On the range of the random walk bridge on the regular tree

by

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Abstract

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Consider a nearest neighbour random walk X_n on the d-regular tree $\mathbb{T}_d$, where $d \geq 2$, conditioned on $X_n = X_0$. This is known as the random walk bridge. We derive Gaussian-like tail bounds for the return probabilities of the random walk bridge on the scale of $n^{1/2}$. This contrasts with the case of the unconditioned random walk, where Gaussian-like tail bounds exists on the scale of n.

We introduce the notion of the infinite bridge, which is known to arise as the distributional limit of the random walk bridge. We also establish some preliminary facts about the infinite bridge.

By showing that the Brownian Continuum Random Tree (BCRT) is characterized by its random self-similarity property, we prove that the range of the random walk bridge converges in distribution to the BCRT when rescaled by $Cn^{-1/2}$ for an appropriate constant C.

To my Grand-dad, who never let me forget that “ya got to get yer education”.

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Chapter 1

Introduction

Since its introduction by Aldous in [1], [2] and [3], the Brownian Continuum Random Tree (CRT) has been identified as the continuum limit of many natural families of random graphs: finite-variance critical Galton-Watson trees conditioned on the total progeny [3] multi-type Galton-Watson trees [19], critical Erdős-Rényi random graphs (after removing a constant order number of edges) [14], and random graphs with a prescribed degree sequence [20]. Indeed Aldous [2] conjectures that for any 'reasonable' model for random trees on n vertices, which has diameter $\approx n^{1/2}$, the scaling limit is either the BCRT or can be simply derived from it or its related processes.

Let X_n be a random walk on a transitive graph or a symmetric random walk on a Cayley graph, and let B_n be the related random walk bridge: $B_n = (X_0, X_1, \dots, X_n)$ given $X_n = X_0$. Let T_n be the range of B_n: the subtree with vertices $\{B_n(0), B_n(1), \dots, B_n(n)\}$. Let R_n be the number of vertices in T_n: $R_n := |\{B_n(0), B_n(1), \dots, B_n(n)\}|$. In [5], it is shown that, under certain assumptions about the asymptotic return probability $\mathbb{P}[X_n = X_0]$, R_n satisfies a weak law of large numbers:

$$\frac{R_n}{n} \xrightarrow{p} 1 - F(R), \tag{1.1}$$

where F is the generating function of the first return time of X_n to X_0, and R is its radius of convergence. Moreover, as a consequence of [6], in the case of a simple random walk on a regular tree, the diameter of T_n is on the order of $n^{1/2}$. Hence, it is reasonable to believe that the range T_n of a random walk bridge on a regular tree converges in distribution to the BCRT. We give a positive answer in the case of a nearest neighbour random walk bridge on the free product $\Gamma_d = \mathbb{Z}_2 * \cdots * \mathbb{Z}_2$ of d copies of $\mathbb{Z}_2$. The case of a nearest neighbour random walk bridge on the free group will follow. Note that the Cayley graph of Γ_d with the generating set $A = \{a_1, \dots, a_d\}$ is a d-regular tree.

Theorem 1.0.1. *Let B_n be a nearest neighbour random walk bridge on Γ_d, and let T_n be the subtree with vertices $\{B_n(0), B_n(1), \dots, B_n(n)\}$. Then $\frac{T_n}{\sqrt{n}}$ converges in distribution to a BCRT, rescaled by a constant depending on the law of X_n.*

We specify the topology of this convergence in Section 2.0.3.

We prove Theorem 1.0.1 by using the self-similarity property of the BCRT from [4]. We take **Brownian scaling** by λ of a measured metric space (X, μ) to mean rescaling the measure μ by λ, and the metric d_X by $\sqrt{\lambda}$.

Theorem 1.0.2 ([4]). *Given a doubly-roote BCRT (T, μ, v_1, v_2), sample an independent vertex v_3 from μ. Let $v = b(v_1, v_2, v_3)$ be the branchpoint of these three vertices, and let T_i be the component of $T/\{v\}$ containing v_i. Set $\Delta_i = \mu(T_i)$. Then*

(a) $\Delta = (\Delta_1, \Delta_2, \Delta_3)$ *is a Dirichlet-*$(1/2, 1/2, 1/2)$ *random variable.*

(b) The doubly-rooted CRT $\tau_i = \left(\frac{T_i}{\sqrt{\Delta_i}}, \frac{\mu|_{T_i}}{\Delta_i}, v, v_i\right)$ *is a BCRT.*

(c) $(\Delta, \tau_1, \tau_2, \tau_3)$ *are independent.*

We show that, under a suitable moment condition on the typical distance of two points, the self-similar property of Theorem 1.0.2 characterizes the BCRT:

Theorem 1.0.3. *Let (T, μ) be a continuum random tree satisfying the conclusion of Theorem 1.0.2. Assume further than $\mathbb{E}d(v_1, v_2) = \sqrt{\frac{\pi}{2}}$ and $Var(d(v_1, v_2)) = \sigma^2 < \infty$, where v_1, v_2 are iid samples from μ. Then (T, μ) is the BCRT.*

Roughly speaking, splitting at the branchpoint of three random points of the range of the random walk bridge T_n gives an approximate decomposition of T_n in terms of the range of three shorter random walk bridges. The well-known return probabilities

$$\mathbb{P}[X_n = X_0] \sim C \frac{R^{-n}}{n^{3/2}} \tag{1.2}$$

for a random walk on a tree give the required Dirichlet-$(1/2, 1/2, 1/2)$ statistics for the joint distribution of the measure of the smaller bridges. Using finer, uniform asymptotics of $\mathbb{P}[X_n = x_n]$ due to Lalley [15], we show that the limit of $T_n/\sqrt{n}$ satisfies the hypotheses of Theorem 1.0.3.

Remark 1.0.1. In [6] something close to Theorem 1.0.1 is proved for the range of a *simple* random walk bridge on a regular tree: it concludes that the distance function $\frac{|B_n|}{\sqrt{n}}$ of a *simple* random walk bridge on $\mathbb{T}_d$, with Brownian scaling, converges to a Brownian excursion. To turn this into a proof, one would simply need to show that the distance function $\frac{B_n}{\sqrt{n}}$ is a good enough approximation of the depth-first search on the range. Nevertheless, [6] relied on finding a simple random walk excursion on the integers contained in $(|B(i)|)$; for a general random walk, the process $(|B(i)|)$ is not a Markov process, and such an exact method does not work.

Remark 1.0.2. Duquesne proves a related result to Theorem 1.0.1 in [8]. Specifically, they consider the range τ_ϵ of a random walk $X_n^{(\epsilon)}$ on a d-ary tree such that X_n^ϵ moves towards the root with probability $1/2 - \epsilon$ and away from the root with probability $1/2 + \epsilon$. Then $\epsilon\tau_\epsilon$ converges in distribution to a sin tree – that is, a continuum random tree encoded by two independent linear Brownian motions with drift conditioned to stay positive forever, rescaled in time by a constant factor γ depending on the downward jump distribution.

Remark 1.0.3. A stronger version of Theorem 1.0.3 was proven independently in [18]. They show that the BCRT is the unique fixed point of a particular operator $\mathscr{F}$ related to its recursive self-similarity. Their result includes Theorem 1.0.3, but doesn't seem to be easily adaptable to prove the stronger version, Theorem 3.2.1. They also show that under appropriate technical assumptions, the orbit of a CRT (T, μ) under $\mathscr{F}$ converges to the BCRT.

Remark 1.0.4. We offer two additional proofs of Theorem 1.0.1 in Appendices A and B.

First, we compute directly the limiting distribution of the k-point statistics. The mathematical content is not very different from the proof of Theorem 3.2.1, aside from some additional technical details which vanish in the limit. The main difference is that the 2-point statistics must be computed first. This can of course be remedied by mimicking the derivation of the 2-point statistics in the proof of Theorem 3.2.1.

The second considers the distance function $|B_n(\cdot)|$ as the contour function of a coloured tree, which we recognize as a special case of a multi-type Galton-Watson tree. By [19], the height process of such trees converge to the BCRT. We relate the height process to the contour function of T_n through the distance function $|B_n(\cdot)|$.

In the context of random walks, the notion of the 'infinite bridge' as a time-homogeneous Markov process was introduced in [17] for a simple random walk on a d-regular tree. The infinite bridge $(X_0^\infty, X_1^\infty, \ldots)$ is defined as the distributional limit as $n \to \infty$ of B_n. In fact, the existence of the infinite bridge on any Gromov hyperbolic group follows from asymptotics of $\mathbb{P}[X_n = X_0]$ obtained in [11] (See also [22]). We prove this in Chapter 5.

The contour function of the self-similar continuum random tree (SSCRT) is the two-sided Bessel(3) process $\tilde{W}^+$: that is, $(\tilde{W}_t^+)_{t\geq 0}$ and $(\tilde{W}_{-t}^+)_{t\leq 0}$ are independent Bessel(3) processes. We call the continuum tree encoded by a one-sided Bessel(3) process a 'half' a SSCRT.

Conjecture 1. *Let B_∞ be the infinite random walk bridge on a non-degenerate Gromov hyperbolic group. Let $T_\infty = \{B_\infty(0), B_\infty(1), \ldots\}$ be the range of B_∞. Then ϵT_∞ converges in distribution to half a SSCRT as $\epsilon \to 0$.*

Chapter 2

Preliminaries

define $\xrightarrow{p}$, $\Rightarrow$

2.0.1 General notation

With respect to estimates, we typically disregard the particular value of constants. In these cases, we will use the notation

$$A \preceq B$$

to mean that there is a constant C such that $A \leq C \cdot B$. The constant C should be universal over the parameters of the expressions A and B, but may depend on the law of the random walk.

We use

$$A_n \sim B_n$$

to mean there is a constant $C > 0$ such that $\lim_{n\to\infty} \frac{A_n}{B_n} = C$, and

$$A \asymp B$$

to mean there is a constant $C > 0$ such that $A/C \leq B \leq C \cdot A$.

We use the notation $f(x) = o\left(g(x);\ x \to a\right)$ to mean

$$\lim_{x\to a} \frac{f(x)}{g(x)} = 0.$$

Likewise, we say that $f(x) = O\left(g(x);\ x \to a\right)$ when there is a $\delta > 0$ and $C < \infty$ such that

$$\left|\frac{f(x)}{g(x)}\right| \leq C$$

for some constant C, whenever $|x-a| \leq \delta$ if a is finite, or whenever $x > 1/\delta$ when a is ∞.

To avoid cumbersome notation, when t is real, we write $f(t)$ instead of $f(\lfloor t \rfloor)$ when f takes integer arguments.

We will sometimes use the Gromov-Hausdorff metric applied to marked metric spaces. Let X, Y be subsets of some metric space M, and $x_1, \ldots, x_k$ (resp. $y_1, \ldots, y_k$) be elements of X (resp. Y). We define the Hausdorff distance between $X' := (X, x_1, \ldots, x_k)$ and $Y' := (Y, y_1, \ldots, y_k)$ to be

$$d_H(X', Y') := d_H(X, Y) + d(x_1, y_1) + \cdots + d(x_k, y_k),$$

where d_H is the usual Hausdorff metric. The Gromov-Hausdorff metric between general X' and Y' is the obvious extension of the Hausdorff metric to general metric spaces.

2.0.2 Finite trees

We think of a finite tree as a graph-theoretic tree (V, E). Edges $e \in E$ may have some length $\ell(e)$ that varies with e.

We consider ordered trees and labelled trees. An ordered tree is a tree $T = (V, E)$ along with a strict total order $\prec$ that satisfies

1. The minimal vertex ρ is called the **root.**
2. A vertex v is said to be a **descendant** of u if u lies on the path from ρ to v. If u is adjacent to v, in this case v is said to be a *child* of u.
3. If v is a descendant of u, then $u \prec v$.
4. If v is not a descendant of u, and $u \prec v$, then $w \prec v$ for any descendant w of v.

For instance, if we records the order of births in a Galton watson tree conditioned on extinction, we obtain a random ordered tree in a natural way.

If we label the vertices of V, for instance, with $\{1, 2, \ldots, |V|\}$, then we call T a **labelled tree**.

A **measured tree** is a finite tree T_n along with a probability measure p_n on the vertices of T_n. We always assume $\mathrm{supp}(p_n) = V(T_n)$.

For each triple $v_1, v_2, v_3 \in V$, we let $b(v_1, v_2, v_3)$ be the branchpoint of the three vertices, i.e., the unique point located located on each of the three (unique) geodesic paths formed by the three vertices.

A **binary ordered tree** is an ordered tree such that every branchpoint has two children.

A *leaf-labelled* tree is a finite tree whose leaves each have a unique label. When the branchpoints also have a unique label, the tree is called a *labelled tree*.

We use $\mathbb{P}_{\mathscr{T},k}$ to denote the uniform measure on the set of leaf-labelled, binary trees with k leaves.

2.0.3 Continuum trees

A **real tree**[1] is a closed, rooted geodesic space T such that between every two points x, $y \in T$, there is a unique geodesic between x and y, denoted as $[[x, y]]$. That is, there is a unique isometry $f : [0, d(x, y)] \to T$ with $f(0) = x$ and $f(d(x, y)) = y$. Call $]]x, y[[:= f((0, d(x, y)))$ the **interior** of the geodesic $[[x, y]]$.

We define the **skeleton** S of T to be the collection of all points on the interior of some geodesic:

$$S := \bigcup \{f((0, x)) \mid \ f : [0, x] \to T \text{ is an isometric embedding}\}.$$

We call elements of T/S **leaves**. A point $b \in T$ is a **branchpoint** if b lies on at least two geodesics. T is **binary** if every branchpoint separates exactly two connected components from ρ.

A **continuum random tree** (CRT) is a real tree T along with a non-atomic probability measure μ satisfying

1. $\mu(T/S) = 1$;
2. $\mu\{y \mid x \in]]\rho, y[[\} > 0$ for each $x \in S$, where ρ is the root.

Any real tree T equipped with a probability measure μ satisfying $\mu(T/S) = 1$ can be turned into a CRT by pruning off branches of measure 0, ensuring the second condition is satisfied.

When one special vertex v are specified, the triple (τ, μ, v) is called a *rooted* CRT. When two special vertices v_1, v_2 are specified, the quadruple (τ, μ, v_1, v_2) is called a *doubly rooted* CRT. We always enforce that $v \sim \mu$ and $(v_1, v_2) \sim \mu^{*(2)}$ in these cases.

[1] Also known as $\mathbb{R}$-trees, or metric trees.

2.0.4 Ordered trees, labelled trees and contour functions

Given a finite ordered tree T (with unit edge lengths), we can define the **contour function** as follows: have a particle start at the root and traverse the tree at unit speed, visiting the leaves according to their order. The contour function $T(t)$ is the distance to the root of the particle at time t. This map is a bijection between ordered trees and nonnegative nearest neighbour walk paths – thinking now about a particle constructing the tree, for each upward step of the walk, the particle creates a new edge and vertex which is later in the order than everything already created, and for each downward step, the particle moves one vertex closer to the root. [2]

If T does not necessarily have unit edge lengths, we define the contour function in the same way, except that the particle moves along the edge e at speed equal to the length $\ell(e)$ of e.

In the continuum case, Theorem 15 of [3] implies that, assuming a certain compactness condition on T, there is a 1-1 correspondence between measures on ordered continuum trees T and continuous functions $f : [0,1] \to \mathbb{R}+$ satisfying Condition 3.1.1. We only deal with trees which satisfy the required compactness condition.

In either case, as there is a bijection between contour functions and ordered trees, we use the symbol T to refer to both the tree and its contour function.

2.0.5 Reduced subtrees

Given a finite subset V of vertices of a (finite or continuum) tree T, we use the notation $r(V)$, to denote the reduced subtree associated with V. The reduced subtree $r(V)$ is the tree which has as its vertices V along with all branchpoints of points $b(v_1, v_2, v_3)$ for $v_i \in V$, and which has as its graph structure the induced graph structure from T: i.e., there is an edge between two vertices v_1, v_2 in $R(V)$ if there is a path in T from v_1 to v_2 which passes through no other vertices of $R(V)$. Each edge (v_1, v_2) is given a length equal to the distance in T between v_1 and v_2.

Such a tree is characterized by its combinatorial type as a labelled tree and the edge lengths. We will use the notation $(\mathscr{T}, x_1, \dots, x_k)$ to describe such trees, where $\mathscr{T}$ is the combinatorial shape of the tree, and $x_1, \dots, x_k$ are the edge lengths. As we will be working with reduced subtrees with exchangeable edge length distributions, we do not worry about the order in which we record the edge lengths x_i.

If T is an ordered tree, then we may consider $R(V)$ to be an ordered tree by inheriting

[2] A cute alternative description: apply glue underneath the nearest neighbour path, press it together like an accordion, let the glue dry and unfold.

the order from T.

If $(R(k), k \geq 1)$ is a family of proper trees with k leaves, we say that $R(k)$ forms a **consistent family** if, after deleting a uniform vertex from $R(k)$, the resulting tree is equal in distribution to $R(k-1)$.

If (T, μ) is a continuum tree, we define its k**-point statistics** as follows: let v_1, v_2, $\ldots$, v_k be iid samples from μ, and let $V = \{v_1, \ldots, v_k\}$. Then the k-point statistics are $R_k(T, \mu) := R(V)$. Note that the family $R_k(T, \mu)$ is automatically consistent.

2.0.6 Random walks

For a group Γ with finite generating set S, the (right) Cayley graph G of Γ is the graph with vertex set $V(G) = \Gamma$, and with edge set $E(G) = \{(g, gs) \mid g \in \Gamma, s \in S\}$. Note that G is invariant under left multiplication by any group element.

Let $A = \{a_1, \ldots, a_d\}$ be an alphabet, and consider the group

$$\Gamma = \langle a_1, \ldots, a_d \mid a_1^2, \ldots, a_d^2 \rangle,$$

the free product of d copies of $\mathbb{Z}_2$. The (right) Cayley graph of G with respect to the generating set A is the d-regular tree $\mathbb{T}_d$. Denote the root, i.e. the identity element, by ρ. We equip $\mathbb{T}_d$ with the usual graph metric – we extend it from a discrete metric space to a continuous metric space by making each edge disjoint unit intervals, and gluing the unit intervals at the endpoints.

Remark 2.0.1. We restrict to the case of random walks on free products of $\mathbb{Z}_2$ to match the assumption of [15]. However, the same techniques should apply to random walks on free groups.

Note that the law of a symmetric random walk on a free group of rank d can be obtained from random walks on the free product of $2d$ copies of $\mathbb{Z}_2$. More yet, the law of the asymmetric random walk bridge with transition probability $\mathbb{P}[X_i = a_{\pm i}] = p_{\pm i}$ is equal in distribution to the symmetric random walk bridge with transition probabilities $\mathbb{P}[X_1 = a_{\pm i}] = \sqrt{p_i p_{-i}}$, so there is no loss of generality in the restriction to free products of $\mathbb{Z}_2$.

We will use X_n to denote the n-th step distribution of a lazy random walk on $\mathbb{T}_d$, i.e.

$$\mathbb{P}[X_{n+1} = x \mid X_n = x] = p_0, \;\; \mathbb{P}[X_{n+1} = xa_i \mid X_n = x] = p_i,$$

where $p_0 + p_1 + \cdots + p_d = 1$. We insist that $p_0 > 0$: adding laziness doesn't change the problem at all, but simplifies notation, and avoids the present parity issue. For $n \geq 0$, let B_n be the n-step random walk bridge, i.e.

$$B_n = (B_n(0), \dots, B_n(n))$$

has the distribution of $(X_0, \dots, X_n)$ conditioned on $X_n = 0$. We sometimes refer to random walk bridges simply as 'bridges', not to be confused with the Brownian bridge.

We say that n the **duration** of an n-step random walk bridge B_n. We denote the measure on bridges of duration n by $\mathbb{E}_n$.

The random walk bridge is invariant in distribution under time reversal and translation by certain times. Namely,

$$B_n(t) \stackrel{d}{=} B_n(n-t),$$

and

$$B_n(t) \stackrel{d}{=} B_n(T)^{-1} B_n \left(t + T \mod n\right)$$

if T is independent of B_n.

When discussing random walk bridges of length n, we will identify the times n and 0, and apply a 'clockwise' order on $\{0, 1, \dots, n-1\}$: say that $a \leq b \leq c$ if $b \in \{a, a+1, \dots, c\}$ (mod n).

We use T_n to mean the **range** of the random walk bridge B_n, that is $T_n = \{B_n(0), \dots, B_n(n)\}$. By ordering the vertices $v \in T_n$ according to the first hitting times by the process B_n, there is a natural ordering on T_n. As mentioned in Subsection 2.0.4, we also use T_n to refer to its contour function.

We often make use of the *initial part* of a random walk bridge. The initial part $\text{Init}_M(B_n)$ of a random walk bridge B_n is the process

$$B_n(0)^{-1} \left(B_n(0), B_n(1), \dots, B_n(M)\right) \times B_n(0)^{-1} \left(B_n(n-0), B_n(n-1), \dots, B_n(n-M)\right)$$

In other words, $\text{Init}_M(B_n)$ is the first and last M steps taken by the random walk.

Chapter 3

Characterization of the BCRT

3.1 Contour functions and random finite-dimensional distributions

Recall that a finite ordered tree is determined by its contour function. In fact, the metric on the tree is determined from its nearest neighbour path $S.$ as follows: if the walk is at vertices v_1, v_2 at respective times i_1, i_2 in the walk $S.$, then

$$d_T(v_1, v_2) = S_{i_1} + S_{i_2} - 2 \min_{i_1 \le i \le i_2} S_i.$$

If i^* is a minimizer of $S_{i_1} + S_{i_2} - 2\min_{i_1 \le i \le i_2} S_i$, then the particle performing the contour walk is visiting the branchpoint of the vertices visited at times i_1 and i_2.

It is well known that a suitably rescaled non-negative simple random walk bridge converges in distribution to a Brownian excursion e_t of duration 1. In some sense, then, we can think of the Brownian excursion as the contour function of the BCRT. In fact, one construction of the BCRT is the metric space $([0,1]/\sim, d_{2e})$, where $t_1 \sim t_2$ if both $e_{t_1} = e_{t_2}$ and $e_t \ge e_{t_1}$ for $t \in (t_1, t_2)$, and

$$d_{2e}(t_1, t_2) := 2\left(e_{t_1} + e_{t_2} - 2 \min_{t_1 \le t \le t_2} e_t\right).$$ [1]

In addition, the Lebesgue measure m on $[0,1]$ induces a measure μ on $[0,1]/\sim$, and the order on $[0,1]$ induces a natural total order $\prec$ on $[0,1]/\sim$. Namely, when $[t]$ is the equivalence class of t, we say $[t_1] \prec [t_2]$ if $\min[t_1] < \min[t_2]$ – ie. the contour process e_t

[1] Whether one uses e_t or $2e_t$ is a matter of convention, with the latter arising more naturally as the 'standard' BCRT.

'visits' the point $[t_1]$ before the point $[t_2]$.

More generally, we can consider continuum trees constructed as above from functions $f : [0,1] \to [0,\infty)$ satisfying the following conditions (Condition 1 of [3]):

Condition 3.1.1.

- f is continuous.
- $f(0) = f(1) = 0$ and $f(t) = 0$ for at most one t in $(0,1)$.
- The set of times of strict local minima is dense.
- If t_1, t_2 are strict local minima with $f(t_1) = f(t_2)$, then

$$\inf_{t_1 \le t \le t_2} f(t) < f(t_1).$$

- The set of one-sided local minima has Lebesgue measure 0.

Under some technical assumptions, which we do not discuss, any ordered continuum random tree can be encoded by some (random) continuous function satisfying Condition 3.1.1.

Theorem 3.1.2 ([3], Theorem 15)**.** *Let $(R(k), k \ge 1)$ be a consistent family of random binary trees. The following are equivalent.*

1. *There exists an ordered continuum random tree $(T, \mu, \prec)$, which is 'order-leaf-dense' whose k-point reduced subtrees are distributed as $R(k)$.*

2. *There exists a random function f satisfying Condition 3.1.1 such that $([0,1]/\sim, d_f, \prec)$ is an ordered continuum random tree whose k-point reduced subtrees are distributed as $R(k)$.*

Moreover, we can infer convergence of the contour function of a family of trees by the convergence of k-point statistics.

Theorem 3.1.3 ([3], Theorem 20)**.** *Let (T_n, μ_n) be a sequence of ordered graph-theoretic trees T_n along with the uniform measure μ_n on $V(T_n)$. Then the rescaled contour functions $\frac{T_n(nt)}{\sqrt{n}}$ converge to twice a Brownian excursion in the sense of weak convergence in $C[0,1]$ if and only if*

(a) The k-point statistics $R_k(T_n, \mu_n)$ converge in distribution to the k-point statistics $R_k(T, \mu)$ of the BCRT.

(b) $\delta(k) := \limsup_{n\to\infty} \max_{v\in V(T_n)} \min_{r\in R_k(T_n,\mu_n)} d(v,r) \xrightarrow{p} 0$ *as* $k \to \infty$.

There is a slight difference in our setup: for our proofs, we do not choose uniform vertices as in Theorem 20 of [3], but rather from a random weighted measure p_n induced by the . local time of the random walk bridge.

We can recover Theorem 3.1.3 for general p_n satisfying an asymptotic nonatomicity condition (see Remark 4 of Section 3.3 of [3]) by distorting the contour function accordingly. Let (T_n, p_n) be a sequence of measured trees, where $|V(T_n)| = n$, and consider the non- rescaled contour function $T_n := T_n$. Rescale the height of the contour function by $\sqrt{n}$ as normal, but rescale the interval $[i, i+1]$ by the measure $p_n(v)$, where v is the child in the $T_n(i)$, $T_n(i+1)$ relationship. Then the proof of Theorem 3.1.3 goes through unchanged.

Since we would like to prove convergence of the undistorted contour function, our approach is to couple a vertex V_n sampled from p_n with a vertex U_n sampled from the uniform measure μ_n of T_n such that $d(U_n, V_n) \xrightarrow{p} 0$. Then one can show that the Hausdorff distance of $R_k(T_n, p_n)$ and $R_k(T_n, \mu_n) \xrightarrow{p} 0$, which implies convergence of the true contour function.

Remark 3.1.1. Le Gall showed in [9] that the Gromov-Hausdorff distance of two real trees is bounded by the sup-norm of the difference between their contour functions. Hence, either of the above imply that the trees T_n converge in distribution to the Brownian CRT in the Gromov-Hausdorff metric on metric spaces.

However, convergence of the contour function of ordered trees is a stronger form of convergence than Gromov-Hausdorff convergence: for instance, given a measured tree T_n, on n vertices, connect ϵn new vertices to any fixed vertex v of T_n to obtain a new tree T'_n. Then T'_n converges in distribution to the BCRT in terms of Gromov-Hausdorff convergence, but the contour function of T'_n will have a flat piece of length $\epsilon/(1+\epsilon)$.

3.2 Self-similar characterization

In [3], it is shown that the distribution of the k-point statistics of the BCRT,

$$R_k(T, \mu) = (\mathscr{T}, X_1, \ldots, X_{2k-3}),$$

is the product measure of $\mathbb{P}_{\mathscr{T},k}$ and the vector with density $se^{-s}ds$, where $s = x_1 + \ldots, x_{2k-3}$. We give an alternate description of its distribution: $(X_1, \ldots, X_{2k-3})$ is equal in distribution distribution to

$$2\left(R_1\sqrt{\Delta_1},\ldots,R_{2k-3}\sqrt{\Delta_{2k-3}}\right), \tag{3.1}$$

where R_i are Rayleigh distributed, $(\Delta_1,\ldots,\Delta_{2k-3})$ has the Dirichlet-$(1/2;k)$ [2] distribution, and everything is independent. The main ingredient is the recursive self-similarity of the BCRT (Theorem 1.0.2). Indeed, it follows immediately by recursively applying Theorem 1.0.2 along with Proposition 6.1.1.

In fact, under suitable moment conditions, a weaker version of this self-similarity property characterizes the BCRT, since we can obtain the k-point statistics from the 3-point statistics and self-similarity.

Theorem 3.2.1. *Let $\mathscr{P}=\{\mathbb{P}_\alpha\}_{\alpha\in A}$ be a family of probability measures on doubly-rooted ordered continuum trees. Assume that for any probability measure $\mathbb{P}\in\mathscr{P}$, if $\tau=(T,\mu,V_1,V_2)$ is a sample from $\mathbb{P}$, that*

A1. Given (T,μ), the joint distribution of (V_1,V_2) are independent samples from μ.

Let $V_3\in T$ be a third sample from μ, independent from (V_1,V_2).

A2. Splitting T at the branchpoint v of V_1, V_2 V_3 disconnects T into three sub-trees $T_1^{(3)}$, $T_2^{(3)}$, $T_3^{(3)}$.

A3. The sizes $\Delta^{(3)}=\left(\mu\left(T_1^{(3)}\right),\mu\left(T_2^{(3)}\right),\mu\left(T_3^{(3)}\right)\right)$ are jointly distributed as a Dirichlet-$(1/2,1/2,1/2)$ random variable.

Let

$$\tau_i^{(3)}=\left(\frac{T_i^{(3)}}{\sqrt{\mu\left(T_i^{(3)}\right)}},\frac{\mu\mid_{\mu\left(T_i^{(3)}\right)}}{\mu\left(T_i^{(3)}\right)},V_i,b\right)$$

be the Brownian-rescaled doubly-rooted versions of $T_i^{(3)}$.

A4. There is a one-parameter family of measures $(\mathbb{P}_x)_{x\in[0,1]}\subset\mathscr{P}$ such that given $\Delta^{(3)}$,

$$\left(\tau_1^{(3)},\tau_2^{(3)},\tau_3^{(3)}\right)\sim\mathbb{P}_{\Delta_1^{(3)}}\times\mathbb{P}_{\Delta_2^{(3)}}\times\mathbb{P}_{\Delta_3^{(3)}}.$$

Furthermore, assume that

[2] We use Dirichlet-$(1/2;k)$ as a short-hand for the Dirichlet-$\overbrace{(1/2,\ldots,1/2)}^{k}$ distribution.

B1. $\mathbb{E}[d(V_1, V_2)] = \sqrt{\pi/2}$.

B2. *Var*$(d(V_1, V_2)) \le \sigma^2$ *for some* $\sigma^2 > 0$*, uniformly over* $\mathscr{P}$.

B3. $\mathbb{P}_x$ *is continuous in* x*:* $\mathbb{E}_x f(\tau) = \mathbb{E}_y f(\tau) + o(1;\ x \to y)$.

B4. For each x*,* $\mathbb{P}_{x,y}$ *is multiplicative in the following sense: for each* x*, the one-parameter family* $(\mathbb{P}_{x,y})_{y\in[0,1]}$ *satisfies* $\mathbb{P}_{x,y} = \mathbb{P}_{xy}$.

Then (T, μ) *is the BCRT.*

Remark 3.2.1. Our argument is more complicated than that of [18], but we don't need to assume exact self-similarity, which will be useful when applying to convergent subsequences of discrete trees.

Proof. We need to show that, for the k-point reduced subtree $R_k = R_k(T, \mu)$,

(i) The shape of R_k is distributed as $\mathbb{P}_{\mathscr{T},k}$.

(ii) The edge statistics R_k of T are distributed according to (3.1).

(iii) The shape and edge statistics of R_k are independent.

To do this, we will use the self similarity of an actual BCRT (T, μ). We know that the shape of a BCRT is distributed as $\mathbb{P}_{\mathscr{T},k}$. Consider the k-point reduced subtree $R_k(T, \mu)$ of the BCRT. Let $(V_1, V_2, \ldots, V_k)$ be a uniform ordering of the leaves V of the reduced subtree. Then V_1, V_2, V_3 are iid choices from μ, and hence, splitting at the branchpoint $v := b(V_1, V_2, V_3)$ breaks T into three smaller trees $T_1^{(3)}, T_2^{(3)}, T_3^{(3)}$ whose Brownian rescaled versions are distributed as independent BCRTs. Moreover, v, V_i are distributed according to $\mu_i = \mu|_{T_i^{(3)}}$, and are all conditionally independent with respect to Δ.

Since the remaining vertices of V are independent of everything so far, will lie in $T_i^{(3)}$ with probability $\mu\left(T_i^{(3)}\right)$. Hence, the number of leaves L_i of $T_i^{(3)}$ are distributed as

$$(L_1, L_2, L_3) = (K_1 + 2, K_2 + 2, K_3 + 2),$$

where K_i are Multinomial-$(k - 3, \Delta)$.

Finally, since the V_i are iid choices from μ, conditioned on L_i, the vertices $\left\{V_j^{(i)}\right\}_{i=1}^{L_j} = \{V_1, \ldots, V_k\} \cap V\left(T_i^{(3)}\right)$ are iid , μ_i distributed random variables, independent of everything else.

We have thus discovered an exact self-similarity property of labelled binary trees: $\mu\left(T_i^{(3)}\right)^{-1/2} R_{L_i}\left(T_i^{(3)}, \mu_i\right)$ are uniform labelled binary trees with L_i leaves uniformly labelled from $\{v\} \cup V_i$, and conditioned on $K_i = k_i$, edge statistics with density $s_i e^{-s_i^2/2}$ where $s_i = x_1 + \cdots + x_{2k_i+1}$. Moreover, $\mu\left(T_i^{(3)}\right)^{-1/2} R_{L_i}\left(T_i^{(3)}, \mu_i\right)$ are conditionally independent given L_i.

Reversing the perspective, a uniform labelled binary tree with k leaves can be built recursively from smaller uniform labelled binary trees. If we then allow the edge lengths to be random, this gives an inductive construction of the full k-point statistics from the 2- point length statistics and the k-point shape statistics:

Construction 1.

1. *Let $\Delta = (\Delta_1, \Delta_2, \Delta_3)$ be a Dirichlet-$(1/2; 3)$ random variable.*
2. *Let K_i be independent Multinomial-$(k-3, \Delta)$.*
3. *Let (I_1, I_2, I_3) be a uniform partition of $\{4, \ldots, k\}$ into three sets of size K_1, K_2, K_3, respectively.*
4. *Let T_i be independent samples from $R_{K_i+2}(T, \mu)$, with respective labels $\{v, i\} \cup I_i$.*
5. *Rescale edges in T_i by $\sqrt{\Delta_i}$, and glue the re-scaled trees T_i together at the special vertex v in each of the three trees.*

Then the resulting tree is equal in distribution to the k-point reduced subtree $R_k(T, \mu)$ of the BCRT.

Next, we consider a sample $\tau = (T, \mu, V_1, V_2)$ from the distribution $\mathbb{P} \in \mathscr{P}$. Sample $V_3, V_4, \ldots, V_k$ independently from μ, and split T at the branchpointss of the V_i. This results in $2k-3$ smaller trees $T_i^{(k)}$, each with two marked points $V_1^{(i)}, V_2^{(i)}$. We inductively show the following:

Claim.

1. The joint distribution of the sizes

$$\Delta^{(k)} = \left(\mu\left(T_1^{(k)}\right), \ldots, \mu\left(T_{2k-3}^{(k)}\right)\right)$$

   of the $2k-3$ sub-trees forms a Dirichlet-$(1/2; 2k-3)$ random vector.

2. The Brownian-rescaled trees

$$\tau_i^{(k)} = \left(\frac{T_i^{(k)}}{\sqrt{\Delta_i}}, \mu_i, V_1^{(i)}, V_2^{(i)}\right)$$

are jointly distributed as the product measure $\mathbb{P}_{\times \Delta_i^{(k)}}$ given $\Delta^{(k)}$.

3. The type $\mathscr{T}$ of $R_k(T,\mu)$ is distributed as $\mathbb{P}_{\mathscr{T},k}$, and is independent of everything else.

As we are yet uncertain of the exchangeability of the $\tau_i^{(3)}$'s, we choose the ordering which best respects the history: if V_k lies in $T_j^{(k-1)}$, then we set

$$\tau_i^{(k)} = \begin{cases} \tau_i^{(k-1)} & \text{i < j} \\ \tau_{i-2}^{(k-1)} & \text{i > j + 2} \end{cases} \tag{3.2}$$

and choose a uniform ordering for $\left(\tau_j^{(k)}, \tau_{j+1}^{(k)}, \tau_{j+2}^{(k)}\right)$.

The assumptions of the theorem provide the base case of the claim. Suppose, then, that the claim is true for up to $k-1$. Sample $V_3, \ldots, V_k$ independently from μ. Then $V_k \in T_i^{(k-1)}$ with probability $\Delta_i^{(k-1)}$; since $T_i^{(k-1)}$ is in $\mathscr{P}$ by induction, the new point V_k splits $T_i^{(k-1)}$ into pieces of size $\Delta_i^{(k-1)}\left(\Delta_1^{(3)}, \Delta_2^{(3)}, \Delta_3^{(3)}\right)$ for some Dirichlet-$(1/2, 1/2, 1/2)$ random variable $\Delta^{(3)}$. It follows from Proposition 6.1.1 that

$$\left(\Delta_1^{(k-1)}, \Delta_2^{(k-1)}, \ldots, \Delta_{i-1}^{(k-1)}, \Delta_1^{(3)}\Delta_i^{(k-1)}, \Delta_2^{(3)}\Delta_i^{(k-1)}, \Delta_3^{(3)}\Delta_i^{(k-1)}, \Delta_{(k-1)i+1}, \ldots, \Delta_{(k-1)2k-5}\right) \tag{3.3}$$

is a Dirichlet-$(1/2; 2k-3)$ random variable.

We split τ at the set of branchpoints of the points $(V_1, \ldots, V_k)$, and try to understand the dependence structure.

Decomposition 1.

1. *Sample* $\tau = (T, \mu, V_1, V_2)$ *from* $\mathbb{P}$.

2. *Sample* V_3 *from* μ. *Split* τ *at the branchpoint* $v = b(V_1, V_2, V_3)$, *yielding*

   - *three trees* $T_1^{(3)}$, $T_2^{(3)}$, $T_3^{(3)}$,
   - *their measures* $\Delta^{(3)} = \left(\mu\left(T_1^{(3)}\right), \mu\left(T_2^{(3)}\right), \mu\left(T_3^{(3)}\right)\right)$,
   - *the Brownian-rescaled doubly-rooted CRTs* $\tau_i^{(3)} = \left(\frac{T_i^{(3)}}{\sqrt{\Delta_i^{(3)}}}, \frac{\mu|_{T_i^{(3)}}}{\Delta_i^{(3)}}, V_i, V\right)$

3. *Pick* $(V_4, V_5, \ldots, V_k)$ *from* $\mu^{\times(k-3)}$.

4. *Let* $I_j = \left\{i \mid V_i \in T_j^{(3)}\right\}$.

The outcome is a triple of marked CRTs

$$\left(\tau_j^{(3)}, V_3^{(j)}, \ldots, V_{k_j}^{(j)}\right)$$

which, given $\Delta^{(3)}$ *and* k_j*, we claim to be distributed according to the product measure*

$$\left(\mathbb{P}_{\Delta_1^{(3)}} \times \mu_1^{\times k_1}\right) \times \left(\mathbb{P}_{\Delta_2^{(3)}} \times \mu_2^{\times k_2}\right) \times \left(\mathbb{P}_{\Delta_3^{(3)}} \times \mu_3^{\times k_3}\right).$$

By Assumption A4, $\tau_j^{(3)}$ are independent given $\Delta^{(3)}$, and are independent of k_j and hence I_j given $\Delta^{(3)}$. By definition of μ_j, $(V_i)_{i\in I_j}$ are independent of each other given k_j. (To be convinced of this, consider the contour representation of τ, and the representation of V_i as independent Uniform-$[0,1]$ random variables.) This verifies step 4 of Decomposition 1.

By induction, the type $\mathscr{T}_j$ of the reduced subtree $R\left(V_1^{(j)}, \ldots, V_{k_j}^{(j)}\right)$ of $T_j^{(3)}$ is a uniform labelled binary tree with k_j+2 leaves. Since this only depends on k_j, $\mathscr{T}_j$ are independent given k_j. Moreover, since k_j are jointly Multinomial-$\left(k-3, \Delta^{(3)}\right)$, upon gluing at a randomly chosen vertex, the resulting type $\mathscr{T}$ is a uniform labelled binary tree with k leaves, and is thus independent of everything.

Again by induction, splitting $\tau_j^{(3)}$ at the points $\left(V_1^{(j)}, \ldots, V_{k_j}^{(j)}\right)$ yields the law

$$\mathbb{P}_{\mathscr{T},k_j+2}\mathbb{P}_{\mathscr{T},k_j+2} \times \left(\times_{i=1}^{k_j+2}\mathbb{P}_{\Delta^{(3)}}\right).$$

Finally, we will deduce the 2-point statistics from the self-similar structure of $\mathscr{P}$. Let $V_1, V_2, \ldots,$ be iid selections from μ, and let $Y = d(V_1, V_2)$. Let N be the number of edges between V_1 and V_2, and let $Y_i = Y_i^{(k)}$ be the Brownian-rescaled edge lengths along the path between them in $\mathscr{T}$.

Let X be the 2-point distance in the BCRT. Since a uniform labelled binary tree is a conditioned Galton-Watson tree with a Bernoulli-$(1/2)$ offspring distribution, it follows that

$$\frac{N}{\sqrt{k}} \Rightarrow X. \tag{3.4}$$

By induction we find a sequence of random variables Y_i with common mean $\sqrt{\pi/2}$ and variance bounded by σ^2, and a Dirichlet-$(1/2; 2k-3)$ random variable $(\Delta_1, \ldots, \Delta_{2k-3})$ such that $Y \overset{d}{=} \sum_{i=1}^N \sqrt{\Delta_i} Y_i$. Define $S_n = \sum_{j=1}^n \sqrt{\Delta_i} Y_i$ for $n = 1, 2, \ldots, 2k-3$.

Let Δ_+^d be the d-dimensional unit simplex. Taking advantage of the densities of Dirichlet-$(1, 1/2, \ldots, 1/2)$ and Dirichlet-$(1, 1, 1/2, \ldots, 1/2)$ random variables, and applying Stirling's approximation,

$$\begin{aligned}
\mathbb{E}\sqrt{\Delta_i} &= \frac{\Gamma(k/2)}{\Gamma(1/2)^k}\int_{\Delta_+^d}\frac{1}{\sqrt{x_2x_3\cdots x_k}}dx_1\cdots dx_k\\
&= \frac{\Gamma(k/2)}{\Gamma(\frac{k}{2})\Gamma(1/2)}\\
&\sim \sqrt{\frac{2}{\pi k}}\left(1+O(1/k)\right);\\
\mathbb{E}\sqrt{\Delta_i\Delta_j} &= \frac{\Gamma(k/2)}{\Gamma(1/2)^k}\int_{\Delta_+^d}\frac{1}{\sqrt{x_3x_4\cdots x_k}}dx_1\cdots dx_k\\
&= \frac{\Gamma(k/2)}{\Gamma(k/2+1)\Gamma(1/2)^2}\\
&= \frac{2}{\pi k}\left(1+O(1/k)\right); && (3.5)\\
\mathrm{Cov}(\sqrt{\Delta_1},\sqrt{\Delta_2}) &= O\left(\frac{1}{k^2}\right). && (3.6)
\end{aligned}$$

We prove a weak law of large numbers to show that $S_N - N\mathbb{E}\left[\sqrt{\Delta_1}Y_1\right] = S_N - \frac{N}{\sqrt{k}} \xrightarrow{p} 0$ as $k\to\infty$, i.e. $Y \overset{d}{=} X$.

Preconditioning on Δ_i, Δ_j, the conditional independence of Y_i and Y_j together with Assumption B1 implies that

$$\begin{aligned}
\mathrm{Cov}(\sqrt{\Delta_i}Y_i,\sqrt{\Delta_j}Y_j) &= \mathrm{Cov}\left(\sqrt{\Delta_1}\mathbb{E}\left[Y_i\,|\,\Delta\right],\sqrt{\Delta_2}\mathbb{E}\left[Y_j\,|\,\Delta\right]\right)\\
&= \frac{\pi}{2}\mathrm{Cov}\left(\sqrt{\Delta_1},\sqrt{\Delta_2}\right)\\
&= O\left(\frac{1}{k^2}\right).
\end{aligned}$$

This gives the following variance bound:

$$\begin{aligned}
\mathbb{P}\left[\left|S_n - n\mathbb{E}\sqrt{\Delta_1}Y_1\right| > \epsilon\right] &\le \frac{\mathrm{Var}\left(S_n\right)}{\epsilon^2}\\
&= \epsilon^{-2}\left\{\sum_{i=1}^n \mathbb{E}\left[\sqrt{\Delta_i}Y_i - \sqrt{\frac{\pi}{2k}}\right]^2\right\} \quad +\epsilon^{-1}\sum_{i=1}^n\sum_{j=i+1}^n \mathrm{Cov}\left(\sqrt{\Delta_i}Y_i,\sqrt{\Delta_j}Y_j\right)\Bigg\}\\
&\preceq \epsilon^{-2}\left(\frac{n}{k}+\frac{n^2}{k^2}\right)\\
&\preceq \frac{n}{k\epsilon^2}.
\end{aligned}$$

Since $\frac{N}{\sqrt{k}} \to X$ in distribution, which is a positive random variable, we can find k_ϵ

$\epsilon_0 > 0$ such that for $k \geq k_\epsilon$,

$$\mathbb{P}\left[\epsilon_0\sqrt{k} \leq N \leq \epsilon_0^{-1}\sqrt{k}\right] \geq 1 - \epsilon.$$

Hence, for $k \geq k_\epsilon$,

$$\begin{aligned}
\mathbb{P}\left[\left|S_N - N\mathbb{E}\sqrt{\Delta_1}Y_1\right| > \epsilon\right] &= \sum_{n=\epsilon_0\sqrt{k}}^{\epsilon_0^{-1}\sqrt{k}} \mathbb{P}\left[\left|S_n - n\mathbb{E}\sqrt{\Delta_1}Y_1\right| > \epsilon\right]\mathbb{P}\left[N = n\right] + o\left(1;\ \epsilon \to 1\right) \\
&= \sum_{n=\epsilon_0\sqrt{k}}^{\epsilon_0^{-1}\sqrt{k}} \frac{n}{k\epsilon^2}\mathbb{P}\left[N = n\right] + o\left(1;\ \epsilon \to 1\right) \\
&= \frac{n}{k\epsilon^2} + o\left(1;\ \epsilon \to 1\right).
\end{aligned}$$

□

Chapter 4

Self-similar decomposition of random walk bridges

Let T_n be the range of an n-step random walk bridge B_n. Define the measure p_n on $V(T_n)$ as follows:

$$p_n(v) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}(B_n(i) = v).$$

In other words, to sample from p_n, if we choose a uniform time t in $\{1, 2, \dots, n\}$, then $B_n(t)$ is distributed as p_n.

Collecting some facts below about finite bridges, we'll verify that we can apply Theorem 3.2.1 to the set of subsequential limits, which proves Theorem 1.0.1.

Define a 3-marked bridge of duration n to be a pair (B_n, s_1, s_2, s_3), where B_n is an n-step random walk bridge, and s_i are a independent uniform choices from $\{1, 2, \dots, n\}$, conditioned to be unique for technical reasons. (This does not have a significant effect on the measure, as the s_i are unique with probability $1 - o\,(1;\ n \to \infty)$.) Define $\mathbb{E}_{n,3}$ be the measure on k-marked bridges (B_n, s_1, s_2, s_3) of duration n.

Let (B_n, s_1, s_2, s_3) be a sample from $\mathbb{E}_{n,3}$, and let T_n be its range. Let t'_1, t'_2, t'_3 be the order statistics of s_1, s_2, s_3 – then (t'_1, t'_2, t'_3) is a uniform choice of ordered times. Let $v_i = B_n(t'_i)$, and $V = \{v_1, v_2, v_3\}$. We decompose the range of a 3-marked bridge B_n into smaller trees in two ways:

- First, by splitting at the branchpoint $v' := b(v_1, v_2, v_3)$, yielding up to d smaller trees $T'_1, \dots, T'_k$, which are the respective range of various random walk *excursions* $B'_1, \dots, B'_k$.
- Second, according to the following algorithm.

Algorithm 1. *Choose a triple* (r_1, r_2, r_3) *uniformly satisfying*

- $t'_3 \le r_1 \le t'_1 \le r_2 \le t'_2 \le r_3 \le t'_3$, *where we think of the times as lying on a circle, and* $x \le y \le z$ *means* y *is on the clockwise path from* x *to* z,
- $B_n(r_1) = B_n(r_2) = B_n(r_3) =: v$.

Such a triple is guaranteed, perhaps, for instance, with $v = v'$. *Define* $B_i(t) = B_n(r_i + t)$, $t = 0, 1, \ldots, r_{i+1} - r_i$, *and* $t_i = t'_i - r_i (\mod n)$.

Given the durations n_i of B_i, (B_i, t_i) are 'almost' independent marked bridges. Specifically, let $\mathbb{P}^N_{n,3}$ denote the measure on B_n biased by N, and let $\mathbb{E}^N_{n,3}$ be expectation with respect to $\mathbb{P}^N_{n,3}$. That is, for any continuous, bounded function f of, we have

$$\mathbb{E}_{n,3}\left[f\left(\frac{n_1}{n}, \frac{n_2}{n}, \frac{n_3}{n}\right)\right] = \mathbb{E}^N_{n,3}\left[\frac{f\left(\frac{n_1}{n}, \frac{n_2}{n}, \frac{n_3}{n}\right)}{N\mathbb{E}^N_{n,3}N^{-1}}\right]. \tag{4.1}$$

By 6.5.1, under $\mathbb{E}^N_{n,3}$ given the durations n_i, the bridges (B_i, t_i) are independent 1-marked bridges.

Without loss of generality, assume that v_i happens to lie in B'_i for $i = 1, 2, 3$. By understanding the structure of B_i, we will see that with high probability:

- (B'_i, v_i) and (B_i, v_i) are 'close' for $i = 1, 2, 3$.
- B'_i are 'small' for $i > 3$.

The notions of 'close' and 'small' roughly mean that asymptotically, the assumptions of Theorem 3.2.1 hold.

For simplicity of notation, use n to denote a convergent subsequence of τ_n – that is, the distribution of τ_n converges to some measure $\mathbb{P} \in \mathscr{P}$. Recall or make the following definitions.

- Let τ'_n be the 3-marked ordered tree corresponding to $\left(\frac{T_n}{\sqrt{n}}, s_1, s_2, s_3\right)$. Then $T_n(s_i)$ are adjacent to three independent uniform vertices of T_n – see Lemma 12 of [3].
- Let τ_n be the 3-marked ordered tree corresponding to $\left(\frac{T_n}{\sqrt{n}}, \tilde{s}_1, \tilde{s}_2, \tilde{s}_3\right)$, where $\tilde{s}_i$ are uniformly chosen from those times s such that $T_n(s) = B_n(s_i)$.
- Let T'_i be the contour function of B'_i. Let $\tau'_i = \left(\frac{T'_i}{\sqrt{n'_i}}, t'_i\right)$.
- Let T_i be the contour function of B_i. Let $\tau_i = \left(\frac{T_i}{\sqrt{n_i}}, t_i\right)$.

Using Propositions 4.0.2- 4.0.5, we show the following, thus proving Theorem 1.0.1.

Proposition 4.0.1. *Let* $\tau_n = \left(\left(\frac{T_n(nt)}{\sqrt{n}}\right), p_n, t_1, t_2\right)$, *where* (t_1, t_2) *are indepenent Uniform* $[0,1]$ *random variables. Then the assumptions of Theorem 3.2.1 hold for the set* $\mathscr{P}$ *of subsequential limits of* τ_n.

Proposition 4.0.2. *The normalized contour function* $\frac{|T_n(nt)|}{\sqrt{n}}$ *is tight in the topology of uniform convergence.*

Proposition 4.0.3. *There is a coupling between a uniform vertex* U_n *and a vertex* V_n *sampled from* p_n *such that*

$$\frac{d(U_n, V_n)}{\sqrt{n}} \xrightarrow{p} 0.$$

Proposition 4.0.4. *The random variables* $(p_n(T_1), p_n(T_2), p_n(T_3))$ *converge in distribution to a Dirichlet-*$(1/2, 1/2, 1/2)$ *random variable.*

Proposition 4.0.5. $\frac{\left\|T_i - T_i'\right\|_\infty}{\sqrt{n}} \xrightarrow{p} 0.$

Proof of Proposition 4.0.1. Consider the distribution of τ_n under $\mathbb{E}_{n,3}$.

1. By Proposition 4.0.3, the law of τ_n under $\mathbb{E}_{n,3}$ is close to the law of τ_n' under the measure $\mathbb{E}^U_{n,3}$ where s_i are iid uniform $[0,1]$ random variables, in the sense that $\mathbb{E}_{n,3} f(\tau_n) - \mathbb{E}^U_{n,3} f(\tau_n') = o\,(1;\, n \to \infty)$. In particular they have the same distributional limit.

2. By Proposition 4.0.4 together with 6.2.6, the joint measures of $(p_n(T_1'), p_n(T_2'), p_n(T_3'))$ converge to a Dirichlet-$(1/2, 1/2, 1/2)$ random variable Δ. This proves Assumptions A2 and A3 of Theorem 3.2.1.

3. By Proposition 4.0.5, the joint distributions of τ_i' under $\mathbb{E}_{n,3}$ is close to the joint distributions of τ_i in the same sense as above. Hence, we instead exhibit the required asymptotic self-similarity properties A4, B1-B4 between the distribution of τ_n and τ_i.

4. For the coup de grâce, we use Proposition 6.7.1 and Proposition 6.7.1. Let f_i be bounded continuous functions on 1-marked ordered trees. Let τ be any subsequential limit of τ_n, and let τ_i be the three subtrees from Assumption 4 of Theorem

3.2.1. Then

$$\begin{aligned}
\mathbb{E}\left[\prod_i f_i(\tau_i) \,|\, \Delta^{(3)} = \vec{x}\right] &= \mathbb{E}\left[\prod_i f_i(\tau_i) \,|\, |\Delta^{(3)} - \vec{x}| < \epsilon\right] + o\,(1;\ \epsilon \to 0) \\
&= \mathbb{E}_{n,3}\left[\prod_i f_i(\tau_{i,n}) \,|\, \left|\frac{n_i}{n} - x_i\right| < \epsilon\right] + o\,(1;\ \epsilon \to 0) + o\,(1;\ n \to \infty) \\
&= \mathbb{E}^N_{n,3}\left[\frac{\prod_i f_i(\tau_{i,n})}{N\mathbb{E}^N_{n,3}N^{-1}} \;\Big|\; \left|\frac{n_i}{n} - x_i\right| < \epsilon\right] + o\,(1;\ \epsilon \to 0) + o\,(1;\ n \to \infty) \\
&= \mathbb{E}^N_{n,3}\left[\prod_i f_i(\tau_{i,n}) \;\Big|\; \left|\frac{n_i}{n} - x_i\right| < \epsilon\right] + o\,(1;\ \epsilon \to 0) + o\,(1;\ n \to \infty)
\end{aligned}$$

by Proposition A.0.2. Using Proposition 6.7.1

$$\begin{aligned}
\mathbb{E}^N_{n,3}\left[\prod_i f_i(\tau_{i,n}) \;\Big|\; \left|\frac{n_i}{n} - x_i\right| < \epsilon\right] &= \mathbb{E}^N_{n,3}\left[\prod_i f_i(\tau_{i,n}) \,|\, n_i = x_i n\right] + o\,(1;\ \epsilon \to 0) + o\,(1;\ n \to \infty) \\
&= \prod_i \mathbb{E}_{x_i n}\left[f_i(\tau_{x_i n})\right] + o\,(1;\ \epsilon \to 0) + o\,(1;\ n \to \infty)
\end{aligned}$$

This shows three things:

(a) By setting $f_j \equiv 1$ for $j \neq i$, $\mathbb{E}_{x_i n}$ converges in distribution, ie. λn is a convergent subsequence for each $\lambda \in (0,1)$.

(b) The subsequential limit $\mathbb{E}$ satisfies assumption A4 of Theorem 3.2.1.

(c) More so, by Proposition 6.7.1, the measures $\mathbb{E}_x$ are continuous in x in the required sense. Applying the same argument to the convergent subsequence $x_i n$, we get that

$$\mathbb{E}_{x_i}\left[\prod_j f_j(\tau_j) \,|\, \Delta^{(3)} = \vec{y}\right] = \prod_j \mathbb{E}_{x_i y_j} f_j(\tau),$$

thus verifying assumption B3 and B4.

We prove directly in Section 6.3 that, up to a constant factor, the rescaled two point statistics $d(v_1, v_2)/\sqrt{n}$ of two independent p_n-distributed vertices scale to a Raileigh distribution, thus concluding the proof. □

4.1 Proofs of Propositions 4.0.2-4.0.5

Proof of Proposition 4.0.5. Define $R_n^{(t)}$ to be the cardinality of $\{B_n(0), \ldots, B_n(nt)\}$. As a part of the proof of Theorem 3 of [5], it is shown that

$$\frac{R_n^{(t)}}{n} \xrightarrow{p} (1 - F(R))t \tag{4.2}$$

as $n \to \infty$. It follows that

$$I_n(t) := \frac{R_n^{(t)}}{R_n} \xrightarrow{p} t \tag{4.3}$$

Since this is a non-increasing process in t, simultaneous convergence in probability along the dyadics $\frac{i}{2^k}$ implies uniform convergence in probability:

$$\|I_n(t) - t\|_\infty \xrightarrow{p} 0. \tag{4.4}$$

Let $\{v_1, v_2, \ldots, v_{R_n}\}$ be the vertices of T_n, listed in order of the first visit by B_n, and let T be an independent Uniform-$[0, 1]$ random variable. Then the law of $V := v_{TR_n}$ is uniform on the vertices of T_n, while $V' := v_{I_n(T)R_n}$ has the law of $B_n(nT)$.

From (4.4), we know that that $T \in [I_n(T - \epsilon), I_n(T + \epsilon)]$ with probability $1 - o(1)$. On this event, the bridge took at most ϵn steps between V and V'. By the invariance of the law of B_n, the distance between V and V' is stochastically dominated by

$$M_{n,\epsilon} = \max_{t \in \{1,2,\ldots,\epsilon n\}} |B_n(t)|,$$

By Proposition 6.2.5, $\limsup_{n\to\infty} \frac{M_{n,\epsilon}}{\sqrt{n}} \xrightarrow{p} 0$ as $\epsilon \to 0$. □

Proof of Proposition 4.0.2. We prove in Section **??** that the distance function $\frac{|B_n(nt)|}{\sqrt{n}}$ is tight in the topology of uniform convergence. Recall the uniform convergence (4.4)

$$I_n(t) := \frac{R_n^{(t)}}{R_n} \to 0$$

from (4.4). This implies

$$\frac{T_n(nt) - |B_n(nt)|}{\sqrt{n}} \to 0 \tag{4.5}$$

in the Skorokhod topology. Since the Skorokhod topology restricted to $C[0, 1]$ coincides with the uniform topology, $\frac{T_n(nt)}{\sqrt{n}}$ is also tight in the uniform topology. □

Proof of Proposition 4.0.4. We start by considering the joint distribution of the durations of B_i. We take advantage of a simplification via a change of measure on 3-marked bridges. First, the decomposition from Algorithm 1 of a single 3-marked bridge of $n+2$ steps into shorter 1-marked bridges induces a measure on triples $B_n = ((B_1, t_1), (B_2, t_2), (B_3, t_3))$ of 1-marked bridges with a total of $n + 2$ steps – accounting for double counting the intersection times. Let n_i be the number of steps in B_i. Then $n = n_1 + n_2 + n_3 + 2$.

Let N be the number of choices of the triple (r_1, r_2, r_3) used in the decomposition. Sampling B_n biased by N is a very simple measure.

If N and n_i were independent under $\mathbb{E}^N_{n,3}$, we would practically be finished by using (1.2). While they are not independent, we show in Section 6.4 that they are asymptotically independent [1], and in particular, under $\mathbb{P}^N_{n,3}$, the random variable $\left(\frac{n_1}{n}, \frac{n_2}{n}, \frac{n_3}{n}\right)$ converges in distribution to a Dirichlet-$(1/2, 1/2, 1/2)$ random variable Δ.

Since N converges in distribution under $\mathbb{E}^N_{n,3}$, $\mathbb{E}^N_{n,3}N^{-1}$ is uniformly bounded away from 0. By asymptotic independence,

$$\begin{aligned}
\mathbb{E}^N_{n,3}\left[\frac{f\left(\frac{n_1}{n}, \frac{n_2}{n}, \frac{n_3}{n}\right)}{N\mathbb{E}^N_{n,3}N^{-1}}\right] &= \frac{1}{\mathbb{E}^N_{n,3}N^{-1}}\left(\mathbb{E}^N_{n,3}N^{-1}\mathbb{E}^N_{n,3}\left[f\left(\frac{n_1}{n}, \frac{n_2}{n}, \frac{n_3}{n}\right)\right] + o(1)\right) \\
&= \mathbb{E}\left[f(\Delta_1, \Delta_2, \Delta_3)\right] + o\left(\frac{1}{\mathbb{E}^N_{n,3}N^{-1}}\right) \\
&= \mathbb{E}\left[f(\Delta_1, \Delta_2, \Delta_3)\right] + o(1)
\end{aligned}$$

This shows that under $\mathbb{E}_{n,3}$, the durations of B_i jointly converge to a Dirichlet-$(1/2, 1/2, 1/2)$ random variable. It remains to show that the duration of B'_i is sufficiently close to the duration of B_i.

Let p_i, q_i be the respective 'start' and 'finish' times of B'_i; ie. $B'_i(t) = B(p_i + t)$ for $t = 0, \ldots, q_i - p_i$. Let $v'_i = b(v, v', v_i)$ be the branchpoint of v, v', and v_i. Then v'_i is in both the range of B'_i and of B_i for $i = 1, 2, 3$. By Lemma 6.2.10, $d(v, v'_i) \leq d(v, v')$ is tight.

Consider the following times:

$$\tau_i := \sup\left\{t \mid r_i \leq t \leq t_i,\, B_n(t) = v'_i\right\}; \quad \sigma_i := \inf\left\{t \mid t_i \leq t \leq r_{i_1},\, B_n(t) = v'_i\right\}.$$

By the invariance in distribution under time reversal, τ_i and σ_i share the same distribution; by Corollary 6.2.9, they are tight. Hence, $\frac{\sigma_i - \tau_i}{n}$ jointly converge to the same Dirichlet-$(1/2, 1/2, 1/2)$ random variable as $\frac{n_i}{n}$. As $q_i - p_i \geq \sigma_i - \tau_i$, and $\sum_{i=1}^3 q_i - p_i \leq n$,

[1] We say that X_n, Y_n are asymptotically independent if for any continuous bounded functions f, g, $\mathbb{E}[f(X_n)g(Y_n)] - \mathbb{E}[f(X_n)]\mathbb{E}[g(Y_n)] \to 0$.

the same is true for $q_i - p_i$. □

Proof of Proposition 4.0.5. First, by Corollary 6.3.4, we know that with probability $1 - o(1)$, $d(v, v_i)$ are on the order of $\sqrt{n}$. By Proposition 6.2.10, the branchpoint $b(v_i, v_j, v')$ is at distance $O(1)$ from v', so the only way for v_i to belong to two B'_i is for at least one of the B_i to return sufficiently close to v near the 'middle' of the bridge. By Lemma 6.2.10, this happens with probability $o\,(1;\ n \to \infty)$. Hence, v_i belong to exactly one B'_i with probability $1 - o\,(1;\ n \to \infty)$.

Recall the times τ_i, σ_i defined in the proof of Proposition 4.0.4. Let m_i be the number of steps in B'_i. The contour process of B'_i and of B_i are identical between times τ_i and σ_i – ie. $B'_i(\tau_i - r_i + t) = B(\tau_i - p_i + t)$ for $t = 0, 1, \ldots, \tau_i - \sigma_i$. Since any contour process is 1-Lipschitz, it follows that for any $t \in [0, 1]$, if $n_i t \in [\tau_i, \sigma_i]$,

$$\begin{aligned}
|T'_i(n_i t) - T_{B_i}(m_i t)| &= |T'_i(n_i t - (\tau_i - r_i) + (\tau_i - r_i))) - T_i(m_i t)| \\
&= |T_i(n_i t - (\tau_i - r_i) + (\tau_i - p_i)) - T_i(m_i t)| \\
&\leq t\,|m_i - n_i| + |r_i - p_i| \\
&\leq M := 2 \sum_{i=1}^{3} \tau_i - \sigma_{i-1}.
\end{aligned}$$

If $n_i t \in [0, \tau_i]$, then

$$\begin{aligned}
|T'_i(n_i t) - T_i(m_i t)| &\leq |T'_i(n_i t)) - T_{B_i}(\tau_i))| + |T_i(\tau_i) - T_i(n_i t)| \\
&\leq \tau_i - r_i + \tau_i - p_i \\
&\leq M.
\end{aligned}$$

Similarly, if $n_i t \in [\sigma_i, r_{i+1}]$, $|T'_i(n_i t) - T_i(m_i t)| \leq M$, so

$$\|T'_i(n_i t) - T_i(m_i t)\|_\infty \leq M.$$

Note that $\tau_i - r_i$ has the same law as $r_i - \sigma_{i-1}$, so by Corollary 6.2.8 together with the tightness of $d(v, v'_i)$, M is tight, which concludes the proof. □

Chapter 5

The infinite bridge

Let B_n be a random walk bridge of duration n. In [17], i is shown that when X_n is a *simple* random walk, then the distance $(|B(0)|, |B(1)|, \ldots, |B(n)|)$ of the simple random walk bridge converges as a process to a time-homogeneous Markov process on $\mathbb{Z}$. In fact, the bridge itself converges to a time-homogeneous Markov process on $\mathbb{T}_d$. This phenomena is much more common: Given a random walk X_n on a transitive graph Γ, define the function

$$h(x) := \lim_{n\to\infty} \frac{\mathbb{P}[X_n = x]}{\mathbb{P}[X_n = X_0]}, \tag{5.1}$$

if the limit exists. Let N be the number of returns of X_n to X_0, let

$$G(z) = \mathbb{E}\left[z^N\right],$$

be the Greens function G of X_n, and let R be its radius of convergence. It is easy to see that h is R-harmonic. In particular, the h-transform of X_n is a time-homogeneous Markov process B_∞ on Γ. The process B_∞ is the limit of the random walk bridge:

Theorem 5.0.1. *Let X_n be an aperiodic, irreducible random walk on a transitive graph Γ such that the function h defined in* (5.1) *exists. Let B_∞ be the Doob h-transform of the random walk X_n. Then the random walk bridge B_n as well as its time reversal $B'_n(k) = B_n(n-k)$ converges in distribution to the B_∞ processes.*

Furthermore, when Γ is the free product $\mathbb{Z}_2 * \mathbb{Z}_2 * \cdots * \mathbb{Z}_2$ or the free group of rank $d \geq 2$, with the canonical generating set,

$$(B_n(k), B_n(n-k))_{k=0}^n$$

jointly converge to two half infinite bridges $((B_\infty, B'_\infty)$ which are independent given their limiting end.

Remark 5.0.1. Since $\mathbb{P}[X_{2n} = x]$ is maximal at $x = X_0$, subsequential limits of (5.1) exist. It's clear by the definition of the spectral radius that any subsequential limit is $1/R$-harmonic. The subsequential limits correspond exactly to the subsequential limits of the random walk bridge $B_n(\cdot)$, which also clearly exist since $(B_n(0), B_n(1), \ldots, B_n(k))$ take d^k values for each $n \geq k$.

When Γ has the Liouville property – that is, when every bounded harmonic function of Γ is constant – then any subsequential limit is constant, and therefore the infinite bridge measure coincides with the measures on two independent random walks.

Proof. It's clear that the distribution of $B_n(k)$ converges to the distribution of $B_\infty(k)$ when h exists. Moreoer, given $B_n(k)$, the path $B_{n,k} := (B_n(0), \ldots, B_n(k))$ is a random walk bridge of length k from $B_n(0)$ to $B_n(k)$. Hence,

$$\begin{aligned}
\mathbb{P}\left[B_{n,k} = (x_0, \ldots, x_k)\right] &= \frac{\mathbb{P}\left[(X_0, \ldots, X_k) = (x_0, \ldots, x_k)\right] \mathbb{P}\left[B_n(k) = x_k\right]}{\mathbb{P}\left[X_k = x_k\right]} \\
&= \frac{\mathbb{P}\left[(X_0, \ldots, X_k) = (x_0, \ldots, x_k)\right] \mathbb{P}\left[B_\infty(k) = x_k\right]}{\mathbb{P}\left[X_k = x_k\right]} + o\left(1;\ n \to \infty\right) \\
&= \mathbb{P}\left[(B_\infty(0), \ldots, B_\infty(k)) = (x_0, \ldots, x_k)\right] + o\left(1;\ n \to \infty\right)
\end{aligned}$$

since B_∞ is a Markov process.

Let B_n be a random walk bridge on $\mathbb{T}_d$. Fix k, and $\ell > k$. Define the following.

- For an edge e in the Cayley graph, e^- is the vertex at the end of e closest to the root ρ, and e^+ is the vertex at the end of e furthest from ρ.
- $M_n = B_n(n/2)$.
- $e_{\ell,n}$ is the edge on the geodesic from ρ to M_n such that the vertex $e_{\ell,n}^-$ is at distance k.
- f_v is the first positive hitting time of v.
- $\sigma_{\ell,n} = \max\left(i \leq n/2 \mid B_n(i) = e_{\ell,n}^-\right)$.
- $\tau_{\ell,n} = \max\left(i \leq n/2 \mid B_n(n-i) = e_{\ell,n}^-\right)$.

Given the quantities $Y_n := (e_{\ell,n}, \sigma_{\ell,n}, \tau_{\ell,n})$, the distribution on the first and last steps k steps of B_n does not depend on n, and are independent of each other. Moreover, the edge $e_{\ell,n}$ converges in distribution to the branch at distance ℓ that the infinite bridge ends up in. As the distribution of $(B_n(n-k), \ldots, B_n(0), \ldots, B_n(k))$ is a function of Y_n, it suffices to show that Y_n converges in distribution.

Given $(e_{\ell,n}, \sigma_{\ell,n}, \tau_{\ell,n})$, the random walk bridge is composed of three independent parts:

- A $\sigma_{\ell,n}$-step random walk bridge from X_0 to $e^-_{\ell,n}$.
- An $n - (\sigma_{\ell,n} + \tau_{\ell,n})$-step random walk excursion from $e^-_{\ell,n}$ to itself, conditioned on the first step being $e^+_{\ell,n}$.
- A $\tau_{\ell,n}$-step random walk bridge from $e^-_{\ell,n}$ to X_0.

Hence,

$$\begin{aligned}\mathbb{P}_n\left[e_{\ell,n} = e, \sigma_{\ell,n} = s, \tau_{\ell,n} = t\right] &= \frac{\mathbb{P}[X_s = e^-]\mathbb{P}[f_{e^-} = n-(t+s)\,|\,X_0 = e^-, X_1 = e^+]\mathbb{P}[X_t = e^-]}{\mathbb{P}[X_n = X_0]} \\ &= \mathbb{P}[X_s = e^-]\mathbb{P}[X_t = e^-]R^{s+t}c_e + o\left(1;\ n \to \infty\right)\end{aligned}$$

for some constant c_e depending only on the generator associated to e, following Proposition 4 from [10]. This implies that $(\sigma_{\ell,n}, \tau_{\ell,n}, e_{\ell,n})$ converges in distribution to random variables $(\sigma_\ell, \tau_\ell, e_\ell)$, where (σ_ℓ, τ_ℓ) are independent given e_ℓ. □

The analogous result for a Brownian bridge on a Riemannian manifold is much harder, and many facts about the infinite Brownian bridge can be found in [12], where the term infinite Brownian loop is used. In particular, let (B^0_t) be the infinite Brownian loop in a symmetric space $\mathbb{M}$ of rank d, and let j be the number of positive indivisible roots. Then the process

$$\frac{d\left(B^0_{Tt}, B^0_0\right)}{\sqrt{T}} \Rightarrow |W^{(d)}_t|, \tag{5.2}$$

where $|W^{(d)}_t|$ is the Bessel process of dimension $D = d + 2j$. In particular, when $\mathbb{M}$ is the hyperbolic plane, $D = 3$.

The very general result of [11] (see also [22]) imply that h exists when Γ is a Gromov-hyperbolic group. Motivated by (5.2), we make the following conjectures:

Conjecture 2. *Let X_n be a symmetric, aperiodic, irreducible random walk on a non-degenerate Gromov-hyperbolic group, and let $B_\infty(\cdot)$ be the infinite bridge. Then*

$$\frac{d(B_\infty(nt), B_\infty(0))}{\sqrt{n}} \Rightarrow |W_t|, \tag{5.3}$$

where $|W_t|$ is a 3-dimensional Bessel process.

Since $|W_t|$ is the contour process of the non-compact BCRT, and in addition, is the Brownian excursion near 0, this would suggest

Conjecture 3. *Let X be a either symmetric, aperiodic, irreducible random walk on a non-degenerate Gromov-hyperbolic group, or a Brownian motion in a symmetric space of rank* 1. *Let $B_n(\cdot)$ be $X(\cdot)$ conditioned on $X(n) = X(0)$, and let $B_\infty(\cdot)$ be the infinite bridge.*

1. *The range of $B_n(\cdot)$, rescaled by $n^{-1/2}$, converges in distribution to the compact BCRT.*

2. *The range of $B_\infty(\cdot)$, rescaled by ϵ, converges in distribution to the non-compact BCRT.*

Chapter 6

Technical results

In this section we will collect some technical results used to prove Propositions 4.0.1 - 4.0.5.

6.1 Size-biased splitting of Dirichlet random variables

The simplest Dirichlet-($\boldsymbol{\alpha}$) random variable is the Dirichlet-$(1; k)$, which has the following interpretation: choose $k-1$ independent uniform points on the unit interval, and break the interval at each chosen point. Then the joint distribution of the lengths of each segment is a Dirichlet-$(1; k)$ random variable. From this interpretation, we immediately see the following. If $(\Delta_1, \ldots, \Delta_k)$ is Dirichlet-$(1; k)$, and we choose a random index I to be i with probability Δ_i and an independent Uniform-$[0, 1]$ U, then

$$(\Delta_1,\ \Delta_2, \ldots, \Delta_{i-1},\ U\Delta_i,\ (1-U)\Delta_i,\ \Delta_{i+1}, \ldots, \Delta_k)$$

is a Dirichlet-$(1; k+1)$ random variable. There is a related "self-similarity" identity for Dirichlet-$(1/2; k)$ random variables involving a 3-way splitting.

Proposition 6.1.1. *Let* $\Delta = (\Delta_1, \Delta_2, \ldots, \Delta_k)$ *be a Dirichlet-*$(1/2; k)$ *random variable, and let* $I = i$ *with probability* Δ_i*. Let* $(\Delta'_1, \Delta'_2, \Delta'_3)$ *be an independent Dirichlet-*$(1/2, 1/2, 1/2)$ *random variable. Then*

$$(\Delta_1, \ldots, \Delta_{I-1},\ \Delta_I\Delta'_1,\ \Delta_I\Delta'_2,\ \Delta_I\Delta'_3,\ \Delta_{I+1}, \ldots, \Delta_k)$$

is a Dirichlet-$(1/2; k+2)$ *random variable.*

Proof. Consider a k-dimensional Dirichlet-$(1/2, \ldots, 1/2)$ random variable. Conditioned on $I = i$, Δ is a Dirichlet-$(1/2, 1/2, \ldots, 3/2, \ldots, 1/2)$ random variable, where the $3/2$ is in coordinate i. By exchangeability of Δ, we can assume that $i = k$ without loss of generality.

Let $\Gamma_1, \ldots, \Gamma_{k+2}$ be iid Gamma-$(1/2, 1)$ random variables, and set $\Gamma := \sum_{i=1}^{k+2} \Gamma_i$ and $\Gamma' = \Gamma_k + \Gamma_{k+1} + \Gamma_{k+2}$. Then Γ' is a Gamma-$(3/2, 1)$ random variable, and $\Delta^{(k)} := \frac{1}{\Gamma}(\Gamma_1, \ldots, \Gamma_{k-1}, \Gamma')$ is Dirichlet-$(1/2, \ldots, 1/2, 3/2)$, and $\Delta^{(3)} := \left(\frac{\Gamma_k}{\Gamma'}, \frac{\Gamma_{k+1}}{\Gamma'}, \frac{\Gamma_{k+2}}{\Gamma'}\right)$ is a Dirichlet-$(1/2, 1/2, 1/2)$ random variable. Moreover, it's clear from their densities that $\Delta^{(3)}$ is independent from Γ', and hence from $\Delta^{(k)}$. Hence, the splitting

$$\frac{1}{\Gamma}(\Gamma_1, \ldots, \Gamma_{k-1}, \Gamma') \to \frac{1}{\Gamma}(\Gamma_1, \ldots, \Gamma_{k-1}, \Gamma_k, \Gamma_{k+1}, \Gamma_{k+2})$$

exhibits the required distributional relationship. □

6.2 Return probabilities and related results

6.2.1 Asymptotics of return probabilities

Sharp asymptotics are derived in [15] for the return probabilities of nearest neighbour random walks on free product $G := \mathbb{Z}_2 * \cdots * \mathbb{Z}_2$ of d copies of $\mathbb{Z}_2$. For a vertex x in the Cayley graph of G, let $x = x_1 x_2 \cdots x_m$ be the reduced word representing x. Define the **profile** of x to be the function $\pi : G \to \mathbb{Z}_+^d$ with[1]

$$\pi_i(x) = \sum_{k=0}^{m} \mathbf{1}(x_k = a_i) = \#\{k \mid x_k = a_i\}.$$

Notice that $d(x, \rho) = \|\pi(x)\|_1$, where $\|\cdot\|_1$ is the L^1 norm on $\mathbb{R}^d$.

Let F_i be the generating function of the first hitting time τ_i of a_i, ie. $F_i(z) := \mathbb{E}\left[z_i^{\tau}; \tau_i < \infty\right]$. By (2.1) of [15], F_i all have the same radius of convergence; call it R. Define $\psi_i(s) = \log F_i(e^s)$ for $s \in (-\infty, \log R)$,

$$\phi(\vec{x}) := \inf_{-\infty \leq s \leq \log R} \left(\sum_{i=1}^{d} x_i \psi_i(s) - s \right), \tag{6.1}$$

$$\beta_n(\pi(x)/n) := \mathbb{P}\left[X_n = x\right] e^{-n\phi(\pi(x)/n)},$$

and $s(\vec{x})$ to be the minimizer of $\sum_{i=1}^{d} x_i \phi_i(s) - s$.

[1]Our notation differs slightly from the notation in [15].

Theorem 6.2.1 (Theorem 1 of [15]). *The function β_n satisfies*

$$\lim_{n\to\infty} \sup_{\vec{x}} \frac{\left|\log \beta_n\left(\frac{\vec{x}}{n}\right)\right|}{n} = 0,$$

where the supremum is taken over $\left\{\vec{x} \in \mathbb{Z}_+^d \mid \sum_{i=1}^d x_i \leq n\right\}$.

Moreover, for any fixed $\alpha \in (1/2, 1)$, the asymptotics of β_n are as follows.

1. *There exist a positive constant e_0 and positive vector $\vec{e}$ such that uniformly in the regime $\|\vec{x}\|_1 \leq n^\alpha$:*

$$\beta_n(\vec{x}) \sim n^{-3/2}\left(\vec{x}\cdot\vec{e} - e_0\right) \preceq \frac{\|\vec{x}\|_1 + 1}{n^{3/2}}. \tag{6.2}$$

2. *Uniformly in the regime $n^\alpha \leq \|\vec{x}\|_1 \leq n - n^\alpha$:*

$$\beta_n(\vec{x}) \sim G\left(e^{s(\vec{x})}\right) \cdot \left\{2\pi n \sum_{i=1}^d x_i \psi_i''(s(\vec{x}))\right\}^{-1/2} \preceq \frac{1}{\sqrt{n\,\|\vec{x}\|_1}}. \tag{6.3}$$

3. *Uniformly in the regime $n - n^\alpha \leq \|\vec{x}\|_1 \leq n$:*

$$\beta_n(\vec{x}) \sim k^k e^{-k}/k! \preceq \frac{1}{\sqrt{k}}, \tag{6.4}$$

where $k = n - \|\vec{x}\|_1$.

Remark 6.2.1. There are two mistakes in the statement of (3. 1) of [15]; the symbol Ψ should be replaced with $\times$, and the factor $\{R/4\pi n^3\}$ should be $\left\{\frac{R}{4\pi n^3}\right\}^{1/2}$.

With some effort, we can better understand the behaviour of the exponential rate function ϕ.

From the identity $e^{\psi_i(s)} = \mathbb{E}\left[e^{s\tau_i}; \tau_i < \infty\right]$, we have

$$\psi_i'(s) = \frac{\mathbb{E}\left[\tau_i e^{s\tau_i}; \tau_i < \infty\right]}{\mathbb{E}\left[e^{s\tau_i}; \tau_i < \infty\right]}, \tag{6.5}$$

$$\psi_i''(s) = \frac{\mathbb{E}\left[\tau_i^2 e^{s\tau_i}; \tau_i < \infty\right]}{\mathbb{E}\left[e^{s\tau_i}; \tau_i < \infty\right]} - \left(\frac{\mathbb{E}\left[\tau_i e^{s\tau_i}; \tau_i < \infty\right]}{\mathbb{E}\left[e^{s\tau_i}; \tau_i < \infty\right]}\right)^2. \tag{6.6}$$

As this is the mean and variance of τ_i under exponential tilting, ψ_i', ψ_i'' are positive for each i and for each $s \in (-\infty, \log R)$. Moreover, $\psi_i''(e^s) \to 0$ as $s \to -\infty$, since $\tau_i > 0$.

We use Δ_+^d to mean the set of vectors $\vec{x} \in \mathbb{R}_+^d$ with $\sum_{i=1}^d x_i = 1$. For such a vector $\vec{x} \in \Delta_+^d$, define $\phi_{\vec{x}}(t) := \phi(t\vec{x})$. Let $s_{\vec{x}}(t) = s(t\vec{x})$ be the minimizer of $\sum_{i=1}^d t x_i \psi_i(s) - s$.

By the definition of ϕ and $s_{\vec{x}}(t)$,

$$\begin{aligned}
\phi_{\vec{x}}(0) &= -\log R, \\
\phi'_{\vec{x}}(t) &= \sum_{i=1}^{d} x_i \psi_i(s_{\vec{x}}(t)), \\
\phi'_{\vec{x}}(0) &= \sum_{i=1}^{d} x_i \psi_i(\log(R)), \\
1 &= \sum_{i=1}^{d} t x_i \psi'_i(s_{\vec{x}}(t)). \qquad (6.7)
\end{aligned}$$

Differentiating (6.7),

$$s'_{\vec{x}}(t) = \frac{-1}{t^2 \sum_{i=1}^{d} x_i \psi''_i(s_{\vec{x}}(t))} < 0. \qquad (6.8)$$

In particular, $s_{\vec{x}}(t)$ is strictly decreasing.

We arrive at the formula

$$\phi''_{\vec{x}}(t) = \sum_{i=1}^{d} x_i \psi'_i(s_{\vec{x}}(t)) s'_{\vec{x}}(t) = \frac{s'_{\vec{x}}(t)}{t} = \frac{-1}{t^3 \sum_{i=1}^{d} x_i \psi''_i(s_{\vec{x}}(t))}. \qquad (6.9)$$

The following result is fundamental to proving Gaussian-like tail bounds.

Proposition 6.2.2. *There is a constant $\epsilon_0 > 0$ such that*

$$\phi''_{\vec{x}}(t) \leq -\epsilon_0$$

uniformly over $\vec{x} \in \Delta^d_+$, $t \in [0,1]$.

Proof. We need to understand both $s_{\vec{x}}(t)$ and $\psi''_i(s)$. By Proposition 1 of [15], there are functions H_i, K_i which are analytic in some neighbourhood of $\{|z| \leq R\}$ such that $F_i(z) = H_i(z) + \sqrt{R - z} K_i(z)$.

Notice

$$\begin{aligned}\psi_i'(s) &= \frac{F_i'(e^s)e^s}{F_i(e^s)} \\ &= \frac{H_i(R)e^s - \frac{K_i(R)e^s}{2\sqrt{R-e^s}} + O\left(\sqrt{R-e^s};\ s \to \log R^-\right)}{H_i(R) + O\left(R - e^s;\ s \to \log R^-\right)} \\ &= \frac{-K_i(R)e^s}{2H_i(R)\sqrt{R-e^s}} + O\left((;)\ s \to \log R^-\right)\sqrt{R-e^s}.\end{aligned} \tag{6.10}$$

For brevity, define $\vec{A}$ by

$$A_i = \frac{-K_i(R)}{2H_i(R)};$$

From (6.7),

$$\frac{\sqrt{R - e^{s_{\vec{x}}(t)}}}{t} = e^{s_{\vec{x}}(t)}\vec{x} \cdot \vec{A} + O\left((;)\ t \to 0\right)\sqrt{R - e^{s_{\vec{x}}t}}, \tag{6.11}$$

where the constant is uniformly bounded over $\vec{x}$. Solving for $s_{\vec{x}}(t)$, there are constants $0 < c < C$ and $\epsilon_0 > 0$ such that

$$\log(R - Ct^2) \leq s_{\vec{x}}(t) \leq \log(R - ct^2) \tag{6.12}$$

whenever $t \leq \epsilon_0$. In particular, $s_{\vec{x}}(t) \to \log R$ *uniformly* over $\vec{x} \in \Delta_+^d$ as $t \to 0$.

Again for brevity, define $\vec{B}$ by

$$B_i = \frac{-K_i'(R)}{4F_i(R)}.$$

Differentiating (6.10), we get

$$\begin{aligned}\psi_i''(s) &= \frac{F_i''(e^s)e^{2s}}{F_i(e^s)} + \frac{F_i'(e^s)e^s}{F_i(e^s)} - \frac{F_i'(e^s)^2e^{2s}}{2F_i(e^s)^2} \\ &= (R - e^s)^{-3/2}\frac{-K_i'(R)e^{2s}}{4F_i(R)} + O\left(\frac{1}{R-e^s}\right),\end{aligned}$$

so

$$t^3 \sum_{i=1}^{d} x_i \psi_i''(s_{\vec{x}}(t)) = \left(\frac{t}{\sqrt{R - e^{s_{\vec{x}}(t)}}} \right)^3 \vec{x} \cdot \vec{B} e^{2 s_{\vec{x}}(t)} + t^3 O\left(\frac{t}{R - e^{s_{\vec{x}}(t)}} \right)$$
$$= \left(\vec{x} \cdot \vec{A} + O\left(\sqrt{R - e^{s_{\vec{x}}(t)}} \right) \right)^{-3} \vec{x} \cdot \vec{B} e^{2 s_{\vec{x}}(t)}$$
$$+ t^3 O\left(\frac{1}{\vec{x} \cdot \vec{A} + O\left(\sqrt{R - e^{s_{\vec{x}}(t)}} \right)} \right).$$

Now consider $t \geq \epsilon_0$. By (6.12) and the monotonicity of $s_{\vec{x}}(t)$, there exists a $\delta_0 > 0$ such that $s_{\vec{x}}(t) \leq \log R - \delta_0$ whenever $t \geq \epsilon_0/2$. Since $\psi_i''(s) \to 0$ as $s \to -\infty$, it follows that

$$\sup_{s \leq \log R - \delta_0} \psi_i''(s) < \infty.$$

In particular,

$$t^3 \sum_{i=1}^{d} x_i \psi_i''(s_{\vec{x}}(t)) \leq \vec{x} \cdot \vec{C} \preceq \left\| \vec{C} \right\|_1$$

for $t \geq \epsilon_0/2$, uniformly over $\vec{x}$. □

Lemma 6.2.3. *There exists constants $\epsilon_0, C > 0$ such that*

$$\mathbb{P}\left[B_n(i) = x \right] \leq C \exp\left\{ 2\pi(x) \cdot \nabla\phi(0) - \epsilon_0 |x|^2 \frac{n}{i(n-i)} \right\}$$

uniformly over n, x, i. On the range $|x| \leq i^{3/5}$, $i \leq n/2$ or $|x| \leq (n-i)^{3/5}$, $i \geq n/2$,

$$\mathbb{P}\left[B_n(i) = x \right] \leq \frac{C|x|^2}{i^{3/2}} \exp\left\{ 2\pi(x) \cdot \nabla\phi(0) - \epsilon_0 |x|^2 \frac{n}{i(n-i)} \right\}.$$

Proof. By to the invariance of the bridge distribution under time reversal, we may assume without loss of generality that $i \le n/2$. Consider an arbitrary group element x. We have

$$\begin{aligned}
\mathbb{P}\left[B_n(i) = x\right] &= \mathbb{P}\left[X_i = x\right]\mathbb{P}\left[X_{n-i} = x\right]\mathbb{P}\left[X_n = X_0\right]^{-1} \\
&= \frac{\beta_i\left(\frac{\pi(x)}{i}\right)\beta_{n-i}\left(\frac{\pi(x)}{n-i}\right)}{\beta_n\left(\vec{0}\right)} \exp\left\{i\phi\left(\frac{\pi(x)}{i}\right) + (n-i)\phi\left(\frac{\pi(x)}{n-i}\right) - n\phi(0)\right\} \\
&\preceq \beta_i\left(\frac{\pi(x)}{i}\right)\exp\left\{i\phi\left(\frac{\pi(x)}{i}\right) + (n-i)\phi\left(\frac{\pi(x)}{n-i}\right)\right\} R^n \\
&= \beta_i\left(\frac{\pi(x)}{i}\right) R^n \\
&\quad \times \exp\left\{i\phi(0) + \pi(x)\cdot\nabla\phi(0) + i\frac{\langle \pi(x), \pi(x)\rangle_{D^2\phi(\xi_1)}}{2i^2}\right\} \\
&\quad \times \exp\left\{(n-i)\phi(0) + \pi(x)\cdot\nabla\phi(0) + \frac{\langle \pi(x), \pi(x)\rangle_{D^2\phi(\xi_2)}}{2(n-i)}\right\}
\end{aligned}$$

for some vectors ξ_1, ξ_2 of the form $t_i\vec{x}$ for some $t_i \in [0,1]$. By Proposition 6.2.2, there is some $\epsilon_0 > 0$, independent of $\vec{x}$, such that $\langle \pi(x), \pi(x)\rangle_{D^2\phi(\xi)} \le -2\epsilon_0|x|^2$ uniformly over $\xi \in \mathbb{R}^d_+$ such that $\sum_{i=1}^d \xi_i \le 1$. As $\phi(0) = -\log R$, it follows that

$$\begin{aligned}
\mathbb{P}\left[B_n(i) = x\right] &\le \beta_i(\pi(x)/i)\exp\left\{2\pi(x)\cdot\nabla\phi(0) + \frac{\langle \pi(x), \pi(x)\rangle_{D^2\phi(\xi_1)}}{2i} + \frac{\langle \pi(x), \pi(x)\rangle_{D^2\phi(\xi_2)}}{2(n-i)}\right\} \\
&\le \beta_i(\pi(x)/i)\exp\left\{2\pi(x)\cdot\nabla\phi(0) - \epsilon_0|x|^2\frac{n}{i(n-i)}\right\}
\end{aligned}$$

uniformly over n and the choices of x and i satisfying the hypothesis.

Since $\beta_i(\pi(x)/i)$ is always bounded, and $n - i \ge n/2$,

$$\mathbb{P}\left[B_n(i) = x\right] \le C\exp\left\{2\pi(x)\cdot\nabla\phi(0) - \epsilon_0|x|^2/2n\right\}$$

uniformly over n, $i \le n/2$, and $|x| \le \delta_0 n$. If $|x| \le i^{3/5}$, then $\beta_i(\pi(x)/i) \preceq \frac{|x|}{i^{3/2}}$, giving the sharper bound. □

6.2.2 Tail estimates and tightness

We apply the results of the previous two sections to deduce uniform superexponential tail bounds for $|B_n(nt)|/\sqrt{n}$.

Proposition 6.2.4. *There exists $C, \epsilon_0 > 0$ such that uniformly over $i \in \{0, 1, \ldots, n\}$,*

$$\mathbb{P}\left[|B_n(i)| \geq x\sqrt{n}\right] \leq Ce^{-\epsilon_0 x^2 \frac{n}{i}}.$$

Proof. Using the weaker asymptotics of Lemma 6.2.3 as well as Equation (6.15),

$$\begin{aligned}
\sum_{k=x\sqrt{n}}^{i} \mathbb{P}\left[|B_n(i)| = k\right] &\preceq \sum_{k=x\sqrt{n}}^{i} ke^{-\epsilon_0 k^2 \frac{n}{i(n-i)}} \sum_{|x|=k} e^{2\pi(x)\cdot\nabla\phi(0)} \\
&\preceq \sum_{k=x\sqrt{n}}^{i} ke^{-\epsilon_0 k^2/i} \\
&\leq \int_{x\sqrt{n}}^{\infty} ue^{-\epsilon_0 u^2/i} du \\
&= \frac{i}{2\epsilon_0} e^{-\epsilon_0 x^2 \frac{n}{i}}.
\end{aligned}$$

□

We use the tail estimates from Proposition 6.2.4 to deduce Gaussian-like tail estimates for the modulus of continuity.

It easily follows from Proposition 6.2.5 that $\mathbb{E}[\tilde{M}_{n,\epsilon}] = o\left(1;\ \epsilon \to 0\right)$.

Proposition 6.2.5. *Let $\omega_n(\delta) = \max\left\{\left|\frac{||B_n(nt)|-|B_n(ns)||}{\sqrt{n}}\right| \mid |t-s| \leq \delta\right\}$ be the modulus of continuity. There exists C, $\epsilon_1 > 0$ such that*

$$\mathbb{P}\left[\omega_n(\delta) \geq x\right] \leq Ce^{-\epsilon_1 x^2/\delta}.$$

Proof. Let $M_{n,\delta} = n^{-1/2} \max_{i=1,\ldots,n\delta} |B_n(i)|$. For each k, $i \leq 2^k$, define

$$\begin{aligned}
D_{i,k} &:= \frac{d\left(B_n\left(n\delta\frac{i}{2^k}\right), B_n\left(n\delta\frac{i+1}{2^k}\right)\right)}{\sqrt{n}} \\
M_k &:= \max_i D_{i,k}.
\end{aligned}$$

By Proposition 6.2.4

$$\mathbb{P}\left[D_{i,k} \geq x\right] = \mathbb{P}\left[|B_n(n\delta i 2^{-k})| \geq x\sqrt{n}\right] \leq Ce^{-\epsilon_0 x^2 2^k/\delta}.$$

Hence,

$$\mathbb{P}\left[M_k \geq x\right] \leq 2^k \mathbb{P}\left[D_{1,k} \geq x\right] \leq C2^k e^{-\epsilon_0 x^2 2^k/\delta}.$$

Since $M \leq \sum_{k=1}^{\log_2 n} M_k$, for any $\epsilon > 0$,

$$\begin{aligned}\mathbb{P}\left[M_{n,\delta} \geq x\right] &\leq \mathbb{P}\left[M_k \geq x(1-\epsilon)\epsilon^{k-1}\ \exists k\right] \\ &\leq \sum_{k=1}^{\log_2 n} \mathbb{P}\left[M_k \geq (1-\epsilon)x\epsilon^{k-1}\right] \\ &\preceq \sum_{k=1}^{\infty} 2^k e^{-\epsilon_0 x^2 (2\epsilon^2)^k/\delta} \\ &\preceq \int_1^{\infty} 2^u e^{-\epsilon_0 x^2 (2\epsilon^2)^u/\delta} dx.\end{aligned}$$

Choosing $\epsilon < 1/\sqrt{2}$, this is at most $Ce^{-\epsilon_1 x^2/\delta}$ for some $C, \epsilon_1 > 0$.

Finally, note that

$$M_{n,i,\delta} := \max\left(\frac{||B_n(j)| - |B_n(ni\delta)||}{\sqrt{n}} \middle| ni\delta \leq j \leq n(i+1)\delta\right)$$

has the same law as $M_{n,\delta}$ Since $\omega_n(\delta) \leq 2\max_{j=1,2,\dots,\delta^{-1}} M_{n,i,\delta}$,

$$\begin{aligned}\mathbb{P}\left[\omega_n(\delta) \geq x\right] &\leq \sum_{i=1}^{1/\delta} \mathbb{P}\left[M_{n,i,\delta} \geq x/2\right] \\ &\leq \delta^{-1} C e^{-\epsilon_1 x^2/4\delta} \\ &\preceq C e^{-\epsilon_1 x^2/8\delta}.\end{aligned}$$

□

6.2.3 Local time of bridges

We show that a random walk bridge spends a small amount of time near its starting point. Recall the function ϕ from Section 6.2.1.

Lemma 6.2.6. *Consider a random walk bridge B_n of duration n. Let $L_x = \sum_{i=0}^{n} \mathbf{1}(B_n(i) = x)$ be the local time of the process $B_n(\cdot)$ at vertex x. (x may depend on n.) Then*

$$\mathbb{E}\left[L_x\right] \asymp |x| e^{2\pi(x)\cdot\nabla\phi(0)}$$

and hence by (6.15),

$$\sum_{|x|=k} \mathbb{E}\left[L_x\right] \asymp |x|.$$

(When $x = \rho$, $\mathbb{E}[L_x]$ is easily seen to converge to a positive constant.)

Proof. Write

$$\begin{aligned}\mathbb{E}\left[\, L_x \,\right] &= \mathbb{E}\left[\sum_{i=0}^{n} \mathbf{1}(B(i) = x)\right] \\ &\leq 2\left(\sum_{i=|x|}^{|x|+|x|^{3/4}} + \sum_{i=|x|+|x|^{3/4}}^{|x|^{5/3}} + \sum_{i=|x|^{5/3}}^{n/2}\right)\mathbb{P}\left[B(i) = x\right].\end{aligned}$$

(This only lacks equality because the ranges of summation may overlap.)

We apply Lemma 6.2.3:

1. On the range $|x| \leq i \leq |x| + |x|^{3/4}$, (6.4) implies $\beta_i\left(\frac{\pi(x)}{i}\right) \preceq \frac{1}{\sqrt{i-|x|}}$, so this sum is at most

$$\sum_{i=|x|}^{|x|+|x|^{3/4}} |x| \frac{e^{2\pi(x)\cdot\nabla\phi(0)} e^{-C|x|^2/i}}{\sqrt{i - |x|}} \preceq e^{2\pi(x)\cdot\nabla\phi(0)} e^{-C|x|} |x|^{11/8} \preceq |x| e^{2\pi(x)\cdot\nabla\phi(0)}.$$

2. On the range $|x| + |x|^{3/4} \leq i \leq |x|^{5/3}$, (6.3) implies $\beta_i\left(\frac{\pi(x)}{i}\right) \preceq \frac{1}{\sqrt{n|x|}}$, so this sum is at most

$$\sum_{i=|x|+|x|^{3/2}}^{|x|^{5/3}} \sqrt{\frac{|x|}{i}} e^{2\pi(x)\cdot\nabla\phi(0)} e^{-Cx^2/i} \preceq |x|^{7/6} e^{2\pi(x)\cdot\nabla\phi(0)} e^{-C|x|^{1/5}} \preceq |x| e^{2\pi(x)\cdot\nabla\phi(0)}.$$

3. On the range $|x|^{5/3} \leq i \leq n/2$, we can use (6.2), so the sum is at most

$$\begin{aligned}\sum_{i=|x|^{5/3}}^{n/2} \frac{|x|^2 e^{2\pi(x)\cdot\nabla\phi(0)}}{i^{3/2}} e^{-\frac{\langle \pi(x), \pi(x)\rangle_E}{i}} &\asymp |x|^2 e^{2\pi(x)\cdot\nabla\phi(0)} \sum_{i=|x|^{5/3}}^{n/2} \frac{e^{-C|x|^2/i}}{i^{3/2}} \\ &\asymp |x|^2 e^{2\pi(x)\cdot\nabla\phi(0)} \int_{|x|^{5/3}}^{n/2} \frac{e^{-C|x|^2/t}}{t^{3/2}} dt \\ &\asymp |x| e^{2\pi(x)\cdot\nabla\phi(0)}.\end{aligned}$$

□

We now prove a version of Lemma 6.2.6 where the radius of the sphere grows slowly in n, but we only consider times in the 'middle section' of the bridge.

Lemma 6.2.7. *Let $L^{(n)}_{j,k} := \sum_{i=l}^{n-l} \mathbf{1}(|B_n(i)| \le k)$ be the local time of a random walk bridge B_n of duration n in the ball of radius k, restricted to times $\{j, j+1, \dots, n-j\}$. Then if $k = o\,(\min(j, n-j)^2)$, we have*

$$\mathbb{E}\left[L^{(n)}_{j,k} \right] = O\left(\frac{(k+1)^{9/4}}{j^{1/2}} \right) + o\left(\frac{(k+1)}{j^2};\ j \to \infty \right)$$

Proof. Without loss of generality we assume $j \le n/2$.

Recall

$$\mathbb{P}\left[\, B_n(i) = x \,\right] \sim C(\pi(x) \cdot \mu + O(1))^2 e^{2\pi(x)\cdot \nabla\phi(0)} \frac{n^{\frac{3}{2}}}{i^{3/2}(n-i)^{3/2}}.$$

whenever $|x|^2 = o(\min(j, n-j))$. Define $L^{(n)}_{j,x}$ to be the local time of B_n at the site x restricted to the time $\{j, \dots, n-j\}$. For large enough n,

$$\begin{aligned}\sum_{|x|=k} \sum_{i=j}^{n-j} \mathbb{P}\left[\, B_n(i) = x \,\right] &\preceq (k+1)^2 \sum_{i=j}^{n/2} \frac{e^{-Ck^2/i}}{i^{3/2}} \\ &\preceq (k+1)^2 \sum_{i=j}^{\infty} \frac{e^{-Ck^2/i}}{i^{3/2}} \\ &\preceq \frac{(k+1)^{9/4}}{j^{1/2}}.\end{aligned}$$

□

This gives the following very crude result, as well as tightness for the last return time to the root.

Corollary 6.2.8.

$$\mathbb{P}\left[|B_n(i)| \le k \text{ for some } i \in \{j, j+1, \dots, n-j\} \right] \preceq \frac{k^{9/4}}{j^{1/2}}.$$

Corollary 6.2.9. *Let B be a random walk bridge of duration n. Let $T = \min\{i \le n/2 \mid |B(i)| = r\}$. Then*

$$\mathbb{P}[T \ge x] \to 0 \text{ as } x \to \infty$$

uniformly over n.

Next, we show that two independent bridges only intersect near their starting point.

Lemma 6.2.10. *Consider two independent random walk bridges B (resp. B'). Let A_k be the event that B and B' intersect outside of the ball of radius k. There are constants $C, \epsilon_0 > 0$ such that*

$$\mathbb{P}\left[\, A_k \,\right] \le Ce^{-\epsilon_0 k}.$$

Proof. Let n (resp. n') be the duration of B (resp. B'). Let $\tilde{A}_k$ be the event that there is an intersection at distance k. Then

$$\begin{aligned}
\mathbb{P}\left[\, \tilde{A}_k \,\right] &\le \sum_{i=1}^{n}\sum_{j=1}^{n'}\sum_{|x|=k} \mathbb{P}\left[\, B(i) = x, B'(j) = x \,\right] \\
&= \sum_{|x|=k}\sum_{i=1}^{n}\sum_{j=1}^{n'} \mathbb{P}\left[\, B(i) = x \,\right]\mathbb{P}\left[\, B'(j) = x \,\right] \\
&= \sum_{|x|=k}\left(\sum_{i=1}^{n} \mathbb{P}\left[\, B(i) = x \,\right]\right)\left(\sum_{i=1}^{n'} \mathbb{P}\left[\, B'(i) = x \,\right]\right) \\
&= \sum_{|x|=k} \mathbb{E}\left[\, L_x \,\right]\mathbb{E}\left[\, L'_x \,\right] \\
&\preceq k^2 \sum_{|x|=k} e^{4\pi(x)\cdot\nabla\phi(0)}
\end{aligned}$$

by Lemma 6.2.6. Since $\pi(x)$, $\nabla\phi(0)$ belong to $\mathbb{R}^d_+$, there is a $C > 0$ such that $\pi(x) \cdot \nabla\phi(0) \le -Ck$. Hence,

$$\mathbb{P}\left[\, \tilde{A}_k \,\right] \preceq k^2 e^{-Ck} \sum_{|x|=k} e^{2\pi(x)\cdot\nabla\phi(0)} \preceq e^{-Ck/2}.$$

The union bound gives

$$\mathbb{P}\left[\, A_k \,\right] \le \sum_{j=1}^{n} \mathbb{P}\left[\, \tilde{A}_j \,\right] \preceq \sum_{j=k}^{n} e^{-Ck/2} \preceq e^{-Ck/4}.$$

□

6.3 The distribution of the word profile

Recall the definitions of $F_i(z)$, R from Section 6.2.1.

Proposition 6.3.1. *Let* $\mu_i = F_i(R)^2$. *Then*

$$\sum_i \frac{\mu_i}{\mu_i + 1} = 1 \tag{6.13}$$

Proof. The following is a result of [10]. Define σ as in Chapter 5 by

$$\sigma(x) = \lim_{n\to\infty} \frac{p^{(n)}(x)}{p^{(n)}(e)}.$$

Then σ is $1/R$-harmonic:

$$\sum_y p(x,y)\sigma(y) = \frac{1}{R}\sigma(x).$$

To each positive z-harmonic function ϕ on a hyperbolic group G, there exists a probability measure ν on the Poisson boundary Ω of G such that

$$\phi(x) = \int_\Omega K_z(x,\omega)d\nu(\omega),$$

where $K_z(x,\omega)$ is the Poisson kernel corresponding to the random walk kernel $p(x,y)$. [2]

In our case, the Poisson boundary is homeomorphic to the space of ends, ie. the set of infinite, reduced words $x_1x_2x_3\cdots$. Let ν be the boundary measure corresponding to σ, and let E_x be the set of (infinite, reduced) words in Ω beginning with x. From [10], Theorem 4,

$$\nu_i := \nu(E_{a_i}) = \frac{\mu_i}{\mu_i + 1},$$

As the collection $\{E_{a_1}, \ldots, E_{a_d}\}$ is a partition of Ω, it follows that

$$\sum_i \frac{\mu_i}{1+\mu_i} = \sum_i \nu_i = 1.$$

□

In fact, it is shown in [10] that the harmonic measure ν is multiplicative: if $x = a_{i_1}a_{i_2}\cdots a_{i_k}$, then

$$\nu(E_x) = \mu_{i_1}\mu_{i_2}\cdots\mu_{i_{k-1}}\nu_{i_k}. \tag{6.14}$$

[2] We refer you to [23] for the definition of the Poisson kernel.

In particular, this implies that

$$\sum_{|x|=k} e^{2\pi(x)\cdot\nabla\phi(0)} \cdot \nu_{i_k}/\mu_{i_k} = 1 \tag{6.15}$$

when a_{i_k} is the last letter of x.

To avoid cumbersome notation, for the remainder of this section, we use the following convention: when u and v are vectors and $f : \mathbb{R} \to \mathbb{R}$ is a function, we say that $f(u)$ is the vector with components $f(u_i)$: $f(u)_i := f(u_i)$; and uv is the componentwise product of u and v: $(uv)_i := u_i v_i$.

Make the following definitions.

- N_π is the number of words x in the tree $\mathbb{T}_d$ with $\pi(x) = \pi$.
- Ball_n is the ball of radius n in $\mathbb{Z}_+^d$ in the L^1 norm.
- When m is supported in $\mathbb{R}_+^d$, $L[m](\vec{y}) := \int_{\mathbb{R}_+^d} e^{-x\cdot\vec{y}} dm(x)$ is the Laplace transform of the measure m.
- $C_0^+(\mathbb{R}_+^d)$ is the space of positive continuous functions on $\mathbb{R}_+^d$ decaying at infinity.
- Define the vector $v_\mu = \frac{1-\mu}{1+\mu}$.

Define the family of measures

$$\begin{aligned} m_n(\vec{y}) &:= \frac{1}{\sqrt{n}} \sum_{\pi\in\mathbb{Z}_d^+} N_\pi e^{2\pi\cdot\nabla\phi(0)} \delta_{\pi/\sqrt{n}}(\vec{y}) \\ &= \frac{1}{\sqrt{n}} \sum_{x\in\mathbb{T}_d} e^{2\pi(x)\cdot\nabla\phi(0)} \delta_{\frac{\pi(x)}{\sqrt{n}}}(\vec{y}). \end{aligned}$$

Theorem 6.3.2. *The measure m_n converges in the weak* topology to the Lebesgue measure on the infinite ray $v_\mu\mathbb{R}_+$.*

Proof. We will see that the Laplace transform can be computed via the eigenvalues of a particular non-negative matrix $M_{\vec{y},n}$. We use the identity (6.13) to conclude that the Laplace transform of m_n converges to the Laplace transform of the Lebesgue measure concentrated on a particular line in $\mathbb{R}_+^d$.

The Laplace transform of m_n is:

$$\mathscr{L}[m_n](\vec{y}) = \frac{1}{\sqrt{n}} \sum_{x\in\mathbb{T}_d} e^{\pi(x)\cdot\left(2\nabla\phi(0)-\frac{\vec{y}}{\sqrt{n}}\right)}. \tag{6.16}$$

Consider the complete graph K_d on d vertices $\{1, 2, \ldots, d\}$. There is a 1-1 correspondence between paths p in K_d and vertices x in $\mathbb{T}_d$: the path p through vertices $(p_1, p_2, \ldots, p_n)$ corresponds to the word $\prod_{i=1}^n a_{p_i}$. Hence, the expression (6.16) is equal to

$$\frac{1}{\sqrt{n}} \sum_{\text{paths } p \text{ in } K_d} e^{\pi(p)\cdot\left(2\nabla\phi(0) - \vec{y}/\sqrt{n}\right)}.$$

Recall that $\mu_i = F_i(R)^2$, so $e^{\pi\cdot 2\nabla\phi(0)} = \prod_i \mu_i^{\pi_i}$. Define the matrices

- $D_\mu := \operatorname{diag}(\mu_i)$,
- $M_0 := (1 - I)D_\mu$,
- $D_{\vec{y},n} := \operatorname{diag}\left(\mu e^{-\frac{\vec{y}}{\sqrt{n}}}\right)$,
- $M_{\vec{y},n} := (1 - I)D_{\vec{y},n} = (1 - I)\left(D_\mu - D_{\frac{\mu\vec{y}}{\sqrt{n}}} + O(1/n)\right)$,

where the last inequality is the first-order Taylor expansion. Then

$$\left\|M_{\vec{y},n}^k\right\|_1 = \sum_{\text{paths } p \text{ of length } k} e^{\pi(p)\cdot\left(2\nabla\phi(0) - \frac{\vec{y}}{\sqrt{n}}\right)},$$

and hence

$$\begin{aligned} L[m_n](\vec{y}) &= \frac{1}{\sqrt{n}} \sum_{\text{paths } p \text{ in } K_d} e^{\pi(p)\cdot\left(2\nabla\phi(0) - \frac{\vec{y}}{\sqrt{n}}\right)} \\ &= \frac{1}{\sqrt{n}} \left\|(I - M_{\vec{y},n})^{-1}\right\|_1 \\ &= \frac{1}{\sqrt{n}} \left\|\left(I + D_{\vec{y},n} - \vec{1}^{\,T} \mu e^{\frac{-\vec{y}}{\sqrt{n}}}\right)^{-1}\right\|_1 \end{aligned}$$

By the ShermanMorrison formula,

$$\begin{aligned} \left\|\left(I + D_{\vec{y},n} - \vec{1}^{\,T}\mu e^{\frac{-\vec{y}}{\sqrt{n}}}\right)^{-1}\right\|_1 &= \left\|(I + D_{\vec{y},n})^{-1}\right\| \\ &\quad + \left\|\frac{(I - D_{\vec{y},n})^{-1}\,\vec{1}\left(\mu e^{\frac{-\vec{y}}{\sqrt{n}}}\right)(I - D_{\vec{y},n})^{-1}}{1 - \left(\mu e^{\frac{-\vec{y}}{\sqrt{n}}}\right)(I + D_{\vec{y},n})^{-1}\,\vec{1}}\right\|_1 \\ &= \frac{\sqrt{n}}{\vec{y}\cdot\frac{1-\mu}{1+\mu}} + O\left(1/\sqrt{n}\right) \end{aligned}$$

by Proposition 6.3.1.

To conclude, we use the following variant of e.g. Theorem 4.3 from [13].

Theorem 6.3.3. *Let μ_n, μ be (possibly infinite) measures on $\mathbb{R}_+^d$. Then $\mu_n \to \mu$ vaguely if and only if $L(\mu_n) \to L(\mu)$.*

While the proof in [13] is for the Fourier transform of probability measures, it is essentially the same for the Laplace transform of measures supported on $\mathbb{R}_+^d$ with one exception: the Stone-Weierstrass Theorem only guarantees uniform approximation of functions in $C_0^+(\mathbb{R}_+^d)$ by the family $\left\{e^{-\vec{y}\cdot\vec{x}} : \vec{y} \in \mathbb{R}_+^d\right\}$. Since the tail of eg. the Lebesgue measure cannot be disregarded, we cannot use bounded continuous functions as our test functions, as is the case for probability measures, and so we don't get the 'weak' convergence of probability theory.

□

Corollary 6.3.4. *Consider the location $v = B_n(nT)$ of a random walk bridge at an independent uniform time T. Let $\pi(v)$ be the word profile of v. Then*

1. $\frac{\pi(v)}{|\pi(v)|}$ *converges in probability to the constant vector $\vec{v}_\mu$.*
2. $\frac{|\pi(v)|}{\sqrt{n}}$ *converges in distribution to a random variable with density*

$$4Z_\mu x e^{-2C_\mu x^2}, \tag{6.17}$$

where Z_μ is the normalizing constant.

Proof. Recall that the Laplace transforms of the measures

$$m_n = \frac{1}{\sqrt{n}} \sum_{\pi \in \mathrm{Ball}_{n/2}} N_\pi e^{2\pi\cdot\nabla\phi(0)} \delta_{\frac{\pi}{\sqrt{n}}}.$$

converge point-wise to the Laplace transform of the Lebesgue measure m_μ of the line $\mathbb{R}_+\vec{v}_\mu$. Hence, m_n converges vaguely to m_μ. Define

$$f(\pi) = \int_0^1 \frac{(\pi\cdot\vec{e})^2}{[t(1-t)]^{3/2}} e^{-C_d \frac{<\pi,\pi>_E}{t(1-t)}} dt,$$

where $E = D^2\phi(0)$. We claim that the measures

$$\tilde{m}_n(\pi) := f\left(\frac{\pi}{\sqrt{n}}\right) m_n$$

converge weakly to the Raileigh distribution. Consider some continuous function $g : \mathbb{R}_+^d \to \mathbb{R}$ with bounded support. Then $g(\pi) f\left(\frac{\pi}{\sqrt{n}}\right)$ is also continuous with bounded support, which means

$$\int_{\mathbb{R}_+^d} g(\pi) d\tilde{m}_n(\pi) = \int_{\mathbb{R}_+^d} g(\pi) f\left(\frac{\pi}{\sqrt{n}}\right) dm_n(\pi) \to \int_0^\infty g(xv_\mu) f(C'_\mu x v_\mu) e^{-C_\mu \frac{x^2}{t(t-1)}} dx.$$

Hence, the measures $\tilde{m}_n$ converge *weakly* to the probability measure

$$Z_\mu \left(\int_0^1 \frac{x^2}{[t(1-t)]^{3/2}} e^{-C_\mu \frac{x^2}{t(1-t)}} dt \right) dx. \tag{6.18}$$

Modulo tightness, (6.18) is the density of a (rescaled) Brownian excursion at a uniform time T. From [16] (see also (3.4) of [21] or (6.2) of [7] for probabilistic proofs), (6.18) is actually the density of a (rescaled) Rayleigh distribution.

Finally, the probabilities

$$\begin{aligned}\mathbb{P}\left[\frac{|B_n(nT)|}{\sqrt{n}} \in [a,b]\right] &= \sum_{|v|=a\sqrt{n}}^{b\sqrt{n}} \int_0^1 \mathbb{P}\left[B_n(nt) = v\right] dt \\ &= \int_0^1 \sum_v \mathbf{1}(\|\pi(v)\|_1 \in [a,b]) \frac{(\pi(v)\cdot\mu)^2}{[t(1-t)]^{3/2}} e^{-C_d \frac{<\pi(v),\pi(v)>_E}{t(1-t)}} dt + o\left(1;\ n\to\infty\right),\end{aligned}$$

where the $o(1)$ term is *uniform* over $|\pi(v)| \in [a,b]$. Hence, we have

$$\begin{aligned}\mathbb{P}\left[\frac{|B_n(nT)|}{\sqrt{n}} \in [a,b]\right] &= \int_{\mathbb{R}_+^d} \mathbf{1}(\|\vec{x}\|_1 \in [a,b]) d\tilde{m}_n(\pi) + o\left(1;\ n\to\infty\right) \\ &= Z_\mu \int_a^b \left(\int_0^1 \frac{x^2}{[t(1-t)]^{3/2}} e^{-C_\mu \frac{x^2}{t(1-t)}} dt \right) dx + o\left(1;\ n\to\infty\right)\end{aligned}$$

Tightness follows from the tail estimate in Propostion 6.2.4.

□

6.4 Asymptotic independence of N and n_i

Consider a triple $((B_1,t_1),(B_2,t_2),(B_3,t_3))$ of 1-marked bridges. In this section, let B be the 3-marked bridge obtained by gluing B_i together at their starting point, and shifting time to t_1.

Recall that $N = N(B)$ is the number of ways of decomposing the 3-marked bridge $B = (B_n, t_1, t_2, t_3)$ into three 1-marked bridges (B_1, t_1'), (B_2, t_2'), (B_3, t_3').

We show asymptotic independence under $\mathbb{P}^N_{n,3}$ in two steps: first, we show that with high probability, N is a function of only the first and last few steps of the B_i; second, we show that n_i are asymptotically independent of the first and last few steps.

For a triple $((B_1, t_1), (B_2, t_2), (B_3, t_3))$ of 1-marked bridges, the initial part is defined as

$$\mathrm{Init}_M(B) := (\mathrm{Init}_M(B_1), \mathrm{Init}_M(B_2), \mathrm{Init}_M(B_3)).$$

Define $N_M = N_M(B)$ to be the number of ways of decomposing B into three 1-marked bridges *using only the times in the initial part*. Then, $N_M(B)$ is a function of $\mathrm{Init}_M(B)$.

Proposition 6.4.1. $\mathbb{P}^N_{n,3}[N \neq N_M] = O(M^{-1/5})$.

Proof. Combining Corollary 6.2.8 with $j = M$ and $k = M^{1/10}$, and Lemma 6.2.10, implies that with probability $1 - O\left(\frac{M^{3/10}}{M^{1/2}} + e^{-CM^{1/10}/2}\right) = 1 - O(M^{-1/5})$ there is no decomposition of a 3-labelled bridge B involving times outside of Init_M. □

Lemma 6.4.2. *Let $Init^{(i)}_M$ be any choice of the initial part of a random walk bridge, for $i = 1, 2, 3$. Let $Init = \left(Init^{(1)}_M, Init^{(2)}_M, Init^{(3)}_M\right)$. Let $B = (B_1, B_2, B_3)$ chosen according to the conditional measure $\mathbb{P}^N_{n,Init} := \mathbb{P}^N_{n,3}[\cdot \mid Init_M(B) = Init]$.*

Then

$$\Delta_n := \left(\frac{|B_1|}{n}, \frac{|B_2|}{n}, \frac{|B_3|}{n}\right)$$

converges in distribution to a Dirichlet-$(1/2, 1/2, 1/2)$ random variable. In particular, $\Delta_n(B)$ is asymptotically independent of $Init_M(B)$, and hence of $N_M(B)$, $N(B)$.

Proof. Let $B_n = ((B_1, t_1), (B_2, t_2), (B_3, t_3))$ be chosen according to the measure $\mathbb{P}^N_{n,\mathrm{Init}}[\cdot]$. That is, the probability of a particular triple $((B_1, t_1), (B_2, t_2), (B_3, t_3))$ is proportional to

$$\prod_{i=1}^{3} n_1 \mathbb{P}\left[(X_0, X_1, \ldots, X_{n_i}) = B_i\right]$$

for those triples with the specified initial part, and 0 otherwise. In particular,

$$\begin{aligned}
\mathbb{P}^N_{n,\mathrm{Init}}\left[(|B_1|, |B_2|, |B_3|) = (n_1, n_2, n_3)\right] &\propto \sum_{|B_i| = n_i, i=1,2,3} \prod_{i=1,2,3} n_i \mathbb{P}\left[(X_0, X_1, \ldots, X_{n_i}) = B_i \mid \mathrm{Init}\right] \\
&= \prod_{i=1,2,3} n_i \mathbb{P}\left[X_{n_i} = X_0 \mid \mathrm{Init}\right] \\
&= C^3_{\mathrm{Init}} \frac{R^{-n}}{\sqrt{n_1 n_2 n_3}}(1 + o(1)),
\end{aligned}$$

where the $o(1)$ term is uniformly small as n_1, n_2, n_3 simultaneously tend to ∞. Moreover, the normalizing factor is

$$Z_{n,\text{Init}} := \sum_{n_1+n_2+n_3=n} \prod_{i=1,2,3} \mathbb{P}\left[X_{n_i} = X_0 \,|\, \text{Init}\right]. \tag{6.19}$$

Since B_n given $\text{Init}_M(B_n) = \text{Init}$ is a random walk bridge between the (fixed) endpoints of Init,

$$\mathbb{P}\left[X_n = X_0 \,|\, \text{Init}_M = \text{Init}\right] = C_{\text{Init}} \frac{R^{-n}}{n^{3/2}} \left(1 + o\left(1;\ n \to \infty\right)\right),$$

giving

$$\begin{aligned} Z_{n,\text{Init}} = C_{\text{Init}}^3 R^{-n} &\sum_{\substack{n_1+n_2+n_3=n \\ n_i > N_0 \forall i}} \frac{1}{\sqrt{n_1 n_2 n_3}} \left(1 + o\left(1;\ N_0 \to \infty\right)\right) \\ &+ O\left(\sum_{\substack{n_1+n_2+n_3=n \\ n_3 \le N_0 \exists i}} \prod_{i=1,2,3} \mathbb{P}\left[X_{n_i} = X_0 \,|\, \text{Init}_M(X_n) = \text{Init}\right] \right) \\ &+ O\left(\sum_{\substack{n_1+n_2+n_3=n \\ n_2, n_3 \le N_0 \exists i}} \prod_{i=1,2,3} \mathbb{P}\left[X_{n_i} = X_0 \,|\, \text{Init}_M(X_n) = \text{Init}\right] \right). \end{aligned}$$

Denote these sums respectively by Σ_1, Σ_2, Σ_3. By recognizing a Riemann sum,

$$\begin{aligned} \Sigma_1 &= C_{\text{Init}}^3 R^{-n} \sum_{\substack{n_1+n_2+n_3=n \\ n_i > N_0\ \forall i}} \frac{1}{\sqrt{n_1 n_2 n_3}} \left(1 + o\left(1;\ N_0 \to \infty\right)\right) \\ &= C_{\text{Init}}^3 R^{-n} n^{1/2} \sum_{\substack{n_1+n_2+n_3=n \\ n_i > N_0\ \forall i}} \frac{1}{\sqrt{\frac{n_1}{n}\frac{n_2}{n}\frac{n_3}{n}}} \frac{1}{n^2} \left(1 + o\left(1;\ N_0 \to \infty\right)\right) \\ &= C_{\text{Init}}^3 R^{-n} n^{1/2} \int_{\Delta_+^3} \frac{1}{\left(x_1 x_2 x_3\right)^{1/2}} dx_1 dx_2 dx_3 \left(1 + o\left(1;\ N_0 \to \infty\right)\right). \end{aligned}$$

On the other hand, if exactly 1 of n_1, n_2, n_3 is at most N_0 – without loss of generality, say, n_3 – then

$$\sum_{\substack{n_1+n_2+n_3=n\\ n_1,n_2>N_0,n_3\leq N_0}} \prod_{i=1,2,3} \mathbb{P}\left[X_{n_i}=X_0 \,|\, \mathrm{Init}_M(X_n)=\mathrm{Init}\right] = O\left(R^{-n}\sum_{\substack{n_1+n_2+n_3=n\\ n_1,n_2>N_0,n_3<N_0}} \frac{1}{\sqrt{n_1n_2n_3}}\right)$$
$$= O\left(R^{-n}n^{-1/2}\int_{\Delta_+^2}\frac{1}{\sqrt{x_1x_2}}dx_1dx_2\right)$$
$$= o(\Sigma_1).$$

Similar analysis shows that $\Sigma_3 = o(\Sigma_1)$. We conclude that

$$Z_{n,\mathrm{Init}} = C_{\mathrm{Init}}^3 R^{-n}n^{1/2}\int_{\Delta_+^3}\frac{1}{(x_1x_2x_3)^{1/2}}dx_1dx_2dx_3\left(1+o\left(1;\ N_0\to\infty\right)\right),$$

and hence for (x_1,x_2,x_3) in the 3-dimensional unit simplex,

$$n^2\mathbb{P}\left[|B_i|=nx_i,\ i=1,2,3 \,|\, \mathrm{Init}_M(B_n)=\mathrm{Init}\right] = \frac{(x_1x_2x_3)^{-1/2}}{\int_{\Delta_+^3}(x_1x_2x_3)^{-1/2}dx_1dx_2dx_3}\left(1+o\left(1;\ N_0\to\infty\right)\right),$$

which is the density of a Dirichlet-$(1/2,1/2,1/2)$ random variable.

In particular, the distributional limit of $\Delta_n := \left(\frac{n_1}{n},\frac{n_2}{n},\frac{n_3}{n}\right)$ doesn't depend on Init_M, which easily implies that Δ_n is asymptotically independent of N_M.

Since $\lim_{M\to\infty}\limsup_{n\to\infty}\mathbb{P}_{n,3}^N\left[N\neq N_M\right]=0$, it follows that Δ_n and N_n are asymptotically independent. □

6.5 Recursive decomposition of marked bridges

To avoid writing $2k-3$ everywhere, in this section we fix k' and set $k=2k'-3$.

To prove continuity of the bridge measure, we push further the self-similar decomposition from Chapter 4.

Given a random walk bridge B_n, we will pick k' ordered times $s_1,\ldots,s_{k'}$ in $\{1,2,\ldots,n\}$ uniformly at random to obtained a k'-marked random walk bridge B_n. For technical reasons, choose them without replacement to ensure distictness. As this happens with probability $1-o\left(1;\ n\to\infty\right)$, asymptotically this has no effect. Let $V=\{B_n(s_1),\ldots,B_n(s_{k'})\}$. By labelling the vertex $B_n(s_i)$ with the integer i, we obtain a measure on labelled trees with k' leaves, namely the distribution of the reduced subtree $r(V)$ with these labels.

We consider the order statistics of $(s_1,\ldots,s_{k'})$. Let $t_1\leq t_2\leq\ldots\leq t_{k'}$ be the order statistics. To recover uniform times $s_1, \ldots, s_k$, simply randomize the indices. Since the random walk bridge is translation invariant, upon shifting the random walk bridge by $(B_n(t_1))^{-1}$ and shifting, we may assume that $t_1 = 0$.

We will make use of ordered tuples of independent 1-marked random walk bridges, glued together at the marked points in some fashion. For each $n \geq 0$, let $S_{n,k}$ be the set of all ordered tuples of the form

$$B = (\mathscr{T}, B_1, \ldots, B_k),$$

where B_i are 1-marked bridges whose durations sum to n, and $\mathscr{T}$ is an ordered binary tree with k' leaves. Ordered tuples of this form are called *decomposed bridges*.

We give an algorithm which, given a k'-marked random walk bridge B, outputs a collection of decomposed bridges induced by B.

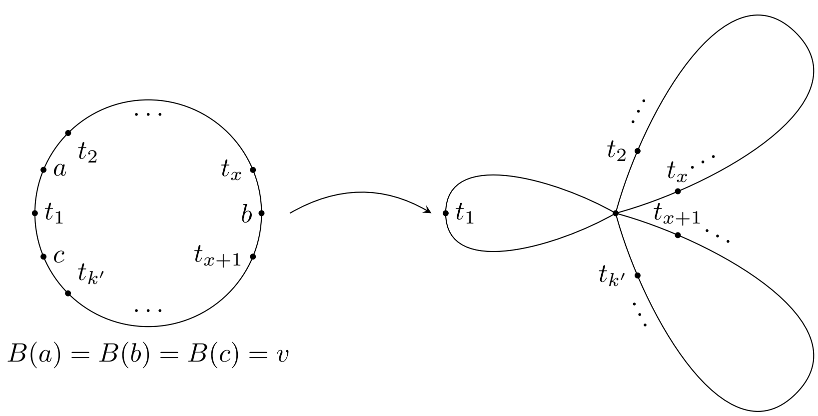

Figure 6.1: The decomposition of the "timeline" in Algorithm 2.

Algorithm 2. *Let B be a bridge with k' marked ordered times $t_1, \ldots, t_{k'}$. Let a, b, c be such that*

$$\begin{aligned} &\bullet \quad B(a) = B(b) = B(c) \\ &\bullet \quad t_1 \leq a \leq t_2 \leq b \leq t_{k'} \leq c \leq t_1 \end{aligned} \tag{6.20}$$

Set $v = B(a) = B(b) = B(c)$, and

$$B_1 = (B(t_1), \ldots, B(a) = B(c), \ldots, B(t_1 - 1), B(t_1)) \times (a - t_1).$$

If x is minimal such that $t_x \le b \le t_{x+1}$, then set

$$B_1' = (B(a), \ldots, B(b)) \times (t_2 - a, \ldots, t_x - a)$$
$$B_2' = (B(b), \ldots, B(c)) \times (t_{x+1} - b, \ldots, t_{k'} - b).$$

Recursively apply Algorithm 2 to both B_1' and B_2'. For each pair of outputs

$$(\mathscr{T}_1, B_1^{(1)}, \ldots, B_{2x-3}^{(1)}),\ (\mathscr{T}_2, B_1^{(2)}, \ldots, B_{2(k'-x)-1}^{(2)}),$$

define $\mathscr{T}$ to be the tree having root x_1, with, in clockwise order, an edge from v to x_1, a copy of $\mathscr{T}_1$ attached to v, and then a copy of $\mathscr{T}_2$ attached to v, the root being x_1. Define $B_{j+1} = B_j^{(1)}$ for $j = 1, \ldots, 2x-3$, and $B_{j+2x-2} = B_j^{(2)}$ for $j = 1, \ldots, 2(k'-x)-1$. Add the element

$$(\mathscr{T}, B_1, \ldots, B_k)$$

to the output. Repeat for each choice of (a, b, c) satisfying (6.20).

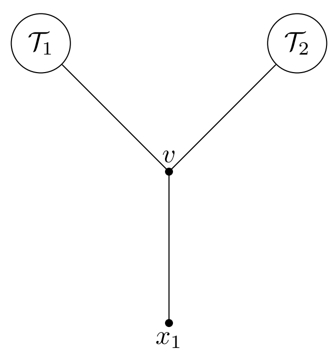

Figure 6.2: Constructing the type $\mathscr{T}$.

Remark 6.5.1.

1. It's clear that there always exist a, b, c satisfying the constraints, for the bridges must pass through the branchpoint of $B_n(t_1)$, $B_n(t_2)$ and $B_n(t_{k'})$.
2. It's easy to see that if we start with two different k'-marked bridges B_1, B_2, then the outputs will be disjoint. Indeed, given a decomposed bridge $(\mathscr{T}, B_1, \ldots, B_k)$, we can glue together $B_1, \ldots, B_k$ in a manner indicated by the type $\mathscr{T}$: when walking around $\mathscr{T}$, replace the edges by successive 1-marked bridges, starting the next

bridge at the marked time in the previous bridge, and assigning times appropriately. Applying Algorithm 2 to a k'-marked bridge B, and then applying the reverse "gluing" algorithm to any of the outputs will give back the original marked bridge B.

3. Moreover, it's clear that for a fixed B, different choices of the times a, b, c at any point in the recursion will also yield different elements of $S_{n,k}$, since the new 1-marked bridge formed at that step will be different.

4. The type $\mathscr{T}$ is an ordered binary tree with k' leaves corresponding to the times $t_1, t_2, \ldots, t_{k'}$. By invariance $\mathscr{T}$ is uniform among these trees.

5. The small bridges B_{n_i} are almost independent, given their sizes, as we see below.

If we choose $(B_n, s_1, \ldots, s_{k'})$ from $\mathbb{E}_{n,k}$ and select a uniform output, this gives us a measure on $S_{n,k}$. A more natural measure is the following.

$$\mathbb{P}^N_{n,k}\left[(\mathscr{T}, (B_{n_1}, t_1), \ldots, (B_{n_k}, t_k))\right] \propto \prod_i \mathbb{P}_{n_i}[B_{n_i}].$$

In other words, (B_{n_i}, t_i) are independent given their sizes, and t_i are independent uniform on $\{0, 1, \ldots, n_i\}$. When $k = 3$, this definition agrees with the definition in Chapter 4.

Proposition 6.5.1. *The measure $\mathbb{E}^N_{n,k}$ on k'-marked bridges $(B_n, t_1, \ldots, t_{k'})$ is $\mathbb{E}_{n,k}$ biased by N:*

$$\mathbb{E}_{n,k}\left[f(B_n, t_1, \ldots, t_{k'})\right] = \frac{\mathbb{E}^N_{n,k}[f(B_n, t_1, \ldots, t_{k'})N^{-1}]}{\mathbb{E}^N_{n,k} N^{-1}}.$$

for any bounded continuous function f.

Proof. The probability of a particular k'-marked bridge $(B_n, t_1, \ldots, t_{k'})$ under $\mathbb{E}_{n,k}$ is proportional to $\mathbb{P}_n[B_n]$.

The probability of $((B_{n_1}, t_1), \ldots, (B_{n_k}, t_k))$ under $\mathbb{E}^N_{n,k}$ is also proportional to $\mathbb{P}_n[B_n]$, where B_n is the resulting bridge after the inverse gluing operation of Algorithm 2. Since the inverse of Algorithm 2 is injective, ie. gluing distinct decomposed bridges according to $\mathscr{T}$ gives distinct k'-marked bridges, the probability of $(B_n, t_1, \ldots, t_{k'})$ under $\mathbb{E}^N_{n,k}$ is proportional to $N\mathbb{P}_n[B_n]$ where $N = N(B_n, t_1, \ldots, t_{k'})$ is the size of the output of Algorithm 2. □

The 1-marked bridges B_i are exchangeable. In particular, their durations are as well, and they sum to n.

$$\mathbb{E}^N_{n,k}\frac{n_i}{n} = \frac{1}{k} \tag{6.21}$$

6.6 On N

For a random walk bridge B, Let $N = N(B)$ be the size of the output of Algorithm 2.

Proposition 6.6.1. *Fix a positive integer M. Let $(\mathscr{T}, (B_1, t_1), \ldots, (B_k, t_k))$ be a sample from $\mathbb{E}^N_{n,k}$. For each branchpoint v of $\mathscr{T}$, let B^v_i be the three bridges intersecting at v, and let $\tilde{N}_v$ be the number of intersections of B^v_1, B^v_2 and B^v_3. Let $\tilde{N} = \prod_v \tilde{N}_v$. Then*

$$\mathbb{P}^N_{n,k}[N \neq \tilde{N}] = o\,(1;\ M \to \infty)\,.$$

Proof. This follows by induction on k, since the number of ways of decomposing B_n into three shorter bridges, involving times outside of Init_M, is 0 with probability $1 - o\,(1;\ M \to \infty)$. □

Proposition 6.6.2. *The probability $\mathbb{P}^N_{n,k}[N = 1]$ is uniformly bounded from below. Consequently, $\mathbb{E}^N_{n,k} N^{-1}$ is uniformly bounded away from 0.*

Proof. Sample $(\mathscr{T}, (B_{n_1}, t_1), \ldots, (B_{n_k}), t_k)$ from $\mathbb{E}^N_{n,k}$. Let v be any branchpoint of $\mathscr{T}$. Let B^1, B^2 be any two of the three bridges intersecting at v. Then the number of intersections $\tilde{N}$ of B^1, B^2 converge in distribution to the number of intersections of two independent infinite bridges according to the following argument:

1. For fixed M, the number of intersections $\tilde{N}_{v,M}$ in the first and last steps of B^i starting at v converge in distribution to the number of intersections in the first and last M steps of two independent infinite bridges.

2. The number of intersections of two independent bridges B^1, B^2, ignoring those from the first and last M steps of B^1, is equal to 0 with probability $1 - o\,(1;\ M \to \infty)$, from Lemma 6.2.10 and Corollary 6.2.8.

3. By choosing M arbitrarily large and then sending $n \to \infty$, the number $\tilde{N}_v$ is arbitrarily close in distribution to the number of intersections of two independent infinite bridges.

Since the infinite bridge is a transient Markov process, there is a non-zero chance that any two independent infinite bridges never intersect.

Finally, we define the event E as follows.

- For each branchpoint v, associate the two bridges $B_1^{(v)}$ and $B_2^{(v)}$ intersecting at v, furthest from the "first" leaf v_1.
- Let E_v be the event that $\tilde{N}_v = 1$.
- Let $E = \cap_v E_v$.

The events E_v are independent under $\mathbb{E}_{n,k}^N$, and hence $\mathbb{P}_{n,k}^N[E]$ is uniformly bounded away from 0. The event that $N = 1$ contains the event E. □

6.7 Continuity of the bridge measure

In this section, we show that the distribution of the function $D_n(t) = \frac{B_{\lambda n}(\lambda n t)}{\sqrt{\lambda n}}$ is continuous in λ in the L_∞ topology.

Proposition 6.7.1. *Let f be a bounded continuous function on the space of $C[0,1]$ functions with the $\|\cdot\|_\infty$ norm. Then*

$$\mathbb{E}_n f(D_n) - \mathbb{E}_m f(D_m) = o\left(1;\ m/n \to 1\right).$$

Using the following two facts, we couple the measures $\mathbb{E}_{n,k}^N$ and $\mathbb{E}_{m,k}^N$ so they're 'almost' equal.

Lemma 6.7.2. *Let $n_1, \ldots, n_k$ be the durations of the bridges sampled from $\mathbb{E}_{n,k}^N$, and let $m_1, \ldots, m_k$ be those from $\mathbb{E}_{m,k}^N$. Then*

$$d_{TV}\left((n_1, \ldots, n_{k-1}), (m_1, \ldots, m_{k-1})\right) = o\left(1;\ m/n \to 1\right).$$

Proof. This follows from the fact that the probability mass functions for $(n_1, \ldots, n_{k-1})$ and $(m_1, \ldots, m_{k-1})$ are uniformly close on a set of sufficiently high probability. We can prove this using the return probabilities. Specifically,

$$\mathbb{P}_{n,k}^N[(n_1, \ldots, n_k) = (x_1, \ldots, x_k)] \propto \left(x_1 x_2 \cdots x_k\right)^{1/2},$$

from (1.2) and so

$$\frac{\mathbb{P}^N_{n,k}\left[(n_1,\ldots,n_k)=(x_1,\ldots,x_k)\right]}{\mathbb{P}^N_{m,k}\left[(m_1,m_2,\ldots,m_k)=(x_1,\ldots,x_{k-1},x'_k)\right]}=\sqrt{\frac{x_k}{x'_k}}+o\left(1;\ \min(x_i)\to\infty\right).$$

The conclusion easily follows. □

Lemma 6.7.3. *Two bridges B_n and B_m can be coupled so that the initial part $Init_M(B_n)$ and $Init_M(B_m)$ agree with probability $o\,(1;\ m,n\to\infty)$.*

Proof. Since by the proof of Theorem 5.0.1, the initial parts converge in distribution to a discrete random variable, they can be coupled to equal with high probability. Given the initial part $\mathrm{Init}(B_n)$, we can sample from B_n by sampling a random walk bridge $\tilde{B}_n$ on $n-2M$ steps from the two endpoints of $\mathrm{Init}(B_n)$. Sample B_m similarly. □

Proof of Proposition 6.7.1. Proceeding, we couple $\mathbb{E}^N_{n,k}$ and $\mathbb{E}^N_{m,k}$ together so that the resulting bridges are almost equal, and $N(B_n)=N(B_m)$, with high probability. Denote $N_n=N(B_n)$ and $N_m=N(B_m)$.

1. Couple the durations n_i and m_i as in Lemma 6.7.2.

2. Sample B_{n_i} independently from $\mathbb{E}_{n_i}$.

3. On the event that $n_i=m_i$ for $i<k$, set $B_{m_i}=B_{n_i}$. Call this event E_1. Otherwise, sample B_{m_i} in any manner.

4. Finally, couple B_{n_k} with B_{m_k} as in Lemma 6.7.3 so that their initial parts agree with probability $1+o\,(1;\ m,n\to\infty)$. Call this event E_2.

Recall from Proposition 6.2.5 that there are constants $C<\infty$ and $\epsilon>0$ such that the modulus of continuity $\omega_n(\delta)=\max\{|D_n(t)-D_n(s)|\mid |t-s|\le\delta\}$ satisfies

$$\mathbb{P}\left[\omega_n(\delta)\ge x\right]\le Ce^{-\epsilon x^2/\delta}.$$

Define $\delta_{m,n}(t)=|D_n(t)-D_m(t)|$.

On the event E_1, $B_n(i)=B_m(i)$ for $i=0,1,\ldots,n-n_1-n_k+t_1$, and $B_n(n-i)=B_m(m-i)$ for $i=0,1,\ldots,n_1-t_1$. Hence, if $mt<m-m_k-m_1+t_1$, then

$$
\begin{aligned}
\delta_{m,n}(t) &= \left| \frac{|B_n(nt)|}{\sqrt{n}} - \frac{|B_m(mt)|}{\sqrt{m}} \right| \\
&= \left| \frac{|B_n(nt)| - |B_m(mt)|}{\sqrt{n}} \right| + o\left(1;\ m/n \to 1\right) \|D_n\|_\infty \\
&= \left| \frac{|B_n(nt)| - |B_n(mt)|}{\sqrt{n}} \right| + o\left(1;\ m/n \to 1\right) \|D_n\|_\infty \\
&= |D_n(t) - D_n(t')| + o\left(1;\ m/n \to 1\right) \|D_n\|_\infty \\
&\le \omega_n(\delta) + o\left(1;\ m/n \to 1\right) \|D_n\|_\infty \qquad (6.22)
\end{aligned}
$$

where $t' = t \cdot \frac{m}{n}$, and $\delta = \sup\left(|t - t'|\right) = o\left(1;\ m/n \to 1\right)$. Hence, by Proposition 6.2.5,

$$
\begin{aligned}
\mathbb{E}\left[\sup_t \{\delta_{m,n}(t)\} \mathbf{1}(mt < m - m_k - m_1 + t_1, E_1)\right] &\le \mathbb{E}\left[\omega_n(\delta) + o\left(1;\ m/n \to 1\right) \|D_n\|_\infty\right] \\
&= o\left(1;\ \delta \to 0\right) + o\left(1;\ m/n \to \infty\right).
\end{aligned}
$$

If $mt \ge m - m_k - m_1 + t_1$, then $t > 1 - \frac{n_1 + n_k}{n} + o\left(1;\ m/n \to 1\right)$, so

$$
\left| \frac{|B_n(nt)| - |B_n(n - n_k)|}{\sqrt{n}} \right| \le \left[\max\left(\frac{|B_{n_k}(i)|}{\sqrt{n_k}}\right) + \max\left(\frac{|B_{n_{k-1}}(i)|}{\sqrt{n_k}}\right)\right] \sqrt{\frac{n_k}{n}}
$$

so by Proposition 6.2.5,

$$
\mathbb{E}\left[\left| \frac{|B_n(nt)| - |B_n(n - n_k)|}{\sqrt{n}} \right|\right] \le \sqrt{\mathbb{E}\left[\omega_n(\delta)^2\right] \left\{\mathbb{E}\left[\frac{n_k}{n}\right]\right\}} \preceq \sqrt{\frac{1}{k}}.
$$

Similarly,

$$
\mathbb{E}\left[\left| \frac{|B_m(mt)| - |B_m(m - m_k)|}{\sqrt{m}} \right|\right] \preceq \sqrt{\frac{1}{k} + o\left(1;\ m/n \to 1\right)}.
$$

Since $D_n(n - n_k) = D_m(m - m_k)$ on E_1, this implies that

$$
\mathbb{E}\left[\sup_t \{|D_n(nt) - D_m(mt)|\} \mathbf{1}(mt \ge m - m_k - m_1 + t_1, E_1)\right] = o\left(1;\ k \to \infty\right) + o\left(1;\ m/n \to 1\right). \qquad (6.23)
$$

On E_1^c, we can bound $\|D_n - D_m\|_\infty$ as follows.

$$\mathbb{E}^N_{m,n,k}\left[\|D_n - D_m\|_\infty \mathbf{1}(E_1^c)\right] \leq \sqrt{\mathbb{E}^N_{m,n,k}\left[\|D_n - D_m\|_\infty^2\right]\mathbb{P}^N_{n,m,k}[E_1^c]} = o\left(1;\ m/n \to 1\right).$$

Hence, it follows that

$$\mathbb{E}^N_{m,n}\left\|D_n - D_m\right\|_\infty = o\left(1;\ m/n \to 1\right) + o\left(1;\ k \to \infty\right). \tag{6.24}$$

Since the marginal of D_n under $\mathbb{E}^N_{m,n,k}$ is $\mathbb{E}^N_{n,k}$, it follows that

$$\mathbb{E}_n f(D_n) = \mathbb{E}^N_{n,m,k}\frac{f(D_n)}{N\mathbb{E}^N_{n,m,k}N_n^{-1}}.$$

The marginal of D_m under $\mathbb{E}^N_{m,n,k}$ is $\mathbb{E}^N_{m,k}$, so

$$\mathbb{E}_n f(D_n) - \mathbb{E}_m f(D_m) = \mathbb{E}^N_{m,n,k}\left[\frac{f(D_n)}{N_n\mathbb{E}^N_{m,n,k}N_n^{-1}} - \frac{f(D_m)}{N_m\mathbb{E}^N_{m,n,k}N_m^{-1}}\right] \tag{6.25}$$

$$= \mathbb{E}^N_{m,n,k}\left[\frac{f(D_n) - f(D_m)}{N_n\mathbb{E}^N_{m,n,k}N_n^{-1}}\right] \tag{6.26}$$

$$+ \mathbb{E}^N_{m,n,k}\left[f(D_m)\frac{N_n - N_m}{N_m\mathbb{E}^N_{m,n,k}N_m^{-1}}\right] \tag{6.27}$$

The first term is $o\left(1;\ m/n \to 1\right) + o\left(1;\ k \to \infty\right)$ by (6.24). The second term is

$$O\left(1;\ m,n \to \infty\right) o\left(1;\ x \to \infty\right) + o\left(1;\ m/n \to 1\right),$$

sotnce on E_2, $N_n = N_m$ with probability $1 - o\left(1;\ x \to \infty\right)$, and

$$\mathbb{P}^N_{m,n,k}[E_2^c] = O\left(1;\ m/n \to 1\right) o\left(1;\ x \to \infty\right).$$

□

6.8 Independence of the intial part

Define

$$\begin{aligned}\sigma_{k,n} &= \sup\left\{i \leq n/2 \mid B_n(i) = 0\right\}, \\ \tau_{k,n} &= \sup\left\{n - i \leq n/2 \mid B_n(i) = 0\right\}, \\ \tilde{n} &= n - (\sigma_{k,n} + \tau_{k,n}).\end{aligned}$$

Decompose a random walk bridge B_n into $(B_{\sigma_{k,n}}, E_{\tilde{n}}, B_{\tau_{k,n}})$, where

$$\begin{aligned}B_{\sigma_{k,n}} &:= (B_n(0), B_n(1), \ldots, B_n(\sigma_{k,n})), \\ E_{\tilde{n}} &:= (B_n(\sigma_{k,n}), B_n(\sigma_{k,n}+1), \ldots, B_n(\tau_{k,n})), \\ B_{\tau_{k,n}} &:= (B_n(n), B_n(n-1), \ldots, B_n(\tau_{k,n}))\end{aligned}$$

The processes $B_{\sigma_{k,n}}$, $B_{\tau_{k,n}}$ are random walk bridges from ρ to e^-, and $E_{\tilde{n}}$ is a random walk excursion.a Moreover, they are independent given $\sigma_{k,n}$, $\tau_{k,n}$.

Call the excursion $\mathbb{E}_{\tilde{n}}$ the *grand excursion* of B_n.

Lemma 6.8.1. *Let E_n be a random walk excursion. There is a coupling between E_n and a random walk bridge $B_{n'}$ such that*

- *E_n is the grand excursion of $B_{n'}$ with probability $1 - o\,(1;\ n \to \infty)$.*
- $\frac{n}{n'} \xrightarrow{p} 1$.

Proof. Since $(\sigma_{k,n}, \tau_{k,n})$ jointly converge to the last return times (s,t) of an infinite bridge, $(B_{\sigma_{k,n}}, B_{\tau_{k,n}})$ jointly converge in distribution to two independent random time-length random walk bridges (B_s, B_t). As this is a discrete random variable, we can couple $(B_{\sigma_{k,n}}, B_{\tau_{k,n}})$ together with (B_s, B_t) such that they are equal with probability $1 - o\,(1;\ n \to \infty)$.

Recall that the first step $E_{\tilde{n}}(1)$ converges in distribution to a random generator g_∞. We can approximate the bridge B_n in total variation by

- Sample (B_s, B_t) and g_∞ from the infinite bridge measure.
- Let $\tilde{n} = n - (s+t)$.
- Let $E'_{\tilde{n}}$ be an excursion of length $\tilde{n}$ with first step g_∞.

- Let B'_n be B_s, followed by $E_{\tilde{n}}$, followed by B_t.

Since $(B_{\sigma_{k,n}}, B_{\tau_{k,n}}, E_{\tilde{n}}(1))$ can be coupled to be equal to (B_s, B_t, g_∞) with probability $1 - o(1;\ n \to \infty)$, on this event we use the same excursions E and E', and hence $B_n = B'_n$ with probability $1 - o(1;\ n \to \infty)$.

Reversing the point of view, given an random walk excursion E_n, sample (B_s, B_t) from the infinite bridge measure given $g_\infty = E_n(1)$. Let $B'_{n'}$ be B_s, followed by E_n, followed by B_t. Then $B'_{n'}$ can be coupled with a random walk bridge $B_{n'}$ of duration n' to be equal with probability $1 - o(1;\ n \to \infty)$ □

Proposition 6.8.2. *Let f_i be bounded continuous functions. Let $Init = Init_M(B_1)$ be the initial part of B_1. Then*

$$\mathbb{E}_{\times n_i}\left[\frac{\prod_i f_i(T_i)}{N} \mid Init_1\right] = \mathbb{E}_{\times n_i}\left[\frac{\prod_i f_i(T_i)}{N}\right] + o(1;\ \min(n_i) \to \infty)$$

real: define this measure!

Proof. Sample an independent g_∞ from the first edge of the end of an infinite bridge. Let e be the first edge on the geodesic from $B_1(0)$ to $B_1(n_i/2)$ of type g_∞ that is at distance at least M from $B_1(0)$. Let v be the endpoint of e closest to $B_1(0)$. Since the initial part Init_M converges in distribution (for fixed M) to the initial part of an infinite bridge, the weak LLN for the word profile (part 2 of Corollary 6.3.4) implies that $\mathbb{P}_{n_1}[|v| > x] = o(1;\ x \to \infty)$.

Since $\sigma_{k,n}$ and $\tau_{k,n}$ converge in distribution, it follows that $\frac{\tilde{n}_1}{n_1} \xrightarrow{p} 1$ as $n_1 \to \infty$.

The process $\tilde{E}_{\tilde{n}_1}(i) = (v^{-1}B_1(\sigma_{k,n} + i))_{i=0}^{\tilde{n}_1}$ is a random walk excursion on $\tilde{n}_1$ steps conditioned on the first step being g_∞, and is independent of Init_M given $\tilde{n}_1$. By Lemma 6.8.3, $\tilde{E}_{\tilde{n}_1}$ can be coupled with a random walk excursion $E_{\tilde{n}_1}$ such that they are equal with probability $1 - o(1;\ n_1 \to \infty)$: we can couple the first steps to be the same with probability $1 - o(1;\ \tilde{n} \to \infty)$, and if they agree use the same excursion.

Finally, using the coupling from Lemma 6.8.1 we can couple $E_{\tilde{n}_1}$ with a random walk bridge B'_1 on n'_1 steps such that $E_{\tilde{n}_1}$ is the grand excursion of B'_1. This implies that $f_1(T_i) - f_1(T'_i) \xrightarrow{p} 0$.

Letting $a_{\tilde{n}_1}$ be the number of added steps to the beginning of $E_{\tilde{n}_1}$, define $t'_1 = t_1 + (a_{\tilde{n}_1} - \sigma_{\tilde{n}_1}$. This way, $\mathrm{Init}_M(B_1) = \mathrm{Init}_M(B'_1)$ with probability $1 - o(1;\ n_1 \to \infty)$.

While the bridge B'_1 is distributed as $\mathbb{E}_{n'_1}$, as a marked bridge (B'_1, t'_1) it's not necessarily distributed as $\mathbb{E}_{n'_1}$. However, if we apply a random time-shift:

- let U be uniform from $\{1, 2, \ldots, n'_1\}$, independent of everything;

- replace B'_1 with $B'_1(U)^{-1}B'_1(\cdot + U)$;
- replace t'_1 with $t'_1 - U \pmod{n'_1}$;

then (B'_1, t'_1) is distributed as a 1-marked random walk bridge.

Note that each of these couplings can be done with random variables that are independent of $(B_2, B_3, \dots, B_k)$. Hence, we can couple $(B_1 \,|\, \mathrm{Init}_1, B_2, \dots, B_k)$ with $(B'_1, B_2, B_3, \dots, B_k)$ such that

(a) $(B'_1, B_2, \dots, B_k)$ is distributed as $\mathbb{E}_{n'_1} \times \mathbb{E}_{n_2} \times \cdots \times \mathbb{E}_{n_k}$ given n'_1,

(b) $\frac{\prod_{i>1} f_i(T_i)}{N} - \frac{f_1(T'_i)\prod_{i>1} f_i(T_i)}{N} \xrightarrow{p} 0$ as $n_1 \to \infty$, since

$$(B'_1(t'_1 - k, \dots, B'_1(t'_1 + k)) = (B_1(t_1 - k, \dots, B_1(t_1 + k))$$

and hence $N(B) = N(B')$ with probability $1 - o\,(1;\ \tilde{n} \to \infty)$.

By (a) and (b) and the fact that $n'_i/n_i \xrightarrow{p} 1$, we are ensured that

$$\begin{aligned}\mathbb{E}_{\times n_i}\left[\frac{\prod_i f_i(T_i)}{N} \,|\, \mathrm{Init}_1\right] &= \mathbb{E}_{n'_1} \times \mathbb{E}_{n_2} \times \cdots \times \mathbb{E}_{n_k}\left[\frac{\prod_i f_i(T_i)}{N}\right] + o\,(1;\ \min(n_i) \to \infty) \\ &= \mathbb{E}_{n_1} \times \cdots \times \mathbb{E}_{n_k}\left[\frac{\prod_i f_i(T_i)}{N}\right] + o\,(1;\ \min(n_i) \to \infty)\,.\end{aligned}$$

by Proposition 6.7.1.

□

Lemma 6.8.3. *Let E_n be a random walk excursion of length n. Then $g_n = E_n(1) = E_n(n-1)$ converges in distribution.*

Proof. This follows from the proof of Theorem 5.0.1. □

Appendices

Appendix A

Direct convergence of the k-point statistics

In this section, we give a direct proof that the k-point statistics of the range of the random walk bridge converge in distribution, using Corollary 6.3.4 more directly.

To avoid writing $2k-3$ everywhere, fix k' and set $k = 2k'-3$. We continue to refer to the statistics of $R_{k'}(T)$ as the k-point statistics.

Sample $(\mathscr{T}, (B_1, t_1), \ldots, (B_k, t_k))$ from $\mathbb{E}^N_{n,k}$. Let f_i, g_i be any bounded continuous functions. Define

$$D_i = \frac{|B_i(t_i)|}{\sqrt{n_i}}, \qquad F_i = f_i(D_i)$$
$$\Delta_{i,n} = \frac{n_i}{n}, \qquad G_i = g_i(\Delta_{i,n}).$$

Since N converges in distribution, by the proof of Proposition 6.6.2, $\nu := \lim_{n\to\infty} \mathbb{E}^N_{n,3} N^{-1}$ exists.

Lemma A.0.1.

$$\mathbb{E}_{\times n_i}\left[\frac{\prod_i F_i}{N}\right] = \prod_i \mathbb{E}\left[f_i(X)\right]\nu^{k'-2} + o\left(1;\ \min\left(n_i\right) \to \infty\right).$$

Proof. Make the following definitions.

- Let v be a uniform branchpoint of $\mathscr{T}$.
- Label the three branches of $\mathscr{T}$ intersecting at v by $1, 2, 3$ in any order.

- Let J_j be the index set of the bridges B_i which lie in branch j of $\mathscr{T}$; let $k_j = \#\{J_j\}$, ie. k_j is the number of 1-marked bridges in branch j.
- Let $B_i^{(j)}$ be the 1-marked bridges in banch j; let $n_i^{(j)}$ be their durations and $t_i^{(j)}$ be their marked times.
- Without loss of generality, let $B_1^{(j)}$ be the 1-marked bridge in branch j which passes through v.
- Let $\mathscr{T}_j$ be the shape of the j-th branch of $\mathscr{T}$.
- Let $\left(B_j', t_1^{(j)}, \ldots, t_{k_j'}^{(j)}\right)$ be the k_j'-marked tree obtained by gluing $\left(B_i^{(j)}, t_i^{(j)}\right)$ according to $\mathscr{T}_j$.
- Let N_j be the size of the output of Algorithm 2 for $\left(\mathscr{T}_j, B_j', t_i^{(j)}\right)$.
- Let $\mathrm{Init}_{j,M}$ be the initial part of just $B_1^{(j)}$. Let $\mathrm{Init}_v = (\mathrm{Init}_{1,M}, \mathrm{Init}_{2,M}\mathrm{Init}_{3,M})$ be the initial part of the three bridges $B_1^{(j)}$ intersecting at v.

Then

$$\begin{aligned}\mathbb{E}_{\times n_i}\left[\frac{\prod_i F_i}{N}\right] &= \mathbb{E}_{\times n_i}\left[\mathbb{E}_{\times n_i}\left[\frac{\prod_i F_i}{N} \mid \mathrm{Init}_v, n_i^{(j)}\right]\right] \\ &= \mathbb{E}_{\times n_i}\left[\frac{1}{N_v}\mathbb{E}_{\times n_i}\left[\frac{\prod_i F_i}{N_1 N_2 N_3} \mid \mathrm{Init}_v, n_i^{(j)}\right]\right] + o\left(1;\ M \to \infty\right)\end{aligned}$$

since $B_1^{(j)}$ do not intersect outside of the times involving Init_v with probability $1 - o\left(1;\ M \to \infty\right)$. Next, since $\left(B_i^{(j)}, N_j, \mathscr{T}_j\right)$ are independent given $n_i^{(j)}$, Init_v,

$$\mathbb{E}_{\times n_i}\left[\frac{1}{N_v}\mathbb{E}_{\times n_i}\left[\frac{\prod_i F_i}{N_1 N_2 N_3} \mid \mathrm{Init}_v, n_i^{(j)}\right]\right] = \mathbb{E}_{\times n_i}\left[\frac{1}{N_v}\prod_j \mathbb{E}_{\times n_i^{(j)}}\left[\frac{\prod_{i\in J_j} F_i}{N_j} \mid \mathrm{Init}_j\right]\right].$$

By Lemmas 6.8.2 and A.0.1,

$$\begin{aligned}\mathbb{E}_{\times n_i}\left[\frac{1}{N_v}\prod_j \mathbb{E}_{\times n_i^{(j)}}\left[\frac{\prod_{i\in J_j} F_i}{Nj} \mid \mathrm{Init}_j\right]\right] &= \mathbb{E}_{\times n_i}\left[\frac{1}{N_v}\prod_i \mathbb{E}\left[f_i(X)\right]\nu^{k_j'-2} + o\left(1;\ \min\left(n_i\right) \to \infty\right)\right] \\ &= \nu^{k'-2}\prod_i \mathbb{E}\left[f_i(X)\right] + o\left(1;\ \min\left(n_i\right) \to \infty\right).\end{aligned}$$

□

Proposition A.0.2. *Let X be a Raleigh-distributed random variable, and let Δ be a Dirichlet-$(1/2; k)$ random variable. Then*

$$\mathbb{E}^N_{n,k}\left[\prod_i f_i(D_i) g_i(\Delta_{i,n})/N\right] = \mathbb{E}\left[\prod_i g_i(\Delta_i)\right]\prod_i \mathbb{E}\left[f_i(X)\right]\mathbb{E}^N_{n,k}\left[\frac{1}{N}\right] + o\left(1;\ n\to\infty\right).$$

Proof. Conditioning on the durations n_i, the bridges B_i are independent. Hence,

$$\begin{aligned}
\mathbb{E}^N_{n,k}\left[\frac{\prod_i F_i G_i}{N}\right] &= \mathbb{E}^N_{n,k}\left[\prod_i G_i \mathbb{E}^N_{n,k}\left[\frac{\prod_i F_i}{N}\middle|\, n_i\right]\right] \\
&= \mathbb{E}^N_{n,k}\left[\prod_i G_i \mathbb{E}_{\times n_i}\left[\frac{\prod_i F_i}{N}\right]\right] \\
&= \mathbb{E}^N_{n,k}\left[\prod_i G_i\left(\prod_i \mathbb{E}\left[f_i(X)\right]\nu^{k'-2} + o\left(1;\ \min(n_i)\to\infty\right)\right)\right] && \text{by Lemma A.0.1} \\
&= \mathbb{E}^N_{n,k}\left[\prod_i G_i\right]\prod_i \mathbb{E}\left[f_i(X)\right]\nu^{k'-2} + o\left(1;\ n\to\infty\right)
\end{aligned}$$

since $\frac{n_i}{n}$ converges to a positive random variable. By adapting the proof of Lemma 6.4.2 to k bridges,

$$\begin{aligned}
\mathbb{E}^N_{n,k}\left[\prod_i G_i\right]\prod_i \mathbb{E}\left[f_i(X)\right]\nu^{k'-2} + o\left(1;\ n\to\infty\right) = \mathbb{E}\left[\prod_i g_i(\Delta_i^{(k)})\right]\prod_i \mathbb{E}\left[f_i(X)\right]\mathbb{E}^N_{n,k}\left[\frac{1}{N}\right] \\
+ o\left(1;\ n\to\infty\right)
\end{aligned}$$

□

Corollary A.0.3. *The rescaled k-point reduced subtree $R_{k,n} := R_k(B_n, p_n)$ of a random walk bridge B_n sampled from $\mathbb{E}_n$ converges in distribution to the k-point statistics of the BCRT.*

Proof.
Let $\Delta_n := \frac{1}{n}(n_1, n_2, \ldots, n_k)$, and $X_{i,n} := \frac{|B_i(t_i)|}{\sqrt{n_i}}$.

1. By Lemma 6.2.10, the expected distance from the branchpoint b of $B_n(t_1)$, $B_n(t_2)$, $B_n(t_3)$ in $\mathbb{T}_d$ and their branchpoint v in $\mathscr{T}$ is $O\left(1;\ n\to\infty\right)$. Hence, the distances in the rescaled reduced subtree $R_{k,n}$ are asymptotically equal to the distance $\frac{|B_i(t_i)|}{\sqrt{n_i}}$ with probability $1 - o\left(1;\ n\to\infty\right)$.

2. For bounded continuous f_i, g_i,

$$\mathbb{E}_{n,k}\left[\prod_i f_i(D_i)g_i(\Delta_i)\right] = \mathbb{E}_{n,k}^N\left[\frac{\prod_i f_i(D_i)g_i(\Delta_i)}{N\mathbb{E}_{n,k}^N N^{-1}}\right]$$
$$= \mathbb{E}\left[\prod_i g_i(\Delta_i)\right]\prod_i \mathbb{E}\left[f_i(X)\right] + o\left(1;\ n \to \infty\right)$$

   by Proposition A.0.2. Hence, $(\Delta_n, X_{1,n}, \ldots, X_{k,n})$ converge in distribution to $(X_1, X_2, \ldots, X_k, \Delta)$ under $\mathbb{E}_{n,k}$.

3. Under $\mathbb{E}_{n,k}^N$, $\mathscr{T}$ is a uniform ordered binary tree with k edges, which is the shape of the reduced subtree $R_{k'}(\tau, \mu)$ of the BCRT. Since $\mathscr{T}$ and N are independent under $\mathbb{E}_{n,k}^N$, $\mathscr{T}$ has the same distribution under $\mathbb{E}_{n,k}$. Moreover, since the edge-lengths in the rescaled reduced subtree $R_{k,n}$ converge to a positive random variable, while the distances between the branchpoints in $\mathscr{T}$ and the branchpoints in T are $o\left(1;\ n \to \infty\right)$, the shape of $R_{k,n}$ is equal to $\mathscr{T}$ with probability $1 - o\left(1;\ n \to \infty\right)$, and is asymptotically independent of $(X_1, X_2, \ldots, X_k, \Delta)^{(n)}$.

Hence, the rescaled reduced subtree $R_{k,n}$ converges in distribution to the reduced subtree $R_k\tau, \mu$ of a BCRT.

□

Appendix B

A short proof of Theorem 1.0.1 for the simple random walk bridge

B.1 Simple random walk

It is proved in [6] that the rescaled distance function of a simple random walk bridge

$$D_n(t) := \frac{|B_n(nt)|}{C_d\sqrt{n}} \Rightarrow e_t \tag{B.1}$$

where e_t is a Brownian excursion.

To prove Theorem 1.0.1 for the case of the simple random walk bridge, it suffices to show that $\|D_n(t) - T_n(t)\| \xrightarrow{p} 0$. This is proved in the proof of Proposition 4.0.2, which relies only (1.1).

B.2 The general case

THIS SECTION WILL REMAIN UNFINISHED FOR THE PURPOSE OF THIS DRAFT.

In this case, we consider non-lazy random walk *excursions*. Given a $2n$-step excursion E_{2n}, we construct a contour function $\tilde{E}_{2n}$ as follows:

- When $|E_{2n}(i+1)| = |E_{2n}(i)| + 1$, $\tilde{E}_{2n}$ creates a new edge whose colour is the generator $E_{2n}(i)^{-1}E_{2n}(i+1)$.
- When $|E_{2n}(i+1)| = |E_{2n}(i)| - 1$, $\tilde{E}_{2n}$ takes a step towards the root.

Hence, $|\tilde{E}_{2n}(i)| = |E_{2n}(i)|$.

The weight of $\tilde{E}_{2n}$ is given by

$$\mathbb{P}_n[\tilde{E}_{2n}] \asymp \prod_{v \in T_n} \left(\prod_{i \neq c(v)} p_i^{N_i^{(v)}} \right)$$

This is the special case of the measure on multi-type Galton Watson trees ([19], p. 1132) where the measure $\mu^{(i)}$ is Geometric with parameter p_j^2 in each coordinate $j \neq i$, and is the measure supported on $\{0\}$ in coordinate i, conditioned to have n vertices.

The height process H_n of T_n is defined as follows. List the vertices v_0, v_1, $\ldots$, v_n in order. Then $H_n(k) = d(v_k, v_0)$. Theorem 2 of [19] proves that $\frac{H_n}{C\sqrt{n}} \Rightarrow e_t$ for an appropriate constant C. In this case, we show that the height function H_n and the contour function $|\tilde{E}_{2n}|$ of T_n are close by taking advantage of the already known LLN.

For each k, let J_k be the first time the contour process $|\tilde{E}_{2n}|$ visits v_k. It's easy to see that

$$J_k + H_n(k) = 2k.$$

Indeed, when $\left|\tilde{E}_{2n}\right|$ decreases for i consecutive steps, then increases by 1, creating a new vertex, J_k increases by $i+1$ steps, and $H_n(k)$ decreases by $i-1$, netting an increase by 2. Since $\frac{H_n(\cdot)}{\sqrt{n}}$ is tight, it follows that

$$\frac{J_{nt}}{2n} \xrightarrow{p} t.$$

Since J_{nt} is monotone in t, this implies convergence in probability in the uniform $\|\cdot\|_\infty$ topology.

Note that $\left|\tilde{E}_{2n}(k)\right| = H_n(J_k) - i$ for $i = 0, 1, \ldots, J_{k+1}$. Hence,

$$\begin{aligned} \frac{|E_{2n}(2nt)|}{C\sqrt{n}} &= \frac{\left|\tilde{E}_{2n}(2nt)\right|}{C\sqrt{n}} \\ &= \frac{\left|\tilde{E}_{2n}(2nt)\right| - \left|\tilde{E}_{2n}(J_{nt})\right|}{C\sqrt{n}} + \frac{H_n(nt)}{C\sqrt{n}} \end{aligned}$$

Since $\frac{|\tilde{E}_{2n}|}{\sqrt{n}}$ is tight, and $\frac{J_{nt}}{2n} \xrightarrow{p} t$ uniformly, the first term uniformly converges to 0 in probability. It follows that $\frac{|E_{2n}(2nt)|}{C\sqrt{n}} \Rightarrow e_t$ in $\|\cdot\|_\infty$.

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