$\theta_{km}(-\mathbf{a}, -\mathbf{b}) = (-1)^{nq} \theta_{km}(\mathbf{a}, \mathbf{b}).$$

(iii) For $\mathbf{a}, \mathbf{b} \in L^{*n}$ the series θ_{km} satisfies the following modular transformation identity:

$$\theta_{km}(\mathbf{a}, \mathbf{b}; \gamma\tau) = \lambda(\gamma)\mathfrak{g}(b, d; L) e^{\pi \mathrm{i}\,\mathrm{tr}[\langle \mathbf{a}, \mathbf{a}\rangle b a^\top]} J(\gamma, \tau)^{p+q} \theta_{km}(\mathbf{a}a, \mathbf{b}d; \tau),$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N) \subseteq \mathrm{Sp}(n; \mathbb{Z})$ satisfies $\mathrm{tr}\langle xa, yb\rangle \in 2\mathbb{Z}$, $\mathrm{tr}\langle xc, yd\rangle \in 2\mathbb{Z}$ for all $x, y \in L^n$, $N \geqslant 1$ satisfies $NL^* \subseteq L$ and $N\langle v, v'\rangle \in 2\mathbb{Z}$ for all $v, v' \in L^*$, $\lambda(\gamma)^8 = 1$, $\mathfrak{g}(b, d; L)$ is a Gauss sum given by the formula

$$\mathfrak{g}(b, d; L) = |\det d|^{\frac{p+q}{2}} \sum_{x \in L^n / L^n d^\top} e^{\pi \mathrm{i}\,\mathrm{tr}[\langle x, x\rangle b d^{-1}]},$$

and $J(\gamma,\tau)$ is a continuous function on $\mathcal{H}_n$ satisfying $J(\gamma,\tau)^2 = \det(c\tau+d)$.

(iv) *If additionally $p+q=2k$ and $\langle v,v'\rangle \in 2\mathbb{Z}$ for all $v,v' \in L$ then*

$$\theta_{km}(\mathbf{a},\mathbf{b};\gamma\tau) = \chi(d)e^{\pi \mathrm{i}\,\mathrm{tr}[\langle \mathbf{a},\mathbf{a}\rangle ba^\top]}\det(c\tau+d)^k\theta_{km}(\mathbf{a}a,\mathbf{b}d;\tau),$$

where $\gamma \in \Gamma_0(N)$ as in (iii), and

$$\chi(d) = (\operatorname{sign}\det d)^k\left(\frac{(-1)^k\Delta}{|\det d|}\right),$$

where $\Delta = \det(\langle v_i,v_j\rangle)_{i,j=1}^{p+q}$ for any $\mathbb{Z}$-basis $\{v_i\}$ of L.

(v) *Let $\frac{\partial}{\partial\overline{\tau}}$ be a differential operator defined as*

$$\frac{\partial}{\partial\overline{\tau}} = \sum \frac{\partial}{\partial\overline{\tau}_\alpha}e_\alpha^*,$$

where $\{e_\alpha\}$ is a basis of the space of symmetric complex $n\times n$-matrices, $\{e_\alpha^*\}$ is the dual basis with respect to the trace form, i.e. such that $\mathrm{tr}[e_\alpha e_\beta^*] = \delta_{\alpha,\beta}$, and $\tau = \sum\limits_\alpha \tau_\alpha e_\alpha$.

Then the vector-valued form $\frac{\partial}{\partial\overline{\tau}}\theta_{km}$ is exact.

(vi) *For $\mathbf{a},\mathbf{b} \in L^{*n}$ the form θ_{km} is Γ-invariant, i.e. $\gamma^*\theta_{km} = \theta_{km}$ for all $\gamma\in\Gamma$. Hence, in this case θ_{km} descends to a nq-form on the quotient space $\Gamma\backslash\mathcal{D}$.*

For any oriented submanifold Δ in $\mathcal{D}$ of dimension nq we can define

$$\Theta_{km}^\Delta(\tau) = 2^{\frac{nq}{2}}\int\limits_\Delta \theta_{km}(\tau)$$

the integral of θ_{km} over Δ, provided it converges. The **key observation** here is that Θ_{km}^Δ inherits the modular transformation law in τ and becomes, in general, a genus n Siegel nonholomorphic modular form. Many interesting modular forms arise in this way.

The property (v) of θ_{km} combined with the Stokes' Theorem gives another result:

Theorem 1.2 ([KM4])**.** *If $C \in H_{nq}(\Gamma\backslash\mathcal{D})$ is a cycle then the corresponding theta function $\Theta_{km}^C(\tau)$ is holomorphic.*

In the case $\operatorname{sig} V = (m,1)$ the symmetric space $\mathcal{D}$ coincides with the hyperbolic space $\mathcal{H}^m$. We identify $\mathcal{D}$ with one component of the hyperboloid given by equation

$\langle z,z\rangle = -1$. In his Ph.D. thesis [Zw], while working on mock modular forms, Zwegers introduced nonholomorphic theta series attached to such V. Zwegers defined

$$\widehat{\Theta}^{c,c'}_{\mathbf{a},\mathbf{b}}(\tau) = \sum_{x\in L+\mathbf{a}} \left[E\Big(\frac{\langle c,x\rangle\sqrt{v}}{\sqrt{-Q(c)}}\Big) - E\Big(\frac{\langle c',x\rangle\sqrt{v}}{\sqrt{-Q(c')}}\Big)\right] e^{2\pi \mathrm{i}\tau Q(x)} e^{2\pi\mathrm{i}\langle x,\mathbf{b}\rangle},$$

where $\tau = u + \mathrm{i}v \in \mathcal{H}$, $Q(x) = \frac{1}{2}\langle x,x\rangle$, $\mathbf{a},\mathbf{b}\in V$, c,c' belong to the same component of the cone $\{x\in V | Q(x) < 0\}$, and E is the *error function* defined by

$$E(z) = 2\int_0^z e^{-\pi x^2} dx.$$

Zwegers proved that these functions transform in τ just like classic theta functions. We prove that in fact Zwegers theta functions are integrals of θ_{km} for $n = 1$.

Theorem A. *The following identity holds:*

$$\Theta^{z_{c,c'}}_{km}(\tau) = -\frac{1}{2}\widehat{\Theta}^{c,c'}_{\mathbf{a},\mathbf{b}}(\tau),$$

where $c,c'\in\mathcal{D}$ and $z_{c,c'}$ is a geodesic path between c and c'.

Thus, the modularity of Zwegers' series follows from the key observation, whereas Zwegers' proof is indirect.

As a higher genus generalization of Zwegers theta functions we consider integrals of θ_{km} over geodesic simplices.

Theorem B. *Let $\Delta = \Delta(c_0, c_1, \ldots, c_n)$ be an oriented geodesic simplex constructed on vertices $c_0, c_1, \ldots, c_n \in \mathcal{D}$. Then*

$$\Theta^{\Delta}_{km}(\tau) = \sum_{x\in L^n+\mathbf{a}} E(x\sqrt{2v},\Delta) e^{\pi\mathrm{i}\,\mathrm{tr}[\tau\langle x,x\rangle]} e^{2\pi\mathrm{i}\,\mathrm{tr}\langle x,\mathbf{b}\rangle},$$

where E is a matrix generalization of the Zwegers' error function defined by the formula

$$E(y,\Delta) = \int_{\rho_y(\Delta)} \exp(-\pi(r_1^2 + \cdots + r_n^2)) dr_1 \ldots dr_n,$$

and for $y = (y_1,\ldots,y_n)\in V^n$, $\rho_y : \mathcal{D}\to\mathbb{R}^n$ is the map $z\mapsto(\langle z,y_1\rangle,\ldots,\langle z,y_n\rangle)$.

Note that in some special cases the image of Δ folds over itself, and we need to slightly modify the region $\rho_y(\Delta)$.

In the case $n = 2$ we give an explicit formula for our matrix error function using the integral

$$F(\alpha, \beta; z) = \int_0^z \exp\left(-\frac{\pi}{\alpha \cos^2 x - \beta}\right) dx.$$

We consider next the case $n = p$ when θ_{km} is a top degree form on $\mathcal{D}$ and could be written as a function times the invariant volume form μ on $\mathcal{D}$. First, we derive an explicit formula for θ_{km}.

Theorem C. *For $n = p$ the following identity holds:*

$$\theta_{km}(\tau) = \sum_{x \in L^p + \mathbf{a}} \mathcal{P}_{p,q}(\sqrt{v}\langle x, z\rangle) e^{\pi \mathrm{i} \operatorname{tr}\langle x,x\rangle_{\tau,Z}} e^{2\pi \mathrm{i} \operatorname{tr}\langle x, \mathbf{b}\rangle} \cdot \mu,$$

where $\langle x, x\rangle_{\tau,Z} = \tau\langle x, x\rangle + 2\mathrm{i}v\langle x, z\rangle\langle z, x\rangle$, z is any orthonormal basis of $Z \in \mathcal{D}$, and $\mathcal{P}_{p,q} : \mathrm{M}_p(\mathbb{R}) \to \mathbb{R}$ is a matrix generalization of the Hermite polynomial.

We define a matrix generalization of the Hermite polynomial as follows. For $x = (x_{ij}) \in \mathrm{M}_p(\mathbb{R})$ let $\mathcal{P}_{p,q}(x)$ be a polynomial in x_{ij} satisfying

$$\mathcal{P}_{p,q}(x) e^{-2\pi \operatorname{tr}[x^\top x]} = (-4\pi)^{-pq} D^q e^{-2\pi \operatorname{tr}[x^\top x]},$$

where $D = \det(\frac{\partial}{\partial x_{ij}})$ is a differential operator on $C^\infty(\mathrm{M}_p(\mathbb{R}))$.

In general θ_{km} is not integrable over an arbitrary noncompact region Δ. The case of most interest for us is $\Delta = \Gamma\backslash\mathcal{D}$ when $\mathbf{a}, \mathbf{b} \in L^{*p}$. We always consider the orientation of $\Gamma\backslash\mathcal{D}$ for which the volume form μ is positive. Theorem 1.2 does not hold in this case, and the integral may be nonholomorphic. However, in the case $\operatorname{sig} V = (m, 1)$ the integral is still holomorphic.

Theorem 1.3 ([K1, K2])**.** *For $\operatorname{sig} V = (m, 1)$ the following identity holds:*

$$\Theta_{km}^{\Gamma\backslash\mathcal{D}}(\tau) = \Theta_{km}^{+}(\tau) + \Theta_{km}^{0}(\tau),$$

where

$$\Theta_{km}^{+}(\tau) = \sum_{\substack{x \in \Gamma\backslash L^m + \mathbf{a} \\ \langle x,x\rangle > 0}} \frac{1}{|\Gamma_x|} \varepsilon(x) e^{\pi \mathrm{i} \operatorname{tr}[\tau\langle x,x\rangle]} e^{2\pi \mathrm{i} \operatorname{tr}\langle x, \mathbf{b}\rangle},$$

$$\Theta_{km}^{0}(\tau) = \frac{1}{2}(-1)^m \sum_{\substack{x \in \Gamma\backslash L^m + \mathbf{a} \\ x \text{ reduced}}} \varepsilon(x) \mathbf{B}_1(\nu(x)) e^{\pi \mathrm{i} \operatorname{tr}[\tau\langle x,x\rangle]} e^{2\pi \mathrm{i} \operatorname{tr}\langle x, \mathbf{b}\rangle},$$

where $\mathbf{B}_1$ *is the Bernoulli* 1*-periodic polynomial defined on* $[0,1]$ *by the formula*

$$\mathbf{B}_1(x) = \begin{cases} \frac{1}{2} - x, & x \in (0,1), \\ 0, & x = 0,1, \end{cases}$$

$\varepsilon(x) = \pm 1$ *depending on the orientation of* x*, and* Γ_x *is the stabilizer of the frame* x *in* Γ*. Summation in* Θ^0_{km} *is restricted to the* m*-frames generating degenerate subspaces that are* ***reduced*** *according to Definition 5.10.*

We explain the notation in detail in Chapter 5, when we provide an alternative proof of this result, which is more direct than the original one.

For $\operatorname{sig} V = (1,q)$ only the cases $q = 1$ and $q = 2$ were previously considered. The case $q = 1$ follows from Theorem 1.3. For $q = 2$ the integral was computed [F] by Funke (see formulas (4.1), (4.2)). Funke showed that for $q = 2$ the theta function $\Theta^{\Gamma\backslash\mathcal{D}}_{km}(\tau)$ is nonholomorphic in general.

We prove another result.

Theorem D. *For* $\operatorname{sig} V = (1,q)$ *and* $q \geqslant 3$ *the following identity holds:*

$$\Theta^{\Gamma\backslash\mathcal{D}}_{km}(\tau) = \Theta^{+}_{km}(\tau) + \Theta^{0}_{km}(\tau),$$

where

$$\Theta^{+}_{km}(\tau) = \sum_{\substack{x \in \Gamma\backslash L + \mathbf{a} \\ \langle x,x\rangle > 0}} \frac{1}{|\Gamma_x|} \varepsilon(x) e^{\pi i \tau \langle x,x\rangle} e^{2\pi i \langle x, \mathbf{b}\rangle},$$

$$\Theta^{0}_{km}(\tau) = \begin{cases} 0, & q \text{ odd, or } \mathbf{a} \notin L, \\ (-1)^k \frac{(2k-1)!!}{2^k \pi^k} \mu(\Gamma\backslash\mathcal{D}), & q = 2k \text{ and } \mathbf{a} \in L, \end{cases}$$

and $\varepsilon(x) = \pm 1$ *depending on the orientation of* x*.*

We consider the general case next. We obtain two results.

Theorem E. *For arbitrary pseudoeuclidean space* V *of signature* (p,q)*, lattice* $L \subset V$ *satisfying* $L \subseteq L^*$*, and* $\mathbf{a}, \mathbf{b} \in L^{*p}$ *the integral*

$$\int_{\Gamma\backslash\mathcal{D}} \theta_{km}(\tau)$$

converges.

More precisely, the function $\|\theta_{km}(\tau)\| : \mathcal{D} \to \mathbb{R}$ *satisfying* $|\theta_{km}(\tau)| = \|\theta_{km}(\tau)\| \cdot \mu$ *belongs to the class* $\mathrm{L}^1(\Gamma\backslash\mathcal{D})$.

Hence, the theta function $\Theta_{km}^{\Gamma\backslash\mathcal{D}}(\tau)$ is always well-defined. However, it is not in general termwise integrable.

Finally, the parameter $\mathbf{a}$ is called *nonsingular* if every frame $x \in L^p + \mathbf{a}$ generates a nondegenerate pseudoeuclidean subspace of dimension p. Equivalently, $\mathbf{a}$ is nonsingular iff the matrix $\langle x, x\rangle$ is nonsingular for all $x \in L^p + \mathbf{a}$. We compute the integral for all $\mathbf{a}$ nonsingular and q odd.

Theorem F. *If q is odd and $\mathbf{a}$ is nonsingular then*

$$\Theta_{km}^{\Gamma\backslash\mathcal{D}}(\tau) = \sum_{\substack{x \in \Gamma\backslash L^p + \mathbf{a} \\ \langle x,x\rangle > 0}} \frac{1}{|\Gamma_x|} \varepsilon(x) e^{\pi \mathrm{i}\,\mathrm{tr}[\tau\langle x,x\rangle]} e^{2\pi \mathrm{i}\,\mathrm{tr}\langle x,\mathbf{b}\rangle}.$$

We prove this theorem using the properties of the generalized Hermite polynomial $\mathcal{P}_{p,q}$. We split the summation in θ_{km} into parts depending on the spectrum of $\langle x, x\rangle$. We show that for $\mathbf{a}$ nonsingular θ_{km} is in fact termwise integrable, and from the properties of $\mathcal{P}_{p,q}$ for q odd we derive that all terms with $\langle x, x\rangle$ not positive-definite vanish. The special case $q = 1$ of this theorem was proved in [K1].

For q even we expect the situation to be different. In this case the frames with linearly dependent vectors may give nonzero contribution. In particular, we have that $\varphi_{km}(0)$ is equal to $c\mu$ for some nonzero constant $c = \mathcal{P}_{p,q}(0)$, and its integral over $\Gamma\backslash\mathcal{D}$ is nonzero.

Notation

$\langle \cdot, \cdot \rangle$ A nondegenerate symmetric inner product on V and its matrix-valued generalization.

1_n The identity matrix of size $n \times n$.

$\mathbf{a}, \mathbf{b}$ Elements of V^n, the *constraints*.

$\mathbf{B}_1$ The Bernoulli 1-periodic polynomial.

$\mathcal{B}^m$ An open unit ball in $\mathbb{R}^m$.

$\mathcal{B}^{p,q}$ An open unit ball in $\mathrm{M}_{p,q}(\mathbb{R})$.

c, c' Points in $\mathcal{D}$.

$\mathcal{D}$ The symmetric space attached to G, viewed and the Positive, or the Negative Grassmannian.

Δ A simplex in $\mathcal{D}$.

Γ_L A discrete subgroup of G that preserves L.

Γ A finite index subgroup of Γ_L satisfying the property $\Gamma|_{L*/L} = \mathrm{id}_{L*/L}$.

G The identity component of $\mathrm{O}(V)$.

GL The general linear group.

$\mathrm{Gr}_r(V)$ The Grassmannian of r-dimensional subspaces of a vector space V.

$\mathrm{Gr}_p^+(V)$, $\mathrm{Gr}_q^-(V)$ The Positive and the Negative Grassmannians of a pseudoeuclidean space V.

$\mathcal{H}$ The upper half-plane.

$\mathcal{H}^m$ The m-dimensional hyperbolic space.

$\mathcal{H}_n$ The Siegel upper half-space of genus n.

θ_{km} The Kudla-Millson theta series.

$\Theta^{c,c'}_{\mathbf{a},\mathbf{b}}$, $\widehat{\Theta}^{c,c'}_{\mathbf{a},\mathbf{b}}$ Zwegers theta functions.

Θ^{Δ}_{km} The normalized integral of θ_{km} over a region Δ.

i $\sqrt{-1}$.

L A full rank lattice in V.

μ The invariant volume form on $\mathcal{D}$.

$\mathrm{M}_{p,q}(\mathbb{R})$ The space of real $p \times q$ matrices.

Mp The metaplectic group.

n The genus of θ_{km}.

O The orthogonal group.

(p, q) The signature of V.

$\mathcal{P}_{p,q}$ The matrix generalization of the Hermite polynomial.

$\mathbb{Q}$ The rational numbers.

$\mathbb{R}$ The real numbers.

$\mathbf{R}$ The Weil representation.

S_n The symmetric group.

SL The special linear group.

SO The special orthogonal group.

Sp The symplectic group.

$\mathrm{St}_G(x)$ The stabilizer of the element x under the action of group G. Also denoted G_x.

$\mathcal{S}$ The Schwartz space.

$\mathfrak{S}$ The Siegel domain for Γ.

τ An element of $\mathcal{H}_n$.

T_z A tangent space at point z.

u, v The real and the imaginary parts of τ.

U The unitary group.

UT The unitriangular group.

V A nondegenerate pseudoeuclidean vector space.

$\mathbb{Z}$ The integers.

Chapter 2

Auxiliary results

2.1 Pseudoeuclidean spaces

Let V be a real vector space equipped with a symmetric bilinear form $\langle \cdot, \cdot \rangle$. This is a *pseudoeuclidean space*. We consider vectors in V only as column-vectors and linear operators act on V on the left. The group of all operators in $\mathrm{GL}(V)$ preserving the bilinear form will be denoted $\mathrm{O}(V)$. Let $m = \dim V$.

Every vector $x \in V$ is either *positive* if $\langle x, x \rangle > 0$, *negative* if $\langle x, x \rangle < 0$, or *isotropic* if $\langle x, x \rangle = 0$. A basis $x_1, \ldots, x_p, y_1, \ldots, y_q, z_1, \ldots, z_r$ of V is *pseudoorthonormal* if it is orthogonal, and $\langle x_i, x_i \rangle = 1$, $\langle y_j, y_j \rangle = -1$, $\langle z_k, z_k \rangle = 0$ for all i, j, k. The pair (p, q) is the *signature* $\operatorname{sig} V$ of V. It could be shown that the signature does not depend on the choice of a pseudoorthonormal basis for V. If $p + q < m$ (i.e. if $r > 0$) then V is called *degenerate*. The most degenerate case is $(p, q) = (0, 0)$. In this case V is a *null* space. If V is null then $\mathrm{O}(V) = \mathrm{GL}(V)$. The space V is Euclidean if it is nondegenerate and $\operatorname{sig} V = (m, 0)$.

Let V be a nondegenerate pseudoeuclidean space of signature (p, q). If $W \subseteq V$ is a subspace then the restriction of the form $\langle \cdot, \cdot \rangle$ gives it a structure of a pseudoeuclidean subspace as well.

Proposition 2.1. *Let* $\dim W = d$ *and* $\operatorname{sig} W = (p_1, q_1)$. *Then* $p_1 \leqslant p$, $q_1 \leqslant q$, *and* $d - p_1 - q_1 \leqslant \min(p, q)$. *In particular, if* W *is null then* $d \leqslant \min(p, q)$. *All these cases are feasible.*

For example, the subspace generated by vectors $x_1 + y_1, \ldots, x_q + y_q$ for $q \leqslant p$ is a maximum null subspace for arbitrary pseudoorthonormal basis $x_1, \ldots, y_q$.

Proposition 2.2. *If* W *is a nondegenerate subspace of* V, *then its pseudoorthogonal complement* $W^\perp$ *is nondegenerate as well. Moreover, it holds that* $\operatorname{sig} W + \operatorname{sig} W^\perp = \operatorname{sig} V$.

A subspace W is degenerate iff $W \cap W^\perp \neq 0$. In this case $W \cap W^\perp$ is a nonzero null subspace of V.

If W is a subspace of V, there is a way to compute its signature, but first we need to extend the inner product to couples of vectors. For $x \in V^r$ and $y \in V^s$ let $\langle x, y\rangle = (\langle x_i, y_i\rangle)_{i,j=1}^{r,s}$; this is a real $r \times s$-matrix. Thus, we consider the inner product also as a matrix-valued function $\langle\cdot,\cdot\rangle : V^r \times V^s \to \mathrm{M}_{r,s}(\mathbb{R})$. Moreover, every real matrix $a \in \mathrm{M}_{r,s}(\mathbb{R})$ defines a linear map $V^r \to V^s$ by right multiplication.

Lemma 2.3. *Let $x \in V^r$, $y \in V^s$ and let $a \in \mathrm{M}_r(\mathbb{R})$ and $b \in \mathrm{M}_s(\mathbb{R})$ be real matrices. Then $\langle xa, yb\rangle = a^\top \langle x, y\rangle b$.*

Proof. The ijth entry of $\langle xa, yb\rangle$ is equal to

$$\langle x_1 a_{1i} + \cdots + x_r a_{ri}, y_1 b_{1j} + \cdots + y_s b_{sj}\rangle = \sum_{k,l=1}^{r,s} a_{ki}\langle x_k, y_l\rangle b_{lj}.$$

This coincides with the corresponding entry of the matrix $a^\top\langle x, y\rangle b$. □

Lemma 2.4. *Let $x = (x_1, \dots, x_m)$ be any basis of a subspace W. Then $\operatorname{sig} W = (p, q)$ iff the symmetric matrix $\langle x, x\rangle \in \mathrm{M}_m(\mathbb{R})$ has p positive and q negative eigenvalues. In particular, W is null iff $\langle x, x\rangle = 0$.*

Proof. Since $\langle x, x\rangle$ is a symmetric matrix, there exists an orthogonal matrix $\sigma \in \mathrm{O}(m)$ such that $\sigma^\top\langle x, x\rangle\sigma = \langle x\sigma, x\sigma\rangle$ is diagonal. Then $x\sigma$ is an orthogonal basis of W. After a normalization we get a pseudoorthonormal basis for W. The number of positive and negative vectors in it coincides with the number of positive and negative entries of $\langle x\sigma, x\sigma\rangle$. On the other hand, since $\sigma^\top = \sigma^{-1}$, it follows that $\langle x\sigma, x\sigma\rangle$ has the same spectrum as $\langle x, x\rangle$. □

We call a nondegenerate subspace W *positive* if $\operatorname{sig} W = (p_1, 0)$, and *negative* if $\operatorname{sig} W = (0, q_1)$. Moreover, W is *definite* if it is either positive or negative.

For a positive-semidefinite symmetric matrix v we write $\sqrt{v} = v^{\frac{1}{2}}$ for its *square root* – another symmetric positive-semidefinite matrix satisfying $\sqrt{v}\sqrt{v} = v$. The square root is a smooth function on the space of symmetric positive-semidefinite real matrices.

Lemma 2.5. *Let W be a definite subspace of V. Let $x = (x_1, \dots, x_m)$ be a basis of W. Then xh is an orthonormal basis for W, where $h = (\pm\langle x, x\rangle)^{-\frac{1}{2}}$ and the $\pm$ sign depends on whether W is positive or negative.*

Proof. Consider the negative case. We have that $h = (-\langle x, x\rangle)^{-\frac{1}{2}}$, so $\langle x, x\rangle = -h^{-2}$ Therefore,

$$\langle xh, xh\rangle = h\langle x, x\rangle h = h(-h^{-2})h = -1_m.$$

Therefore, xh is orthonormal. The positive case is similar. □

2.2 The Negative Grassmannian

Let V be a pseudoeuclidean space of signature (p, q). Let $(x, y) = (x_1, \dots, x_p, y_1, \dots, y_q)$ be a pseudoorthogonal basis of V. This basis defines an orientation on V and on all maximum definite subspaces of V. Let G be the identity component of $\mathrm{O}(V)$; this is an index 4 subgroup when p and q are nonzero, and an index 2 subgroup otherwise. We say that a orthonormal basis $z = (z_1, \dots, z_q)$ of some maximal negative subspace Z is *properly oriented* if there exists $g \in G$ such that $gz = y$. Let $K \cong \mathrm{SO}(p) \times \mathrm{SO}(q)$ be a subgroup of G that acts separately on $x_1, \dots, x_p$ and $y_1, \dots, y_q$. This is a maximum compact subgroup of G. In this section we describe some properties of the symmetric space G/K.

Consider the Grassmannian $\mathrm{Gr}_r(V)$. It is a real manifold of dimension $r(n-r)$. The signature then becomes a function

$$\mathrm{sig} : \mathrm{Gr}_r(V) \to \mathbb{Z} \times \mathbb{Z}.$$

The Grassmannian breaks down to a union of subsets corresponding to all possible signatures (p_1, q_1) with $p_1 \leqslant p$, $q_1 \leqslant q$. It could be shown that degenerate subspaces form a closed set of measure zero, and every nondegenerate signature (p_1, q_1), $p_1 + q_1 = r$ defines an open subset in $\mathrm{Gr}_r(V)$. Note, that the signature is *not* continuous on $\mathrm{Gr}_r(V)$. Let

$$\mathrm{Gr}_r^+(V) = \mathrm{sig}^{-1}(r, 0), \quad \mathrm{Gr}_r^0(V) = \mathrm{sig}^{-1}(0, 0), \quad \mathrm{Gr}_r^-(V) = \mathrm{sig}^{-1}(0, r)$$

be the *Positive*, the *Null*, and *Negative Grassmannians* respectively. The Negative Grassmannian $\mathrm{Gr}_q^-(V)$ is, therefore, the collection of all maximal negative subspaces.

Proposition 2.6. *The Negative Grassmannian $\mathrm{Gr}_q^-(V)$ is a noncompact real manifold of dimension pq.*

Note that the manifolds $\mathrm{Gr}_q^-(V)$ and $\mathrm{Gr}_p^+(V)$ are naturally diffeomorphic under the map $Z \mapsto Z^\perp$. Therefore, the properties of the Positive Grassmannian are completely identical.

Lemma 2.7. *Let $Z \in \mathrm{Gr}_q^-(V)$. The tangent space $\mathsf{T}_Z \mathrm{Gr}_q^-(V)$ is naturally isomorphic to $\mathrm{Hom}(Z, Z^\perp)$.*

Proof. Fix $Z \in \mathrm{Gr}_q^-(V)$. Consider a smooth map $\varphi : G \to \mathrm{Gr}_q^-(V)$, $g \mapsto gZ$. Its differential at the identity is $d\varphi : \mathfrak{o}(V) \to \mathsf{T}_Z \mathrm{Gr}_q^-(V)$. The Lie algebra $\mathfrak{o}(V)$ of the group $\mathrm{O}(V)$ is naturally identified with the set of operators $a : V \to V$ satisfying $a + a^* = 0$. Every such operator has a decomposition $a = \left(\begin{smallmatrix} b & w \\ -w^* & c \end{smallmatrix}\right)$, where $b : Z \to Z$, $c : Z^\perp \to Z^\perp$ are skew-adjoint, and $w : Z \to Z^\perp$ is arbitrary.

The kernel $\mathrm{Ker}\, d\varphi$ is the Lie algebra of the stabilizer $\mathrm{St}_{\mathrm{O}(V)}(Z)$ which consists of operators of the form $\left(\begin{smallmatrix} b & 0 \\ 0 & c \end{smallmatrix}\right)$. Therefore, the tangent space $\mathsf{T}_Z \mathrm{Gr}_q^-(V)$ is naturally identified with the subalgebra $\mathfrak{p} = \{\left(\begin{smallmatrix} 0 & w \\ -w^* & 0 \end{smallmatrix}\right) | w \in \mathrm{Hom}(Z, Z^\perp)\}$.

□

There is a natural inner product on $\mathrm{Hom}(Z, Z^\perp)$ inherited from V. It allows us to naturally define an invariant Riemannian metric on $\mathrm{Gr}_q^-(V)$.

Lemma 2.8. *Let $v, w \in \mathsf{T}_Z \mathrm{Gr}_q(V)$ be two tangent vectors identified with operators of the form $Z \to Z^\perp$. Then their inner product $\langle v, w\rangle$ is equal to $-\mathrm{tr}[v^* w]$.*

Proof. Let $x_1, \dots, x_p, y_1, \dots, y_q$ be a pseudoorthonormal basis of V such that $Z = \langle y_1, \dots, y_q\rangle \in \mathrm{Gr}_q^-(V)$ and $Z^\perp = \langle x_1, \dots, x_p\rangle$. Let $v = (v_{ij})_{i,j=1}^{q,p}$, $w = (w_{ij})_{i,j=1}^{q,p}$. Then the inner product defined on $V \otimes V$ and restricted to $Z \otimes Z^\perp$ gives

$$\langle v, w\rangle = \sum_{i,j=1}^{q,p} v_{ij} w_{ij}.$$

Then we can see that the inner product identifies the adjoint operator w^* with the matrix $(-w_{ji})_{i,j=1}^{p,q}$. Then we conclude that $\mathrm{tr}[v^* w] = -\sum v_{ij} w_{ij} = -\langle v, w\rangle$. This completes the proof. □

In the case $(p, q) = (m, 1)$ the negative Grassmannian is the *hyperbolic space* $\mathcal{H}^m$. In the sequel we write $\mathcal{D}$ for $\mathrm{Gr}_q^-(V)$.

Construct a parametrization for $\mathcal{D}$. It turns out that as a smooth manifold $\mathcal{D}$ is diffeomorphic to $\mathbb{R}^{pq}$. Fix a pseudoorthonormal basis $(x, y) = (x_1, \dots, x_p, y_1, \dots, y_q)$ for V. For $t = (t_{ij})_{i,j=1}^{p,q} \in \mathrm{M}_{p,q}(\mathbb{R})$ let $y(t) = (y_1(t), \dots, y_q(t)) = xt + y$, i.e.

$$\begin{aligned} y_1(t) =& y_1 + x_1 t_{11} + \dots + x_p t_{p1}, \\ y_2(t) =& y_2 + x_1 t_{12} + \dots + x_p t_{p2}, \\ & \dots\dots\dots\dots\dots \\ y_q(t) =& y_q + x_1 t_{1q} + \dots + x_p t_{pq}. \end{aligned}$$

We have that

$$\langle y(t), y(t)\rangle = -1 + t^\top t.$$

To obtain a parametrization of $\mathcal{D}$, we need this matrix to be negative-definite. First, we recall the properties of a matrix norm. For a vector $s = (s_1, \dots, s_n)^\top \in \mathbb{R}^n$ we denote $\|s\|$ for its usual Euclidean norm $\sqrt{s_1^2 + \cdots + s_n^2}$. For a matrix $t \in \mathrm{M}_{m,n}(\mathbb{R})$ let

$$\|t\| = \sup_{s \neq 0} \frac{\|ts\|}{\|s\|} = \sup_{s \in S^{n-1}} \|ts\|.$$

be its usual L^2-norm. Hence, for any $t \in \mathrm{M}_{m,n}(\mathbb{R})$ and $s \in \mathbb{R}^n$ it holds that $\|ts\| \leqslant \|t\|\|s\|$. Moreover, it could be proved that $\|t\|^2$ coincides with the largest eigenvalue of the positive-semidefinite symmetric matrix $t^\top t$. We have also $\|t^\top\| = \|t\|$ for any t. For a square complex matrix a we denote $\sigma(a)$ for its spectrum.

Lemma 2.9. *The matrix $-1 + t^\top t$ is negative-definite iff $\|t\| < 1$.*

Proof. The matrix $-1 + t^\top t$ is negative-definite iff $\sigma(-1 + t^\top t) \in (-\infty, 0)$. This is equivalent to $\sigma(t^\top t) \in (-\infty, 1)$. This identity holds iff the maximum eigenvalue of $t^\top t$, which is exactly $\|t\|^2$, is less than 1. □

Let $\mathcal{B}^{p,q} = \{t \in \mathrm{M}_{p,q}(\mathbb{R}) | \|t\| < 1\}$ be the *open matrix unit ball.* This is an open set in the space of all matrices. We get a parametrization of $\mathcal{D}$ by $\mathcal{B}^{p,q}$. This is the *Klein model* of $\mathcal{D}$.

Lemma 2.10. *The map $\mathcal{B}^{p,q} \to \mathcal{D}$, $t \mapsto \langle y_1(t), \dots, y_q(t)\rangle$ is a diffeomorphism.*

Proof. So far we have that $t \mapsto \langle y(t)\rangle$ is a smooth injective map. Now we prove it is surjective.

Let Z be a negative subspace. Let $Y = \langle y\rangle$. Assume that the orthogonal projection of Z onto Y does not coincide with Y. Then the kernel of this projection is non-trivial, and there exists a vector $z' \in Z$ such that $z' \perp Y$. This means that z' is positive, contradicting the assumption that Z was negative.

Hence, Z projects onto Y. By taking pre-images of base vectors $y_1, \dots, y_q$ we obtain a basis of Z of the form $xt + y$. □

Moreover, let $x(t) = x + yt^\top$. Then we have that $t \mapsto \langle x(t)\rangle$ gives the parametrization of the Positive Grassmannian $\mathrm{Gr}_p^+(V)$, and $x(t) \perp y(t)$. Indeed,

$$\langle x(t), y(t)\rangle = \langle x + yt^\top, xt + y\rangle = \begin{pmatrix} 1_p & t \end{pmatrix} \begin{pmatrix} 1_p & \\ & -1_q \end{pmatrix} \begin{pmatrix} t \\ 1_q \end{pmatrix} = t - t = 0.$$

From the basis $y(t)$ we can obtain, unlike the complete Grassmannian, an orthonormal base that smoothly depends on the point, just take $z(t) = y(t)(-\langle y(t), y(t)\rangle)^{-\frac{1}{2}}$.

Consider another way to parametrize $\mathcal{D}$. Let $z(t)$ be as above. Consider a map $r_{ij} : \mathcal{D} \to \mathbb{R}$ given by $r_{ij} = \langle x_i, z_j(t)\rangle$. Let $\rho = (r_{ij}) : \mathcal{D} \to \mathrm{M}_{p,q}(\mathbb{R})$.

Proposition 2.11. *The map ρ defines a differmorphism between $\mathcal{D}$ and $\mathrm{M}_{p,q}(\mathbb{R})$.*

Proof. We have that in t coordinates $\rho(\langle z(t)\rangle) = t(1 - t^\top t)^{-\frac{1}{2}}$. Therefore, it suffices to prove that the map

$$t \mapsto t(1 - t^\top t)^{-\frac{1}{2}}$$

is a diffeomorphism between $\mathcal{B}^{p,q}$ and $\mathrm{M}_{p,q}(\mathbb{R})$.

Construct the opposite map. Let $\psi : \mathrm{M}_{p,q}(\mathbb{R}) \to \mathrm{M}_{p,q}(\mathbb{R})$ be defined by $\psi(s) = s(1 + s^\top s)^{-\frac{1}{2}}$. Assume that $s = t(1 - t^\top t)^{-\frac{1}{2}}$. Then $ss^\top = t(1 - t^\top t)^{-1}t^\top$, but

$$(1 - t^\top t)^{-1} = 1 + t^\top t + (t^\top t)^2 + \cdots,$$

and the matrix series converges since $\sigma(t^\top t) \in [0, 1)$. Therefore,

$$ss^\top = t(1 + t^\top t + (t^\top t)^2 + \cdots)t^\top = tt^\top + (tt^\top)^2 + \cdots = (1 - tt^\top)^{-1} - 1,$$

and

$$1 + s^\top s = (1 - t^\top t)^{-1}.$$

Taking a square root we obtain that $\psi(s) = t$.

Finally, we show that $\psi(\mathrm{M}_{p,q}(\mathbb{R})) \subseteq \mathcal{B}^{p,q}$. Indeed,

$$\psi(s)^\top\psi(s) = (1 + s^\top s)^{-1/2}s^\top s(1 + s^\top s)^{-1/2} = s^\top s(1 + s^\top s)^{-1} = 1 - (1 + s^\top s)^{-1}.$$

Let $f(x) = 1 - (1 + x)^{-1}$. Whatever the spectrum of $s^\top s$ is we have that

$$\sigma(\psi(s)^\top\psi(s)) \subseteq f([0, \infty)) = [0, 1).$$

This implies that $\|\psi(s)\| < 1$. □

Therefore, we have another parametrization of $\mathcal{D}$ given by the map

$$s \mapsto \langle z(s)\rangle = \langle xs + y\sqrt{1 + s^\top s}\rangle,$$

where $s \in \mathrm{M}_{p,q}(\mathbb{R})$.

Note, that both parametrizations require a fixed pseudoorthonormal basis (x, y).

Proposition 2.12 ([W]). *The invariant volume form μ on $\mathcal{D}$ is given by the formula*

$$\mu = \frac{dt}{\det(1 - t^{\top}t)^{\frac{p+q}{2}}},$$

or

$$\mu = \frac{ds}{\det(1 + s^{\top}s)^{\frac{1}{2}}},$$

where

$$dt = \bigwedge_{i=1}^{p}\bigwedge_{j=1}^{q} dt_{ij}, \quad \textit{and} \quad ds = \bigwedge_{i=1}^{p}\bigwedge_{j=1}^{q} ds_{ij}.$$

We can also view the symmetric space $\mathcal{D}$ as the space of *majorants* of V. The inner product $\langle \cdot, \cdot \rangle$ on V is not positive-definite in general, but there are ways to make it so. Fix a subspace $Z \in \mathrm{Gr}_q^-(V)$. Let

$$\langle \cdot, \cdot \rangle_Z = \begin{cases} \langle \cdot, \cdot \rangle, & \text{on} \quad Z^{\perp}, \\ -\langle \cdot, \cdot \rangle, & \text{on} \quad Z, \end{cases}$$

that is,

$$\langle x, y \rangle_Z = \langle x_1, y_1 \rangle - \langle x_2, y_2 \rangle,$$

where $x = x_1 + x_2$, $y = y_1 + y_2$, with $x_1, y_1 \in Z^{\perp}$, and $x_2, y_2 \in Z$. We extend this product to matrix values similarly by letting $\langle x, y \rangle_Z = (\langle x_i, y_j \rangle_Z)_{i,j=1}^{r,s}$ for $x \in V^r$, $y \in V^s$.

Proposition 2.13. *Fix an orthonormal basis $z_1, \ldots, z_q$ of $Z \in \mathrm{Gr}_q^-(V)$. Then*

$$\langle x, y \rangle_Z = \langle x, y \rangle - 2\langle x, z \rangle \langle z, y \rangle.$$

for all $x \in V^r$, $y \in V^s$.

Proposition 2.14. *The bilinear form $\langle \cdot, \cdot \rangle_Z$ is a positive-definite inner product on V.*

Proof. Since $\langle x_1, x_1 \rangle \geqslant 0$ and $\langle x_2, x_2 \rangle \leqslant 0$ we have that for any vector $x = x_1 + x_2$ it holds that $\langle x, x \rangle_Z \geqslant 0$. □

For $Z \in \mathrm{Gr}_p^+(V)$ let $\langle x, y \rangle_Z = \langle x, y \rangle_{Z^{\perp}}$.

Proposition 2.15. *Fix an orthonormal basis $z_1, \ldots, z_p$ of $Z \in \mathrm{Gr}_p^+(V)$. Then*

$$\langle x, y \rangle_Z = -\langle x, y \rangle + 2\langle x, z \rangle \langle z, y \rangle.$$

for all $x \in V^r$, $y \in V^s$.

For $(p,q) = (m,0)$ we assume for convenience that $\mathrm{Gr}_0^-(V)$ is a singleton, and $\langle x, y\rangle_0 = \langle x, y\rangle$, where the unique point $0 \in \mathrm{Gr}_0^-(V)$ is the zero subspace $\{0\}$ of V.

Finally, for $\tau = u + \mathrm{i}v \in \mathcal{H}_n$ and $x, y \in V^n$ let $\langle x, y\rangle_{\tau,Z} = u\langle x, y\rangle + \mathrm{i}v\langle x, y\rangle_Z$.

2.3 The upper half-space model

In the case $(p,q) = (m,1)$ or $(p,q) = (1,m)$ the symmetric space $\mathcal{D}$ is isomorphic to the hyperbolic space $\mathcal{H}^m$. In the case of signature $(m,1)$ it is convenient to view $\mathcal{D}$ as a set of negative lines, i.e., as $\mathrm{Gr}_1^-(V)$, and as the set of positive lines in the case of signature $(1,m)$. For the hyperbolic space $\mathcal{H}^m$ we make use of the well-known *upper half-space model*. We identify $\mathcal{H}^m$ with the set $\{(y,t) | t > 0\}$, where $y = (y_1, \dots, y_{m-1}) \in \mathbb{R}^{m-1}$. The isomorphism with $\mathcal{D}$ in the signature $(1,m)$ is given by the map

$$(y,t) \mapsto \frac{1}{2t}\begin{pmatrix} 1 + \sum_{i=1}^{m-1} y_i^2 + t^2 \\ -2y_1 \\ \vdots \\ -2y_{m-1} \\ 1 - \sum_{i=1}^{m-1} y_i^2 - t^2 \end{pmatrix},$$

where we identify $\mathrm{Gr}_1^+(V)$ with the right component of the hyperboloid $\langle x, x\rangle = 1$ in V.

Indeed, we have that

$$\left(1 + \sum_{i=1}^{m-1} y_i^2 + t^2\right)^2 - \left(1 - \sum_{i=1}^{m-1} y_i^2 - t^2\right)^2 - \sum_{i=1}^{m-1} (-2y_i)^2 = 4t^2.$$

The similar formula for $\mathrm{Gr}_1^-(V)$ in the case of signature $(m,1)$ is

$$(y,t) \mapsto \frac{1}{2t}\begin{pmatrix} 1 - \sum_{i=1}^{m-1} y_i^2 - t^2 \\ 2y_1 \\ \vdots \\ 2y_{m-1} \\ 1 + \sum_{i=1}^{m-1} y_i^2 + t^2 \end{pmatrix},$$

Proposition 2.16. *The invariant volume form in the upper half-space model is*

$$\mu = \frac{dy_1 \wedge \ldots \wedge dy_{m-1} \wedge dt}{t^m}.$$

We now construct another model for the Negative Grassmannian that generalizes the upper half-space model to the case of arbitrary signature.

Assume that $p \geqslant q$. The operator

$$\begin{pmatrix} \frac{1}{\sqrt{2}}1_q & & \frac{1}{\sqrt{2}}1_q \\ & 1_{p-q} & \\ \frac{1}{\sqrt{2}}1_q & & -\frac{1}{\sqrt{2}}1_q \end{pmatrix}$$

sends the standard pseudoorthogonal basis to a Witt basis with the Gram matrix

$$\begin{pmatrix} & & 1_q \\ & 1_{p-q} & \\ 1_q & & \end{pmatrix}.$$

Let $u_1, \ldots, u_q, v_1, \ldots, v_{p-q}, w_1, \ldots, w_q$ be a Witt basis. Let $u = (u_1, \ldots, u_q) \in V^q$ be a frame generating a maximum null subspace.

Proposition 2.17. *The stabilizer* $\mathrm{St}_G(u)$ *consists of operators* $n(a,b,c)$ *of the form*

$$n(a,b,c) = \begin{pmatrix} 1_q & -b^\top a & -\frac{1}{2}b^\top b + c \\ & a & b \\ & & 1_q \end{pmatrix},$$

where $a \in \mathrm{SO}(p-q)$, $b \in \mathrm{M}_{p-q,q}(\mathbb{R})$, *and* $c \in \mathrm{M}_q(\mathbb{R})$ *is skew-symmetric.*

Let $N = \{n(1,b,c)\}$. This is an abelian subgroup of G of dimension $q(p-q) + \binom{q}{2}$.

In a pseudoorthonormal basis, every space $Z \in \mathrm{Gr}_q^-(V)$ has a unique basis of the form $\left(\begin{smallmatrix} t \\ 1_q \end{smallmatrix}\right)$, where $t = \left(\begin{smallmatrix} t_1 \\ t_2 \end{smallmatrix}\right) \in \mathcal{B}^{p,q}$ according to the Klein model. In the Witt coordinates this basis is

$$\begin{pmatrix} \frac{1}{\sqrt{2}}(t_1 + 1_q) \\ t_2 \\ \frac{1}{\sqrt{2}}(t_1 - 1_q) \end{pmatrix}.$$

Since the norm of the matrix t_1 does not exceed 1, the matrix $t_1 - 1_q$ is invertible, and the space Z is also generated by the frame

$$w(s) = \begin{pmatrix} s_1 \\ s_2 \\ 1_q \end{pmatrix} = \begin{pmatrix} (t_1 + 1_q)(t_1 - 1_q)^{-1} \\ \sqrt{2} t_2 (t_1 - 1_q)^{-1} \\ 1_q \end{pmatrix}.$$

For arbitrary $s = \left(\begin{smallmatrix} s_1 \\ s_2 \end{smallmatrix}\right) \in \mathrm{M}_{p,q}(\mathbb{R})$ we have

$$\langle w(s), w(s) \rangle = \begin{pmatrix} s_1^\top & s_2^\top & 1_q \end{pmatrix} \begin{pmatrix} & & 1_q \\ & 1_{p-q} & \\ 1_q & & \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ 1_q \end{pmatrix} = s_1^\top + s_1 + s_2^\top s_2.$$

Therefore, for $w(s)$ to generate a negative subspace, we need the *symmetric part* of $s_1 + \frac{1}{2} s_2^\top s_2$ to be negative-definite. Let $s_1 + s_1^\top = -2t^2$, where $t > 0$. Let $s_2 = \sqrt{2} y$. Then $s_1 = -t^2 - y^\top y - c$ for some skew-symmetric c. We have that

$$w(s) = w(t, y, c) = \begin{pmatrix} -t^2 - y^\top y - c \\ \sqrt{2} y \\ 1_q \end{pmatrix}.$$

In the corresponding pseudoorthonormal coordinates this basis is

$$w(t, y, c) = \begin{pmatrix} \frac{1}{\sqrt{2}}(1_q - t^2 - y^\top y - c) \\ \sqrt{2} y \\ \frac{1}{\sqrt{2}}(-1_q - t^2 - y^\top y - c) \end{pmatrix}.$$

We have that $\langle w(s), w(s) \rangle = -2t^2$. Thus by multiplying on the right by $(\sqrt{2} t)^{-1}$ we obtain an orthonormal basis. We also negate the last q rows to get a better looking formulas. We obtain the following result.

Theorem 2.18. *The following map*

$$(t, y, c) \mapsto z(t, y, c) = \begin{pmatrix} \frac{1}{2}(1_q - t^2 - y^\top y - c) t^{-1} \\ y t^{-1} \\ \frac{1}{2}(1_q + t^2 + y^\top y + c) t^{-1} \end{pmatrix},$$

where $y \in \mathrm{M}_{p-q,q}(\mathbb{R})$, $t \in \mathrm{M}_q(\mathbb{R})$ symmetric and positive-definite, and $c \in \mathrm{M}_q(\mathbb{R})$ skew-symmetric, defines a parametrization of $\mathrm{Gr}_q^-(V)$ by providing an orthonormal basis for every element $Z \in \mathrm{Gr}_q^-(V)$.

We can see that in the special case $\operatorname{sig} V = (m,1)$ this coincides with the upper half-space model for $\mathcal{H}^m$.

Note also that the group N freely acts on $\mathcal{D}$. We have that

$$n(1, \sqrt{2}y_1, c_1) \cdot z(t, y_2, c_2) = z(t, y_1 + y_2, c_1 + c_2).$$

Now we derive the formula for the invariant volume μ form in the new coordinates. For

$$t = \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1q} \\ t_{12} & t_{22} & \cdots & t_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ t_{1q} & t_{2q} & \cdots & t_{qq} \end{pmatrix}, y = \begin{pmatrix} y_{11} & \cdots & y_{1q} \\ \vdots & \ddots & \vdots \\ y_{p-q,1} & \cdots & y_{p-q,q} \end{pmatrix}, c = \begin{pmatrix} 0 & c_{12} & \cdots & c_{1q} \\ -c_{12} & 0 & \cdots & c_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ -c_{1q} & -c_{2q} & \cdots & 0 \end{pmatrix}$$

let

$$dt = \bigwedge_{1 \leqslant i \leqslant j \leqslant q} dt_{ij}, \quad dy = \bigwedge_{i=1}^{p-q} \bigwedge_{j=1}^{q} dy_{ij}, \quad dc = \bigwedge_{1 \leqslant i < j \leqslant q} dc_{ij}.$$

Theorem 2.19. *The following identity holds:*

$$\mu = \frac{s(t)}{\det(t)^{p+q-1}} dt \wedge dy \wedge dc,$$

where

$$s(t) = \prod_{1 \leqslant i < j \leqslant q} \frac{\lambda_i + \lambda_j}{2},$$

and $\sigma(t) = \{\lambda_1, \ldots, \lambda_q\}$ *is the spectrum of* t.

Proof. We compute the volume form according to the formula

$$\mu = \sqrt{|G|} dt \wedge dy \wedge dc,$$

where G is the Gram matrix for the corresponding tangent vectors. From the action of N it follows that $|G|$ does not depend on y and c. Moreover, we have that $\mathrm{SO}(q)$ acts on $\mathcal{D}$ by isometries. The map

$$\sigma \mapsto \begin{pmatrix} \sigma & & \\ & 1_{p-q} & \\ & & \sigma \end{pmatrix}$$

given in Witt coordinates is a well defined Lie group homomorphism $\mathrm{SO}(q) \to G$. The action on the frames $z(t, y, c)$ is the following:

$$\sigma \cdot z(t, y, c) = z(\sigma t \sigma^\top, y\sigma^\top, \sigma c \sigma^\top).$$

Therefore, it suffices to compute the Gram matrix for $t = \operatorname{diag}\{\lambda_1, \dots, \lambda_q\}$, $y = 0$, $c = 0$.

Note that if we have a curve $s \mapsto \langle z(s) \rangle = Z(s)$ in a Grassmannian, such that $z(s)$ is an orthonormal basis for every s, then $z'(s)$ could not be immediately identified with the tangent vector to this curve. By differentiating $\langle z(s), z(s) \rangle = -1_q$ we obtain only that $\langle z(s), z'(s) \rangle$ is skew-symmetric. The tangent vector is in fact an operator $\frac{\partial}{\partial s} \in \mathsf{T}_{Z(s)}\mathcal{D}$ sending $z(s)$ to $z'(s) + z(s)\langle z(s), z'(s) \rangle$. For a fixed point $Z(s)$ we identify the tangent vector $\frac{\partial}{\partial s}$ with the positive frame $z'(s) + z(s)\langle z(s), z'(s) \rangle$.

Now we compute the tangent vectors. By a straight-forward computation we obtain that

$$\frac{\partial}{\partial y_{ij}} = \begin{pmatrix} 0 \\ \frac{1}{\lambda_j} E_{ij} \\ 0 \end{pmatrix},$$

$$\frac{\partial}{\partial t_{ii}} = \begin{pmatrix} -\frac{1}{2}(\frac{1}{\lambda_i^2} - 1) E_{ii} \\ 0 \\ -\frac{1}{2}(\frac{1}{\lambda_i^2} + 1) E_{ii} \end{pmatrix},$$

$$\frac{\partial}{\partial t_{ij}} = \frac{1}{4}\left(\frac{1}{\lambda_i} + \frac{1}{\lambda_j}\right) \begin{pmatrix} (-\lambda_i - \frac{1}{\lambda_i}) E_{ij} + (-\lambda_j - \frac{1}{\lambda_j}) E_{ji} \\ 0 \\ (\lambda_i - \frac{1}{\lambda_i}) E_{ij} + (\lambda_j - \frac{1}{\lambda_j}) E_{ji} \end{pmatrix},$$

$$\frac{\partial}{\partial c_{ij}} = \frac{1}{4} \begin{pmatrix} (-\frac{1}{\lambda_i^2 \lambda_j} - \frac{1}{\lambda_j}) E_{ij} + (\frac{1}{\lambda_i \lambda_j^2} + \frac{1}{\lambda_i}) E_{ji} \\ 0 \\ (-\frac{1}{\lambda_i^2 \lambda_j} + \frac{1}{\lambda_j}) E_{ij} + (\frac{1}{\lambda_i \lambda_j^2} - \frac{1}{\lambda_i}) E_{ji} \end{pmatrix},$$

where E_{ij} is the corresponding elementary matrix.

We can see that all these tangent vectors are pairwise orthogonal. For example, we have that

$$\begin{aligned} \left\langle \frac{\partial}{\partial t_{ij}}, \frac{\partial}{\partial c_{ij}} \right\rangle &= \frac{1}{16}\left(\frac{1}{\lambda_i} + \frac{1}{\lambda_j}\right)\left(\left(-\lambda_i - \frac{1}{\lambda_i}\right)\left(-\frac{1}{\lambda_i^2 \lambda_j} - \frac{1}{\lambda_j}\right) - \left(\lambda_i - \frac{1}{\lambda_i}\right)\left(-\frac{1}{\lambda_i^2 \lambda_j} + \frac{1}{\lambda_j}\right)\right) \\ &+ \frac{1}{16}\left(\frac{1}{\lambda_i} + \frac{1}{\lambda_j}\right)\left(\left(-\lambda_j - \frac{1}{\lambda_j}\right)\left(\frac{1}{\lambda_i \lambda_j^2} + \frac{1}{\lambda_i}\right) - \left(\lambda_j - \frac{1}{\lambda_j}\right)\left(\frac{1}{\lambda_i \lambda_j^2} - \frac{1}{\lambda_i}\right)\right) \\ &= \frac{1}{16}\left(\frac{1}{\lambda_i} + \frac{1}{\lambda_j}\right)\left(\frac{2}{\lambda_i \lambda_j} + \frac{2}{\lambda_i \lambda_j} - \frac{2}{\lambda_i \lambda_j} - \frac{2}{\lambda_i \lambda_j}\right) = 0. \end{aligned}$$

Hence, the matrix G is diagonal. Therefore, we have then

$$\begin{aligned}\sqrt{|G|} &= \prod_{i=1}^{p-q}\prod_{j=1}^{q}\sqrt{\left\langle \frac{\partial}{\partial y_{ij}}, \frac{\partial}{\partial y_{ij}}\right\rangle}\cdot \prod_{1\leqslant i\leqslant j\leqslant q}\sqrt{\left\langle \frac{\partial}{\partial t_{ij}}, \frac{\partial}{\partial t_{ij}}\right\rangle}\cdot \prod_{1\leqslant i<j\leqslant q}\sqrt{\left\langle \frac{\partial}{\partial c_{ij}}, \frac{\partial}{\partial c_{ij}}\right\rangle}\\ &= \prod_{i=1}^{p-q}\prod_{j=1}^{q}\frac{1}{\lambda_j}\cdot\prod_{j=1}^{q}\frac{1}{\lambda_j}\cdot\prod_{1\leqslant i<j\leqslant q}\frac{1}{\sqrt{2}}\left(\frac{1}{\lambda_i}+\frac{1}{\lambda_j}\right)\cdot\prod_{1\leqslant i<j\leqslant q}\frac{1}{\sqrt{2}\lambda_i\lambda_j}\\ &= |t|^{-(p-q)}\cdot|t|^{-1}\cdot s(t)|t|^{-(q-1)}\cdot|t|^{-(q-1)} = s(t)|t|^{-(p+q-1)}.\end{aligned}$$

This completes the proof. □

Moreover, it is known [M] that for a positive-definite $q\times q$-matrix t it holds that $d(t^2) = 2^{\frac{q(q+1)}{2}}|t|s(t)dt$. Thus, if we introduce $w = t^2$, then in the new coordinates the volume form becomes

$$\mu = \frac{dw\wedge dy\wedge dc}{2^{\frac{q(q+1)}{2}}\det(w)^{\frac{p+q}{2}-1}}.$$

Note that w also runs over the space of symmetric positive-definite $q\times q$-matrices.

2.4 Explicit formulas for φ_{km} and θ_{km}

We construct the explicit formula following [KM2] for the differential forms φ_{km} and θ_{km} on $\mathcal{D}$ in the top degree case $n = p$ and in the case $q = 1$.

Let W be $\mathbb{R}^{2n}$ equipped with a standard symplectic inner product. Let $\mathbb{W} = V\otimes W$. The tensor product of inner forms defines a symplectic form on $\mathbb{W}$. We have, therefore, an embedding

$$\rho : \mathrm{O}(V)\times\mathrm{Sp}(n;\mathbb{R})\to\mathrm{Sp}(\mathbb{W}).$$

The groups $\mathrm{O}(V)$ and $\mathrm{Sp}(n;\mathbb{R})$ form a dual reductive pair in the sense of Howe. The embedding ρ lifts [KM1] to a homomorphism

$$\widetilde{\rho} : \mathrm{O}(V)\times\mathrm{Mp}(n;\mathbb{R})\to\mathrm{Mp}(\mathbb{W}).$$

The Weil representation $\mathbf{R}$ is a representation of $\mathrm{Mp}(\mathbb{W})$ on the space $\mathrm{L}^2(V^n)$, and its restriction is a well-defined representation of $\mathrm{Mp}(n;\mathbb{R})$ on $\mathrm{L}^2(V^n)$. The Schwartz space $\mathcal{S}(V^n)$ is a dense invariant subspace of $\mathrm{L}^2(V^n)$ under the action of $\mathrm{Mp}(n;\mathbb{R})$.

The form φ_{km} is an element of $[\mathcal{S}(V^n)\otimes\Omega^{nq}(\mathcal{D})]^G$, i.e. it is a G-invariant form in the sense that

$$\varphi_{km}(x) = (g^*\varphi_{km})(gx)$$

for all $g \in G$.

Fix a point $Z_0 \in \mathcal{D}$, and its stabilizer $K = G_{Z_0}$ in G. The tangent space $\mathsf{T}_{Z_0}\mathcal{D}$ is naturally identified with the subalgebra

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & w \\ w^* & 0 \end{pmatrix} \Big| w \in \mathrm{Hom}(Z_0, Z_0^\perp) \right\}$$

of the Lie algebra $\mathfrak{o}(V)$ of G. Then we have the following isomorphism

$$[\mathcal{S}(V^n) \otimes \Omega^{nq}(\mathcal{D})]^G \cong [\mathcal{S}(V^n) \otimes \Lambda^{nq}\mathfrak{p}^*]^K.$$

Therefore, we can define φ_{km} on $\mathfrak{p}$ and then spread it to the whole space $\mathcal{D}$ using the invariance property.

Let $e_1, \ldots, e_p, f_1, \ldots, f_q$ be the corresponding pseudoorthonormal basis of V. Then $e_i \otimes f_j$ form a basis of $\mathsf{T}_{Z_0}\mathcal{D}$. Let α_{ij} be a dual basis of $\mathsf{T}^*_{Z_0}\mathcal{D}$, i.e. $\alpha_{ij}(e_k \otimes f_l) = \delta_{ik}\delta_{jl}$. Let $w = (w_1, \ldots, w_n) \in \mathsf{T}_{Z_0}\mathcal{D}^n$.

The form $\varphi_{km}(x)$ then is equal [KM2, p.370] to

$$\prod_{i=1}^{n} \prod_{j=1}^{q} \nabla_{ij} \cdot \varphi_0,$$

where

$$\varphi_0(x) = e^{-\pi \sum\limits_{i=1}^{p+q} \sum\limits_{j=1}^{p} x_{ij}^2},$$

and

$$\nabla_{ij} = \frac{1}{2} \sum_{k=1}^{p} \left(x_{ki} - \frac{1}{2\pi} \frac{\partial}{\partial x_{ki}} \right) E_{kj},$$

and E_{ij} is exterior multiplication by α_{ij}.

Let

$$D_{ki} = \frac{1}{2} \left(x_{ki} - \frac{1}{2\pi} \frac{\partial}{\partial x_{ki}} \right) \tag{2.1}$$

be a differential operator. The operators D_{ij} pairwise commute; hence, the operator

$$D = \det(D_{ij}) = \sum_{\sigma \in \mathrm{S}_p} (-1)^\sigma D_{1,\sigma(1)} \ldots D_{p,\sigma(p)}. \tag{2.2}$$

is well-defined. We define a generalization $\mathcal{P}_{p,q}$ of the Hermite polynomial as a function on $\mathrm{M}_p(\mathbb{R})$ that satisfies the equation

$$D^q \varphi_0(x) = \mathcal{P}_{p,q}(x_0)\varphi_0(x), \tag{2.3}$$

where x_0 is a matrix constructed from the first p rows of x.

Now we can give the formulas for φ_{km} and θ_{km} in the top degree case.

Theorem 2.20. *The forms φ_{km} and θ_{km} in the top degree case $n = p$ are given by the formulas*

$$\begin{aligned} \varphi_{km}(x) &= \mathcal{P}_{p,q}(\langle x, z\rangle) e^{-\pi \operatorname{tr}\langle x,x\rangle_Z} \cdot \mu, \\ \theta_{km}(\tau) &= \sum_{x \in L^p + \mathbf{a}} \mathcal{P}_{p,q}(\sqrt{v}\langle x, z\rangle) e^{\pi \mathrm{i} \operatorname{tr}\langle x,x\rangle_{\tau,Z}} e^{2\pi \mathrm{i} \operatorname{tr}\langle x,\mathbf{b}\rangle} \cdot \mu, \end{aligned}$$

where $Z \in \mathcal{D} \cong \mathrm{Gr}_p^+(V)$, z is any properly oriented orthonormal basis of Z, and μ is the invariant volume form.

Proof. We have that

$$\varphi_{km} = \prod_{i=1}^{p} \prod_{j=1}^{q} \sum_{k=1}^{p} D_{ki} E_{kj} \cdot \varphi_0.$$

The exterior product

$$\alpha_{11} \wedge \cdots \wedge \alpha_{pq} = E_{11} \cdot \ldots \cdot E_{pq} \cdot 1$$

coincides with the invariant volume form μ on $\mathcal{D}$ since it equals 1 on every properly oriented orthonormal frame. By opening the brackets we obtain

$$\begin{aligned} \varphi_{km} &= \prod_{i=1}^{p} \prod_{j=1}^{q} \sum_{k=1}^{p} D_{ki} E_{kj} \cdot \varphi_0 \\ &= \sum_{k_{11},\ldots,k_{pq}=1}^{p} \left[D_{k_{11}1} \ldots D_{k_{1q}1} \ldots D_{k_{p1}p} \ldots D_{k_{pq}p} \cdot \varphi_0 \right] \cdot E_{k_{11}1} \ldots E_{k_{1q}q} \ldots E_{k_{p1}1} \ldots E_{k_{pq}q}. \end{aligned}$$

For the wedge product to be nonzero, we need the indices $k_{1j}, \ldots, k_{pj}$ to form a permutation of $1, \ldots, p$. Let $k_{ij} = \sigma_j(i)$ for some $\sigma_j \in \mathrm{S}_p$. Then the sum becomes

$$\sum_{\sigma_1,\ldots,\sigma_q \in \mathrm{S}_p} \left[D_{\sigma_1(1),1} \ldots D_{\sigma_q(1),1} \ldots D_{\sigma_1(p),p} \ldots D_{\sigma_q(p),p} \cdot \varphi_0 \right] \cdot E_{\sigma_1(1),1} \ldots E_{\sigma_q(1),q} \ldots E_{\sigma_1(p),1} \ldots E_{\sigma_q(p),q}.$$

Moreover, we have further

$$E_{\sigma_1(1),1} \ldots E_{\sigma_q(1),q} \ldots E_{\sigma_1(p),1} \ldots E_{\sigma_q(p),q} = (-1)^{\sigma_1 \ldots \sigma_q} E_{11} \ldots E_{1q} \ldots E_{p1} \ldots E_{pq},$$

and

$$\sum_{\sigma_1,\dots,\sigma_q\in \mathrm{S}_p}(-1)^{\sigma_1\dots\sigma_q}D_{\sigma_1(1),1}\dots D_{\sigma_q(1),1}\dots D_{\sigma_1(p),p}\dots D_{\sigma_q(p),p}$$
$$=\left[\sum_{\sigma\in \mathrm{S}_p}(-1)^{\sigma}D_{\sigma(1),1}\dots D_{\sigma(p),p}\right]^q.$$

Therefore, we have that $\varphi_{km}(x)=D^q\varphi_0(x)\cdot\mu$, where D is an operator given by

$$D=\det(D_{ij})=\sum_{\sigma\in \mathrm{S}_p}(-1)^{\sigma}D_{1,\sigma(1)}\dots D_{p,\sigma(p)}. \tag{2.4}$$

According to (2.3) we have

$$D^q\varphi_0(x)=\mathcal{P}_{p,q}(x_0)\varphi_0(x),$$

where $x_0=(x_{ij})_{i,j=1}^p$ is the square matrix constructed from the first p rows of x.

The first p rows of x are given by the matrix $\langle x,z\rangle$. Moreover, we have that

$$\varphi_0(x)=e^{-\pi\,\mathrm{tr}[x^\top x]}=e^{-\pi\,\mathrm{tr}[-\langle x,x\rangle+2\langle x,z\rangle\langle z,x\rangle]}=e^{-\pi\,\mathrm{tr}\langle x,x\rangle_Z}.$$

This gives the formula for φ_{km}.

Let $\tau=u+\mathrm{i}v\in\mathcal{H}_n$, and

$$g_\tau=\begin{pmatrix}1&u\\0&1\end{pmatrix}\begin{pmatrix}v^{\frac12}&0\\0&v^{-\frac12}\end{pmatrix}.$$

By slightly abusing the notation we write $\begin{pmatrix}a&0\\0&a^{-\top}\end{pmatrix}$ and $\begin{pmatrix}1&b\\0&1\end{pmatrix}$ for the pre-images of these elements in $\mathrm{Mp}(n;\mathbb{R})$ whose action of the Weil representation is given by

$$\left(\mathbf{R}\begin{pmatrix}a&0\\0&a^{-\top}\end{pmatrix}f\right)(x)=|a|^{\frac12}f(xa), \tag{2.5}$$
$$\left(\mathbf{R}\begin{pmatrix}1&b\\0&1\end{pmatrix}f\right)(x)=e^{\pi\mathrm{i}\langle x,xb\rangle}f(x). \tag{2.6}$$

We have that

$$\theta_{km}(\tau)=\sum_{x\in L^p+\mathbf{a}}|v|^{-\frac{p+q}{4}}(\mathbf{R}(g_\tau)\varphi_{km})(x)e^{2\pi\mathrm{i}\,\mathrm{tr}\langle x,\mathbf{b}\rangle}.$$

Futhermore,

$$\begin{aligned}(\mathbf{R}(g_\tau)\varphi_{km})(x) &= |v|^{\frac{p+q}{4}} e^{\pi \mathrm{i}\,\mathrm{tr}\langle x, xu\rangle}\varphi_{km}(x\sqrt{v}) \\ &= |v|^{\frac{p+q}{4}} e^{\pi \mathrm{i}\,\mathrm{tr}[u\langle x,x\rangle]} \cdot \mathcal{P}_{p,q}(\langle x\sqrt{v}, z\rangle) e^{-\pi\,\mathrm{tr}\langle x\sqrt{v}, x\sqrt{v}\rangle_Z}\mu \\ &= |v|^{\frac{p+q}{4}} \mathcal{P}_{p,q}(\sqrt{v}\langle x, z\rangle) e^{\pi \mathrm{i}\,\mathrm{tr}[u\langle x,x\rangle + \mathrm{i}v\langle x,x\rangle_Z]}\mu \\ &= |v|^{\frac{p+q}{4}} \mathcal{P}_{p,q}(\sqrt{v}\langle x, z\rangle) e^{\pi \mathrm{i}\,\mathrm{tr}\langle x,x\rangle_{\tau,Z}}\mu.\end{aligned}$$

This gives the formula for θ_{km}. □

Let

$$\varphi_{km}(\tau, x) = |v|^{-\frac{p+q}{4}}(\mathbf{R}(g_\tau)\varphi_{km})(x) = \mathcal{P}_{p,q}(\sqrt{v}\langle x, z\rangle) e^{\pi \mathrm{i}\,\mathrm{tr}\langle x,x\rangle_{\tau,Z}}\mu.$$

Then we have the formula

$$\theta_{km}(\tau) = \sum_{x \in L^p + \mathbf{a}} \varphi_{km}(\tau, x) e^{2\pi \mathrm{i}\,\mathrm{tr}\langle x, \mathbf{b}\rangle}.$$

We now describe some properties of the generalized Hermite polynomial $\mathcal{P}_{p,q}$. Let $x = (x_{ij})$ be a real $p \times p$-matrix.

Proposition 2.21. *The following properties of $\mathcal{P}_{p,q}$ hold.*

(i) *$\mathcal{P}_{p,q}(x)$ is a polynomial of degree pq in p^2 variables $x = (x_{ij})$. It is even or odd in every row and column depending on whether q is even or odd.*

(ii) *$\mathcal{P}_{p,q}(x)$ in invariant under transposition and left and right multiplication by elements of $\mathrm{SO}(p)$:*

$$\begin{aligned}\mathcal{P}_{p,q}(x^\top) &= \mathcal{P}_{p,q}(x), \\ \mathcal{P}_{p,q}(\sigma_1 x \sigma_2) &= \det(\sigma_1)^q \det(\sigma_2)^q \mathcal{P}_{p,q}(x).\end{aligned}$$

(iii) *If q is odd and x is singular then $\mathcal{P}_{p,q}(x) = 0$.*

(iv) *The following identity holds:*

$$\mathcal{P}_{p,q}(x) e^{-2\pi\,\mathrm{tr}[x^\top x]} = (-4\pi)^{-pq} \det\left(\tfrac{\partial}{\partial x_{ij}}\right)^q e^{-2\pi\,\mathrm{tr}[x^\top x]}.$$

Proof. We prove (iv) first. For a smooth function f we have that

$$\begin{aligned}&\tfrac{\partial}{\partial x}(f(x)e^{-2\pi x^2}) = (f'(x) - 4\pi x f(x))e^{-2\pi x^2} = -4\pi(xf(x) - \tfrac{1}{4\pi}f'(x))e^{-\pi x^2}\cdot e^{-\pi x^2}\\ &= -4\pi\cdot\tfrac{1}{2}\left(x - \tfrac{1}{2\pi}\tfrac{\partial}{\partial x}\right)(f(x)e^{-\pi x^2})\cdot e^{-\pi x^2}.\end{aligned}$$

Therefore, for $x = (x_{ij})$ and a function $f(x)$ on $\mathrm{M}_p(\mathbb{R})$ it holds that

$$D_{ij}(f(x)e^{-\pi\,\mathrm{tr}[x^\top x]})e^{-\pi x_{ij}^2} = (-4\pi)^{-1}\tfrac{\partial}{\partial x_{ij}}(f(x)e^{-2\pi\,\mathrm{tr}[x^\top x]}),$$

and

$$D(f(x)e^{-\pi\,\mathrm{tr}[x^\top x]})e^{-\pi\,\mathrm{tr}[x^\top x]} = (-4\pi)^{-p}\det(\tfrac{\partial}{\partial x})(f(x)e^{-2\pi\,\mathrm{tr}[x^\top x]}),$$

where $\frac{\partial}{\partial x}$ is the differential operator matrix $(\frac{\partial}{\partial x_{ij}})_{i,j=1}^p$.

Then by induction on q we show (iv). The base case is trivial, and

$$\begin{aligned}&\mathcal{P}_{p,q+1}(x)e^{-2\pi\,\mathrm{tr}[x^\top x]} = D(\mathcal{P}_{p,q}(x)e^{-\pi\,\mathrm{tr}[x^\top x]})e^{-\pi\,\mathrm{tr}[x^\top x]}\\ &= (-4\pi)^{-p}\det(\tfrac{\partial}{\partial x})(\mathcal{P}_{p,q}(x)e^{-2\pi\,\mathrm{tr}[x^\top x]})\\ &= (-4\pi)^{-p}\det(\tfrac{\partial}{\partial x})\left[(-4\pi)^{-pq}\det(\tfrac{\partial}{\partial x})^q e^{-2\pi\,\mathrm{tr}[x^\top x]}\right]\\ &= (-4\pi)^{-p(q+1)}\det(\tfrac{\partial}{\partial x})^{q+1}e^{-2\pi\,\mathrm{tr}[x^\top x]}.\end{aligned}$$

Thus, we have that

$$\begin{aligned}&\mathcal{P}_{p,q}(x^\top)e^{-2\pi\,\mathrm{tr}[xx^\top]} = (-4\pi)^{-pq}\det(\tfrac{\partial}{\partial x}^\top)e^{-2\pi\,\mathrm{tr}[xx^\top]}\\ &= (-4\pi)^{-pq}\det(\tfrac{\partial}{\partial x})e^{-2\pi\,\mathrm{tr}[x^\top x]} = \mathcal{P}_{p,q}(x)e^{-2\pi\,\mathrm{tr}[x^\top x]}.\end{aligned}$$

implying that $\mathcal{P}_{p,q}(x) = \mathcal{P}_{p,q}(x^\top)$.

For the second part of (ii) let $y = (y_{ij}) = \sigma_1 x \sigma_2$. Let $\frac{\partial}{\partial y}$ be the differential operator matrix $(\frac{\partial}{\partial y_{ij}})_{i,j=1}^p$. Then $\frac{\partial}{\partial y} = \sigma_1(\frac{\partial}{\partial x})\sigma_2$ and $\det(\frac{\partial}{\partial y}) = \det(\sigma_1)\det(\sigma_2)\det(\frac{\partial}{\partial x_{ij}})$. Then, since $\mathrm{tr}[x^\top x] = \mathrm{tr}[y^\top y]$ by (iv) we have that

$$\begin{aligned}\mathcal{P}_{p,q}(y)e^{-2\pi\,\mathrm{tr}[y^\top y]} &=(-4\pi)^{-pq}\det(\tfrac{\partial}{\partial y})^q e^{-2\pi\,\mathrm{tr}[x^\top x]}\\ &=(-4\pi)^{-pq}\det(\sigma_1)^q\det(\sigma_2)^q\det(\tfrac{\partial}{\partial x})^q e^{-2\pi\,\mathrm{tr}[x^\top x]}\\ &=\det(\sigma_1)^q\det(\sigma_2)^q\mathcal{P}_{p,q}(x)e^{-2\pi\,\mathrm{tr}[x^\top x]}.\end{aligned}$$

After cancelling $e^{-2\pi\,\mathrm{tr}[x^\top x]}$ we obtain (ii).

By choosing $\sigma_1 = \mathrm{diag}(1,\dots,-1,\dots,1)$ we obtain from (ii) that $\mathcal{P}_{p,q}$ is odd in every row for q odd and even for q even. This implies (i).

Finally, we obtain (iii). Let x be singular. According to (ii), we can assume x is diagonal. Then one of its diagonal entries must be zero, and we have a zero row here. But for q odd we have that $\mathcal{P}_{p,q}$ is odd in every row according to (i). Thus, $\mathcal{P}_{p,q}(x) = 0$. $\square$

Consider some special cases depending on the signature (p, q) of V.

$(p, 1)$ If $q = 1$ then $D\varphi_0(x) = \det(x_{ij})\varphi_0(x)$ since $D_{ij}\varphi_0(x) = x_{ij}\varphi_0(x)$. Therefore, $\mathcal{P}_{p,1}(x) = \det(x)$.

$(1, q)$ If $p = 1$ then $x = x_{11}$, $D = D_{11}$, and $\mathcal{P}_q = \mathcal{P}_{1,q}$ is a single variable polynomial of degree q. The property (iii) of Proposition 2.21 implies the following connection to the Hermite polynomials $H_q(x)$:

$$\mathcal{P}_q(x) = 2^{-q}(2\pi)^{-q/2}H_q(\sqrt{2\pi}x).$$

In particular, we have that

$$\begin{aligned}
\mathcal{P}_0(x) &= 1, \\
\mathcal{P}_1(x) &= x, \\
\mathcal{P}_2(x) &= x^2 - \tfrac{1}{4\pi}, \\
\mathcal{P}_3(x) &= x^3 - \tfrac{3}{4\pi}x, \\
\mathcal{P}_4(x) &= x^4 - \tfrac{3}{2\pi}x^2 + \tfrac{3}{16\pi^2}.
\end{aligned}$$

For $q = 2k$ even we have

$$\mathcal{P}_{2k}(0) = (-1)^k \frac{(2k-1)!!}{2^{2k}\pi^k}.$$

$(2, 2)$ We have that $D = D_{11}D_{22} - D_{12}D_{21}$, and $D^2 = D_{11}^2D_{22}^2 + D_{12}^2D_{21}^2 - 2D_{11}D_{12}D_{21}D_{22}$. By a straight-forward computation we obtain

$$\begin{aligned}
\mathcal{P}_{2,2}(x) =& (x_{11}^2 - \tfrac{1}{4\pi})(x_{22}^2 - \tfrac{1}{4\pi}) + (x_{12}^2 - \tfrac{1}{4\pi})(x_{21}^2 - \tfrac{1}{4\pi}) - 2x_{11}x_{12}x_{21}x_{22} \\
=& (x_{11}x_{22} - x_{12}x_{21})^2 - \tfrac{1}{4\pi}(x_{11}^2 + x_{12}^2 + x_{21}^2 + x_{22}^2) + \tfrac{1}{8\pi^2} \\
=& \det(x)^2 - \tfrac{1}{4\pi}\operatorname{tr}[x^\top x] + \tfrac{1}{8\pi^2}.
\end{aligned}$$

Moreover, we can compute some Fourier transforms featuring $\mathcal{P}_{p,q}$.

Lemma 2.22. *Let* $f(x) = \mathcal{P}_{p,q}(x)e^{-2\pi\operatorname{tr}[x^\top x]}$. *Then*

$$\widehat{f}(w) = \mathrm{i}^{-pq}2^{-pq-\frac{p^2}{2}}\det(w)^q e^{-\frac{1}{2}\pi\operatorname{tr}[w^\top w]}.$$

Proof. Let $g(x) = e^{-2\pi\operatorname{tr}[x^\top x]}$. Then $\widehat{g}(w) = 2^{-\frac{p^2}{2}}e^{-\frac{1}{2}\pi\operatorname{tr}[w^\top w]}$. We have that $f(x) = (-4\pi)^{-pq}\det(\frac{\partial}{\partial x_{ij}})^q g(x)$. The partial derivative $\frac{\partial}{\partial x_{ij}}$ corresponds to multiplication by $2\pi\mathrm{i}w_{ij}$. Thus, the differential operator $(-4\pi)^{-p}\det(\frac{\partial}{\partial x_{ij}})$ corresponds to multiplication by $(-4\pi)^{-p}(2\pi\mathrm{i})^p\det(w) = (2\mathrm{i})^{-p}\det(w)$. After applying it q times we obtain the claim. □

Lemma 2.23. *Let* $f(x) = \mathcal{P}_{p,q}(\left(\begin{smallmatrix} x \\ w_0 \end{smallmatrix}\right))e^{-2\pi\operatorname{tr}[x^\top x]}$ *for* $x \in \mathrm{M}_{q,p}(\mathbb{R})$, *and* $w_0 \in \mathrm{M}_{p-q,p}(\mathbb{R})$ *fixed. Then*

$$\widehat{f}(w) = \mathcal{Q}_{p,q}(\left(\begin{smallmatrix} w \\ w_0 \end{smallmatrix}\right))e^{-\frac{1}{2}\pi\operatorname{tr}[w^\top w]},$$

where $\mathcal{Q}_{p,q}$ *is a polynomial depending on* p^2 *variables that satisfies the formula*

$$\mathcal{Q}_{p,q}(\left(\begin{smallmatrix} w \\ w_0 \end{smallmatrix}\right))e^{-\frac{1}{2}\pi\operatorname{tr}[w^\top w]-2\pi\operatorname{tr}[w_0^\top w_0]} = (-1)^{pq}\mathrm{i}^{q^2}2^{q^2-\frac{5pq}{2}}\pi^{q^2-pq}\cdot D_1^q e^{-2\pi\operatorname{tr}[w^\top w+w_0^\top w_0]},$$

where

$$D_1 = \det\begin{pmatrix} w_{11} & w_{12} & \cdots & w_{1p} \\ \vdots & \vdots & \ddots & \vdots \\ w_{q1} & w_{q2} & \cdots & w_{qp} \\ \frac{\partial}{\partial w^0_{11}} & \frac{\partial}{\partial w^0_{12}} & \cdots & \frac{\partial}{\partial w^0_{1p}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial}{\partial w^0_{p-q,1}} & \frac{\partial}{\partial w^0_{p-q,2}} & \cdots & \frac{\partial}{\partial w^0_{p-q,p}} \end{pmatrix},$$

$w = (w_{ij})_{i,j=1}^{q,p}$, *and* $w_0 = (w^0_{ij})_{i,j=1}^{p-q,p}$.

Proof. Let $g(x) = e^{-2\pi\operatorname{tr}[x^\top x]-2\pi\operatorname{tr}[w_0^\top w_0]}$. Then $\widehat{g}(w) = 2^{-\frac{pq}{2}}e^{-\frac{1}{2}\pi\operatorname{tr}[w^\top w]-2\pi\operatorname{tr}[w_0^\top w_0]}$. We have that $f(x) = e^{2\pi\operatorname{tr}[w_0^\top w_0]}(-4\pi)^{-pq}D^q g(x)$, where

$$D = \det\begin{pmatrix} \frac{\partial}{\partial x_{11}} & \frac{\partial}{\partial x_{12}} & \cdots & \frac{\partial}{\partial x_{1p}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x_{q1}} & \frac{\partial}{\partial x_{q2}} & \cdots & \frac{\partial}{\partial x_{qp}} \\ \frac{\partial}{\partial w^0_{11}} & \frac{\partial}{\partial w^0_{12}} & \cdots & \frac{\partial}{\partial w^0_{1p}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial}{\partial w^0_{p-q,1}} & \frac{\partial}{\partial w^0_{p-q,2}} & \cdots & \frac{\partial}{\partial w^0_{p-q,p}} \end{pmatrix}.$$

As before, the partial derivative $\frac{\partial}{\partial x_{ij}}$ corresponds to multiplication by $2\pi\mathrm{i}w_{ij}$. Hence, after we decompose D along the first q rows, we obtain that D corresponds to the operator

$(2\pi \mathrm{i})^q D_1$. Therefore,

$$\begin{aligned}\widehat{f}(w) =&2^{-\frac{pq}{2}}(-4\pi)^{-pq}(2\pi \mathrm{i})^{q^2} e^{2\pi \operatorname{tr}[w_0^\top w_0]} D_1^q e^{-2\pi \operatorname{tr}[w^\top w + w_0^\top w_0]} \\ =&(-1)^{pq} \mathrm{i}^{q^2} 2^{q^2-\frac{5pq}{2}} \pi^{q^2-pq} e^{2\pi \operatorname{tr}[w_0^\top w_0]} D_1^q e^{-2\pi \operatorname{tr}[w^\top w + w_0^\top w_0]}.\end{aligned}$$

This completes the proof. □

In particular, we can see that $\mathcal{Q}_{p,q}(\left(\begin{smallmatrix} w \\ w_0 \end{smallmatrix}\right))$ is zero if any row of the matrix w is zero.

Now assume that $q = 1$ and $1 \leqslant n \leqslant p$. In this case it is convenient to view the symmetric space $\mathcal{D}$ as the set of negative lines in V, i.e. $\mathrm{Gr}_1^-(V)$. Every negative line has a unique properly oriented generator z satisfying $\langle z, z\rangle = -1$. Hence, we identify $\mathcal{D}$ with one component of the hyperboloid $\langle z, z\rangle = -1$. The tangent space $\mathsf{T}_z\mathcal{D}$ is identified with the positive subspace $z^\perp$. We identify $\mathsf{T}_z^*\mathcal{D}$ with $z^\perp$ as well.

Theorem 2.24. *The forms φ_{km} and θ_{km} in the case $q = 1$ and $1 \leqslant n \leqslant p$ are given by the formulas*

$$\begin{aligned}\varphi_{km}(x)(w) &= e^{-\pi \operatorname{tr}\langle x,x\rangle_z} \det\langle x, w\rangle, \\ \theta_{km}(\tau)(w) &= |v|^{\frac{1}{2}} \sum_{x \in L^n + \mathbf{a}} e^{\pi \mathrm{i} \operatorname{tr}\langle x,x\rangle_{\tau,z}} e^{2\pi \mathrm{i} \operatorname{tr}\langle x, \mathbf{b}\rangle} \det\langle x, w\rangle,\end{aligned}$$

where $z \in \mathcal{D}$, $w = (w_1, \ldots, w_n) \in \mathsf{T}_z\mathcal{D}^n$.

Proof. We have that

$$\varphi_{km} = \prod_{i=1}^{n} \sum_{k=1}^{p} D_{ki} E_k \cdot \varphi_0,$$

where $D_{ki} = \frac{1}{2}\left(x_{ki} - \frac{1}{2\pi}\frac{\partial}{\partial x_{ki}}\right)$, and E_k is an exterior multiplication by α_k, where $\alpha_1, \ldots, \alpha_p$ is an orthonormal basis of $\mathsf{T}_z\mathcal{D}$.

After opening brackets we obtain the sum

$$\begin{aligned}&\sum_{k_1=1}^{p} \cdots \sum_{k_n=1}^{p} [D_{k_1 1} \cdot \ldots \cdot D_{k_n n} \cdot \varphi_0] \cdot \alpha_{k_1} \wedge \cdots \wedge \alpha_{k_n} \\ =&\varphi_0(x) \sum_{k_1=1}^{p} \cdots \sum_{k_n=1}^{p} [x_{k_1 1} \cdot \ldots \cdot x_{k_n n}] \cdot \alpha_{k_1} \wedge \cdots \wedge \alpha_{k_n} \\ =&\varphi_0(x) \cdot \beta_1 \wedge \cdots \wedge \beta_n,\end{aligned}$$

where

$$\beta_i = \sum_{k=1}^{p} x_{ki} \alpha_k.$$

For $w \in \mathsf{T}_z\mathcal{D}$ we have that $\beta_i(w) = \langle x_i, w\rangle$. Then

$$(\beta_1 \wedge \cdots \wedge \beta_n)(w_1, \ldots, w_n) = \det(\langle x_i, w_j\rangle).$$

This completes the proof for φ_{km}. We omit the computation for θ_{km} since it follows the same steps as in the top degree case. $\square$

Consider also another special case $\operatorname{sig} V = (n, 0)$. In this case $\mathcal{D}$ is just a point and $\theta_{km}(\tau)$ is simply a function of τ. We can see that it coincides with the classic Siegel genus n theta series

$$\Theta(\tau) = \sum_{x \in L^n + \mathbf{a}} e^{\pi \mathrm{i} \operatorname{tr}[\tau\langle x, x\rangle]} e^{2\pi \mathrm{i} \operatorname{tr}\langle x, \mathbf{b}\rangle}.$$

Therefore, we can view Θ_{km} as a generalization of the classic theta functions.

Chapter 3

Integrals of θ_{km} for $\operatorname{sig} V = (m, 1)$

Let V be a pseudoeuclidean space of signature $(m, 1)$, let $Q(x) = \frac{\langle x,x\rangle}{2}$. In his Ph.D. thesis [Zw] Zwegers introduced functions

$$\Theta^{c,c'}_{\mathbf{a},\mathbf{b}}(\tau) = \sum_{x\in L+\mathbf{a}} \big(\operatorname{sign}\langle c, x\rangle - \operatorname{sign}\langle c', x\rangle\big) e^{2\pi i\tau Q(x)} e^{2\pi i\langle x,\mathbf{b}\rangle},$$

$$\widehat{\Theta}^{c,c'}_{\mathbf{a},\mathbf{b}}(\tau) = \sum_{x\in L+\mathbf{a}} \left[E\Big(\frac{\langle c, x\rangle\sqrt{v}}{\sqrt{-Q(c)}}\Big) - E\Big(\frac{\langle c', x\rangle\sqrt{v}}{\sqrt{-Q(c')}}\Big)\right] e^{2\pi i\tau Q(x)} e^{2\pi i\langle x,\mathbf{b}\rangle},$$

where $\tau = u + iv \in \mathcal{H}$, $\mathbf{a}, \mathbf{b} \in V$, c, c' belong to the same component of the cone $\{x \in V|Q(x) < 0\}$, and E is the *error function* defined by

$$E(z) = 2\int_0^z e^{-\pi x^2}dx, \quad E(\pm\infty) = \pm 1.$$

Zwegers proved the following results regarding his theta functions.

Theorem 3.1 ([Z])**.** *The non-holomorphic function $\widehat{\Theta}^{c,c'}_{\mathbf{a},\mathbf{b}}$ satisfies the same transformation equations expressing $\widehat{\Theta}^{c,c'}_{\mathbf{a},\mathbf{b}}(\tau+1)$ and $\widehat{\Theta}^{c,c'}_{\mathbf{a},\mathbf{b}}(-1/\tau)$ as finite linear combinations of functions $\widehat{\Theta}^{c,c'}_{\mathbf{a}',\mathbf{b}'}(\tau)$ as in the positive-definite case, and in particular is a non-holomorphic modular form of weight $(m+1)/2$.*

The function $\Theta^{c,c'}_{\mathbf{a},\mathbf{b}}$ is the *holomorphic part* of $\widehat{\Theta}^{c,c'}_{\mathbf{a},\mathbf{b}}$. It does not satisfy the modular transformation law. The exact relation with $\widehat{\Theta}^{c,c'}_{\mathbf{a},\mathbf{b}}$ is the following. Let

$$\beta(x) = \int_x^\infty u^{-\frac{1}{2}} e^{-\pi u} du = 2\int_{\sqrt{x}}^\infty e^{-\pi t^2} dt = 1 - E(\sqrt{x}),$$

for $x \geqslant 0$ be the *complementary error function*, or *incomplete gamma function*. Let

$$\Phi^c_{\mathbf{a},\mathbf{b}}(\tau) = \sum_{x \in L+\mathbf{a}} \operatorname{sign}\langle x, c\rangle \beta(v\langle x, c\rangle^2) e^{2\pi i Q(x)} e^{2\pi i \langle x, \mathbf{b}\rangle}.$$

be the *Phi function*. The series above is rapidly convergent with summands bounded by $e^{-A\|x\|^2}$ for some constant $A > 0$. Then

$$\widehat{\Theta}^{c,c'}_{\mathbf{a},\mathbf{b}}(\tau) = \Theta^{c,c'}_{\mathbf{a},\mathbf{b}}(\tau) - \Phi^c_{\mathbf{a},\mathbf{b}}(\tau) + \Phi^{c'}_{\mathbf{a},\mathbf{b}}(\tau).$$

If $c' = \gamma c$ for some $\gamma \in \Gamma$ then

$$\begin{aligned}\Phi^{c'}_{\mathbf{a},\mathbf{b}}(\tau) &= \sum_{x \in L+\mathbf{a}} \operatorname{sign}\langle x, \gamma c\rangle \beta(v\langle x, \gamma c\rangle^2) e^{2\pi i Q(x)} e^{2\pi i \langle x, \mathbf{b}\rangle} \\ &= \sum_{x \in L+\mathbf{a}} \operatorname{sign}\langle \gamma^{-1} x, c\rangle \beta(v\langle \gamma^{-1} x, c\rangle^2) e^{2\pi i Q(\gamma^{-1} x)} e^{2\pi i \langle \gamma^{-1} x, \mathbf{b}\rangle} \\ &= \sum_{x \in \gamma^{-1}(L+\mathbf{a})} \operatorname{sign}\langle x, c\rangle \beta(v\langle x, c\rangle^2) e^{2\pi i Q(x)} e^{2\pi i \langle x, \mathbf{b}\rangle} = \Phi^c_{\mathbf{a},\mathbf{b}}(\tau),\end{aligned}$$

since $\gamma^{-1}(L+\mathbf{a}) = L+\mathbf{a}$. This implies that $\widehat{\Theta}^{c,c'}_{\mathbf{a},\mathbf{b}}(\tau) = \Theta^{c,c'}_{\mathbf{a},\mathbf{b}}(\tau)$ and we have a holomorphic modular form in this case.

We show that the Zwegers' theta function $\widehat{\Theta}^{c,c'}_{\mathbf{a},\mathbf{b}}(\tau)$ could be obtained as an integral of $\theta_{km}(\tau)$ for V in the case $n = 1$. More precisely, if we identify $\mathcal{D}$ with one component of the hyperboloid $\langle x, x\rangle = -1$, we get the following result.

Theorem 3.2. *Let $c, c' \in \mathcal{D}$ and let $z_{c,c'}$ be a geodesic path between c and c'. Then*

$$\Theta^{z_{c,c'}}_{km}(\tau) = \sqrt{2} \int_c^{c'} \theta_{km}(\tau) = -\frac{1}{2} \widehat{\Theta}^{c,c'}_{\mathbf{a},\mathbf{b}}(\tau).$$

For $n \geqslant 2$ we consider integrals of θ_{km} over geodesic simplices as a higher-dimensional generalization of the Zwegers theta functions. In the case $n = 2$, we compute the explicit formula for $\Theta^{\Delta}_{km}(\tau)$ where Δ is a geodesic triangle in $\mathcal{D}$. In the final formula we use the function

$$F(\alpha, \beta; z) = \int_0^z e^{-\frac{\pi}{\alpha \cos^2 x - \beta}} dx. \qquad (F)$$

For a different approach to higher error functions see, e.g. [ABMP].

3.1 Zwegers' results

Assume $\operatorname{sig} V = (m,1)$. Then for $n = 1$ the theta series θ_{km} is a closed 1-form on the hyperbolic space $\mathcal{H}^m$ given by the formula

$$\theta_{km}(\tau)(w) = \sum_{x \in L+\mathbf{a}} \sqrt{v} e^{\pi \mathrm{i} \tau \langle x,x \rangle - 2\pi v \langle x,z \rangle^2} e^{2\pi \mathrm{i} \langle x, \mathbf{b} \rangle} \langle x, w \rangle,$$

where $w \in \mathsf{T}_z \mathcal{D}$.

Proof of Theorem 3.2. The geodesic $z_{c,c'}$ in the hyperbolic space is given by a path $z = z_{c,c'} : [0,1] \to \mathcal{D}$, $z(t) = \frac{tc+(1-t)c'}{\sqrt{-\langle tc+(1-t)c', tc+(1-t)c' \rangle}}$. The theta series θ_{km} is termwise integrable over $z_{c,c'}$. We have that

$$\begin{aligned} \Theta_{km}^{z_{c,c'}}(\tau) = \sqrt{2} \int\limits_{z_{c,c'}} \theta_{km}(\tau) &= \sum_{x \in L+\mathbf{a}} \left[\int\limits_{z_{c,c'}} \sqrt{2v} e^{-2\pi v \langle x,z \rangle^2} \langle x, w \rangle \right] e^{\pi \mathrm{i} \tau \langle x,x \rangle} e^{2\pi \mathrm{i} \langle x, \mathbf{b} \rangle} \\ &= \sum_{x \in L+\mathbf{a}} \left[\int\limits_0^1 \sqrt{2v} e^{-2\pi v \langle x, z(t) \rangle^2} \langle x, z'(t) \rangle dt \right] e^{\pi \mathrm{i} \tau \langle x,x \rangle} e^{2\pi \mathrm{i} \langle x, \mathbf{b} \rangle}. \end{aligned}$$

Let $s = \sqrt{2v}\langle x, z(t) \rangle$. Then $ds = \sqrt{2v}\langle x, z'(t) \rangle$ and the sum is equal to

$$\begin{aligned} &\sum_{x \in L+\mathbf{a}} \left[\int\limits_{\sqrt{2v}\langle x,c \rangle}^{\sqrt{2v}\langle x,c' \rangle} e^{-\pi s^2} ds \right] e^{\pi i \tau \langle x,x \rangle} e^{2\pi \mathrm{i} \langle x, \mathbf{b} \rangle} \\ &= \frac{1}{2} \sum_{x \in L+\mathbf{a}} \left[E(\sqrt{2v}\langle x, c' \rangle) - E(\sqrt{2v}\langle x, c \rangle) \right] e^{\pi i \tau \langle x,x \rangle} e^{2\pi \mathrm{i} \langle x, \mathbf{b} \rangle} = -\frac{1}{2} \widehat{\Theta}_{\mathbf{a},\mathbf{b}}^{c,c'}(\tau). \end{aligned}$$

□

As a corollary we obtain Theorem 3.1 since $\Theta_{km}^{z_{c,c'}}$ inherits the modular transformation law from θ_{km}.

Moreover, if $c' = \gamma c$ for some $\gamma \in \Gamma$ then the path $z_{c,c'}$ is closed in the locally symmetric space $\Gamma \backslash \mathcal{D}$, and it represents a homology class $[z_{c,c'}] \in H_1(\Gamma \backslash \mathcal{D})$. Therefore, Theorem 1.2 implies then that $\Theta_{km}^{z_{c,c'}}$ is holomorphic explaining the equality $\widehat{\Theta}_{\mathbf{a},\mathbf{b}}^{c,c'}(\tau) = \Theta_{\mathbf{a},\mathbf{b}}^{c,c'}(\tau)$ in this case.

3.2 Integral over a geodesic simplex in $\mathcal{H}^m$

In general the theta series θ_{km} is a closed n-form

$$\theta_{km}(\tau)(w) = \sum_{x \in L^n + \mathbf{a}} |v|^{\frac{1}{2}} e^{\pi \mathrm{i}\, \mathrm{tr}[\tau\langle x,x\rangle] - 2\pi\, \mathrm{tr}[v\langle x,z\rangle\langle z,x\rangle]} e^{2\pi \mathrm{i}\, \mathrm{tr}\langle x,\mathbf{b}\rangle} \det\langle x, w\rangle$$

on $\mathcal{D}$. Fix $n+1$ points $c_0, c_1, \ldots, c_n \in \mathcal{D}$. They form a unique geodesic simplex $\Delta = \Delta(c_0, \ldots, c_n)$. As before, since the simplex Δ is compact, θ_{km} is termwise integrable over Δ. We obtain that

$$\Theta_{km}^{\Delta}(\tau) = 2^{\frac{n}{2}} \int\limits_{\Delta} \theta_{km}(\tau) = \sum_{x \in L^n + \mathbf{a}} \left[\int\limits_{\Delta} |2v|^{\frac{1}{2}} e^{-2\pi\, \mathrm{tr}[v\langle x,z\rangle\langle z,x\rangle]} \det\langle x, w\rangle \right] e^{\pi \mathrm{i}\, \mathrm{tr}[\tau\langle x,x\rangle]} e^{2\pi \mathrm{i}\, \mathrm{tr}\langle x,\mathbf{b}\rangle}.$$

For a fixed $y = (y_1, \ldots, y_n) \in V^n$ consider a map $\rho = (r_1, \ldots, r_n) : \mathcal{D} \to \mathbb{R}^n$ defined by $z \mapsto \langle y, z\rangle = (\langle y_1, z\rangle, \ldots, \langle y_n, z\rangle)^\top$. Then

$$\mathrm{tr}[\langle y, z\rangle\langle z, y\rangle] = \langle y_1, z\rangle^2 + \cdots + \langle y_n, z\rangle^2 = \sum_{i=1}^{n} r_i^2,$$

and $dr_1(w) = \langle y_1, w\rangle$ for all $w \in \mathsf{T}_z\mathcal{D}$. Therefore, for $y = x\sqrt{2v}$ we have that

$$\begin{aligned} |2v|^{\frac{1}{2}} e^{-2\pi\, \mathrm{tr}[v\langle x,z\rangle\langle z,x\rangle]} \det\langle x, w\rangle &= e^{-\pi\, \mathrm{tr}[\langle x\sqrt{2v},z\rangle\langle z,x\sqrt{2v}\rangle]} \det\langle x\sqrt{2v}, w\rangle \\ &= e^{-\pi\, \mathrm{tr}[\langle y,z\rangle\langle z,y\rangle]} \det\langle y, w\rangle = \rho^*\Big(e^{-\pi \sum_{i=1}^{n} r_i^2} dr_1 \wedge \cdots \wedge dr_n\Big). \end{aligned}$$

Let Δ_y be an oriented curvilinear simplex in $\mathbb{R}^n$ bounded by the images under the map ρ of $n+1$ geodesic surfaces joining the vertices $c_0, \ldots, c_n$, i.e. satisfying $\partial\Delta_y = \rho(\partial\Delta)$. Note that in some cases Δ_y is not equal to the image $\rho(\Delta)$. However, according to the Stokes' Theorem, the following integral is completely defined by the boundary of Δ and the following equality holds:

$$\int\limits_{\Delta} |2v|^{\frac{1}{2}} e^{-2\pi\, \mathrm{tr}[v\langle x,z\rangle\langle z,x\rangle]} \det\langle x, w\rangle = \int\limits_{\Delta_{x\sqrt{2v}}} e^{-\pi \sum_{i=1}^{n} r_i^2} dr_1 \wedge \cdots \wedge dr_n.$$

Let

$$E(y, \Delta) = \int\limits_{\Delta_y} e^{-\pi \sum_{i=1}^{n} r_i^2} dr_1 \wedge \cdots \wedge dr_n. \tag{3.1}$$

We obtain the following result.

Theorem 3.3. *For a geodesic simplex Δ the following formula holds:*

$$\Theta^{\Delta}_{km}(\tau) = \sum_{x \in L^m + \mathbf{a}} E(x\sqrt{2v}, \Delta) e^{\pi \mathrm{i} \operatorname{tr}[\tau \langle x, x \rangle]} e^{2\pi \mathrm{i} \operatorname{tr} \langle x, \mathbf{b} \rangle}.$$

Hence, the coefficients of the theta series $\Theta^{\Delta}_{km}(\tau)$ are integrals of the Gaussian over some special curvilinear simplices.

Geodesic surfaces of dimension n in $\mathcal{D}$ are intersections of $\mathcal{D}$ with $(n+1)$-dimensional subspaces of V, which are central hyperboloids lying on these subspaces. Since the map ρ is linear we obtain that every simplex Δ_y is bounded by $n+1$ central hyperboloids in $\mathbb{R}^n$.

3.3 Explicit formulas for a geodesic triangle

We now specialize to the case $n = 2$ and continue the computation. For every x we have that $\Delta_{x\sqrt{2v}}$ is a curvilinear triangle bounded by three hyperbolas.

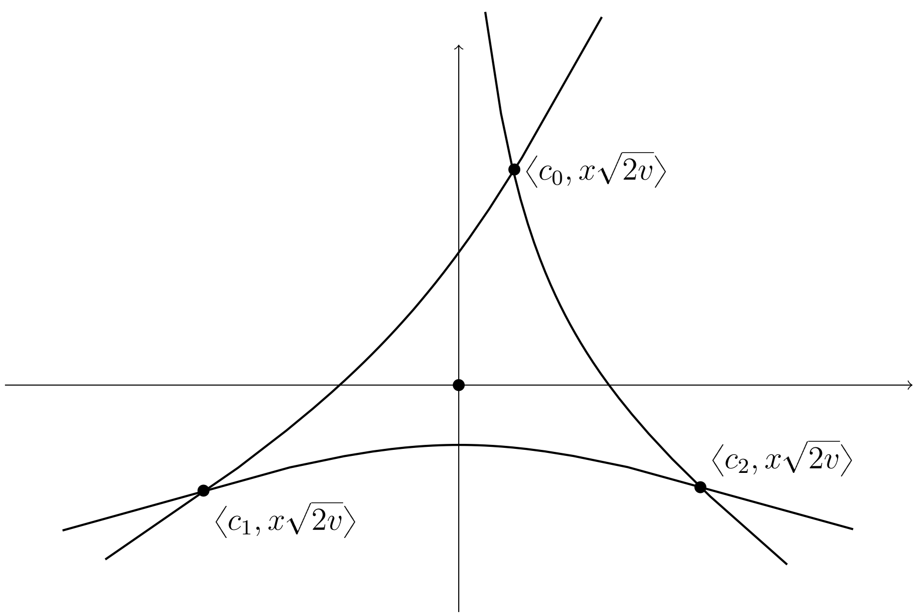

Figure 3.1: The curvilinear triangle $\Delta_{x\sqrt{2v}}$.

Consider the polar substitution $r_1 = r\cos\varphi$, $r_2 = r\sin\varphi$. The vertex $\langle c_i, x\sqrt{2v}\rangle$ has then φ-coordinate equal to $\varphi_i = \arctan(\langle c_i, y_2\rangle, \langle c_i, y_1\rangle)$, where $\arctan(y, x)$ is the usual two-argument arctangent function returning the angle $\varphi \in (-\pi, \pi]$ satisfying $\cos\varphi = x/\sqrt{x^2 + y^2}$ and $\sin\varphi = y/\sqrt{x^2 + y^2}$.

The central hyperbola passing through the points $\langle c_i, x\sqrt{2v}\rangle$ and $\langle c_j, x\sqrt{2v}\rangle$ could be given by the usual polar equation

$$r^2 = \frac{1}{\alpha\cos^2(\varphi - \varphi_0) - \beta}$$

for $\varphi - \varphi_0 \in (-\arccos\sqrt{\frac{\beta}{\alpha}}, \arccos\sqrt{\frac{\beta}{\alpha}})$ and parameters $\alpha = \alpha_{ij}$, $\beta = \beta_{ij}$, $\varphi_0 = \varphi_{ij}$ satisfying $\alpha > \beta > 0$. The angle φ_0 is the φ-coordinate of the hyperbola's vertex.

Consider the function

$$F(\alpha, \beta; z) = \int_0^z \exp\left(-\frac{\pi}{\alpha\cos^2 x - \beta}\right) dx. \tag{F}$$

It is originally defined on $[-\arccos\sqrt{\frac{\beta}{\alpha}}, \arccos\sqrt{\frac{\beta}{\alpha}}]$, but we extend it to a 2π-periodic function defined on

$$\bigcup_{k=-\infty}^{\infty} \left[2\pi k - \arccos\sqrt{\frac{\beta}{\alpha}}, 2\pi k + \arccos\sqrt{\frac{\beta}{\alpha}}\right].$$

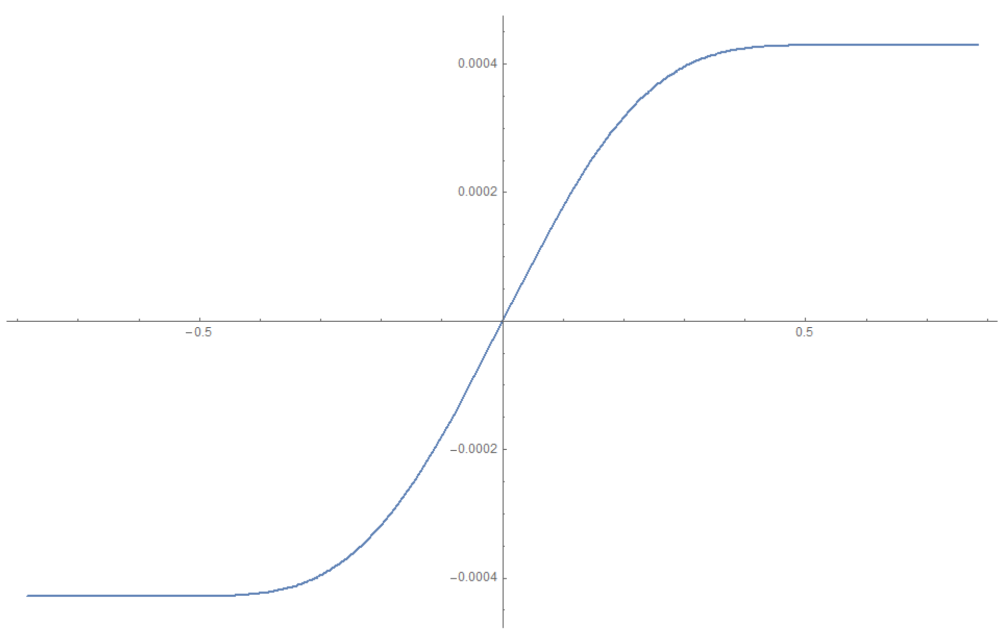

Figure 3.2: The graph of $F(1, 0.5; z)$ on $[-\frac{\pi}{4}, \frac{\pi}{4}]$.

We obtain

$$E(x\sqrt{2v}, \Delta) = \int\limits_{\Delta_{x\sqrt{2v}}} e^{-\pi(r_1^2+r_2^2)} dr_1 \wedge dr_2 = \int\limits_{\Delta_{x\sqrt{2v}}} re^{-\pi r^2} dr d\varphi$$

$$= \sum_{i=0}^{2} \int\limits_{\varphi_i}^{\varphi_{i+1}} \int\limits_{0}^{\frac{1}{\sqrt{\alpha_{i,j+1}\cos^2(\varphi-\varphi_{i,i+1})-\beta_{i,i+1}}}} re^{-\pi r^2} dr d\varphi$$

$$= \frac{1}{2\pi} \sum_{i=0}^{2} \int\limits_{\varphi_i}^{\varphi_{i+1}} \left[1 - \exp\left(-\frac{\pi}{\alpha_{i,i+1}\cos^2(\varphi - \varphi_{i,i+1}) - \beta_{i,i+1}}\right)\right] d\varphi$$

$$= \frac{1}{2\pi} \sum_{i=0}^{2} \angle_{i,i+1}(x\sqrt{2v}, \Delta) +$$

$$+ \frac{1}{2\pi} \sum_{i=0}^{2} [F(\alpha_{i,i+1}, \beta_{i,i+1}; \varphi_i - \varphi_{i,i+1}) - F(\alpha_{i,i+1}, \beta_{i,i+1}; \varphi_{i+1} - \varphi_{i,i+1})],$$

where $\angle_{ij}(y, \Delta)$ denotes the oriented angle from which the side joining $\langle c_i, y\rangle$ to $\langle c_j, y\rangle$ of the simplex Δ_y is seen from the origin. Let

$$\varepsilon(y, \Delta) = \frac{1}{2\pi} \sum_{i=0}^{2} \angle_{i,i+1}(y, \Delta). \qquad (\varepsilon)$$

Unfortunately, there is no simple expression for $\varepsilon(x\sqrt{2v}, \Delta)$ in general. Trivially, if the origin lies strictly inside the triangle $\Delta_{x\sqrt{2v}}$, then $\varepsilon(x\sqrt{2v}, \Delta) = \pm 1$ depending on x. If it lies outside, then $\varepsilon(x\sqrt{2v}, \Delta) = 0$. However, if the origin lies on the border $\partial\Delta_{x\sqrt{2v}}$ (e.g., when $\langle c_i, x\rangle = 0$) then it is nontrivial and depends on v.

Therefore, it is left to find the constants $\alpha_{ij}, \beta_{ij}, \varphi_{ij}$.

Consider the geodesic η between two vertices c_i and c_j in $\mathcal{D}$. It could be given by an equation

$$\eta(t) = \frac{c_i t + c_j(1-t)}{\sqrt{-\langle c_i t + c_j(1-t), c_i t + c_j(1-t)\rangle}} = \frac{c_i t + c_j(1-t)}{\sqrt{1 - 2t(1-t)(1 + \langle c_i, c_j\rangle)}},$$

$t \in [0, 1]$. Let $x\sqrt{2v} = (y_1, y_2)$. Then the image under ρ of this arc is the curve

$$t \mapsto \frac{(\langle c_i, y_1\rangle t + \langle c_j, y_1\rangle(1-t), \langle c_i, y_2\rangle t + \langle c_j, y_2\rangle(1-t))}{\sqrt{1 - 2t(1-t)(1 + \langle c_i, c_j\rangle)}}.$$

in $\mathbb{R}^2$. Along η we have

$$r^2 = \frac{(\langle c_i - c_j, y_1\rangle t + \langle c_j, y_1\rangle)^2 + (\langle c_i - c_j, y_2\rangle t + \langle c_j, y_2\rangle)^2}{1 - 2t(1-t)(1 + \langle c_i, c_j\rangle)},$$

and

$$\cos\varphi = \frac{\langle c_i - c_j, y_1\rangle t + \langle c_j, y_1\rangle}{\sqrt{(\langle c_i - c_j, y_1\rangle t + \langle c_j, y_1\rangle)^2 + (\langle c_i - c_j, y_2\rangle t + \langle c_j, y_2\rangle)^2}},$$
$$\sin\varphi = \frac{\langle c_i - c_j, y_2\rangle t + \langle c_j, y_2\rangle}{\sqrt{(\langle c_i - c_j, y_1\rangle t + \langle c_j, y_1\rangle)^2 + (\langle c_i - c_j, y_2\rangle t + \langle c_j, y_2\rangle)^2}}.$$

We have that

$$\alpha\cos^2(\varphi - \varphi_{ij}) - \beta = \frac{1 - 2t(1-t)(1 + \langle c_i, c_j\rangle)}{(\langle c_i - c_j, y_1\rangle t + \langle c_j, y_1\rangle)^2 + (\langle c_i - c_j, y_2\rangle t + \langle c_j, y_2\rangle)^2}$$

Thus, the expression

$$\begin{aligned}
1 - 2t(1-t)(1 + \langle c_i, c_j\rangle) &+ \beta\big[(\langle c_i - c_j, y_1\rangle t + \langle c_j, y_1\rangle)^2 + (\langle c_i - c_j, y_2\rangle t + \langle c_j, y_2\rangle)^2\big] \\
=& t^2(2 + 2\langle c_i, c_j\rangle + \beta\langle c_i - c_j, y_1\rangle^2 + \beta\langle c_i - c_j, y_2\rangle^2) \\
&+ 2t(-1 - \langle c_i, c_j\rangle + \beta\langle c_i - c_j, y_1\rangle\langle c_j, y_1\rangle + \beta\langle c_i - c_j, y_2\rangle\langle c_j, y_2\rangle) \\
&+ 1 + \beta\langle c_j, y_1\rangle^2 + \beta\langle c_j, y_2\rangle^2.
\end{aligned}$$

must be a full square. This is equivalent to

$$\begin{aligned}
&(\beta\langle c_i - c_j, y_1\rangle\langle c_j, y_1\rangle + \beta\langle c_i - c_j, y_2\rangle\langle c_j, y_2\rangle - 1 - \langle c_i, c_j\rangle)^2 \\
=&(\beta\langle c_i - c_j, y_1\rangle^2 + \beta\langle c_i - c_j, y_2\rangle^2 + 2 + 2\langle c_i, c_j\rangle)(1 + \beta\langle c_j, y_1\rangle^2 + \beta\langle c_j, y_2\rangle^2),
\end{aligned}$$

or

$$\delta^2\beta^2 + \gamma\beta - \langle c_i, c_j\rangle^2 + 1 = 0,$$

and holds for positive

$$\beta = \frac{-\gamma + \sqrt{4(\langle c_i, c_j\rangle^2 - 1)\delta^2 + \gamma^2}}{2\delta^2}, \qquad (\beta)$$

where

$$\gamma = \gamma_{ij} = \langle c_i - c_j, y_1\rangle^2 + \langle c_i - c_j, y_2\rangle^2 + 2(\langle c_i, y_1\rangle\langle c_j, y_1\rangle + \langle c_i, y_2\rangle\langle c_j, y_2\rangle)(1 + \langle c_i, c_j\rangle) \quad (\gamma)$$

and

$$\delta = \delta_{ij} = \begin{vmatrix} \langle c_i, y_1 \rangle & \langle c_i, y_2 \rangle \\ \langle c_j, y_1 \rangle & \langle c_j, y_2 \rangle \end{vmatrix}. \qquad (\delta)$$

Moreover, we have that

$$1 - 2t(1-t)(1 + \langle c_i, c_j \rangle) + \beta\big[(\langle c_i - c_j, y_1 \rangle t + \langle c_j, y_1 \rangle)^2 + (\langle c_i - c_j, y_2 \rangle t + \langle c_j, y_2 \rangle)^2\big] =$$

$$\frac{\big((2t-1)(1 + \langle c_i, c_j \rangle) + \beta\big[(\langle c_i - c_j, y_1 \rangle t + \langle c_j, y_1 \rangle)\langle c_i - c_j, y_1 \rangle + (\langle c_i - c_j, y_2 \rangle t + \langle c_j, y_2 \rangle)\langle c_i - c_j, y_2 \rangle\big]\big)^2}{2(1 + \langle c_i, c_j \rangle) + 4\beta(\langle c_i - c_j, y_1 \rangle^2 + \langle c_i - c_j, y_2 \rangle^2)}$$

Solving linear equation

$$2t - 1 = (\langle c_i - c_j, y_1 \rangle t + \langle c_j, y_1 \rangle)\alpha_1 + (\langle c_i - c_j, y_2 \rangle t + \langle c_j, y_2 \rangle)\alpha_2$$

we obtain

$$\alpha_1 = \frac{\langle c_i + c_j, y_2 \rangle}{\delta}, \quad \alpha_2 = -\frac{\langle c_i + c_j, y_1 \rangle}{\delta}$$

Therefore, $\alpha \cos^2(\varphi - \varphi_{ij})$ is equal to

$$\frac{(\cos\varphi \cdot [\frac{\langle c_i + c_j, y_1 \rangle}{\delta}(1 + \langle c_i, c_j \rangle) + \beta\langle c_i - c_j, y_2 \rangle] - \sin\varphi \cdot [\frac{\langle c_i + c_j, y_2 \rangle}{\delta}(1 + \langle c_i, c_j \rangle) - \beta\langle c_i - c_j, y_1 \rangle])^2}{2(1 + \langle c_i, c_j \rangle) + 4\beta(\langle c_i - c_j, y_1 \rangle^2 + \langle c_i - c_j, y_2 \rangle^2)}$$

and

$$\frac{(\cos\varphi[\langle c_i + c_j, y_1 \rangle(1 + \langle c_i, c_j \rangle) + \beta\delta\langle c_i - c_j, y_2 \rangle] - \sin\varphi[\langle c_i + c_j, y_2 \rangle(1 + \langle c_i, c_j \rangle) - \beta\delta\langle c_i - c_j, y_1 \rangle])^2}{2\delta^2(1 + \langle c_i, c_j \rangle) + \beta\delta^2(\langle c_i - c_j, y_1 \rangle^2 + \langle c_i - c_j, y_2 \rangle^2)}.$$

Therefore, we have formulas for the missing parameters. We have that

$$\varphi_{ij} = \arctan\big(\langle c_i + c_j, y_2 \rangle(1 + \langle c_i, c_j \rangle) - \beta\delta\langle c_i - c_j, y_1 \rangle, \langle c_i + c_j, y_1 \rangle(1 + \langle c_i, c_j \rangle) + \beta\delta\langle c_i - c_j, y_2 \rangle\big), \qquad (\varphi)$$

and α_{ij} is equal to

$$\frac{\sqrt{[\langle c_i + c_j, y_1 \rangle(1 + \langle c_i, c_j \rangle) + \beta\delta\langle c_i - c_j, y_2 \rangle]^2 + [\langle c_i + c_j, y_2 \rangle(1 + \langle c_i, c_j \rangle) - \beta\delta\langle c_i - c_j, y_1 \rangle]^2}}{2\delta^2(1 + \langle c_i, c_j \rangle) + \beta\delta^2(\langle c_i - c_j, y_1 \rangle^2 + \langle c_i - c_j, y_2 \rangle^2)} \qquad (\alpha)$$

We obtain the following result.

Theorem 3.4. *The theta function Θ^{Δ}_{km} for a geodesic triangle $\Delta = \Delta(c_0, c_1, c_2)$ is equal* *to the series*

$$\sum_{x\in L^2+\mathbf{a}} \left[\varepsilon(x\sqrt{2v},\Delta) + \sum_{i=0}^{2} \big(F(\alpha_{i,i+1},\beta_{i,i+1};\varphi_i-\varphi_{i,i+1}) - F(\alpha_{i,i+1},\beta_{i,i+1};\varphi_{i+1}-\varphi_{i,i+1})\big)\right] \times$$
$$\times e^{\pi \mathrm{i}\,\mathrm{tr}[\tau\langle x,x\rangle]} e^{2\pi \mathrm{i}\,\mathrm{tr}\langle x,\mathbf{b}\rangle},$$

where parameters $\alpha_{ij},\beta_{ij},\gamma_{ij},\delta_{ij},\varphi_i,\varphi_{ij}$ *and the functions* ε*,* F *are given by the formulas* (α)*,* (β)*,* (γ)*,* (δ)*,* (φ)*,* (ε)*, and* (F) *above.*

Chapter 4

Integrals of θ_{km} for $\operatorname{sig} V = (1, q)$

Assume now $\operatorname{sig} V = (1, q)$. Then θ_{km} is a q-form on the hyperbolic space $\mathcal{H}^q$ that we identify with $\mathrm{Gr}_1^+(V)$. Any integral of θ_{km} will be an elliptic modular form in $\tau \in \mathcal{H}$.

Assume further $\mathbf{a} \in L^*$, $\mathbf{b} = 0$. We compute the integrals of θ_{km} over the fundamental region $\Gamma \backslash \mathcal{D}$. Special cases $q = 1$ and $q = 2$ were considered in [K1] and [F] respectively. In particular, it was proven that in this case the integral converges. However, it is not necessarily termwise integrable. For $q \geqslant 3$ it follows from the general theory [We2, Prop. 8 on p.75] that the integral absolutely converges and is termwise integrable. We have that

$$\theta_{km}(\tau) = \sum_{x \in L + \mathbf{a}} \mathcal{P}_q(\sqrt{v}\langle x, z\rangle) e^{\pi \mathrm{i} \langle x, x\rangle_{\tau, z}} \cdot \mu,$$

and

$$\langle x, x\rangle_{\tau,z} = u\langle x, x\rangle + \mathrm{i}v\langle x, x\rangle_z = u\langle x, x\rangle - \mathrm{i}v\langle x, x\rangle + 2\mathrm{i}v\langle x, z\rangle^2 = \overline{\tau}\langle x, x\rangle + 2\mathrm{i}v\langle x, z\rangle^2$$

where

$$\tau = u + \mathrm{i}v \in \mathcal{H}.$$

We split the form θ_{km} into three parts

$$\theta_{km} = \theta_{km}^+ + \theta_{km}^0 + \theta_{km}^-,$$

where we restrict the summation to the positive x in the first case, to isotropic (including $x = 0$) in the second, and to the negative x in the final case. The theta integral then decomposes to the sum

$$\Theta_{km}^{\Gamma \backslash \mathcal{D}}(\tau) = \Theta_{km}^+(\tau) + \Theta_{km}^0(\tau) + \Theta_{km}^-(\tau).$$

Kudla and Millson showed [KM3] that

$$\Theta^+_{km}(\tau) = 2^{\frac{q}{2}} \int\limits_{\Gamma\backslash\mathcal{D}} \theta^+_{km}(\tau) = \sum_{\substack{x\in\Gamma\backslash L+\mathbf{a}\\ \langle x,x\rangle>0}} \frac{1}{|\Gamma_x|}\varepsilon(x)e^{\pi i\tau\langle x,x\rangle},$$

where $\varepsilon(x) = 1$ for q even and $\varepsilon(x) = \pm 1$ for q odd depending on whether x forms a properly oriented basis of $\langle x\rangle$ or not.

For $q = 1$ and $q = 2$ the singular part $\Theta^0_{km}(\tau)$ may be nonzero. We present an alternative computation of the singular part in the case $q = 1$. Funke showed that for $q = 2$ the negative part $\Theta^-_{km}(\tau)$ is nonzero and nonholomorphic.

We prove that in the remaining case $q \geqslant 3$ the integral $\Theta^-_{km}(\tau)$ vanishes. We obtain the following result

Theorem 4.1. *In the case $p = 1$ and $q \geqslant 3$ the theta function $\Theta^{\Gamma\backslash\mathcal{D}}_{km}(\tau)$ is a holomorphic elliptic modular form given by the formula*

$$\Theta^{\Gamma\backslash\mathcal{D}}_{km}(\tau) = \Theta^+_{km}(\tau) + \Theta^0_{km}(\tau),$$

where

$$\Theta^0_{km}(\tau) = \begin{cases} 0, & q \text{ odd, or } \mathbf{a}\notin L,\\ (-1)^k\frac{(2k-1)!!}{2^k\pi^k}\mu(\Gamma\backslash\mathcal{D}), & q = 2k \text{ and } \mathbf{a}\in L.\end{cases}$$

Thus, the case $\operatorname{sig} V = (1,q)$ is now completely settled.

4.1 The holomorphic part

In this section we compute the integral $\int\limits_{\Gamma\backslash\mathcal{D}} \theta^+_{km}(\tau)$ originally solved in [KM3]. We have that

$$\begin{aligned}\int\limits_{\Gamma\backslash\mathcal{D}} \theta^+_{km}(\tau) &= \sum_{\substack{x\in L+\mathbf{a}\\ \langle x,x\rangle>0}} \int\limits_{\Gamma\backslash\mathcal{D}} \varphi_{km}(\tau,x) = \sum_{\substack{x\in\Gamma\backslash L+\mathbf{a}\\ \langle x,x\rangle>0}} \sum_{\gamma\in\Gamma_x\backslash\Gamma} \int\limits_{\Gamma\backslash\mathcal{D}} \varphi_{km}(\tau,\gamma x)\\
&= \sum_{\substack{x\in\Gamma\backslash L+\mathbf{a}\\ \langle x,x\rangle>0}} \sum_{\gamma\in\Gamma_x\backslash\Gamma} \int\limits_{\gamma^{-1}\Gamma\backslash\mathcal{D}} \varphi_{km}(\tau,x) = \sum_{\substack{x\in\Gamma\backslash L+\mathbf{a}\\ \langle x,x\rangle>0}} \int\limits_{\Gamma_x\backslash\mathcal{D}} \varphi_{km}(\tau,x)\\
&= \sum_{\substack{x\in\Gamma\backslash L+\mathbf{a}\\ \langle x,x\rangle>0}} \frac{1}{|\Gamma_x|}\int\limits_{\mathcal{D}} \varphi_{km}(\tau,x) = \sum_{\substack{x\in\Gamma\backslash L+\mathbf{a}\\ \langle x,x\rangle>0}} \frac{1}{|\Gamma_x|} e^{\pi i\,\overline{\tau}\langle x,x\rangle}\int\limits_{\mathcal{D}} \mathcal{P}_q(\sqrt{v}\langle x,z\rangle)e^{-2\pi v\langle x,z\rangle^2}\mu.\end{aligned}$$

Consider the integral

$$\int_{\mathcal{D}} \mathcal{P}_q(\sqrt{v}\langle x, z\rangle) e^{-2\pi v\langle x,z\rangle^2} \mu.$$

Since φ_{km} is invariant, without loss of generality we can assume $x = (x_0, 0 \ldots, 0)$ for some $x_0 \neq 0$. Let $\varepsilon(x) = \operatorname{sign}(x_0)^q$. In this case we make use of the Klein model of the hyperbolic space. Identify $\mathcal{H}^q$ with the open unit disk

$$\mathcal{B}^q = \{(t_1, \ldots, t_q) | r^2 := t_1^2 + \cdots + t_q^2 < 1\}$$

The invariant form is

$$\mu = (1 - r^2)^{-\frac{q+1}{2}} dt_1 \ldots dt_q.$$

Then $\langle x, z\rangle = \frac{x_0}{\sqrt{1-t_1^2-\cdots-t_q^2}} = \frac{x_0}{\sqrt{1-r^2}}$ and

$$\int_{\mathcal{D}} \mathcal{P}_q(\sqrt{v}\langle x, z\rangle) e^{-2\pi v\langle x,z\rangle^2} \mu = \int_{\mathcal{B}^q} \mathcal{P}_q\left(\frac{\sqrt{v}x_0}{\sqrt{1-r^2}}\right) e^{-2\pi \frac{vx_0^2}{1-r^2}} \frac{dt_1 \ldots dt_q}{(1-r^2)^{\frac{q+1}{2}}}.$$

After the polar substitution the integral becomes

$$\mu_{q-1} \int_0^1 \mathcal{P}_q\left(\frac{\sqrt{v}x_0}{\sqrt{1-r^2}}\right) e^{-2\pi \frac{vx_0^2}{1-r^2}} \frac{r^{q-1} dr}{(1-r^2)^{\frac{q+1}{2}}},$$

where μ_{q-1} is the volume of the unit sphere S^{q-1}.

Let further $s = \frac{1}{\sqrt{1-r^2}}$ and $ds = \frac{1}{(1-r^2)^{\frac{3}{2}}} dt$. Then the integral becomes

$$\mu_{q-1} \int_1^\infty \mathcal{P}_q(\sqrt{v}x_0 s) e^{-2\pi v x_0^2 s^2} \left(s^2 - 1\right)^{\frac{q-2}{2}} ds,$$

Let

$$I_q(a) = \int_1^\infty \mathcal{P}_q(as) e^{-2\pi a^2 s^2} \left(s^2 - 1\right)^{\frac{q-2}{2}} ds.$$

Clearly, from the properties of $\mathcal{P}_q$ it follows that

$$I_q(-a) = (-1)^q I_q(a).$$

Assume further that $a > 0$. Then the Mellin transform of $I_q(a)$ is

$$\begin{aligned}
&\int_0^\infty \int_1^\infty a^{u-1} \mathcal{P}_q(as) e^{-2\pi a^2 s^2} \left(s^2 - 1\right)^{\frac{q-2}{2}} ds da \\
&= \int_1^\infty \left(s^2 - 1\right)^{\frac{q-2}{2}} \left[\int_0^\infty a^{u-1} \mathcal{P}_q(as) e^{-2\pi a^2 s^2} da \right] ds \\
&= \int_1^\infty \left(s^2 - 1\right)^{\frac{q-2}{2}} s^{-u} ds \cdot \int_0^\infty \mathcal{P}_q(b) e^{-2\pi b^2} b^{u-1} db,
\end{aligned}$$

where $b = as$.

In the first integral we apply the substitution $t = 1/s^2$, $s = 1/\sqrt{t}$ in order to get

$$\frac{1}{2} \int_0^1 \left(\frac{1}{t} - 1\right)^{\frac{q-2}{2}} t^{\frac{u}{2} - \frac{3}{2}} dt = \frac{1}{2} \int_0^1 (1-t)^{\frac{q-2}{2}} t^{\frac{u-q-1}{2}} dt.$$

In the last integral we recognize the definition of the Euler beta function. The integral is

$$\frac{1}{2} \mathrm{B}\left(\tfrac{q}{2}, \tfrac{u-q+1}{2}\right) = \frac{1}{2} \frac{\Gamma(\frac{q}{2}) \Gamma(\frac{u-q+1}{2})}{\Gamma(\frac{u+1}{2})}.$$

In the second integral we apply integration by parts

$$\begin{aligned}
&\int_0^\infty \mathcal{P}_q(b) e^{-2\pi b^2} b^{u-1} db = (-4\pi)^{-q} \int_0^\infty (e^{-2\pi b^2})^{(q)} b^{u-1} db \\
&= (-4\pi)^{-q} (-1)^q \int_0^\infty e^{-2\pi b^2} (u-1) \ldots (u-q) b^{u-q-1} db \\
&= (4\pi)^{-q} \tfrac{\Gamma(u)}{\Gamma(u-q)} \int_0^\infty e^{-2\pi b^2} b^{u-q-1} db = (4\pi)^{-q} \tfrac{1}{2} \tfrac{\Gamma(u)}{\Gamma(u-q)} (2\pi)^{-\frac{u-q}{2}} \int_0^\infty e^{-c} c^{\frac{u-q-2}{2}} dc \\
&= (4\pi)^{-q} \tfrac{1}{2} (2\pi)^{-\frac{u-q}{2}} \tfrac{\Gamma(u) \Gamma(\frac{u-q}{2})}{\Gamma(u-q)},
\end{aligned}$$

where $c = 2\pi b^2$.

Therefore, the product of integrals is

$$\begin{aligned}
&\frac{1}{4}(4\pi)^{-q}(2\pi)^{-\frac{u-q}{2}}\frac{\Gamma(\frac{q}{2})\Gamma(\frac{u-q+1}{2})}{\Gamma(\frac{u+1}{2})}\frac{\Gamma(u)\Gamma(\frac{u-q}{2})}{\Gamma(u-q)} \\
&= \frac{1}{4}(4\pi)^{-q}(2\pi)^{-\frac{u-q}{2}}\frac{\Gamma(\frac{q}{2})}{\Gamma(\frac{u+1}{2})}\frac{\Gamma(u)}{\Gamma(u-q)}\cdot\Gamma(u-q)\sqrt{\pi}2^{1-(u-q)} \\
&= \frac{1}{4}(4\pi)^{-q}(2\pi)^{-\frac{u-q}{2}}\frac{\Gamma(\frac{q}{2})\Gamma(\frac{u}{2})\Gamma(\frac{u+1}{2})2^{-1+u}}{\Gamma(\frac{u+1}{2})}\cdot 2^{1-(u-q)} \\
&= \frac{1}{4}2^{q}(4\pi)^{-q}(2\pi)^{-\frac{u-q}{2}}\Gamma(\tfrac{q}{2})\Gamma(\tfrac{u}{2}) \\
&= 2^{q-2-2q+\frac{q}{2}}\pi^{-q+\frac{q}{2}}\Gamma(\tfrac{q}{2})\cdot(2\pi)^{-\frac{u}{2}}\Gamma(\tfrac{u}{2}) \\
&= \tfrac{1}{4}(2\pi)^{-\frac{q}{2}}\Gamma(\tfrac{q}{2})\cdot(2\pi)^{-\frac{u}{2}}\Gamma(\tfrac{u}{2}).
\end{aligned}$$

Applying inverse Mellin transform we obtain Cahen-Mellin integral

$$I_q(a) = \tfrac{1}{4}(2\pi)^{-\frac{q}{2}}\Gamma(\tfrac{q}{2})\cdot\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}(\sqrt{2\pi}a)^{-u}\Gamma(\tfrac{u}{2})du = \tfrac{1}{2}(2\pi)^{-\frac{q}{2}}\Gamma(\tfrac{q}{2})\cdot e^{-2\pi a^2}.$$

Therefore,

$$\begin{aligned}
&\int_{\mathcal{D}}\mathcal{P}_q(\sqrt{v}\langle x,z\rangle)e^{-2\pi v\langle x,z\rangle^2}d\mu = \mu_{q-1}I_q(\sqrt{v}x_0) = \mu_{q-1}\varepsilon(x)I_q(\sqrt{v}|x_0|) \\
=&\varepsilon(x)\mu_{q-1}\tfrac{1}{2}(2\pi)^{-\frac{q}{2}}\Gamma(\tfrac{q}{2})\cdot e^{-2\pi v x_0^2} = 2^{-\frac{q}{2}}\varepsilon(x)e^{-2\pi v x_0^2} = 2^{-\frac{q}{2}}\varepsilon(x)e^{-2\pi v\langle x,x\rangle}
\end{aligned}$$

since $\mu_{q-1} = \frac{2\pi^{q/2}}{\Gamma(\frac{q}{2})}$. Hence, we have that

$$\Theta^{+}_{km}(\tau) = 2^{\frac{q}{2}}\int_{\Gamma\backslash\mathcal{D}}\theta^{+}_{km}(\tau) = \sum_{x\in\Gamma\backslash L+\mathbf{a}}\frac{1}{|\Gamma_x|}\varepsilon(x)e^{\pi\mathrm{i}\overline{\tau}\langle x,x\rangle}e^{-2\pi v\langle x,x\rangle} = \sum_{x\in\Gamma\backslash L+\mathbf{a}}\frac{1}{|\Gamma_x|}\varepsilon(x)e^{\pi\mathrm{i}\tau\langle x,x\rangle}.$$

4.2 Small-dimensional cases

Assume now $(p,q) = (1,1)$. The symmetric space $\mathcal{D}$ in this case is one branch of a hyperbola. The existence of the null vectors implies that V is a split space. Therefore, we can identify V with the standard pseudoeuclidean space $\mathbb{R}^{1,1}$. The lattice L has rational generators. Moreover, in this case Γ is a trivial group, and the fundamental region for Γ is $\mathcal{D}$ itself and the singular part $\Theta^0_{km}(\tau)$ is nonzero in general.

Theorem 4.2 ([K2])**.** *In the case* $\operatorname{sig} V = (1,1)$ *the following formula holds for the* *integral of the singular part*

$$\Theta^0_{km}(\tau) = -\frac{1}{2} \sum_{\substack{x \in \Gamma \backslash L + \mathbf{a} \\ x \text{ reduced}}} \varepsilon(x) \mathbf{B}_1(\nu(x)),$$

where $\mathbf{B}_1$ *is the Bernoulli* 1*-periodic function defined on* $[0,1]$ *by*

$$\mathbf{B}_1(\alpha) = \begin{cases} 0, & \alpha = 0, \ \alpha = 1, \\ \alpha - \frac{1}{2}, & 0 < \alpha < 1, \end{cases}$$

and reduced means the following: $x = \pm\nu(x) u_x$*, where* $u_x \in L$ *is primitive,* $\langle u_x, u_x \rangle = 0$*,* $\langle u_x, z \rangle < 0$ *for all* $z \in \mathcal{D}$*, and* $\nu(x) \in [0,1)$ *is a constant. The sign* $\varepsilon(x) = \pm 1$ *satisfies* $x = \varepsilon(x)\nu(x)u_x$*.*

We give an alternative proof of this result. First, we show that in fact there are at most four reduced vectors. The null vectors x lie on the two isotropic lines ℓ_1, ℓ_2 in V and form two rank 1 sublattices in L. Let $\ell_1 = \{(x,x) | x \in \mathbb{R}\} \subset V$ and $\ell_2 = \{(x,-x) | x \in \mathbb{R}\} \subset V$; let $L_1 = (L + \mathbf{a}) \cap \ell_1$ and $L_2 = (L + \mathbf{a}) \cap \ell_2$. Then we split θ^0_{km} to a sum $\theta^{(1)}_{km} + \theta^{(2)}_{km}$, where

$$\theta^{(1)}_{km}(\tau) = \sum_{x \in L_1} \varphi_{km}(\tau, x) \quad \text{and} \quad \theta^{(2)}_{km}(\tau) = \sum_{x \in L_2} \varphi_{km}(\tau, x).$$

Assume that $L_1 = \{(r(k+a), r(k+a)) | k \in \mathbb{Z}\}$ for some $a, r \in \mathbb{Q}$, $a \in [0,1)$. Then $x' = (ra, ra)$ and $x'' = (r(a-1), r(a-1))$ are reduced. Indeed, we have $u_{x'} = u_{x''} = (r,r) \in V_1$ is primitive, $\nu(x') = a$, $\varepsilon(x') = 1$, and $\nu(x'') = a - 1$, $\varepsilon(x'') = -1$.

We have the following formula

$$\theta^{(1)}_{km}(\tau) = \sum_{\substack{k \in \mathbb{Z} \\ x = (r(k+a), r(k+a))}} \sqrt{v} e^{-2\pi v \langle x, z \rangle^2} \langle x, w \rangle.$$

Assuming $z = \big(\frac{1}{2}(\frac{1}{t} + t), \frac{1}{2}(\frac{1}{t} - t)\big)$, $t \in (0, \infty)$ we obtain

$$\begin{aligned} \langle x, z \rangle &= \tfrac{1}{2} r(k+a)(\tfrac{1}{t} + t) - \tfrac{1}{2} r(k+a)(\tfrac{1}{t} - t) = r(k+a)t, \\ \langle x, w \rangle &= \tfrac{1}{2} r(k+a)(-\tfrac{1}{t^2} + 1)dt - \tfrac{1}{2} r(k+a)(-\tfrac{1}{t^2} - 1)dt = r(k+a)dt. \end{aligned}$$

Therefore, we have

$$\theta^{(1)}_{km}(\tau) = \sum_{k=-\infty}^{\infty} \sqrt{v} e^{-2\pi v r^2 (k+a)^2 t^2} r(k+a) dt.$$

Integrating this over $\mathcal{D}$ gives

$$\Theta^{(1)}_{km}(\tau) = \sqrt{2}\int_{\mathcal{D}} \theta^{(1)}_{km} = \int_0^\infty \left[\sum_{k=-\infty}^{\infty} e^{-2\pi v r^2 (k+a)^2 t^2} \sqrt{2v} r (k+a)\right] dt$$
$$= \int_0^\infty \left[\sum_{k=-\infty}^{\infty} e^{-\pi (k+a)^2 t^2} (k+a)\right] dt,$$

where in the last integral we replace t and $\sqrt{2v}rt$.

Lemma 4.3. *The following identity holds:*

$$\int_0^\infty \sum_{k\in\mathbb{Z}+a} k e^{-\pi k^2 t^2} dt = -\mathbf{B}_1(a).$$

Proof. Indeed, according to the Poisson summation the sum

$$\sum_{k\in\mathbb{Z}+a} k e^{-\pi k^2 t^2}$$

is equal to

$$\sum_{k\in\mathbb{Z}} e^{2\pi \mathrm{i} ka} \cdot \frac{\mathrm{i}}{2\pi} \cdot \frac{d}{dk}\left(\frac{1}{t} e^{-\pi \frac{k^2}{t^2}}\right) = \sum_{k\in\mathbb{Z}} \mathrm{i} e^{2\pi \mathrm{i} ka} \cdot -\tfrac{k}{t^3} e^{-\pi\frac{k^2}{t^2}}$$
$$= \sum_{k=1}^\infty \mathrm{i}(e^{-2\pi \mathrm{i} ka} - e^{2\pi \mathrm{i} ka}) \tfrac{k}{t^3} e^{-\pi\frac{k^2}{t^2}} = \sum_{k=1}^\infty 2\sin 2\pi ka \cdot \tfrac{k}{t^3} e^{-\pi\frac{k^2}{t^2}}.$$

This series could be integrated termwise. Furthermore,

$$\sum_{k=1}^\infty 2\sin 2\pi ka \int_0^\infty \tfrac{k}{t^3} e^{-\pi\frac{k^2}{t^2}} dt = \sum_{k=1}^\infty 2\sin 2\pi ka \cdot \frac{1}{2\pi k} = \frac{1}{\pi}\sum_{k=1}^\infty \frac{\sin 2\pi ka}{k} = -\mathbf{B}_1(a).$$

The last equation uses a well-known Fourier expansion of $\mathbf{B}_1$. □

Applying the lemma we obtain

$$\Theta^{(1)}_{km}(\tau) = -\mathbf{B}_1(a) = -\varepsilon(x')\mathbf{B}_1(\nu(x')) = -\varepsilon(x'')\mathbf{B}_1(\nu(x''))$$
$$= -\frac{1}{2}(\varepsilon(x')\mathbf{B}_1(\nu(x')) + \varepsilon(x'')\mathbf{B}_1(\nu(x''))).$$

for x', x'' reduced on the line ℓ_1.

Similarly, if $L_2 = \{(r(k+b), -r(k+b)) | k \in \mathbb{Z}\}$ for some $r, b \in \mathbb{Q}$, $b \in [0,1)$ then the reduced vectors on this line are $x' = (rb, -rb)$ and $x'' = (r(b-1), -r(b-1))$. We have also $u_{x'} = u_{x''} = (-r, r)$, $x' = -bu_x$, $x'' = (1-b)u_x$, $\varepsilon(x') = -1$, $\varepsilon(x'') = 1$, $\nu(x') = -b$, $\nu(x'') = 1-b$. After similar computations we obtain

$$\begin{aligned}\Theta^{(2)}_{km}(\tau) &= \mathbf{B}_1(b) = -\varepsilon(x')\mathbf{B}_1(\nu(x')) = -\varepsilon(x'')\mathbf{B}_1(\nu(x''))\\ &= -\frac{1}{2}(\varepsilon(x')\mathbf{B}_1(\nu(x')) + \varepsilon(x'')\mathbf{B}_1(\nu(x''))).\end{aligned}$$

This concludes the computation of $\Theta^0_{km}(\tau)$ in the case $\operatorname{sig} V = (1,1)$.

The case $\operatorname{sig} V = (1,2)$ was completely solved by Funke in his Ph.D thesis. We state his results without proof. It turns out that the integral may indeed be nonholomorphic.

Theorem 4.4 ([F]). *In the case* $\operatorname{sig} V = (1,2)$ *the theta integral* $\Theta^{\Gamma\backslash\mathcal{D}}_{km}(\tau)$ *is equal to the sum*

$$\Theta^+_{km}(\tau) + \Theta^-_{km}(\tau) + \Theta^0_{km}(\tau),$$

where

$$\Theta^-_{km}(\tau) = \sum_{\substack{x\in\Gamma\backslash L+\mathbf{a}\\ x^\perp \text{ splits}}} \frac{1}{4\pi\sqrt{-v\langle x,x\rangle}}\beta(-4\pi v\langle x,x\rangle)e^{\pi i\tau\langle x,x\rangle}, \tag{4.1}$$

$$\Theta^0_{km}(\tau) = -\frac{1}{2\pi}\mu(\Gamma\backslash\mathcal{D})\delta_{\mathbf{a}} + \frac{1}{2\pi\sqrt{v}}\epsilon(L,\mathbf{a},\Gamma), \tag{4.2}$$

and

$$\beta(s) = \int_0^\infty t^{-\frac{3}{2}}e^{-st}dt,$$

$\epsilon(L,\mathbf{a},\Gamma)$ *is the total volume of cusps of* Γ *that intersect with* $L+\mathbf{a}$, *and*

$$\delta_{\mathbf{a}} = \begin{cases} 1, & \mathbf{a}\in L,\\ 0, & \mathbf{a}\notin L.\end{cases}$$

Funke showed that for some choice of L the function $\frac{1}{2}\Theta^{\Gamma\backslash\mathcal{D}}_{km}(\tau)$ coincides with the well-known Zagier Einsenstein series of weight 3/2 first described in [ZE]:

$$\mathcal{F}(\tau) = \sum_{n=1}^{\infty} h(n)e^{2\pi i n\tau} + \frac{1}{16\pi\sqrt{v}}\sum_{n=-\infty}^{\infty}\beta(4\pi n^2 v)e^{-2\pi i n^2\tau}, \tag{4.3}$$

where $h(n)$ is the the class number of positive-definite binary quadratic forms of discriminant $-n^2$.

4.3 Integral of θ^-_{km}

Assume now $q \geqslant 3$. In this case the whole theta series θ_{km} is termwise integrable and we can apply the same unfolding procedure. We have that

$$\int\limits_{\Gamma\backslash\mathcal{D}} \theta^-_{km}(\tau) = \sum_{\substack{x\in\Gamma\backslash L+\mathbf{a}\\ \langle x,x\rangle<0}} e^{\pi \mathrm{i}\,\overline{\tau}\langle x,x\rangle} \int\limits_{\Gamma_x\backslash\mathcal{D}} \mathcal{P}_q(\sqrt{v}\langle x,z\rangle)e^{-2\pi v\langle x,z\rangle^2}\mu.$$

We prove the following result.

Lemma 4.5. *For $q \geqslant 3$ and x negative*

$$\int\limits_{\Gamma_x\backslash\mathcal{D}} \mathcal{P}_q(\sqrt{v}\langle x,z\rangle)e^{-2\pi v\langle x,z\rangle^2}\mu = 0.$$

In this case consider the upper half-space model for $\mathcal{D}$. Every point $(y_1,\dots,y_{q-1},t)$ corresponds to the unit vector $\left(\frac{1+\sum y_i^2+t^2}{2t}, -\frac{y_1}{t},\dots,-\frac{y_{q-1}}{t},\frac{1-\sum y_i^2-t^2}{2t}\right)^\top \in V$.

Let $x \in L+\mathbf{a} \subset V$ be a negative vector, let $Y = x^\perp$. Then Y is a subspace of V of signature $(1,q-1)$. Without loss of generality assume that $x = (0,x_1,0,\dots,0)$ for some $x_1 \neq 0$. Let $\mathcal{D}_1 = \mathcal{D}\cap Y$ be a hyperbolic geodesic hyperplane of $\mathcal{D}$. Then $\mathcal{D}_1$ is given by the equation $y_1 = 0$ in $\mathcal{D}$. Moreover, for the volume forms μ, μ_1 on $\mathcal{D}$ and $\mathcal{D}_1$ we have formulas

$$\mu = \frac{dy_1\dots dy_{q-1}dt}{t^q}, \quad \mu_1 = \frac{dy_2\dots dy_{q-1}dt}{t^{q-1}},$$

in particular, every point $z\in\mathcal{D}$ could be identified with the pair (y_1,z_1) for some $y_1\in\mathbb{R}$ and $z_1\in\mathcal{D}_1$, and

$$\mu = \frac{1}{t}dy_1\mu_1.$$

Let $L_0 = L\cap Y$ and $L_1 = L\cap\langle x\rangle$. Then L_0 is a maximum rank lattice in Y. Moreover, Γ_x stabilizes L_0 in Y. Therefore, Γ_x is an arithmetic subgroup of $\mathrm{O}(Y)$. In particular, since $q\geqslant 3$ we have that $\mathrm{O}(Y)$ is semisimple; thus, there exists a finite volume fundamental domain $\mathcal{F}$ for the action of Γ_x on $\mathcal{D}_1$. In the case $q=1$ and $q=2$ this is not true. Since the action of Γ_x does not affect the y_1 coordinate, we have the following result.

Lemma 4.6. *The set $\mathbb{R}\times\mathcal{F}$ is a fundamental domain for Γ_x in $\mathcal{D}$.*

Now we can complete the proof of lemma.

Proof of Lemma 4.5. We have that

$$\int\limits_{\Gamma_x\backslash\mathcal{D}} \mathcal{P}_q(\sqrt{v}\langle x,z\rangle)e^{-2\pi v\langle x,z\rangle^2}\mu = \int\limits_{\mathcal{F}}\int\limits_{-\infty}^{\infty} \mathcal{P}_q(\sqrt{v}\tfrac{x_1y_1}{t})e^{-2\pi v\frac{x_1^2y_1^2}{t^2}}\tfrac{1}{t}dy_1\mu_1.$$

Let $u = \sqrt{v}\frac{x_1y_1}{t}$. Then the integral becomes

$$\tfrac{1}{\sqrt{v}x_1}\int\limits_{\mathcal{F}}\int\limits_{-\infty}^{\infty} \mathcal{P}_q(u)e^{-2\pi u^2}du\mu_1 = \tfrac{\mu_1(\mathcal{F})}{\sqrt{v}x_1}\int\limits_{-\infty}^{\infty} \mathcal{P}_q(u)e^{-2\pi u^2}du.$$

For the inner integral we have trivially

$$\int\limits_{-\infty}^{\infty} \mathcal{P}_q(u)e^{-2\pi u^2}du = -\tfrac{1}{4\pi}\mathcal{P}_{q-1}(u)e^{-2\pi u^2}\Big|_{-\infty}^{\infty} = 0.$$

This completes the proof. □

Hence, for $q \geqslant 3$ it holds that $\Theta_{km}^{-}(\tau) = 0$.

4.4 Integral of θ_{km}^0

Consider now the singular part θ_{km}^0. As before, for $q \geqslant 3$ we can apply the unfolding procedure once again to get

$$\int\limits_{\Gamma\backslash\mathcal{D}} \theta_{km}^0(\tau) = \sum_{\substack{x\in\Gamma\backslash L+\mathbf{a}\\ \langle x,x\rangle=0}}\int\limits_{\Gamma_x\backslash\mathcal{D}} \mathcal{P}_q(\sqrt{v}\langle x,z\rangle)e^{-2\pi v\langle x,z\rangle^2}\mu.$$

In this case we also make use of the upper half-space model for $\mathcal{D}$. We prove another result.

Lemma 4.7. *For $q \geqslant 3$ and $x \neq 0$ isotropic*

$$\int\limits_{\Gamma_x\backslash\mathcal{D}} \mathcal{P}_q(\sqrt{v}\langle x,z\rangle)e^{-2\pi v\langle x,z\rangle^2}\mu = 0.$$

Proof. Without loss of generality, assume that $x = (x_0, 0, \ldots, 0, -x_0)$ for some $x_0 \neq 0$. Then its stabilizer $\mathrm{St}_G(x)$ is isomorphic to $\mathbb{R}^{q-1}$ and acts by translations in variables $y_1, \ldots, y_{q-1}$ in $\mathcal{D}$. The stabilizer $\Gamma_x = \mathrm{St}_\Gamma(x)$ is, therefore, a full rank lattice in $\mathbb{R}^{q-1}$. We can extend the action of Γ_x to the boundary $\partial\mathcal{D}$ of $\mathcal{D}$, which we identify with $\mathbb{R}^{q-1}$ in variables $y_1, \ldots, y_{q-1}$.

The fundamental region for Γ_x acting on $\mathcal{D}$ is then $\mathcal{F} \times (0, \infty)$, where $\mathcal{F} = \Gamma_x \backslash \partial\mathcal{D}$ is a fundamental region for Γ acting on $\partial\mathcal{D}$. The form μ extends to the measure $\overline{\mu} = dy_1 \ldots dy_{q-1}$ on $\partial\mathcal{D}$.

Moreover, for $z \in \mathcal{D}$ we have that

$$\langle x, z \rangle = x_0 \left(\tfrac{1+\sum y_i^2}{2t} + \tfrac{1-\sum y_i^2}{2t} \right) = \tfrac{x_0}{t}.$$

We have further

$$\int\limits_{\Gamma_x \backslash \mathcal{D}} \mathcal{P}_q(\sqrt{v}\langle x, z \rangle) e^{-2\pi v \langle x, z \rangle^2} \mu = \int\limits_{\mathcal{F}} \int\limits_0^\infty \mathcal{P}_q(\sqrt{v}\tfrac{x_0}{t}) e^{-2\pi v \frac{x_0^2}{t^2}} \tfrac{dt}{t^q} dy_1 \ldots dy_{q-1}.$$

Let $w = \sqrt{v}\frac{x_0}{t}$. Then $dw = -\sqrt{v}\frac{x_0 dt}{t^2}$ and the integral becomes

$$(\sqrt{v}x_0)^{-q+1}\overline{\mu}(\mathcal{F}) \int\limits_0^\infty \mathcal{P}_q(w) e^{-2\pi w^2} w^{q-2} dw.$$

By applying integration by parts, we obtain

$$\begin{aligned} &\int\limits_0^\infty \mathcal{P}_q(w) e^{-2\pi w^2} w^{q-2} dw \\ &= \sum_{i=1}^{q-1} (-1)^{i-1} (-4\pi)^{-i} \mathcal{P}_{q-i}(w) e^{-2\pi w^2} \tfrac{d^{i-1}}{dw} w^{q-2} \Big|_0^\infty + \\ &(-1)^{q-1} \int\limits_0^\infty (-4\pi)^{-(q-1)} \mathcal{P}_1(w) e^{-2\pi w^2} \tfrac{d^{q-1}}{dw} w^{q-2} dw. \end{aligned}$$

In the sum above all terms are zero. Therefore, the integral is zero as well. This concludes the proof. □

We have only the zero term left. For q odd it also vanishes since $\mathcal{P}_q$ is odd. Hence, $\Theta^0_{km}(\tau) = 0$ for q odd. For $q = 2k$ even we have that

$$\varphi_{km}(\tau, 0) = \mathcal{P}_{2k}(0)\mu = (-1)^k \frac{(2k-1)!!}{2^{2k}\pi^k}\mu.$$

Note that 0 belongs to $L + \mathbf{a}$ only when $L + \mathbf{a} = L$, i.e. $\mathbf{a} \in L$. Therefore, for $q = 2k$ and

$\mathbf{a} \in L$ we have that

$$\Theta^0_{km}(\tau) = 2^k \int\limits_{\Gamma\backslash\mathcal{D}} (-1)^k \frac{(2k-1)!!}{2^{2k}\pi^k}\mu = (-1)^k \frac{(2k-1)!!}{2^k\pi^k}\mu(\Gamma\backslash\mathcal{D}).$$

Hence, Theorem 4.1 is proved.

Chapter 5

The general case

In this chapter we consider the integral of θ_{km} for $n = p$ over the fundamental region $\Gamma\backslash\mathcal{D}$ in the case of arbitrary signature (p,q). We prove the following result

Theorem 5.1. *The form θ_{km} belongs to the class $\mathrm{L}^1(\Gamma\backslash\mathcal{D})$. In particular, the integral*

$$\Theta_{km}^{\Gamma\backslash\mathcal{D}}(\tau) = 2^{\frac{pq}{2}} \int\limits_{\Gamma\backslash\mathcal{D}} \theta_{km}(\tau)$$

converges.

In some cases we compute the integral. In general we have a decomposition

$$\theta_{km} = \theta_{km}^{+} + \theta_{km}^{-} + \theta_{km}^{\pm} + \theta_{km}^{0},$$

where in the first three terms we restrict the summation over x to frames generating a positive, negative, or indefinite subspace of dimension q respectively. Finally, θ_{km}^0 is a sum over the *singular* frames x, i.e. such that the matrix $\langle x, x\rangle$ is singular. We denote the integrals of the these series over $\Gamma\backslash\mathcal{D}$, multiplied by $2^{\frac{pq}{2}}$, respectively by Θ_{km}^{+}, Θ_{km}^{-}, $\Theta_{km}^{\pm}$, Θ_{km}^{0}.

In [KM3] the following formula for $\Theta_{km}^{+}(\tau)$ was established:

$$\Theta_{km}^{+}(\tau) = \sum_{\substack{x \in \Gamma\backslash L^p + \mathbf{a} \\ \langle x,x\rangle > 0}} \frac{1}{|\Gamma_x|} \varepsilon(x) e^{\pi \mathrm{i}\,\mathrm{tr}[\tau\langle x,x\rangle]},$$

where $\varepsilon(x) = 1$ for q even and $\varepsilon(x) = \pm 1$ for q odd depending on whether the frame x forms a properly oriented basis of $X = \langle x\rangle$ or not.

The parameter $\mathbf{a}$ is called *nonsingular* if every frame $x \in L^p + \mathbf{a}$ generates a nondegenerate pseudoeuclidean subspace of dimension p. Equivalently, this means that $\theta_{km}^0 = 0$.

We prove the following result.

Theorem 5.2. *For q odd and $\mathbf{a}$ nonsingular the theta integral $\Theta_{km}^{\Gamma\backslash\mathcal{D}}(\tau)$ is equal to the holomorphic series $\Theta_{km}^{+}(\tau)$; in particular, the latter is a holomorphic Siegel genus p modular form.*

The special case $q = 1$ of this theorem was proved in [K1]. Note that nonsingular $\mathbf{a}$ exist. For example, take $V = \mathbb{R}^{p,q}$ to be the standard pseudoeuclidean space. Let $L = M\mathbb{Z}^{p+q}$ for some positive integer M. Then $L^* = \frac{1}{M}\mathbb{Z}^{p+q}$. Take $\mathbf{a} \in \mathbb{Z}^{p+q}$ such that $\langle \mathbf{a}, \mathbf{a}\rangle$ is invertible modulo M, i.e. $\langle \mathbf{a}, \mathbf{a}\rangle \in \mathrm{GL}(p; \mathbb{Z}/M\mathbb{Z})$. Then $\mathbf{a}$ is nonsingular, since for every $x \in L^p + \mathbf{a}$ we have that

$$\det\langle x, x\rangle \equiv \det\langle \mathbf{a}, \mathbf{a}\rangle \not\equiv 0 \pmod M$$

and $\langle x, x\rangle$ is invertible. For such $\mathbf{a}$ we can take $\mathbf{a} = \left(\begin{smallmatrix} 1_p \\ 0 \end{smallmatrix}\right)$. Then $\langle \mathbf{a}, \mathbf{a}\rangle = 1_p$ belongs to $\mathrm{GL}(p; \mathbb{Z}/M\mathbb{Z})$ for every $M \geqslant 2$.

For q even the situation is quite different, since terms with linearly dependent vectors can make a nonzero contribution.

Theorem 5.3 ([K2])**.** *In the case $\operatorname{sig} V = (m, 1)$ the singular integral is given by the formula*

$$\Theta_{km}^{0}(\tau) = \frac{1}{2}(-1)^m \sum_{\substack{x \in \Gamma\backslash L^m + \mathbf{a} \\ x \text{ reduced}}} \mathbf{B}_1(\nu(x))\varepsilon(x)e^{\pi \mathrm{i}\, \mathrm{tr}[\tau\langle x, x\rangle]},$$

where $\mathbf{B}_1$ is the Bernoulli 1-periodic polynomial.

The definitions of $\nu(x)$, $\varepsilon(x)$ and of term *reduced*, as well as an alternative proof of this result is presented in the final section.

5.1 The Iwasawa decomposition and Siegel domains

The condition $L \subseteq L^*$ implies that we can define a rational structure on V. If we choose any $\mathbb{Z}$-basis of L, then its Gram matrix would be integral. We write $V(\mathbb{Q})$ for the $\mathbb{Q}$-subspace generated by L. We have then $V = V(\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R}$. Moreover, G as an algebraic group is defined over $\mathbb{Q}$ as well. Assume that $p \leqslant q$ and $\operatorname{rank}_{\mathbb{Q}} G = \operatorname{rank}_{\mathbb{R}} G$. Then we can choose a Witt basis

$$v_1', \ldots, v_p', v_1, \ldots, v_{q-p}, v_1'', \ldots, v_p''$$

for $V(\mathbb{Q})$, satisfying $\langle v_i, v'_j\rangle = \langle v_i, v''_j\rangle = \langle v'_i, v'_j\rangle = \langle v''_i, v''_j\rangle = 0$ and $\langle v'_i, v''_i\rangle = -\frac{1}{2}$. In this basis,

$$\langle\cdot,\cdot\rangle \sim \begin{pmatrix} & & -\frac{1}{2}1_p \\ & Q_0 & \\ -\frac{1}{2}1_p & & \end{pmatrix},$$

and Q_0 is a symmetric negative-definite matrix defining a $\mathbb{Q}$-space V_0 on vectors $v_1, \dots, v_{q-p}$.

The vectors $v'_1, \dots, v'_p$ generate a maximum null subspace V' in V, and the vectors $v''_1, \dots, v''_p$ form a maximal null subspace V''. Moreover, we have a direct decomposition $V = V' \oplus V_0 \oplus V''$. Every vector $x \in V$ decomposes as $x = x' + x^0 + x''$.

Let P be a stabilizer in G of the flag of null subspaces

$$\langle v'_1\rangle \subset \langle v'_1, v'_2\rangle \subset \cdots \subset \langle v'_1, \dots, v'_p\rangle.$$

Then P is a minimal parabolic subgroup of G. In the above basis P consists of pseudoorthogonal matrices of the form $\begin{pmatrix} a & * & * \\ & o & * \\ & & a^{-\top} \end{pmatrix}$, where $a \in \mathrm{GL}(q;\mathbb{R})$ is upper triangular, $o \in \mathrm{SO}(V_0)$ is orthogonal.

Let N be the unipotent radical of P. Then

$$N = \{n(a,b,c) = \begin{pmatrix} a & 2ab^\top Q_0 & a(b^\top Q_0 b + c) \\ & 1_{q-p} & b \\ & & a^{-\top} \end{pmatrix}\}$$

where $a \in \mathrm{UT}(p;\mathbb{R})$ is unitriangular, $b \in \mathrm{M}_{q-p,p}(\mathbb{R})$ is arbitrary, and $c \in \mathrm{M}_p(\mathbb{R})$ is skew-symmetric.

Let

$$A = \{a(t) = \begin{pmatrix} t & & \\ & 1_{q-p} & \\ & & t^{-1} \end{pmatrix}\},$$

where

$$t = \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & \ddots & \\ & & & t_p \end{pmatrix} \in \mathrm{GL}(p;\mathbb{R})$$

is a diagonal matrix with $t_1, \dots, t_p \in \mathbb{R}^*$. Then A is a maximal $\mathbb{Q}$-split torus in P. For $R > 0$ let

$$A_R = \{a(t) | t_1/t_2 \geqslant R, \dots, t_{p-1}/t_p \geqslant R, t_p \geqslant R\}.$$

Moreover, let $Z_0 = \langle v'_1 - v''_1, \dots, v'_p - v''_p\rangle$. Then Z_0 is a maximum positive subspace, and $\{v'_i - v''_i\}$ is its orthonormal basis since $\langle v'_i - v''_i, v'_i - v''_i\rangle = -2\langle v'_i, v''_i\rangle = 1$. Let $K = \{g \in G | gZ_0 = Z_0\}$ be the stabilizer of Z_0. This is a maximal compact subgroup of G.

We have that $G = NAK$ is the Iwasawa decomposition of G. The *Siegel domain* for this decomposition is a set $\mathfrak{S} = \omega A_R K$ (and $\mathfrak{S}' = \omega A_R K \cdot Z_0$) for some compact $\omega \subseteq N$

and $R > 0$.

Then [B] there exists a Siegel domain $\mathfrak{S}$, such that there exist finitely many shifts of $\mathfrak{S}$ that together form a *coarse fundamental domain* for Γ, i.e. there exists a finite set $C \subset G$ satisfying

$$\Gamma \cdot C \cdot \mathfrak{S} = G,$$
$$\Gamma \cdot C \cdot \mathfrak{S}' = \mathcal{D},$$

and $\gamma\mathfrak{S} \cap \mathfrak{S}$ is nonempty only for finitely many $\gamma \in \Gamma$.

Moreover, the double quotient set $\Gamma\backslash G(\mathbb{Q})/P$ is finite and could be considered as a set of cusps of Γ. Then we can choose the representatives $g_1, \ldots, g_s \in G(\mathbb{Q})$ of these double cosets, and there exists a Siegel domain $\mathfrak{S}$ for $C = \{g_1, \ldots, g_s\}$.

There is an analogous construction of the Iwasawa decomposition and Siegel domain in the case $p \geqslant q$.

5.2 Convergence of the integral

In this section we prove Theorem 5.1 using the Iwasawa decomposition.

Let

$$\theta(\mathbf{a}, L)(Z) = \sum_{x \in L^p + \mathbf{a}} \mathcal{P}_{p,q}(\sqrt{v}\langle x, z\rangle) e^{\pi \mathrm{i}\, \mathrm{tr}\langle x,x\rangle_{\tau,Z}}$$

be a complex function on $\mathcal{D}$ satisfying $\theta_{km}(\tau) = e^{2\pi \mathrm{i}\, \mathrm{tr}\langle \mathbf{a},\mathbf{b}\rangle}\theta(\mathbf{a}, L) \cdot \mu$ for $\mathbf{a}, \mathbf{b} \in L^{*p}$. In order to prove Theorem 5.1 it suffices to show that $\theta(\mathbf{a}, L) \in \mathrm{L}^1(\mathfrak{S}'; \mu)$.

Since $L \subseteq L^*$, we can choose a basis of V such that L and L^* consist of rational vectors. Moreover, we can choose a positive integer M such that $M\mathbb{Z}^{p+q} \subseteq L$ is a finite index sublattice. Then $\theta(\mathbf{a}, L)$ is a finite sum of terms of the form $\theta(M\mathbb{Z}^{p+q}, \mathbf{a}')$ and it suffices to show that $\theta(M\mathbb{Z}^{p+q}, \mathbf{a}')$ is of class L^1. Therefore, assume further that $L = M\mathbb{Z}^{p+q}$ and $\mathbf{a}$ is rational. Furthermore, we identify L^p with $\mathrm{M}_{p+q,p}(M\mathbb{Z})$. We consider the cases $p \leqslant q$ and $p \geqslant q$ separately.

Assume first that $p \leqslant q$. In this case $z_0 = \begin{pmatrix} 1_q \\ 0 \\ -1_q \end{pmatrix}$ is a properly oriented orthonormal basis of some positive subspace $Z_0 \in \mathcal{D}$. Then for $Z = gZ_0$, where $g = n(a, b, c)^{-1}a(t) \in \mathfrak{S}$ we have

$$\theta(\mathbf{a}, L)(Z) = \sum_{x \equiv \mathbf{a}(L)} \mathcal{P}_{p,q}(\sqrt{v}\langle x, gz_0\rangle) e^{\pi \mathrm{i}\, \mathrm{tr}\langle x,x\rangle_{\tau,gZ_0}} = \sum_{x \equiv \mathbf{a}(L)} \mathcal{P}_{p,q}(\sqrt{v}\langle y, z_0\rangle) e^{\pi \mathrm{i}\, \mathrm{tr}\langle y,y\rangle_{\tau,Z_0}},$$

where $y = g^{-1}x = a(t)^{-1}n(a,b,c)x$. If $x = \begin{pmatrix} x' \\ x^0 \\ x'' \end{pmatrix}$ then

$$y = \begin{pmatrix} y' \\ y^0 \\ y'' \end{pmatrix} = \begin{pmatrix} t^{-1} & & \\ & 1_{q-p} & \\ & & t \end{pmatrix} \begin{pmatrix} a & 2ab^\top Q_0 & a(b^\top Q_0 b + c) \\ & 1_{q-p} & b \\ & & a^{-\top} \end{pmatrix} \begin{pmatrix} x' \\ x^0 \\ x'' \end{pmatrix} = \begin{pmatrix} t^{-1}a(x' + 2b^\top Q_0 x^0 + b^\top Q_0 b x'' + cx'') \\ x^0 + bx'' \\ ta^{-\top}x'' \end{pmatrix}.$$

We have that

$$\langle y, y\rangle = \begin{pmatrix} y'^\top & y^{0\top} & y''^\top \end{pmatrix} \begin{pmatrix} & & -\frac{1}{2} \\ & Q_0 & \\ -\frac{1}{2} & & \end{pmatrix} \begin{pmatrix} y' \\ y^0 \\ y'' \end{pmatrix} = -\frac{1}{2}y'^\top y'' - \frac{1}{2}y''^\top y' + y^{0\top} Q_0 y^0,$$

and

$$\begin{aligned}\langle y, z_0\rangle &= \begin{pmatrix} t^{-1}a(x' + 2b^\top Q_0 x^0 + b^\top Q_0 b x'' + cx'') \\ x^0 + bx'' \\ ta^{-\top}x'' \end{pmatrix}^\top \begin{pmatrix} & & -\frac{1}{2} \\ & Q_0 & \\ -\frac{1}{2} & & \end{pmatrix} \begin{pmatrix} 1_p \\ 0 \\ -1_p \end{pmatrix} \\ &= \begin{pmatrix} t^{-1}a(x' + 2b^\top Q_0 x^0 + b^\top Q_0 b x'' + cx'') \\ x^0 + bx'' \\ ta^{-\top}x'' \end{pmatrix}^\top \begin{pmatrix} \frac{1}{2}1_p \\ 0 \\ -\frac{1}{2}1_p \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}t^{-1}a(x' + 2b^\top Q_0 x^0 + b^\top Q_0 b x'' + cx'') - \frac{1}{2}ta^{-\top}x'' \end{pmatrix}^\top = \begin{pmatrix} \frac{1}{2}y' - \frac{1}{2}y'' \end{pmatrix}^\top.\end{aligned}$$

Let $w = (\frac{1}{2}y' - \frac{1}{2}y'')\sqrt{v} = \frac{1}{2}t^{-1}a(x' + 2b^\top Q_0 x^0 + b^\top Q_0 b x'' + cx'')\sqrt{v} - \frac{1}{2}ta^{-\top}x''\sqrt{v}$. Then

$$\langle y\sqrt{v}, z_0\rangle\langle z_0, y\sqrt{v}\rangle = w^\top w.$$

Hence,

$$\begin{aligned}&\pi \mathrm{i}\,\mathrm{tr}\langle y, y\rangle_{\tau, Z_0} = \pi \mathrm{i}\,\mathrm{tr}[\overline{\tau}\langle y, y\rangle + 2\mathrm{i}v\langle y, z_0\rangle\langle z_0, y\rangle] \\ &= -\pi \mathrm{i}\,\mathrm{tr}[\overline{\tau}\, y''^\top y'] + \pi \mathrm{i}\,\mathrm{tr}[\overline{\tau}\, y^{0\top} Q_0 y^0] - 2\pi\,\mathrm{tr}[w^\top w] \\ &= -\pi \mathrm{i}\,\mathrm{tr}[\overline{\tau}\, y''^\top y''] + \pi \mathrm{i}\,\mathrm{tr}[\overline{\tau}\, y^{0\top} Q_0 y^0] - 2\pi \mathrm{i}\,\mathrm{tr}[v^{-\frac{1}{2}}\,\overline{\tau}\, y''^\top w] - 2\pi\,\mathrm{tr}[w^\top w].\end{aligned}$$

While x' runs over $\mathrm{M}_p(M\mathbb{Z}) + \mathbf{a}'$ we have that w runs over $L_0 + \mathbf{c}$, where

$$L_0 = \frac{1}{2}t^{-1}a \cdot \mathrm{M}_p(M\mathbb{Z}) \cdot \sqrt{v}$$

is a full rank lattice in $\mathrm{M}_p(\mathbb{R})$ and

$$\mathbf{c} = \frac{1}{2}t^{-1}a(\mathbf{a}' + 2b^\top Q_0 x^0 + b^\top Q_0 b x'' + cx'')\sqrt{v} - \frac{1}{2}ta^{-\top}x''\sqrt{v}.$$

The dual lattice, therefore, is

$$L_0^* = 2ta^{-\top} \cdot \mathrm{M}_p(\tfrac{1}{M}\mathbb{Z}) \cdot v^{-\frac{1}{2}}.$$

We have further

$$\theta(\mathbf{a},L)(Z)=\sum_{\substack{x^0\equiv\mathbf{a}^0\\ x''\equiv\mathbf{a}''}} e^{\pi\mathrm{i}\,\mathrm{tr}[\overline{\tau}\,y^{0\top}Q_0y^0]}e^{\pi\mathrm{i}\,\mathrm{tr}[\overline{\tau}\,y''^\top y'']}\sum_{w\in L_0+\mathbf{c}}\mathcal{P}_{p,q}(w)e^{-2\pi\,\mathrm{tr}[w^\top w]}e^{-2\pi\mathrm{i}\,\mathrm{tr}[v^{-\frac12}\,\overline{\tau}\,y''^\top w]}.$$

In the inner sum we apply the Poisson summation formula. According to Lemma 2.22 we obtain

$$\sum_{w\in L_0+\mathbf{c}}\mathcal{P}_{p,q}(w)e^{-2\pi\,\mathrm{tr}[w^\top w]}e^{-2\pi\mathrm{i}\,\mathrm{tr}[v^{-\frac12}\,\overline{\tau}\,y''^\top w]}=C\sum_{w\in L_0^*+y''\,\overline{\tau}\,v^{-\frac12}}|w|^q e^{-\frac12\pi\,\mathrm{tr}[w^\top w]}e^{2\pi\mathrm{i}\,\mathrm{tr}[\mathbf{c}^\top w]}$$
$$=C\sum_{w\in L_0^*}|w+y''\,\overline{\tau}\,v^{-\frac12}|^q e^{-\frac12\pi\,\mathrm{tr}[(w+y''\,\overline{\tau}\,v^{-\frac12})^\top(w+y''\,\overline{\tau}\,v^{-\frac12})]}e^{2\pi\mathrm{i}\,\mathrm{tr}[\mathbf{c}^\top(w+y''\,\overline{\tau}\,v^{-\frac12})]},$$

where

$$C=\mathrm{i}^{-pq}2^{-pq-\frac{p^2}{2}}\,\mathrm{vol}^{-1}(\mathrm{M}_p(\mathbb{R})/L_0)e^{-2\pi\mathrm{i}\,\mathrm{tr}[\mathbf{c}^\top y''\,\overline{\tau}\,v^{-\frac12}]}.$$

Trivially, we have

$$\mathrm{vol}(\mathrm{M}_p(\mathbb{R})/L_0)=|\tfrac12 t^{-1}a|^p|v|^{\frac p2}M^{p^2}=2^{-p^2}M^{p^2}|t|^{-p}|v|^{\frac p2}.$$

Moreover, we have further

$$\begin{aligned}&(w+y''\,\overline{\tau}\,v^{-\frac12})^\top(w+y''\,\overline{\tau}\,v^{-\frac12})\\=&(w+y''uv^{-\frac12})^\top(w+y''uv^{-\frac12})-(y''\mathrm{i}v^{\frac12})^\top w-w^\top(y''\mathrm{i}v^{\frac12})-v^{\frac12}y''^\top y''v^{\frac12}.\end{aligned}$$

Therefore,

$$\begin{aligned}&e^{-\frac12\pi\,\mathrm{tr}[(w+y''\,\overline{\tau}\,v^{-\frac12})^\top(w+y''\,\overline{\tau}\,v^{-\frac12})]}e^{2\pi\mathrm{i}\,\mathrm{tr}[\mathbf{c}^\top w]}\\=&e^{2\pi\mathrm{i}\,\mathrm{tr}[(\mathbf{c}+\frac12 y''v^{\frac12})^\top w]}e^{-\frac12\pi\,\mathrm{tr}[(w+y''u)^\top v^{-1}(w+y''u)]}e^{\frac12\pi\,\mathrm{tr}[vy''^\top y'']}.\end{aligned}$$

Finally, assuming $w=2ta^{-\top}w'v^{-\frac12}$ where $w'\in\mathrm{M}_p(\frac1M\mathbb{Z})$ we obtain the following result.

Lemma 5.4. *For $p\leqslant q$ the following identity holds:*

$$\begin{aligned}\theta(\mathbf{a},L)(Z)=&\frac{\mathrm{i}^{-pq}2^{\frac{p^2}{2}-pq}|t|^{p+q}}{M^{p^2}|v|^{\frac{p+q}{2}}}\sum_{\substack{x^0\equiv\mathbf{a}^0\\ x''\equiv\mathbf{a}''}}e^{\pi\mathrm{i}\,\mathrm{tr}[\overline{\tau}\,y^{0\top}Q_0y^0]}e^{\pi\mathrm{i}\,\mathrm{tr}[uy''^\top y'']}\sum_{w'\in\mathrm{M}_p(\frac1M\mathbb{Z})}|2w'+x''\,\overline{\tau}\,|^q\times\\&\times e^{2\pi\mathrm{i}\,\mathrm{tr}[(\mathbf{a}'+2b^\top Q_0x^0+b^\top Q_0bx''+cx'')^\top w']}\times\\&\times e^{-2\pi\,\mathrm{tr}[ta^{-\top}(w'+\frac12x''u)v^{-1}(w'+\frac12x''u)^\top a^{-1}t]}e^{-\frac12\pi\,\mathrm{tr}[ta^{-\top}x''vx''^\top a^{-1}t]}.\end{aligned}$$

The sum in the exponent

$$2\,\mathrm{tr}[ta^{-\top}(w'+\tfrac{1}{2}x''u)v^{-1}(w'+\tfrac{1}{2}x''u)^\top a^{-1}t]+\tfrac{1}{2}\,\mathrm{tr}[ta^{-\top}x''vx''^\top a^{-1}t]$$

is a positive semidefinite quadratic form in the variables $t_1,\dots,t_p$ and could be written in the form

$$b_1t_1^2+b_2t_2^2+\cdots+b_pt_p^2,$$

for some $b_i\geqslant 0$. If for some x^0,x'',w' a term of the sum is not rapidly decreasing in the variable t_i then $b_i=0$ which implies that the whole ith row of both matrices $a^{-\top}(w'+\frac{1}{2}x''u)v^{-\frac{1}{2}}$ and $a^{-\top}x''v^{\frac{1}{2}}$ is zero. Combining this together we get that the ith row of the matrix $a^{-\top}(w'+\frac{1}{2}x''\,\overline{\tau})v^{-\frac{1}{2}}$ is zero as well. But then the coefficient $|2w'+x''\,\overline{\tau}\,|^q$ is zero. Therefore, every nonzero term in the sum *is rapidly decreasing in each variable* $t_1,\dots,t_p$ *of* t. This proves Theorem 5.1 for $p\leqslant q$.

In the case $p\geqslant q$ we have a similar, but more technical computation. Consider the rational Witt basis of V of the form

$$\begin{pmatrix} & & -\frac{1}{2}1_q\\ & Q_0 & \\ -\frac{1}{2}1_q & & \end{pmatrix}.$$

Let $z_0=\begin{pmatrix}1_q & 0\\ 0 & T\\ -1_q & 0\end{pmatrix}$ be an orthonormal basis of some positive subspace $Z_0\in\mathcal{D}$, where $T\in\mathrm{GL}^+(p-q;\mathbb{R})$ is a matrix satisfying $T^\top Q_0T=1_{p-q}$. Then for $g=n(a,b,c)^{-1}a(t)\in\mathfrak{S}$ we have

$$\theta(\mathbf{a},L)(Z)=\sum_{x\equiv\mathbf{a}(L)}\mathcal{P}_{p,q}(\sqrt{v}\langle y,z_0\rangle)e^{\pi\mathrm{i}\,\mathrm{tr}\langle y,y\rangle_{\tau,Z_0}},$$

where $y=g^{-1}x=a(t)^{-1}n(a,b,c)x$ as before. More precisely,

$$y=\begin{pmatrix}y'\\ y^0\\ y''\end{pmatrix}=\begin{pmatrix}t^{-1} & & \\ & 1_{p-q} & \\ & & t\end{pmatrix}\begin{pmatrix}a & 2ab^\top Q_0 & a(b^\top Q_0b+c)\\ & 1_{p-q} & b\\ & & a^{-\top}\end{pmatrix}\begin{pmatrix}x'\\ x^0\\ x''\end{pmatrix}=\begin{pmatrix}t^{-1}a(x'+2b^\top Q_0x^0+b^\top Q_0bx''+cx'')\\ x^0+bx''\\ ta^{-\top}x''\end{pmatrix}$$

Furthermore,

$$\langle y,y\rangle=-\frac{1}{2}y'^\top y''-\frac{1}{2}y''^\top y'+y^{0\top}Q_0y^0,$$

and

$$\begin{aligned}\langle y, z_0\rangle &= \begin{pmatrix} t^{-1}a(x'+2b^\top Q_0x^0+b^\top Q_0bx''+cx'') \\ x^0+bx'' \\ ta^{-\top}x'' \end{pmatrix}^\top \begin{pmatrix} & & -\frac{1}{2} \\ & Q_0 & \\ -\frac{1}{2} & & \end{pmatrix} \begin{pmatrix} 1_q & 0 \\ 0 & T \\ -1_q & 0 \end{pmatrix} \\ &= \begin{pmatrix} t^{-1}a(x'+2b^\top Q_0x^0+b^\top Q_0bx''+cx'') \\ x^0+bx'' \\ ta^{-\top}x'' \end{pmatrix}^\top \begin{pmatrix} \frac{1}{2}1_p & 0 \\ 0 & T^{-1} \\ -\frac{1}{2}1_p & 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}t^{-1}a(x'+2b^\top Q_0x^0+b^\top Q_0bx''+cx'')-\frac{1}{2}ta^{-\top}x'' \\ T^{-1}(x^0+bx'') \end{pmatrix}^\top = \begin{pmatrix} \frac{1}{2}y'-\frac{1}{2}y'' \\ T^{-1}y^0 \end{pmatrix}^\top .\end{aligned}$$

Let $w = (\frac{1}{2}y' - \frac{1}{2}y'')\sqrt{v} = \frac{1}{2}t^{-1}a(x' + 2b^\top Q_0x^0 + b^\top Q_0bx'' + cx'')\sqrt{v} - \frac{1}{2}ta^{-\top}x''\sqrt{v}$ and $w_0 = T^{-1}y^0\sqrt{v}$. Then

$$\langle y\sqrt{v}, z_0\rangle\langle z_0, y\sqrt{v}\rangle = w^\top w + w_0^\top w_0,$$

and

$$\begin{aligned}&\pi\mathrm{i}\,\mathrm{tr}\langle y, y\rangle_{\tau,Z_0} = \pi\mathrm{i}\,\mathrm{tr}[\overline{\tau}\langle y, y\rangle + 2\mathrm{i}v\langle y, z_0\rangle\langle z_0, y\rangle] \\ &= -\pi\mathrm{i}\,\mathrm{tr}[\overline{\tau}\, y''^\top y'] + \pi\mathrm{i}\,\mathrm{tr}[\overline{\tau}\, y^{0\top} Q_0 y^0] - 2\pi\,\mathrm{tr}[w^\top w + w_0^\top w_0] \\ &= -\pi\mathrm{i}\,\mathrm{tr}[\overline{\tau}\, y''^\top y''] + \pi\mathrm{i}\,\mathrm{tr}[\overline{\tau}\, y^{0\top} Q_0 y^0] - 2\pi\mathrm{i}\,\mathrm{tr}[v^{-\frac{1}{2}}\,\overline{\tau}\, y''^\top w] - 2\pi\,\mathrm{tr}[w^\top w + w_0^\top w_0].\end{aligned}$$

For $p \geqslant q$ we have that x' runs over $\mathrm{M}_{q,p}(M\mathbb{Z}) + \mathbf{a}'$ and w runs over $L_0 + \mathbf{c}$, where

$$L_0 = \frac{1}{2}t^{-1}a \cdot \mathrm{M}_{q,p}(M\mathbb{Z}) \cdot \sqrt{v}$$

is a full rank lattice in $\mathrm{M}_{q,p}(\mathbb{R})$ and

$$\mathbf{c} = \frac{1}{2}t^{-1}a(\mathbf{a}' + 2b^\top Q_0x^0 + b^\top Q_0bx'' + cx'')\sqrt{v} - \frac{1}{2}ta^{-\top}x''\sqrt{v}.$$

The dual lattice is

$$L_0^* = 2ta^{-\top} \cdot \mathrm{M}_{q,p}(\tfrac{1}{M}\mathbb{Z}) \cdot v^{-\frac{1}{2}}.$$

We have further

$$\begin{aligned}\theta(\mathbf{a}, L)(Z) = \sum_{\substack{x^0\equiv\mathbf{a}^0 \\ x''\equiv\mathbf{a}''}} & e^{\pi\mathrm{i}\,\mathrm{tr}[\overline{\tau}\, y^{0\top} Q_0 y^0]} e^{\pi\mathrm{i}\,\mathrm{tr}[\overline{\tau}\, y''^\top y'']} e^{-2\pi\,\mathrm{tr}[w_0^\top w_0]} \times \\ & \times \sum_{w\in L_0+\mathbf{c}} \mathcal{P}_{p,q}(\left(\begin{smallmatrix} w \\ w_0 \end{smallmatrix}\right)) e^{-2\pi\,\mathrm{tr}[w^\top w]} e^{-2\pi\mathrm{i}\,\mathrm{tr}[v^{-\frac{1}{2}}\,\overline{\tau}\, y''^\top w]}.\end{aligned}$$

As before, we apply the Poisson summation to the inner sum. According to Lemma 2.23

we obtain

$$\sum_{w\in L_0+\mathbf{c}} \mathcal{P}_{p,q}(\left(\begin{smallmatrix} w \\ w_0 \end{smallmatrix}\right))e^{-2\pi\,\mathrm{tr}[w^\top w]}e^{-2\pi\mathrm{i}\,\mathrm{tr}[v^{-\frac{1}{2}}\,\overline{\tau}\,y''^\top w]}$$
$$=\frac{e^{-2\pi\mathrm{i}\,\mathrm{tr}[\mathbf{c}^\top y''\,\overline{\tau}\,v^{-\frac{1}{2}}]}}{\mathrm{vol}(\mathrm{M}_{q,p}(\mathbb{R})/L_0)} \sum_{w\in L_0^*+y''\,\overline{\tau}\,v^{-\frac{1}{2}}} \mathcal{Q}_{p,q}(\left(\begin{smallmatrix} w \\ w_0 \end{smallmatrix}\right))e^{-\frac{1}{2}\pi\,\mathrm{tr}[w^\top w]}e^{-2\pi\mathrm{i}\,\mathrm{tr}[\mathbf{c}^\top w]}$$
$$=\frac{1}{\mathrm{vol}(\mathrm{M}_{q,p}(\mathbb{R})/L_0)} \sum_{w\in L_0^*} \mathcal{Q}_{p,q}\left(\left(\begin{smallmatrix} w+y''\,\overline{\tau}\,v^{-\frac{1}{2}} \\ w_0 \end{smallmatrix}\right)\right) e^{-\frac{1}{2}\pi\,\mathrm{tr}[(w+y''\,\overline{\tau}\,v^{-\frac{1}{2}})^\top(w+y''\,\overline{\tau}\,v^{-\frac{1}{2}})]}e^{-2\pi\mathrm{i}\,\mathrm{tr}[\mathbf{c}^\top(w+y''\,\overline{\tau}\,v^{-\frac{1}{2}})]}.$$

We have that $\mathrm{vol}(\mathrm{M}_{q,p}(\mathbb{R})/L_0) = 2^{-pq}M^{pq}|t|^{-p}|v|^{\frac{q}{2}}$. Assuming $w = 2ta^{-\top}w'v^{-\frac{1}{2}}$ for some $w' \in \mathrm{M}_{q,p}(\frac{1}{M}\mathbb{Z})$ we obtain another formula.

Lemma 5.5. *For $p \geqslant q$ the following identity holds:*

$$\theta(\mathbf{a}, L) = \frac{2^{pq}|t|^p}{M^{pq}|v|^{\frac{q}{2}}} \sum_{\substack{x^0\equiv\mathbf{a}^0 \\ x''\equiv\mathbf{a}''}} e^{\pi\mathrm{i}\,\mathrm{tr}[\overline{\tau}\,y^{0\top}Q_0y^0]}e^{\pi\mathrm{i}\,\mathrm{tr}[uy''^\top y'']}e^{-2\pi\,\mathrm{tr}[w_0^\top w_0]}\times$$
$$\times \sum_{w'\in\mathrm{M}_p(\frac{1}{M}\mathbb{Z})} \mathcal{Q}_{p,q}\left(\left(\begin{smallmatrix} ta^{-\top}(2w'+x''\,\overline{\tau})v^{-\frac{1}{2}} \\ w_0 \end{smallmatrix}\right)\right) e^{2\pi\mathrm{i}\,\mathrm{tr}[(\mathbf{a}'+2b^\top Q_0x^0+b^\top Q_0bx''+cx'')^\top w']}\times$$
$$\times e^{-2\pi\,\mathrm{tr}[ta^{-\top}(w'+\frac{1}{2}x''u)v^{-1}(w'+\frac{1}{2}x''u)^\top a^{-1}t]}e^{-\frac{1}{2}\pi\,\mathrm{tr}[ta^{-\top}x''vx''^\top a^{-1}t]}.$$

Similarly, if the exponent does not rapidly decrease in the variable t_i then the ith row of the matrix $ta^{-\top}(2w'+x''\,\overline{\tau})v^{-\frac{1}{2}}$ is zero. However, from the determinant-like properties of the polynomial $\mathcal{Q}_{p,q}$ it follows that in this case the coefficient $\mathcal{Q}_{p,q}\left(\left(\begin{smallmatrix} ta^{-\top}(2w'+x''\,\overline{\tau})v^{-\frac{1}{2}} \\ w_0 \end{smallmatrix}\right)\right)$ vanishes. Therefore, in this case again every nonzero term is rapidly decreasing in each variable $t_1, \dots, t_q$. This concludes Theorem 5.1 in the case $p \geqslant q$.

There is also a similar computation for the case $\mathrm{rank}_{\mathbb{Q}}\, G < \mathrm{rank}_{\mathbb{R}}\, G$ which we omit.

Hence, the theta integral $\Theta_{km}^{\Gamma\backslash\mathcal{D}}(\tau)$ is well-defined. Moreover, it has a Fourier decomposition

$$\Theta_{km}^{\Gamma\backslash\mathcal{D}}(\tau) = \sum_m \theta_m(\tau)e^{\pi\mathrm{i}\,\mathrm{tr}[\tau m]},$$

where the summation goes over symmetric rational $p \times p$ matrices m. For fixed m we have that

$$\theta_m(\tau)e^{\pi\mathrm{i}\,\mathrm{tr}[\tau m]} = \int_{\Gamma\backslash\mathcal{D}} \sum_{x\in\mathcal{L}_m} \varphi_{km}(\tau, x),$$

where $\mathcal{L}_m = \{x \in L^p + \mathbf{a} | \langle x, x\rangle = m\}$.

For some m the above sum may be termwise integrable, and this would simplify the computation. For $m \in \mathrm{GL}(n;\mathbb{Q})$ nonsingular the set $\Gamma\backslash\mathcal{L}_m$ is finite.

Applying formally the unfolding process we obtain

$$\theta_m(\tau)e^{\pi\mathrm{i}\,\mathrm{tr}[\tau m]} = \sum_{x\in\Gamma\backslash\mathcal{L}_m} \int\limits_{\Gamma_x\backslash\mathcal{D}} \varphi_{km}(\tau,x). \tag{5.1}$$

Therefore, if the integral

$$\int\limits_{\Gamma_x\backslash\mathcal{D}} \|\varphi_{km}(\tau,x)\|\mu,$$

where

$$\|\varphi_{km}(\tau,x)\| = |\mathcal{P}_{p,q}(\sqrt{v}\langle x,v\rangle)e^{\pi\mathrm{i}\,\mathrm{tr}\langle x,x\rangle_{\tau,Z}}| = e^{\pi\,\mathrm{tr}[v\langle x,x\rangle]}|\mathcal{P}_{p,q}(\sqrt{v}\langle x,v\rangle)|e^{-2\pi\,\mathrm{tr}[v\langle x,z\rangle\langle z,x\rangle]}$$

converges for arbitrary $x\in\mathcal{L}_m$, then the series for $\theta_m(\tau)$ is absolutely convergent, and we can integrate it termwise. In particular, the formula (5.1) holds.

Moreover, we can apply the invariance property of φ_{km}. We have that

$$\int\limits_{\Gamma_x\backslash\mathcal{D}} \varphi_{km}(\tau,x) = \int\limits_{\Gamma_{gx}\backslash\mathcal{D}} \varphi_{km}(\tau,gx)$$

for all $g\in G$. The same holds for integrals of $\|\varphi_{km}(\tau,x)\|$.

Kudla and Millson showed [KM3] that for x generating a positive subspace

$$\int\limits_{\mathcal{D}} \varphi_{km}(x) = 2^{-\frac{pq}{2}}\varepsilon(x)e^{-\pi\,\mathrm{tr}\langle x,x\rangle},$$

and this integral absolutely converges. Then we have that

$$\int\limits_{\Gamma_x\backslash\mathcal{D}} \varphi_{km}(\tau,x) = \frac{1}{|\Gamma_x|}e^{\pi\mathrm{i}\,\mathrm{tr}[u\langle x,x\rangle]}\int\limits_{\mathcal{D}} \varphi_{km}(x\sqrt{v})$$
$$=2^{-\frac{pq}{2}}\frac{1}{|\Gamma_x|}\varepsilon(x)e^{\pi\mathrm{i}\,\mathrm{tr}[u\langle x,x\rangle+\mathrm{i}\langle x\sqrt{v},x\sqrt{v}\rangle]} = 2^{-\frac{pq}{2}}\frac{1}{|\Gamma_x|}\varepsilon(x)e^{\pi\mathrm{i}\,\mathrm{tr}[\tau\langle x,x\rangle]}.$$

This implies that

$$\Theta^+_{km}(\tau) = 2^{\frac{pq}{2}}\int\limits_{\Gamma\backslash\mathcal{D}} \theta^+_{km}(\tau) = \sum_{\substack{x\in\Gamma\backslash L^p+\mathbf{a}\\ \langle x,x\rangle>0}} \frac{1}{|\Gamma_x|}\varepsilon(x)e^{\pi\mathrm{i}\,\mathrm{tr}[\tau\langle x,x\rangle]}.$$

5.3 Integral of θ^-_{km}

Assume that $x \in V^p$ generates a negative subspace of dimension p. This is possible only for $p \leqslant q$.

Lemma 5.6. *For all x negative, except the case when $(p,q) = (1,2)$ and $\langle x\rangle^\top$ is split, the integral*

$$\int\limits_{\Gamma_x \backslash \mathcal{D}} \varphi_{km}(\tau, x)$$

absolutely converges. If q is odd, then the integral is equal to zero.

Without loss of generality, assume that $x = \left(\begin{smallmatrix} 0 \\ c \end{smallmatrix}\right)$, where $c \in \mathrm{GL}(p;\mathbb{R})$ is nonsingular. Let $X = \langle x\rangle \in \mathrm{Gr}^-_p(V)$, let $Y = X^\perp$, $\operatorname{sig} Y = (p, q-p)$. The stabilizer Γ_x preserves the spaces X and Y, and it is an arithmetic subgroup of $\mathrm{O}(Y)$. Let $\mathcal{D}_1 = \mathrm{Gr}^+_p(Y)$ be the symmetric space attached to Y. Let $\mathcal{F}$ be a fundamental domain for Γ_x acting on $\mathcal{D}_1$.

The domain $\mathcal{F}$ has infinite volume only if $\operatorname{sig} Y = (p, q-p) = (1,1)$ and Y is split. Then $q = p + q - p = 1 + 1 = 2$. Hence, if $q \neq 2$, $\mathcal{F}$ is of finite volume.

Consider the Klein model for $\mathcal{D}$ and $\mathcal{D}_1$. Let $t = (t_1\ t_2)$ where $t_1 \in \mathcal{B}^{p,q-p}$ parametrizes $\mathcal{D}_1$, and $t_2 \in \mathrm{M}_p(\mathbb{R})$ satisfies $t_1t_1^\top + t_2t_2^\top < 1_p$. We have the following decomposition of the invariant measure μ on $\mathcal{D}$:

$$\mu = \frac{\det(1 - t_1t_1^\top)^{\frac{q}{2}}}{\det(1 - tt^\top)^{\frac{p+q}{2}}}\mu_1 dt_2,$$

where

$$\mu_1 = \frac{dt_1}{\det(1 - t_1t_1^\top)^{\frac{q}{2}}}.$$

is the invariant measure on $\mathcal{D}_1$.

Moreover, the orthonormal frame $z(t)$ attached to t is

$$z = z(t) = t^\top(1 - tt^\top)^{-\frac{1}{2}} = \begin{pmatrix} t_1^\top(1 - tt^\top)^{-\frac{1}{2}} \\ t_2^\top(1 - tt^\top)^{-\frac{1}{2}} \end{pmatrix},$$

and

$$\langle x, z\rangle = \begin{pmatrix} 0 & c^\top \end{pmatrix} \begin{pmatrix} 1_p & 0 \\ 0 & -1_q \end{pmatrix} \begin{pmatrix} t_1^\top(1 - tt^\top)^{-\frac{1}{2}} \\ t_2^\top(1 - tt^\top)^{-\frac{1}{2}} \end{pmatrix} = -c^\top t_2^\top(1 - tt^\top)^{-\frac{1}{2}}.$$

Therefore,

$$\begin{aligned}
&\int\limits_{\Gamma_x\backslash\mathcal{D}} |\mathcal{P}_{p,q}(\sqrt{v}\langle x,z\rangle)|e^{-2\pi\,\mathrm{tr}[v\langle x,z\rangle\langle z,x\rangle]}\mu \\
=&\int\limits_{\Gamma_x\backslash\mathcal{D}} \left|\mathcal{P}_{p,q}\Big(-\sqrt{v}c^\top t_2^\top(1-tt^\top)^{-\frac{1}{2}}\Big)\right|e^{-2\pi\,\mathrm{tr}[cvc^\top t_2^\top(1-tt^\top)^{-1}t_2]}\frac{dt}{(1-tt^\top)^{\frac{p+q}{2}}} \\
=&\int\limits_{\mathcal{F}}\int\limits_{t_1t_1^\top+t_2t_2^\top<1} \left|\mathcal{P}_{p,q}\Big(-\sqrt{v}c^\top t_2^\top(1-tt^\top)^{-\frac{1}{2}}\Big)\right|e^{-2\pi\,\mathrm{tr}[cvc^\top t_2^\top(1-tt^\top)^{-1}t_2]}\frac{\det(1-t_1t_1^\top)^{\frac{q}{2}}}{\det(1-tt^\top)^{\frac{p+q}{2}}}dt_2\,\mu_1.
\end{aligned}$$

We consider the inner integral and show it does not depend on t_1. Indeed, we have that

$$\begin{aligned}
&t_2^\top(1-tt^\top)^{-1}t_2 = t_2^\top(1-t_1t_1^\top-t_2t_2^\top)^{-1}t_2 \\
=&t_2^\top(1-t_1t_1^\top)^{-\frac{1}{2}}\Big(1-(1-t_1t_1^\top)^{-\frac{1}{2}}t_2t_2^\top(1-t_1t_1^\top)^{-\frac{1}{2}}\Big)^{-1}(1-t_1t_1^\top)^{-\frac{1}{2}}t_2 \\
=&s^\top(1-ss^\top)^{-1}s,
\end{aligned}$$

where

$$s=(1-t_1t_1^\top)^{-\frac{1}{2}}t_2.$$

Moreover, we have that $s^\top(1-ss^\top)^{-\frac{1}{2}}$ is not a symmetric matrix, but it differs by a $\mathrm{SO}(p)$-matrix from $t_2^\top(1-tt^\top)^{-\frac{1}{2}}$. Since $\mathcal{P}_{p,q}$ is $\mathrm{SO}(p)$-invariant, we have that

$$\mathcal{P}_{p,q}\Big(-\sqrt{v}c^\top t_2^\top(1-tt^\top)^{-\frac{1}{2}}\Big)=\mathcal{P}_{p,q}\Big(-\sqrt{v}c^\top s^\top(1-ss^\top)^{-\frac{1}{2}}\Big).$$

Furthermore, we have that $t_2t_2^\top<1-t_1t_1^\top$ implies $ss^\top<1$. The Jacobian $|\frac{\partial s}{\partial t_2}|$ is equal to $\det(1-t_1t_1^\top)^{-\frac{p}{2}}$. Finally, it holds that

$$\det(1-tt^\top)=\det(1-t_1t_1^\top)\det(1-ss^\top).$$

Hence, after this substitution the inner integral becomes

$$\int\limits_{\mathcal{B}^{p,p}} \left|\mathcal{P}_{p,q}\Big(-\sqrt{v}c^\top s^\top(1-ss^\top)^{-\frac{1}{2}}\Big)\right|e^{-2\pi\,\mathrm{tr}[cvc^\top s^\top(1-ss^\top)^{-1}s]}\frac{ds}{\det(1-ss^\top)^{\frac{p+q}{2}}}.$$

Finally, let $w=s^\top(1-ss^\top)^{-\frac{1}{2}}$. Then $dw=\det(1-ss^\top)^{-\frac{2p+1}{2}}ds$ and

$$1+w^\top w=(1-ss^\top)^{-1}.$$

We get the integral

$$\int_{\mathrm{M}_p(\mathbb{R})} |\mathcal{P}_{p,q}(-\sqrt{v}c^\top w)| e^{-2\pi \operatorname{tr}[cvc^\top w^\top w]} \det(1+w^\top w)^{\frac{q-p-1}{2}} dw.$$

Clearly, this integral converges since its argument rapidly decreases. Let

$$I(v,x) = \int_{\mathrm{M}_p(\mathbb{R})} \mathcal{P}_{p,q}(-\sqrt{v}c^\top w) e^{-2\pi \operatorname{tr}[cvc^\top w^\top w]} \det(1+w^\top w)^{\frac{q-p-1}{2}} dw$$

be the *fiber integral*. Then we have that

$$\int_{\Gamma_x \backslash \mathcal{D}} \varphi_{km}(\tau,x) = e^{\pi \mathrm{i} \operatorname{tr}[\overline{\tau}\langle x,x\rangle]} \mu_1(\Gamma_x \backslash \mathcal{D}_1) I(v,x).$$

Lemma 5.7. *For q odd the fiber integral $I(v,x)$ is equal to zero.*

Proof. This follows from the properties of the polynomial $\mathcal{P}_{p,q}$. Let $\sigma = \operatorname{diag}(-1,1,\dots,1)$ and $w' = w\sigma$. Then

$$\begin{aligned} I(v,x) &= \int_{\mathrm{M}_p(\mathbb{R})} \mathcal{P}_{p,q}(-\sqrt{v}c^\top w) e^{-2\pi \operatorname{tr}[cvc^\top w^\top w]} \det(1+w^\top w)^{\frac{q-p-1}{2}} dw = \\ &= \int_{\mathrm{M}_p(\mathbb{R})\cdot\sigma} \mathcal{P}_{p,q}(-\sqrt{v}c^\top w'\sigma) e^{-2\pi \operatorname{tr}[cvc^\top w'^\top w']} \det(1+w'^\top w')^{\frac{q-p-1}{2}} \cdot (-1)^p dw' \\ &= -\int_{\mathrm{M}_p(\mathbb{R})} \mathcal{P}_{p,q}(-\sqrt{v}c^\top w') e^{-2\pi \operatorname{tr}[cvc^\top w'^\top w']} \det(1+w'^\top w')^{\frac{q-p-1}{2}} dw' = -I(v,x). \end{aligned}$$

□

We conjecture this integral to be zero in all cases.

5.4 Integral of $\theta^{\pm}_{km}$

Assume now that $x \in V^p$ generates a nondegenerate indefinite space X of dimension p.

Lemma 5.8. *For all x nonsingular generating an indefinite space X, except the case when* $\operatorname{sig} X^\perp = (1,1)$ *and $X^\perp$ is split, the integral*

$$\int_{\Gamma_x \backslash \mathcal{D}} \varphi_{km}(\tau,x)$$

absolutely converges. If q is odd, then the integral is equal to zero.

Assume that $\operatorname{sig} X = (p-r, r)$ for some $1 \leqslant r < p$. Let $Y = X^{\perp}$, then $\operatorname{sig} Y = (r, q-r)$. Consider an pseudoorthonormal basis (x', x'', y', y'') of V, where $x' \in V^r, x'' \in V^{p-r}$ are positive, and $y' \in V^r$, $y'' \in V^{q-r}$ are negative. Without loss of generality we can assume that $X = \langle x'', y'' \rangle$, $Y = \langle x', y' \rangle$. Then, we have that

$$x = \begin{pmatrix} 0 \\ c \\ 0 \\ d \end{pmatrix} = x''c + y''d,$$

for some $\left(\begin{smallmatrix} c \\ d \end{smallmatrix}\right) \in \mathrm{GL}(p;\mathbb{R})$ nonsingular.

Let also $X' = \langle x' \rangle$, $X'' = \langle x'' \rangle$, $Y' = X'^{\perp} = \langle x'', y', y'' \rangle$ and $Y'' = X''^{\perp} = \langle x', y', y'' \rangle$. Then $Y \subset Y'' \subset V$; let further $\mathcal{D}_1 = \mathrm{Gr}^-_{q-r}(Y)$ and $\mathcal{D}_2 = \mathrm{Gr}^-_{q-r}(Y'') \cong \mathrm{Gr}^+_p(Y'')$.

As before, consider the Klein model for $\mathcal{D}$. We decompose the parameter t into four blocks as follows $t = \left(\begin{smallmatrix} t_1 & t_3 \\ t_2 & t_4 \end{smallmatrix}\right)$, where $t_1 \in \mathcal{B}^{r,q-r}$ parametrizes $\mathcal{D}_1$, and $t' = \left(\begin{smallmatrix} t_1 \\ t_2 \end{smallmatrix}\right) \in \mathcal{B}^{p,q-r}$ parametrizes $\mathcal{D}_2$. Moreover, as in the previous case Γ_x is an arithmetic subgroup of $\mathrm{O}(Y)$. It has a finite volume fundamental region $\mathcal{F}$ iff $q \neq 2$, or if $q = 2$ and $Y(\mathbb{Q})$ is anisotropic,

We have that $z = z(t) = \left(\begin{smallmatrix} 1_p \\ t^\top \end{smallmatrix}\right)(1 - tt^\top)^{-1/2}$ and

$$\begin{aligned}
\langle x, z \rangle &= \begin{pmatrix} 0 & c^\top & 0 & d^\top \end{pmatrix} \begin{pmatrix} 1_r & 0 & 0 & 0 \\ 0 & 1_{p-r} & 0 & 0 \\ 0 & 0 & -1_{q-r} & 0 \\ 0 & 0 & 0 & -1_r \end{pmatrix} \begin{pmatrix} 1_r & 0 \\ 0 & 1_{p-r} \\ t_1^\top & t_2^\top \\ t_3^\top & t_4^\top \end{pmatrix} (1 - tt^\top)^{-1/2} \\
&= \begin{pmatrix} -d^\top t_3^\top & c^\top - d^\top t_4^\top \end{pmatrix} (1 - tt^\top)^{-1/2} = \left((0 \;\; c^\top) - d^\top t''^\top \right) (1 - tt^\top)^{-1/2},
\end{aligned}$$

where $t' = \left(\begin{smallmatrix} t_1 \\ t_2 \end{smallmatrix}\right)$, $t'' = \left(\begin{smallmatrix} t_3 \\ t_4 \end{smallmatrix}\right)$ and $t = (t' \; t'')$. We have that

$$\begin{aligned}
&\int\limits_{\Gamma_x \backslash \mathcal{D}} |\mathcal{P}_{p,q}(\sqrt{v}\langle x, z \rangle)| e^{-2\pi \operatorname{tr}[v\langle x,z\rangle\langle z,x\rangle]} \mu \\
&= \int\limits_{\Gamma_x \backslash \mathcal{D}_1} \int\limits_{t_1^\top t_1 + t_2^\top t_2 < 1} \int\limits_{t' t'^\top + t'' t''^\top < 1} |\mathcal{P}_{p,q}(\sqrt{v}\langle x, z \rangle)| e^{-2\pi \operatorname{tr}[v\langle x,z\rangle\langle z,x\rangle]} \frac{\det(1 - t_1 t_1^\top)^{\frac{q}{2}}}{\det(1 - tt^\top)^{\frac{p+q}{2}}} dt'' dt_2 \mu_1.
\end{aligned}$$

First, we apply the substitution $s_2 = (1 - t' t'^\top)^{-\frac{1}{2}} t''$ to the most inner integral and another

$s_1 = t_2(1 - t_1^\top t_1)^{-\frac{1}{2}}$ to the middle one. Then $s_2 \in \mathcal{B}^{p,r}$,

$$ds_2 = \det(1 - t't'^\top)^{-\frac{r}{2}} dt'', \quad ds_1 = \det(1 - t_1 t_1^\top)^{-\frac{p-r}{2}} dt_2,$$

and

$$\det(1 - tt^\top) = \det(1 - t't'^\top) \det(1 - s_2 s_2^\top),$$

and

$$(1 - tt^\top)^{-1} = (1 - t't'^\top - t''t''^\top)^{-1} = (1 - t't'^\top)^{-\frac{1}{2}} (1 - s_2 s_2^\top)^{-1} (1 - t't'^\top)^{-\frac{1}{2}}.$$

Therefore, matrices $(1 - tt^\top)^{-\frac{1}{2}}$ and $(1 - t't'^\top)^{-\frac{1}{2}}(1 - s_2 s_2^\top)^{-\frac{1}{2}}$ differ by an $\mathrm{SO}(p)$ element on the right. Thus, we have that

$$\mathcal{P}_{p,q}\left(\sqrt{v}\left(\begin{pmatrix}0 & c^\top\end{pmatrix} - d^\top t''^\top\right)(1 - tt^\top)^{-\frac{1}{2}}\right) = \mathcal{P}_{p,q}\left(\sqrt{v}\left(\begin{pmatrix}0 & c^\top\end{pmatrix}(1 - t't'^\top)^{-\frac{1}{2}} - d^\top s_2^\top\right)(1 - s_2 s_2^\top)^{-\frac{1}{2}}\right).$$

Then we obtain the following

$$\begin{aligned}
\frac{\det(1 - t_1 t_1^\top)^{\frac{q}{2}}}{\det(1 - tt^\top)^{\frac{p+q}{2}}} dt'' dt_2 &= \frac{\det(1 - t_1 t_1^\top)^{\frac{q}{2}}}{\det(1 - t't'^\top)^{\frac{p+q-r}{2}} \det(1 - s_2 s_2^\top)^{\frac{p+q}{2}}} ds_2 dt_2 \\
= \frac{ds_2 dt_2}{\det(1 - s_1 s_1^\top)^{\frac{p+q-r}{2}} \det(1 - s_2 s_2^\top)^{\frac{p+q}{2}} \det(1 - t_1 t_1^\top)^{\frac{p-r}{2}}} &= \frac{ds_2 ds_1}{\det(1 - s_1 s_1^\top)^{\frac{p+q-r}{2}} \det(1 - s_2 s_2^\top)^{\frac{p+q}{2}}}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
(1 - t't'^\top)^{-1} &= 1 + t't'^\top + t't'^\top t't'^\top + \cdots = 1 + t'(1 + t'^\top t' + t'^\top t' t'^\top t' + \cdots)t'^\top \\
&= 1 + t'(1 - t'^\top t')^{-1} t'^\top = 1 + \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} (1 - t_1^\top t_1 - t_2^\top t_2)^{-1} \begin{pmatrix} t_1^\top & t_2^\top \end{pmatrix} \\
&= \begin{pmatrix} 1 + t_1(1 - t_1^\top t_1 - t_2^\top t_2)^{-1} t_1^\top & t_1(1 - t_1^\top t_1 - t_2^\top t_2)^{-1} t_2^\top \\ t_2(1 - t_1^\top t_1 - t_2^\top t_2)^{-1} t_1^\top & 1 + t_2(1 - t_1^\top t_1 - t_2^\top t_2)^{-1} t_2^\top \end{pmatrix}.
\end{aligned}$$

We can construct a triangular matrix a satisfying $aa^\top = (1 - t't'^\top)^{-1}$. We obtain

$$a = \begin{pmatrix} * & * \\ 0 & \left(1 + t_2(1 - t_1^\top t_1 - t_2^\top t_2)^{-1} t_2^\top\right)^{\frac{1}{2}} \end{pmatrix}.$$

Then there exists an element $o \in \mathrm{SO}(q)$ depending on t_1 and t_2 that satisfies $ao =$

$(1-t't'^{\top})^{-\frac{1}{2}}$. We have further

$$\begin{aligned}
&\mathcal{P}_{p,q}\left(\sqrt{v}\left(\scriptstyle (0 \quad c^{\top})(1-t't'^{\top})^{-\frac{1}{2}}-d^{\top}s_2^{\top}\right)(1-s_2s_2^{\top})^{-\frac{1}{2}}\right)\\
=&\mathcal{P}_{p,q}\left(\sqrt{v}\left(\scriptstyle (0 \quad c^{\top})ao-d^{\top}s_2^{\top}\right)(1-s_2s_2^{\top})^{-\frac{1}{2}}\right)\\
=&\mathcal{P}_{p,q}\left(\sqrt{v}\left(\scriptstyle (0 \quad c^{\top})a-d^{\top}s_2^{\top}o^{\top}\right)(1-os_2s_2^{\top}o^{\top})^{-\frac{1}{2}}o\right)\\
=&\mathcal{P}_{p,q}\left(\sqrt{v}\left(\scriptstyle (0 \quad c^{\top})a-d^{\top}(os_2)^{\top}\right)(1-(os_2)(os_2)^{\top})^{-\frac{1}{2}}\right)
\end{aligned}$$

We can replace further os_2 with s_2. Finally, we have that

$$(0 \quad c^{\top})a = (0 \quad c^{\top}\big(1+t_2(1-t_1^{\top}t_1-t_2^{\top}t_2)^{-1}t_2^{\top}\big)^{\frac{1}{2}}),$$

and

$$\begin{aligned}
&t_2(1-t_1^{\top}t_1-t_2^{\top}t_2)^{-1}t_2^{\top}\\
=&t_2(1-t_1^{\top}t_1)^{-\frac{1}{2}}\big(1-(1-t_1^{\top}t_1)^{-\frac{1}{2}}t_2^{\top}t_2(1-t_1^{\top}t_1)^{-\frac{1}{2}}\big)^{-1}(1-t_1^{\top}t_1)^{-\frac{1}{2}}t_2^{\top}\\
=&s_1(1-s_1^{\top}s_1)^{-1}s_1^{\top}.
\end{aligned}$$

Hence, as in the previous case we obtain the integral that is independent on t_1:

$$\int\limits_{\mathcal{B}^{p-r,q-r}}\int\limits_{\mathcal{B}^{p,r}} |\mathcal{P}_{p,q}(s)|e^{-2\pi\operatorname{tr}[ss^{\top}]}\det(1-s_1s_1^{\top})^{-\frac{p+q-r}{2}}\det(1-s_2s_2^{\top})^{-\frac{p+q}{2}}ds_2ds_1,$$

where

$$s=\sqrt{v}\left(\scriptstyle (0 \quad c^{\top}(1+s_1(1-s_1^{\top}s_1)^{-1}s_1^{\top})^{\frac{1}{2}})-d^{\top}s_2^{\top}\right)(1-s_2s_2^{\top})^{-\frac{1}{2}}$$

After another substitution $w_1=s_1(1-s_1^{\top}s_1)^{-\frac{1}{2}}$ and $w_2=s_2^{\top}(1-s_2s_2^{\top})^{-\frac{1}{2}}$ we obtain

$$dw_1=\det(1-s_1^{\top}s_1)^{-\frac{q+p-2r+1}{2}}ds_1,\quad dw_2=(1-s_2s_2^{\top})^{-\frac{p+r+1}{2}}ds_2,$$

and

$$1+w_1w_1^{\top}=(1-s_1^{\top}s_1)^{-1},\quad 1+w_2^{\top}w_2=(1-s_2s_2^{\top})^{-1},$$

and the integral becomes

$$\int\limits_{\mathrm{M}_{p-r,q-r}(\mathbb{R})}\int\limits_{\mathrm{M}_{r,p}(\mathbb{R})} |\mathcal{P}_{p,q}(w)|e^{-2\pi\operatorname{tr}[ww^{\top}]}\det(1+w_1w_1^{\top})^{\frac{r-1}{2}}\det(1+w_2^{\top}w_2)^{\frac{q-r-1}{2}}dw_2dw_1,$$

where

$$w = \sqrt{v}\left(\begin{smallmatrix} 0 & c^\top(1+w_1w_1^\top)^{\frac{1}{2}} \end{smallmatrix} \right)(1+w_2^\top w_2)^{\frac{1}{2}} - d^\top w_2 \right).$$

As before, we can see that the integrand rapidly decreases and the integral converges. Therefore,

$$\int_{\Gamma_x\backslash\mathcal{D}} \varphi_{km}(\tau, x) = e^{\pi \mathrm{i}\,\mathrm{tr}[\overline{\tau}\langle x,x\rangle]}\mu_1(\Gamma_x\backslash\mathcal{D}_1)I(v,x),$$

where

$$I(v,x) = \int_{\mathrm{M}_{p-r,q-r}(\mathbb{R})}\int_{\mathrm{M}_{r,p}(\mathbb{R})} \mathcal{P}_{p,q}(w)e^{-2\pi\,\mathrm{tr}[ww^\top]}\det(1+w_1w_1^\top)^{\frac{r-1}{2}}\det(1+w_2^\top w_2)^{\frac{q-r-1}{2}}dw_2dw_1,$$

is the fiber integral.

Lemma 5.9. *For q odd the fiber integral $I(v,x)$ is equal to zero.*

Proof. Let $\sigma = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$ be an elementary matrix. Consider a substitution $w_2 = w_2'\sigma$. Then we have that $\sigma^{-1} = \sigma^\top = \sigma$ and

$$\begin{aligned} &\left((0 \;\; c^\top(1+w_1w_1^\top)^{\frac{1}{2}})(1+w_2^\top w_2)^{\frac{1}{2}} - d^\top w_2 \right) = \left((0 \;\; c^\top(1+w_1w_1^\top)^{\frac{1}{2}})(1+\sigma w_2'^\top w_2'\sigma)^{\frac{1}{2}} - d^\top w_2'\sigma \right) \\ &= \left((0 \;\; c^\top(1+w_1w_1^\top)^{\frac{1}{2}})\sigma(1+w_2'^\top w_2')^{\frac{1}{2}}\sigma - d^\top w_2'\sigma \right) = \left((0 \;\; c^\top(1+w_1w_1^\top)^{\frac{1}{2}})(1+w_2'^\top w_2')^{\frac{1}{2}} - d^\top w_2' \right)\sigma, \end{aligned}$$

since

$$(0 \;\; c^\top(1+w_1w_1^\top)^{\frac{1}{2}})\sigma = (0 \;\; c^\top(1+w_1w_1^\top)^{\frac{1}{2}})$$

because multiplication on the right by σ simply negates the first column of the matrix, which is zero in this case.

Furthermore, for q odd we have that $\mathcal{P}_{p,q}$ is odd in every column; thus,

$$\mathcal{P}_{p,q}\left(\left((0 \;\; c^\top(1+w_1w_1^\top)^{\frac{1}{2}})(1+w_2'^\top w_2')^{\frac{1}{2}} - d^\top w_2' \right)\sigma\right) = -\mathcal{P}_{p,q}\left(\left((0 \;\; c^\top(1+w_1w_1^\top)^{\frac{1}{2}})(1+w_2'^\top w_2')^{\frac{1}{2}} - d^\top w_2' \right)\right).$$

The remaining parts of the integral remain unchanged. Therefore, the integral is zero. □

This implies Theorem 5.2. Indeed, for q odd and $\mathbf{a}$ nonsingular we have that $\theta_{km} = \theta_{km}^+ + \theta_{km}^- + \theta_{km}^\pm$. However, we have showed that the series θ_{km}^- and $\theta_{km}^\pm$ are absolutely convergent and after integration they vanish. Hence, we obtain that in this case $\Theta_{km}^{\Gamma\backslash\mathcal{D}}(\tau) = \Theta_{km}^+(\tau)$.

5.5 Integral of θ^0_{km} for $\operatorname{sig} V = (m, 1)$

Assume now once again $\operatorname{sig} V = (m, 1)$. As before, we identify $\mathcal{D}$ with one component of the hyperboloid $\langle z, z\rangle = -1$ in V.

The form θ^0_{km} is a sum over frames $x \in L^m + \mathbf{a}$ satisfying $\langle x\rangle = \ell^\perp$ for some isotropic line $\ell \in \mathrm{Gr}^0_1(V(\mathbb{Q}))$. Note that the stabilizer $\mathrm{St}_G(x)$ of any such x is trivial.

If $\langle x\rangle = \ell^\perp$ then there exists a combination $x_1 a_1 + \cdots + x_m a_m$ lying in ℓ, where $a = (a_1, \ldots, a_m) \in \mathbb{Q}^m$. Moreover, we can assume that $a \in \mathbb{Z}^m$ and that a is primitive. Such vector is unique up to sign.

Moreover, on every line $\ell \in \mathrm{Gr}^0_1(V(\mathbb{Q}))$ we can choose a unique vector u_ℓ satisfying $\langle u_\ell, z\rangle < 0$ for all $z \in \mathcal{D}$ and $L \cap \ell = \mathbb{Z} \cdot u_\ell$.

Let

$$S_m(\ell, a) = \{x \in V(\mathbb{Q})^m | \langle x\rangle = \ell^\perp,\ xa \in \ell\},$$

and

$$\Xi_\ell(\mathbf{a}, L) = \{a \in \mathbb{Z}^m | a \neq 0 \text{ primitive},\ (L^m + \mathbf{a}) \cap S_m(\ell, a) \neq \varnothing\}.$$

Definition 5.10. *For $x \in S_m(\ell, a)$ and $a \in \Xi_\ell(\mathbf{a}, L)$ let*

$$xa = \nu u_\ell,$$

*where $\nu \in \mathbb{Q}$. A frame x is **reduced** if there exists a choice of a satisfying $\nu \in [0, 1)$. For x reduced let $\nu(x) = \nu$.*

Trivially, we have that if x is reduced, then γx is reduced as well, and $\nu(\gamma x) = \nu(x)$ for all $\gamma \in \Gamma$.

For an isotropic line ℓ choose another line ℓ' such that $H = \ell + \ell'$ is a hyperbolic space. We have then $\ell^\perp = \ell + H^\perp$. We can define a distinguished orientation on $H^\perp$. The basis $h_1, \ldots, h_{m-1}$ of $H^\perp$ is properly oriented if it could be completed to the properly oriented basis $u_\ell, h_1, \ldots, h_{m-1}, u_{\ell'}$ of $V(\mathbb{Q})$. Moreover, this orientation does not depend on the choice of ℓ'.

For $\langle x\rangle = \ell^\perp$ let

$$\varepsilon(x) = \begin{cases} 1, & x \text{ and } u_\ell, h_1, \ldots, h_{m-1} \text{ determine the same orientation on } \langle x\rangle, \\ -1, & \text{otherwise.} \end{cases}$$

Let $e_0 = (0, \ldots, 0, 1)^\top$. For every $a \in \Xi_\ell(\mathbf{a}, L)$ choose a matrix $s \in \mathrm{SL}(m; \mathbb{Z})$ satisfying $e_0 = sa$. Then $S_m(\ell, a) = S_m(\ell, e_0)s$. If $x_0 \in S_m(\ell, e_0)$ is reduced, then so is $x_0 s$, and $\nu(x_0) = \nu(x_0 s)$.

For $x_0 \in S_m(\ell, e_0) \cap (L^m + \mathbf{a})$, and $k \in \mathbb{Z}$ we have that

$$x_0 + ku_\ell e_0^\top \in S_m(\ell, e_0) \cap (L^m + \mathbf{a}),$$

and $(x_0 + ku_\ell e_0^\top)e_0 = (\nu(x_0) + k)u_\ell$. Among the frames $x_0 + ku_\ell e_0^\top$ exactly one is reduced for e_0. For $x_0 \in S_m(\ell, e_0) \cap (L^m + \mathbf{a})$ reduced and $s \in \mathrm{SL}(m; \mathbb{Z})$ let further

$$\varphi'_{km}(\tau, x_0 s) = \sum_{k \in \mathbb{Z}} \varphi_{km}\big(\tau, x_0 s + ku_\ell e_0^\top s\big). \tag{5.2}$$

Thus, $\varphi'_{km}(\tau, x)$ is well-defined for every reduced $x \in S_m(\ell, a) \cap (L^m + \mathbf{a})$.

Note that every singular frame x appears twice in such sums since there are two choices for a. We have therefore

$$\theta^0_{km}(\tau) = \frac{1}{2} \sum_{\substack{x \in L^m + \mathbf{a} \\ x \text{ reduced}}} \varphi'_{km}(\tau, x).$$

Note that the summands in any $\varphi'_{km}(\tau, x)$ are not Γ-equivalent since all of them have distinct value $\nu(x)$ for fixed a. Moreover, we have also $\gamma^* \varphi'_{km}(\tau, \gamma x) = \varphi'_{km}(\tau, x)$.

Theorem 5.3 then follows from the next two lemmas.

Lemma 5.11. *Let x be reduced. Then*

$$\|\varphi'_{km}(\tau, x)\| = O(e^{-Ct^2})$$

for some $C > 0$ as $t \to \infty$.

Lemma 5.12. *Let x be reduced. Then the following identity holds:*

$$\int_{\mathcal{D}} \varphi'_{km}(\tau, x) = (-1)^m 2^{-\frac{m}{2}} \varepsilon(x) \mathbf{B}_1(\nu(x)) e^{\pi \mathrm{i}\, \mathrm{tr}[\tau \langle x, x \rangle]}.$$

We prove both lemmas at once.

Proof. We make use of the upper half-space model of $\mathcal{D}$. In every point (y, t) of $\mathrm{Gr}_1^-(V)$

construct a properly oriented orthonormal basis

$$z=\begin{pmatrix} -y & \frac{-1-yy^\top+t^2}{2t} \\ 1_{m-1} & -\frac{1}{t}y^\top \\ y & \frac{-1+yy^\top-t^2}{2t}\end{pmatrix}=\begin{pmatrix} -y_1 & \cdots & -y_{m-1} & \frac{-1-\sum_{i=1}^{m-1}y_i^2+t^2}{2t} \\ 1 & \cdots & 0 & -\frac{y_1}{t} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & -\frac{y_{m-1}}{t} \\ y_1 & \cdots & y_{m-1} & \frac{-1+\sum_{i=1}^{m-1}y_i^2-t^2}{2t}\end{pmatrix}.$$

Without loss of generality we may assume that $\ell=\langle x\rangle\cap\langle x\rangle^\top$ is generated by the vector $(1,0,\ldots,0,-1)\in V$. Assume that $u_\ell=(-r,0,\ldots,0,r)^\top$ for some $r>0$. Finally, assume that $x=x_0s$ for some $x_0\in S_m(\ell,e_0)$ and $s\in\mathrm{SL}(m;\mathbb{Z})$.

Every frame $x_0\in(L^m+\mathbf{a})\cap S_m(\ell,e_0)$ is of the form

$$x_0=\begin{pmatrix} -c_0 & -\nu r \\ c' & 0 \\ c_0 & \nu r\end{pmatrix},$$

where $c'\in\mathrm{GL}(m-1;\mathbb{R})$, and $\nu=\nu(x_0)$. For this x_0 we have that

$$\varepsilon(x_0)=\operatorname{sign}\det\begin{pmatrix} c_0 & \nu r \\ c' & 0\end{pmatrix}=(-1)^{m-1}\operatorname{sign}\det c'\cdot\operatorname{sign}\nu.$$

We have that

$$\int_{\mathcal{D}}\varphi'_{km}(\tau,x)=\int_{\mathcal{D}}\sum_{\nu\in\mathbb{Z}+\nu(x)}\varphi_{km}(\tau,x_0s).$$

where

$$x_0=\begin{pmatrix} -c_0 & -\nu r \\ c' & 0 \\ c_0 & \nu r\end{pmatrix}.$$

We have that

$$\langle x_0,x_0\rangle=\begin{pmatrix} c'^\top c' & 0 \\ 0 & 0\end{pmatrix},$$

and

$$\langle x\sqrt{v},z\rangle=\langle x_0s\sqrt{v},z\rangle=\sqrt{v}s^\top\begin{pmatrix} c'^\top & \frac{1}{t}(c_0-yc')^\top \\ 0 & \frac{1}{t}\nu\end{pmatrix}=d\begin{pmatrix} 1_{m-1} & -\frac{1}{t}y'^\top \\ 0 & \frac{1}{t}\nu\end{pmatrix},$$

where $d = \sqrt{v}s^\top \left(\begin{smallmatrix} c'^\top & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and $y' = y - c_0 c'^{-1}$. Note that $dy' = dy$, and y' has the same domain $\mathbb{R}^{m-1}$ as y. Thus, for simplicity in the sequel we write y for y'.

Assume that $d = \begin{pmatrix} d' & d_0 \end{pmatrix}$ where $d' \in \mathrm{M}_{m,m-1}(\mathbb{R})$, $d_0^\top \in \mathbb{R}^m$. Note that $d'^\top d_0 = 0$. Furthermore,

$$d^\top d = \begin{pmatrix} d'^\top d' & 0 \\ 0 & d_0^\top d_0 \end{pmatrix}.$$

Taking determinant we obtain $|d|^2 = |d'^\top d'||d_0^\top d_0|$. On the other hand,

$$\operatorname{sign}\det d = \operatorname{sign}\det c' = (-1)^{m-1}\varepsilon(x).$$

We have further

$$\det\langle x\sqrt{v}, z\rangle = |d|\nu t^{-1}.$$

Moreover,

$$d \begin{pmatrix} 1_{m-1} & -\frac{1}{t}y^\top \\ 0 & \frac{1}{t}\nu \end{pmatrix} = \begin{pmatrix} d' & -\frac{1}{t}d'y^\top + \frac{1}{t}\nu d_0 \end{pmatrix}.$$

Therefore,

$$\begin{aligned} \operatorname{tr}[\langle x\sqrt{v}, z\rangle\langle z, x\sqrt{v}\rangle] &= \operatorname{tr}[d'^\top d'] + \frac{1}{t^2}(-d'y^\top + \nu d_0)^\top(-d'y^\top + \nu d_0) \\ &= \operatorname{tr}[d'^\top d'] + \frac{1}{t^2}(\nu^2 d_0^\top d_0 + y^\top d'^\top d' y). \end{aligned}$$

We have that

$$\begin{aligned} \operatorname{tr}[v\langle x, x\rangle] = \operatorname{tr}[svs^\top\langle x_0, x_0\rangle] = \operatorname{tr}[svs^\top \left(\begin{smallmatrix} c'^\top c' & 0 \\ 0 & 0 \end{smallmatrix}\right)] \\ = \operatorname{tr}[\sqrt{v}s^\top \left(\begin{smallmatrix} c'^\top \\ 0 \end{smallmatrix}\right) \left(\begin{smallmatrix} c' & 0 \end{smallmatrix}\right) s\sqrt{v}] = \operatorname{tr}[d'd'^\top] = \operatorname{tr}[d'^\top d']. \end{aligned}$$

Hence, we have that

$$\varphi'_{km}(\tau, x) = e^{\pi \mathrm{i} \operatorname{tr}[\overline{\tau}\langle x,x\rangle]} e^{-2\pi \operatorname{tr}[d'^\top d']} |d| \sum_{\nu \in \mathbb{Z}+\nu(x)} \frac{\nu}{t} e^{-\frac{2\pi}{t^2}(\nu^2 d_0^\top d_0 + y^\top d'^\top d' y)} \cdot \frac{dy dt}{t^m}.$$

Applying the Poisson summation we obtain

$$\sum_{\nu \in \mathbb{Z}+\nu(x)} \nu e^{-\frac{2\pi}{t^2}\nu^2 d_0^\top d_0} = \left(\frac{t}{\sqrt{2d_0^\top d_0}}\right)^3 \sum_{k=1}^{\infty} 2\sin(2\pi k\nu(x)) \cdot k e^{-\pi \frac{k^2 t^2}{2d_0^\top d_0}}.$$

We can see that in this sum every term rapidly decreases in t as $t \to \infty$. This proves

Lemma 5.11.

Now we proceed to the integration. We have

$$\int_{\mathcal{D}} \varphi'_{km}(\tau, x) = e^{\pi \mathrm{i}\, \mathrm{tr}[\overline{\tau}\langle x,x\rangle]} e^{-2\pi\, \mathrm{tr}[d'^\top d']} |d| \int_0^\infty \int_{\mathbb{R}^{m-1}} \sum_{\nu \in \mathbb{Z}+\nu(x)} \frac{\nu}{t^{m+1}} e^{-2\pi \frac{\nu^2 d_0^\top d_0}{t^2}} e^{-2\pi \frac{y^\top d'^\top d' y}{t^2}} dy dt.$$

Let $w = \frac{1}{t} y'$. Then $dw = \frac{1}{t^{m-1}} dt$ and after this substitution the integral decomposes into product

$$\int_0^\infty \sum_{\nu \in \mathbb{Z}+\nu(x)} \nu e^{-2\pi \frac{\nu^2 d_0^\top d_0}{t^2}} \frac{dt}{t^2} \cdot \int_{\mathbb{R}^{m-1}} e^{-2\pi w^\top d'^\top d' w} dw.$$

The second integral is standard. We have that

$$\int_{\mathbb{R}^{m-1}} e^{-2\pi w^\top d'^\top d' w} dw = 2^{-\frac{m-1}{2}} |d'^\top d'|^{-\frac{1}{2}}.$$

In the first integral let $s = \frac{1}{t}\sqrt{2 d_0^\top d_0}$. The integral becomes

$$\frac{1}{\sqrt{2 d_0^\top d_0}} \int_0^\infty \sum_{\nu \in \mathbb{Z}+\nu(x)} \nu e^{-\pi \nu^2 s^2} ds$$

According to Lemma 4.3, it is equal to

$$-2^{-\frac{1}{2}} |d_0^\top d_0|^{-\frac{1}{2}} \mathbf{B}_1(\nu(x)).$$

Therefore, we obtain

$$\begin{aligned} \int_{\mathcal{D}} \varphi'_{km}(\tau, x) &= -\frac{|d|}{\sqrt{|d'^\top d'| |d_0^\top d_0|}} 2^{-\frac{m}{2}} \mathbf{B}_1(\nu(x)) e^{\pi \mathrm{i}\, \mathrm{tr}[\overline{\tau}\langle x,x\rangle]} e^{-2\pi \mathrm{i}\, \mathrm{tr}[v\langle x,x\rangle]} \\ &= -\operatorname{sign}\det d \cdot 2^{-\frac{m}{2}} \mathbf{B}_1(\nu(x)) e^{\pi \mathrm{i}\, \mathrm{tr}[\tau\langle x,x\rangle]} \\ &= (-1)^m 2^{-\frac{m}{2}} \varepsilon(x) \mathbf{B}_1(\nu(x)) e^{\pi \mathrm{i}\, \mathrm{tr}[\tau\langle x,x\rangle]}. \end{aligned}$$

This concludes the proof. □

Theorem 5.3 now easily follows. Lemma 5.11 implies that the series

$$\theta_{km}(\tau) = \frac{1}{2} \sum_{\substack{x \in L^m + \mathbf{a} \\ x \text{ reduced}}} \varphi'_{km}(\tau, x)$$

could be integrated termwise. By applying the usual unfolding procedure we get

$$\int\limits_{\Gamma\backslash\mathcal{D}} \theta^0_{km}(\tau) = \frac{1}{2} \sum_{\substack{x\in L^m+\mathbf{a} \\ x \text{ reduced}}} \int\limits_{\Gamma\backslash\mathcal{D}} \varphi'_{km}(\tau, x) = \frac{1}{2} \sum_{\gamma\in\Gamma} \sum_{\substack{x\in\Gamma\backslash L^m+\mathbf{a} \\ x \text{ reduced}}} \int\limits_{\Gamma\backslash\mathcal{D}} \varphi'_{km}(\tau, \gamma x)$$

$$= \frac{1}{2} \sum_{\gamma\in\Gamma} \sum_{\substack{x\in\Gamma\backslash L^m+\mathbf{a} \\ x \text{ reduced}}} \int\limits_{\gamma^{-1}\Gamma\backslash\mathcal{D}} \varphi'_{km}(\tau, x) = \frac{1}{2} \sum_{\substack{x\in\Gamma\backslash L^m+\mathbf{a} \\ x \text{ reduced}}} \int\limits_{\mathcal{D}} \varphi'_{km}(\tau, x).$$

Then we apply Lemma 5.12 and obtain the theorem.

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