RANDOM SCHROEDINGER OPERATORS WITH CONNECTIONS TO SPECTRAL PROPERTIES
OF GROUPS AND DIRECTED POLYMERS

by

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Abstract

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This thesis studies random Schroedinger operators with connections to group theory and models from statistical physics. First, we study 1D operators obtained as perturbations of the standard adjacency operator on $\mathbb{Z}$ by putting random i.i.d. noise with finite logarithmic variance on the edges. We study their expected spectral measures μ_H near zero. We prove that the measure exhibits a spike of the form $\mu_H(-\varepsilon, \varepsilon) \sim \frac{C}{|\log \varepsilon|^2}$, which was first observed by Dyson for a specific choice of the edge weight distribution. We prove the result in generality, without assuming any regularity of edge weights. We also identify the limiting local eigenvalue distribution, obtained by counting crossings of the Brownian motion derived from the operator. The limiting distribution is different from Poisson and the usual random matrix statistics. The results also hold in the setting where the edge weights are not independent, but are sufficiently ergodic, e.g. exhibit mixing. In conjunction with group theoretic tools, we then use the result to compute Novikov-Shubin invariants, which are group invariants related to the spectral measure, for various groups, including lamplighter groups and lattices in the Lie group Sol.

Second, we study similar operators in the two dimensional setting. We construct a random Schroedinger operator on a subset of the hexagonal lattice and study its smallest eigenvalues. Using an asymptotic mapping, we relate these eigenvalues to the partition function of the directed polymer model on the square lattice. For a specific choice of the edge weight distribution, we obtain a model known as the log-Gamma polymer, which is integrable. Recent results about the fluctuations of free energy for the log-Gamma polymer allow us to prove Tracy-Widom type fluctuations for the smallest eigenvalue of the original random Schroedinger operator.

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Contents

Chapter 1

Introduction

The topic studied in this thesis are random Schroedinger operators and their connection to spectral properties of groups and to directed polymers. Below, we provide an overview of the results obtained in the thesis and explain their relevance in the context of statistical physics and group theory. The content of this thesis is based on two joint papers with Bálint Virág: [KV16b], [KV16a].

In Chapter 2, we study random Schroedinger operators on $\mathbb{Z}$ with random noise on the edges. Such an operator H acts on functions on $\mathbb{Z}$ by:

$$Hf(n) = a_{n-1}f(n-1) + a_n f(n+1) \tag{1.1}$$

where a_n is a sequence of random real-valued edge weights. The model was first studied in 1953 by Dyson [Dys53], who used it to model a linear chain of coupled harmonic oscillators with random coupling constants. The simplest physical example of such a chain would be a sequence of masses connected by elastic springs. It is straightforward to reduce studying the equations of motion for such a system to studying the spectrum of an operator of the form given by equation 1.1. Since the spectral measure of such an operator is random object, it makes sense to average over the randomness and study a deterministic object, the expected spectral measure of H, which we shall denote by μ_H.

Dyson asked about the asymptotic density of eigenvalues of such a system near zero, which often encodes useful physical information about the qualitative properties of the system. Formally, we're studying the the asymptotics of the quantity $\mu_H(-\varepsilon, \varepsilon)$ as $\varepsilon \to 0$. For a specific choice of edge weights, Dyson proved the existence of a spike of the form:

$$\mu_H(-\varepsilon, \varepsilon) \sim \frac{C}{|\log \varepsilon|^2} \tag{1.2}$$

This is in sharp contrast to the noiseless case, i.e. the standard adjacency operator on $\mathbb{Z}$, whose spectral measure remains bounded near zero. The existence of such a spike is also believed to be typical of other physical systems, e.g. 1D supersymmetric quantum mechanics (see [CT98]).

It has been long conjectured that the spike of the form 1.2 should be universal, regardless of the particular choice for the randomness of the edge weights. Indeed, heuristic and numerical arguments supporting this claim have been given in the physics literature [ER78]. However, despite over 60 years from Dyson's original

paper, the only rigorous general result in this direction was [CP89], where authors prove a lower bound $\mu_H(-\varepsilon, \varepsilon) \geq \frac{C}{|\log \varepsilon|^3}$ for independent weights with bounded continuous density supported away from zero.

In Chapter 2 of this thesis, we settle the question, proving:

Theorem 1.0.1. *Let a_i be i.i.d. random variables such that $\sigma^2 = \mathrm{Var} \log |a_i| < \infty$. Then for the random Schroedinger operator H defined by:*

$$(Hf)(i) = a_{i-1}f(i-1) + a_i f(i+1)$$

we have

$$\mu_H(-\varepsilon, \varepsilon) = \frac{\sigma^2}{\left|\log^2 \varepsilon\right|}(1 + o_\varepsilon(1)) \tag{1.3}$$

Under the same assumptions, the analysis also yields an explicit limit of the local eigenvalue statistics around 0. The limiting distribution has a simple description in terms of up- and down-crossings, or wells, made by the underlying Brownian Motion (see Theorem 2.3.3). We highlight that the limiting distribution is different from Poisson or the usual random matrix statistics. Both results also hold in the case of non-i.i.d. variables, provided they are sufficiently ergodic. An example of such a system is given by hyperbolic dynamics on a torus determined by a hyperbolic matrix.

The result about the spectral measure can be then applied in the group theoretic setting. Let us briefly describe the background about the spectral measures of groups. Let G be a group with a finite generating set S. A self-adjoint element element $T \in \mathbb{C}[G]$, for example the adjacency operator of the Cayley graph $Cay(G, S)$, defines a bounded self-adjoint operator on $\ell^2(G)$ which can be equipped with a spectral measure μ_T. Of particular interest is the behavior of μ_T around 0, that is, bounds on the quantity $\mu_T((0, \varepsilon))$ as $\varepsilon \to 0$. Bounds on this quantity are strongly related to various notions from the theory of topological L^2-invariants, including L^2-torsion and the Determinant Approximation Conjecture.

The most general upper bound comes in the form of the so called Lück bound ([Lüc02], see [ATV] for a more general treatment), which claims that for G a sofic group and $T \in \mathbb{Q}[G]$ for some constant C we have:

$$\mu_T((0, \varepsilon)) \leq \frac{C}{|\log \varepsilon|} \tag{1.4}$$

It is not known if this bound is the best possible. In fact, for a long time it had been conjectured that the actual behavior of μ_T is far from the bound claimed in (1.4). Define the Novikov–Shubin invariant $\alpha(T)$ as:

$$\alpha(T) = \liminf_{\varepsilon \to 0} \frac{\log \mu_T((0, \varepsilon))}{\log \varepsilon}$$

Informally, $\alpha(T) = \alpha > 0$ means that $\mu_T((0, \varepsilon))$ behaves like $\sim \varepsilon^\alpha$. The Lott–Lück Conjecture [LL95] states that for any group G and $T \in \mathbb{C}[G]$, the Novikov–Shubin invariant $\alpha(T)$ is positive. This has been established only for virtually abelian and virtually free groups.

The conjecture was disproved in [**?**], where the authors construct a lamplighter group G and a group ring element T with Novikov–Shubin invariant equal to 0. The construction relies on a correspondence between the spectral measure of T and spectral measures of a certain random Schroedinger operator on $\mathbb{Z}$. For certain families of groups one can, given $T \in \mathbb{C}[G]$, construct a random Schroedinger operator H on $\mathbb{Z}$ whose

expected spectral measure μ_H is equal to μ_T (a construction due to Grabowski [Gra14]).

By applying the translation from groups to random Schroedinger operators and using our results about the spectral measure of the latter, we arrive at new counterexamples to the Lott–Lück conjecture (Theorem 2.1.4). In particular, the counterexamples are made of finitely presented groups which are lattices in the Lie group Sol. This also disproves the lattice variant of the Lott–Lück conjecture, which was not settled before.

The theorem above concerns semidirect products with $\mathbb{Z}$, which give rise to random Schroedinger operators defined also on $\mathbb{Z}$. It would be interesting to extend this analysis to other base graphs. To our knowledge, there are no results available for any base graphs other than $\mathbb{Z}$.

Question 1.0.1. *Describe the spectral measure around zero for base graphs different than $\mathbb{Z}$, for example $\mathbb{Z}^d$ or free groups F_k. What exponents of $|\log \varepsilon|$ can be obtained in this way?*

The second part of the thesis, contained in Chapter 3, concerns connections between eigenvalues of 2D random Schroedinger operators and random directed polymers. The reader is invited to consult e.g. [CY06] for a comprehensive treatment of random polymers. In statistical physics, the model is usually set up as follows. Let G be the square lattice from $(1, 1)$ to (n, n). Each edge e of the lattice is equipped with a random wieght w_e. Let Π denote the set of all directed up-right path π connecting $(1, 1)$ to (n, n). To every $\pi \in Pi$ we associate its energy $E(\pi)$, equal to the sum of weights of its edges, $E(\pi) = \sum_{e \in \pi} w_e$. For a fixed quantity $\beta > 0$, known as the inverse temperature, one the considers the Gibbs measure on the set of all paths, given by:

$$\mu_{n,\beta}(\pi) := e^{-\beta E(\pi)}/Z_n$$

where the normalizing constant Z_n is known as the partition function of the model:

$$Z_{n,\beta} := \sum_{\pi \in \Pi} e^{-\beta E(\pi)}$$

This setup can be thought of as a finite temperature version of the model known as Last Passage Percolation, where instead of the weighted sum over all paths we are simply interested in the weight of the heaviest path:

$$L_n = \max_{\pi \in \Pi}\{\sum_{e \in \pi} w_e\}$$

As usual in statistical physics, the knowledge of the partition function allows one to determine various thermodynamic properties of the system – free energy, existence of phase transitions, the global shape of the most typical polymer configuration etc.

A mathematical model related to the above, known as the log-Gamma polymer model, has recently been studied [Sep12, BCR13]. In this model, each edge is equipped with a random weight drawn from the so-called inverse Gamma distribution $\Gamma^{-1}(\gamma)$ for some parameter $\gamma > 0$, which plays the role analogous to the inverse temperature β. The partition function is then equal to:

$$Z_n = \sum_{\pi \in \Pi} \prod_{e \in \pi} w_e$$

The feature which makes this particular model attractive is its integrability. The properties of the Gamma

distribution allow one to produce exact formulas, which has led to explicit descriptions of the leading order and fluctuations of the partition function [BCR13]. In particular, it turns out that the fluctuations of $\log Z_n$ fall into the KPZ universality class, as they follow the Tracy-Widom $n^{1/3}$ scaling.

We discover a connection between random Schroedinger operators and polymers, set up as follows. We construct a certain subset of the hexagonal lattice and construct a random Schroedinger operator on that lattice, with edge weights given by i.i.d. random variables drawn from $\Gamma^{-1}(\gamma)$ for some parameter $\gamma > 0$. Studying the eigenvalues of the operator is equivalent, via a simple combinatorial mapping, to studying the singular values of a certain directed graph associated to the lattice. In our case, the directed graph is exactly the directed square lattice.

We then connect the singular values to certain quantities in the log-Gamma polymer model on the square lattice, related to the partition function Z_n. In the simplest case, the smallest eigenvalue of the random Schroedinger operator is asymptotically the same as the partition function of the log-Gamma polymer. Thanks to recent results about the fluctuations of the free energy [BCR13], we are able to establish Tracy-Widom fluctuations for the smallest eigenvalue (Theorem 3.1.2). To our knowledge, this is the first example of Tracy-Widom type fluctuations occuring in the setting of random Schroedinger operators.

In light of the correspondence between the random Schroedinger operator and the directed polymer model, it is natural to try to exploit it to study the spectral measure of the operator, analogously to the 1D case. We plan to pursue this direction further. In particular, any result about the spectral measure of the 2D operator would shed light on Question 1.0.1.

Chapter 2

Dyson's spike for random Schroedinger operators and Novikov–Shubin invariants of groups

2.1 Introduction

In this paper, we study a class of Schroedinger operators with random edge weights given by:

$$(Hf)(i) = a_{i-1}f(i-1) + a_i f(i+1)$$

where a_i form a stationary process. One can think of such an operator as a perturbation of the standard adjacency operator on $\mathbb{Z}$ (with all $a_i = 1$) by random noise. It is well known that the presence of even small amount of noise can dramatically influence the spectral properties of H. While the spectral measure of the standard adjacency operator is absolutely continuous, the noisy variant typically has a fully discrete spectrum with exponentially localized eigenfunctions. This phenomenon is known as Anderson localization and is well studied in mathematical physics.

A natural object of study is the *expected spectral measure*, denoted μ_H, which is the spectral measure of H averaged over all random instances. Here, another phenomenon occurs, related to the behavior of μ_H at zero. The standard adjacency operator on $\mathbb{Z}$ has spectral measure μ with bounded density near zero, so in particular $\mu(-\varepsilon, \varepsilon) \sim \frac{\varepsilon}{\pi}$. In contrast, μ_H can exhibit behavior of the form:

$$\mu_H(-\varepsilon, \varepsilon) \sim \frac{C}{|\log \varepsilon|^2},$$

which goes to 0 as $\varepsilon \to 0$ slower than any power of ε. If μ_H happens to have a density, this means it must have a sharp spike near 0 of the form $\frac{1}{\varepsilon} \cdot \frac{C}{|\log \varepsilon|^3}$. This phenomenon was first observed in the famous paper by Dyson [Dys53], who proved $\mu_H(-\varepsilon, \varepsilon) \sim \frac{C}{|\log \varepsilon|^2}$ for a specific choice of edge weight distribution. One could expect that such behavior should be typical, independent of any particular properties of the distribution, and

indeed, heuristic and numerical arguments supporting this claim have been given in the physics literature [ER78]. However, despite over 60 years from Dyson's original paper, the only rigorous general result in this direction was [CP89], where authors prove a lower bound $\mu_H(-\varepsilon, \varepsilon) \geq \frac{C}{|\log \varepsilon|^3}$ for independent weights with bounded continuous density supported away from zero.

In this paper, we settle the question, proving:

Theorem 2.1.1. *Let a_i be i.i.d. random variables such that $\sigma^2 = \operatorname{Var} \log |a_i| < \infty$. Then for the random Schroedinger operator H defined by:*

$$(Hf)(i) = a_{i-1} f(i-1) + a_i f(i+1)$$

we have

$$\mu_H(-\varepsilon, \varepsilon) = \frac{\sigma^2}{\left|\log^2 \varepsilon\right|} (1 + o_\varepsilon(1)) \tag{2.1}$$

Notably, we do not assume any regularity of the distribution: it can even take finitely many values. A version of this theorem holds also for edge weights which are not independent, as long as they satisfy suitable correlation decay (see Theorem 3.1.2 for precise statement). Our theorem easily reproduces Dyson's result from [Dys53], see the discussion after Theorem 3.1.2.

The crucial ingredient in the proof is truncating the operator H to a finite interval and finding a discrete process that counts its eigenvalues. We then proceed to identify a scaling limit of this process, which involves the Brownian Motion arising as the limit of the discrete random walk with steps $\log \left| \frac{a_{2i-1}}{a_{2i}} \right|$.

In the course of proving the theorem, we also establish eigenvalue bounds for finite Schroedinger operators. These are expressed in terms of up- and down-crossings, see Sections 2.2.3 and 2.2.4 for details and precise definitions:

Proposition 2.1.2. *Let H_n be a Schroedinger operator of size n, with n odd, with edge weights a_i. Let M_n^λ denote the number of eigenvalues of H_n inside the interval $(0, \lambda)$. Let $D_n^{1 \pm \delta}$ denote the number of $(1 \pm \delta) \left| \log \lambda \right|$-downcrossings made by the process $\sum_{i=1}^k \log \left| \frac{a_{2i-1}}{a_{2i}} \right|$. Let B_n^δ be denote the number of k for which $|\log |a_k|| > \frac{\delta}{8} |\log \lambda|$. Then for any $\delta \in (0, 1)$ and $\lambda < \left(\frac{\delta}{16n} \right)^{\frac{2}{\delta}}$ we have:*

$$D_n^{1+\delta} - 2B_n^\delta \leq M_n^\lambda \leq D_n^{1-\delta} + 2B_n^\delta$$

Under the same assumptions, the analysis also yields an explicit limit of the local eigenvalue statistics around 0. The limiting distribution has a simple description in terms of up- and down-crossings, or wells, made by the underlying Brownian Motion. Figure 2.1 shows an example of wells made by a Brownian Motion. We highlight that the limiting distribution is different from Poisson or the usual random matrix statistics.

Theorem 2.1.3. *Let a_i be i.i.d. random variables such that $\sigma^2 = \operatorname{Var} \log |a_i| < \infty$. Let H be the random Schroedinger operator defined by:*

$$(Hf)(i) = a_{i-1} f(i-1) + a_i f(i+1)$$

and let H_n the restricton of H to an interval of length $n + 1$. Let $\{\Lambda_n(t, \eta), t, \eta > 0\}$ be the process equal to
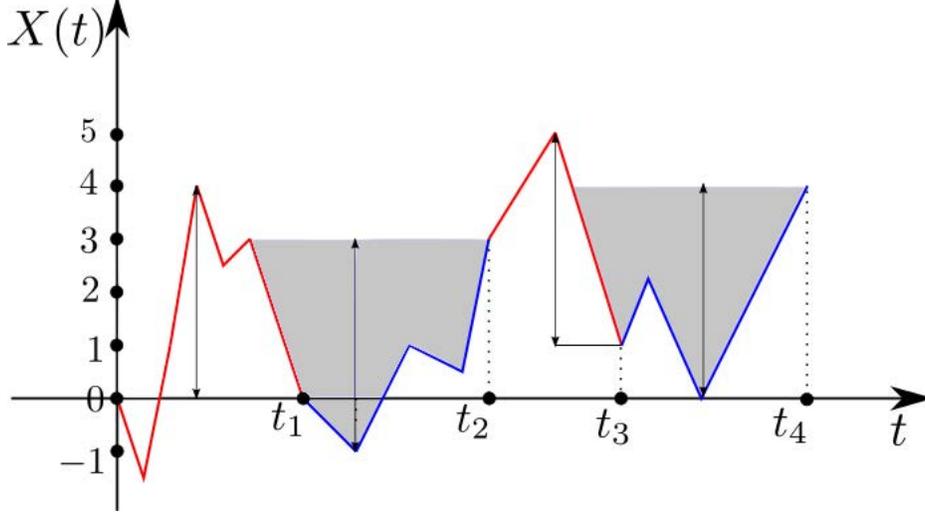
Figure 2.1: A random walk path with two wells of depth 4

the number of eigenvalues of $H_{\lfloor tn \rfloor}$ inside the interval $(0, e^{-\eta \sqrt{n}})$. Let $\{\Lambda(t, \eta), t, \eta > 0\}$ be the process equal to the total number of disjoint η-wells that a Brownian Motion with variance σ^2 makes inside the interval $[0, t]$ (see Definition 2.2.16). Consider a subsequence of n such that $\lfloor tn \rfloor$ is odd.

Then the process $\Lambda_n(t, \eta)$ converges to the process $\Lambda(t, \eta)$, i.e. all finite dimensional distributions of Λ_n converge weakly to finite dimensional distributions of $\Lambda(t, \eta)$.

As a corrollary, for the smallest positive eigenvalue $\lambda_0^{(n)}$ of H_n, we have:

$$\frac{-\log \lambda_0^{(n)}}{\sqrt{n}} \implies \sigma \cdot \sup_{t \in [0,1]} |B(t)|$$

in distribution, where $B(t)$ is a Brownian Motion of variance 1 (see Remark 2.3.4).

We then proceed to apply our result to computation of topological invariants of groups called the Novikov–Shubin invariants. Let G be a finitely generated group and $H \in \mathbb{C}[G]$ a group ring element that determines a self-adjoint operator $H : \ell^2(G) \to \ell^2(G)$ with spectral measure μ_H. The Novikov–Shubin invariant of H, denoted $\alpha(H)$, is determined by the behavior of μ_H at zero:

$$\alpha(H) := \liminf_{\varepsilon \to 0} \frac{\log(\mu_H(-\varepsilon, \varepsilon) - \mu_H(\{0\}))}{\log \varepsilon}$$

Informally, if $\alpha(H) = \alpha$ and $\alpha > 0$, this means that $\mu_H(-\varepsilon, \varepsilon)$ behaves like $\sim \varepsilon^\alpha$. In general, Novikov–Shubin invariants are rather difficult to compute, but carry interesting topological information about manifolds with fundamental group G – see [Eck00], [Lüc02] for further background on $\alpha(H)$.

Of particular interest is the question of positivity of $\alpha(H)$. Note, for example, that behavior of the form $\mu_H \sim \frac{C}{|\log \varepsilon|^2}$ implies $\alpha(H) = 0$. It was conjectured by Lott and Lück [LL95] that $\alpha(H) > 0$ for any group G and any $H \in \mathbb{C}[G]$. This has been recently disproved in [GV]. Our paper provides a new counterexample:

Theorem 2.1.4. *There exists a group $G = \mathbb{Z}^2 \rtimes_A \mathbb{Z}$, with A a hyperbolic matrix, and $H \in \mathbb{C}[G]$ corresponding to a random walk on G such that $\alpha(H) = 0$.*

In comparison to previous work, this gives a simple counterexample which is not only finitely presented, but also a lattice in a Lie group (Sol), see Section 2.4.2 for a more precise statement. In conjunction with [GV], our technique can be also used to prove $\alpha(H) = 0$ for lamplighter groups $\Gamma \wr \mathbb{Z}$ with Γ arbitrary, see Section 2.4.3.

The connection between group theory and random Schroedinger operators comes in the form of a construction due to [Gra14] that allows one, for certain semidirect products G, to build, given $H \in \mathbb{C}[G]$, a random Schroedinger operator whose expected spectral measure is equal to μ_H (see Section 2.4.2 for details). To compute this measure, one needs the full power of our main theorem. For lamplighters, one obtains i.i.d. edge weights, but the underlying distribution can be discrete (e.g. for $\mathbb{Z}_2 \wr \mathbb{Z}$), so it is important that we have (2.1) without any smoothness assumption about the edge weights. In the case $G = \mathbb{Z}^2 \rtimes_A \mathbb{Z}$, the resulting operator has dependent edge weights, which we can handle thanks to the hyperbolic nature of the underlying map.

The paper is organized as follows. In Section 2.2, we restrict H to a finite interval and derive a process that counts the eigenvalues of this restriction. This part is completely deterministic. In Section 2.3, we use that process to prove the main theorem which establishes equation (2.1) under suitable assumptions. Then, in Section 2.4, we describe the connection to group theory and proceed to apply the main theorem to computations for the groups mentioned above.

2.2 The eigenvalue process and its limit

2.2.1 The expected spectral measure of H

Let Ω be a probability space and let $(\dots, a_{-1}, a_0, a_1, \dots) : \Omega \to \mathbb{R}^{\mathbb{Z}}$ be a bi-infinite sequence of real numbers drawn from some joint shift-invariant probability distribution. We will always assume that almost surely, none of a_i are equal to 0. The distribution defines a random Schroedinger operator H given by:

$$(Hf)(i) = a_{i-1}f(i-1) + a_i f(i+1)$$

In this section, we define what is meant by the expected spectral measure of H. There are standard definitions of this object, but we want to avoid technicalities coming from (i) the fact that the moments of the a_i may not exist; and (ii) having to find the domain of H on which it is self-adjoint.

Let μ, ν be probability measures. The Kolmogorov distance $d_K(\mu, \nu)$ is defined as $d_K(\mu, \nu) = \sup_x |\mu(-\infty, x] - \nu(-\infty, x]|$. Note that if a sequence of measures μ_n forms a Cauchy sequence with respect to d_K, it converges weakly to some measure μ.

Let H_n denote the finite dimensional operator equal to the restriction of H to the set $\{1, \dots, n+1\}$ by setting $a_i = 0$ for $i \notin \{1, \dots, n\}$. In this way, we obtain a random finite dimensional operator H_n whose (random) spectral measure μ_{H_n} is defined as its empirical eigenvalue distribution. Let μ_n denote the expected spectral measure of H_n, i.e. the average of μ_{H_n} take over the randomness of edge weights.

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Proposition 2.2.1. *The sequence of measures $\{\mu_{2^k-1}\}_{k=1}^{\infty}$ converges weakly to some limit measure μ that satisfies $d_K(\mu, \mu_{2^k-1}) \leq \frac{1}{2^k}$.*

Proof. For $k \geq 1$, consider the operators H_{2^k-1} and $H_{2^{k+1}-1}$. Let $\widetilde{H}_{2^{k+1}-1}$ denote the operator obtained from $H_{2^{k+1}-1}$ by setting $a_{2^k} = 0$. Since the matrix $H_{2^{k+1}-1} - \widetilde{H}_{2^{k+1}-1}$ has a possibly nonzero entry only at a_{2^k}, we have $\text{rank}(H_{2^{k+1}-1} - \widetilde{H}_{2^{k+1}-1}) \leq 1$. By a standard matrix inequality [BS10, Theorem A.43]:

$$d_K(\mu_{H_{2^{k+1}-1}}, \mu_{\widetilde{H}_{2^{k+1}-1}}) \leq \frac{1}{2^{k+1}} \cdot \text{rank}(H_{2^{k+1}-1} - \widetilde{H}_{2^{k+1}-1}) \leq \frac{1}{2^{k+1}} \tag{2.2}$$

Note that the matrix $\widetilde{H}_{2^{k+1}-1}$ consists of two disjoint blocks corresponding to vertices $\{1, \ldots, 2^k\}$ and $\{2^k + 1, \ldots, 2^{k+1}\}$. This and the shift-invariance of the distribution of $(\ldots, a_{-1}, a_0, a_1, \ldots)$ implies that the expected spectral measure of $\widetilde{H}_{2^{k+1}-1}$ is the same as the expected spectral measure of H_{2^k-1}. Thus, inequality (2.2) holds after taking expectations:

$$d_K(\mu_{2^k}, \mu_{2^{k+1}}) \leq \frac{1}{2^{k+1}} \tag{2.3}$$

Altogether, (2.3) implies that the sequence of measures μ_{2^k} is a Cauchy sequence, so it converges weakly to a measure μ. The bounds used above easily imply that $d(\mu, \mu_{2^k}) \leq \sum_{i=k}^{\infty} \left(\frac{1}{2}\right)^{i+1} = \frac{1}{2^k}$ as claimed. $\qquad \square$

Definition 2.2.2. *The expected spectral measure μ_T of the random Schroedinger operator H is defined as the limit measure μ constructed in Proposition 2.2.1.*

Note that in the case where all a_i are bounded, H is a bounded operator on $\ell^2(\mathbb{Z})$. One can then define the spectral measure of an instance H_ω as follows. Let $\delta_0(0) = 1, \delta_0(i) = 0$ for $i \neq 0$. The spectral measure μ_{H_ω} is defined via specifying its moments:

$$m_k = \langle \delta_0, (H_\omega)^k \delta_0 \rangle = \int_{\mathbb{R}} x^k d\mu_{H_\omega} \tag{2.4}$$

Since the operator H_ω is bounded, the moment sequence m_k specifies the measure uniquely. One then defines the expected spectral measure μ_H simply as the expectation of the measures μ_{H_ω}. The approximation by finite operators claimed in Proposition 2.2.1 is then obtained by a bounded operator analogue [BSV13, Lemma 6.1] of inequality (2.2).

2.2.2 Transfer matrices

Throughout the following sections, we are concerned with a single instance H_ω, which we will call H from now on. Therefore, H is a fixed, deterministic operator – we introduce the probabilistic part of the analysis in Section 2.3.

To control the spectral measure of H, we will approximate H by operators H_n supported on finite intervals. In Section 2.2.2, we use the standard transfer matrix approach to derive a process that counts the eigenvalues of H_n. After the setup contained in Section 2.2.3, we analyze the process in Section 2.2.4 and find its limiting behavior in Section 2.2.5.

Let H_n be the restriction of H to the set $\{1, \ldots, n+1\}$, i.e. the operator obtained from H by putting all weights outside the interval $[1, n+1]$ equal to 0. For any given $\lambda \in \mathbb{R}$, we are interested in computing the

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number of eigenvalues of H_n inside the interval $[0, \lambda]$. We shall perform this computation using the transfer matrix approach.

We start with the eigenvalue equation. For the first and last equation we set $a_0 = a_{n+1} = 1, \phi_0 = \phi_{n+2} = 0$. The eigenvalue equation can be then written as:

$$H_n \phi = \lambda \phi$$

$$a_{k-1} \phi_{k-1} - \lambda \phi_k + a_k \phi_{k+1} = 0, \ k = 1, 2, \ldots, n+1$$

Letting:

$$T_{k-1}^\lambda = \begin{pmatrix} \frac{\lambda}{a_k} & -\frac{a_{k-1}}{a_k} \\ 1 & 0 \end{pmatrix}, \ k = 1, 2, \ldots, n+1$$

we can write the recursion as:

$$\begin{pmatrix} \phi_{k+1} \\ \phi_k \end{pmatrix} = T_{k-1}^\lambda \begin{pmatrix} \phi_k \\ \phi_{k-1} \end{pmatrix}$$

In particular, λ is an eigenvalue if and only if for some c we have:

$$\begin{pmatrix} 0 \\ c \end{pmatrix} = T_n^\lambda \cdots \cdots T_0^\lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.5}$$

We will be interested in the evolution of $(T_k^0 \cdots \cdots T_0^0)^{-1} \cdot T_k^\lambda \cdots \cdots T_0^\lambda$ as k changes from 0 to n. Let:

$$R_k^\lambda := (T_k^0)^{-1} T_k^\lambda = \begin{pmatrix} 0 & 1 \\ -\frac{a_{k+1}}{a_k} & 0 \end{pmatrix} \begin{pmatrix} \frac{\lambda}{a_{k+1}} & -\frac{a_k}{a_{k+1}} \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{\lambda}{a_k} & 1 \end{pmatrix}$$

Now we rewrite:

$$(T_k^0 \cdots \cdots T_0^0)^{-1} \cdot T_k^\lambda \cdots \cdots T_0^\lambda = (R_k^\lambda)^{T_{k-1}^0 \cdots \cdots T_0^0} \cdot (R_{k-1}^\lambda)^{T_{k-2}^0 \cdots \cdots T_0^0} \cdots \cdots R_0^\lambda \tag{2.6}$$

where $R_k^A = A^{-1} R_k A$.

It is desirable to express (2.6) in a more tractable way. Define:

$$S_k = \sum_{i=1}^k 2 \log \left| \frac{a_{2i-1}}{a_{2i}} \right|$$

We first compute products of odd and even numbers of T_k^0. We have:

$$T_{k+1}^0 T_k^0 = \begin{pmatrix} 0 & -\frac{a_{k+1}}{a_{k+2}} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\frac{a_k}{a_{k+1}} \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -\frac{a_{k+1}}{a_{k+2}} & 0 \\ 0 & -\frac{a_k}{a_{k+1}} \end{pmatrix} \tag{2.7}$$

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so:

$$T^0_{2k} \cdot T^0_{2k-1} \cdots \cdots T^0_1 \cdot T^0_0 = (-1)^k \begin{pmatrix} 0 & -\frac{\varepsilon_k \cdot e^{-\frac{1}{2}S_k}}{a_{2k+1}} \\ \varepsilon_k \cdot e^{\frac{1}{2}S_k} & 0 \end{pmatrix} \tag{2.8}$$

$$T^0_{2k+1} T^0_{2k} \cdot T^0_{2k-1} \cdots \cdots T^0_1 \cdot T^0_0 = (-1)^{k+1} \begin{pmatrix} \varepsilon_{k+1} \cdot e^{\frac{1}{2}S_{k+1}} & 0 \\ 0 & \frac{\varepsilon_k \cdot e^{-\frac{1}{2}S_k}}{a_{2k+1}} \end{pmatrix} \tag{2.9}$$

where we have written $\prod_{i=1}^k \frac{a_{2i-1}}{a_{2i}}$ as $\varepsilon_k \cdot e^{\frac{1}{2}S_k}$ with $\varepsilon_k = \pm 1$.

We now have:

$$Q^\lambda_k := (R^\lambda_{2k+1})^{T^0_{2k} \cdots \cdots T^0_0} (R^\lambda_{2k})^{T^0_{2k-1} \cdots \cdots T^0_0} = \begin{pmatrix} 1 - \frac{\lambda^2}{a^2_{2k+1}} & \lambda \frac{e^{-S_k}}{a^2_{2k+1}} \\ -\lambda e^{S_k} & 1 \end{pmatrix} \tag{2.10}$$

We will now prove a lemma that justifies the usefulness of the representation (2.6). Let $\mathbb{H} = \{z \in \mathbb{C} : \Im z \geq 0\} \cup \{\infty\}$ be the upper half plane. We can identify a vector $\binom{a}{b}$ with a point $z \in \mathbb{H}$ by letting $z = \frac{a}{b}$. In this identification, the vector $\binom{1}{0}$ is mapped to ∞. We can translate the action of matrices on vectors into action on $\mathbb{H}$. Recall that matrices $A \in SL(2, \mathbb{R})$ act by isometries of the hyperbolic plane $\mathbb{H}$ in the upper half plane model. A matrix:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

corresponds to the map $\mathcal{T}_A : \mathbb{H} \to \mathbb{H}$ such that $\mathcal{T}_A(z) = \frac{az+b}{cz+d}$. To simplify notation, we will write A instead of $\mathcal{T}_A$.

From now on we assume that n is odd. This implies that

$$(T^0_n \cdots \cdots T^0_0)^{-1}(0) = 0 \tag{2.11}$$

Lemma 2.2.3. *Let M^λ_n denote the number of eigenvalues of H_n inside the interval $[0, \lambda]$ and let J^λ_n denote the number of times the process $(R^\lambda_k)^{T^0_{k-1} \cdots \cdots T^0_0} \cdots \cdots R^\lambda_0(\infty)$ passes 0 or ∞ as k ranges from 0 to n. Then $M^\lambda_n = \lceil \frac{1}{2} J^\lambda_n \rceil$.*

Proof. Let $A = [0, \lambda] \times [0, n]$. We define $f : [0, \lambda] \times \{0, \ldots, n\} \to \partial \mathbb{H}$ by:

$$f(\lambda^*, k) = (R^{\lambda^*}_k)^{T^0_{k-1} \cdots \cdots T^0_0} \cdots \cdots R^{\lambda^*}_0(\infty)$$

and interpolate f linearly to obtain a continuous map $f : A \to S^1$. Note that by (2.5) and (2.11), λ^* is an eigenvalue whenever:

$$(R^{\lambda^*}_k)^{T^0_{k-1} \cdots \cdots T^0_0} \cdots \cdots R^{\lambda^*}_0(\infty) = 0$$

Let S_1, S_2, S_3, S_4 be the sides of the rectangle A (see Figure 2.2). Since A is contractible and f is continuous, the loop $f : S_1 \cup S_2 \cup S_3 \cup S_4 \to \partial \mathbb{H}$ has winding number zero. Since $f(S_1) = f(S_4) = \{\infty\}$, the same is true for the loop $f : S_2 \cup S_3 \to \partial \mathbb{H}$. Since f is monotone on both of these intervals, with opposite direction, the number of times it passes 0 on them is the same, which easily finishes the proof. $\square$
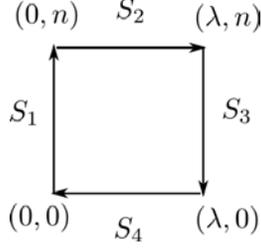
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Figure 2.2: The loop from Lemma 2.2.3

2.2.3 Crossings and the rotation process

In Section 2.2.2 we have shown that in order to study the number of eigenvalues of H_n, we have to study the time evolution of a process given by composing rotations (see equation 2.10 and remark thereafter). In the following sections, we study this process in detail and derive its continuous-time scaling limit.

We consider a fixed, deterministic operator H_n on the interval of size n, given by edge weights $\{a_k\}_{k=1}^n$. The process constructed in the previous section can be described informally as follows. Starting from the initial point $v_0 \in \mathbb{R}$, each point v_k will be moved by a rotation R_k to a new point v_{k+1}. The center of each rotation, equal to $i\frac{e^{-S_k}}{|a_{2k+1}|}$, is obtained as a product of a_i and λ represents the speed of rotation.

Formally, we study the process defined by:

$$
Q_k^\lambda = \begin{pmatrix} 1 - \frac{\lambda^2}{a_{2k+1}^2} & \lambda \frac{e^{-S_k}}{a_{2k+1}^2} \\ -\lambda e^{S_k} & 1 \end{pmatrix},
$$
$$
v_k^\lambda = Q_{k-1}^\lambda(v_{k-1}^\lambda)
$$
$$
v_0^\lambda = \infty
$$

The process described in (2.10) is exactly of this form.

Let $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z \geq 0\} \cup \{\infty\}$ and $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$. Since we would like to study the points $\log v_k$ for $v_k \in \partial\mathbb{H}$, it is natural to introduce the following setup. For $k \in \mathbb{Z}$, let $\mathbb{A}_k = \{\mathbb{R} + k \cdot i\pi\} \cup \{-\infty, +\infty\}$. Let $\mathbb{A} = \bigsqcup_{k=1}^\infty \mathbb{A}_k$ be the union of lines plus points $\pm\infty$, connected in such a way that $\mathbb{A}_{2k} \cap \mathbb{A}_{2k+1} = \{+\infty\}, \mathbb{A}_{2k-1} \cap \mathbb{A}_{2k} = \{-\infty\}$. Considering $\exp : \mathbb{A} \to \partial\mathbb{H}$, we can treat its inverse as a (multi-valued) map $\log : \partial\mathbb{H} \to \mathbb{A}$, where $\mathbb{A}_{2k} \subseteq \log(\{z \geq 0\})$ and $\mathbb{A}_{2k+1} \subseteq \log(\{z \leq 0\})$.

Remark 2.2.4. *Note that $\mathbb{A}$ has a natural ordering inherited from the real line, which we will denote by $\geq_\mathbb{A}$. If $x \in \mathbb{A}_j$ and $y \in \mathbb{A}_i$ for $j > i$, then $x \geq_A y$. If x, y are in the same component, $x \geq_\mathbb{A} y$ means $x \geq y$ if $x, y \in \mathbb{A}_{2i}$ or $x \leq y$ if $x, y \in \mathbb{A}_{2i+1}$. The time evolution governed by Q_k is monotone with respect to this ordering, i.e. if $y \geq_\mathbb{A} y'$, then $Q_k(y) \geq_\mathbb{A} Q_k(y')$.*

The processes studied below will consist of points starting at $+\infty \in \mathbb{A}_1$ and decreasing monotoneously until they jump past $-\infty$ to $\mathbb{A}_2$, whereupon they increase until they jump to $\mathbb{A}_3$, and so on.

We introduce the scaled version of the processes, defined as follows:

$$X_k = \frac{1}{|\log \lambda|} S_k$$

$$Y_k = \frac{1}{|\log \lambda|} \log v_k^\lambda$$

X_k is supposed to represent the rescaled motion of the rotation center, while Y_k is the rescaled trajectory of v_k. With this setup, Y_k takes values in $\mathbb{A}$, either in $\mathbb{A}_{2i}$ if $v_k^\lambda \geq 0$ or $\mathbb{A}_{2i+1}$ if $v_k^\lambda \leq 0$. It jumps from $\mathbb{A}_i$ to $\mathbb{A}_{i+1}$ whenever v_k^λ changes sign.

Note that for $v_{k-1}^\lambda > 0$, so that $Y_{k-1} \in \mathbb{A}_{2i}$ for some i, we can write:

$$Y_k = \frac{1}{|\log \lambda|} \log \left| \frac{(1 - \frac{\lambda^2}{a_{2k-1}^2})\lambda^{-Y_{k-1}} + \frac{1}{a_{2k-1}^2}\lambda^{1+X_{k-1}}}{1 - \lambda^{1-Y_{k-1}-X_{k-1}}} \right|$$

where $Y_k \in \mathbb{A}_{2i}$ if the expression under the absolute value is nonnegative and $Y_k \in \mathbb{A}_{2i+1}$ otherwise.

Remark 2.2.5. *Assume that $v_{k-1} > 0$ and $1 - \frac{\lambda^2}{a_{2k-1}^2} > 0$. Then v_k makes a jump, i.e. $v_k \leq 0$, if and only if $1 - X_{k-1} - Y_{k-1} \leq 0$.*

We now introduce the concept of crossings, which relate to the number of jumps made by the process. Let $X(t)$ be a real valued function from $\mathbb{R}$ or $\mathbb{Z}$. Define

$$M(t_1, t_2) = \max_{t \in [t_1, t_2]} X(t)$$

$$m(t_1, t_2) = \min_{t \in [t_1, t_2]} X(t)$$

Definition 2.2.6. *Let $\tau_0 = 0$ and define for $i \geq 1$:*

$$\tau_{2i-1} = \inf\{s \geq \tau_{2i-2} \mid M(\tau_{2i-2}, s) - X(s) \geq \alpha\}$$

$$\tau_{2i} = \inf\{s \geq \tau_{2i-1} \mid m(\tau_{2i-1}, s) - X(s) \leq -\alpha\}$$

We will say that X has made a α-downcrossing at time τ_{2i-1} and a α-upcrossing at time τ_{2i}, for $i \geq 1$. Both down- and upcrossings will be called crossings.

Definition 2.2.7. *Given a function $X(t)$ and $\alpha > 0$, the α-crossing process associated to X is defined as:*

$$Z^\alpha(t) = \begin{cases} -M(\tau_{2i}, t) + \frac{\alpha}{2} \in \mathbb{A}_{2i+1}, & \text{for } t \in [\tau_{2i}, \tau_{2i+1}) \\ -m(\tau_{2i-1}, t) - \frac{\alpha}{2} \in \mathbb{A}_{2i}, & \text{for } t \in [\tau_{2i-1}, \tau_{2i}) \end{cases}$$

where τ_k are times of subsequent crossings as in Definition 2.2.6.

Informally, the proces $Z^\alpha(t)$ evolves as follows. It is started in $\mathbb{A}_1$ at $\frac{\alpha}{2}$ and decreases monotoneously as it is "pushed" by $-M(\tau_0, t) + \frac{\alpha}{2}$. The moment $M(\tau_0, t) - X(t) \geq \alpha$, $Z^\alpha(t)$ jumps to $\mathbb{A}_2$ and everything starts afresh, only with $-M(\tau_0, t) + \frac{\alpha}{2}$ replaced with $-m(\tau_1, t) - \frac{\alpha}{2}$ and moving in the opposite direction. The process jumps to $\mathbb{A}_3$ when $X(t) - m(\tau_1, t) \geq \alpha$ and so on.

Definition 2.2.8. *We say that a process Y has made a jump at time k if $Y_k \in \mathbb{A}_j$ and $Y_{k-1} \in \mathbb{A}_{j-1}$.*

With these definitions, the first crossing is always a downcrossing. Note that the times of successive jumps of the α-crossing process $Z^\alpha(t)$ are exactly the times of α-crossings of the underlying function X.

2.2.4 Upper and lower bounds on the number of jumps

In this Section, we prove upper and lower bounds on the number of jumps of Y in terms of crossings made by X. The Propositions below will be used for $\lambda = e^{-\sqrt{n}}$ but are stated and proved in generality, with λ arbitrary. Proposition 2.1.2 follows by recalling that by Lemma 2.2.3, the number of eigenvalues is $\lceil \frac{1}{2} J_n \rceil$ and that is simply the number of downcrossings.

In both Propositions below, we consider the processes Y and Z^α, the α-crossing process associated to X for appropriate value of α. We will only consider the case when $Y, Z \in \mathbb{A}_{2i}$, as the case of the odd numbered component is handled in a similar way. We can write the one step recursion for Z as $Z_{k+1} = z'(X_{k+1}, Z_k)$, where z' is given by:

$$z'(x, z) = \begin{cases} -x + \frac{\alpha}{2} \in \mathbb{A}_{2i+1}, & \text{if } z \geq -x + \frac{\alpha}{2} \\ \max\{z, -x - \frac{\alpha}{2}\} \in \mathbb{A}_{2i}, & \text{otherwise} \end{cases}$$

Likewise, we can write the one step recursion for Y as $Y_{k+1} = y'(X_k, Y_k, a_{2k+1})$, where y' is given by:

$$y'(x, y, a) = \frac{1}{|\log \lambda|} \log \frac{(1 - \frac{\lambda^2}{a^2})\lambda^{-y} + \frac{1}{a^2}\lambda^{1+x}}{1 - \lambda^{1-y-x}}$$

In both proofs, we will repeatedly use the following estimates, which hold if $|\log|a_k|| \leq \frac{\delta}{16}|\log \lambda|$, $\delta < 2$ and $\lambda < 1$:

$$\lambda^{\frac{\delta}{8}} < \frac{1}{a_k^2} < \lambda^{-\frac{\delta}{8}} \tag{2.12}$$

$$1 - \frac{\lambda^2}{a_k^2} > 1 - \lambda^{2-\frac{\delta}{8}} > 0 \tag{2.13}$$

Proposition 2.2.9. *Pick $\delta \in (0, 2)$. Let J_n be equal to the number of jumps made by $\{Y_k\}_{k=1}^n$. Let C_n be equal to the number of $(2 - \delta)|\log \lambda|$-up- or down-crossings made by S_k and let B_n be the number of k such that $|\log|a_k|| > \frac{\delta}{16}|\log \lambda|$. Then for all $\lambda < \left(\frac{\delta}{32n}\right)^{\frac{4}{\delta}}$ we have $J_n \leq 2B_n + C_n$.*

Proof. First, we replace Y by a process defined as follows. Suppose that $Y_k \in \mathbb{A}_i$. Whenever $|\log|a_k|| > \frac{\delta}{16}|\log \lambda|$, instead of following its usual evolution, the modified process jumps to $Y_k \in \mathbb{A}_{i+2}$, i.e. the same point, but two components ahead. Since Y can make at most one jump in one step, the modified process is always ahead of Y, in particular, it makes at least as many jumps. Thus, it suffices to bound the number of jumps of Y by C_n, under the assumption that $|\log|a_k|| \leq \frac{\delta}{16}|\log \lambda|$, since the remaining jumps are taken care of by the term $2B_n$.

Let Z be the $(2 - \delta)$-crossing process associated to X (Definition 2.2.7). Recall that C_n, the number of $(2 - \delta)$-crossings made by X, is equal to the number of jumps made by Z. Thus we need to prove that Z makes at least as many jumps as Y.

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Recall the one step recursion for Y and Z. Suppose that $Z_k \in \mathbb{A}_i$ and let $W_k = Z_k + (-1)^i \frac{\delta}{4n} \cdot k$. The evolution of W_k is governed by the recursion:

$$w'(x, w, k) = z'\left(x, w - (-1)^i \frac{\delta}{4n} \cdot k\right) + (-1)^i \frac{\delta}{4n} \cdot (k+1)$$

Note that w' makes jumps at the same time as z'. We claim that in order to prove that Z makes at least as many jumps as Y, it suffices to prove that for any values of x, w, a, k such that $k \leq n$ and a satisfies (2.12), (2.13) we have:

$$w'(x, w, k) \geq_{\mathbb{A}} y'(x, w, a) \tag{2.14}$$

We prove inductively that $W_k \geq_{\mathbb{A}} Y_{k+1}$, from which the claim about the number of jumps follows as W_k makes a jump if and only if Z_k makes a jump. For the base case $k = 0$, we have:

$$y'(\infty) = \frac{1}{|\log \lambda|} \log \frac{1 - \frac{\lambda^2}{a^2}}{\lambda} > 1 + \frac{1}{|\log \lambda|} \log(1 - \lambda^{2 - \frac{\delta}{8}}) > 1 - \frac{\delta}{8} \tag{2.15}$$

where the last estimate follows from $\lambda^{\frac{\delta}{8}} < \frac{1}{2}$, easily implied by $\lambda < \left(\frac{\delta}{32n}\right)^{\frac{4}{\delta}}$. Thus, $Y_1 > 1 - \frac{\delta}{8} > 1 - \frac{\delta}{2} = Z_0$. The map y' is monotone, meaning that $y_1 \leq_{\mathbb{A}} y_2$ implies $y'(x, y_1, a) \leq_{\mathbb{A}} y'(x, y_2, a)$. The inductive step then follows by applying (2.14) and then monotonicity::

$$W_{k+1} = w'(X_{k+1}, W_k, k) \geq_{\mathbb{A}} y'(X_{k+1}, W_k, a_{2k+3}) \geq_{\mathbb{A}} y'(X_{k+1}, Y_{k+1}, a_{2k+3}) = Y_{k+2}$$

Note that the increments of w' and y' are translation invariant, so without loss of generality we can put $x = 0$. The results hold for all values of a_k satisfying the assumptions, so we suppress the variable a and write $y(w)$ for $y(0, w, a)$.

We only consider the case when $y, z \in \mathbb{A}_{2i}$ as the case of the odd numbered component is handled in a similar way. We first claim that if $1 - \frac{\delta}{4} \geq u \geq -1 + \frac{\delta}{2}$, then $y'(u) - u < \frac{\delta}{4n}$. We have:

$$y'(u) - u = \frac{1}{|\log \lambda|} \log \frac{(1 - \frac{\lambda^2}{a^2})\lambda^{-u} + \frac{1}{a^2}\lambda}{1 - \lambda^{1-u}} - u \overset{(2.12)}{\leq} \frac{1}{|\log \lambda|} \log \frac{1 + \lambda^{1+u-\frac{\delta}{8}}}{1 - \lambda^{1-u}} \leq$$

$$\frac{1}{|\log \lambda|} \log \frac{1 + \lambda^{\frac{3}{8}\delta}}{1 - \lambda^{\frac{\delta}{4}}} \leq \frac{1}{|\log \lambda|} \cdot \frac{\lambda^{\frac{\delta}{4}} + \lambda^{\frac{3}{8}\delta}}{1 - \lambda^{\frac{\delta}{4}}} \leq \frac{1}{|\log \lambda|} \cdot \frac{2 \cdot \lambda^{\frac{\delta}{4}}}{1 - \lambda^{\frac{\delta}{4}}}$$

The assumptions $\lambda < \left(\frac{\delta}{32n}\right)^{\frac{4}{\delta}}$ and $\delta < 2$ easily imply $\frac{1}{|\log \lambda|} < 2$ and $\frac{1}{1 - \lambda^{\frac{\delta}{4}}} < 2$, so we get:

$$y'(u) - u < 8 \cdot \lambda^{\frac{\delta}{4}} < \frac{\delta}{4n}$$

again by $\lambda < \left(\frac{\delta}{32n}\right)^{\frac{4}{\delta}}$.

We now prove (2.14). The first case to consider is when y' does not jump, which by Remark 2.2.5 and (2.13) implies $w < 1$. Then, either w' jumps and the claim is trivial, or it does not. In that case, we have $w < 1 - \frac{\delta}{2} + \frac{\delta}{4n} \cdot k < 1 - \frac{\delta}{4}$ and $z' = \max\{w - \frac{\delta}{4n} \cdot k, -1 + \frac{\delta}{2}\}$. If $w \geq -1 + \frac{\delta}{2} + \frac{\delta}{4n} \cdot k$, then $z' = w - \frac{\delta}{4n} \cdot k$, so

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$w' = w + \frac{\delta}{4n}$. Also $1 - \frac{\delta}{4} \geq w \geq -1 + \frac{\delta}{2}$, so $y'(w) - w < \frac{\delta}{4n} = w' - w$ as desired. If $w < -1 + \frac{\delta}{2} + \frac{\delta}{4n} \cdot k$, then:

$$y'(w) < y'\left(-1 + \frac{\delta}{2} + \frac{\delta}{4n} \cdot k\right) < -1 + \frac{\delta}{2} + \frac{\delta}{4n} \cdot k + \frac{\delta}{4n} = z' + \frac{\delta}{4n} \cdot (k+1) = w'$$

The other remaining case is when y' makes a jump, so $w \geq 1$. This means that $z \geq 1 - \frac{\delta}{4n} \cdot k > 1 - \frac{\delta}{4} > 1 - \frac{\delta}{2}$, so z' makes a jump as well. Also, $z' = 1 - \frac{\delta}{2} \in \mathbb{A}_{2i+1}$ and $w' > z'$. It thus suffices to prove that $y' \leq_{\mathbb{A}} 1 - \frac{\delta}{2}$ in $\mathbb{A}_{2i+1}$, which translates to $y' \geq 1 - \frac{\delta}{2}$ since we are in an odd numbered component. We have $y'(w) > y'(\infty)$ and $y'(\infty) > 1 - \frac{\delta}{2}$ by (2.15). $\qquad \square$

Proposition 2.2.10. *Pick $\delta \in (0,2)$. Let J_n be equal to the number of jumps made by $\{Y_k\}_{k=1}^n$. Let C_n be equal to the number of $(2 + \delta) |\log \lambda|$-up- or down-crossings made by S_k and let B_n be the number of k such that $|\log |a_k|| > \frac{\delta}{16} |\log \lambda|$. Then for all $\lambda < 1$ we have $C_n - 2B_n \leq J_n$.*

Proof. As in Proposition 2.2.9, we replace Y by a process which makes two jumps whenever $|\log |a_k|| > \frac{\delta}{16} |\log \lambda|$. Now it suffices to bound the number of jumps of Y from below by C_n, under the assumption that $|\log |a_k|| \leq \frac{\delta}{16} |\log \lambda|$, since the remaining jumps are taken care of by the term $2B_n$.

Let Z be the $(2 + \delta)$-crossing process associated to X (Definition 2.2.7). We only consider the case when $Y_k, Z_k \in \mathbb{A}_{2i}$ as the other one is handled in a similar way. Recall that C_n, the number of $(2 + \delta)$-crossings made by X, is equal to the number of jumps made by Z. Thus we need to prove that Y makes at least as many jumps as Z.

Recall the one step recursion for Y and Z. We claim that in order to prove that Y makes at least as many jumps as Z, it suffices to prove that for any values of x, z, a such that a satisfies (2.12), (2.13) we have:

$$z'(x,z) \leq_{\mathbb{A}} y'(x,z,a) \tag{2.16}$$

We prove inductively that $Z_k \leq_{\mathbb{A}} Y_{k+1}$ for all $k \geq 0$. The base case is clear since $Z_0 = 1 + \frac{\delta}{2}$ and $Y_1 < 1$. The map y' is monotone, meaning that $y_1 \leq_{\mathbb{A}} y_2$ implies $y'(x, y_1, a) \leq_{\mathbb{A}} y'(x, y_2, a)$. The inductive step then follows by applying (2.16) and then monotonicity::

$$Z_{k+1} = z'(X_{k+1}, Z_k) \leq_{\mathbb{A}} y'(X_{k+1}, Z_k, a_{2k+3}) \leq_{\mathbb{A}} y'(X_{k+1}, Y_{k+1}, a_{2k+3}) = Y_{k+2}$$

Note that the increments of z' and y' are translation invariant, so without loss of generality we can put $x = 0$. The results hold for all values of a_k satisfying the assumptions, so we suppress the variable a and write $y(w)$ for $y(0, w, a)$.

To prove (2.16), consider first the case when $y'(z)$ does not make a jump, which by Remark 2.2.5 and (2.13) means $z < 1$. This in particular means that z' does not make a jump, so $z' = \max\{z, -1 - \frac{\delta}{2}\}$. Since y' is increasing, we have $y'(z) > z$. On the other hand, since $z \geq_{\mathbb{A}} -\infty$, we have:

$$y'(z) \geq y'(-\infty) = \frac{1}{|\log \lambda|} \log \frac{1}{a^2} \lambda \geq -1 - \frac{\delta}{8}$$

Thus, $y' \geq \max\{z, -1 - \frac{\delta}{8}\} \geq z'$, finishing the claim.

The other case is when y' makes a jump from $\mathbb{A}_{2i}$ to $\mathbb{A}_{2i+1}$, so that $z \geq 1$. Then either $z < 1 + \frac{\delta}{2}$, so

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z' remains in $\mathbb{A}_{2i}$ and the claim is trivial, or $z \geq 1 + \frac{\delta}{2} \in \mathbb{A}_{2i}$. In that case $z' = 1 + \frac{\delta}{2} \in \mathbb{A}_{2i+1}$ and we need to prove $y' \geq_{\mathbb{A}} 1 + \frac{\delta}{2} \in \mathbb{A}_{2i+1}$, that is, $y' \leq 1 + \frac{\delta}{2}$ since we are in the odd numbered component. Since $z \geq_{\mathbb{A}} 1 + \frac{\delta}{2} \in \mathbb{A}_{2i}$, by monotonicity of y' we have:

$$y'(z) \leq y'\left(1 + \frac{\delta}{2}\right) = \frac{1}{|\log \lambda|} \log \frac{(1 - \frac{\lambda^2}{a^2})\lambda^{-1-\frac{\delta}{2}} + \frac{1}{a^2}\lambda}{\lambda^{-\frac{\delta}{2}} - 1} \leq$$
$$1 + \frac{\delta}{2} + \frac{1}{|\log \lambda|} \log \frac{1 + \frac{1}{a^2}\lambda^{2+\frac{\delta}{2}}}{\lambda^{-\frac{\delta}{2}} - 1} < 1 + \frac{\delta}{2} + \frac{1}{|\log \lambda|} \log \frac{1 + \lambda^{2+\frac{3}{8}\delta}}{\lambda^{-\frac{\delta}{2}} - 1}$$

It suffices to make the expression under the logarithm smaller than 1. This is easily implied by $\lambda < 1$ and $\lambda^{-\frac{\delta}{2}} > 3$, which follows from the assumption that $\lambda < \left(\frac{1}{3}\right)^{\frac{2}{\delta}}$. $\qquad\square$

We note that for even values of n, we obtain similar statements, but with roles of down- and upcrossings reversed. In particular, we obtain an analogue of Proposition 2.1.2 for even n, which is identical but bounds the number of jumps in terms of upcrossings, rather than downcrossings of S_k.

2.2.5   Convergence to the scaling limit

We now study the continuous-time scaling limit of the processes Y. To this end, we put $\lambda = e^{-\sqrt{n}}$ and consider a sequence of processes $Y^{(n)}$ such that the underlying functions $X^{(n)}$ converge to some function X. The reason for this particular scaling is related to Brownian scaling which we shall use in Section 2.3 when introducing the probabilistic part of the analysis. In the formula below the reader should recognize the same scaling as in the Central Limit Theorem.

We introduce the continuous time version of the discrete process X with scaling $\lambda = e^{-\sqrt{n}}$. For any η, we can write:

$$X_{\eta n}^{(n)}(t) := \frac{1}{\sqrt{\eta^2 n}} S_{\eta^2 nt}^{(n)}$$

We introduce the superscript n to emphasize that for different values of n, the sums $S_k^{(n)}$ depend on different sets of edge weights $\{a_k^{(n)}\}_{k=1}^n$ for each n. While it may seem natural to consider sequences obtained from a single infinite operator H restricted to finite intervals, we will need the main theorem of this section stated in generality.

We will establish the main theorem of this section under the following assumptions.

Condition 2.2.11. $X_n^{(n)}(t)$ *converge uniformly on some interval* $[0, T]$ *to a function* $X(t)$ *such that* $X(0) = 0$.

Condition 2.2.12. *Fix some* $\nu > 0$. *Whenever* $M(t) - X(t) = 2\nu$ *(resp.* $m(t) - X(t) = -2\nu$*), for any* $\varepsilon > 0$ *there exists* $t' < t + \varepsilon$ *such that* $M(t') - X(t') > 2\nu$ *(resp.* $m(t') - X(t') < -2\nu$*).*

Condition 2.2.13. *We have:*

$$\lim_{n \to \infty} \max_{k=1,\ldots,n} \frac{\left|\log |a_k^{(n)}|\right|}{\sqrt{n}} = 0 \tag{2.17}$$

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Note that Condition 2.2.13 is not implied by Condition 2.2.11 since Condition 2.2.11 only implies that $\left|\log\left|\frac{a_{2k-1}^{(n)}}{a_{2k}^{(n)}}\right|\right|$ are small in the limit and implies nothing about $\left|\log|a_k^{(n)}|\right|$.

Theorem 2.2.14. *Let $X_n^{(n)}(t)$ be a sequence of piecewise linear functions that converge uniformly on some interval $[0,T]$ to a function $X(t)$ such that $X(0) = 0$ (Condition 2.2.11). Suppose that $X(t)$ satisfies Condition 2.2.12 with $\nu = 1$ and Condition 2.2.13 is satisfied. Let J_n denote the number of jumps made by $\{Y_k^{(n)}\}_{k=1}^n$ and let J denote the number of 2-crossings made by X. Then J_n converges to J as $n \to \infty$.*

Proof. Let C_n^α denote the number of α-crossings made by X_n, let C^α denote the number of α-crossings made by X and let B_n^δ denote the number of k such that $\left|\log\left|a_k^{(n)}\right|\right| > \frac{\delta}{16}\sqrt{n}$. Pick $\delta > 0$. By applying Propositions 2.2.9 and 2.2.10 with $\lambda = e^{-\sqrt{n}}$, for large enough n we obtain:

$$C_n^{2+\delta} - 2B_n^\delta \leq J_n \leq C_n^{2-\delta} + 2B_n^\delta \tag{2.18}$$

Since X_n converge to X uniformly, it is easy to see that for large enough n we have $C_n^{2+\delta} \geq C^{2+2\delta}$ and $C_n^{2-\delta} \leq C^{2-2\delta}$. Moreover, Condition 2.2.12 implies that $C^{2-2\delta}$ and $C^{2+2\delta}$ converge to $C^2 = J$ as $\delta \to 0$. Also, by Condition 2.2.13 we have that B_n^δ converges to 0 as $n \to \infty$. Thus, we can make the upper and lower bound in (2.18) arbitrarily close to J by first picking small enough δ and then large enough n, depending on δ, which finishes the proof. $\qquad\square$

Remark 2.2.15. *The proofs of Proposition 2.2.9 and 2.2.10 imply that the discrete process Y is always bounded from both sides by $(2 \pm \delta)$-crossing processes $Z^{2\pm\delta}$, outside of jump times τ_i. This can be actually used to prove the convergence of the processes Y to the 2-crossing process Z^2. We do not spell out the details as we only need the convengence of the number of jumps.*

We are now ready to prove the Lemma which summarizes the connection between the asymptotic number of eigenvalues of Schroedinger operators described in Section 2.2.2 and the processes introduced above.

Definition 2.2.16. *We say that an interval $[a,b] \subseteq [0,T]$ is an s-well of X for some $s > 0$ if: 1. $b = T$ or $X(a) = X(b)$; 2. $X(a) > X(x)$ for all $x \in (a,b)$ and $X(a) - X(x) \geq s$ for some $x \in (a,b)$.*

Remark 2.2.17. *Note that the maximal number of disjoint s-wells made by X inside $[0,T]$ is equal to the number of s-downcrossings.*

Lemma 2.2.18. *Let $\{H^{(n)}\}_{n=1}^\infty$ be a family of Schroedinger operators, each given by edge weights $\{a_i^{(n)}\}_{i=1}^n$. Let $H_m^{(n)}$ denote the operator $H^{(n)}$ restricted to the interval $[1, m+1]$. For $t, \eta > 0$, let $\Lambda_n(t,\eta)$ denote the number of eigenvalues of $H_{\lfloor tn \rfloor}^{(n)}$ inside the interval $[0, e^{-\eta\sqrt{n}}]$. Consider a subsequence of n such that $\lfloor tn \rfloor$ is odd. Suppose that Conditions 2.2.11. 2.2.12 and 2.2.13 are satisfied, including Condition 2.2.12 with $\nu = \eta$. In particular, $X_n^{(n)}$ converge to some function X. Let $\Lambda(t,\eta)$ denote the number of 2η-downcrossings of X inside the interval $[0, \frac{1}{2}t]$. Then $\Lambda_n(t,\eta)$ converges to $\Lambda(t,\eta)$ as $n \to \infty$.*
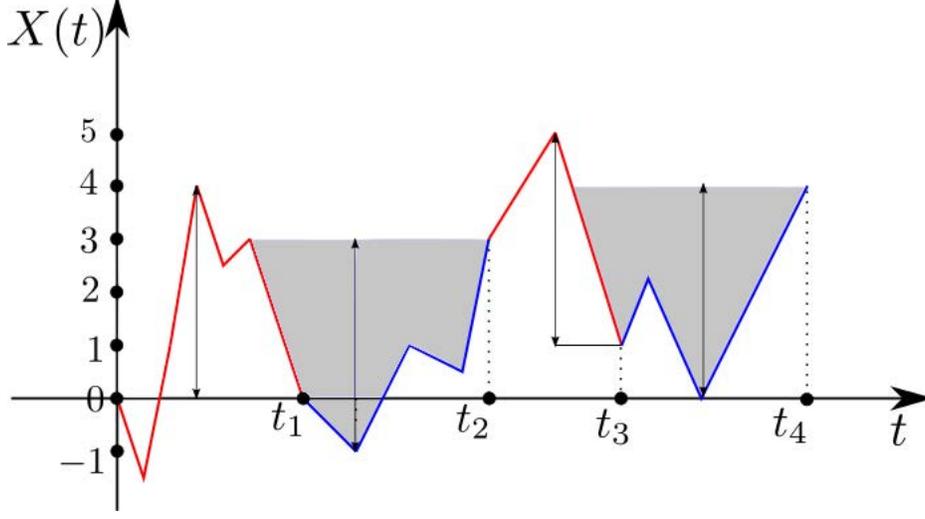
Figure 2.3: $X(t)$ makes a 4-downcrossing at t_1 and t_3 and 4-upcrossing at t_2 and t_4. The resulting 4-wells are shown in grey.

Proof. Recall that:

$$S_k^{(n)} = \sum_{i=1}^{k} U_i^{(n)} = \sum_{i=1}^{k} 2 \log \left| \frac{a_{2i-1}^{(n)}}{a_{2i}^{(n)}} \right|$$

$$X_n^{(n)}(t) = \frac{1}{\sqrt{n}} S_{nt}^{(n)}$$

where the superscript n emphasizes that, for different n, the sums $S^{(n)}$ depend on a different set of edge weights $a_i^{(n)}$.

Put $T = \frac{1}{\eta^2}t$. By letting $n' = \eta^2 n$, $\Lambda_n(t,\eta)$ is equal to the number of eigenvalues of $H_{\lfloor Tn' \rfloor}^{(n)}$ inside the interval $[0, e^{-\sqrt{n'}}]$. By Lemma 2.2.3, this is equal to $\lceil \frac{1}{2} J_n \rceil$, where J_n is the number of jumps made up to time $\frac{1}{2}Tn'$ by the process $\{Y^{(n),\lambda}\}_{k=1}^{\frac{1}{2}Tn'}$, with $\lambda = e^{-\sqrt{n'}}$ and the underlying random walk being $X_{n'}^{(n)}$. Note that the scaling factor n' is now different than the sample number n. The factor of $\frac{1}{2}$ appears since each step of the discrete process used in Section 2.2.3 actually corresponds to two steps of the eigenvalue counting process from Section 2.2.2(see equation (2.10)).

We have:

$$X_{n'}^{(n)}(t) = \frac{1}{\sqrt{n'}} S_{n't}^{(n)} = \frac{1}{\eta} \cdot \frac{1}{\sqrt{n}} S_{\eta^2 nt}^{(n)}$$

Since by assumption $X_n^{(n)}(t) \to X(t)$, we have $X_{n'}^{(n)}(t) \to X_\eta(t) := \frac{1}{\eta} \cdot X(\eta^2 t)$.

Since X satisfies Condition 2.2.12 with $\nu = \eta$, it follows that X_η satisfies the same condition with $\nu = 1$. Since the assumptions of Theorem 2.2.14 are satisfied for X_η, the number of jumps made by $\{Y^{(n),\lambda}\}_{k=1}^{\frac{1}{2}Tn'}$ converges to the number of 2-down- or upcrossings that X_η makes inside the interval $[0, \frac{1}{2}T]$. This is the same as the number of 2η-crossings made by X inside $[0, \frac{1}{2}t]$. This finishes the proof by noting that if C is the number of 2-down- or upcrossings, then $\lceil \frac{1}{2}C \rceil$ is the number of 2-downcrossings since the first crossing is always a downcrossing. □

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2.3 Local statistics and the expected spectral measure at zero

In this section, we use the limiting process from the previous section to derive the limit of local eigenvalue statistics of H_n and compute μ_H, the expected spectral measure of the random Schroedinger operator H, near zero. The main focus of this section are Theorem 2.3.3, Lemma 2.3.9 and Theorem 3.1.2. We shall use the notation from Section 2.2. Whenever we speak of convergence in distribution of processes, we shall mean weak convergence of measures on $C([0,T])$.

2.3.1 Local eigenvalue statistics

In this Section, we consider a random Schroedinger operator with edge weights given by a sequence of random variables $\{a_i\}_{i=1}^{\infty}$. We define:

$$U_i = 2\log\left|\frac{a_{2i-1}}{a_{2i}}\right|, \;\; S_n = \sum_{i=1}^{n} U_i, \;\; X_n(t) = \frac{1}{\sqrt{n}}S_{\lfloor nt \rfloor}$$

Definition 2.3.1. *Suppose that $\{U_i\}_{i=1}^{\infty}$ satisfy $\mathbb{E}U_i = 0$ and $\mathbb{E}U_i^2 < \infty$. We say that U_i satisfy the functional Central Limit Theorem if X_n converge in distribution on some interval $[0,T]$ to a Brownian motion X with mean zero.*

Proposition 2.3.2. *Suppose that $\{a_i\}_{i=0}^{\infty}$ are identically distributed and $\mathbb{E}(\log|a_i|)^2 < \infty$. Then for any $\varepsilon > 0$ we have:*

$$\mathbb{P}\left(\max_{k=1,\ldots,n}\frac{|\log|a_k||}{\sqrt{n}} > \varepsilon\right) \to 0$$

i.e. the sequence of variables $\{\max_{k=1,\ldots,n}\frac{\log|a_k|}{\sqrt{n}}\}_{n=1}^{\infty}$ converges to 0 in probability.

Proof. Let $X_k = \log^2|a_i|$, so that $\mathbb{E}X_k < \infty$. A simple exercise in probability (see e.g. [Bil95], Problem 21.3) shows that when variables X_k are identically distributed with $\mathbb{E}X_1 < \infty$, we have $\mathbb{E}\max_{k=1,\ldots,n} X_k = o(n)$. This together with Markov's inequality proves the desired claim. $\square$

If the functional Central Limit Theorem is satisfied, we can use Lemma 2.2.18 and Skorokhod almost sure representation to obtain convergence in distribution of the local eigenvalue statistics. This describes the limiting local eigenvalue distribution around 0 in terms of crossings made by a Brownian motion, recall Definition 2.2.6. We note that the limiting distribution is not Poisson, as Proposition 2.3.5 shows that it exhibits a Gaussian tail.

Theorem 2.3.3. *Suppose that U_i satisfy the functional Central Limit Theorem and let $\mathrm{Var}\,\log|a_i| = \sigma^2 < \infty$. Let H_m be the operator H restricted to the interval $[1, m+1]$. For $t, \eta > 0$, let $\{\Lambda_n(t,\eta), t, \eta > 0\}$ denote the random process equal to the number of eigenvalues of $H_{\lfloor tn \rfloor}$ inside the interval $[0, e^{-\eta\sqrt{n}}]$ and let $\{\Lambda(t,\eta), t, \eta > 0\}$ denote the random process equal to the maximal number of disjoint η-wells made by a Brownian motion with variance σ^2 inside the interval $[0,t]$. Consider a subsequence of n such that $\lfloor tn \rfloor$ is odd.*

Then the process $\Lambda_n(t,\eta)$ converges to the process $\Lambda(t,\eta)$, i.e. all finite dimensional distributions of Λ_n converge weakly to finite dimensional distributions of Λ.

Proof. By assumption, X_n converge weakly to X, which is a Brownian Motion with variance $8\sigma^2$. By Skorokhod representation theorem ([Kal02], Theorem 4.30), we can find a common probability space such that $X_n \to X$ uniformly almost surely. The maximum process $M - X$ is equal in distribution to a reflected Brownian motion and it is standard ([MP10]) that almost surely the Brownian motion has no isolated zeros. These two facts easily imply that X almost surely satisfies Condition 2.2.12 for any value of ν. Moreover, by Proposition 2.3.2 and almost sure representation of variables converging in probability, we obtain that Condition 2.2.13 is satisfied almost surely. Thus, almost surely the conditions of Theorem 2.2.14 are satisfied.

By Lemma 2.2.18, for any t, η it holds that $\Lambda_n(t, \eta)$ converges to the number of 2η-wells that X makes inside the interval $[0, \frac{1}{2}t]$. By Brownian scaling this has the same distribution as the number of η-wells that a Brownian motion with variance σ^2 makes inside the interval $[0, t]$, which is exactly $\Lambda(t, \eta)$. The same statement holds for Λ_n treated as a process - for any finite number of $\{(t_1, \eta_1), \dots, (t_k, \eta_k)\}$ convergence holds simultaneously almost surely for all t_i, η_i, which implies weak convergence of all finite dimensional distributions.

Note that the limiting 2-crossing process Z^2 is discontinuous, but is right continuous with left limits. Then Remark 2.2.15 also implies that the processes $Y^{(n)}$ converge weakly to Z^2. We do not describe this in detail as below we only need the weak convergence of the number of eigenvalues and not of the processes $Y^{(n)}$ themselves. $\qquad\square$

Remark 2.3.4. *Various questions about the eigenvalues of H_n can be phrased in terms of crossings made by the limiting Brownian Motion X. For example, let $\lambda_0^{(n)}$ be the smallest positive eigenvalue of H_n. It follows from the preceding Theorem that $\frac{-\log \lambda_0^{(n)}}{\sqrt{n}}$ converges in distribution to the maximal size of a downcrossing made by X inside the interval $[0, 1]$, that is, the maximal η such that $M(t) - X(t) = \eta$ for some $t \in [0, 1]$. Since $M(t) - X(t)$ is equal in distribution to a reflected Brownian motion $|X(t)|$, we obtain that:*

$$\frac{-\log \lambda_0^{(n)}}{\sqrt{n}} \implies \sigma \cdot \sup_{t \in [0,1]} |B(t)|$$

in distribution, where $B(t)$ is the standard Brownian Motion of variance 1.

Proposition 2.3.5. *With $\Lambda(t, \eta)$ defined as in Theorem 2.3.3, for some constants $a, b > 0$ depending on t, η, σ we have*

$$\mathbb{P}(\Lambda(t, \eta) > m) \le a e^{-bm^2}$$

Proof. Suppose that $\Lambda(t, \eta) > m$, so $X(t)$ has made more than $2m$ η-up- or down-crossings inside the interval $[0, t]$. By Brownian scaling, we can put $\eta' = \frac{\eta\sqrt{t}}{\sigma}$ and instead consider η'-crossings of a standard Brownian motion inside the interval $[0, 1]$ Let $T_1, \dots, T_{2m}$ be the times between subsequent η-up- or down-crossings. The times T_i are independent and identically distributed. Moreover, note that $R(t) = M(t) - X(t)$ has the distribution of a reflected Brownian motion, so by the reflection principle we have:

$$\mathbb{P}(T_i \le x) = \mathbb{P}(\max_{[0,x]} R(t) \ge \eta') \le 2 \cdot \mathbb{P}(\max_{[0,x]} B(t) \ge \eta') = 4 \cdot \mathbb{P}(B(x) \ge \eta') \le 4e^{-\frac{\eta'^2}{2x}}$$

Since there are more than $2m$ crossings inside $[0, 1]$, this means that $T_1 + \dots T_{2m} < 1$. Let $\Delta_{2m} =$

$\{(x_1, \ldots, x_{2m}) \in [0,1]^{2m} | \sum_{i=1}^{2m} x_i \leq 1\}$. We have:

$$\mathbb{P}(T_1 + \ldots T_{2m} < 1) \leq \int_{\Delta_{2m}} \prod_{i=1}^{2m} \mathbb{P}(T_i \in dx_i) \leq 4 \int_{\Delta_{2m}} \prod_{i=1}^{2m} e^{-\frac{\eta'^2}{2x_i}} = 4 \int_{\Delta_{2m}} e^{-\frac{\eta'^2}{2} \sum_{i=1}^{2m} \frac{1}{x_i}} \leq$$

$$4 \int_{\Delta_{2m}} e^{-2m^2 \eta'^2 \frac{1}{\sum_{i=1}^{2m} x_i}} \leq 4 \cdot \text{vol}(\Delta_{2m}) \cdot e^{-2\eta'^2 \cdot m^2} = 4 \cdot \frac{e^{-2\eta'^2 \cdot m^2}}{(2m)!}$$

$\square$

2.3.2 The expected spectral measure at zero

Having established weak convergence of the local eigenvalue statistics, we now set the background to compute the expected spectral measure of H. We shall consider sequences of U_i satisfying the following technical condition:

Condition 2.3.6. *At least one of the following holds:*

1. *for some $\gamma > 2$ we have $\mathbb{E} |\log|a_i||^{\gamma} < \infty$ and for some constant C and any $n \geq 1$:*

$$\mathbb{E} |U_1 + \cdots + U_n|^{\gamma} \leq Cn^{\frac{\gamma}{2}}, \tag{2.19}$$

2. *The a_i are i.i.d. and $\mathbb{E} |\log|a_i||^2 < \infty$.*

Note that in the i.i.d. case a standard computation verifies the inequality (2.19) with $\gamma = 2$. However, we will later work in the setting when U_i are dependent, but exhibit a suitable correlation decay (see Section 2.4.2).

Having established weak convergence of the number of jumps, we would like to establish that the convergence also holds in expectation. This will be implied by the upcoming Lemma 2.3.8. An important element in the proof is the maximal inequality of [Mó76], whose special case says the following:

Proposition 2.3.7. *Let $S(i,j) = \sum_{k=i}^{j} U_k, M(i,j) = \max_{i \leq k \leq j} S(i,k)$. If for some $\gamma > 2$ and $A > 0$ we have:*

$$\mathbb{E} |S(i,j)|^{\gamma} \leq A(j - i + 1)^{\frac{\gamma}{2}}$$

for all $j > i \geq 1$, then:

$$\mathbb{E} |M(i,j)|^{\gamma} \leq c_{\gamma} \cdot A(j - i + 1)^{\frac{\gamma}{2}}$$

for all $j > i \geq 1$, with the constant c_{γ} depending only on γ.

Lemma 2.3.8. *Suppose that U_i satisfy Condition 2.3.6. Let J_n be the random variable equal to the number of jumps made by $Y^{(n)}$ in the interval $[1, n]$. Then the family of variables $\{J_n\}_{n \geq 1}$ is uniformly integrable.*

Proof. By Proposition 2.2.9 with $\delta = 1$ and $\lambda = e^{-\sqrt{n}}$, we obtain $J_n \leq 2B_n + C_n$ for all n greater than some global constant n_0, where:

$$B_n = \sum_{i=1}^{n} \mathbb{1}_{\{|\log|a_i|| > \frac{1}{16} \sqrt{n}\}}$$

is the number of times i such that $|\log |a_i|| > \frac{1}{16}\sqrt{n}$ and C_n is the number of 1-crossings made by X_n. Thus, it suffices to show that B_n and C_n are uniformly integrable. We will first prove the general case. Then we give a different argument for the i.i.d. case under weaker moment assumptions.

To prove that B_n is uniformly integrable, it suffices to show that $\mathbb{E}B_n \to 0$. By Condition 2.3.6 we can estimate:

$$\mathbb{E}B_n = n \cdot \mathbb{P}\left(|\log|a_i|| > \frac{1}{16}\sqrt{n}\right) \leq n \cdot \frac{\mathbb{E}|\log|a_i||^\gamma}{n^{\frac{\gamma}{2}}(\frac{1}{16})^\gamma}.$$

Since $\gamma > 2$, this converges to 0 as $n \to \infty$.

For a crossing $[s, t]$, its length is $t - s$. To show that C_n are uniformly integrable, it suffices to show that the expected number of 1-crossings $[s, t]$ of length at most $n \cdot 2^{-\ell_0}$ is at most $\varepsilon(\ell_0)$ for all n, where $\varepsilon(\ell_0) \to 0$ as $\ell_0 \to \infty$. Indeed, for given ℓ_0 the number of 1-crossings that are longer is bounded independently of n.

We first bound the expected number $e(n, \ell)$ of 1-crossings that are of length at least $n \cdot 2^{-\ell}$ and at most $2n \cdot 2^{-\ell}$. To each such crossing $[s, t]$ we associate the first time of the form $k \cdot \lfloor 2^{-\ell}n \rfloor$ contained in $[s, t]$ for some integer k. Each such time is contained in at most one such crossing, so the number of such crossings is bounded above by the sum of the indicators over such times that a crossing contains that time.

As a consequence, $e(n, \ell)$ is bounded by the number of such times, which is 2^ℓ, times the maximal probability that a crossing contains a given time. If this happens, then the range (i.e. the maximum minus the minimum) of the process S_k over the interval $[i, j]$ of size $4n \cdot 2^{-\ell}$ centered at that time is at least $\sqrt{n}$. In particular, either $M(i, j) \geq \sqrt{n}/2$, or the same holds for the absolute minimum. By Markov's inequality, Proposition 2.3.7 and Condition 2.3.6 we get

$$\mathbb{P}(M(i, j) \geq \sqrt{n}/2) \leq \frac{2^{\gamma/2}\mathbb{E}|M(i, j)|^\gamma}{n^{\gamma/2}} \leq \frac{c \cdot \mathbb{E}|S(i, j)|^\gamma}{n^{\gamma/2}} \leq \frac{c' \cdot (4n \cdot 2^{-\ell})^{\gamma/2}}{n^{\gamma/2}}$$

giving $e(n, \ell) \leq 4^{\gamma/2}c' \cdot 2^\ell 2^{-\ell\gamma/2}$. As a result, the expected number of 1-crossings of length at most $n \cdot 2^{-\ell_0}$ is at most

$$c \sum_{\ell=\ell_0}^{\lceil \log_2 n \rceil} 2^{\ell(1-\gamma/2)} < \frac{c \cdot 2^{\ell_0(1-\gamma/2)}}{1 - 2^{(1-\gamma/2)}} =: \varepsilon(\ell_0) \to 0$$

as long as $\gamma > 2$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of the i.i.d. case. We now prove the i.i.d. case assuming only $\mathbb{E}|\log|a_i||^2 < \infty$. It suffices to show that the variance of B_n is uniformly bounded in n. The B_n are Bernoulli random variables, so their variance is bounded by their mean. We have

$$\mathbb{E}(B_n) = n \cdot \mathbb{P}\left(|\log|a_1|| > \frac{1}{16}\sqrt{n}\right) \leq n \cdot \frac{\mathbb{E}|\log|a_i||^2}{n(\frac{1}{16})^2},$$

which is bounded.

To show that C_n is uniformly integrable, it suffices to show exponential tails uniform in n, namely a bound of the form $\mathbb{P}(C_n \geq k) < c^k$ for some $c < 1$. Let a be a large constant to be chosen later. We divide the interval $[1, n]$ into a intervals $I_1, \ldots, I_a$ of size $\frac{n}{a}$. Let $N_{n,j}$ denote the number of crossings with starting points in I_j. It suffices to show that each $N_{n,j}$ has exponential tails uniform in n.

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If the walk makes at least 2 crossings inside $I_j = [s, t]$, then the maximal absolute increment between times s, t in this interval is at least $\sqrt{n}$. With the notation of Proposition 2.3.7 either $M(s, t) \geq \sqrt{n}/2$ or the same holds for the absolute minimum. By the classical Kolmogorov maximal inequality for i.i.d variables

$$\mathbb{P}(M(s, t) > \lambda) \leq \frac{(s - t) \cdot \text{Var } U_i}{\lambda^2},$$

which gives

$$P(N_{i,j} \geq 2) \leq 2 \cdot \frac{n/a \cdot \text{Var } U_i}{n/4}. \tag{2.20}$$

By choosing a large enough, we can make the right hand side smaller than some global constant $b < 1$, which we fix from now on.

We can write for any i:

$$\mathbb{P}(N_{n,j} \geq 2i) = \prod_{k=1}^{i} \mathbb{P}(N_{n,j} \geq 2k \mid N_{n,j} \geq 2k - 2)$$

If we stop the random walk in the time interval I_j after the $2k - 2$nd crossing (when it exists), the conditional law of the remaining stretch is another independent random walk in a smaller interval. The conditional probability of the remaining walk making a double crossing is at most b again. This shows that all terms in the above product are bounded above by b, so $N_{n,j}$ has exponential tails uniform in n, as required. $\qquad\square$

The crucial step in computing the spectral measure of H is approximation by an operator H_n on a finite interval. The spectral measure of H_n is computed in the following lemma.

Lemma 2.3.9. *Fix $K > 0$ and let $n = \lfloor K^2 |\log \varepsilon|^2 \rfloor$ be odd (in particular, $n \to \infty$ as $\varepsilon \to 0$). Let H_n be equal to the operator H restricted to the interval $\{1, \ldots, n + 1\}$ and let μ_n be its expected spectral measure. Assume that for all i $\text{Var } \log |a_i| = \sigma^2 < \infty$. Suppose that:*

1. *U_i satisfy the functional Central Limit Theorem (Definition 2.3.1)*

2. *U_i satisfy Condition 2.3.6*

Then there exists c_K such that $\lim_{K \to \infty} c_K = 0$ and for sufficiently small ε:

$$\left| \mu_n(-\varepsilon, \varepsilon) - \frac{\sigma^2}{|\log^2 \varepsilon|} \right| \leq \frac{c_K}{|\log^2 \varepsilon|}$$

Proof. Let $M_{K,n}$ denote the number of eigenvalues of H_n inside the interval $[-\varepsilon, \varepsilon]$ and let $M_{K,n}^+$ denote the number of eigenvalues inside the interval $[0, \varepsilon]$. We start by ruling out zero eigenvalue of H_n. The equation for eigenvectors (see Section 2.2.2) with $\lambda = 0$ can be solved recursively and for odd n the only solution is identically zero. Since we assumed n is odd, all the eigenvalues of H_n are thus nonzero. Since the underlying graph is bipartite, every eigenvalue $\lambda_+ \in (0, \varepsilon]$ has a corresponding eigenvalue $\lambda_- = -\lambda_+ \in [-\varepsilon, 0)$, so:

$$M_{K,n} = 2M_{K,n}^+$$

Let J_K denote the number of $\frac{1}{K}$-crossings that a Brownian motion with variance σ^2 makes inside the interval $[0,1]$. By Brownian scaling this is the same as the number of 1-crossings that a Brownian motion with variance σ^2 makes inside the interval $[0,K^2]$. By assumption, $U_i = 2\log\left|\frac{a_{2i-1}}{a_{2i}}\right|$ satisfy the functional Central Limit Theorem. Since $\varepsilon = e^{-\frac{1}{K}\sqrt{n}}$, by putting $t = 1, s = \frac{1}{K}$ in Theorem 2.3.3 we obtain that $M_{K,n}^+$ converges weakly to $\left\lceil \frac{1}{2}J_K \right\rceil$. Thus we obtain:

$$M_{K,n} = 2M_{K,n}^+ \Rightarrow 2\left\lceil \frac{1}{2}J_K \right\rceil = J_K + \mathbb{1}_{\{J_K = 2k+1\}}$$

By Lemma 2.3.8 the family $\{M_{K,n}\}_{n\geq 1}$ is uniformly integrable since $M_{K,n}$ differs from the number of jumps by at most one. Weak convergence thus implies convergence in expectation, so by letting:

$$a_{K,n} = \mathbb{E}M_{K,n} - \mathbb{E}J_K - \mathbb{E}\mathbb{1}_{\{J_K = 2k+1\}} \tag{2.21}$$

for each fixed K we have $\lim_{n\to\infty} a_{K,n} = 0$.

We shall now compute $\mathbb{E}J_K$. We shall use standard properties of Brownian motion, which can be found e.g. in [MP10]. We first compute τ, the expected time to make an up or down crossing. Let X be a Brownian motion with variance σ^2. Consider $\tau_1 = \inf\{t \geq 0 : M(0,t) - X(t) = 1\}$. Since $M(0,t) - X(t)$ has the distribution of a reflected Brownian motion, τ_1 has the same distribution as $\tau_2 = \inf\{t \geq 0 : X(t) = 1 \vee X(t) = -1\}$. It is standard that $\mathbb{E}\tau_2 < \infty$. If $X(t)$ is stopped at time τ_2, by Wald identity we obtain $\mathbb{E}X(\tau_2)^2 = \sigma^2 \cdot \mathbb{E}\tau_2$, which gives $\mathbb{E}\tau_2 = \frac{1}{\sigma^2}$. Clearly we can treat the event $\{m(0,t) - X(t) = -1\}$ in the same way, so we have:

$$\tau = \mathbb{E}\tau_2 = \frac{1}{\sigma^2} \tag{2.22}$$

Now, let T_i be i.i.d. random variables equal to the times between successive up or down crossings made by $X(t)$. Note that $\mathbb{E}T_i = \tau$ and we can write J_K as:

$$J_K = \sup\{n : \sum_{i=1}^{n} T_i \leq K^2\}$$

In other words, J_K is equal to the number of jumps up to time K^2, made by a renewal process with expected renewal time τ. By the law of large numbers for renewal processes [Dur10], it follows that $\lim_{K\to\infty} \frac{\mathbb{E}J_K}{K^2} = \frac{1}{\tau}$. Let:

$$b_K = \frac{\mathbb{E}J_K}{K^2} - \frac{1}{\tau} \tag{2.23}$$

so that $\lim_{K\to\infty} b_K = 0$.

In this way we obtain:

$$\mu_n(-\varepsilon, \varepsilon) = \frac{\mathbb{E}M_{K,n}}{n} \overset{(2.21)}{=} \frac{\mathbb{E}J_K + \mathbb{E}\mathbb{1}_{\{J_K = 2k+1\}} + a_{K,n}}{n} =$$

$$\frac{\frac{\mathbb{E}J_K}{K^2}}{\left|\log^2 \varepsilon\right|} + \frac{\mathbb{E}\mathbb{1}_{\{J_K = 2k+1\}} + a_{K,n}}{n} \overset{(2.23)}{=} \frac{\frac{1}{\tau} + b_K}{\left|\log^2 \varepsilon\right|} + \frac{\mathbb{E}\mathbb{1}_{\{J_K = 2k+1\}} + a_{K,n}}{n} \overset{(2.22)}{=}$$

$$\frac{\sigma^2}{\left|\log^2 \varepsilon\right|} + \frac{1}{\left|\log^2 \varepsilon\right|}\left(b_K + \frac{1}{K^2}\left(\mathbb{E}\mathbb{1}_{J_K = 2k+1} + a_{K,n}\right)\right)$$

Recall that $\lim_{n\to\infty} a_{K,n} = 0$, and $n = K^2 \left|\log \varepsilon\right|^2$, so for fixed K for sufficiently small ε $a_{K,n} < 1$. Also, $\mathbb{E}\mathbb{1}_{\{J_K = 2k+1\}} \leq 1$. Thus, we can let $c_K = b_K + \frac{3}{K^2}$ and the lemma is proved. $\qquad\square$

We are now ready to prove our main theorem.

Theorem 2.3.10. *Let μ_H be the expected spectral measure of H as defined by Definition 2.2.2. With notation and assumptions of Lemma 2.3.9 the following holds:*

$$\mu_H(-\varepsilon, \varepsilon) = \frac{\sigma^2}{\left|\log^2 \varepsilon\right|}(1 + o_\varepsilon(1))$$

Proof. Let $n = K^2 \left|\log \varepsilon\right|^2$ be equal to $n = 2^k$ for some k, with $K > 0$ to be chosen later.

By Proposition 2.2.1 and choice of $n = 2^k$, we have $d_K(\mu_H, \mu_n) \leq \frac{1}{n}$. By the definition of the Kolmogorov distance, this implies:

$$\left|\mu_n(-\varepsilon, \varepsilon) - \mu_H(-\varepsilon, \varepsilon)\right| \leq \frac{2}{n} = \frac{2}{K^2 \left|\log^2 \varepsilon\right|}$$

By Lemma 2.3.9, there exists some $\varepsilon(K)$ such that for all $\varepsilon < \varepsilon(K)$ we have:

$$\left|\mu_n(-\varepsilon, \varepsilon) - \frac{\sigma^2}{\left|\log \varepsilon\right|^2}\right| \leq \frac{c_K}{\left|\log^2 \varepsilon\right|}$$

Using the triangle inequality and multiplying by $\left|\log \varepsilon\right|^2/\sigma^2$ we get

$$\left|\mu_H(-\varepsilon, \varepsilon) \cdot \frac{\left|\log \varepsilon\right|^2}{\sigma^2} - 1\right| \leq \frac{1}{\sigma^2}\left(\frac{2}{K^2} + c_K\right).$$

Since $\lim_{K\to\infty} c_K = 0$, for every $\delta > 0$ we can find $K(\delta)$ such that the right hand side is smaller than δ for all $\varepsilon < \varepsilon(K(\delta))$, which proves the claim. $\qquad\square$

We now discuss how the above result relates to Dyson's results from [Dys53]. Dyson considers random variables λ_j and the matrix Λ defined by $\Lambda_{j+1,j} = -\Lambda_{j,j+1} = i\lambda_j^{\frac{1}{2}}$ and zero otherwise. He then proceeds to compute the function $M(z)$, defined as the fraction of eigenvalues of Λ inside the interval $(-\sqrt{z}, \sqrt{z})$. To translate to our setting, we put $a_i = \lambda_i^{\frac{1}{2}}$ in the definition of the operator H_n and note that Λ is conjugate to H_n by a diagonal matrix A with $A_{k,k} = (-i)^{k-1}$. We then get $M(z) = \mu_H(-\varepsilon, \varepsilon)$ for $\varepsilon = \sqrt{z}$.

In Section VI, Dyson computes the asymptotics of $M(z)$ explicitly for λ_j drawn from the probability distribution $G_n(\lambda) = \frac{n^n}{(n-1)!}\lambda^{n-1}e^{-n\lambda}$, where $n \geq 1$ is an integer parameter. These asymptotics can be easily recovered from Theorem 3.1.2 by simply computing the variance Var $\log |a_i|$ with $a_i = \sqrt{\lambda_i}$. For example, for $n = 1$ λ_j are exponential random variables and by computing the variance of their logarithm we obtain $\mu(-\varepsilon, \varepsilon) \sim \frac{C}{\left|\log \varepsilon\right|^2}$ for $C = \frac{1}{4} \cdot \frac{\pi^2}{6}$, which is in agreement with Dyson's explicit computation (formula (72) in [Dys53]; the factor $\frac{1}{4}$ comes from $\varepsilon = \sqrt{z}$ in the formula for $M(z)$).

2.4 Spectral measures for groups

2.4.1 Random Schroedinger operators from group ring elements

In this section, we use random Schroedinger operators to study spectral measures of group ring elements. The construction and exposition below is based on [Gra14]. We assume familiarity with Pontryagin duality for Abelian groups.

Let Γ be a discrete group, M a discrete Abelian group and $\rho : \Gamma \curvearrowright M$ an action of Γ on M. Let $X = \widehat{M}$ denote the Pontryagin dual of M. Each $m \in M$ determines a function $\widehat{m} : X \to \mathbb{C}$, given by $m(x) = x(m)$, and by linearity we can extend this to $\mathbb{C}[M]$, i.e. to any $f \in \mathbb{C}[M]$ we associate $\widehat{f} : X \to \mathbb{C}$, which we shall call the Fourier transform of f.

Given ρ, we also have a dual action $\widehat{\rho} : \Gamma \curvearrowright X$, given by $(\widehat{\rho}(\gamma)(x))(m) = x(\rho(\gamma^{-1}) \cdot m)$. If we choose a generating set S for Γ, we can consider the Schreier graph $\mathrm{Schr}(\Gamma, X, S)$ associated to the dual action $\widehat{\rho}$.

We consider the semidirect product corresponding to the action ρ, i.e. the group $G = M \rtimes \Gamma$. Let $H \in \mathbb{C}[G]$ be a self-adjoint group ring element, which we identify with the corresponding self-adjoint operator $H : \ell^2(G) \to \ell^2(G)$. Every such element can be written as:

$$H = \sum_{\gamma} \gamma \cdot f_\gamma$$

where $f_\gamma \in \mathbb{C}[M]$. Let $\widehat{f_\gamma} : X \to \mathbb{C}$ denote the dual of f_γ.

For $x \in X$, let $\mathrm{Schr}(x)$ denote the connected component of the Schreier graph $\mathrm{Schr}(\Gamma, X, S)$ containing x. We define $H_x : \ell^2(\mathrm{Schr}(x)) \to \ell^2(\mathrm{Schr}(x))$ as the convolution operator on $\mathrm{Schr}(x)$ with the following edge labels. The label of edge from y to $\widehat{\rho}(\gamma) \cdot y$ is given by $\widehat{f_\gamma}(y)$. Note that since X is equipped with the Haar measure ν, we can treat H_x as a random operator, where x is chosen randomly according to ν.

For a self-adjoint $H : \ell^2(G) \to \ell^2(G)$, μ_H will denote its spectral measure, i.e. the unique measure such that for any $k \geq 0$, we have:

$$\langle \delta_e, H^k \delta_e \rangle = \int_{\mathbb{R}} x^k d\mu_H(x)$$

where $\delta_e \in \ell^2(G)$ equals 1 at e (the identity element) and zero otherwise.

In order to compute μ_H, we invoke the following theorem [Gra14], which gives the correspondence between the spectral measure of H and the expected spectral measure of H_x.

Theorem 2.4.1. *The spectral measure μ_H is equal to the expected spectral measure of the family H_x, i.e. for any Borel subset A:*

$$\mu_H(A) = \int_X \mu_{H_x}(A) d\nu(x)$$

In the examples we study, we shall take $\Gamma = \mathbb{Z}$ and $f_\gamma \neq 0$ only for $\gamma = a, a^{-1}$, where $a = 1$ is the standard generator of $\mathbb{Z}$. In that case, the corresponding operator H_x can be easily described. If $x \in X$ is chosen from the Haar measure ν on X, for almost every x $\mathrm{Schr}(x)$ will be isomorphic to $\mathbb{Z}$, with the edge weight from n to $n+1$ given by $\widehat{f_\gamma}(\widehat{\rho}(a^n)(x))$. Therefore H_x is a random Schroedinger operator on $\mathbb{Z}$, where the randomness in edge weights $\widehat{f_\gamma}(\widehat{\rho}(a^n)(x))$ comes from the random choice of x.

The above correspondence shows that in order to compute μ_H, it suffices to analyze the expected spectral

measure of the random Schroedinger operator H_x. We shall now perform this computation for specific examples, using results derived in previous sections.

2.4.2 Semidirect products by hyperbolic matrices

In this section we consider groups G of the following form. Let $\mathbb{Z}$ act on $\mathbb{Z} \times \mathbb{Z}$ by a hyperbolic matrix $A \in SL(2, \mathbb{Z})$. For concreteness we take $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

Let $G = \mathbb{Z} \times \mathbb{Z} \rtimes_A \mathbb{Z}$ be the corresponding semidirect product. Let s, t be the standard generators of $\mathbb{Z} \times \mathbb{Z}$ and let a denote the generator of $\mathbb{Z}$. Note that groups of this type correspond precisely to lattices in the Sol group [MS12].

We consider the switch-walk operator $H \in \mathbb{C}[G]$ shifted by $5e$:

$$H = a \cdot (s + t + s^{-1} + t^{-1} + 5e) + (s + t + s^{-1} + t^{-1} + 5e) \cdot a^{-1}$$

We shall prove the following theorem:

Theorem 2.4.2. *The spectral measure of $H \in \mathbb{C}[G]$ satisfies:*

$$\mu_H(-\varepsilon, \varepsilon) = \frac{C}{\left| \log^2 \varepsilon \right|} (1 + o_\varepsilon(1))$$

for some constant $C > 0$. In particular, the Novikov–Shubin invariant of H is equal to 0.

We start with a technical lemma needed to ensure that assumptions of the main theorem are satisfied.

Lemma 2.4.3. *Let $\{U_i\}_{i=1}^\infty$ be a stationary sequence of random variables such that $\mathbb{E}U_i = 0$, $\mathbb{E}U_i^4 < \infty$. Let $U_{i_1, \ldots, i_k} = U_{i_1} \cdot \cdots \cdot U_{i_k}$. Assume that $U_{i_1, \ldots, i_k}$ satisfy uniformly exponential correlation decay, i.e. there exist $c > 0$ and $\lambda < 1$ such that for any $l \geq 2$ and any $i_1 \leq \cdots \leq i_k \leq i_{k+1} \leq \cdots \leq i_l$ we have:*

$$\left| \mathbb{E}U_{i_1, \ldots, i_k, i_{k+1}, \ldots, i_l} - \mathbb{E}U_{i_1, \ldots, i_k} \cdot \mathbb{E}U_{i_{k+1}, \ldots, i_l} \right| \leq c\lambda^{i_{k+1} - i_k}$$

Then U_i satisfy Condition 2.3.6 with $\gamma = 4$, i.e. for some constant $C > 0$:

$$\mathbb{E}(U_1 + \cdots + U_n)^4 \leq Cn^2$$

Proof. Let us expand:

$$\mathbb{E}(U_1 + \cdots + U_n)^4 = \sum_{1 \leq i_1, \ldots, i_4 \leq n} \mathbb{E}U_{i_1, \ldots, i_4}$$

Let us order the indices so that $i_1 \leq \cdots \leq i_4$. We define a block of size l to be a maximal set of indices $I = \{i_k, i_{k+1}, \ldots, i_{k+l-1}\}$ such that $|i_{k+j} - i_{k+j-1}| \leq b \log n$ for $j = 1, \ldots, l-1$, with b chosen so that $c\lambda^{b \log n} = n^{-2}$.

Consider a term in which there is a block containing only a single index i_k. The number of such terms is at most n^4. By correlation decay, for each such term we can write:

$$\mathbb{E}U_{i_1, \ldots, i_{k-1}, i_k, \ldots, i_4} \leq \mathbb{E}U_{i_1, \ldots, i_{k-1}} \cdot \mathbb{E}U_{i_k, \ldots, i_4} + n^{-2} \leq \mathbb{E}U_{i_1, \ldots, i_{k-1}} \cdot \left(\mathbb{E}U_{i_k} \cdot \mathbb{E}U_{i_{k+1}, \ldots, i_4} + n^{-2} \right) + n^{-2}$$

Since $\mathbb{E}U_i = 0$, the contribution from a single such term is $O(n^{-2})$, so the total contribution from such terms is $O(n^2)$.

If there is no block with only a single index, then there is either one block $\{i_1, i_2, i_3, i_4\}$ or two blocks $\{i_1, i_2\}, \{i_3, i_4\}$. The number of one block terms is at most $n \cdot (b \log n)^3 = O(n^2)$, so we only need to bound the contribution from the two block terms. Note that for any k:

$$\mathbb{E}U_{i_k i_{k+1}} \leq c\lambda^{i_{k+1} - i_k}$$

Since $i_3 - i_2 > b \log n$, we have:

$$\mathbb{E}U_{i_1, i_2, i_3, i_4} \leq \mathbb{E}U_{i_1, i_2} \cdot \mathbb{E}U_{i_3, i_4} + n^{-2} \leq c^2 \cdot \lambda^{i_2 - i_1} \cdot \lambda^{i_4 - i_3} + n^{-2}$$

The second term is again bounded by the number of possible terms n^4 times n^{-2}, which gives $O(n^2)$. For the first term on the right hand side, there are n choices for i_1 and i_3, and once these are chosen, summation over possible values of i_2 and i_4 gives a geometric series bounded by $\frac{1}{1-\lambda}$, so we obtain that the total contribution is also $O(n^2)$. $\square$

Proof of Theorem 2.4.2. We apply the construction from Section 2.4.1. The Pontriagin dual of $\mathbb{Z} \times \mathbb{Z}$ is equal to $\mathbb{T} = S^1 \times S^1$, with the dual action $\widehat{\rho}$ given by $\widehat{\rho}(a)(x) = A^T x$. To simplify expressions we define $B = A^T$. Let $f = s + t + s^{-1} + t^{-1} + 5e$. The dual $\widehat{f} : \mathbb{T} \to \mathbb{C}$ is given by:

$$\widehat{f}(z_1, z_2) = 2\mathrm{Re}z_1 + 2\mathrm{Re}z_2 + 5$$

By construction from Section 2.4.1, the random Schroedinger operator corresponding to H is given as follows. The random edge weight from n to $n+1$ is given by $a_n = \widehat{f}(B^n x)$, where $x \in \mathbb{T}$ chosen uniformly from the Haar measure.

It suffices to check that the random Schroedinger operator defined as above satisfies the assumptions of Theorem 3.1.2. Writing $U_i = \log \left| \frac{a_{2i-1}}{a_{2i}} \right|$ explicitly, we obtain:

$$U_i = \log \left| \frac{\widehat{f}(B^{2i-1}x)}{\widehat{f}(B^{2i}x)} \right|$$

Letting $\phi : \mathbb{T} \to \mathbb{R}$ be equal to:

$$\phi(y) = \log \left| \frac{2\mathrm{Re}(B^{-1}y_1) + 2\mathrm{Re}(B^{-1}y_2) + 5}{2\mathrm{Re}(y_1) + 2\mathrm{Re}(y_2) + 5} \right| \tag{2.24}$$

and $S = B^2$ we can write $U_i = \phi(S^i x)$. Note that, thanks to the shift by $5e$, ϕ is a well defined C^∞-function on $\mathbb{T}$.

We first check that U_i satisfy the functional Central Limit Theorem. We say that $\phi : \mathbb{T} \to \mathbb{R}$ is a coboundary if there exists a measurable h such that $\phi = h - h \circ S$. We can invoke the functional Central Limit Theorem proved for actions of toral automorphisms in [LB99], which can be stated as follows:

Proposition 2.4.4. *Let $S : \mathbb{T} \to \mathbb{T}$ be a toral map generated by a hyperbolic matrix. Let $\phi \in L^2(\mathbb{T})$ with Fourier series:*

$$\phi(\cdot) = \sum_{k \in \mathbb{Z}^2} c_k \cdot e^{2i\pi\langle k, \cdot \rangle}$$

such that $c_0 = \int_{\mathbb{T}} \phi = 0$.

If ϕ is not a coboundary and the Fourier coefficients c_k satisfy for some $A > 0$ and $\theta > 2$:

$$\left| c_{(k_1,k_2)} \right| \leq A \prod_{i=1}^{2} \frac{1}{(1 + |k_i|)^{\frac{1}{2}} \log^{\theta}(2 + |k_i|)} \tag{2.25}$$

then the functional Central Limit Theorem holds for the sequence $\{S^i \phi\}_{i=0}^{\infty}$.

This statement is implied by the main theorem of [LB99] via Remark 1 therein.

Since ϕ as defined is (2.24) is a smooth function, its Fourier coefficients c_k decay faster than any polynomial $|k|^{-\alpha}$, so in particular condition (2.25) is satisfied.

It remains to check that ϕ is not a coboundary. To this end, it suffices to exhibit a periodic orbit $\{x, Sx, \ldots, S^{k-1}x\}$ such that $\sum_{i=0}^{k-1} \phi(S^i x) \neq 0$. Recalling that $S = B^2$ and $\phi(y) = g(B^{-1}y) - g(y)$ for $g(y) = \log|2\mathrm{Re}y_1 + 2\mathrm{Re}y_2 + 5|$, this is equivalent to:

$$g(x) + g(B^2 x) + \cdots + g(B^{2k}x) \neq g(Bx) + g(B^3 x) + \cdots + g(B^{2k+1}x) \tag{2.26}$$

Recall that $B = A^T = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. It is readily checked that the set $\{(\frac{1}{3}, 0), (\frac{2}{3}, \frac{1}{3}), (\frac{2}{3}, 0), (\frac{1}{3}, \frac{2}{3})\}$ is periodic and corresponds to the set of points $\{(e^{\frac{4}{3}i\pi}, 1), (e^{\frac{4}{3}i\pi}, e^{\frac{3}{3}i\pi}), (e^{\frac{4}{3}i\pi}, 1), (e^{\frac{2}{3}i\pi}, e^{\frac{2}{3}i\pi})\} \subseteq \mathbb{T}$. After computing both sides of (2.26) we conclude that they are not equal, which proves that ϕ is not a coboundary. Note that this is the only step of the proof where we use the specific form of the matrix A and the same proof will hold for different choices of A, provided one can prove that ϕ is not a coboundary (e.g. by exhibiting a suitable periodic orbit).

The second step is to verify that U_i satisfy Condition 2.3.6. This is taken care of by Lemma 2.4.3, provided we can establish uniform exponential correlation decay. Actions of hyperbolic matrices are well known to satisfy such exponential decay of correlations for smooth observables. We use the main theorem from [Rue76], which implies the following as a special case:

Proposition 2.4.5. *Let $S : \mathbb{T} \to \mathbb{T}$ be a toral map generated by a hyperbolic matrix. Then there exist $C, k > 0$ such that if $\phi', \phi'' : \mathbb{T} \to \mathbb{R}$ are C^1 functions, we have:*

$$\left| \mu((\phi' \circ S^{-m'}) \cdot (\phi'' \circ S^{-m''})) - \mu(\phi') \cdot \mu(\phi'') \right| \leq C \|\phi'\|_{C^1} \cdot \|\phi''\|_{C^1} \cdot e^{-k|m' - m''|}$$

where $\mu(f) = \int_{\mathbb{T}} f d\mu$ and μ is the Lebesgue measure.

We use the notation of Lemma 2.4.3. Let $l \geq 2$ and let $i_1 \leq \cdots \leq i_k \leq i_{k+1} \leq \cdots \leq i_l$ be such that

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$|i_{k+1} - i_k| \geq n$. Define:

$$\phi'(x) = U_{i_1+n,\ldots,i_k+n} = \prod_{j=1}^{k} U_{i_j+n} = \prod_{j=1}^{k} \phi(S^{i_j+n}x)$$

$$\phi''(x) = U_{i_{k+1},\ldots,i_l} = \prod_{j=k+1}^{l} U_{i_j} = \prod_{j=k+1}^{l} \phi(S^{i_j}x)$$

so that $\mathbb{E}U_{i_1,\ldots,i_k} = \mu(\phi' \circ S^{-n}) = \mu(\phi')$ and $\mathbb{E}U_{i_{k+1},\ldots,i_l} = \mu(\phi'')$.

With this notation, we have:

$$U_{i_1,\ldots,i_l} = \prod_{j=1}^{l} U_{i_j} = (\phi' \circ S^{-n}) \cdot \phi''$$

and $\mathbb{E}U_{i_1,\ldots,i_l} = \mu((\phi' \circ S^{-n}) \cdot \phi'')$. Note that $\|\phi'\|_{C^1}, \|\phi''\|_{C^1}$ are bounded by a global constant that depends only on l and $\|\phi\|_{C^1}$. Applying Proposition 2.4.5 for $m' = 0, m'' = n$ proves that the assumptions of Lemma 2.4.3 are satisfied, so U_i satisfy Condition 2.3.6. $\qquad\square$

2.4.3 Lamplighter groups

We shall now describe how a similar approach can be used for computing spectral measures of lamplighter groups. We start with the standard lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z} = \oplus_{\mathbb{Z}}\mathbb{Z}_2 \rtimes \mathbb{Z}$.

Let $\mathbb{Z}_2 = \{e, t\}$. Let e_0 denote the element of $\oplus_{\mathbb{Z}}\mathbb{Z}_2$ that has e at every position and let t_i denote the element of $\oplus_{\mathbb{Z}}\mathbb{Z}_2$ that has t at position zero and e elsewhere. Consider the switch-walk operator $H \in \mathbb{C}[\mathbb{Z}_2 \wr \mathbb{Z}]$ given by:

$$H = a \cdot (p \cdot e_0 + (1-p) \cdot t_0) + (p \cdot e_0 + (1-p) \cdot t_0) \cdot a^{-1}$$

so letting $f_a = p \cdot e_0 + (1-p) \cdot t_0$ we have:

$$H = a \cdot f_a + f_a \cdot a^{-1}$$

This operator corresponds to the random walk on G where at each step we move either left or right and then either leave the current lamp intact with probability p or change it with probability $1 - p$.

The Pontriagin dual of $M = \oplus_{\mathbb{Z}}\mathbb{Z}_2$ is equal to $X = \prod_{\mathbb{Z}}\mathbb{Z}_2$, with the Haar measure on X being the usual product measure. Since $\rho : \mathbb{Z} \curvearrowright M$ is action by translations, the dual action $\widehat{\rho} : \mathbb{Z} \curvearrowright X$ is given by $(\widehat{\rho}(a)(y))_j = y_{j-1}$, i.e. the Bernoulli shift. Therefore, the edge weights in the corresponding random Schroedinger operator will be i.i.d. with distribution determined by $\widehat{f_a} : \prod_{\mathbb{Z}}\mathbb{Z}_2 \to \mathbb{C}$. Since $\widehat{e_0} = \mathbb{1}, \widehat{t_0} = (-1)^{y_0}$, for $y \in \prod_{\mathbb{Z}}\mathbb{Z}_2$ we have:

$$\widehat{f_a}(y) = p + (1-p)(-1)^{y_0}$$

so the edge weight is equal to 1 or $2p - 1$ with probability $\frac{1}{2}$ each.

We consider $p \neq \frac{1}{2}$, as otherwise the relevant random variables $\log\left|\frac{a_{2i-1}}{a_{2i}}\right|$ are infinite with positive probability (note that $p = \frac{1}{2}$ gives edge weights 0 or 1, i.e. the edge percolation on $\mathbb{Z}$). For such an operator we can apply Theorem 3.1.2 and obtain the spectral measure at zero to be $\mu_H(-\varepsilon, \varepsilon) \approx \frac{C}{|\log \varepsilon|^2}$,

with $C = \frac{1}{4}(\log|2p-1|)^2$.

A similar approach can be used for general lamplighter groups of the form $G = \Lambda \wr \mathbb{Z}$, where the lamp group Λ is not necessarily Abelian or finite. Let Λ be generated by a set S closed under inverses and let $\lambda = \frac{1}{|S|} \sum_{s \in S} s_0$. Consider the switch-walk operator $H \in \mathbb{C}[G]$ given by:

$$H = a \cdot \lambda + \lambda \cdot a^{-1}$$

Such an operator corresponds to the random walk on G where at each step we move either left or right and then change the lamp by performing a step of a simple random walk in Λ.

In this setting, it is known (Grabowski and Virág, unpublished) that the random Schroedinger operator corresponding to H is obtained by putting i.i.d. weights on the edges, each drawn from the distribution given by μ_λ, the spectral measure of the simple random walk on Λ. As before, we are in position to use Theorem 3.1.2 and get $\mu_H(-\varepsilon, \varepsilon) \approx \frac{C}{|\log \varepsilon|^2}$.

Note that this method can be extended to H of more general form, where instead of $\lambda = \frac{1}{|S|} \sum_{s \in S} s_0$ we consider general elements $f \in \mathbb{C}[\oplus_{\mathbb{Z}} \Lambda]$, in particular, involving s_i for $i \neq 0$. In that case, the edge weights in the corresponding random Schroedinger operator will not be independent anymore. However, what remains true is the edge weights are obtained by a factor of i.i.d. process. More precisely, if f contains only s_i for $|i| \leq k$, the edge weights in the operator will be obtained as follows - we put i.i.d. labels on the edges and then each edge is assigned a weight that depends only on the labels in a neighborhood of that edge of size k. Thus, the obtained weights are weakly dependent and the same technique as in Section 2.4.2 can be used to obtain the theorem in this case.

Chapter 3

2D random Schroedinger operators and directed log-Gamma polymers

3.1   Introduction

In this paper, we study the correspondence between certain random Schroedinger operators defined on a subset of the 2D hexagonal lattice and a statistical physics model known as the log-Gamma polymer model. Directed log-gamma polymer model on a square lattice is obtained by putting random weights on the edges of the lattice, drawn from the inverse Gamma distribution, and considering up-right paths connecting the opposite corners of the square, where each path is weighted by the product of its edges. One is then interested in various statistical properties of such paths. The model has recently attracted considerable attention [Sep12],[BCR13], as it is integrable, i.e. allows explicit computations.

We construct a 2D random Schroedinger operator H and a mapping which maps its eigenvalues onto certain quantities in the directed polymer model, called the partition functions. Using results about the fluctuations of free energy for log-Gamma polymers [BCR13], we prove Tracy–Widom GUE fluctuations for the smallest positive eigenvalue of H (Theorem 3.1.2). To our knowledge this is the first known example of such fluctuations for a random Schroedinger operator. Moreover, we provide a description of higher eigenvalues in terms of partition functions related to non-intersecting paths. Such objects arise naturally in the technique knows as geometric Robinson-Schoensted-Knuth correspondence [COSZ14].

We consider a random Schroedinger operator defined on a hexagonal lattice in the shape of a rhombus. Formally, let G_n be a subset of the hexagonal lattice consisting of $2n - 1$ levels, with level k, for $k = 0, \ldots, 2n - 2$, containing $\min\{k, 2n - k - 2\}$ hexagons. The first and last level contain only a single edge. An example of such lattice for $n = 4$ is shown in figure 3.1.

We consider edges equipped with random real-valued weights, where the weight of e is denoted by w_e. The random Schroedinger operator H_n, acting on functions $f : G_n \to \mathbb{R}$, is the weighted adjacency operator on G_n:

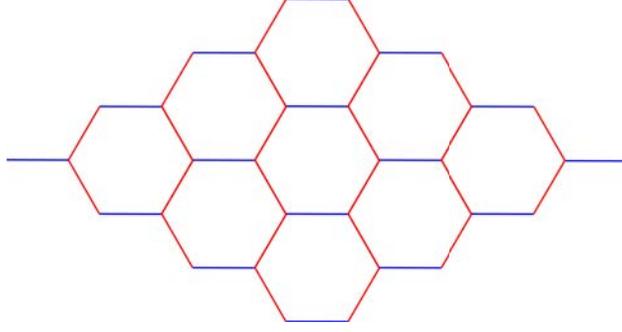$$(H_n f)(v) = \sum_{e=(v,w)} w_e f(w)$$

Figure 3.1: The lattice G_4.

where the sum is over all edges adjacent to v.

We will restrict the choice of weights to a particular distribution:

Definition 3.1.1. *A random variable X has inverse Gamma distribution with parameter $\gamma > 0$, denoted $\Gamma^{-1}(\gamma)$, if its probability distribution is supported on positive reals with density:*

$$\mathbb{P}(X \in dx) = \frac{1}{\Gamma(\gamma)} x^{-\gamma-1} \exp\left(-\frac{1}{x}\right) dx$$

Pick some parameter $\gamma > 0$. We consider two models defined on the lattice G_n:

1. **(i.i.d. model)** All edge weights are drawn independently at random from some distribution that satisfies $\mathbb{E}e^{-t \log X}, \mathbb{E}e^{t \log X} < \infty$ for some $t > 0$

2. **(mixed model)** The blue edges are given weight -1. The red edge weights are drawn independently at random from the inverse Gamma distribution $\Gamma^{-1}(\gamma)$

In the mixed model, we prove the following theorem about the smallest positive eigenvalue of H_n:

Theorem 3.1.2. *Then there exists some $\gamma^* > 0$ such that for any $\gamma \in (0, \gamma^*)$ if we let λ_n be the smallest positive eigenvalue of H_n in the mixed model with parameter γ, then:*

$$\mathbb{P}\left(\frac{-\log \lambda_n - \bar{f}_\gamma n}{n^{1/3}} \leq r\right) \to F_{\text{GUE}}\left(\left(\frac{\bar{g}_\gamma}{2}\right)^{-1/3} r\right)$$

where $f_\gamma = -2\Psi(\gamma/2)$ and $g_\gamma = -2\Psi''(\gamma/2)$, where Ψ is the digamma function.

The i.i.d. model is arguably more natural and we also expect the theorem to hold in that case:

Conjecture 3.1.1. *Theorem 3.1.2 holds also in the i.i.d. model for appropriate choice of constants.*

In both models, we also make the following conjecture generalizing Tracy–Widom fluctuations also to higher eigenvalues:

Conjecture 3.1.2. *For any fixed k, the fluctuations of $\log \prod_{i=1}^{k} \lambda_{n-i+1}$, rescaled by $n^{1/3}$, converge in distribution to the top k points of the Airy point process.*

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We make a step toward Conjecture 3.1.2 by proving that in the i.i.d.model, the leading order of the top k eigenvalues is related to the partition functions for k-tuples of non-intersecting paths (Proposition 3.3.1). Such objects appear naturally while studying exact formulas related to the geometric RSK correspondence [COSZ14].

We end this Section with a remark about properties of the inverse gamma distribution we shall use later:

Remark 3.1.3. *Let $X \sim \Gamma^{-1}(\gamma)$ and let Ψ be the digamma function. Then $\mathbb{E} \log X = -\Psi(\gamma)$ and $\mathrm{Var} \log X = \Psi'(\gamma)$. In particular, from the properties of the digamma function, for sufficiently small γ we have $\mathbb{E} \log X > 0$. Also, for small enough $t > 0$ we have $\mathbb{E} e^{-t \log X} < \infty$.*

3.2 Eigenvalues and polymers

The results in this Section are deterministic - we introduce the probabilistic part of the analysis in Section 3.3. In order to study the eigenvalues of a random Schroedinger operator on a graph G, we first prove a lemma allowing us to study instead singular values of a certain directed graph derived from G.

Lemma 3.2.1. *Let $G = (V, E)$ be a weighted bipartite graph on $2n$ vertices with bipartition $V = A \sqcup B$. Let w_e denote the weight of edge e. Suppose that G admits a perfect matching $M \subseteq E$ with edges $e_i = (a_i, b_i), a_i \in A, b_i \in B, i = 1, \ldots, n$. Let $\widetilde{G}$ be a weighted directed graph on n vertices, with vertex set M and with edges defined as follows. For each $e_i \in M$, we have a loop (e_i, e_i) with weight w_{e_i}. For each edge $f = (a_i, b_j) \notin M$, we have a directed edge (e_i, e_j) with weight w_f.*

Let A be the adjacency matrix of G and let $\widetilde{A}$ be the adjacency matrix of $\widetilde{G}$. Then the eigenvalues λ_i of A are equal to $\pm \sigma_i$, where σ_i are the singular values of $\widetilde{A}$.

Proof. Let $A = \{a_1, \ldots, a_n\}, B = \{b_1, \ldots, b_n\}$. Let us index the rows and columns of A with $(a_1, \ldots, a_n, b_1, \ldots, b_n)$. Then A has the block form:

$$A = \begin{pmatrix} 0 & \widetilde{A} \\ \widetilde{A}^T & 0 \end{pmatrix}$$

Indeed, each edge (a_i, b_i) in G corresponds to the edge (e_i, e_i) in $\widetilde{G}$, giving the diagonal entries of $\widetilde{A}$. Each edge (a_i, b_j) for $i \neq j$ corresponds to an edge (e_i, e_j) in $\widetilde{G}$, giving the off-diagonal entries. Clearly, the eigenvalues of A are equal to $\pm$ the square roots of eigenvalues of $\widetilde{A} \widetilde{A}^T$, which are simply the singular values of $\widetilde{A}$. □

We now construct a general mapping between singular values of a directed graph G and partition functions of the polymer model on the same graph. The results are stated in generality, but will be used for directed graphs derived from the particular lattice G_n described in Section 3.1.

Let G be an directed acyclic weighted graph on n vertices and let A denote its adjacency matrix. Assume that every vertex has a loop with nonzero weight, which implies that A is invertible. For $v, w \in G$, a path π from v to w is a sequence of edges connecting distinct vertices $(v = u_1, \ldots, u_n = w)$, with weight defined as:

$$\mathrm{wt}(\pi) := \prod_{i=1}^{n-1} \frac{A_{u_k, u_{k+1}}}{-A_{u_k, u_k}} \tag{3.1}$$

Definition 3.2.2. *Fix any $k = 1, \ldots, n$ and two subsets of vertices $S = \{u_1, \ldots, u_k\}$, $V = \{v_1, \ldots, v_k\}$. Consider k-tuples of vertex-disjoint paths $\pi = (\pi_1, \ldots, \pi_k)$, with π_i connecting u_i and $v_{\sigma(i)}$ for some permutation σ. Denote this permutation by $\sigma(\pi)$. We define:*

$$Z_{S,T}^{(k)} := \sum_{\pi = (\pi_1, \ldots, \pi_k)} \operatorname{sgn}(\sigma(\pi)) \prod_{i=1}^{k} \operatorname{wt}(\pi_i)$$

For $k = 1$ we will simply write:

$$Z_{u,v} = \sum_{\pi : v \to w} \operatorname{wt}(\pi)$$

In the case where the edge weights are random, the random paths are called polymers and the quantities $Z_{u,v}$ are called the partition functions of the polymer model on G.

For $u \in G$, let f_u denote the function defined by $f_u(u) = 1$ and $f_u(v) = Z_{u,v}$ for $u \neq v$. Let δ_u be the function equal to 1 on u and 0 otherwise.

Proposition 3.2.3. *The functions f_u satisfy $A f_u = \delta_u$.*

Proof. We have:

$$(A f_v)(w) = \sum_{u \to w} A_{u,w} f_v(u) + A_{w,w} f_v(w) = \sum_{u \to w} \sum_{\pi \in \Pi(v,u)} A_{u,w} \operatorname{wt}(\pi) + A_{w,w} \sum_{\pi \in \Pi(v,w)} \operatorname{wt}(\pi) =$$

$$- A_{w,w} \sum_{\sigma \in \Pi(v,w)} \operatorname{wt}(\sigma) + A_{w,w} \sum_{\pi \in \Pi(v,w)} \operatorname{wt}(\pi) = 0$$

$\square$

The quantities $Z_{S,T}^{(k)}$ can be related to f_u using the well known Lindstrom-Gessel-Viennot formula for expressing sums over non-intersecting paths as determinants:

Proposition 3.2.4. *For $S = \{u_1, \ldots, u_k\}, T = \{v_1, \ldots, v_k\}$ we have:*

$$Z_{S,T}^{(k)} = \det(f_{u_i}(v_j))_{i,j=1}^{k}$$

Let $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$ denote the singular values of A^{-1}. Note that by Proposition 3.2.3, the matrix A^{-1} expressed in the basis consisting of δ_u has the functions f_u as its columns.

Theorem 3.2.5. *For any $k = 1, \ldots, n$, we have:*

$$\max_{S,T} \left| Z_{S,T}^{(k)} \right| \leq \prod_{i=1}^{k} \sigma_i(A^{-1}) \leq \binom{n}{k}^2 \cdot \max_{S,T} \left| Z_{S,T}^{(k)} \right|$$

where the maximum ranges over all pairs of sets $S = \{u_1, \ldots, u_k\}, T = \{v_1, \ldots, v_k\}$.

Proof. We first use the following formula for the product of the singular values [Hog06]:

$$\prod_{i=1}^{k} \sigma_i(A^{-1}) = \max\{ \left| \det(U^* A^{-1} V) \right| : U, V \in \mathbb{C}^{n \times k}, U U^* = V V^* = I_k \} \tag{3.2}$$

For a set $S \subseteq \{1, \ldots, n\}$ of size k and a matrix $B \in \mathbb{C}^{n \times k}$, let B_S denote the submatrix obtained by taking rows with indices from S. With this notation I_S is the matrix having columns equal to δ_s for $s \in S$, i.e. the coordinate vectors corresponding to points in S. We have $B_S = I_S^* B$.

For the lower bound, for any $S = \{u_1, \ldots, u_k\}, T = \{v_1, \ldots, v_k\}$ we plug $U = I_S, V = I_T$ into (3.2). Since A^{-1} has the vectors f_i as columns, by Proposition 3.2.4 we have $\det((I_S)^* A^{-1} I_T) = Z_{S,T}^{(k)}$, from which the lower bound follows.

For the upper bound, for any U, V we use the Cauchy-Binet formula twice:

$$\det(U^* A^{-1} V) = \sum_S \det(U_S^*) \det((A^{-1} V)_S) = \sum_S \det(U^* I_S) \det(I_S^* A^{-1} V) =$$

$$\sum_{S,T} \det(U^* I_S) \cdot \det(I_S^* A^{-1} I_T) \cdot \det(I_T^* V)$$

Clearly, we have $|\det(U^* I_S)|, |\det(I_T^* V)| \leq 1$, so:

$$\max_{U,V} |\det(U^* A^{-1} V)| \leq \binom{n}{k}^2 \cdot \max_{S,T} |\det(I_S^* A^{-1} I_T)| = \binom{n}{k}^2 \cdot \left| \max_{S,T} Z_{S,T} \right|$$

$\square$

We will now apply the construction above to the hexagonal lattice G_n from the previous Section. In the case of G_n, the perfect matching in Lemma 3.2.1 consists of blue edges. The corresponding directed graph $\widetilde{G_n}$ is a directed square lattice with a loop added to each vertex. Both lattices are shown in Figure 3.2.
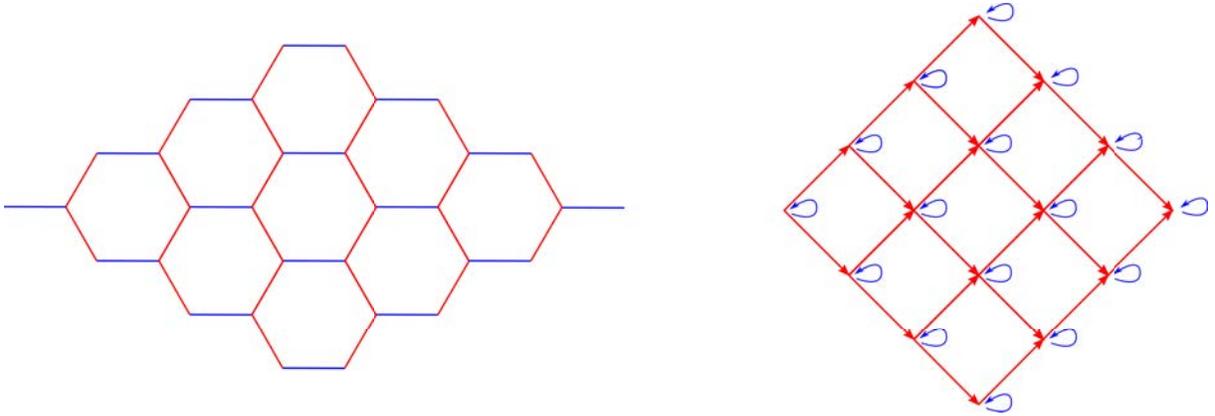

Figure 3.2: The lattice G_n and the corresponding directed lattice $\widetilde{G_n}$.

In the mixed model, blue loops have weights -1 and red edges have independent weights $w_e \sim \Gamma^{-1}(\gamma)$.

In the i.i.d. model, all edges and loops are independent drawn from the same distribution. In light of formula 3.1, this is equivalent to the following. For a loop with weight Z and outgoing edges with weights X, Y, put weight 1 on the loop and weights $X/Z, Y/Z$ on the edges. Note that in this case for an edge e with weight w_e we have that $\mathbb{E} \log w_e = 0$. Also, the edges are not independent, although they are independent along every path.

In the case of directed square lattice from $(1,1)$ to (n,n), we will write $Z_n^{(k)}$ for $Z_{S_0,T_0}^{(k)}$ with $S_0 = \{(1,1),(1,2),\ldots,(1,k)\}$, $T_0 = \{(n,n-k+1),(n,n-k+2),\ldots,(n,n)\}$.

3.3 Probabilistic results

We now introduce the probabilistic part of the analysis for the square lattice with random edge weights w_e. We do not require edge weights to be independent, only that for each path its edges are independent. For an edge e let $X_e := \log|w_e|$. For a path π we then have $\log|\mathrm{wt}(\pi)| = \sum_{e\in\sigma} X_e$. We furthermore assume that $\mathbb{E}X_e \geq 0$ and $\mathbb{E}e^{-tX_e} < \infty$ for some $t > 0$. If $\mathbb{E}X_e = 0$, we also assume that $\mathbb{E}e^{tX_e} < \infty$ for some $t > 0$.

In this Section, we prove two results. First, in Section 3.3.1, for the case $\mathbb{E}X_e = 0$, corresponding to the i.i.d. model, we show that the leading order of the top singular values of A_n^{-1} is with high probability determined by the leading order behavior of the quantities $Z_n^{(k)}$ (Proposition 3.3.1). Second, in Section 3.3.2, for the case $\mathbb{E}X_e > 0$, corresponding to the mixed model, we show that for $k = 1$ this also holds for the fluctuations, which are of order $n^{1/3}$. By known results about fluctuations of the polymer partition function, this in particular implies (Theorem 3.3.8) Tracy–Widom fluctuations of the smallest singular value of A_n, for the weights drawn from the inverse Gamma distribution.

We shall rely on the following large deviation inequalities [Dur10]. Pick a path π of length at most n. Since $\mathbb{E}X_e \geq 0$ and $\mathbb{E}e^{-tX_e} < \infty$ for some $t > 0$, we have:

$$\mathbb{P}(\sum_{e\in\pi} X_e < -\delta n) \leq e^{-I(\delta)n} \tag{3.3}$$

for some rate function I. If $\mathbb{E}X_e = 0$ and $\mathbb{E}e^{tX_e} < \infty$ for some $t > 0$, we have:

$$\mathbb{P}(\sum_{e\in\pi} X_e > \delta n) \leq e^{-I(\delta)n} \tag{3.4}$$

Whenever we say that an event holds with high probability (w.h.p.), it will mean that the probability that it does not hold is exponentially small in n.

3.3.1 Non-intersecting paths

Assume that $\mathbb{E}X_e = 0$. The goal of this Section is the proof of the following statement:

Proposition 3.3.1. *For any fixed $k \geq 1$ and $\delta > 0$, we have:*

$$\mathbb{P}\left(\frac{1}{n}\left|\log\left|Z_n^{(k)}\right| - \max_{S,T}\left|\log Z_{S,T}^{(k)}\right|\right| > \delta\right) \to 0$$

In view of Theorem 3.2.5, this means that the leading order behavior of the smallest k singular values of A_n is typically the same as of the quantities $Z_n^{(k)}$.

From now on we fix k. We will freely use factors of the form $e^{-C\delta n}$, with C being a global constant depending only on k and which may change from line to line. The large deviation inequalities 3.3, 3.4 imply that we can union bound over any subexponential number of paths to ensure that they all have weights at least $e^{-\delta n}$ and at most $e^{\delta n}$.

For a pair of sets $S = \{s_1, \ldots, s_k\}, T = \{t_1, \ldots, t_k\}$ and a permutation σ, let:

$$Z_{S,T}^{\sigma} := \sum_{\pi = (\pi_1, \ldots, \pi_k)} \prod_{i=1}^{k} \mathrm{wt}(\pi_i)$$

where the path π_i connects s_i to $t_{\sigma(i)}$. With this notation we have:

$$Z_{S,T}^{(k)} = \sum_{\sigma} \mathrm{sgn}(\sigma) Z_{S,T}^{\sigma}$$

We aim to prove that for any $\delta > 0$, we have:

$$\left| Z_n^{(k)} \right| \geq \max_{S,T} |Z_{S,T}| \cdot e^{-C\delta n}$$

Clearly we have:

$$|Z_{S,T}| \leq k! \cdot \max_{\sigma} \left| Z_{S,T}^{\sigma} \right|$$

so to prove the result it suffices to prove that w.h.p for all σ, S, T we have:

$$\left| Z_n^{(k)} \right| \geq \left| Z_{S,T}^{\sigma} \right| \cdot e^{-C\delta n}$$

Lemma 3.3.2. *Fix σ, S, T. Suppose that for some i we have that s_i and $t_{\sigma(i)}$ differ in both coordinates by at least $30k$. Then there exists a point s_i' such that s_i' is at least $10k$ to the right of s_i and, letting S' be equal to S with s_i replaced by s_i', we have w.h.p.:*

$$\left| Z_{S,T}^{\sigma} \right| \leq e^{C\delta n} \cdot \left| Z_{S',T}^{\sigma} \right|$$

The same conclusion holds if we require s_i' to be at least $10k$ above s_i. The same statement holds for moving points from T left or down.

Proof. The assumption about s_i and $t_{\sigma(i)}$ implies that s_i is away from the top and right boundary by at least $10k$. Consider the points $\{x_1, \ldots, x_n\}$ lying on the vertical line $10k$ to the right of s_i. Any path joining s_i to $t_{\sigma(i)}$ must pass through one of the points x_j. For each j, the number of paths from s_i to x_j is polynomial, so by union bound we can assume that they all have weights at most $e^{\delta n}$. Let S_j denote S with s_i replaced by x_j. In this way, we obtain:

$$\left| Z_{S,T}^{\sigma} \right| \leq n^C \cdot e^{\delta n} \cdot \sum_j \left| Z_{S_j,T}^{\sigma} \right| \leq e^{C\delta n} \cdot \max_j \left| Z_{S_j,T}^{\sigma} \right|$$

so the conclusion holds by taking s_i' to be equal to x_j achieving the maximum on the right hand side. $\qquad\square$

We now state the following Condition on σ, S, T, which will be later shown to hold w.h.p.:

Condition 3.3.3. *We have that σ, S, T are such that there is no pair of points $(s_i, t_{\sigma(i)})$ which are closer in one of the coordinates by less than $30k$.*

Lemma 3.3.4. *Let σ, S, T satisfy Condition 3.3.3 and either $S \neq S_0$ or $T \neq T_0$. Then w.h.p. there exist σ', S', T' such that:*

$$\left| Z_{S',T'}^{\sigma'} \right| \geq \left| Z_{S,T}^{\sigma} \right| \cdot e^{-C\delta n}$$

and $|S_0 \backslash S'| < |S_0 \backslash S|$ or, if $S = S_0$, $|T_0 \backslash T'| < |T_0 \backslash T|$.

Proof. Assume that $S \backslash S_0 \neq \emptyset$, as the case of modifying T is analogous. Let s be the first point from $S_0 \backslash S$. Pick any point $s' \in S \backslash S_0$ and draw a path $\tau : s \to s'$ that first goes right and then up. Such a path exists because Condition 3.3.3 there are no points in S closer to the bottom or right boundary than $30k$. Let S' be the set S with s' replaced by s.

Let $N(\tau)$ be the neighborhood of τ of size $5k$, i.e. the set of all points within distance at most $5k$ of some point on τ. By applying Lemma 3.3.2, we can assume that $N(\tau)$ contains no points from $S \backslash S_0$ or T, other than s'. Otherwise, we move offending points from S to the right and offending points from T to the left. This can be done since $N(\tau)$ is of distance at least $25k$ from the top of right border.

Pick any non-intersecting tuple Π connecting S to T. We claim that there exists a tuple Π' connecting S' to T such that $\mathrm{wt}(\Pi') \geq \mathrm{wt}(\Pi) \cdot e^{-C\delta n}$. Moreover, the mapping from Π to Π' has at most polynomially many preimages. This suffices to prove the proposition.

Let $\Pi = (\pi_1, \ldots, \pi_k)$, with π_i starting at s'. Consider the tuple $\Pi^{(0)} = (\pi_1^{(0)}, \ldots, \pi_k^{(0)})$, where $\pi_i^{(0)} = \pi_i \circ \tau$ and $\pi_j^{(0)} = \pi_j$ for $j \neq i$. The tuple $\Pi^{(0)}$ now connects S' to T. Note that $\pi_i^{(0)}$ may intersect other paths from this tuple.

A crossing between two paths π, π' is a connected set of vertices in $\pi \cap \pi'$ such that π enters and exits the intersection at different sides of π'. A tuple is non-crossing if there are no crossings between paths, although they may still be intersections which are non-crossing. For each crossing intersection in $\Pi^{(0)}$, we perform a switch, as shown in Figure 3.3, which makes the intersection non-crossing. In this way we can remove all crossings and obtain a non-crossing tuple $\Pi^{(1)} = (\pi_1^{(1)}, \ldots, \pi_k^{(1)})$. Note that the permutation σ may change to a different permutation σ'.
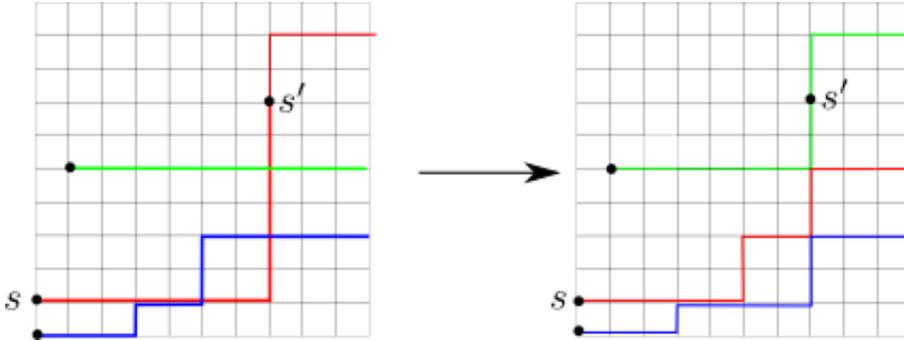

Figure 3.3: Switching paths to make them non-crossing. The path $\pi_i^{(0)}$ on the left is red.

We choose an ordering on paths from $\Pi^{(1)}$, with $\pi \geq \pi'$ if π lies above π' and list paths from $\Pi^{(1)}$ according to this ordering, with the largest element being $\pi_1^{(1)}$. We now inductively remove intersection between paths. At step i we will obtain a tuple $\Pi^{(i)}$ such that all paths $\pi_j^{(i)} \in \Pi^{(i)}$ with $j < i$ do not intersect any paths. After k steps we obtain a tuple $\Pi^{(k)}$ in which all paths are disjoint, which the required tuple $\Pi' := \Pi^{(k)}$

To pass from $\Pi^{(i)}$ to $\Pi^{(i+1)}$, consider the path $\pi_i^{(i)}$. For each intersection $\pi_i^{(i)} \cap \pi_j^{(i)}$, for some $j > i$, we remove the intersection by pushing the path $\pi_j^{(i)}$ down and to the right, obtaining a new path $\pi_j^{(i+1)}$. This is shown in Figure 3.4. Note that this may create new intersections between paths below $\pi_i^{(i)}$, but not between paths above or equal to $\pi_i^{(i)}$. We retain all the paths which were not modified, obtaining a tuple $\Pi^{(i+1)}$ in which now $\pi_j^{(i+1)}$ for $j \leq i$ do not intersect any paths.
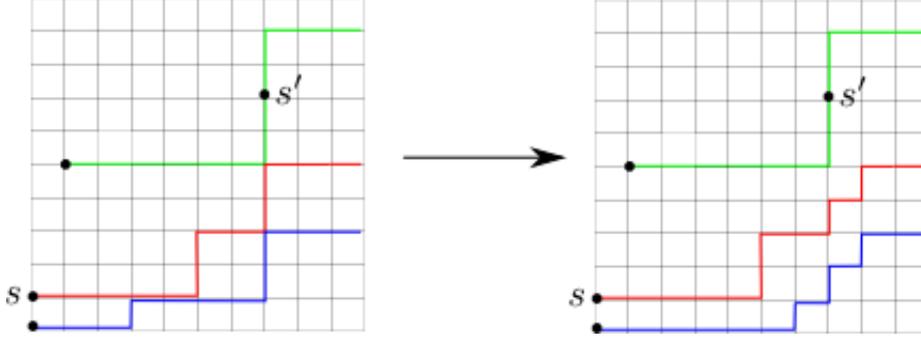

Figure 3.4: Pushing paths away to make them non-intersecting.

Let E be the set of all edges used by the initial tuple $\Pi^{(0)}$, counted with multiplicities, and let E' be the set of all edges used by $\Pi^{(k)}$. Note that $E \backslash E'$ and $E' \backslash E$ are both unions of a bounded number of connected path segments lying inside $N(\tau)$. Moreover, $|E| = |E'|$. There is a polynomial number of such segments, so we can union bound over them to ensure that the total difference in weights between $\Pi^{(0)}$ and $\Pi^{(k)}$ is at most $e^{-C\delta n}$. Likewise, the path τ chosen at the beginning has w.h.p. weight at least $e^{-\delta n}$, which proves the required inequality $\mathrm{wt}(\Pi') \geq \mathrm{wt}(\Pi) \cdot e^{-C\delta n}$. The same reasoning shows that the map taking Π to Π' has polynomially many preimages, which overall implies that:

$$\left| Z_{S',T'}^{\sigma'} \right| \geq \left| Z_{S,T}^{\sigma} \right| \cdot e^{-C\delta n}$$

as required. $\qquad\square$

Lemma 3.3.5. *With high probability we have:*

$$\left| Z_n^{(k)} \right| \geq \left| Z_n^{(k-1)} \right| \cdot e^{-C\delta n}$$

Proof. We only sketch the proof as it is similar to that of Lemma 3.3.4. Consider any tuple $\Pi = (\pi_1, \ldots, \pi_{k-1})$ contributing to $Z_n^{(k-1)}$. We augment Π to a tuple Π' with a path π_k joining $(1, k)$ to $(n, n - k + 1)$, first going right and then up. The resulting tuple Π' clearly has weight at least $\mathrm{wt}(\Pi) \cdot e^{-\delta n}$ w.h.p. It suffices to make Π' non-intersecting, which is done in a similar way as in Lemma 3.3.4, i.e. by first switching the paths to make them non-crossing and them removing the intersection by pushing the paths away. The only difference is that we start from the bottom-most path and push the paths up and to the left instead of down and right. This is needed since the path π_{k-1} may enter $(n, n - k + 2)$ from below and thus containt the point $(n, n - k + 1)$. $\qquad\square$

Proposition 3.3.6. *With high probability for all k and for all σ, S, T we have:*

$$\left|Z_n^{(k)}\right| \geq \left|Z_{S,T}^\sigma\right| \cdot e^{-C\delta n} \tag{3.5}$$

Proof. We proceed by induction on k. The case $k = 1$ is clear since for $S = \{s\}, T = \{t\}$ we can connect $(1,1)$ to s and t to (n,n) with arbitrarily chosen paths that w.h.p. have weights at least $e^{-\delta n}$.

For the inductive step, we first show that it suffices to prove 3.5 for σ, S, T that satisfy Condition 3.3.3. Indeed, suppose that σ, S, T are such that there exists a pair $(s_i, t_{\sigma(i)})$ such that on one of the coordinates the difference is less than $30k$. The number of paths from s_i to $t_{\sigma(i)}$ is at most polynomial and by union bound all of them have weights at most $e^{\delta n}$. Let $\widetilde{S}, \widetilde{T}$ denote the sets with $s_i, t_{\sigma(i)}$ removed. We thus have:

$$\left|Z_{\widetilde{S},\widetilde{T}}^{\widetilde{\sigma}}\right| \geq n^{-C} e^{-C\delta n} \cdot \left|Z_{S,T}^\sigma\right|$$

By the inductive assumption, we have $\left|Z_{\widetilde{S},\widetilde{T}}^{\widetilde{\sigma}}\right| \cdot e^{-C\delta n} \leq \left|Z_n^{(k-1)}\right|$ and by Lemma 3.3.5 we have $\left|Z_n^{(k-1)}\right| \cdot e^{-C\delta n} \leq \left|Z_n^{(k)}\right|$, which finishes the claim.

To finish the proof we repeatedly apply Lemma 3.3.4 to change any initial configuration σ, S, T into id, S_0, T_0. $\qquad\square$

3.3.2 Fluctuations of the smallest singular value

We now prove, that if $\mathbb{E}X_e > 0$, so that the weight of a typical path grows exponentially, we can establish the behavior of fluctuations of the smallest singular value of A_n. We state the following Lemma keeping in mind the special case of weights drawn from the inverse Gamma distribution, when the fluctuations are of order $n^{1/3}$.

Lemma 3.3.7. *Suppose that $\mathbb{E}X_e = \mu > 0$. Then for any $\varepsilon > 0$ we have:*

$$\mathbb{P}\left(\left|\log \max_{x,y \in G_n} |Z_{x,y}| - \log |Z_n|\right| > \varepsilon n^{1/3}\right) \to 0$$

Proof. We will prove that for every $x, y \in G_n$ such that $x \leq y$, with probability $\geq 1 - 2e^{cn^{1/3}}$ there exist two paths $\sigma : (1,1) \to x$ and $\tau : y \to (n,n)$ such that $\min\{\log \mathrm{wt}(\sigma), \log \mathrm{wt}(\tau)\} > -\frac{\varepsilon}{2} n^{1/3}$. Then every path $\pi : x \to y$ contributing to $Z_{x,y}$ can be extended to a path $\pi' = \tau \circ \pi \circ \sigma : (1,1) \to (n,n)$ and $\log \mathrm{wt}(\pi') > \log \mathrm{wt}(\pi) - \varepsilon n^{1/3}$. This implies that $Z_n \geq \sum_{\pi'} \mathrm{wt}(\pi') > e^{-\varepsilon n^{1/3}} \sum_\pi \mathrm{wt}(\pi) = e^{-\varepsilon n^{1/3}} Z_{x,y}$. Performing the union bound over possible pairs x, y shows that this holds simultaneously for all pairs x, y with probability $1 - o(1)$, thus establishing the claim.

Fix $x, y \in G_n$ and let σ be the leftmost path joining $(1,1)$ to x. We claim that with high probability $\log \mathrm{wt}(\sigma) > -\frac{\varepsilon}{2} n^{1/3}$. Let m be the length of σ and let $\mu := \mathbb{E}X_e > 0$. We let $Y_e = X_e - \mu$. By the large deviation inequality:

$$\mathbb{P}\left(\sum_{e \in \sigma} X_e \leq -\frac{\varepsilon}{2} n^{1/3}\right) \leq \mathbb{P}\left(\sum_{e \in \sigma} Y_e \leq -m\mu - \frac{\varepsilon}{2} n^{1/3}\right) \leq \exp(-mI(\mu) - \frac{\varepsilon}{2} n^{1/3}) \leq \exp(-\frac{\varepsilon}{2} n^{1/3})$$

The desired path τ joining y to (n, n) is constructed in the same way, thus proving the claimed estimate. □

We note that the proof does not carry on to the $\mathbb{E}X = 0$ case, since in that case typical paths have logarithmic weights of order $\sqrt{n}$, which is larger than the fluctuation scaling $n^{1/3}$.

We now establish the fluctuations of the smallest singular value in the case where the edge weights are drawn from the inverse Gamma distribution.

Theorem 3.3.8. *Let $\sigma_n(A_n)$ be the smallest singular value of the adjacency matrix A_n of the lattice G_n with independent inverse Gamma weights $w_e \sim \Gamma^{-1}(\gamma)$. Then there exists some $\gamma^* > 0$ such that for any $\gamma \in (0, \gamma^*)$ we have:*

$$\mathbb{P}\left(\frac{-\log \sigma_n(A_n) - \bar{f}_\gamma n}{n^{1/3}} \le r\right) \to F_{\text{GUE}}\left(\left(\frac{\bar{g}_\gamma}{2}\right)^{-1/3} r\right)$$

where $f_\gamma = -2\Psi(\gamma/2)$ and $g_\gamma = -2\Psi''(\gamma/2)$.

Proof. By Theorem 1 of [BCR13], we have for sufficiently small γ:

$$\mathbb{P}\left(\frac{\log Z_n - \bar{f}_\gamma n}{n^{1/3}} \le r\right) \to F_{\text{GUE}}\left(\left(\frac{\bar{g}_\gamma}{2}\right)^{-1/3} r\right)$$

By Remark 3.1.3, for small enough γ the inverse Gamma edge weights satisfy the assumptions of Lemma 3.3.7 and $\mathbb{E}e^{-tX_e} < \infty$ for small enough $t > 0$. Thus, we have $n^{-1/3}(\log Z_n - \log \max_{x,y \in G_n} Z_{x,y})$ converges to 0 in probability. By Theorem 3.2.5 applied for $k = 1$, for sufficiently small γ we have:

$$\max_{x,y \in G_n} Z_{x,y} \le \sigma_1(A_n^{-1}) \le n^4 \cdot \max_{x,y \in G_n} Z_{x,y}$$

Since $\sigma_n(A_n) = \frac{1}{\sigma_1(A_n^{-1})}$, this implies that $|-\log \sigma_n(A_n) - \max_{x,y \in G_n} Z_{x,y}| \le 4 \log n$, so that $n^{-1/3}(-\log \sigma_n(A_n) - \max_{x,y \in G_n} Z_{x,y})$ also converges to 0 in probability. Combining these two proves the desired convergence. □

Theorem 3.1.2 follows from Theorem 3.3.8 by invoking Lemma 3.2.1 and noting that in the mixed model, the dual graph corresponds exactly to the log gamma polymer from Theorem 3.3.8.

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