Quantomorphisms and Quantized Energy Levels for Metaplectic-c Quantization

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Abstract

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Metaplectic-c quantization was developed by Robinson and Rawnsley as an alternative to the classical Kostant-Souriau quantization procedure with half-form correction. This thesis extends certain properties of Kostant-Souriau quantization to the metaplectic-c context. We show that the Kostant-Souriau results are replicated or improved upon with metaplectic-c quantization.

We consider two topics: quantomorphisms and quantized energy levels. If a symplectic manifold admits a Kostant-Souriau prequantization circle bundle, then its Poisson algebra is realized as the space of infinitesimal quantomorphisms of that circle bundle. We present a definition for a metaplectic-c quantomorphism, and prove that the space of infinitesimal metaplectic-c quantomorphisms exhibits all of the same properties that are seen in the Kostant-Souriau case.

Next, given a metaplectic-c prequantized symplectic manifold (M, ω) and a function $H \in C^\infty(M)$, we propose a condition under which E, a regular value of H, is a quantized energy level for the system (M, ω, H). We prove that our definition is dynamically invariant: if two functions on M share a regular level set, then the quantization condition over that level set is identical for both functions. We calculate the quantized energy levels for the n-dimensional harmonic oscillator and the hydrogen atom, and obtain the quantum mechanical predictions in both cases. Lastly, we generalize the quantization condition to a level set of a family of Poisson-commuting functions, and show that in the special case of a completely integrable system, it reduces to a Bohr-Sommerfeld condition.

Contents

Chapter 1

Introduction

In physics, a classical mechanical system is described by a phase space, which contains position and momentum coordinates for the component particles. The physically observable quantities for the system are functions on phase space. To form the quantum mechanical version of such a system, we construct a Hilbert space of quantum states, and we identify observables with self-adjoint operators on the Hilbert space.

Geometric quantization is a branch of symplectic geometry that generalizes the concept of quantization to abstract symplectic manifolds. Given a symplectic manifold (M, ω) and a function $H \in C^\infty(M)$, the corresponding Hamiltonian vector field ξ_H on M has integral curves that satisfy Hamilton's equations in local canonical coordinates. Thus we can view (M, ω) as a classical phase space, and H as a Hamiltonian energy function. To quantize (M, ω), we require a recipe for constructing a suitable Hilbert space of quantum states, and a Lie algebra isomorphism from $C^\infty(M)$, or a subalgebra of it, to operators on those states.

The original formulation of geometric quantization is due to Kostant [12] and Souriau [20]. Their quantization procedure requires that (M, ω) admit a prequantization circle bundle and a metaplectic structure. In Chapter 2, after we review the necessary elements of symplectic geometry and principal bundles, we present the Kostant-Souriau quantization procedure with half-form correction.

The metaplectic-c group is a circle extension of the symplectic group. Metaplectic-c quantization, which was developed by Robinson and Rawnsley [15], is a variant of Kostant-Souriau quantization in which the prequantization circle bundle and metaplectic structure are replaced by a single object called a metaplectic-c prequantization. Robinson and Rawnsley proved that metaplectic-c quantization can be applied to all systems that admit metaplectic quantizations, and to some where the Kostant-Souriau process fails. In the final section of Chapter 2, we give a detailed description of a metaplectic-c prequantization and its properties.

The objective of this document is to explore some of the ways in which metaplectic-c quantization replicates or improves upon known results for Kostant-Souriau quantization. We begin in Chapter 3 by examining the notion of a quantomorphism: that is, an isomorphism of prequantization bundles that preserves all of their structures. In the context of a Kostant-Souriau prequantization circle bundle for (M, ω), it is known that the space of infinitesimal quantomorphisms is isomorphic, as a Lie algebra, to $C^\infty(M)$. We formulate a definition of a metaplectic-c quantomorphism, and prove that the space of infinitesimal metaplectic-c quantomorphisms is again isomorphic to $C^\infty(M)$.

The remainder of the document focuses on the concept of a quantized energy level. In quantum mechanics, the energy spectrum for a spatially confined particle is discrete: only certain energy levels are permitted. Various interpretations of this phenomenon can be found in the literature for different forms of geometric quantization. Given (M, ω) and a function $H \in C^\infty(M)$, we propose a new definition for a quantized energy level E of H in the case where (M, ω) admits a metaplectic-c prequantization. Our definition is evaluated over a regular level set of H, and it does not require either symplectic reduction or a choice of polarization.

We present our definition in Chapter 4, and show that its properties compare favourably with others that have been studied. Our main result, Theorem 4.3.5, states that if two functions $H_1, H_2 \in C^\infty(M)$ are such that $H_1^{-1}(E_1) = H_2^{-1}(E_2)$ for regular values E_1, E_2, then E_1 is a quantized energy level for (M, ω, H_1) if and only if E_2 is a quantized energy level for (M, ω, H_2). That is, our condition measures a geometric property of the level set, and does not depend on the dynamics of a specific choice of H. As such, we refer to Theorem 4.3.5 as the dynamical invariance theorem. Also in Chapter 4, we demonstrate a computational technique for lifting a local change of coordinates on M to the level of metaplectic-c prequantizations, and use this technique to evaluate the quantized energy levels for the n-dimensional harmonic oscillator.

In Chapter 5, we calculate the quantized energy levels of the hydrogen atom, using the physical model that is identical to the Kepler problem. This calculation is more technically demanding than that for the harmonic oscillator because the hydrogen atom has fewer symmetries. We use the Ligon-Schaaf regularization map to transport the problem to TS^3, and apply a further transformation to relate the negative quantized energy levels of the hydrogen atom to those of a free particle on S^3. The latter step depends on the dynamical invariance property. We show that the metaplectic-c quantized energy levels agree with the physical prediction from quantum mechanics.

Lastly, in Chapter 6, we show that our quantized energy condition generalizes from one function on M to a family of Poisson-commuting functions. After proving a generalized version of the dynamical invariance theorem, we consider the special case of a completely integrable system, where the size of the Poisson-commuting family is maximal. In that case, the quantized energy condition simplifies to a Bohr-Sommerfeld condition. Thus our quantized energy condition provides a framework that encompasses both the quantized energy levels of a single function H and the Bohr-Sommerfeld leaves of the real polarization generated by a completely integrable system.

Significant portions of this document have already been published. Chapter 3 appears in [21], Chapter 4 appears in [22], and Chapter 5 appears in [23]. Portions of Chapter 2 and this introduction are amalgams of material from all three papers. Minor truncations and edits have been performed for the sake of internal consistency and to eliminate redundancy. Additional background material has been added to Chapter 2, and the content of Chapter 6 does not appear elsewhere.

Chapter 2

Background

Our starting point is a symplectic manifold (M^{2n}, ω): that is, a $2n$-dimensional manifold equipped with a closed, nondegenerate two-form. This object plays the role of a phase space in classical mechanics. Classical observables are elements of $C^\infty(M)$, the smooth functions on M. One objective of geometric quantization is to build a Hermitian line bundle over M and a Lie algebra homomorphism from $C^\infty(M)$ to operators on sections of the bundle, thereby reproducing the transition from classical to quantum mechanics in which observables become operators on a Hilbert space of quantum states. In Section 2.1, we review the basic facts that we will require concerning symplectic manifolds, circle bundles and complex line bundles, and connections.

In Section 2.2, we describe the Kostant-Souriau quantization procedure. We begin with the prequantization stage, in which (M, ω) is required to admit a prequantization line bundle (L, ∇). Then we introduce a metaplectic structure and a choice of polarization F, and sketch how to construct the complex line bundle of half-forms $\wedge^{1/2}F$. We conclude with the quantization stage, in which the functions in $C^\infty(M)$ whose flows preserve the polarization are mapped to operators on polarized sections of $L \otimes \wedge^{1/2}F$.

Finally, in Section 2.3, we summarize the construction of a metaplectic-c prequantization, as developed by Robinson and Rawnsley [15]. A metaplectic-c prequantization is an object that plays the role of a prequantization line bundle and a metaplectic structure simultaneously. As we will see in later chapters, certain features of Kostant-Souriau quantization with half-form correction can be reproduced by a metaplectic-c prequantization alone, without requiring a choice of polarization. Since our work focuses on the prequantization stage of Robinson and Rawnsley's procedure, we omit a detailed discussion of the quantization stage. Some elements of it are presented in Section 3.3.3 in the context of metaplectic-c quantomorphisms.

Some global remarks concerning notation: for any vector field ξ, the Lie derivative with respect to ξ is written L_ξ. The space of smooth vector fields on a manifold P is denoted by $\mathcal{X}(P)$. Given a smooth map $F : P \to M$ and a vector field $\xi \in \mathcal{X}(P)$, we write $F_*\xi$ for the pushforward of ξ only if the result is a well-defined vector field on M. If P is a bundle over M, $\Gamma(P)$ denotes the space of smooth sections of P, where the base is always taken to be the symplectic manifold M. Planck's constant will only appear in the form $\hbar$.

2.1 Symplectic manifolds and bundles

2.1.1 Hamiltonian vector fields and the Poisson algebra

Let (M, ω) be a symplectic manifold. Given $f \in C^\infty(M)$, define its Hamiltonian vector field $\xi_f \in \mathcal{X}(M)$ by

$$\xi_f \lrcorner \omega = df.$$

Define the Poisson bracket on $C^\infty(M)$ by

$$\{f, g\} = \xi_f g = -\omega(\xi_f, \xi_g), \quad \forall f, g \in C^\infty(M).$$

A standard calculation establishes that for all $f, g \in C^\infty(M)$,

$$[\xi_f, \xi_g] = \xi_{\{f,g\}}.$$

2.1.2 Circle bundles and connection one-forms

Let $Y \xrightarrow{p} M$ be a right principal $U(1)$ bundle over M. For any $\theta \in \mathfrak{u}(1)$, the Lie algebra of $U(1)$, let ∂_θ be the vector field on Y given by

$$\partial_\theta(y) = \left.\frac{d}{dt}\right|_{t=0} y \cdot \exp(t\theta), \quad \forall y \in Y.$$

The vector field ∂_θ is called the vector field generated by the infinitesimal action of $\theta \in \mathfrak{u}(1)$.

A $\mathfrak{u}(1)$-valued one-form γ on Y is called a *connection one-form* if γ is invariant under the right principal action, and for all $\theta \in \mathfrak{u}(1)$, $\gamma(\partial_\theta) = \theta$. If Y is equipped with a connection one-form γ, then there is a two-form ϖ on M, called the *curvature* of γ, such that $d\gamma = p^*\varpi$.

For any $\xi \in \mathcal{X}(M)$, let $\tilde{\xi}$ be the lift of ξ to Y that is horizontal with respect to γ. That is, $p_*\tilde{\xi} = \xi$ and $\gamma(\tilde{\xi}) = 0$. For any $\theta \in \mathfrak{u}(1)$, note that $p_*\partial_\theta = 0$, which implies that $p_*[\tilde{\xi}, \partial_\theta] = [p_*\tilde{\xi}, p_*\partial_\theta] = 0$ and $\gamma([\tilde{\xi}, \partial_\theta]) = -(p^*\varpi)(\tilde{\xi}, \partial_\theta) = 0$. Therefore $[\tilde{\xi}, \partial_\theta] = 0$ for all θ.

2.1.3 Complex line bundles and connections

Now let $L \to M$ be a complex line bundle over M. A *connection* ∇ on L allows us to take the derivative of a section s of L in the direction of a vector field ξ on M. It can be viewed as a map from sections of L to L-valued one-forms on M in the sense that, if s is a section of L and ξ is a vector field on M, then at every point $m \in M$, ∇s acts on the vector $\xi(m)$ to yield a value in L_m, which we interpret as the derivative of s in the direction of that vector. Formally, a connection on L is a map

$$\nabla : \Gamma(L) \to \Gamma(L) \otimes \Omega^1(M)$$

such that, for all $r, s \in \Gamma(L)$, all $\xi \in \mathcal{X}(M)$, and all $f \in C^\infty(M)$,

$$\begin{aligned} \nabla_\xi(r + s) &= \nabla_\xi r + \nabla_\xi s, \\ \nabla_\xi(fs) &= f\nabla_\xi s + (\xi f)s. \end{aligned}$$

If $(Y, \gamma) \xrightarrow{p} M$ is a circle bundle with connection one-form over M, then it has an associated complex line bundle with connection (L, ∇) over M. The complex line bundle L is given by $L = Y \times_{U(1)} \mathbb{C}$. We write an element of L as an equivalence class $[y, z]$ with $y \in Y$ and $z \in \mathbb{C}$. The connection ∇ on L is constructed from the connection one-form γ through the following process.

Given any $s \in \Gamma(L)$, define the map $\tilde{s} : Y \to \mathbb{C}$ so that $[y, \tilde{s}(y)] = s(p(y))$ for all $y \in Y$. Then $\tilde{s}$ has the equivariance property

$$\tilde{s}(y \cdot \lambda) = \lambda^{-1}\tilde{s}(y), \quad \forall\ y \in Y,\ \lambda \in U(1).$$

Conversely, any map $\tilde{s} : Y \to \mathbb{C}$ with the above equivariance property can be used to construct a section s of L by setting $s(m) = [y, \tilde{s}(y)]$ for all $m \in M$ and any $y \in Y$ such that $p(y) = m$.

Let $\xi \in \mathcal{X}(M)$ be given, and let $\tilde{\xi}$ be its horizontal lift to Y. If $\tilde{s} : Y \to \mathbb{C}$ is an equivariant map, then so is $\tilde{\xi}\tilde{s}$. This follows from the fact that $[\tilde{\xi}, \partial_\theta] = 0$ for all $\theta \in \mathfrak{u}(1)$. Define the connection ∇ on L so that for any $\xi \in \mathcal{X}(M)$ and $s \in \Gamma(L)$, $\nabla_\xi s$ is the section of L that satisfies

$$\widetilde{\nabla_\xi s} = \tilde{\xi}\tilde{s}.$$

2.1.4 Holonomy

Let (Y, γ) be a circle bundle with connection one-form over M. Let $u : \mathbb{R} \to M$ be a path in M such that $u(t) = u(t+1)$ for all t, and let $\mathcal{C} \subset M$ be the image of u in M. At every point $y \in Y_{u(t)}$, the tangent vector $\dot{u}(t)$ lifts horizontally to a tangent vector $\widetilde{\dot{u}}(t) \in T_yY$, resulting in a vector field on $Y|_{\mathcal{C}}$.

Fix a starting point $y_0 \in Y_{u(0)}$. Let $\widetilde{u}(t)$ be the integral curve for the lifted vector field such that $\widetilde{u}(0) = y_0$. Then $\widetilde{u}(t)$ must be a lift of $u(t)$. In particular, since $u(0) = u(1)$, we have $\widetilde{u}(1) \in Y_{u(0)}$. The resulting map $y_0 \mapsto \widetilde{u}(1)$ is an automorphism of the fiber $Y_{u(0)}$, called the *holonomy*[1] of γ over the closed curve $u(t)$. If $\widetilde{u}(1) = y_0$, then γ is said to have trivial holonomy over $u(t)$.

The concept of holonomy can be translated to the corresponding line bundle with connection (L, ∇). Let $\widetilde{u}(t)$ be a lift of $u(t)$ to L. The image of $\widetilde{u}(t)$ can be thought of as a section s of L, defined only over $\mathcal{C}$. The lift $\widetilde{u}(t)$ is called horizontal if $\nabla_{\dot{u}(t)} s = 0$ for all t. Given a starting point $\widetilde{u}(0)$ in the fiber $L_{u(0)}$, the condition of being horizontal uniquely determines the rest of the lift. As before, $\widetilde{u}(1) \in L_{u(0)}$, so we obtain a map $\widetilde{u}(0) \mapsto \widetilde{u}(1)$ from the fiber $L_{u(0)}$ to itself, called the holonomy of ∇ over $u(t)$.

2.2 Kostant-Souriau quantization and the half-form correction

Geometric quantization in its original form was developed in the 1960's by Kostant [12] and Souriau [20]. The half-form correction was added by Blattner, Kostant and Sternberg [2]. The

[1] This material is standard, but we are particularly indebted to the clarity of exposition in [11]

following overview is based on the detailed treatments that can be found in [10], [18], and [24].

2.2.1 Prequantization

Let (M, ω) be a symplectic manifold. In the prequantization stage, we require (M, ω) to admit a prequantization circle bundle.

Definition 2.2.1. A **prequantization circle bundle** for (M, ω) is a right principal $U(1)$ bundle $Y \xrightarrow{p} M$, together with a connection one-form γ on Y satisfying $d\gamma = \frac{1}{i\hbar} p^*\omega$.

If (M, ω) admits a prequantization circle bundle, then the associated complex line bundle with connection (L, ∇) is called a *prequantization line bundle*. Note that if we view L as $Y \times_{U(1)} \mathbb{C}$, then the usual Hermitian inner product on $\mathbb{C}$ can be used to define a Hermitian structure on L, and that Hermitian structure is compatible with the connection ∇. The manifold (M, ω) admits a prequantization circle bundle (equivalently, a prequantization line bundle) if and only if the cohomology class $\left[\frac{1}{h}\omega\right] \in H^2(M, \mathbb{R})$ is integral.

Assume that (M, ω) admits a prequantization circle bundle (Y, γ), with corresponding prequantization line bundle (L, ∇). One of the goals of the prequantization process is to produce a representation $r : C^\infty(M) \to \operatorname{End}\Gamma(L)$. To be consistent with quantum mechanics in the case of a physically realizable system, the map r is required to satisfy the following axioms, which are based on an analysis by Dirac [5] on the relationship between classical and quantum mechanical observables.

(1) $r(1)$ is the identity map on $\Gamma(L)$,

(2) for all $f, g \in C^\infty(M)$, $[r(f), r(g)] = i\hbar r(\{f, g\})$ (up to sign convention).

In the context of Kostant-Souriau prequantization, a suitable map is given by

$$r(f) = i\hbar\nabla_{\xi_f} + f, \quad \forall f \in C^\infty(M).$$

The fact that this map satisfies condition (2) can be verified by direct computation; as we will show in Section 3.2.3, it can also be viewed as a consequence of the Lie algebra isomorphism between $C^\infty(M)$ and the space of infinitesimal quantomorphisms of (Y, γ).

2.2.2 Quantization and the half-form correction

Recall that geometric quantization is motivated by the transition from classical to quantum mechanics. The phase space for a classical system consists of all possible combinations of position and momentum coordinates. However, the wave functions for the corresponding quantum system depend only on the position coordinates (or, more generally, on a complete set of commuting observables). This observation motivates the introduction of a structure called a polarization.

Assume that (M, ω) admits a prequantization line bundle (L, ∇). A *polarization*[2] F is an involutive Lagrangian subbundle of the complexified tangent bundle $TM^{\mathbb{C}}$ such that $\dim(F_m \cap \overline{F}_m)$ is constant over all $m \in M$. Given a choice of F, we can restrict our attention to those sections of L that are flat in the directions parallel to F.

Let a choice of F be fixed. The *canonical bundle* for F is the top exterior power of the annihilator of F in $(T^*M)^{\mathbb{C}}$, denoted by K^F. The *half-form bundle*, $\wedge^{1/2}F$, is a choice of square root of K^F, when such a square root exists. A *half-form* is a section of $\wedge^{1/2}F$ over M.

To guarantee that the half-form bundle $\wedge^{1/2}F$ exists, we require that (M, ω) admit a metaplectic structure, which is a lifting of the structure group for (M, ω) from the symplectic group to its unique connected double cover, the metaplectic group. In order for (M, ω) to admit a metaplectic structure, the first Chern class $c_1(TM) \in H^2(M, \mathbb{Z})$ must be even: that is, $\frac{1}{2}c_1(TM) \in H^2(M, \mathbb{Z})$. If this condition is satisfied, then the equivalence classes of metaplectic structures for (M, ω) are in one-to-one correspondence with $H^1(M, \mathbb{Z}_2)$.

Assume that (M, ω) admits a metaplectic structure. Then the metaplectic structure can be used to construct a square root of K^F. From K^F, the half-form bundle $\wedge^{1/2}F$ inherits a partial connection ∇_ζ, defined for all $\zeta \in \mathcal{X}(M)$ such that ζ is parallel to the polarization F. It also inherits a partial Lie derivative L_ξ, defined for all $\xi \in \mathcal{X}(M)$ such that the flow of ξ preserves F.

Let $C^\infty_F(M)$ denote the subalgebra of $C^\infty(M)$ consisting of functions whose Hamiltonian flows preserve the polarization F. The Kostant-Souriau representation with half-form correc-

[2]Other technical constraints, such as positivity, may be imposed on F, but since we will not be performing any explicit computations with polarizations, we will not address these details.

tion, $r_F : C^\infty_F(M) \to \operatorname{End}\Gamma(L \otimes \wedge^{1/2}F)$, is defined by

$$r_F(f) = r(f) \otimes I + I \otimes L_{\xi_f}, \quad \forall f \in C^\infty_F(M).$$

This corrected map is a Lie algebra homomorphism.

A section $s \otimes \nu \in \Gamma(L \otimes \wedge^{1/2}F)$ is called *polarized* if $\nabla_\zeta s = 0$ and $\nabla_\zeta \nu = 0$ for all $\zeta \in \mathcal{X}(M)$ such that ζ is parallel to F. Let $\Gamma_F(L \otimes \wedge^{1/2}F)$ denote the space of polarized sections. The operators $r_F(f)$ restrict to operators on $\Gamma_F(L \otimes \wedge^{1/2}F)$. The polarized sections with compact support form the pre-Hilbert space of quantum states, where the Hermitian inner product arises from the Hermitian structure on L, and a pairing from half-forms to densities. This completes the construction of the algebra representation and the Hilbert space as suggested by quantum mechanics.

2.3 Metaplectic-c quantization

The Kostant-Souriau quantization procedure imposes two conditions on (M, ω): namely, that it admit a prequantization circle bundle and a metaplectic structure. Each of these requirements comes with a cohomological condition to be satisfied. The motivation behind metaplectic-c quantization is to replace the prequantization circle bundle and metaplectic structure with a single bundle that can play both roles simultaneously.

The material in this section is a summary of results originally presented by Robinson and Rawnsley [15]. More detail, including proofs of the properties that we state, can be found in [15]. A similar structure, called a spin-c prequantization, was studied in [4, 7, 8, 9].

2.3.1 Definitions on vector spaces

Let V be an n-dimensional complex vector space with Hermitian inner product $\langle \cdot, \cdot \rangle$. If we view V as a $2n$-dimensional real vector space, then the action of the scalar $i \in \mathbb{C}$ becomes the real automorphism $J : V \to V$. Define the real bilinear form Ω on V by

$$\Omega(v, w) = \operatorname{Im}\langle v, w\rangle, \quad \forall v, w \in V.$$

Then (V, Ω) is a $2n$-dimensional symplectic vector space. The Hermitian and symplectic structures are compatible in the sense that

$$\langle v, w\rangle = \Omega(Jv, w) + i\Omega(v, w), \quad \forall v, w \in V.$$

The symplectic group $\mathrm{Sp}(V)$ is the group of real automorphisms $g : V \to V$ such that $\Omega(gv, gw) = \Omega(v, w)$ for all $v, w \in V$. Since the fundamental group for $\mathrm{Sp}(V)$ is $\mathbb{Z}$, $\mathrm{Sp}(V)$ has a unique double cover called the *metaplectic group*, denoted by $\mathrm{Mp}(V)$. Let $\mathrm{Mp}(V) \xrightarrow{\sigma} \mathrm{Sp}(V)$ denote the covering map. The *metaplectic-c group* $\mathrm{Mp}^c(V)$ is defined to be

$$\mathrm{Mp}^c(V) = \mathrm{Mp}(V) \times_{\mathbb{Z}_2} U(1),$$

where $\mathbb{Z}_2 \subset U(1)$ is the usual subgroup $\{1, -1\}$, and where $\mathbb{Z}_2 \subset \mathrm{Mp}(V)$ consists of the two preimages of $I \in \mathrm{Sp}(V)$ under the covering map σ.

By construction, $\mathrm{Mp}^c(V)$ contains $U(1)$ and $\mathrm{Mp}(V)$ as subgroups. The inclusion of each subgroup into $\mathrm{Mp}^c(V)$ yields a short exact sequence and a group homomorphism on $\mathrm{Mp}^c(V)$. One such sequence is

$$1 \to U(1) \to \mathrm{Mp}^c(V) \xrightarrow{\sigma} \mathrm{Sp}(V) \to 1,$$

where the group homomorphism σ is called the *projection map*, and its restriction to $\mathrm{Mp}(V) \subset \mathrm{Mp}^c(V)$ is the double covering map. The other is

$$1 \to \mathrm{Mp}(V) \to \mathrm{Mp}^c(V) \xrightarrow{\eta} U(1) \to 1,$$

where the group homomorphism η is called the *determinant map*, and it has the property that for any $\lambda \in U(1) \subset \mathrm{Mp}^c(V)$, $\eta(\lambda) = \lambda^2$. At the level of Lie algebras, the map $\sigma_* \oplus \frac{1}{2}\eta_*$ yields the identification

$$\mathfrak{mp}^c(V) = \mathfrak{sp}(V) \oplus \mathfrak{u}(1).$$

Given $g \in \mathrm{Sp}(V)$, let

$$C_g = \frac{1}{2}(g - JgJ).$$

By construction, C_g commutes with J, so it is a complex linear transformation of V. The determinant of C_g as a complex transformation is written $\mathrm{Det}_{\mathbb{C}} C_g$. It can be shown that C_g is always invertible, so $\mathrm{Det}_{\mathbb{C}} C_g$ is a nonzero complex number.

A useful embedding of $\mathrm{Mp}^c(V)$ into $\mathrm{Sp}(V) \times \mathbb{C} \setminus \{0\}$ can be defined as follows. An element $a \in \mathrm{Mp}^c(V)$ is mapped to the pair $(g, \mu) \in \mathrm{Sp}(V) \times \mathbb{C} \setminus \{0\}$, where $g \in \mathrm{Sp}(V)$ satisfies $\sigma(a) = g$, and $\mu \in \mathbb{C} \setminus \{0\}$ satisfies $\eta(a) = \mu^2 \mathrm{Det}_{\mathbb{C}} C_g$. The latter condition defines μ up to a sign, and we adopt the convention that the parameters of $I \in \mathrm{Mp}^c(V)$ are $(I, 1)$. This choice uniquely determines the sign of μ for all $a \in \mathrm{Mp}^c(V)$. Following [15], we refer to (g, μ) as the *parameters* of a.

We note two properties of this parametrization, both of which will be useful in later chapters. First, if $a \in \mathrm{Mp}(V) \subset \mathrm{Mp}^c(V)$, then $\eta(a) = 1$, and so the parameters of a are (g, μ) where $\sigma(a) = g$ and $\mu^2 \mathrm{Det}_{\mathbb{C}} C_g = 1$. Second, if $\lambda \in U(1) \subset \mathrm{Mp}^c(V)$, then the parameters of λ are (I, λ), and for any other element $a \in \mathrm{Mp}^c(V)$ with parameters (g, μ), the parameters of the product $a\lambda$ are $(g, \mu\lambda)$.

2.3.2 Definitions over a symplectic manifold

Let (M, ω) be a $2n$-dimensional symplectic manifold, and assume that a $2n$-dimensional model symplectic vector space (V, Ω) with complex structure J has been fixed. We think of the symplectic frame bundle for (M, ω) as being modeled on $\mathrm{Sp}(V)$, in the following sense.

Definition 2.3.1. The **symplectic frame bundle** $\mathrm{Sp}(M, \omega) \xrightarrow{\rho} M$ is a right principal $\mathrm{Sp}(V)$ bundle over M given by

$$\mathrm{Sp}(M, \omega)_m = \{b : V \to T_m M : b \text{ is a symplectic linear isomorphism}\}, \quad \forall m \in M.$$

The group $\mathrm{Sp}(V)$ acts on the fibers by precomposition.

In the previous section, we defined a metaplectic structure for (M, ω) to be a lifting of the structure group for (M, ω) from $\mathrm{Sp}(V)$ to $\mathrm{Mp}(V)$. Analogously, a metaplectic-c structure for (M, ω) is a lifting of the structure group to $\mathrm{Mp}^c(V)$.

Definition 2.3.2. A **metaplectic-c structure** for (M, ω) consists of a right principal $\mathrm{Mp}^c(V)$

bundle $P \xrightarrow{\Pi} M$ and a map $P \xrightarrow{\Sigma} \mathrm{Sp}(M,\omega)$ such that

$$\Sigma(q \cdot a) = \Sigma(q) \cdot \sigma(a), \quad \forall q \in P,\ \forall a \in \mathrm{Mp}^c(V),$$

and such that the following diagram commutes.

$$\begin{array}{ccc} P & \xrightarrow{\Sigma} & \mathrm{Sp}(M,\omega) \\ \downarrow{\scriptstyle \Pi} & \swarrow{\scriptstyle \rho} & \\ (M,\omega) & & \end{array}$$

To complete the construction of a metaplectic-c prequantization, we equip the metapletic-c structure with a $\mathfrak{u}(1)$-valued one-form that is analogous to the connection one-form on a prequantization circle bundle.

Definition 2.3.3. A **metaplectic-c prequantization** of (M,ω) is a metaplectic-c structure (P,Σ) for (M,ω), together with a $\mathfrak{u}(1)$-valued one-form γ on P such that:

(1) γ is invariant under the principal $\mathrm{Mp}^c(V)$ action;

(2) for any $\alpha \in \mathfrak{mp}^c(V)$, if ∂_α is the vector field on P generated by the infinitesimal action of α, then $\gamma(\partial_\alpha) = \frac{1}{2}\eta_*\alpha$;

(3) $d\gamma = \frac{1}{i\hbar}\Pi^*\omega$.

Note that if (P,Σ,γ) is a metaplectic-c prequantization for (M,ω), then (P,γ) is a circle bundle with connection one-form over $\mathrm{Sp}(M,\omega)$. The circle that acts on the fibers of P is the center $U(1) \subset \mathrm{Mp}^c(V)$. We will view P as a bundle over M or a bundle over $\mathrm{Sp}(M,\omega)$ as the circumstance demands.

For any $\alpha \in \mathfrak{mp}^c(V)$, we can write $\alpha = \kappa \oplus \tau$ under the identification of $\mathfrak{mp}^c(V)$ with $\mathfrak{sp}(V) \oplus \mathfrak{u}(1)$. Then $\gamma(\partial_\alpha) = \tau$, and naturality of the exponential map and equivariance of the map Σ with respect to σ ensure that $\Sigma_*\partial_\alpha = \partial_\kappa$, where ∂_κ is the vector field on $\mathrm{Sp}(M,\omega)$ generated by the infinitesimal action of $\kappa \in \mathfrak{sp}(V)$. In fact, ∂_α is uniquely specified by the two conditions $\gamma(\partial_\alpha) = \tau$ and $\Sigma_*\partial_\alpha = \partial_\kappa$, an observation that will be useful in Chapter 3.

There is a cohomology constraint that must be satisfied in order for (M,ω) to admit a metaplectic-c prequantization, and that condition is very closely related to the ones we saw for the prequantization circle bundle and metaplectic structure. Specifically, (M,ω) is metaplectic-c prequantizable if and only if the cohomology class $\left[\frac{1}{h}\omega\right] + \frac{1}{2}c_1(TM) \in H^2(M,\mathbb{R})$ is integral. Thus, if (M,ω) admits a prequantization circle bundle and a metaplectic structure, then it is metaplectic-c prequantizable, but the converse is not true.

Two metaplectic-c prequantizations for (M,ω) are considered equivalent if there is a diffeomorphism between them that preserves the one-forms and commutes with the respective maps to $\mathrm{Sp}(M,\omega)$. If (M,ω) is metaplectic-c prequantizable, then the set of equivalence classes of metaplectic-c prequantizations for (M,ω) is in one-to-one correspondence with the locally constant cohomology group $H^1(M,U(1))$, which also parametrizes the circle bundles with flat connection over (M,ω). In particular, if $H^1(M,U(1))$ is trivial, then the metaplectic-c prequantization of (M,ω) is unique up to isomorphism.

Chapter 3

Metaplectic-c Quantomorphisms

3.1 Introduction

Recall that a prequantization circle bundle for a symplectic manifold (M, ω) consists of a circle bundle $Y \to M$ and a connection one-form γ on Y such that $d\gamma = \frac{1}{i\hbar}\omega$. Souriau [20] defined a *quantomorphism* between two prequantization circle bundles $(Y_1, \gamma_1) \to (M_1, \omega_1)$ and $(Y_2, \gamma_2) \to (M_2, \omega_2)$ to be a diffeomorphism $K : Y_1 \to Y_2$ such that $K^*\gamma_2 = \gamma_1$. This condition implies that K is equivariant with respect to the principal circle actions. Souriau then defined the *infinitesimal quantomorphisms* of a prequantization circle bundle (Y, γ) to be the vector fields on Y whose flows are quantomorphisms. Kostant [12] proved that the space of infinitesimal quantomorphisms, which we denote $\mathcal{Q}(Y, \gamma)$, is isomorphic to the Poisson algebra $C^\infty(M)$. In Section 3.2, we present an explicit construction of the isomorphism from $\mathcal{Q}(Y, \gamma)$ to $C^\infty(M)$.

The objective of the rest of the chapter is to adapt the concept of an infinitesimal quantomorphism to the case where (M, ω) admits a metaplectic-c prequantization (P, Σ, γ). In Section 3.3, we define a metaplectic-c quantomorphism, which is a diffeomorphism of metaplectic-c prequantizations that preserves all of their structures. Our definition is based on Souriau's, but includes a condition that is unique to the metaplectic-c context. We then use the metaplectic-c quantomorphisms to define $\mathcal{Q}(P, \Sigma, \gamma)$, the space of infinitesimal metaplectic-c quantomorphisms. We show that every property that was proved for $\mathcal{Q}(Y, \gamma)$ has a parallel for $\mathcal{Q}(P, \Sigma, \gamma)$. In particular, $\mathcal{Q}(P, \Sigma, \gamma)$ is isomorphic to the Poisson algebra $C^\infty(M)$. The construction in Section 3.2 is used as a model for the proofs in Section 3.3. We indicate when the calculations are analogous, and when the metaplectic-c case requires additional steps.

3.2 Kostant-Souriau quantomorphisms

In this section, we construct a Lie algebra isomorphism from $C^\infty(M)$ to the space of infinitesimal quantomorphisms. As we have already noted, the fact that these algebras are isomorphic was originally stated by Kostant [12] in the context of line bundles with connection. His proof can be reconstructed from several propositions across Sections 2 – 4 of [12]. Kostant's isomorphism is also stated by Śniatycki [18], but much of the proof is left as an exercise. We are not aware of a source in the literature for a self-contained proof that uses the language of principal bundles, and this is one of our reasons for performing an explicit construction here.

The other goal of this section is to motivate the analogous constructions for a metaplectic-c prequantization, which will be the subject of Section 3.3. Each result that we present for Kostant-Souriau prequantization will have a parallel in the metaplectic-c case. When the proofs are identical, we will simply refer back to the work shown here, thereby allowing Section 3.3 to focus on those features that are unique to metaplectic-c structures.

3.2.1 Infinitesimal quantomorphisms of a prequantization circle bundle

Definition 3.2.1. Let $(Y_1, \gamma_1) \xrightarrow{p_1} (M_1, \omega_1)$ and $(Y_2, \gamma_2) \xrightarrow{p_2} (M_2, \omega_2)$ be prequantization circle bundles for two symplectic manifolds. A diffeomorphism $K : Y_1 \to Y_2$ is called a **quantomorphism** if $K^*\gamma_2 = \gamma_1$.

Let $K : Y_1 \to Y_2$ be a quantomorphism. Notice that for any $\theta \in \mathfrak{u}(1)$, the vector field ∂_θ on Y_1 is completely specified by the conditions $\gamma_1(\partial_\theta) = \theta$ and $d\gamma_1(\partial_\theta) = 0$, and the same is true on Y_2. Since $K^*\gamma_2 = \gamma_1$, we see that $K_*\partial_\theta = \partial_\theta$ for all θ, and so K is equivariant with respect to the principal circle actions.

Definition 3.2.2. Let $(Y, \gamma) \xrightarrow{p} (M, \omega)$ be a prequantization circle bundle. An **infinitesimal quantomorphism** of (Y, γ) is a vector field $\zeta \in \mathcal{X}(Y)$ whose flow ϕ^t on Y is a quantomorphism from its domain to its range for each t. The space of infinitesimal quantomorphisms of (Y, γ) is denoted by $\mathcal{Q}(Y, \gamma)$.

Let $\zeta \in \mathcal{X}(Y)$ have flow ϕ^t. The connection form γ is preserved by ϕ^t if and only if $L_\zeta \gamma = 0$. Therefore the space of infinitesimal quantomorphisms of (Y, γ) is

$$\mathcal{Q}(Y, \gamma) = \{\zeta \in \mathcal{X}(Y) \,|\, L_\zeta \gamma = 0\}.$$

If $K : Y_1 \to Y_2$ is a quantomorphism, then it induces a diffeomorphism (in fact, a symplectomorphism) $K' : M_1 \to M_2$ such that the following diagram commutes.

$$\begin{array}{ccc} Y_1 & \xrightarrow{K} & Y_2 \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} \\ M_1 & \xrightarrow{K'} & M_2 \end{array}$$

This implies that for any $\zeta \in \mathcal{Q}(Y, \gamma)$ with flow ϕ^t, there is a flow ${\phi'}^t$ on M that satisfies $p \circ \phi^t = {\phi'}^t \circ p$. If ζ' is the vector field on M with flow ${\phi'}^t$, then $p_* \zeta = \zeta'$. In other words, elements of $\mathcal{Q}(Y, \gamma)$ descend via p_* to well-defined vector fields on M.

3.2.2 The Lie algebra isomorphism

Let $(Y, \gamma) \xrightarrow{p} (M, \omega)$ be a prequantization circle bundle. We will now present an explicit construction of a Lie algebra isomorphism from $C^\infty(M)$ to $\mathcal{Q}(Y, \gamma)$. Recall that the vector field $\partial_{2\pi i}$ on Y satisfies $\gamma(\partial_{2\pi i}) = 2\pi i \in \mathfrak{u}(1)$ and $p_* \partial_{2\pi i} = 0$.

Lemma 3.2.3. For all $f, g \in C^\infty(M)$,

$$[\tilde{\xi}_f, \tilde{\xi}_g] = \tilde{\xi}_{\{f,g\}} - \frac{1}{2\pi\hbar} p^*\{f, g\} \partial_{2\pi i}.$$

Proof. It suffices to show that

$$p_*[\tilde{\xi}_f, \tilde{\xi}_g] = p_* \left(\tilde{\xi}_{\{f,g\}} - \frac{1}{2\pi\hbar} p^*\{f, g\} \partial_{2\pi i} \right) \quad \text{and}$$
$$\gamma([\tilde{\xi}_f, \tilde{\xi}_g]) = \gamma \left(\tilde{\xi}_{\{f,g\}} - \frac{1}{2\pi\hbar} p^*\{f, g\} \partial_{2\pi i} \right)$$

First, we have

$$p_* \left(\tilde{\xi}_{\{f,g\}} - \frac{1}{2\pi\hbar} p^*\{f, g\} \partial_{2\pi i} \right) = \xi_{\{f,g\}} = [\xi_f, \xi_g].$$

Since $p_*\tilde{\xi}_f = \xi_f$ and $p_*\tilde{\xi}_g = \xi_g$, it follows that $p_*[\tilde{\xi}_f, \tilde{\xi}_g] = [\xi_f, \xi_g]$. Thus the first equation is verified.

Next, note that

$$\gamma\left(\tilde{\xi}_{\{f,g\}} - \frac{1}{2\pi\hbar}p^*\{f,g\}\partial_{2\pi i}\right) = \frac{1}{i\hbar}p^*\{f,g\},$$

and

$$\gamma([\tilde{\xi}_f, \tilde{\xi}_g]) = -\frac{1}{i\hbar}(p^*\omega)(\tilde{\xi}_f, \tilde{\xi}_g) = \frac{1}{i\hbar}p^*\{f,g\}.$$

Therefore the second equation is also verified.

□

Lemma 3.2.4. The map $E : C^\infty(M) \to \mathcal{X}(Y)$ given by

$$E(f) = \tilde{\xi}_f + \frac{1}{2\pi\hbar}p^* f \partial_{2\pi i}, \quad \forall f \in C^\infty(M)$$

is a Lie algebra homomorphism.

Proof. Let $f, g \in C^\infty(M)$ be arbitrary. We need to show that

$$\tilde{\xi}_{\{f,g\}} + \frac{1}{2\pi\hbar}p^*\{f,g\}\partial_{2\pi i} = \left[\tilde{\xi}_f + \frac{1}{2\pi\hbar}p^* f\partial_{2\pi i}, \tilde{\xi}_g + \frac{1}{2\pi\hbar}p^* g\partial_{2\pi i}\right].$$

Using Lemma 3.2.3, the left-hand side becomes

$$[\tilde{\xi}_f, \tilde{\xi}_g] + 2\frac{1}{2\pi\hbar}p^*\{f,g\}\partial_{2\pi i}.$$

Expanding the right-hand side yields

$$[\tilde{\xi}_f, \tilde{\xi}_g] + \left[\tilde{\xi}_f, \frac{1}{2\pi\hbar}p^* g\partial_{2\pi i}\right] + \left[\frac{1}{2\pi\hbar}p^* f\partial_{2\pi i}, \tilde{\xi}_g\right] + \left[\frac{1}{2\pi\hbar}p^* f\partial_{2\pi i}, \frac{1}{2\pi\hbar}p^* g\partial_{2\pi i}\right].$$

The fourth term vanishes because $\partial_\theta(p^*f) = \partial_\theta(p^*g) = 0$ for any $\theta \in \mathfrak{u}(1)$. To evaluate the third term, recall that that $[\partial_\theta, \tilde{\xi}] = 0$ for any $\theta \in \mathfrak{u}(1)$ and $\xi \in \mathcal{X}(M)$. Therefore $[\partial_{2\pi i}, \tilde{\xi}_g] = 0$, so this term reduces to

$$-\frac{1}{2\pi\hbar}(\tilde{\xi}_g p^* f)\partial_{2\pi i} = \frac{1}{2\pi\hbar}p^*\{f,g\}\partial_{2\pi i}.$$

By the same argument, the second term also reduces to

$$\frac{1}{2\pi\hbar} p^*\{f,g\}\partial_{2\pi i}.$$

Combining these results, we find that the right-hand side of the desired equation is

$$[\tilde{\xi}_f, \tilde{\xi}_g] + 2\frac{1}{i\hbar} p^*\{f,g\}\partial_{2\pi i},$$

which equals the left-hand side. □

Lemma 3.2.5. For all $f \in C^\infty(M)$, $E(f) \in \mathcal{Q}(Y,\gamma)$.

Proof. We need to show that $L_{E(f)}\gamma = 0$. We calculate

$$L_{E(f)}\gamma = E(f) \lrcorner\, d\gamma + d(E(f) \lrcorner\, \gamma) = \frac{1}{i\hbar} p^*(\xi_f \lrcorner\, \omega) - \frac{1}{i\hbar} p^* df = 0.$$

□

So far, we have shown that $E : C^\infty(M) \to \mathcal{Q}(Y,\gamma)$ is a Lie algebra homomorphism. We will now construct a map $F : \mathcal{Q}(Y,\gamma) \to C^\infty(M)$, and show that E and F are inverses. This will complete the proof that $C^\infty(M)$ and $\mathcal{Q}(Y,\gamma)$ are isomorphic.

Let $\zeta \in \mathcal{Q}(Y,\gamma)$ be arbitrary. Then $L_\zeta\gamma = \zeta \lrcorner\, d\gamma + d(\gamma(\zeta)) = 0$. This implies that $\partial_\theta \lrcorner\, (\zeta \lrcorner\, d\gamma + d(\gamma(\zeta))) = 0$ for any $\theta \in \mathfrak{u}(1)$. Since $d\gamma(\zeta, \partial_\theta) = \frac{1}{i\hbar}(p^*\omega)(\zeta, \partial_\theta) = 0$, it follows that $\partial_\theta \lrcorner\, d(\gamma(\zeta)) = L_{\partial_\theta}\gamma(\zeta) = 0$. We can therefore define the map $F : \mathcal{Q}(Y,\gamma) \to C^\infty(M)$ so that

$$-\frac{1}{i\hbar} p^* F(\zeta) = \gamma(\zeta), \quad \forall \zeta \in \mathcal{Q}(Y,\gamma).$$

Theorem 3.2.6. The map $E : C^\infty(M) \to \mathcal{Q}(Y,\gamma)$ is a Lie algebra isomorphism with inverse F.

Proof. Let $f \in C^\infty(M)$ and $\zeta \in \mathcal{Q}(Y,\gamma)$ be arbitrary. We will show that $F(E(f)) = f$ and $E(F(\zeta)) = \zeta$. Using the definitions of E and F, we have

$$-\frac{1}{i\hbar} p^* F(E(f)) = \gamma(E(f)) = \gamma\left(\tilde{\xi}_f + \frac{1}{2\pi\hbar} p^* f \partial_{2\pi i}\right) = -\frac{1}{i\hbar} p^* f.$$

This implies that $F(E(f)) = f$.

To show that $E(F(\zeta)) = \zeta$, it suffices to show that $\gamma(E(F(\zeta))) = \gamma(\zeta)$ and $p_*E(F(\zeta)) = p_*\zeta$. By definition,

$$E(F(\zeta)) = \tilde{\xi}_{F(\zeta)} + \frac{1}{2\pi\hbar}p^*F(\zeta)\partial_{2\pi i} = \tilde{\xi}_{F(\zeta)} + \frac{1}{2\pi i}\gamma(\zeta)\partial_{2\pi i}.$$

It is immediate that $\gamma(E(F(\zeta))) = \gamma(\zeta)$, and that $p_*E(F(\zeta)) = \xi_{F(\zeta)}$. Observe that

$$\zeta \lrcorner\, p^*\omega = i\hbar\zeta \lrcorner\, d\gamma = -i\hbar d(\gamma(\zeta)) = p^*(dF(\zeta)),$$

having used $L_\zeta\gamma = 0$. Therefore $(p_*\zeta)\lrcorner\,\omega = dF(\zeta)$, which implies that $p_*\zeta = \xi_{F(\zeta)}$. Thus $p_*E(F(\zeta)) = p_*\zeta$. This concludes the proof that $E(F(\zeta)) = \zeta$.

Since E and F are inverses, and we know from Lemma 3.2.5 that $E : C^\infty(M) \to \mathcal{Q}(Y,\gamma)$ is a Lie algebra homomorphism, it follows that E and F are the desired Lie algebra isomorphisms.

□

The primary goal of Section 3.3 is to duplicate the above construction for the infinitesimal quantomorphisms of a metaplectic-c prequantization. However, before moving on to the metaplectic-c case, we will show how the map E can be used to represent the elements of $C^\infty(M)$ as operators on the space of sections of the prequantization line bundle for (M,ω). This result will also have an analogue in the metaplectic-c case, which we will discuss in Section 3.3.3.

3.2.3 An operator representation of $C^\infty(M)$

Let (L,∇) be the complex line bundle with connection associated to (Y,γ). Recall the association between a section s of L and an equivariant function $\tilde{s} : Y \to \mathbb{C}$ from Section 2.1.2. We note the following properties.

- For any $f \in C^\infty(M)$ and $s \in \Gamma(L)$, the equivariant function corresponding to the section fs is $\widetilde{fs} = p^*f\tilde{s}$.

- The vector field $\partial_{2\pi i}$ is generated by the infinitesimal action of $2\pi i \in \mathfrak{u}(1)$ on Y. Thus, for all $y \in Y$,

$$(\partial_{2\pi i}\tilde{s})(y) = \frac{d}{dt}\bigg|_{t=0} \tilde{s}(y \cdot e^{2\pi i t}) = -2\pi i \tilde{s}(y).$$

Further recall that the Kostant-Souriau representation map $r : C^\infty(M) \to \operatorname{End}\Gamma(L)$ is defined by

$$r(f)s = \left(i\hbar\nabla_{\xi_f} + f\right) s, \quad \forall f \in C^\infty(M),\ s \in \Gamma(L).$$

Using the preceding observations, we see that

$$\widetilde{r(f)s} = \left(i\hbar\tilde{\xi}_f + p^* f\right) \tilde{s} = \left(i\hbar\tilde{\xi}_f - \frac{1}{2\pi i} p^* f \partial_{2\pi i}\right) \tilde{s} = i\hbar E(f)\tilde{s}.$$

Since we proved in Lemma 3.2.4 that $E(\{f, g\}) = [E(f), E(g)]$ for all $f, g \in C^\infty(M)$, the following is immediate.

Theorem 3.2.7. The map $r : C^\infty(M) \to \operatorname{End}\Gamma(L)$ satisfies

$$[r(f), r(g)] = i\hbar r(\{f, g\}), \quad \forall f, g \in C^\infty(M).$$

Thus the same map that provides the isomorphism from $C^\infty(M)$ to $\mathcal{Q}(Y, \gamma)$ also yields the usual Kostant-Souriau representation of $C^\infty(M)$ as a space of operators on $\Gamma(L)$. We will see a similar result in the case of metaplectic-c prequantization.

3.3 Metaplectic-c Quantomorphisms

Having reviewed the properties of infinitesimal quantomorphisms in Kostant-Souriau prequantization, we will now explore their parallels in metaplectic-c prequantization. In Section 3.3.1, we develop our definition for a metaplectic-c quantomorphism, and use it to define an infinitesimal metaplectic-c quantomorphism. The remainder of the chapter is dedicated to proving the metaplectic-c analogues of the results presented in Section 3.2.

Suppose (M, ω) admits a metaplectic-c prequantization (P, Σ, γ). The space of infinitesimal quantomorphisms of (P, Σ, γ) consists of those vector fields on P whose flows preserve all of the structures on (P, Σ, γ). Note that one of these structures is the map $P \xrightarrow{\Sigma} \operatorname{Sp}(M, \omega)$, which does not have a direct analogue in the Kostant-Souriau case. We will show how to incorporate this additional piece of information in the next section.

3.3.1 Infinitesimal metaplectic-c quantomorphisms

As in Section 3.2.1, we begin by developing the idea of a quantomorphism between metaplectic-c prequantizations. Let $(P_1, \Sigma_1, \gamma_1) \xrightarrow{\Sigma_1} \mathrm{Sp}(M_1, \omega_1) \xrightarrow{\rho_1} (M_1, \omega_1)$ and $(P_2, \Sigma_2, \gamma_2) \xrightarrow{\Sigma_2} \mathrm{Sp}(M_2, \omega_2) \xrightarrow{\rho_2} (M_2, \omega_2)$ be metaplectic-c prequantizations for two symplectic manifolds, and let $\Pi_j = \rho_j \circ \Sigma_j$ for $j = 1, 2$. Let $K : P_1 \to P_2$ be a diffeomorphism. We will determine the conditions that K must satisfy in order for it to preserve all of the structures of the metaplectic-c prequantizations. First, by analogy with the Kostant-Souriau definition, assume that K satisfies $K^*\gamma_2 = \gamma_1$.

Fix $m \in M_1$, and consider the fiber P_{1m}. For any $q \in P_{1m}$, notice that

$$T_q P_{1m} = \{\xi \in T_q P_1 \,|\, \Pi_{1*}\xi = 0\} = \ker d\gamma_{1q}.$$

The same property holds for a fiber of P_2 over a point in M_2. By assumption, K_* is an isomorphism from $\ker d\gamma_{1q}$ to $\ker d\gamma_{2K(q)}$ for all $q \in P_1$. Therefore Π_2 is constant on $K(P_{1m})$. Moreover, since K is a diffeomorphism, we can make the analogous argument with K^{-1}, and conclude that $K(P_{1m})$ is in fact a fiber of P_2 over M_2, and every fiber of P_2 is the image of a fiber of P_1. Thus K induces a diffeomorphism $K'' : M_1 \to M_2$ such that the following diagram commutes.

$$\begin{array}{ccc} P_1 & \xrightarrow{K} & P_2 \\ \downarrow{\scriptstyle \Pi_1} & & \downarrow{\scriptstyle \Pi_2} \\ M_1 & \xrightarrow{K''} & M_2 \end{array}$$

Lemma 3.3.1. The map $K'' : M_1 \to M_2$ is a symplectomorphism.

Proof. It suffices to show that $K''^*\omega_2 = \omega_1$. Using the properties of K, γ_1 and γ_2, we calculate

$$\Pi_1^*(K''^*\omega_2) = (K'' \circ \Pi_1)^*\omega_2 = (\Pi_2 \circ K)^*\omega_2 = K^*(i\hbar d\gamma_2) = i\hbar d\gamma_1 = \Pi_1^*\omega_1.$$

Therefore $K''^*\omega_2 = \omega_1$, as required. $\square$

Assume that a model symplectic vector space (V, Ω) has been fixed. Recall from Definition 2.3.1 that an element $b \in \mathrm{Sp}(M_1, \omega_1)_m$ is a map $b : V \to T_m M_1$ such that $b^* \omega_{1m} = \Omega$. Since K'' is a symplectomorphism, the composition $K''_* \circ b : V \to T_{K''(m)} M_2$ satisfies $(K''_* \circ b)^* \omega_{2K''(m)} = \Omega$, which implies that $K''_* \circ b \in \mathrm{Sp}(M_2, \omega_2)_{K''(m)}$. Let $\widetilde{K''} : \mathrm{Sp}(M_1, \omega_1) \to \mathrm{Sp}(M_2, \omega_2)$ be the lift of K'' given by

$$\widetilde{K''}(b) = K''_* \circ b, \quad \forall b \in \mathrm{Sp}(M_1, \omega_1).$$

Then $\widetilde{K''}$ is a diffeomorphism, and it is equivariant with respect to the principal $\mathrm{Sp}(V)$ actions.

Thus, if we assume that $K^* \gamma_2 = \gamma_1$, we obtain the diffeomorphisms $K'' : M_1 \to M_2$ and $\widetilde{K''} : \mathrm{Sp}(M_1, \omega_1) \to \mathrm{Sp}(M_2, \omega_2)$, where both K and $\widetilde{K''}$ are lifts of K''. However, K is not necessarily a lift of $\widetilde{K''}$. Indeed, there might not be any map $K' : \mathrm{Sp}(M_1, \omega_1) \to \mathrm{Sp}(M_2, \omega_2)$ of which K is a lift. A map K for which there is no corresponding K' is constructed in Section 3.4, Example 3.4.1. In Section 3.2.1, we showed that a diffeomorphism of prequantization circle bundles that preserves the connection forms must be equivariant with respect to the principal circle actions. By contrast, Example 3.4.1 demonstrates that it is possible for K to preserve the prequantization one-forms without being equivariant with respect to the principal $\mathrm{Mp}^c(V)$ actions.

Suppose we make the additional assumption that $K(q \cdot a) = K(q) \cdot a$ for all $q \in P_1$ and $a \in \mathrm{Mp}^c(V)$. Then K induces a diffeomorphism $K' : \mathrm{Sp}(M_1, \omega_1) \to \mathrm{Sp}(M_2, \omega_2)$ that satisfies $K' \circ \Sigma_1 = \Sigma_2 \circ K$. Combining this with the map $K'' : M_1 \to M_2$ yields the following commutative diagram.

$$\begin{array}{ccc} P_1 & \xrightarrow{K} & P_2 \\ \downarrow{\scriptstyle \Sigma_1} & & \downarrow{\scriptstyle \Sigma_2} \\ \mathrm{Sp}(M_1, \omega_1) & \xrightarrow{K'} & \mathrm{Sp}(M_2, \omega_2) \\ \downarrow{\scriptstyle \rho_1} & & \downarrow{\scriptstyle \rho_2} \\ M_1 & \xrightarrow{K''} & M_2 \end{array}$$

We now have two maps, K' and $\widetilde{K''}$, which are diffeomorphisms from $\mathrm{Sp}(M_1, \omega_1)$ to $\mathrm{Sp}(M_2, \omega_2)$. By construction, $\rho_2 \circ K' = \rho_2 \circ \widetilde{K''}$, and both K' and $\widetilde{K''}$ are equivariant with respect to the principal $\mathrm{Sp}(V)$ actions. However, it is still possible for K' and $\widetilde{K''}$ to be different. A map K for which $K' \neq \widetilde{K''}$ is given in Example 3.4.2.

As will be shown in Section 3.3.2, this potential discrepancy between K' and $\widetilde{K''}$ must be prevented in order to construct the desired isomorphism between $C^\infty(M)$ and the infinitesimal quantomorphisms. We therefore propose the following definition.

Definition 3.3.2. The diffeomorphism $K : P_1 \to P_2$ is a **metaplectic-c quantomorphism** if

(1) $K^*\gamma_2 = \gamma_1$,

(2) the induced diffeomorphism $K'' : M_1 \to M_2$ satisfies $\widetilde{K''} \circ \Sigma_1 = \Sigma_2 \circ K$.

Let $K : P_1 \to P_2$ be a metaplectic-c quantomorphism. Given our concept of a quantomorphism as a diffeomorphism that preserves all of the structures of a metaplectic-c prequantization, we would expect that K is equivariant with respect to the $\mathrm{Mp}^c(V)$ actions. Equivariance is a consequence of the definition, as we now show.

Let $\alpha \in \mathfrak{mp}^c(V)$ be arbitrary, and write $\alpha = \kappa \oplus \tau$ under the identification of $\mathfrak{mp}^c(V)$ with $\mathfrak{sp}(V) \oplus \mathfrak{u}(1)$. The vector field ∂_α on P_1 is completely specified by the conditions $\gamma_1(\partial_\alpha) = \tau$ and $\Sigma_{1*}\partial_\alpha = \partial_\kappa$, and the same is true on P_2. Notice that

$$\gamma_2(K_*\partial_\alpha) = \gamma_1(\partial_\alpha) = \tau,$$

and

$$\Sigma_{2*}K_*\partial_\alpha = \widetilde{K''}_*\Sigma_{1*}\partial_\alpha = \widetilde{K''}_*\partial_\kappa = \partial_\kappa,$$

where the final equality follows from the fact that $\widetilde{K''}$ is equivariant with respect to $\mathrm{Sp}(V)$. Thus $K_*\partial_\alpha = \partial_\alpha$ for all $\alpha \in \mathfrak{mp}^c(V)$, which implies that K is equivariant with respect to $\mathrm{Mp}^c(V)$, as desired.

Now consider a single metaplectic-c prequantized space $(P, \Sigma, \gamma) \xrightarrow{\Sigma} \mathrm{Sp}(M, \omega) \xrightarrow{\rho} (M, \omega)$ with $\Pi = \rho \circ \Sigma$.

Definition 3.3.3. A vector field $\zeta \in \mathcal{X}(P)$ is an **infinitesimal metaplectic-c quantomorphism** if its flow ϕ^t is a metaplectic-c quantomorphism from its domain to its range for each t.

Let $\zeta \in \mathcal{X}(P)$ have flow ϕ^t. Property (1) of a quantomorphism holds for ϕ^t if and only if $L_\zeta \gamma = 0$. If we assume that ϕ^t satisfies property (1), then there is a flow ϕ''^t on M such that $\Pi \circ \phi^t = \phi''^t \circ \Pi$. The vector field that it generates on M is $\Pi_* \zeta$.

Lemma 3.3.1 shows that ϕ''^t is a family of symplectomorphisms. Therefore we can lift ϕ''^t to a flow on $\mathrm{Sp}(M,\omega)$, denoted by $\widetilde{\phi''^t}$, where $\widetilde{\phi''^t}(b) = (\phi''^t)_* \circ b$ for all $b \in \mathrm{Sp}(M,\omega)$. Let the vector field on $\mathrm{Sp}(M,\omega)$ that has flow $\widetilde{\phi''^t}$ be $\widetilde{\Pi_*\zeta}$. Then property (2) of a quantomorphism holds for ϕ^t if and only if $\Sigma_*\zeta$ is a well-defined vector field on $\mathrm{Sp}(M,\omega)$ and $\Sigma_*\zeta = \widetilde{\Pi_*\zeta}$.

We conclude that the space of infinitesimal metaplectic-c quantomorphisms of (P, Σ, γ) is

$$\mathcal{Q}(P,\Sigma,\gamma) = \{\zeta \in \mathcal{X}(P) \;\Big|\; L_\zeta\gamma = 0 \text{ and } \Sigma_*\zeta = \widetilde{\Pi_*\zeta}\},$$

where it is understood that the condition $\Sigma_*\zeta = \widetilde{\Pi_*\zeta}$ can only be satisfied if $\Sigma_*\zeta$ is well defined. In the next section, we will construct a Lie algebra isomorphism from $C^\infty(M)$ to $\mathcal{Q}(P,\Sigma,\gamma)$.

3.3.2 The Lie algebra isomorphism

We begin with a procedure, given by Robinson and Rawnsley in Section 7 of [15], for lifting a Hamiltonian vector field on M to $\mathrm{Sp}(M,\omega)$ and then to P. These steps will be used in constructing the isomorphism $E : C^\infty(M) \to \mathcal{Q}(P,\Sigma,\gamma)$.

Fix $f \in C^\infty(M)$, and let its Hamiltonian vector field ξ_f have flow φ^t on M. We know that ${\varphi^t}_*$ preserves ω because $L_{\xi_f}\omega = 0$. Let $\tilde{\varphi}^t$ be the lift of φ^t to $\mathrm{Sp}(M,\omega)$ given by

$$\tilde{\varphi}^t(b) = {\varphi^t}_* \circ b, \quad \forall b \in \mathrm{Sp}(M,\omega),$$

and let the vector field on $\mathrm{Sp}(M,\omega)$ with flow $\tilde{\varphi}^t$ be $\tilde{\xi}_f$. We have $\rho_*\tilde{\xi}_f = \xi_f$ by construction. Also, $\tilde{\varphi}^t$ commutes with the right principal $\mathrm{Sp}(V)$ action on $\mathrm{Sp}(M,\omega)$, so $[\tilde{\xi}_f, \partial_\kappa] = 0$ for all $\kappa \in \mathfrak{sp}(V)$. Now let $\hat{\xi}_f$ be the lift of $\tilde{\xi}_f$ to P that is horizontal with respect to γ. Then $\Sigma_*\hat{\xi}_f = \tilde{\xi}_f$ and $\gamma(\hat{\xi}_f) = 0$. A summary of the key properties of ξ_f, $\tilde{\xi}_f$ and $\hat{\xi}_f$ is below.

$$\begin{array}{ccc}
(P,\gamma) & \hat{\xi}_f & \gamma(\hat{\xi}_f) = 0,\ \Sigma_*\hat{\xi}_f = \tilde{\xi}_f,\ \Pi_*\hat{\xi}_f = \xi_f \\
\big\downarrow \Sigma & & \\
\mathrm{Sp}(M,\omega) & \tilde{\xi}_f & [\tilde{\xi}_f, \partial_\kappa] = 0\ \forall \kappa \in \mathfrak{sp}(V),\ \rho_*\tilde{\xi}_f = \xi_f \\
\big\downarrow \rho & & \\
(M,\omega) & \xi_f &
\end{array}$$

In Section 3.2.2, we made use of the vector field $\partial_{2\pi i}$ on Y. The corresponding object in this context is the vector field $\partial_{2\pi i}$ on P, where $2\pi i \in \mathfrak{u}(1) \subset \mathfrak{mp}^c(V)$. This vector field satisfies $\gamma(\partial_{2\pi i}) = 2\pi i$ and $\Sigma_*(\partial_{2\pi i}) = 0$.

Lemma 3.3.4. For all $f, g \in C^\infty(M)$,

$$[\hat{\xi}_f, \hat{\xi}_g] = \hat{\xi}_{\{f,g\}} - \frac{1}{2\pi\hbar}\Pi^*\{f,g\}\hat{\partial}_{2\pi i}.$$

Proof. It suffices to show that

$$\Sigma_*[\hat{\xi}_f, \hat{\xi}_g] = \Sigma_*\left(\hat{\xi}_{\{f,g\}} - \frac{1}{2\pi\hbar}\Pi^*\{f,g\}\hat{\partial}_{2\pi i}\right) \quad \text{and}$$
$$\gamma([\hat{\xi}_f, \hat{\xi}_g]) = \gamma\left(\hat{\xi}_{\{f,g\}} - \frac{1}{2\pi\hbar}\Pi^*\{f,g\}\hat{\partial}_{2\pi i}\right).$$

A calculation establishes that $\widetilde{\xi_{\{f,g\}}} = [\widetilde{\xi_f}, \widetilde{\xi_g}]$. The rest of the proof proceeds identically to that of Lemma 3.2.3. □

Lemma 3.3.5. The map $E : C^\infty(M) \to \mathcal{X}(P)$ given by

$$E(f) = \hat{\xi}_f + \frac{1}{2\pi\hbar}\Pi^* f\hat{\partial}_{2\pi i}, \quad \forall f \in C^\infty(M)$$

is a Lie algebra homomorphism.

Proof. Precisely analogous to Lemma 3.2.4. □

Lemma 3.3.6. For all $f \in C^\infty(M)$, $E(f) \in \mathcal{Q}(P, \Sigma, \gamma)$.

Proof. We need to show that $L_{E(f)}\gamma = 0$ and $\Sigma_* E(f) = \widetilde{\Pi_* E(f)}$. The verification that $L_{E(f)}\gamma = 0$ is the same as that in Lemma 3.2.5. Note that $\Sigma_* E(f) = \tilde{\xi}_f$ and $\Pi_* E(f) = \xi_f$, so $\widetilde{\Pi_* E(f)} = \tilde{\xi}_f = \Sigma_* E(f)$. Thus the necessary conditions are satisfied, and $E(f) \in \mathcal{Q}(P, \Sigma, \gamma)$. □

As before, we will construct an inverse for E, and conclude that E is a Lie algebra isomorphism. Let $\zeta \in \mathcal{Q}(P, \Sigma, \gamma)$ and $\alpha \in \mathfrak{mp}^c(V)$ be arbitrary. Using an identical argument to the one that precedes Theorem 3.2.6, the fact that $\hat{\partial}_\alpha \lrcorner (L_\zeta \gamma) = 0$ implies that $L_{\hat{\partial}_\alpha} \gamma(\zeta) = 0$. Therefore we can define $F : \mathcal{Q}(P, \Sigma, \gamma) \to C^\infty(M)$ so that

$$-\frac{1}{i\hbar}\Pi^* F(\zeta) = \gamma(\zeta), \quad \forall \zeta \in \mathcal{Q}(P, \Sigma, \gamma).$$

Theorem 3.3.7. The map $E : C^\infty(M) \to \mathcal{Q}(P, \Sigma, \gamma)$ is a Lie algebra isomorphism with inverse F.

Proof. Let $f \in C^\infty(M)$ and $\zeta \in \mathcal{Q}(P, \Sigma, \gamma)$ be arbitrary. From the definitions of E and F, $F(E(f))$ satisfies

$$-\frac{1}{i\hbar}\Pi^* F(E(f)) = \gamma(E(f)) = \gamma\left(\hat{\xi}_f + \frac{1}{2\pi\hbar}\Pi^* f \hat{\partial}_{2\pi i}\right) = -\frac{1}{i\hbar}\Pi^* f.$$

Thus $F(E(f)) = f$.

Next, we claim that $\gamma(E(F(\zeta))) = \gamma(\zeta)$ and $\Sigma_* E(F(\zeta)) = \Sigma_* \zeta$. Observe that

$$E(F(\zeta)) = \hat{\xi}_{F(\zeta)} + \frac{1}{2\pi\hbar}\Pi^* F(\zeta)\hat{\partial}_{2\pi i} = \hat{\xi}_{F(\zeta)} + \frac{1}{2\pi i}\gamma(\zeta)\hat{\partial}_{2\pi i}.$$

It is immediate that $\gamma(E(F(\zeta))) = \gamma(\zeta)$ and $\Sigma_* E(F(\zeta)) = \tilde{\xi}_{F(\zeta)}$. From the definition of $\mathcal{Q}(P, \Sigma, \gamma)$, we know that $\Sigma_* \zeta = \widetilde{\Pi_* \zeta}$. It remains to show that $\Pi_* \zeta = \xi_{F(\zeta)}$. We calculate

$$\zeta \lrcorner \Pi^* \omega = \zeta \lrcorner i\hbar d\gamma = -i\hbar d(\gamma(\zeta)) = \Pi^* dF(\zeta).$$

This demonstrates that $(\Pi_* \zeta) \lrcorner \omega = dF(\zeta)$, which implies that $\Pi_* \zeta = \xi_{F(\zeta)}$ as needed. Thus we have shown that $E(F(\zeta)) = \zeta$, and this completes the proof that E and F are inverses. $\square$

If the definition of $\mathcal{Q}(P, \Sigma, \gamma)$ did not include the condition that $\Sigma_* \zeta = \widetilde{\Pi_* \zeta}$, this proof would fail in the final step. We would be able to show that $\Sigma_* E(F(\zeta)) = \tilde{\xi}_{F(\zeta)} = \widetilde{\Pi_* \zeta}$, but this vector field would not necessarily equal $\Sigma_* \zeta$, and so F would not be the inverse of E. This explains why property (2) of a metaplectic-c quantomorphism is necessary in order to obtain a subalgebra of $\mathcal{X}(P)$ that is isomorphic to $C^\infty(M)$.

3.3.3 An operator representation of $C^\infty(M)$

In [15], Robinson and Rawnsley construct an infinite-dimensional Hilbert space $\mathcal{E}'(V)$ of holomorphic functions on $V \cong \mathbb{C}^n$, on which the group $\mathrm{Mp}^c(V)$ acts via the metaplectic representation. They then define the bundle of symplectic spinors for the prequantized system $(P, \Sigma, \gamma) \xrightarrow{\Pi} (M, \omega)$ to be

$$\mathcal{E}'(P) = P \times_{\mathrm{Mp}^c(V)} \mathcal{E}'(V).$$

We omit the details of the metaplectic representation here; the only fact we need is that the subgroup $U(1) \subset \mathrm{Mp}^c(V)$ acts on elements of $\mathcal{E}'(V)$ by scalar multiplication. We write an element of $\mathcal{E}'(P)$ as an equivalence class $[q, \psi]$ with $q \in P$ and $\psi \in \mathcal{E}'(V)$.

Section 7 of [15] contains the following construction. Let $s \in \Gamma(\mathcal{E}'(P))$ be given, and define the map $\tilde{s} : P \to \mathcal{E}'(V)$ so that $[q, \tilde{s}(q)] = s(\Pi(q))$ for all $q \in P$. This map $\tilde{s}$ satisfies the equivariance condition

$$\tilde{s}(q \cdot a) = a^{-1}\tilde{s}(q), \quad \forall q \in P,\, a \in \mathrm{Mp}^c(V),$$

where the action on the right-hand side is that of the metaplectic representation. Conversely, if $\tilde{s} : P \to \mathcal{E}'(V)$ is any map with the equivariance property above, it can be used to define a section $s \in \Gamma(\mathcal{E}'(P))$ by setting $s(m) = [q, \tilde{s}(q)]$ for each $m \in M$ and any $q \in P$ such that $\Pi(q) = m$.

Let $f \in C^\infty(M)$ be arbitrary, and recall the lifting $\xi_f \to \tilde{\xi}_f \to \hat{\xi}_f$ of ξ_f to P. A standard calculation establishes that $[\hat{\xi}_f, \hat{\partial}_\alpha] = 0$ for all $\alpha \in \mathfrak{mp}^c(V)$. Thus, if $\tilde{s} : P \to \mathcal{E}'(V)$ is an equivariant map, then so is $\hat{\xi}_f \tilde{s}$. Define the map $D : C^\infty(M) \to \operatorname{End}\Gamma(\mathcal{E}'(P))$ such that for all $f \in C^\infty(M)$ and $s \in \Gamma(\mathcal{E}'(P))$, $D_f s$ is the section of $\mathcal{E}'(P)$ that satisfies

$$\widetilde{D_f s} = \hat{\xi}_f \tilde{s}.$$

Further, define $\delta : C^\infty(M) \to \operatorname{End}\Gamma(\mathcal{E}'(P))$ by

$$\delta_f s = D_f s + \frac{1}{i\hbar} f s, \quad \forall f \in C^\infty(M),\, s \in \Gamma(\mathcal{E}'(P)).$$

Theorem 7.8 of [15] states that δ is a Lie algebra homomorphism.

We see that the construction of D precisely parallels the construction of the connection ∇ on the prequantization line bundle L associated to a prequantization circle bundle (Y, γ). As in Section 3.2.3, we make two observations.

- For any $s \in \Gamma(\mathcal{E}'(P))$ and $f \in C^\infty(M)$, $\widetilde{fs} = \Pi^* f \tilde{s}$.
- For any equivariant map $\tilde{s} : P \to \mathcal{E}'(V)$, $\hat{\partial}_{2\pi i}\tilde{s} = -2\pi i \tilde{s}$.

Therefore

$$\widetilde{\delta_f s} = \left(\hat{\xi}_f + \frac{1}{2\pi\hbar}\Pi^* f \hat{\partial}_{2\pi i} \right) \tilde{s} = E(f)\tilde{s}.$$

The fact that δ is a Lie algebra homomorphism then follows immediately from Lemma 3.3.5. This construction would apply equally well to any associated bundle where the subgroup $U(1) \subset \mathrm{Mp}^c(V)$ acts on the fiber by scalar multiplication.

3.4 Two diffeomorphisms that are not quantomorphisms

A metaplectic-c quantomorphism K between two metaplectic-c prequantizations $(P_1, \Sigma_1, \gamma_1) \xrightarrow{\Sigma_1} \mathrm{Sp}(M_1, \omega_1) \xrightarrow{\rho_1} (M_1, \omega_1)$ and $(P_2, \Sigma_2, \gamma_2) \xrightarrow{\Sigma_2} \mathrm{Sp}(M_2, \omega_2) \xrightarrow{\rho_2} (M_2, \omega_2)$ is a diffeomorphism $K : P_1 \to P_2$ such that

(1) $K^*\gamma_2 = \gamma_1$,

(2) the induced diffeomorphism $K'' : M_1 \to M_2$ satisfies $\widetilde{K''} \circ \Sigma_1 = \Sigma_2 \circ K$,

where $\widetilde{K''} : \mathrm{Sp}(M_1, \omega_1) \to \mathrm{Sp}(M_2, \omega_2)$ is the lift of K'' given by

$$\widetilde{K''}(b) = K''_* \circ b, \quad \forall b \in \mathrm{Sp}(M_1, \omega_1).$$

We claimed that condition (1) is insufficient to guarantee that K is the lift of some map $K' : \mathrm{Sp}(M_1, \omega_1) \to \mathrm{Sp}(M_2, \omega_2)$. In particular, a diffeomorphism K that only satisfies condition (1) might not be equivariant with respect to the principal $\mathrm{Mp}^c(V)$ actions. We further claimed that K might be equivariant and satisfy condition (1), yet fail to satisfy condition (2). We will now construct examples to support these claims.

Let $M = \mathbb{R}^2 \setminus \{(0,0)\}$ with Cartesian coordinates (p,q) and polar coordinates (r,θ). Equip M with the symplectic form $\omega = dp \wedge dq = rdr \wedge d\theta$, and observe that the one-form $\beta = \frac{1}{2}r^2 d\theta$ satisfies $d\beta = \omega$. Let $V = \mathbb{R}^2$ with basis $\{\hat{x}, \hat{y}\}$ and symplectic form $\Omega = \hat{x}^* \wedge \hat{y}^*$, and consider the global trivialization of the tangent bundle TM such that for all $m \in M$, $T_m M$ is identified with V by mapping $\hat{x} \to \frac{\partial}{\partial p}\Big|_m$ and $\hat{y} \to \frac{\partial}{\partial q}\Big|_m$. Identify $\mathrm{Sp}(M,\omega)$ with $M \times \mathrm{Sp}(V)$ using this trivialization.

Let $P = M \times \mathrm{Mp}^c(V)$, and define the map $\Sigma : P \to \mathrm{Sp}(M,\omega)$ by $\Sigma(m,a) = (m, \sigma(a))$ for all $m \in M$ and $a \in \mathrm{Mp}^c(V)$. Let ϑ_0 be the trivial connection on the product bundle $M \times \mathrm{Mp}^c(V)$, and let $\gamma = \frac{1}{i\hbar}\beta + \frac{1}{2}\eta_* \vartheta_0$. Then (P,γ) is a metaplectic-c prequantization of (M,ω). In both of the examples below, we will give a diffeomorphism $K : P \to P$.

To facilitate the construction in Example 3.4.1, we introduce a more explicit representation for elements of the metaplectic-c group. By definition, $\mathrm{Mp}^c(V) = \mathrm{Mp}(V) \times_{\mathbb{Z}_2} U(1)$. The restriction of the projection map $\mathrm{Mp}^c(V) \xrightarrow{\sigma} \mathrm{Sp}(V)$ to $\mathrm{Mp}(V)$ yields the double covering $\mathrm{Mp}(V) \xrightarrow{\sigma} \mathrm{Sp}(V)$. Write an element of $\mathrm{Mp}^c(V)$ as an equivalence class $[h, e^{2\pi i t}]$ with $h \in \mathrm{Mp}(V)$ and $t \in \mathbb{R}$. In terms of this parametrization, the projection map is given by

$$\sigma[h, e^{2\pi i t}] = \sigma(h),$$

and the determinant map $\mathrm{Mp}^c(V) \xrightarrow{\eta} U(1)$ is given by

$$\eta[h, e^{2\pi i t}] = (e^{2\pi i t})^2.$$

Example 3.4.1. We will define a diffeomorphism $K : P \to P$ that preserves γ, but that does not descend through Σ to a well-defined map on $\mathrm{Sp}(M,\omega)$.

Let $\mu : \mathbb{R} \to \mathrm{Mp}(V)$ be any smooth nonconstant path such that $\mu(t+1) = \mu(t)$ for all $t \in \mathbb{R}$. Note that the composition $\sigma \circ \mu : \mathbb{R} \to \mathrm{Sp}(V)$ is also nonconstant. Now define $F : \mathrm{Mp}(V) \times U(1) \to \mathrm{Mp}(V) \times U(1)$ by

$$F(h, e^{2\pi i t}) = (h\mu(2t), e^{2\pi i t}), \quad \forall h \in \mathrm{Mp}(V),\ t \in \mathbb{R}.$$

This map is a diffeomorphism of $\mathrm{Mp}(V) \times U(1)$, and it descends to a diffeomorphism of $\mathrm{Mp}^c(V)$, which we also denote F. For any $[h, e^{2\pi it}] \in \mathrm{Mp}^c(V)$, observe that

$$\eta(F[h, e^{2\pi it}]) = \eta[h\mu(2t), e^{2\pi it}] = (e^{2\pi it})^2 = \eta[h, e^{2\pi it}].$$

This implies that for any $\alpha \in \mathfrak{mp}^c(V)$, $\frac{1}{2}\eta_* F_* \alpha = \frac{1}{2}\eta_* \alpha$.

Define the diffeomorphism $K : P \to P$ by $K(m, a) = (m, F(a))$ for all $m \in M$ and $a \in \mathrm{Mp}^c(V)$. Since K is the identity on M, it preserves β. From the property of F shown above, K also preserves $\frac{1}{2}\eta_*\vartheta_0$, and thus it preserves γ. Fix $(m, g) \in \mathrm{Sp}(M, \omega)$, and let $h \in \mathrm{Mp}(V)$ be such that $\sigma(h) = g$. Then the fiber of P over (m, g) is $P_{(m,g)} = \{(m, [h, e^{2\pi it}]) \,|\, t \in \mathbb{R}\}$. Notice that

$$\Sigma \circ K(m, [h, e^{2\pi it}])) = \Sigma(m, [h\mu(2t), e^{2\pi it}]) = (m, g\sigma(\mu(2t))),$$

which is not constant with respect to t. Thus $K(P_{(m,g)})$ is not contained within a single fiber of P over $\mathrm{Sp}(M, \omega)$, which shows that there is no map $K' : \mathrm{Sp}(M, \omega) \to \mathrm{Sp}(M, \omega)$ such that $K' \circ \Sigma = \Sigma \circ K$.

□

If $K : P \to P$ is equivariant with respect to $\mathrm{Mp}^c(V)$, then it induces a diffeomorphism $K' : \mathrm{Sp}(M, \omega) \to \mathrm{Sp}(M, \omega)$ that satisfies $K' \circ \Sigma = \Sigma \circ K$. This map and $\widetilde{K''}$ are both lifts of $K'' : M \to M$, but they might not be the same map.

Example 3.4.2. We will define a diffeomorphism $K : P \to P$ that preserves γ and is equivariant with respect to $\mathrm{Mp}^c(V)$, but where $K' \neq \widetilde{K''}$.

Let $T_\lambda : M \to M$ be the map that rotates M about the origin by the angle λ, where λ is not an integer multiple of 2π. Define $K : P \to P$ by

$$K(m, a) = (T_\lambda(m), a), \quad \forall m \in M,\ a \in \mathrm{Mp}^c(V).$$

Then $K^*\gamma = \gamma$, and $K(q \cdot a) = K(q) \cdot a$ for all $q \in P$ and $a \in \mathrm{Mp}^c(V)$. The map $K' : \mathrm{Sp}(M, \omega) \to \mathrm{Sp}(M, \omega)$ is given by

$$K'(m, g) = (T_\lambda(m), g), \quad \forall m \in M,\ g \in \mathrm{Sp}(V),$$

and the map $K'' : M \to M$ is simply T_λ. If we let T_λ also denote the automorphism of V given by rotation about the origin by λ, then under our chosen identification of TM with $M \times V$, we have

$$K''_*(m, v) = (T_\lambda(m), T_\lambda(v)), \quad \forall m \in M,\ v \in V.$$

Therefore $\widetilde{K''} : \mathrm{Sp}(M, \omega) \to \mathrm{Sp}(M, \omega)$ is given by

$$\widetilde{K''}(m, g) = (T_\lambda(m), T_\lambda \circ g), \quad \forall m \in M,\ g \in \mathrm{Sp}(V).$$

Hence $K' \neq \widetilde{K''}$.

□

Chapter 4

Quantized Energy Levels and Dynamical Invariance

4.1 Introduction

It is well known from physics that the quantum mechanical versions of systems such as the harmonic oscillator and the hydrogen atom are restricted to discrete energy levels. More generally, suppose (M, ω) is a quantizable symplectic manifold and $H : M \to \mathbb{R}$ is a function on M. If we think of H as a Hamiltonian energy function, then we can ask what it means for a regular value E of H to be a quantized energy level for the system (M, ω, H).

One possible answer to this question involves the construction of the symplectic reduction. The orbits of the Hamiltonian vector field ξ_H partition the level set $H^{-1}(E)$. If M_E, the space of orbits, is a manifold, then ω induces a symplectic form ω_E on M_E, and (M_E, ω_E) is the symplectic reduction of (M, ω) at E. The values of E for which this new symplectic manifold is quantizable can be taken to be the quantized energy levels of the system. This definition has been applied to the hydrogen atom in the context of Kostant-Souriau quantization [16], and to the harmonic oscillator in the context of metaplectic-c quantization [15]. In both cases, the physically predicted energy spectrum is obtained. However, this definition is limited to cases where the space of orbits is a manifold. It would be more convenient to have a quantized energy condition that is evaluated over the original manifold, rather than the quotient.

In this chapter, we propose a definition for a quantized energy level of (M, ω, H) in the case where (M, ω) admits a metaplectic-c prequantization. Given one additional assumption, which we describe in Section 4.2.2, if E is a quantized energy level of (M, ω, H), then the symplectic reduction (M_E, ω_E) is metaplectic-c quantizable, provided that it is a manifold. The quantized energy condition is evaluated on a bundle over the level set $H^{-1}(E)$.

This scenario was first studied by Robinson [14], and our work is strongly motivated by his results. However, we choose a more flexible condition than the one Robinson considered. As we will show, if H_1 and H_2 are two functions on (M, ω) with regular values E_1 and E_2, respectively, such that $H_1^{-1}(E_1) = H_2^{-1}(E_2)$, then under our definition, E_1 is a quantized energy level of (M, ω, H_1) if and only if E_2 is a quantized energy level of (M, ω, H_2). In other words, the quantization condition only depends on the geometry of the level set, and not on the dynamics of a particular Hamiltonian. This is not true of the definition considered in [14].

Section 4.2 summarizes Robinson's results concerning symplectic reduction, and concludes with our proposed definition for a quantized energy level of a metaplectic-c quantizable system. The proof of the dynamical invariance of our definition is given in Section 4.3. In Section 4.4, we consider the example of the harmonic oscillator. This example serves two purposes. First, we demonstrate a computational technique in which a local change of coordinates on M is lifted to the metaplectic-c prequantization. Second, we state an example of a function on M that has exactly one level set in common with the harmonic oscillator, and show that the quantization condition on that level set is identical for both functions.

Finally, in Section 4.5, we adapt the quantized energy definition to apply to a Kostant-Souriau quantizable symplectic manifold. We show that the Kostant-Souriau and metaplectic-c definitions have almost identical properties, but the metaplectic-c version is the only one that yields the correct quantized energy levels for the harmonic oscillator.

4.2 Quantized Energy Levels

Given a symplectic manifold (M, ω) and a smooth function $H : M \to \mathbb{R}$, the symplectic reduction of (M, ω) at the regular value E of H is constructed by taking the quotient of the level set $H^{-1}(E)$ by the orbits of the Hamiltonian vector field ξ_H. Suppose (M, ω) admits a metaplectic-c prequantization (P, Σ, γ). As discussed in Section 3.3.2, there is a natural lift of ξ_H to a vector field $\widetilde{\xi}_H$ on $\mathrm{Sp}(M, \omega)$; $\widetilde{\xi}_H$ can then be lifted to a vector field $\widehat{\xi}_H$ on P that is horizontal with respect to γ. Robinson [14] constructed a certain associated bundle $P_S \to H^{-1}(E)$, on which $\widehat{\xi}_H$ induces a vector field, and he examined the conditions under which the quotient of P_S by the induced orbits of $\widehat{\xi}_H$ yields a metaplectic-c prequantization for the symplectic reduction of (M, ω) at E.

In this section, we review the key results from [14], then state our alternative definition for a quantized energy level of the system (M, ω, H). Our definition is evaluated on the bundle P_S, and ensures that the symplectic reduction acquires a metaplectic-c prequantization under the same conditions as Robinson's definition. Further, as we will show in Section 4, our definition depends only on the geometry of the level set $H^{-1}(E)$, and not on the specific choice of H. This is an improvement over the definition from [14].

4.2.1 Vector subspaces and quotients

Assume that the model vector space V has been chosen as in Section 2.3.1. Let $W \subset V$ be a real subspace of codimension 1. Define the symplectic orthogonal of W to be

$$W^\perp = \{v \in V : \Omega(v, W) = 0\}.$$

Then $W^\perp$ is a one-dimensional subspace of W. The form Ω induces a symplectic form on the quotient space $W/W^\perp$. This new symplectic vector space has symplectic group $\mathrm{Sp}(W/W^\perp)$ and metaplectic-c group $\mathrm{Mp}^c(W/W^\perp)$. The complex structure J on V induces a complex structure on $W/W^\perp$ in such a way that

$$W/W^\perp \cong (W^\perp \oplus JW^\perp)^\perp, \tag{4.2.1}$$

where the isomorphism respects both the symplectic structures and the complex structures. We write an element of $W/W^\perp$ as an equivalence class $[w]$ for some $w \in W$.

Let $\mathrm{Sp}(V; W)$ be the subgroup of $\mathrm{Sp}(V)$ consisting of those symplectic automorphisms that preserve W:

$$\mathrm{Sp}(V;W) = \{g \in \mathrm{Sp}(V) : gW = W\}.$$

Then elements of $\mathrm{Sp}(V;W)$ also preserve $W^\perp$, and there is a group homomorphism

$$\mathrm{Sp}(V;W) \xrightarrow{\nu} \mathrm{Sp}(W/W^\perp),$$

defined by

$$(\nu g)[w] = [gw], \quad \forall g \in \mathrm{Sp}(V;W), \ \forall w \in W.$$

Let $\mathrm{Mp}^c(V;W) \subset \mathrm{Mp}^c(V)$ be the preimage of $\mathrm{Sp}(V;W)$ under the projection map σ. Robinson and Rawnsley [15] constructed a group homomorphism $\mathrm{Mp}^c(V;W) \xrightarrow{\hat{\nu}} \mathrm{Mp}^c(W/W^\perp)$ such that the diagram below commutes.

$$\begin{array}{ccc} \mathrm{Mp}^c(V) \supset \mathrm{Mp}^c(V;W) & \xrightarrow{\hat{\nu}} & \mathrm{Mp}^c(W/W^\perp) \\ \downarrow \sigma & & \downarrow \sigma \\ \mathrm{Sp}(V) \supset \mathrm{Sp}(V;W) & \xrightarrow{\nu} & \mathrm{Sp}(W/W^\perp) \end{array}$$

The map $\hat{\nu}$ has the following property. Given $a \in \mathrm{Mp}^c(V;W)$, let $\chi(a) = \mathrm{Det}_{\mathbb{R}}(\sigma(a)|W^\perp)$. Then χ is a real-valued character on $\mathrm{Mp}^c(V;W)$, and it satisfies

$$\eta \circ \hat{\nu}(a) = \eta(a)\mathrm{sign}\,\chi(a), \quad \forall a \in \mathrm{Mp}^c(V;W).$$

Since $\mathrm{sign}\,\chi(a)$ is identically 1 on a neighborhood of I, we see that $\eta_* \circ \hat{\nu}_* = \eta_*$ as maps between Lie algebras.

4.2.2 Quotients and reduction over a manifold

Let (M,ω) be a symplectic manifold that admits a metaplectic-c prequantization (P,Σ,γ). The definitions from Section 2.3.2 yield the three-level structure

$$(P,\gamma) \xrightarrow{\Sigma} \mathrm{Sp}(M,\omega) \xrightarrow{\rho} (M,\omega), \quad \rho \circ \Sigma = \Pi.$$

Let $H : M \to \mathbb{R}$ be a smooth function on M. We define the Hamiltonian vector field ξ_H

using the convention that

$$\xi_H \lrcorner \omega = dH.$$

Let the flow of ξ_H be ϕ^t, and note that ϕ^t is a symplectomorphism from its domain to its range for each t. The lift of ϕ^t to a flow $\widetilde{\phi}^t$ on $\mathrm{Sp}(M,\omega)$ is given by

$$\widetilde{\phi}^t(b) = \phi^t_* \circ b, \quad \forall b \in \mathrm{Sp}(M,\omega).$$

Let $\widetilde{\xi}_H$ be the vector field on $\mathrm{Sp}(M,\omega)$ whose flow is $\widetilde{\phi}^t$. Define the vector field $\widehat{\xi}_H$ on P to be the lift of $\widetilde{\xi}_H$ that is horizontal with respect to the one-form γ, and let the flow of $\widehat{\xi}_H$ be $\widehat{\phi}^t$.

We now restrict our attention to a particular level set. Let $E \in \mathbb{R}$ be a regular value of H, and let $S = H^{-1}(E)$. The null foliation $TS^\perp$ is defined fiberwise over S by

$$T_sS^\perp = \{\zeta \in T_sM : \omega(\zeta, T_sS) = 0\}, \quad \forall s \in S.$$

Then $T_sS^\perp$ is a one-dimensional subspace of the tangent space T_sS, and $T_sS^\perp = \mathrm{span}\{\xi_H(s)\}$.

The constructions that follow are taken from [14]. As in Section 4.2.1, fix a subspace $W \subset V$ of codimension 1. By definition, an element $b \in \mathrm{Sp}(M,\omega)_m$ is a symplectic linear isomorphism $b : V \to T_mM$. Let $\mathrm{Sp}(M,\omega;S)$ be the subset of $\mathrm{Sp}(M,\omega)|_S$ given by

$$\mathrm{Sp}(M,\omega;S)_s = \{b \in \mathrm{Sp}(M,\omega)_s : bW = T_sS\}, \quad \forall s \in S.$$

Then $\mathrm{Sp}(M,\omega;S)$ is a principal $\mathrm{Sp}(V;W)$ bundle over S. Note that any $b \in \mathrm{Sp}(M,\omega;S)_s$ maps $W^\perp$ to $T_sS^\perp$.

Viewing P as a circle bundle over $\mathrm{Sp}(M,\omega)$, let P^S be the result of restricting P to $\mathrm{Sp}(M,\omega;S)$. Then P^S is a principal $\mathrm{Mp}^c(V;W)$ bundle over S. Let γ^S be the pullback of γ to P^S. After all of these restrictions, we have constructed the new three-level structure

$$(P^S, \gamma^S) \to \mathrm{Sp}(M,\omega;S) \to S.$$

For each $s \in S$, $T_sS/T_sS^\perp$ is a symplectic vector space with symplectic structure induced by ω_s. Let $\mathrm{Sp}(TS/TS^\perp)$ be the symplectic frame bundle for $TS/TS^\perp$, modeled on $W/W^\perp$:

$$\mathrm{Sp}(TS/TS^\perp)_s = \left\{ b' : W/W^\perp \to T_sS/T_sS^\perp : b' \text{ is a symplectic linear isomorphism} \right\}, \quad \forall s \in S.$$

Recall the group homomorphism $\nu : \mathrm{Sp}(V;W) \to \mathrm{Sp}(W/W^\perp)$ from Section 4.2.1. The bundle associated to $\mathrm{Sp}(M,\omega;S)$ by the map ν can be naturally identified with $\mathrm{Sp}(TS/TS^\perp)$.

Next, let $P_S \to S$ be the bundle associated to $P^S \to S$ by the group homomorphism $\hat{\nu} : \mathrm{Mp}^c(V;W) \to \mathrm{Mp}^c(W/W^\perp)$. Then P_S is a principal $\mathrm{Mp}^c(W/W^\perp)$ bundle over S, and a circle bundle over $\mathrm{Sp}(TS/TS^\perp)$. The one-form γ^S induces a connection one-form γ_S on P_S. This completes the construction of another three-level structure,

$$(P_S, \gamma_S) \to \mathrm{Sp}(TS/TS^\perp) \to S.$$

Now let us consider the actions of ϕ^t, $\widetilde{\phi}^t$ and $\widehat{\phi}^t$ on all of these bundles. First, it is clear that ϕ^t preserves S, since $\xi_H(s) \in T_sS$ at each $s \in S$. Therefore $\widetilde{\phi}^t$ restricts to a flow on $\mathrm{Sp}(M,\omega;S)$, and so $\widehat{\phi}^t$ restricts to a flow on P^S. One can verify that $\widetilde{\phi}^t$ and $\widehat{\phi}^t$ are equivariant with respect to the principal $\mathrm{Sp}(V)$ and $\mathrm{Mp}^c(V)$ actions, respectively, which implies that $\widetilde{\phi}^t$ induces a flow on the associated bundle $\mathrm{Sp}(TS/TS^\perp)$, and $\widehat{\phi}^t$ induces a flow on P_S. We let $\widetilde{\phi}^t$ and $\widehat{\phi}^t$ also denote the respective flows induced on $\mathrm{Sp}(TS/TS^\perp)$ and P_S.

Suppose the null foliation is fibrating, so that the quotient of S by the orbits of ξ_H is a manifold. Let the quotient manifold be M_E, and let ω_E be the symplectic form on M_E induced by ω. As discussed in [14], if the quotient of $\mathrm{Sp}(TS/TS^\perp)$ by $\widetilde{\phi}^t$ is well defined, then it is naturally isomorphic to the symplectic frame bundle $\mathrm{Sp}(M_E,\omega_E)$. To ensure that this quotient is well defined, it is sufficient to require that if $s \in S$ is fixed by ϕ^t, then ϕ^t_* is the identity on T_sS. The culmination of all of these steps is shown below.

$$\begin{array}{ccccccc}
(P,\gamma) & \xleftarrow{\text{incl.}} & (P^S,\gamma^S) & \xrightarrow{\hat{\nu}} & (P_S,\gamma_S) & & \\
\downarrow{\scriptstyle\Sigma} & & \downarrow & & \downarrow & & \\
\mathrm{Sp}(M,\omega) & \xleftarrow{\text{incl.}} & \mathrm{Sp}(M,\omega;S) & \xrightarrow{\nu} & \mathrm{Sp}(TS/TS^\perp) & \xrightarrow{/\widetilde{\phi}^t} & \mathrm{Sp}(M_E,\omega_E) \\
\downarrow{\scriptstyle\rho} & & \downarrow & & \downarrow & & \downarrow \\
(M,\omega) & \xleftarrow{\text{incl.}} & S & \xrightarrow{=} & S & \xrightarrow{/\phi^t} & (M_E,\omega_E)
\end{array}$$

The obvious way to complete the picture is to factor (P_S, γ_S) by $\widehat{\phi}^t$. Robinson [14] addressed the question of when this yields a metaplectic-c prequantization for (M_E, ω_E) with the following theorem.

Theorem 4.2.1. (Robinson, 1990) Assume that the symplectic reduction (M_E, ω_E) is a manifold, and that its symplectic frame bundle $\mathrm{Sp}(M_E, \omega_E)$ can be identified with the quotient of $\mathrm{Sp}(TS/TS^\perp)$ by the induced flow $\widetilde{\phi}^t$. If γ_S has trivial holonomy over all of the closed orbits of $\widetilde{\phi}^t$ on $\mathrm{Sp}(TS/TS^\perp)$, then the quotient of (P_S, γ_S) by $\widehat{\phi}^t$ is a metaplectic-c prequantization for (M_E, ω_E).

To satisfy the holonomy condition in Theorem 4.2.1, it is sufficient – though not necessary – to show that γ^S has trivial holonomy over all closed orbits of $\widetilde{\phi}^t$ on $\mathrm{Sp}(M, \omega; S)$. This is the quantization condition that was explored in the remainder of [14]. However, we argue that a more robust condition arises when we evaluate the holonomy over orbits in $\mathrm{Sp}(TS/TS^\perp)$. As such, we propose the following definition.

Definition 4.2.2. When γ_S has trivial holonomy over all closed orbits of $\widetilde{\phi}^t$ on $\mathrm{Sp}(TS/TS^\perp)$, we say that E is a **quantized energy level** of the system (M, ω, H).

In those cases where E is a quantized energy level and (M_E, ω_E) exists, Theorem 4.2.1 gives a sufficient condition for (M_E, ω_E) to admit a metaplectic-c prequantization. However, Definition 4.2.2 can also be applied to systems where the symplectic reduction does not exist. For the remainder of the chapter, we do not make any assumptions about (M_E, ω_E).

The next section contains the proof of the dynamical invariance property. We will show that if S is a regular level set of two functions H_1 and H_2 on M, then S corresponds to a quantized energy level for (M, ω, H_1) if and only if it does so for (M, ω, H_2).

4.3 Dynamical Invariance

We continue to use all of the definitions established in Section 4.2. To prove that the quantized energy condition is dynamically invariant, we proceed in two stages. In Section 4.3.1, we prove Theorem 4.3.2, which states that if E is a quantized energy level for the system (M, ω, H), then for any diffeomorphism $f : \mathbb{R} \to \mathbb{R}$, the value $f(E)$ is a quantized energy level for the system $(M, \omega, f \circ H)$. While this is a less general statement, it allows us to establish some useful preliminary observations. Then, in Section 4.3.2, we prove Theorem 4.3.5, which states that if the functions $H_1, H_2 : M \to \mathbb{R}$ have regular values E_1 and E_2, respectively, such that $H_1^{-1}(E_1) = H_2^{-1}(E_2)$, then E_1 is a quantized energy level for (M, ω, H_1) if and only if E_2 is a quantized energy level for (M, ω, H_2).

4.3.1 Invariance Under a Diffeomorphism

Let $H : M \to \mathbb{R}$ be given, and fix a regular value E of H. Let $f : \mathbb{R} \to \mathbb{R}$ be a diffeomorphism, and let $S = H^{-1}(E) = (f \circ H)^{-1}(f(E))$.

Denote the Hamiltonian vector field for H by ξ_H, and let it have flow ϕ^t on M. Then ξ_H lifts to $\widetilde{\xi}_H$ on $\mathrm{Sp}(M, \omega)$ and to $\widehat{\xi}_H$ on P, with flows $\widetilde{\phi}^t$ and $\widehat{\phi}^t$, respectively, as in Section 4.2.2. Similarly, denote the Hamiltonian vector field for $f \circ H$ by ξ_{fH}, and let it have flow ρ^t on M. By the same process, we obtain the lifts $\widetilde{\xi}_{fH}$ on $\mathrm{Sp}(M, \omega)$ and $\widehat{\xi}_{fH}$ on P, with flows $\widetilde{\rho}^t$ and $\widehat{\rho}^t$, respectively.

Observe that for any $m \in M$,

$$d(f \circ H)|_m = \left.\frac{df(H)}{dH}\right|_{H(m)} dH|_m,$$

which implies that

$$\xi_{fH}(m) = \left.\frac{df(H)}{dH}\right|_{H(m)} \xi_H(m).$$

Since S is a level set of H, $\left.\frac{df(H)}{dH}\right|_{H(m)}$ is constant over S. Let $c \in \mathbb{R}$ be that constant value on S. Then $\xi_{fH} = c\xi_H$ everywhere on S, which implies that $\rho^t = \phi^{ct}$ everywhere on S.

Recall that $\mathrm{Sp}(TS/TS^\perp)$ can be naturally identified with the bundle associated to $\mathrm{Sp}(M, \omega; S)$ by the group homomorphism $\nu : \mathrm{Sp}(V; W) \to \mathrm{Sp}(W/W^\perp)$. We write an element of $W/W^\perp$ as an equivalence class $[w]$ for some $w \in W$, and we write an element of $T_sS/T_sS^\perp$ as an equivalence class $[\zeta]$ for some $\zeta \in T_sS$. As discussed in Section 4.2.2, the lifted flow $\widetilde{\phi}^t$ on $\mathrm{Sp}(M, \omega)$ maps $\mathrm{Sp}(M, \omega; S)$ to itself and induces a flow $\widetilde{\phi}^t$ on $\mathrm{Sp}(TS/TS^\perp)$. More explicitly, for any $b' \in \mathrm{Sp}(TS/TS^\perp)$, we define $\widetilde{\phi}^t(b')$ by choosing $b \in \mathrm{Sp}(M, \omega; S)$ such that $[bw] = b'[w]$ for all

$w \in W$, and setting

$$\widetilde{\phi}^t(b')[w] = [\widetilde{\phi}^t(b)w], \quad \forall w \in W.$$

These remarks apply equally well to $\widetilde{\rho}^t$.

Lemma 4.3.1. $\widetilde{\rho}^t = \widetilde{\phi}^{ct}$ on $\mathrm{Sp}(TS/TS^{\perp})$.

Proof. Fix $s \in S$ and $b' \in \mathrm{Sp}(TS/TS^{\perp})_s$. Let $b \in \mathrm{Sp}(M,\omega;S)_s$ be such that for all $w \in W$, $[bw] = b'[w]$. Since $\rho^t = \phi^{ct}$ on S, we have $\rho^t_*|TS = \phi^{ct}_*|TS$. Therefore, for any $w \in W$,

$$\widetilde{\rho}^t(b)w = \rho^t_*|_s(bw) = \phi^{ct}_*|_s(bw) = \widetilde{\phi}^{ct}(b)w,$$

where we used the fact that $bw \in T_sS$. Using the definitions of $\widetilde{\phi}^t$ and $\widetilde{\rho}^t$ on $\mathrm{Sp}(TS/TS^{\perp})$, it follows that

$$\widetilde{\rho}^t(b')[w] = [\widetilde{\rho}^t(b)w] = [\widetilde{\phi}^{ct}(b)w] = \widetilde{\phi}^{ct}(b')[w].$$

Thus $\widetilde{\rho}^t = \widetilde{\phi}^{ct}$ on $\mathrm{Sp}(TS/TS^{\perp})$. □

Lemma 4.3.1 implies that $\widetilde{\xi}_{fH} = c\widetilde{\xi}_H$ on $\mathrm{Sp}(TS/TS^{\perp})$. Since $\widetilde{\xi}_{fH}$ is just a constant multiple of $\widetilde{\xi}_H$, it is clear that the flows of both vector fields have the same orbits. Thus, if γ_S has trivial holonomy over all closed orbits of $\widetilde{\phi}^t$ on $\mathrm{Sp}(TS/TS^{\perp})$, then it also has trivial holonomy over all closed orbits of $\widetilde{\rho}^t$. The following is now immediate.

Theorem 4.3.2. If E is a quantized energy level for (M,ω,H) and $f : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism, then $f(E)$ is a quantized energy level for $(M,\omega, f \circ H)$.

4.3.2 Regular Level Set of Multiple Functions

Let $H_1, H_2 : M \to \mathbb{R}$ be two smooth functions, and assume that there are $E_1, E_2 \in \mathbb{R}$ such that E_j is a regular value of H_j for $j = 1, 2$, and $H_1^{-1}(E_1) = H_2^{-1}(E_2)$. Let this shared level set be S. Let the Hamiltonian vector fields ξ_{H_1} and ξ_{H_2} on M have flows ϕ^t and ρ^t, respectively. The induced flows $\widetilde{\phi}^t$ and $\widetilde{\rho}^t$ on $\mathrm{Sp}(TS/TS^{\perp})$ are defined as in Section 4.3.1.

The key observation in this scenario is that TS and $TS^{\perp}$ do not depend on H_1 or H_2. For all $s \in S$, we have

$$T_sS^{\perp} = \mathrm{span}\left\{\xi_{H_1}(s)\right\} = \mathrm{span}\left\{\xi_{H_2}(s)\right\},$$

which implies that there is a map $c : S \to \mathbb{R}$ such that $\xi_{H_2}(s) = c(s)\xi_{H_1}(s)$ for all $s \in S$. The map c need not be constant on S, and ξ_{H_1} and ξ_{H_2} need not be parallel away from S.

In Lemma 4.3.1, we were able to establish a relationship between $\widetilde{\rho}^t$ and $\widetilde{\phi}^t$. In this case, we will not be able to relate $\widetilde{\rho}^t$ and $\widetilde{\phi}^t$ directly, but must instead work in terms of their time derivatives.

For any $s \in S$, $\rho^t_*|_s$ and $\phi^t_*|_s$ preserve TS. Therefore, for all $\zeta \in T_sS$, the tangent vectors $\frac{d}{dt}\big|_{t=0}\left(\rho^t_*|_s\zeta\right)$ and $\frac{d}{dt}\big|_{t=0}\left(\phi^t_*|_s\zeta\right)$ are elements of $T_\zeta TS$. Note that the vector space T_sS can be naturally identified with its tangent space $T_\zeta T_sS$, which is a subspace of $T_\zeta TS$. Thus $\xi_{H_1}(s) \in T_sS$ can also be viewed as an element of $T_\zeta TS$. This identification is used in the statement of the following result.

Lemma 4.3.3. For all $s \in S$ and all $\zeta \in T_sS$,

$$\frac{d}{dt}\bigg|_{t=0}\left(\rho^t_*|_s\zeta\right) = c(s)\,\frac{d}{dt}\bigg|_{t=0}\left(\phi^t_*|_s\zeta\right) + (\zeta c)\xi_{H_1}(s).$$

Proof. We ignore M, and treat S as a $(2n-1)$-dimensional manifold. The vector fields ξ_{H_1} and ξ_{H_2} on S have flows ϕ^t and ρ^t, respectively, and satisfy $\xi_{H_2} = c\xi_{H_1}$.

Fix $s \in S$, and let U be a coordinate neighborhood for s with coordinates $X = (X_1, \ldots, X_{2n-1})$. We write $\phi^t = (\phi^t_1, \ldots, \phi^t_{2n-1})$ and $\rho^t = (\rho^t_1, \ldots, \rho^t_{2n-1})$ with respect to the local coordinates. Then $\phi^t_*|_s$ becomes a $(2n-1) \times (2n-1)$ matrix with the (j,k)th entry given by

$$\left(\phi^t_*|_s\right)_{jk} = \frac{\partial \phi^t_j(X)}{\partial X_k}\bigg|_{X=s},$$

and similarly for $\rho^t_*|_s$. Also, since $\xi_{H_2} = c\xi_{H_1}$, we have

$$\frac{d}{dt}\bigg|_{t=0}\rho^t_j = c\,\frac{d}{dt}\bigg|_{t=0}\phi^t_j, \quad j = 1, \ldots, 2n-1,$$

everywhere on U.

Let $\zeta \in T_sS$ be arbitrary. Then $\zeta = \sum_{k=1}^{2n-1} a_k \frac{\partial}{\partial X_k}$ for some coefficients $a_1, \ldots, a_{2n-1} \in \mathbb{R}$, and

$$\left(\phi^t_*|_s\zeta\right)_j = \sum_{k=1}^{2n-1} a_k \frac{\partial \phi^t_j(X)}{\partial X_k}\bigg|_{X=s} = \zeta\phi^t_j.$$

The same process establishes that $(\rho_*^t|_s\zeta)_j = \zeta\rho_j^t$. When we take the time derivative of the latter equation, we find

$$\left.\frac{d}{dt}\right|_{t=0}\left(\rho_*^t|_s\zeta\right)_j = \zeta\left(\left.\frac{d}{dt}\right|_{t=0}\rho_j^t\right) = \zeta\left(c\left.\frac{d}{dt}\right|_{t=0}\phi_j^t\right).$$

Now we apply the Leibniz rule and recall that $\frac{d}{dt}\big|_{t=0}\phi_j^t(s) = \xi_{H_1}(s)_j$. The result is

$$\left.\frac{d}{dt}\right|_{t=0}\left(\rho_*^t|_s\zeta\right)_j = c(s)\left.\frac{d}{dt}\right|_{t=0}(\phi_*^t|_s\zeta)_j + (\zeta c)\xi_{H_1}(s)_j. \tag{4.3.1}$$

The coordinates on U naturally generate coordinates for $TS|_U$, which in turn provide a basis for $T_\zeta TS$. In terms of that basis,

$$\left.\frac{d}{dt}\right|_{t=0}\left(\rho_*^t|_s\zeta\right) = \left(\xi_{H_2}(s)_1, \ldots, \xi_{H_2}(s)_{2n-1}, \left.\frac{d}{dt}\right|_{t=0}\left(\rho_*^t|_s\zeta\right)_1, \ldots, \left.\frac{d}{dt}\right|_{t=0}\left(\rho_*^t|_s\zeta\right)_{2n-1}\right),$$

with the analogous expression for $\frac{d}{dt}\big|_{t=0}\left(\phi_*^t|_s\zeta\right)$, and

$$\xi_{H_1}(s) = (0, \ldots, 0, \xi_{H_1}(s)_1, \ldots, \xi_{H_1}(s)_{2n-1})$$

as an element of $T_\zeta TS$. Substituting Equation (4.3.1) into the expression for $\frac{d}{dt}\big|_{t=0}\left(\rho_*^t|_s\zeta\right)$ shows that

$$\left.\frac{d}{dt}\right|_{t=0}\left(\rho_*^t|_s\zeta\right) = c(s)\left.\frac{d}{dt}\right|_{t=0}\left(\phi_*^t|_s\zeta\right) + (\zeta c)\xi_{H_1}(s),$$

as desired. □

Lemma 4.3.4. For all $s \in S$ and all $b' \in \mathrm{Sp}(TS/TS^\perp)_s$,

$$\widetilde{\xi}_{H_2}(b') = c(s)\widetilde{\xi}_{H_1}(b').$$

Proof. Within the model vector space V, fix a symplectic basis $(\widehat{x}_1, \ldots, \widehat{x}_n, \widehat{y}_1, \ldots, \widehat{y}_n)$. Let

$$W = \operatorname{span}\{\widehat{x}_1, \ldots, \widehat{x}_n, \widehat{y}_2, \ldots, \widehat{y}_n\},$$

so that

$$W^{\perp} = \operatorname{span}\{\widehat{x}_1\} \quad \text{and} \quad W/W^{\perp} = \operatorname{span}\{[\widehat{x}_2], \ldots, [\widehat{x}_n], [\widehat{y}_2], \ldots, [\widehat{y}_n]\}.$$

For convenience, let $(\widehat{z}_1, \ldots, \widehat{z}_{2n}) = (\widehat{x}_1, \ldots, \widehat{x}_n, \widehat{y}_2, \ldots, \widehat{y}_n, \widehat{y}_1)$. Then

$$W = \operatorname{span}\{\widehat{z}_1, \ldots, \widehat{z}_{2n-1}\}, \quad W^{\perp} = \operatorname{span}\{\widehat{z}_1\}, \quad W/W^{\perp} = \operatorname{span}\{[\widehat{z}_2], \ldots, [\widehat{z}_{2n-1}]\}.$$

For this proof, we will not view an element of the symplectic frame bundle as a map, but as a basis for a tangent space. Explicitly, over $m \in M$, we identify $b \in \operatorname{Sp}(M,\omega)_m$ with the ordered $2n$-tuple $(\zeta_1, \ldots, \zeta_{2n}) \in (T_m M)^{2n}$, where $b\widehat{z}_j = \zeta_j$ for $j = 1, \ldots, 2n$. Similarly, over $s \in S$, we identify $b' \in \operatorname{Sp}(TS/TS^{\perp})_s$ with the ordered $(2n-2)$-tuple $([\zeta_2], \ldots, [\zeta_{2n}]) \in \left(T_s S/T_s S^{\perp}\right)^{2n-2}$, where $b'[\widehat{z}_j] = [\zeta_j]$ for $j = 2, \ldots, 2n-1$.

Let $s \in S$ and $b \in \operatorname{Sp}(M,\omega;S)_s$ be given. Let $b\widehat{z}_j = \zeta_j \in T_s M$ for $j = 1, \ldots, 2n$. Then $\zeta_j \in T_s S$ for $j = 1, \ldots, 2n-1$ and $\zeta_1 \in T_s S^{\perp}$. The flow $\widetilde{\rho}^t$ on $\operatorname{Sp}(M,\omega;S)$, evaluated at b, is given by

$$\widetilde{\rho}^t(b) = \rho^t_*|_m \circ b = \left(\rho^t_*|_m \zeta_1, \ldots, \rho^t_*|_m \zeta_{2n}\right),$$

and the analogous expression holds for $\widetilde{\phi}^t$.

When we descend to $\operatorname{Sp}(TS/TS^{\perp})$, the image of b is the element $b' = ([\zeta_2], \ldots, [\zeta_{2n-1}]) \in (T_s S/T_s S^{\perp})^{2n-2}$. The induced flow $\widetilde{\rho}^t$ on $\operatorname{Sp}(TS/TS^{\perp})$, evaluated at b', is

$$\widetilde{\rho}^t(b') = \left([\rho^t_*|_s \zeta_2], \ldots, [\rho^t_*|_s \zeta_{2n-1}]\right).$$

As t varies, this expression describes a curve in $(TS/TS^{\perp})^{2n-2}$, through b'. The tangent vector to this curve at b' is

$$\widetilde{\xi}_{H_2}(b') = \left.\frac{d}{dt}\right|_{t=0} \widetilde{\rho}^t(b') = \left(\left.\frac{d}{dt}\right|_{t=0} [\rho^t_*|_s \zeta_2], \ldots, \left.\frac{d}{dt}\right|_{t=0} [\rho^t_*|_s \zeta_{2n-1}] \right)$$
$$\in T_{[\zeta_2]}(TS/TS^{\perp}) \times \ldots \times T_{[\zeta_{2n-1}]}(TS/TS^{\perp}).$$

The pushforward of the quotient map $TS \to TS/TS^{\perp}$, based at ζ_j, is a linear surjection $T_{\zeta_j} TS \to T_{[\zeta_j]}(TS/TS^{\perp})$ whose kernel is $T_{\zeta_j} T_s S^{\perp}$. If we identify $T_{\zeta_j} T_s S^{\perp}$ with $T_s S^{\perp}$, then we find a natural isomorphism between $T_{[\zeta_j]}(TS/TS^{\perp})$ and $T_{\zeta_j}TS/T_sS^{\perp}$. Applying that isomorphism to each component of $\widetilde{\xi}_{H_2}(b')$ yields

$$\widetilde{\xi}_{H_2}(b') = \left(\left[\frac{d}{dt}\bigg|_{t=0} (\rho_*^t|_s\zeta_2) \right], \ldots, \left[\frac{d}{dt}\bigg|_{t=0} (\rho_*^t|_s\zeta_{2n-1}) \right] \right)$$
$$\in T_{\zeta_2}TS/T_sS^{\perp} \times \ldots \times T_{\zeta_{2n-1}}TS/T_sS^{\perp}.$$

Since $\zeta_j \in T_sS$ for $j = 2, \ldots, 2n-1$, we can use Lemma 4.3.3:

$$\frac{d}{dt}\bigg|_{t=0} (\rho_*^t|_s\zeta_j) = c(s)\, \frac{d}{dt}\bigg|_{t=0} (\phi_*^t|_s\zeta_j) + (\zeta_j c)\xi_{H_1}(s),$$

where each term in the above equation is viewed as an element of $T_{\zeta_j}TS$. Upon descending to $T_{\zeta_j}TS/T_sS^{\perp}$, the multiple of $\xi_{H_1}(s)$ vanishes and we find that

$$\left[\frac{d}{dt}\bigg|_{t=0} (\rho_*^t|_s\zeta_j) \right] = c(s) \left[\frac{d}{dt}\bigg|_{t=0} (\phi_*^t|_s\zeta_j) \right].$$

Therefore

$$\widetilde{\xi}_{H_2}(b') = \left(c(s) \left[\frac{d}{dt}\bigg|_{t=0} (\phi_*^t|_s\zeta_2) \right], \ldots, c(s) \left[\frac{d}{dt}\bigg|_{t=0} (\phi_*^t|_s\zeta_{2n-1}) \right] \right)$$
$$= c(s)\widetilde{\xi}_{H_1}(b').$$

This completes the proof. □

Thus $\widetilde{\xi}_{H_1}$ and $\widetilde{\xi}_{H_2}$ are parallel on $\mathrm{Sp}(TS/TS^{\perp})$, although the multiplicative factor that relates them is not necessarily constant. This implies that $\widetilde{\phi}^t$ and $\widetilde{\rho}^t$ have identical orbits in $\mathrm{Sp}(TS/TS^{\perp})$, and if γ_S has trivial holonomy over the closed orbits of $\widetilde{\phi}^t$, then the same must also be true for $\widetilde{\rho}^t$. We can now conclude the result that was the objective of this section.

Theorem 4.3.5. If $H_1, H_2 : M \to \mathbb{R}$ are smooth functions such that $H_1^{-1}(E_1) = H_2^{-1}(E_2)$ for regular values E_j of H_j, $j = 1, 2$, then E_1 is a quantized energy level for (M, ω, H_1) if and only if E_2 is a quantized energy level for (M, ω, H_2).

4.4 The Harmonic Oscillator

In this section, we apply our quantized energy condition to the n-dimensional harmonic oscillator. The metaplectic-c quantization of this system has already been examined in [14] and [15], so it will come as no surprise that we obtain the correct quantized energy levels:

$$E_N = \hbar\left(N + \frac{n}{2}\right), \quad N \in \mathbb{Z}.$$

Our first objective in studying this example is to present a useful computational technique: we will locally change coordinates from Cartesian to symplectic polar on the base manifold, then show how to lift this change to the symplectic frame bundle and prequantization bundle. Once symplectic polar coordinates have been established, we will turn to our second objective, which is to construct an explicit example that illustrates the principal of dynamical invariance.

4.4.1 Initial Choices

Let $(\widehat{x}_1, \ldots, \widehat{x}_n, \widehat{y}_1, \ldots, \widehat{y}_n)$ be a basis for the model vector space V such that $\Omega = \sum_{j=1}^n \widehat{x}_j^* \wedge \widehat{y}_j^*$. All elements of V will be written as ordered $2n$-tuples with respect to this basis. Assume that V is identified with $\mathbb{C}^n$ by mapping the point $(a_1, \ldots, a_n, b_1, \ldots, b_n) \in V$ to the point $(b_1 + ia_1, \ldots, b_n + ia_n) \in \mathbb{C}^n$. Then the complex structure J on V is given in matrix form by $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, where I is the $n \times n$ identity matrix.

A note concerning notation: given expressions a_j and b_j, $j = 1, \ldots, n$, we let $(a_j)_{1 \leq j \leq n}$ represent the $n \times n$ diagonal matrix $\mathrm{diag}(a_1, \ldots, a_n)$, and we let $\begin{pmatrix} a_j \\ b_j \end{pmatrix}_{1 \leq j \leq n}$ represent the $2n \times 1$ column vector $(a_1, \ldots, a_n, b_1, \ldots, b_n)^T$.

Let $M = \mathbb{R}^{2n}$, with Cartesian coordinates $(p_1, \ldots, p_n, q_1, \ldots, q_n)$ and symplectic form $\omega = \sum_{j=1}^n dp_j \wedge dq_j$. The energy function for the harmonic oscillator is $H = \frac{1}{2}\sum_{j=1}^n \left(p_j^2 + q_j^2\right)$, which has Hamiltonian vector field

$$\xi_H = \sum_{j=1}^n \left(q_j \frac{\partial}{\partial p_j} - p_j \frac{\partial}{\partial q_j}\right).$$

Let this vector field have flow ϕ^t on M.

There is a global trivialization of TM given by identifying $\widehat{x}_j \mapsto \frac{\partial}{\partial p_j}\Big|_m$, $\widehat{y}_j \mapsto \frac{\partial}{\partial q_j}\Big|_m$ at every point $m \in M$, which induces a global trivialization of $\mathrm{Sp}(M,\omega)$. Let P be the trivial bundle $M \times \mathrm{Mp}^c(V)$, with bundle projection map $P \xrightarrow{\Pi} M$. Define the map $P \xrightarrow{\Sigma} \mathrm{Sp}(M,\omega)$ by

$$\Sigma(m,a) = (m,\sigma(a)), \quad \forall m \in M, \ \forall a \in \mathrm{Mp}^c(V),$$

where the ordered pair on the right-hand side is written with respect to the global trivialization stated above. Then (P,Σ) is a metaplectic-c structure for (M,ω).

On M, define the one-form β by

$$\beta = \frac{1}{2}\sum_{j=1}^{n}(p_j dq_j - q_j dp_j),$$

so that $d\beta = \omega$. Let ϑ_0 be the trivial connection on the product bundle $M \times \mathrm{Mp}^c(V)$, and define the one-form γ on P by

$$\gamma = \frac{1}{i\hbar}\Pi^*\beta + \frac{1}{2}\eta_*\vartheta_0.$$

Then (P,Σ,γ) is a metaplectic-c prequantization for (M,ω). In fact, it is the unique metaplectic-c prequantization up to isomorphism, since M is contractible.

Fix $E > 0$, a regular value of H, and let $S = H^{-1}(E)$. Let $m_0 \in S$ be given. Observe that H and ω are invariant under following transformations: (1) rotation of the $p_j q_j$-plane about the origin by any angle, for any j, and (2) simultaneous rotations of the $p_j p_k$- and $q_j q_k$-planes about the origin by the same angle, for any $j \neq k$. Thus, after suitable rotations, we can assume without loss of generality that $m_0 = (p_{01}, \ldots, p_{0n}, 0, \ldots, 0)$ in Cartesian coordinates, where $p_{0j} \neq 0$ for all j. It is easily established that

$$\phi^t(m_0) = \begin{pmatrix} (\cos t)_{1\le j\le n} & (\sin t)_{1\le j\le n} \\ (-\sin t)_{1\le j\le n} & (\cos t)_{1\le j\le n} \end{pmatrix} \begin{pmatrix} p_{0j} \\ 0 \end{pmatrix}_{1\le j\le n}.$$

This expression describes a periodic orbit with period 2π. Let $\mathcal{C}$ be the orbit of ξ_H through m_0: $\mathcal{C} = \left\{\phi^t(m_0) : t \in \mathbb{R}\right\}$.

4.4.2 From Cartesian to Symplectic Polar Coordinates

On the Manifold

By symplectic polar coordinates, we mean the local coordinates $(s_1, \ldots, s_n, \theta_1, \ldots, \theta_n)$ given by

$$s_j = \frac{1}{2}\left(p_j^2 + q_j^2\right), \quad \theta_j = \tan^{-1}\left(\frac{q_j}{p_j}\right), \quad j = 1, \ldots, n, \tag{4.4.1}$$

whenever these expressions are defined. The polar angles θ_j are all defined modulo 2π. For later reference, the inverse coordinate transformations are

$$p_j = \sqrt{2s_j}\cos\theta_j, \quad q_j = \sqrt{2s_j}\sin\theta_j, \quad j = 1, \ldots, n. \tag{4.4.2}$$

Let

$$U = \left\{(p_1, \ldots, p_n, q_1, \ldots, q_n) \in M : p_j^2 + q_j^2 > 0,\ j = 1, \ldots, n\right\},$$

so that symplectic polar coordinates and the corresponding vector fields $\left\{\frac{\partial}{\partial s_1}, \ldots, \frac{\partial}{\partial s_n}, \frac{\partial}{\partial \theta_1}, \ldots, \frac{\partial}{\partial \theta_n}\right\}$ are defined everywhere on U. Observe that the orbit $\mathcal{C}$ is contained in U. We will construct a local trivialization for $\mathrm{Sp}(M, \omega)$ over U, and a local trivialization for P over $\mathcal{C}$.

When we convert to symplectic polar coordinates on U, we find

$$\omega = \sum_{j=1}^{n} ds_j \wedge d\theta_j,$$

which implies that for all $m \in U$, $\left\{\left.\frac{\partial}{\partial s_1}\right|_m, \ldots, \left.\frac{\partial}{\partial s_n}\right|_m, \left.\frac{\partial}{\partial \theta_1}\right|_m, \ldots, \left.\frac{\partial}{\partial \theta_n}\right|_m\right\}$ is a symplectic basis for $T_m M$. Further,

$$\beta = \sum_{j=1}^{n} s_j d\theta_j, \quad H = \sum_{j=1}^{n} s_j, \quad \xi_H = -\sum_{j=1}^{n} \frac{\partial}{\partial \theta_j}, \tag{4.4.3}$$

and $m_0 = (s_{01}, \ldots, s_{0n}, 0, \ldots, 0)$, where $s_{0j} = \frac{1}{2}p_{0j}^2$, $j = 1, \ldots, n$. The orbit $\mathcal{C}$ has the form

$$\mathcal{C} = \left\{(s_{01}, \ldots, s_{0n}, \tau, \ldots, \tau) : \tau \in \mathbb{R}/2\pi\mathbb{Z}\right\}.$$

The goal is to lift this change of coordinates to the symplectic frame bundle, then to the metaplectic-c prequantization. To facilitate this process, we introduce the following notation.

Let $\Phi_c : U \to \mathbb{R}^{2n}$ be the Cartesian coordinate map, and let $U_c = \Phi_c(U)$. Similarly, let $\Phi_p : U \to \mathbb{R}^n \times (R/2\pi\mathbb{Z})^n$ be the symplectic polar coordinate map, and let $U_p = \Phi_p(U)$. Let F denote the transition map $\Phi_p \circ \Phi_c^{-1}$. Then we have the following commutative diagram.

$$\begin{array}{ccc} & U & \\ \swarrow^{\Phi_c} & & \searrow^{\Phi_p} \\ U_c \subset \mathbb{R}^{2n} & \xrightarrow{\quad F \quad} & U_p \subset \mathbb{R}^n \times (\mathbb{R}/2\pi\mathbb{Z})^n \end{array}$$

This will serve as a model for the changes of coordinates on $\mathrm{Sp}(M,\omega)$ and P.

On the Symplectic Frame Bundle

Recall that an element of the fiber $\mathrm{Sp}(M,\omega)_m$ is a symplectic isomorphism from V to T_mM. Over U, we define the sections b_c and b_p of $\mathrm{Sp}(M,\omega)$ as follows. At each $m \in U$, they are given by

$$b_c(m) : V \to T_mM \quad \text{such that} \quad \begin{pmatrix} \widehat{x}_j \\ \widehat{y}_j \end{pmatrix}_{1\le j\le n} \mapsto \begin{pmatrix} \frac{\partial}{\partial p_j}\Big|_m \\ \frac{\partial}{\partial q_j}\Big|_m \end{pmatrix}_{1\le j\le n},$$

$$b_p(m) : V \to T_mM \quad \text{such that} \quad \begin{pmatrix} \widehat{x}_j \\ \widehat{y}_j \end{pmatrix}_{1\le j\le n} \mapsto \begin{pmatrix} \frac{\partial}{\partial s_j}\Big|_m \\ \frac{\partial}{\partial \theta_j}\Big|_m \end{pmatrix}_{1\le j\le n}.$$

Using these sections, we define two maps $\widetilde{\Phi}_c$, $\widetilde{\Phi}_p$ on $\mathrm{Sp}(M,\omega)$ over U:

$$\widetilde{\Phi}_c : \mathrm{Sp}(M,\omega)|_U \to U_c \times \mathrm{Sp}(V) \quad \text{such that} \quad \widetilde{\Phi}_c(b_c(m)\cdot g) = (\Phi_c(m), g), \quad \forall m \in U,\ \forall g \in \mathrm{Sp}(V),$$

$$\widetilde{\Phi}_p : \mathrm{Sp}(M,\omega)|_U \to U_p \times \mathrm{Sp}(V) \quad \text{such that} \quad \widetilde{\Phi}_p(b_p(m)\cdot g) = (\Phi_p(m), g), \quad \forall m \in U,\ \forall g \in \mathrm{Sp}(V).$$

In other words, the section b_c determines the Cartesian trivialization of $\mathrm{Sp}(M,\omega)$ over U, and b_p determines the symplectic polar trivialization. Lifting the change of coordinates to $\mathrm{Sp}(M,\omega)|_U$ is equivalent to finding a map $\widetilde{F} : U_c \times \mathrm{Sp}(V) \to U_p \times \mathrm{Sp}(V)$ that is a lift of $F : U_c \to U_p$ and such that the following commutes.

$$\begin{array}{ccc} & \mathrm{Sp}(M,\omega)|_U & \\ \swarrow^{\widetilde{\Phi}_c} & & \searrow^{\widetilde{\Phi}_p} \\ U_c \times \mathrm{Sp}(V) & \xrightarrow{\quad \widetilde{F} \quad} & U_p \times \mathrm{Sp}(V) \end{array}$$

If such a map $\widetilde{F}$ exists, then it must satisfy

$$\widetilde{F} \circ \widetilde{\Phi}_c(b_c) = \widetilde{\Phi}_p(b_c),$$

and indeed, once this condition is satisfied, then the value of $\widetilde{F}$ everywhere else will be determined by the $\mathrm{Sp}(V)$ group action and the fact that $\widetilde{F}$ is a lift of F. Let us then evaluate $\widetilde{\Phi}_p(b_c)$.

For all $m \in U$, let

$$G(m) = \begin{pmatrix} \left(\frac{\partial p_j}{\partial s_j}\Big|_m \right)_{1\le j\le n} & \left(\frac{\partial q_j}{\partial s_j}\Big|_m \right)_{1\le j\le n} \\ \left(\frac{\partial p_j}{\partial \theta_j}\Big|_m \right)_{1\le j\le n} & \left(\frac{\partial q_j}{\partial \theta_j}\Big|_m \right)_{1\le j\le n} \end{pmatrix},$$

so that

$$G(m) \begin{pmatrix} \frac{\partial}{\partial p_j}\Big|_m \\ \frac{\partial}{\partial q_j}\Big|_m \end{pmatrix}_{1\le j\le n} = \begin{pmatrix} \frac{\partial}{\partial s_j}\Big|_m \\ \frac{\partial}{\partial \theta_j}\Big|_m \end{pmatrix}_{1\le j\le n}.$$

Then $G(m)$ is a symplectic matrix, and can be viewed as an element of $\mathrm{Sp}(V)$. From the definitions of $b_c(m)$ and the group action, we see that

$$b_c(m) : G(m) \begin{pmatrix} \widehat{x}_j \\ \widehat{y}_j \end{pmatrix}_{1\le j\le n} \mapsto G(m) \begin{pmatrix} \frac{\partial}{\partial p_j}\Big|_m \\ \frac{\partial}{\partial q_j}\Big|_m \end{pmatrix}_{1\le j\le n} = \begin{pmatrix} \frac{\partial}{\partial s_j}\Big|_m \\ \frac{\partial}{\partial \theta_j}\Big|_m \end{pmatrix}_{1\le j\le n}.$$

Thus

$$\widetilde{\Phi}_p(b_c(m)) = (\Phi_p(m), G(m)),$$

and more generally,

$$\widetilde{\Phi}_p(b_c(m) \cdot g) = (\Phi_p(m), G(m)g), \quad \forall g \in \mathrm{Sp}(V).$$

Hence we define the map $\widetilde{F}$ by

$$\widetilde{F}(\Phi_c(m), g) = (\Phi_p(m), G(m)g), \quad \forall m \in U,\ \forall g \in \mathrm{Sp}(V).$$

Observe, using Equation (4.4.2), that the entries of $G(m)$ are singly defined with respect to the angles θ_j, so $G(m)$ is singly defined as m traverses the curve $\mathcal{C}$, or any other closed path through U. This demonstrates that we can change coordinates from Cartesian to polar everywhere on $\mathrm{Sp}(M,\omega)|_U$.

On the Metaplectic-c Prequantization

When we lift the change of variables to P, we restrict our attention further from U to $\mathcal{C}$. Let $\mathcal{C}_c = \Phi_c(\mathcal{C})$ and let $\mathcal{C}_p = \Phi_p(\mathcal{C})$. To emphasize that all of our calculations take place over the closed curve $\mathcal{C}$, we let $m(\tau) = (s_{01}, \dots, s_{0n}, \tau, \dots, \tau) \in \mathcal{C}$ for all $\tau \in \mathbb{R}/2\pi\mathbb{Z}$, and we abbreviate $G(m(\tau))$ by $G(\tau)$. Then $G(\tau)$ is a closed loop through $\mathrm{Sp}(V)$ with period 2π.

Recall that $P = M \times \mathrm{Mp}^c(V)$, and that $\Sigma : P \to \mathrm{Sp}(M,\omega)$ is given by $\Sigma(m,a) = (m, \sigma(a))$ for all $(m,a) \in P$, where the right-hand side is written with respect to the Cartesian trivialization. In other words, P was constructed to be consistent with Cartesian coordinates. To formalize this property, we define

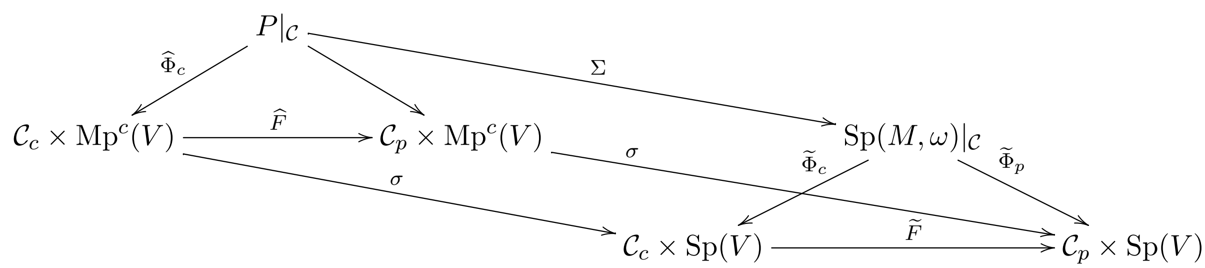
$$\widehat{\Phi}_c : P|_{\mathcal{C}} \to \mathcal{C}_c \times \mathrm{Mp}^c(V) \ \text{ such that } \ \widehat{\Phi}_c(m(\tau),a) = (\Phi_c(m(\tau)),a), \quad \forall (m(\tau),a) \in P|_{\mathcal{C}}.$$

Then $\widehat{\Phi}_c$ is compatible with $\widetilde{\Phi}_c$ in the sense that the following diagram commutes.

$$\begin{array}{ccc} P|_{\mathcal{C}} & \xrightarrow{\Sigma} & \mathrm{Sp}(M,\omega)|_{\mathcal{C}} \\ \downarrow{\scriptstyle \widehat{\Phi}_c} & & \downarrow{\scriptstyle \widetilde{\Phi}_c} \\ \mathcal{C}_c \times \mathrm{Mp}^c(V) & \xrightarrow{\sigma} & \mathcal{C}_c \times \mathrm{Sp}(V) \end{array}$$

In the bottom line, σ maps $\mathcal{C}_c \times \mathrm{Mp}^c(V)$ to $\mathcal{C}_c \times \mathrm{Sp}(V)$ by acting on the second component.

Our goal is to construct a map $\widehat{\Phi}_p : P|_{\mathcal{C}} \to \mathcal{C}_p \times \mathrm{Mp}^c(V)$ that is compatible with $\widetilde{\Phi}_p$ in the same sense. To do so, we will find a map $\widehat{F}$ such that the diagram below commutes, then set $\widehat{\Phi}_p = \widehat{F} \circ \widehat{\Phi}_c$.



Since $\widehat{F}$ must be a lift of $\widetilde{F}$ and therefore of F, we assume that $\widehat{F}$ takes the form

$$\widehat{F}(\Phi_c(m(\tau)), a) = (\Phi_p(m(\tau)), \widehat{G}(\tau)a), \quad \forall m(\tau) \in \mathcal{C},\ \forall a \in \mathrm{Mp}^c(V),$$

where $\widehat{G}(\tau) \in \mathrm{Mp}^c(V)$. From the condition that

$$\sigma \circ \widehat{F} \circ \widehat{\Phi}_c = \widetilde{F} \circ \sigma \circ \widehat{\Phi}_c,$$

it follows that we must have $\sigma(\widehat{G}(\tau)) = G(\tau)$ for all $\tau \in \mathbb{R}/2\pi\mathbb{Z}$. That is, $\widehat{G}(\tau)$ must be a lift of $G(\tau)$ to $\mathrm{Mp}^c(V)$. More specifically, in order for $\widehat{F}$ to be singly defined, $\widehat{G}(\tau)$ must be a closed loop in $\mathrm{Mp}^c(V)$ with period 2π.

There is another consideration: the effect of this change of coordinates on the one-form γ. Recall that γ is defined on P by

$$\gamma = \frac{1}{i\hbar}\Pi^*\beta + \frac{1}{2}\eta_*\vartheta_0,$$

where ϑ_0 is the trivial connection on the product bundle $M \times \mathrm{Mp}^c(V)$. When we change to symplectic polar coordinates, we would like γ to retain the same form, where β can be written in polar coordinates as in Equation (4.4.3), and where ϑ_0 now represents the trivial connection on the product bundle $\mathcal{C}_p \times \mathrm{Mp}^c(V)$. Since $\ker(\eta) = \mathrm{Mp}(V)$, we can accomplish this by requiring that the path $\widehat{G}(\tau)$ lie within $\mathrm{Mp}(V)$.

Recall the parametrization of $\mathrm{Mp}^c(V)$ that was described in Section 2.3.1. If $\widehat{G}(\tau)$ is a lift of $G(\tau)$ to $\mathrm{Mp}(V)$, then the parameters of $\widehat{G}(\tau)$ have the form $(G(\tau), \mu(\tau))$ where $\mu(\tau)^2 \mathrm{Det}_{\mathbb{C}} C_{G(\tau)} = 1$. To determine $\mu(\tau)$, we must examine the matrix $G(\tau) \in \mathrm{Sp}(V)$ explicitly.

Using Equation (4.4.2), we find

$$G(\tau) = \begin{pmatrix} \left(\frac{1}{\sqrt{2s_{0j}}}\cos\tau\right)_{1\le j\le n} & \left(\frac{1}{\sqrt{2s_{0j}}}\sin\tau\right)_{1\le j\le n} \\ \left(-\sqrt{2s_{0j}}\sin\tau\right)_{1\le j\le n} & \left(\sqrt{2s_{0j}}\cos\tau\right)_{1\le j\le n} \end{pmatrix}.$$

Now we calculate

$$C_{G(\tau)} = \frac{1}{2}(G(\tau) - JG(\tau)J)$$

using the complex structure noted in Section 4.4.1, then convert to an $n \times n$ complex matrix. The result is the diagonal matrix

$$C_{G(\tau)} = \frac{1}{2}\left(\left(\sqrt{2s_{0j}} + \frac{1}{\sqrt{2s_{0j}}}\right)e^{i\tau}\right)_{1\le j\le n},$$

which has complex determinant

$$\mathrm{Det}_{\mathbb{C}} C_{G(\tau)} = \prod_{j=1}^{n}\frac{1}{2}\left(\sqrt{2s_{0j}} + \frac{1}{\sqrt{2s_{0j}}}\right)e^{i\tau} = Ke^{in\tau},$$

where K is a positive real value that is constant over $\mathcal{C}$. Thus, a lift of $G(\tau)$ to $\mathrm{Mp}(V)$ is parametrized by

$$\widehat{G}(\tau) \mapsto \left(G(\tau), \frac{1}{\sqrt{K}}e^{-in\tau/2}\right). \tag{4.4.4}$$

Notice that $(G(\tau+2\pi), \mu(\tau+2\pi)) = (G(\tau), \mu(\tau)e^{-in\pi})$. When n is even, we get a closed loop through $\mathrm{Mp}(V)$ with period 2π, which was the optimal outcome. When n is odd, however, traversing $\mathcal{C}$ once multiplies $\mu(\tau)$ by -1. This shows that we cannot always lift $G(\tau)$ to a closed path through $\mathrm{Mp}(V)$. Nevertheless, we choose the path $\widehat{G}(\tau)$ as defined in Equation (4.4.4) because it preserves the form of γ. If we let $\varepsilon_n \in \mathrm{Mp}^c(V)$ be the element whose parameters are $(I, e^{-i\pi n})$, then $\widehat{G}(\tau + 2\pi) = \varepsilon_n \widehat{G}(\tau)$.

To prevent $\widehat{F}$ from being multi-valued, let $\dot{\mathcal{C}} = \{m(\tau) \in \mathcal{C} : \tau \in (0, 2\pi)\}$, and let $\dot{\mathcal{C}}_c$ and $\dot{\mathcal{C}}_p$ be the images of $\dot{\mathcal{C}}$ in Cartesian and symplectic polar coordinates, respectively. Then let $\widehat{F} : \dot{\mathcal{C}}_c \times \mathrm{Mp}^c(V) \to \dot{\mathcal{C}}_p \times \mathrm{Mp}^c(V)$ be given by

$$\widehat{F}(\Phi_c(m(\tau)), a) = (\Phi_p(m(\tau)), \widehat{G}(\tau)a), \quad \forall (m(\tau), a) \in P|_{\dot{\mathcal{C}}},$$

and identify $P|_{\dot{\mathcal{C}}}$ with $\dot{\mathcal{C}}_p \times \mathrm{Mp}^c(V)$ under the map $\widehat{\Phi}_p = \widehat{F} \circ \widehat{\Phi}_c$. This gives us symplectic polar coordinates for P over $\dot{\mathcal{C}}$, and we will manually adjust for the fact that closing the loop multiplies $\widehat{G}(\tau)$ by ε_n. On $\dot{\mathcal{C}}_p \times \mathrm{Mp}^c(V)$, γ takes the form

$$\gamma = \frac{1}{i\hbar}\sum_{j=1}^{n} s_{0j}\, d\theta_j + \frac{1}{2}\eta_* \vartheta_0,$$

where ϑ_0 is the trivial connection on the product bundle, and where the values s_{0j} are constant over $\mathcal{C}_p$.

4.4.3 Quantized Energy Levels of the Harmonic Oscillator

Having established the change of coordinates, we will now drop the explicit use of Φ_p, F and their lifts. Elements of U will be written with respect to the symplectic polar coordinates $(s_1, \ldots, s_n,\, \theta_1, \ldots, \theta_n)$, which for convenience we abbreviate by X_k, $k = 1, \ldots, 2n$. Using the identifications $\widehat{x}_j \mapsto \frac{\partial}{\partial s_j}\Big|_m$, $\widehat{y}_j \mapsto \frac{\partial}{\partial \theta_j}\Big|_m$ for all $m \in U$, $j = 1, \ldots, n$, we write

$$\mathrm{Sp}(M, \omega)|_U = U \times \mathrm{Sp}(V) \quad \text{and} \quad P|_{\dot{\mathcal{C}}} = \dot{\mathcal{C}} \times \mathrm{Mp}^c(V).$$

At the level of tangent spaces, we have

$$T_{(m,I)}\,\mathrm{Sp}(M, \omega) = T_m M \times \mathfrak{sp}(V), \quad \forall m \in U,$$

and

$$T_{(m,I)}P = T_m M \times \mathfrak{mp}^c(V) = T_m M \times \left(\mathfrak{sp}(V) \oplus \mathfrak{u}(1)\right), \quad \forall m \in \dot{\mathcal{C}}.$$

Recall that on U, $\xi_H = -\sum_{j=1}^{n} \frac{\partial}{\partial \theta_j}$. Therefore $T_m S^{\perp} = \mathrm{span}\left\{\sum_{j=1}^{n} \frac{\partial}{\partial \theta_j}\Big|_m\right\}$ at each $m \in U$. Within the model vector space V, let

$$W^{\perp} = \mathrm{span}\left\{\widehat{y}_1 + \ldots + \widehat{y}_n\right\}.$$

Then

$$W = \mathrm{span}\left\{\widehat{x}_1 - \widehat{x}_2, \ldots, \widehat{x}_{n-1} - \widehat{x}_n, \widehat{y}_1, \ldots, \widehat{y}_n\right\},$$

and

$$W/W^{\perp} = \operatorname{span}\left\{[\widehat{x}_1 - \widehat{x}_2], \ldots, [\widehat{x}_{n-1} - \widehat{x}_n], [\widehat{y}_1 - \widehat{y}_2], \ldots, [\widehat{y}_{n-1} - \widehat{y}_n]\right\}.$$

The local trivialization of $\operatorname{Sp}(M,\omega)|_U$ induces the local trivializations $\operatorname{Sp}(M,\omega;S)|_{U\cap S} = U \cap S \times \operatorname{Sp}(V;W)$ and $\operatorname{Sp}(TS/TS^{\perp})|_{U\cap S} = U \cap S \times \operatorname{Sp}(W/W^{\perp})$.

By definition, P_S is the bundle associated to P^S by the group homomorphism $\hat{\nu} : \operatorname{Mp}^c(V) \to \operatorname{Mp}^c(W/W^{\perp})$. Over $\dot{\mathcal{C}}$, we have the local trivializations $P|_{\dot{\mathcal{C}}} = \dot{\mathcal{C}} \times \operatorname{Mp}^c(V)$, $P^S|_{\dot{\mathcal{C}}} = \dot{\mathcal{C}} \times \operatorname{Mp}^c(V;W)$ and $P_S|_{\dot{\mathcal{C}}} = \dot{\mathcal{C}} \times \operatorname{Mp}^c(W/W^{\perp})$, where the associated bundle map $\hat{\nu} : P^S|_{\dot{\mathcal{C}}} \to P_S|_{\dot{\mathcal{C}}}$ acts by

$$\hat{\nu}(m,a) = (m, \hat{\nu}(a)), \quad \forall (m,a) \in P^S|_{\dot{\mathcal{C}}}.$$

Using properties of $\hat{\nu}$ noted in Section 4.2.1, we have $\hat{\nu}(\varepsilon_n) = \varepsilon_n \in \operatorname{Mp}^c(W/W^{\perp})$ for all n. Therefore, within P_S, the same factor of ε_n is required to close the loop from $\dot{\mathcal{C}}$ to $\mathcal{C}$. Further, since $\eta_* \circ \hat{\nu}_* = \eta_*$, the one-form γ_S induced on P_S does not change form under the map $\hat{\nu}$:

$$\gamma_S|_{\dot{\mathcal{C}}} = \frac{1}{i\hbar} \sum_{j=1}^{n} s_{0j}\, d\theta_j + \frac{1}{2}\eta_* \vartheta_0,$$

where ϑ_0 is now the trivial connection on the product bundle $\dot{\mathcal{C}} \times \operatorname{Mp}^c(W/W^{\perp})$. Lastly, the relevant tangent spaces are

$$T_{(m,I)} \operatorname{Sp}(TS/TS^{\perp}) = T_m M \times \mathfrak{sp}(W/W^{\perp}), \ \forall m \in U,$$

and

$$T_{(m,I)} P_S = T_m M \times (\mathfrak{sp}(W/W^{\perp}) \oplus \mathfrak{u}(1)), \quad \forall m \in \dot{\mathcal{C}}.$$

Recall that we chose the initial point $m_0 = (s_{01}, \ldots, s_{0n}, 0, \ldots, 0)$, and that $\mathcal{C}$ is the orbit of ξ_H through m_0. We will show that the orbit of $\widetilde{\xi}_H$ through (m_0, I) is closed in $\operatorname{Sp}(TS/TS^{\perp})$. We first need to lift ϕ^t to the flow $\widetilde{\phi}^t$ on $\operatorname{Sp}(M,\omega)$. By definition, for any $m \in U$, $\widetilde{\phi}^t(m,I) = (\phi^t(m), \phi^t_*|_m)$. This implies that

$$\widetilde{\xi}_H(m,I) = \left.\frac{d}{dt}\right|_{t=0} \widetilde{\phi}^t(m,I) = \left(\xi_H(m), \left.\frac{d}{dt}\right|_{t=0} \phi^t_*|_m\right).$$

We write $\phi^t = (\phi^t_1, \ldots, \phi^t_{2n})$ with respect to the symplectic polar coordinates. Then $\phi^t_*|_m$ is a $2n \times 2n$ matrix, which we interpret as an element of $\mathrm{Sp}(V)$, and its components are given by $(\phi^t_*)_{jk} = \frac{\partial \phi^t_j}{\partial X_k}\Big|_m$. Noting that $\frac{d}{dt}\big|_{t=0} \phi^t_j = (\xi_H)_j$, we compute

$$\left(\frac{d}{dt}\bigg|_{t=0} \phi^t_*|_m \right)_{jk} = \frac{\partial}{\partial X_k}\bigg|_m (\xi_H)_j. \tag{4.4.5}$$

But $\xi_H = -\sum_{j=1}^n \frac{\partial}{\partial \theta_j}$ on U, which has constant components, so $\frac{d}{dt}\big|_{t=0} \phi^t_*$ is identically 0. Thus

$$\widetilde{\xi}_H(m, I) = (\xi_H(m), 0), \quad \forall m \in U.$$

In particular, since the $\mathfrak{sp}(V)$ component of $\widetilde{\xi}_H$ is constant over the orbit $\mathcal{C}$, we can find the flow for $\widetilde{\xi}_H$ on $\mathrm{Sp}(M, \omega)$ through (m_0, I) by exponentiating: it is simply

$$\widetilde{\phi}^t(m_0, I) = (\phi^t(m_0), I).$$

This is clearly a closed orbit with period 2π. The induced flow on the bundle $\mathrm{Sp}(TS/TS^\perp)$ also takes the form

$$\widetilde{\phi}^t(m_0, I) = (\phi^t(m_0), I).$$

Now we lift $\widetilde{\xi}_H$ to P_S, horizontally with respect to γ_S. Over the curve $\dot{\mathcal{C}}$, we calculate that $\xi_H \lrcorner \beta = -\sum_{j=1}^n s_{0j} = -E$. Then, for any $m \in \dot{\mathcal{C}}$, the horizontal lift of $\widetilde{\xi}_H$ to $T_{(m,I)}P_S$ is

$$\widehat{\xi}_H(m, I) = \left(\xi_H(m), 0 \oplus \frac{E}{i\hbar} \right).$$

The $\mathfrak{mp}^c(V)$ component is constant over $\dot{\mathcal{C}}$, so the flow is again calculated by exponentiating:

$$\widehat{\phi}^t(m_0, I) = \left(\phi^t(m_0), \exp\left(0 \oplus \frac{Et}{i\hbar} \right) \right) = (\phi^t(m_0), e^{Et/i\hbar}),$$

where $e^{Et/i\hbar} \in U(1) \subset \mathrm{Mp}^c(W/W^\perp)$.

The quantized energy levels are those values of E for which this orbit closes. However, we must remember that one circuit about $\mathcal{C}$ introduces an extra factor of ε_n. Referring to the properties of the parametrization stated in Section 2.3.1, we see that $\widehat{\phi}^0(m_0, I) = \widehat{\phi}^{2\pi}(m_0, I)$

if and only if $e^{2\pi E/i\hbar} = e^{-in\pi}$, or in other words, when $\frac{2\pi E}{i\hbar} + in\pi = -2\pi i N$ for some $N \in \mathbb{Z}$. Upon rearranging, we find that the quantized energy levels must take the form

$$E = \hbar\left(N + \frac{n}{2}\right), \quad N \in \mathbb{Z}.$$

Thus the quantization condition correctly reproduces the expected energy levels of the harmonic oscillator.

4.4.4 Example of Dynamical Invariance

Fix $n = 2$. It is convenient to use the definitions $s_j = \frac{1}{2}(p_j^2 + q_j^2)$ and $\frac{\partial}{\partial\theta_j} = p_j\frac{\partial}{\partial q_j} - q_j\frac{\partial}{\partial p_j}$ everywhere on M, for $j = 1, 2$. We will point out when a statement only holds on the neighborhood U.

Let $k \in \mathbb{R}$ be a positive constant, and consider the two functions $H_1, H_2 : M \to \mathbb{R}$ given by

$$H_1 = s_1 + s_2 - k, \quad H_2 = (s_1 + s_2 - k)(s_1 + 2s_2 + 1).$$

The function H_1 is just the energy function for the harmonic oscillator, shifted by k: its quantized energy levels are

$$E_N = \hbar N - k, \quad N \in \mathbb{Z}.$$

Notice that $H_1^{-1}(0) = H_2^{-1}(0)$. Let this shared level set be S. The energy $E = 0$ is a quantized energy for the system (M, ω, H_1) if and only if $\frac{k}{\hbar} \in \mathbb{Z}$.

The two Hamiltonian vector fields are

$$\begin{aligned}
\xi_{H_1} &= -\frac{\partial}{\partial\theta_1} - \frac{\partial}{\partial\theta_2}, \\
\xi_{H_2} &= (s_1 + 2s_2 + 1)\xi_{H_1} - H_1\left(\frac{\partial}{\partial\theta_1} + 2\frac{\partial}{\partial\theta_2}\right) \\
&= -(2s_1 + 3s_2 - k + 1)\frac{\partial}{\partial\theta_1} - (3s_1 + 4s_2 - 2k + 1)\frac{\partial}{\partial\theta_2}.
\end{aligned}$$

Since $H_1 = 0$ on S, we see that $\xi_{H_2} = (s_1 + 2s_2 + 1)\xi_{H_1}$ everywhere on S. Thus the vector fields are parallel on S, as expected, and so they share the same orbits in S. However, ξ_{H_1} and ξ_{H_2} are not parallel away from S. Let ϕ^t be the flow of ξ_{H_1}, and let ρ^t be the flow of ξ_{H_2}.

Consider the initial point $m_0 \in S \cap U$, where $m_0 = (s_{01}, s_{02}, 0, 0)$ with $s_{01} + s_{02} = k$ and $s_{01}, s_{02} \neq 0$. The orbit of both ϕ^t and ρ^t through m_0 is

$$\mathcal{C} = \{(s_{01}, s_{02}, \tau, \tau) : \tau \in \mathbb{R}/2\pi\mathbb{Z}\}.$$

From Section 4.4.3, we know that

$$\widetilde{\xi}_{H_1}(m, I) = (\xi_{H_1}(m), 0), \quad \forall m \in \mathcal{C},$$

and therefore

$$\widetilde{\phi}^t(m_0, I) = (\phi^t(m_0), I).$$

By the same calculation, $\widetilde{\xi}_{H_2}(m, I) = (\xi_{H_2}(m), \frac{d}{dt}\big|_{t=0} \rho^t_*|_m)$ for $m \in \mathcal{C}$. Applying Equation (4.4.5) to the components of ξ_{H_2} yields

$$\frac{d}{dt}\bigg|_{t=0} \rho^t_*|_m = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & -3 & 0 & 0 \\ -3 & -4 & 0 & 0 \end{pmatrix},$$

which we interpret as an element of $\mathfrak{sp}(V)$. Let this matrix be denoted by κ. Then

$$\widetilde{\xi}_{H_2}(m, I) = (\xi_{H_2}(m), \kappa),$$

and since the Lie algebra component is constant over $\mathcal{C}$, we obtain the flow on $\mathrm{Sp}(M, \omega)$ through (m_0, I) by exponentiating:

$$\widetilde{\rho}^t(m_0, I) = (\rho^t(m_0), \exp(t\kappa)).$$

A calculation establishes that

$$\exp(t\kappa) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2t & -3t & 1 & 0 \\ -3t & -4t & 0 & 1 \end{pmatrix},$$

which is clearly not periodic. Thus $\widetilde{\rho}^t$ has no closed orbits on $\mathrm{Sp}(M, \omega)$ over $S \cap U$.

Now let us transfer to $\mathrm{Sp}(TS/TS^\perp)$. If we apply the definitions and identifications laid out at the beginning of Section 4.4.3, now setting $n = 2$, then we find $W^\perp = \mathrm{span}\,\{\widehat{y}_1 + \widehat{y}_2\}$, $W/W^\perp = \mathrm{span}\,\{[\widehat{x}_1 - \widehat{x}_2], [\widehat{y}_1 - \widehat{y}_2]\}$, and the identification of $\mathrm{Sp}(M, \omega)|_{\mathcal{C}}$ with $\mathcal{C} \times \mathrm{Sp}(V)$ induces an identification of $\mathrm{Sp}(TS/TS^\perp)|_{\mathcal{C}}$ with $\mathcal{C} \times \mathrm{Sp}(W/W^\perp)$. Notice that

$$\begin{aligned} \exp(t\kappa)(\widehat{x}_1 - \widehat{x}_2) &= \widehat{x}_1 - \widehat{x}_2 + t(\widehat{y}_1 + \widehat{y}_2), \\ \exp(t\kappa)(\widehat{y}_1 - \widehat{y}_2) &= \widehat{y}_1 - \widehat{y}_2. \end{aligned}$$

Therefore the path through $\mathrm{Sp}(W/W^\perp)$ induced by $\exp(t\kappa)$ is

$$\nu(\exp(t\kappa)) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus, on $\mathrm{Sp}(TS/TS^\perp)|_{\mathcal{C}}$,

$$\widetilde{\rho}^t(m_0, I) = \left(\rho^t(m_0), I\right),$$

which coincides with $\widetilde{\phi}^t(m_0, I)$.

The above calculation omits certain cases: namely, if the starting point m_0 has $s_{01} = 0$ or $s_{02} = 0$. We can no longer eliminate such cases by performing a rotation, because H_2 is not symmetric with respect to s_1 and s_2. If $m_0 \notin U$, then we have to modify our approach. For example, if $s_{01} = 0$, then we retain Cartesian coordinates for the p_1q_1-plane and convert to symplectic polar on the p_2q_2-plane. The calculation is more complicated, but the result is similar: over $\mathcal{C}$, we find that $\widetilde{\xi}_{H_2} = (\xi_{H_2}, \kappa)$ for some constant value of $\kappa \in \mathfrak{sp}(V)$. The path $\widetilde{\rho}^t(m_0, I) = (\rho^t(m_0), \exp(t\kappa))$ does not close in $\mathrm{Sp}(M, \omega)$, but the induced path in $\mathrm{Sp}(TS/TS^\perp)$

coincides with $\widetilde{\phi}^t(m_0, I)$. The same pattern holds if we take $s_{02} = 0$.

Thus, if the quantized energy condition were stated in terms of the holonomy of γ^S over closed orbits in $\mathrm{Sp}(M, \omega; S)$, as it was in [14], then the value $E = 0$ would satisfy the condition vacuously for the system (M, ω, H_2), regardless of the value of k. It is only by descending to $\mathrm{Sp}(TS/TS^\perp)$ that we recover the quantization condition $\frac{k}{\hbar} \in \mathbb{Z}$. Hence our definition of a quantized energy level is dynamically invariant, while that in [14] is not.

4.5 Comparison with Kostant-Souriau Quantization

4.5.1 The Quantized Energy Condition and Its Properties

The quantized energy condition in Definition 4.2.2 can be easily adapted to Kostant-Souriau prequantization. Indeed, the Kostant-Souriau case is simpler, since it does not involve the symplectic frame bundle.

Assume that (M, ω) admits a prequantization circle bundle (Y, γ), and let $H : M \to \mathbb{R}$ be a smooth function. We now mimic the constructions in Section 4.2.2, using Y in place of P. Denote the Hamiltonian vector field corresponding to H by ξ_H as before, and let $\widetilde{\xi}_H$ be the lift of ξ_H to Y that is horizontal with respect to γ. Fix E, a regular value of H, and let $S = H^{-1}(E) \subset M$. Let Y^S be the restriction of Y to S, and let γ^S be the pullback of γ to Y^S.

$$\begin{array}{ccc} (Y, \gamma) & \xleftarrow{\text{incl.}} & (Y^S, \gamma^S) \\ \downarrow & & \downarrow \\ (M, \omega) & \xleftarrow{\text{incl.}} & S \end{array}$$

Since there is no group homomorphism with which to construct an associated bundle, the construction stops here and we give the definition of a quantized energy level using (Y^S, γ^S).

Definition 4.5.1. If the connection one-form γ^S has trivial holonomy over all closed orbits of the Hamiltonian vector field ξ_H on S, then E is a **Kostant-Souriau (KS) quantized energy level** for the system (M, ω, H).

It is straightforward to establish that this definition has a dynamical invariance property. Suppose $H_1, H_2 : M \to \mathbb{R}$ are smooth functions such that $H_1^{-1}(E_1) = H_2^{-1}(E_2) = S$ for regular values E_1 and E_2. We argued in Section 4.3.2 that ξ_{H_1} and ξ_{H_2} are parallel on S, which implies that they have the same orbits. Therefore γ^S has trivial holonomy over the orbits of one if and only if it has trivial holonomy over the orbits of the other. The KS version of the dynamical invariance theorem is immediate.

Theorem 4.5.2. If $H_1, H_2 : M \to \mathbb{R}$ are smooth functions such that $H_1^{-1}(E_1) = H_2^{-1}(E_2)$ for regular values E_j of H_j, $j = 1, 2$, then E_1 is a KS quantized energy level for (M, ω, H_1) if and only if E_2 is a KS quantized energy level for (M, ω, H_2).

In Section 4.2.2, we examined the case in which the symplectic reduction of (M, ω) at E is a manifold. Theorem 4.2.1, due to Robinson [14], gives the conditions under which the quantization condition is sufficient to imply that the symplectic reduction admits a metaplectic-c prequantization. A similar theorem can be given in the context of prequantization circle bundles.

First, we state a general result, which was used in [14] to prove Theorem 4.2.1. Let S be an arbitrary manifold, and suppose that (Z, δ) is a principal circle bundle with connection one-form over S. Let the curvature of δ be ϖ. Suppose that F is a foliation of S whose leaf space S_F is a smooth manifold. Denote the leaf projection map by $S \xrightarrow{\pi} S_F$. If δ has trivial holonomy over all of the leaves of F, then Z can be factored to produce a well-defined circle bundle $Z_F \to S_F$. Further, δ descends to a connection one-form δ_F on Z_F, and the curvature ϖ_F of δ_F satisfies $\pi^* \varpi_F = \varpi$.

Now apply this result to the circle bundle $(Y^S, \gamma^S) \to S$, where the foliation is given by the orbits of the vector field ξ_H on the level set S, and the symplectic reduction (M_E, ω_E) is its leaf space.

Theorem 4.5.3. Suppose that the symplectic reduction (M_E, ω_E) for (M, ω) at E is a manifold. If γ^S has trivial holonomy over all closed orbits of ξ_H on S, then the quotient of (Y^S, γ^S) by the orbits of $\widetilde{\xi}_H$ is a prequantization circle bundle for (M_E, ω_E).

Thus the Kostant-Souriau version of the quantized energy condition is sufficient to ensure that the symplectic reduction admits a prequantization circle bundle, whenever the symplectic reduction is a manifold. This is a slight improvement over the metaplectic-c result, since it does not depend on a quotient of the symplectic frame bundle being well defined.

4.5.2 Lack of Half-Shift in the Harmonic Oscillator

In this section, we will determine the KS quantized energy levels of the n-dimensional harmonic oscillator. The calculation will be significantly simpler than that in Section 4.4, but it will yield the wrong answer.

Let $M = \mathbb{R}^{2n}$, with Cartesian coordinates $(p_1, \ldots, p_n, q_1, \ldots, q_n)$ and symplectic form $\omega = \sum_{j=1}^n dp_j \wedge dq_j$. The energy function and corresponding Hamiltonian vector field for the harmonic oscillator are

$$H = \frac{1}{2}\sum_{j=1}^n (p_j^2 + q_j^2), \quad \xi_H = \sum_{j=1}^n \left(q_j \frac{\partial}{\partial p_j} - p_j \frac{\partial}{\partial q_j} \right).$$

Let the flow of ξ_H on M be ϕ^t. We know from Section 4.4.1 that all of the orbits of ξ_H are circles, and that $\phi^{t+2\pi}(m) = \phi^t(m)$ for all $m \in M$.

Let $Y = M \times U(1)$, with projection map $Y \xrightarrow{\Pi} M$. We define

$$\beta = \frac{1}{2}\sum_{j=1}^n (p_j dq_j - q_j dp_j)$$

on M, and let $\gamma = \frac{1}{i\hbar}\Pi^*\beta + \vartheta_0$ on Y, where ϑ_0 is the trivial connection on the product bundle $M \times U(1)$. Then (Y, γ) is a prequantization circle bundle for (M, ω), and it is unique up to isomorphism.

Let $E > 0$ be arbitrary, and let $S = H^{-1}(E)$. At any point $s \in S$, we calculate that $\xi_H \lrcorner \beta = -E$. Therefore the lifted vector field $\widetilde{\xi}_H$ at the point $(s, I) \in Y$ is

$$\widetilde{\xi}_H(s, I) = \left(\xi_H(s), \frac{E}{i\hbar} \right),$$

where we identify the tangent space $T_{(s,I)}Y$ with $T_sM \times \mathfrak{u}(1)$. Note that the $\mathfrak{u}(1)$ component is constant over all of S, so in particular it is constant over an orbit. Let the flow of $\widetilde{\xi}_H$ be $\widetilde{\phi}^t$. By exponentiation, we find that

$$\widetilde{\phi}^t(s, I) = \left(\phi^t(s), e^{Et/i\hbar} \right).$$

From this, it follows that the holonomy of γ^S over the orbits in S is trivial if and only if $E = N\hbar$ for some $N \in \mathbb{N}$. Thus the KS quantized energy levels are inconsistent with the predictions of quantum mechanics when the dimension n is odd.

This shortcoming in the Kostant-Souriau prequantization of the harmonic oscillator is well known, and the standard solution is to proceed from prequantization to quantization while introducing the half-form correction, as described in Section 2.2.2. In this context, the quantized energy levels for the system (M, ω, H) can be taken to be the eigenvalues of the operator corresponding to the energy function H. When the quantization recipe is applied to the harmonic oscillator, the presence of the half-form bundle adds the $\frac{n}{2}$ shift to the energy eigenvalues. However, this correction comes at the cost of introducing a choice of polarization, and the quantized energy definition can no longer be evaluated over a single level set of H.

By comparison, when we use the metaplectic-c formulation of a quantized energy level, we find that the correct harmonic oscillator energies are encoded in the geometry of the level sets of the energy function. The result is dynamically invariant, independent of polarization, and consistent with physical prediction. This example illustrates the benefits of metaplectic-c quantization and our quantized energy definition.

Chapter 5

The Hydrogen Atom

5.1 Introduction

The quantized energy levels of the hydrogen atom have been calculated for various physical models, using various flavors of geometric quantization. Notable examples include:

- Simms [16], who used the observation that the space of orbits corresponding to a fixed negative energy is isomorphic to $S^2 \times S^2$, and determined those energies for which the reduced manifold admits a prequantization circle bundle;
- Sniatycki [19], who looked at the 2-dimensional relativistic Kepler problem and computed a Bohr-Sommerfeld condition for the completely integrable system given by considering the energy and angular momentum functions simultaneously;
- Duval, Elhadad and Tuynman [6], who took the phase space to include the spins of the electron and proton, then chose a polarization and determined the Kostant-Souriau quantized operator corresponding to the energy function with fine and hyperfine interaction terms.

These examples exist at one of two possible extremes. On one hand, the quantized energy condition can be evaluated over the symplectic reduction of a particular level set of the energy function, as in [16]. This definition only looks at one energy level at a time, but it requires constructing the symplectic reduction and establishing that the result is a smooth manifold. On the other hand, the quantized energy levels can be determined from properties of the quantized system as a whole, as in [19] or [6]. These approaches are characterized by requiring a polarization, or the equivalent information – recall that the Hamiltonian vector fields corresponding to the Poisson-commuting functions in a completely integrable system generate a real polarization – and the calculation is not restricted to a single level set of the energy function in question.

Our proposed definition for a quantized energy level acts as a middle ground between these two extremes. The objective of this chapter is to apply the metaplectic-c quantized energy condition to the hydrogen atom, using the physical model that is equivalent to the Kepler problem. We will show that the quantized energy levels are in agreement with the quantum mechanical prediction.

In Section 5.2, we set up our model of the hydrogen atom and construct a metaplectic-c prequantization for its phase space. Section 5.3 presents the Ligon-Schaaf regularization map, which is a symplectomorphism from the negative-energy domain of the hydrogen atom to an open submanifold of TS^3. We show how to relate the energy function for the hydrogen atom to that of a free particle on S^3. Finally, in Section 5.4, we determine the quantized energy levels of a free particle on S^3, and use these to determine the quantized energy levels for the hydrogen atom.

We will apply the constructions that were described in Section 4.2. In addition, we will make repeated use of the notation and conventions that are described below.

5.1.1 Choices for Subsequent Calculations

In the sections that follow, we will need model symplectic vector spaces of several different dimensions. Let us fix some standardized choices.

For any $n \in \mathbb{N}$, let (V_n, Ω_n) be a $2n$-dimensional symplectic vector space. Let $(\widehat{v}_1, \ldots, \widehat{v}_n, \widehat{w}_1, \ldots, \widehat{w}_n)$ be a symplectic basis for V_n, and write all elements of V_n as ordered $2n$-tuples with respect to this basis. The symplectic form can be written in terms of the dual basis as

$$\Omega_n = \sum_{j=1}^{n} \widehat{v}_j^* \wedge \widehat{w}_j^*.$$

Assume that each real vector $(a_1, \ldots, a_n, b_1, \ldots, b_n) \in V_n$ is identified with the complex vector

$(b_1 + ia_1 \ldots, b_n + ia_n) \in \mathbb{C}^n$. The resulting complex structure J on V_n is written in matrix form as $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, where I is the $n \times n$ identity matrix.

When we require a subspace of V_n of codimension 1, we choose

$$W_n = \operatorname{span}\{\widehat{v}_1, \ldots, \widehat{v}_n, \widehat{w}_1, \ldots, \widehat{w}_{n-1}\}.$$

Then

$$W_n^\perp = \operatorname{span}\{\widehat{v}_n\} \quad \text{and} \quad W_n/W_n^\perp = \operatorname{span}\{[\widehat{v}_1], \ldots, [\widehat{v}_{n-1}], [\widehat{w}_1], \ldots, [\widehat{w}_{n-1}]\}.$$

Using equation 4.2.1, it is immediate that $W_n/W_n^\perp$ is isomorphic to V_{n-1} as a symplectic vector space and a complex vector space. The commutative diagram from Section 4.2.1 containing the group homomorphisms ν and $\hat{\nu}$ can be rewritten as follows.

$$\begin{array}{ccc} \mathrm{Mp}^c(V_n) \supset \mathrm{Mp}^c(V_n; W_n) & \xrightarrow{\hat{\nu}} & \mathrm{Mp}^c(V_{n-1}) \\ \downarrow \sigma & & \downarrow \sigma \\ \mathrm{Sp}(V_n) \supset \mathrm{Sp}(V_n; W_n) & \xrightarrow{\nu} & \mathrm{Sp}(V_{n-1}) \end{array}$$

5.2 Quantizing the Hydrogen Atom

5.2.1 Setup

Let $\dot{\mathbb{R}}^3$ represent $\mathbb{R}^3$ with the origin removed, and let $M = T\dot{\mathbb{R}}^3 = \dot{\mathbb{R}}^3 \times \mathbb{R}^3$. We use Cartesian coordinates $q = (q_1, q_2, q_3)$ on $\dot{\mathbb{R}}^3$ and $p = (p_1, p_2, p_3)$ on $\mathbb{R}^3$. Equip M with the symplectic form $\omega = \sum_{j=1}^3 dq_j \wedge dp_j$.

We consider the model of the hydrogen atom that is equivalent to the Kepler problem. Assume that a proton is fixed at the origin in $\mathbb{R}^3$, and that an electron of mass m_e interacts with it via the electrostatic force, which obeys an inverse-square law with constant of proportionality k. Then M is the phase space for the motion of the electron, where q and p represent its position and momentum, respectively. The Hamiltonian energy function is

$$H = \frac{1}{2m_e}|p|^2 - \frac{k}{|q|},$$

and the corresponding Hamiltonian vector field is

$$\xi_H = \sum_{j=1}^{3} \left(\frac{1}{m_e} p_j \frac{\partial}{\partial q_j} - \frac{k}{|q|^3} q_j \frac{\partial}{\partial p_j} \right).$$

Let ξ_H have flow ϕ^t on M.

The solutions to the Kepler problem are well known [1, 3]. The angular momentum vector $L = q \times p$ is a constant of the motion, as is the eccentricity vector $e = \frac{1}{m_e k} p \times L - \frac{q}{|q|}$. In position space, the orbits corresponding to a given energy $E \in \mathbb{R}$ are conic sections with eccentricity

$$|e| = \sqrt{1 + \frac{2E|L|^2}{m_e k^2}}, \tag{5.2.1}$$

having the origin as a focus.

In particular, suppose $E < 0$. Then the value of $|L|$ lies in the interval $\left[0, \sqrt{-\frac{m_e k^2}{2E}}\right]$. If $|L| > 0$, then the orbit is an ellipse, with $|L| = \sqrt{-\frac{m_e k^2}{2E}}$ being the special case of a circle. All elliptical orbits with energy E have period $\frac{2\pi}{\Lambda}$, where $\Lambda = \sqrt{-\frac{8E^3}{m_e k^2}}$. If $|L| = 0$, however, then $e = 1$ and the orbit is a line segment. Physically, this represents the case where the electron begins from rest and collapses on a straight-line trajectory into the proton. Such motion is not periodic, which implies that the level set $H^{-1}(E)$ contains orbits of ξ_H that are not closed. Further, the collapse occurs in finite time, meaning that the vector field ξ_H is not complete.

The objective of this chapter is to determine the quantized energy levels for (M, ω, H). In the next section, we construct a metaplectic-c prequantization for (M, ω), and formulate our approach for performing the quantized energy calculation.

5.2.2 Metaplectic-c Prequantization of (M, ω)

We choose the model symplectic vector space V_3, as described in Section 5.1.1. The tangent bundle TM can be identified with $M \times V_3$ with respect to the global trivialization

$$\widehat{v}_j \mapsto \left.\frac{\partial}{\partial q_j}\right|_m, \quad \widehat{w}_j \mapsto \left.\frac{\partial}{\partial p_j}\right|_m, \quad \forall m \in M,\ j = 1, 2, 3.$$

This yields an identification of the symplectic frame bundle $\mathrm{Sp}(M, \omega)$ with $M \times \mathrm{Sp}(V_3)$.

Let $P = M \times \mathrm{Mp}^c(V_3)$, with bundle projection map $P \xrightarrow{\Pi} M$. Define the map $\Sigma : P \to \mathrm{Sp}(M, \omega)$ by

$$\Sigma(m, a) = (m, \sigma(a)), \quad \forall m \in M, \forall a \in \mathrm{Mp}^c(V_3),$$

where the right-hand side is written with respect to the global trivialization above. Let

$$\beta = \sum_{j=1}^{3} (q_j dp_j + d(q_j p_j)) \tag{5.2.2}$$

on M, so that $d\beta = \omega$. The reason for this choice of β will be made clear in Section 5.3.1. Let γ be the $\mathfrak{u}(1)$-valued one-form on P given by

$$\gamma = \frac{1}{i\hbar}\Pi^*\beta + \frac{1}{2}\eta_*\vartheta_0,$$

where ϑ_0 is the trivial connection on the product bundle. Then (P, Σ, γ) is a metaplectic-c prequantization for (M, ω), and it is unique up to isomorphism.

The quantized energy levels of (M, ω, H) are those regular values E of H such that the holonomy of γ_S is trivial over all closed orbits of $\widetilde{\xi}_H$ on $\mathrm{Sp}(TS/TS^\perp)$, where $S = H^{-1}(E)$. If $E \geq 0$, then the quantization condition can be evaluated immediately. From equation 5.2.1 for the eccentricity of the orbit, we see that if $E = 0$, then the orbits are parabolas in position space, and if $E > 0$, then they are hyperbolas. In these cases, ξ_H has no closed orbits in S, which implies that $\widetilde{\xi}_H$ cannot have any closed orbits in $\mathrm{Sp}(TS/TS^\perp)$. Therefore the holonomy condition is satisfied vacuously, and all nonnegative energy levels are quantized energy levels. This is consistent with the physical prediction from quantum mechanics: a particle that is not spatially confined has a continuous energy spectrum.

It remains to consider the orbits corresponding to negative energy. Let

$$N = \{m \in M : H(m) < 0\}.$$

Then N is an open, simply connected submanifold of M. By restriction, the symplectic form on M induces one on N, and the metaplectic-c prequantization for M induces one for N. We use the same symbols to denote the restricted objects: (N, ω) is a symplectic manifold, and

(P, Σ, γ) is its metaplectic-c prequantization. Since N is simply connected, (P, Σ, γ) is unique up to isomorphism.

Let $E < 0$ be fixed, and let $S = H^{-1}(E) \subset N$. Through the process described in Section 4.2.2, we obtain the three-level structures

$$(P^S, \gamma^S) \to \mathrm{Sp}(N, \omega; S) \to S$$

and

$$(P_S, \gamma_S) \to \mathrm{Sp}(TS/TS^\perp) \to S.$$

Lift ξ_H to $\widetilde{\xi}_H$ on $\mathrm{Sp}(N, \omega; S)$, and let it descend to $\mathrm{Sp}(TS/TS^\perp)$.

To evaluate the quantization condition using these bundles, we would have to determine the closed orbits of $\widetilde{\xi}_H$ on $\mathrm{Sp}(TS/TS^\perp)$, then lift $\widetilde{\xi}_H$ to $\widehat{\xi}_H$ on P_S, horizontally with respect to γ_S, and ensure that every lift of a closed orbit in $\mathrm{Sp}(TS/TS^\perp)$ is closed in P_S. However, this procedure is computationally prohibitive in all but the special case of the circular orbit. Instead, we will use Ligon-Schaaf regularization to transform the quantized energy calculation on (N, ω) into one on an open submanifold of TS^3. This is the subject of Section 5.3.

5.3 Ligon-Schaaf Regularization

Let T^+S^3 represent the result of removing the zero section from TS^3. In Section 5.2.1, we noted that the vector field ξ_H is not complete. The Ligon-Schaaf map is a symplectomorphism from (N, ω) to an open submanifold of T^+S^3, having the property that ξ_H is mapped to a vector field that is complete on T^+S^3. This map was presented in [13], and an in-depth discussion of it can be found in [3].

In this section, we will state the Ligon-Schaaf map and list its relevant properties. Then we will construct a metaplectic-c prequantization for TS^3, and show how to lift the Ligon-Schaaf map to the level of symplectic frame bundles, and of metaplectic-c prequantizations. Using the lifted maps, we will be able to relate the quantized energy levels of the hydrogen atom to those of a free particle on S^3.

Our conventions and notation largely follow those in [3]. We state results without proof;

much more detail can be found in [3].

5.3.1 TS^3 and the Ligon-Schaaf Map

Consider $T\mathbb{R}^4 = \mathbb{R}^4 \times \mathbb{R}^4$ with Cartesian coordinates (x, y), where $x = (x_1, x_2, x_3, x_4)$ and $y = (y_1, y_2, y_3, y_4)$. Let $T\mathbb{R}^4$ have symplectic form

$$\omega_4 = \sum_{j=1}^{4} dx_j \wedge dy_j,$$

and let

$$\beta_4 = -\sum_{j=1}^{4} y_j dx_j \tag{5.3.1}$$

on $T\mathbb{R}^4$, so that $d\beta_4 = \omega_4$. The submanifold TS^3 is given by

$$TS^3 = \left\{(x, y) \in T\mathbb{R}^4 : |x|^2 = 1, x \cdot y = 0\right\} \subset T\mathbb{R}^4,$$

where we take the usual Euclidean inner product on $\mathbb{R}^4$. From now on, we abbreviate $T\mathbb{R}^4$ by Z.

By treating (TS^3, ω_3) as a level set of the constraint functions $c_1(x, y) = \frac{1}{2}(|x|^2 - 1)$ and $c_2(x, y) = x \cdot y$ on (Z, ω_4), it can be shown that the restriction of ω_4 to TS^3 yields a symplectic form on TS^3. Let (TS^3, ω_3) be the resulting symplectic manifold. Let β_3 be the restriction of β_4 to TS^3, so that $d\beta_3 = \omega_3$.

Let T^+S^3 represent TS^3 with the zero section removed:

$$T^+S^3 = \left\{(x, y) \in TS^3 : |y| > 0\right\}.$$

Define the map $D : T^+S^3 \to \mathbb{R}$ by

$$D(x, y) = -\frac{m_e k^2}{2|y|^2}, \quad \forall (x, y) \in T^+S^3.$$

As shown in [3], the Hamiltonian vector field for D on T^+S^3 is

$$\xi_D = \sum_{j=1}^{4} \left(\frac{m_e k^2}{|y|^4} y_j \frac{\partial}{\partial x_j} - \frac{m_e k^2}{|y|^2} x_j \frac{\partial}{\partial y_j} \right),$$

and its flow is

$$\psi_D^t(x,y) = \begin{pmatrix} \cos \frac{m_e k^2}{|y|^3} t & \frac{1}{|y|} \sin \frac{m_e k^2}{|y|^3} t \\ -|y| \sin \frac{m_e k^2}{|y|^3} t & \cos \frac{m_e k^2}{|y|^3} t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The map D is called the Delaunay Hamiltonian, and ξ_D is the Delaunay vector field. Since the orbits of ξ_D must preserve $|y|$, it is clear from the form of ψ_D^t that ξ_D is complete on T^+S^3, and every orbit is closed.

Now let S_n^3 represent S^3 with the north pole $(0,0,0,1)$ removed. The Ligon-Schaaf map $LS : N \to T^+S_n^3$ is given by

$$LS(q,p) = (A \sin\varphi + B\cos\varphi, -\nu A \cos\varphi + \nu B \sin\varphi),$$

where

$$\nu = \sqrt{-\frac{m_e k^2}{2H(q,p)}}, \qquad \varphi = \frac{1}{\nu}(q\cdot p),$$
$$A = \left(\frac{q}{|q|} - \frac{1}{m_e k}(q\cdot p)p, \frac{1}{\nu}(q\cdot p) \right), \quad B = \left(\frac{1}{\nu}|q|p, \frac{1}{m_e k}|p|^2|q| - 1 \right).$$

This map has the following properties:

(1) LS is a diffeomorphism between N and $T^+S_n^3$;

(2) $LS^*\beta_3 = \sum_{j=1}^3 (q_j dp_j + d(q_j p_j)) = \beta$ (recall equation 5.2.2);

(3) $H = D \circ LS$.

Assertions (1) and (2) imply that LS is a symplectomorphism; from that and (3), it follows that $LS_*\xi_H = \xi_D$.

5.3.2 Metaplectic-c Prequantization for TS^3

We begin by constructing a metaplectic-c prequantization for $(Z = T\mathbb{R}^4, \omega_4)$, and let it induce a metaplectic-c prequantization for (TS^3, ω_3). For (Z, ω_4), we proceed in precisely the same way as we constructed (P, Σ, γ) for (M, ω) in Section 5.2.2. Choose the model vector space V_4, and identify the symplectic frame bundle $\mathrm{Sp}(Z, \omega_4)$ with $Z \times \mathrm{Sp}(V_4)$ using the global trivialization for the tangent bundle given by

$$\widehat{v}_j \mapsto \left.\frac{\partial}{\partial x_j}\right|_{(x,y)}, \ \widehat{w}_j \mapsto \left.\frac{\partial}{\partial y_j}\right|_{(x,y)}, \quad \forall (x, y) \in Z,\ j = 1, \ldots, 4.$$

Let $Q_4 = Z \times \mathrm{Mp}^c(V_4)$ with bundle projection map $Q_4 \xrightarrow{\Pi_4} Z$. Define the map $\Gamma_4 : Q_4 \to \mathrm{Sp}(T\mathbb{R}^4, \omega_4)$ by

$$\Gamma_4(x, y, a) = (x, y, \sigma(a)), \quad \forall (x, y) \in Z, \forall a \in \mathrm{Mp}^c(V_4).$$

Lastly, define the $\mathfrak{u}(1)$-valued one-form δ_4 on Q_4 by

$$\delta_4 = \frac{1}{i\hbar}\Pi_4^*\beta_4 + \frac{1}{2}\eta_*\vartheta_0,$$

where ϑ_0 is the trivial connection on Q_4, and where β_4 was defined in equation 5.3.1. Then $(Q_4, \Gamma_4, \delta_4)$ is the unique metaplectic-c prequantization for (Z, ω_4) up to isomorphism.

In order to construct the metaplectic-c prequantization of (TS^3, ω_3) induced by $(Q_4, \Gamma_4, \delta_4)$, we proceed by the process of symplectic reduction. Let $R : Z \to \mathbb{R}$ be given by

$$R(x, y) = \frac{1}{2}|x|^2, \quad \forall (x, y) \in Z.$$

The Hamiltonian vector field for R on Z is

$$\xi_R = -\sum_{j=1}^{4} x_j \frac{\partial}{\partial y_j},$$

and its flow on Z is

$$\psi_R^t(x, y) = (x, y - tx), \quad \forall (x, y) \in Z.$$

It is straightforward to show that TS^3 can be identified with the space of orbits of ξ_R on the level set $A = R^{-1}(\frac{1}{2})$. The orbit projection map $p : A \to TS^3$ is

$$p(x, y) = (x, y - (x \cdot y)x), \quad \forall (x, y) \in A.$$

Let $i : A \to Z$ be the inclusion map. Recall that ω_3 is the symplectic form on TS^3 obtained by restricting ω_4. A calculation shows that $p^*\omega_3 = i^*\omega_4$. Therefore (TS^3, ω_3) is the symplectic reduction of (Z, ω_4) at the level set corresponding to $R = \frac{1}{2}$.

Choose the model subspace W_4 as described in Section 5.1.1, and construct the three-level structure

$$(Q_{4A}, \delta_{4A}) \xrightarrow{\Gamma_{4A}} \mathrm{Sp}(TA/TA^\perp) \to A.$$

Then $\mathrm{Sp}(TA/TA^\perp)$ is a principal $\mathrm{Sp}(V_3)$ bundle over A, and Q_{4A} is a principal $\mathrm{Mp}^c(V_3)$ bundle over A.

Let the symplectic frame bundle $\mathrm{Sp}(TS^3, \omega_3)$ be modeled on V_3, so that $\mathrm{Sp}(TS^3, \omega_3)$ is a principal $\mathrm{Sp}(V_3)$ bundle over TS^3. Then $\mathrm{Sp}(TA/TA^\perp)$ and $\mathrm{Sp}(TS^3, \omega_3)$ have isomorphic fibers. If we were to complete the symplectic reduction process, we would identify $\mathrm{Sp}(TS^3, \omega_3)$ with a quotient of $\mathrm{Sp}(TA/TA^\perp)$. However, since $TS^3 \subset A$, it is more convenient to identify $\mathrm{Sp}(TS^3, \omega_3)$ with a subbundle of $\mathrm{Sp}(TA/TA^\perp)$.

For each $z \in A$, $p_*|_z$ appears in the short exact sequence

$$0 \to T_zA^\perp \to T_zA \xrightarrow{p_*|_z} T_{p(z)}TS^3 \to 0.$$

Note that the projection map p is the identity on $TS^3 \subset A$. It follows that for all $z \in TS^3$, $p_*|_z : T_zA/T_zA^\perp \to T_zTS^3$ is a symplectic isomorphism. Therefore $\mathrm{Sp}(TS^3, \omega_3)$ and $\mathrm{Sp}(TA/TA^\perp)|_{TS^3}$ are isomorphic as principal $\mathrm{Sp}(V_3)$ bundles over TS^3. Using that isomorphism, we view $\mathrm{Sp}(TS^3, \omega_3)$ as a subbundle of $\mathrm{Sp}(TA/TA^\perp)$.

Let (Q, Γ, δ) be the result of restricting $(Q_{4A}, \Gamma_{4A}, \delta_{4A})$ to $\mathrm{Sp}(TS^3, \omega_3) \subset \mathrm{Sp}(TA/TA^\perp)$. Then (Q, Γ, δ) is a metaplectic-c prequantization for (TS^3, ω_3), and it is unique up to isomorphism. We will also use the notation (Q, Γ, δ) to denote the metaplectic-c prequantizations for T^+S^3 and $T^+S^3_n$ obtained by restriction.

5.3.3 Lifting the Ligon-Schaaf Map

Recall that the Ligon-Schaaf map $LS : N \to T^+S^3_n$ is a symplectomorphism. Define the map $\widetilde{LS} : \mathrm{Sp}(N, \omega) \to \mathrm{Sp}(T^+S^3_n, \omega_3)$ by

$$\widetilde{LS}(b) = LS_* \circ b, \quad \forall b \in \mathrm{Sp}(N, \omega),$$

and observe that this is an isomorphism of principal $\mathrm{Sp}(V_3)$ bundles.

In Section 5.2.2, we used the global trivialization

$$\widehat{v}_j \mapsto \left.\frac{\partial}{\partial q_j}\right|_m, \quad \widehat{w}_j \mapsto \left.\frac{\partial}{\partial p_j}\right|_m, \quad \forall m \in N, \ j = 1, 2, 3,$$

to identify $\mathrm{Sp}(N, \omega)$ with $N \times \mathrm{Sp}(V_3)$. From the map $\widetilde{LS}$, we see that $\mathrm{Sp}(T^+S^3_n, \omega_3)$ can be identified with $T^+S^3_n \times \mathrm{Sp}(V_3)$ with respect to the global trivialization

$$\widehat{v}_j \mapsto LS_* \left.\frac{\partial}{\partial q_j}\right|_{LS(m)}, \quad \widehat{w}_j \mapsto LS_* \left.\frac{\partial}{\partial p_j}\right|_{LS(m)}, \quad \forall LS(m) \in T^+S^3_n, \ j = 1, 2, 3.$$

In terms of these trivializations for $\mathrm{Sp}(N, \omega)$ and $\mathrm{Sp}(T^+S^3_n, \omega_3)$, $\widetilde{LS}$ is given simply by

$$\widetilde{LS}(m, g) = (LS(m), g), \quad \forall m \in N, \forall g \in \mathrm{Sp}(V_3). \tag{5.3.2}$$

As we have seen before, once we have a global trivialization for the symplectic frame bundle, it is straightforward to construct a metaplectic-c prequantization. Let $Q' = T^+S^3_n \times \mathrm{Mp}^c(V_3)$ with bundle projection map $Q' \xrightarrow{\Pi'} T^+S^3_n$. Define the map $\Gamma' : T^+S^3_n \to \mathrm{Sp}(T^+S^3_n, \omega_3)$ by

$$\Gamma'(z, a) = (z, \sigma(a)), \quad \forall z \in T^+S^3_n, \forall a \in \mathrm{Mp}^c(V_3).$$

Let

$$\delta' = \frac{1}{i\hbar}\Pi'^*\beta_3 + \frac{1}{2}\eta_*\vartheta_0,$$

where β_3 is the restriction to $T^+S^3_n$ of the one-form defined in equation 5.3.1, and where ϑ_0 is the trivial connection on Q'. Then (Q', Γ', δ') is a metaplectic-c prequantization for $(T^+S^3_n, \omega_3)$.

Recall that $P = M \times \mathrm{Mp}^c(V_3)$. Define $\widehat{LS} : P \to Q'$ by

$$\widehat{LS}(m, a) = (LS(m), a).$$

This is clearly an isomorphism of principal $\mathrm{Mp}^c(V_3)$ bundles. Since $LS^*\beta_3 = \beta$, we have $LS^*\delta' = \gamma$. Lastly, it follows from equation 5.3.2 that $\Gamma' \circ \widehat{LS} = \widetilde{LS} \circ \Sigma$. Therefore $\widehat{LS}$ is an isomorphism of metaplectic-c prequantizations.

All of these observations combine to yield the following commutative diagram.

$$\begin{array}{ccc}
(P, \gamma) & \xrightarrow{\widehat{LS}} & (Q', \delta') \\
\downarrow{\scriptstyle \Sigma} & & \downarrow{\scriptstyle \Gamma'} \\
\mathrm{Sp}(N, \omega) & \xrightarrow{\widetilde{LS}} & \mathrm{Sp}(T^+S^3_n, \omega_3) \\
\downarrow & & \downarrow \\
(N, \omega) & \xrightarrow{LS} & (T^+S^3_n, \omega_3)
\end{array}$$

Each of the maps LS, $\widetilde{LS}$, and $\widehat{LS}$ is an isomorphism. From these isomorphisms and the fact that $D \circ LS = H$, it follows that $E < 0$ is a quantized energy level of (N, ω, H) if and only if it is a quantized energy level of $(T^+S^3_n, \omega_3, D)$.

Recall that we constructed the metaplectic-c prequantization (Q, Γ, δ) for $(T^+S^3_n, \omega_3)$ in the previous section. Since $T^+S^3_n$ is simply connected, (Q', Γ', δ') must be isomorphic to (Q, Γ, δ). Therefore we can calculate the quantized energy levels for the system $(T^+S^3_n, \omega_3, D)$ using (Q, Γ, δ).

In fact, we claim that $E < 0$ is a quantized energy level of (N, ω, H) if and only if it is a quantized energy level of (T^+S^3, ω_3, D). When we replace the north pole in S^3, we acquire more closed orbits: namely, those with x-components that pass through $(0, 0, 0, 1)$. However, due to the rotational symmetry of the system (T^+S^3, ω_3, D) and the one-form β_3, an orbit that passes through $(0, 0, 0, 1)$ can always be transformed into one that does not, without altering the holonomy condition. Thus the quantized energy levels of $(T^+S^3_n, \omega_3, D)$ and (T^+S^3, ω_3, D) are identical.

5.3.4 Rescaling the Delaunay Hamiltonian

So far, we have shown that the negative quantized energy levels of the hydrogen atom are the same as the quantized energy levels of the Delaunay Hamiltonian on T^+S^3. In this section, we will make one final transformation that relates these energies to the quantized energy levels of a free particle on S^3.

Let $K : TS^3 \to \mathbb{R}$ be given by

$$K(x, y) = \frac{1}{2}|y|^2, \quad \forall (x, y) \in TS^3.$$

In the context of classical mechanics, this energy function describes a free particle on S^3. It is shown in [3] that the corresponding Hamiltonian vector field on TS^3 is

$$\xi_K = \sum_{j=1}^{4} \left(y_j \frac{\partial}{\partial x_j} - |y|^2 x_j \frac{\partial}{\partial y_j} \right),$$

and the flow of this vector field is

$$\psi_K^t(x, y) = \begin{pmatrix} \cos|y|t & \frac{1}{|y|}\sin|y|t \\ -|y|\sin|y|t & \cos|y|t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \forall (x, y) \in TS^3 \text{ such that } |y| > 0. \qquad (5.3.3)$$

If $|y| = 0$, then the particle is stationary, and the flow is simply $\psi_K^t(x, 0) = (x, 0)$. Clearly $K = 0$ on the zero section of TS^3, and $K > 0$ on T^+S^3.

Recall that the range of the Delaunay Hamiltonian D is $\mathbb{R}_{<0}$. Let $f : \mathbb{R}_{>0} \to \mathbb{R}_{<0}$ be given by

$$f(z) = -\frac{m_e k^2}{4z}, \quad \forall z \in \mathbb{R}_{>0}. \qquad (5.3.4)$$

Note that f is a diffeomorphism, and $D = f \circ K$ on T^+S^3. Using the dynamical invariance property of quantized energy levels, we see that E is a quantized energy level of (T^+S^3, ω_3, D) if and only if $f^{-1}(E)$ is a positive quantized energy level of (TS^3, ω_3, K). It remains to find the positive quantized energy levels for a free particle on S^3.

5.4 Quantization of a Free Particle on S^3

5.4.1 Orbits of ξ_K and Local Coordinates for TS^3

Let $\mathcal{E} > 0$ be arbitrary, and let $\mathcal{S} = K^{-1}(\mathcal{E}) \subset TS^3$. From the flow of ξ_K in equation 5.3.3, it is apparent that all orbits of ξ_K in $\mathcal{S}$ are closed, with period $\frac{2\pi}{\sqrt{2\mathcal{E}}}$. Let $z_0 = (x_0, y_0) \in \mathcal{S}$ be an arbitrary initial point, and let $\mathcal{C} \subset \mathcal{S}$ be the the orbit of ξ_K through z_0:

$$\mathcal{C} = \left\{\psi_K^t(z_0) : t \in \mathbb{R}\right\} \subset TS^3,$$

We can use the rotational symmetry of the system (TS^3, ω_3, K) to make some simplifying assumptions about $\mathcal{C}$. If we view x_0 and y_0 as two perpendicular vectors in $\mathbb{R}^4$, then there is some rotation about the origin in $\mathbb{R}^4$ that carries them both to the x_3x_4-plane. By performing this rotation in x-space and y-space, we can assume without loss of generality that x_0 and y_0 take the form $x_0 = (0, 0, x_{30}, x_{40})$ and $y_0 = (0, 0, y_{30}, y_{40})$, where $|x|^2 = x_{30}^2 + x_{40}^2 = 1$, $x \cdot y = x_{30}y_{30} + x_{40}y_{40} = 0$, and $|y|^2 = y_{30}^2 + y_{40}^2 = 2\mathcal{E}$. The orbit $\mathcal{C}$ then lies in $x_3x_4y_3y_4$-space.

Thus far, we have treated TS^3 as a submanifold of $Z = T\mathbb{R}^4$, using the coordinates (x, y) on Z to describe points in TS^3. Now we make a local change of coordinates on Z that will yield symplectic coordinates for TS^3 on a neighborhood that contains $\mathcal{C}$. Specifically, we introduce 4-dimensional spherical coordinates and their conjugate momenta. Let $U \subset Z$ be the open set

$$U = \left\{(x, y) \in T\mathbb{R}^4 : x_3^2 + x_4^2 > 0\right\}.$$

On U, let the new spatial coordinates be (a, b, c, r), where

$$\begin{aligned} a &= \arctan \frac{\sqrt{x_2^2 + x_3^2 + x_4^2}}{x_1}, \\ b &= \arctan \frac{\sqrt{x_3^2 + x_4^2}}{x_2}, \\ c &= \arctan \frac{x_4}{x_3}, \\ r &= \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}. \end{aligned}$$

The angles a and b are defined modulo π, and the angle c is defined modulo 2π. Let $\rho_1 =$

$\sqrt{x_2^2+x_3^2+x_4^2}$ and $\rho_2 = \sqrt{x_3^2+x_4^2}$. The conjugate momenta corresponding to the spherical coordinates are

$$\begin{aligned}
p_a =& \frac{x_1}{\rho_1}(x_2y_2+x_3y_3+x_4y_4)-\rho_1y_1,\\
p_b =& \frac{x_2}{\rho_2}(x_3y_3+x_4y_4)-\rho_2y_2,\\
p_c =& x_3y_4-x_4y_3,\\
p_r =& \frac{1}{r}(x_1y_1+x_2y_2+x_3y_3+x_4y_4).
\end{aligned}$$

Later, we will need the inverse transformations, which are

$$\begin{aligned}
x_1 &= r\cos a,\\
x_2 &= r\sin a\cos b,\\
x_3 &= r\sin a\sin b\cos c,\\
x_4 &= r\sin a\sin b\sin c,
\end{aligned} \tag{5.4.1}$$

$$\begin{aligned}
y_1 &= p_r\cos a-\frac{p_a}{r}\sin a,\\
y_2 &= p_r\sin a\cos b+\frac{p_a}{r}\cos a\cos b-\frac{p_b}{r\sin a}\sin b\\
y_3 &= p_r\sin a\sin b\cos c+\frac{p_a}{r}\cos a\sin b\cos c+\frac{p_b}{r\sin a}\cos b\cos c-\frac{p_c}{r\sin a\sin b}\sin c,\\
y_4 &= p_r\sin a\sin b\sin c+\frac{p_a}{r}\cos a\sin b\cos c+\frac{p_b}{r\sin a}\cos b\sin c+\frac{p_c}{r\sin a\sin b}\cos c.
\end{aligned} \tag{5.4.2}$$

For convenience, we let a_j, $j=1,\ldots,4$, range over a,b,c,r.

On U, one can verify that

$$\beta_4 = -\sum_{j=1}^{4} y_j dx_j = -\sum_{j=1}^{4} p_{a_j} da_j, \tag{5.4.3}$$

and so

$$\omega_4 = \sum_{j=1}^{4} dx_j\wedge dy_j = \sum_{j=1}^{4} da_j\wedge dp_j.$$

The submanifold TS^3 is characterized by the constant values $r=1$ and $p_r=0$, which implies that the restrictions to $TS^3 \cap U$ of β_4 and ω_4 are

$$\beta_3 = -\sum_{j=1}^{3} p_{a_j} da_j, \quad \omega_3 = \sum_{j=1}^{3} da_j \wedge dp_{a_j}.$$

Thus (a, b, c, p_a, p_b, p_c) are symplectic coordinates for $TS^3 \cap U$. On this neighborhood, the map K takes the form

$$K = \frac{1}{2}\left(p_a^2 + \frac{p_b^2}{\sin^2 a} + \frac{p_c^2}{\sin^2 a \sin^2 b}\right).$$

Several times now, once we had a set of symplectic coordinates such as $(a, b, c, r, p_a, p_b, p_c, p_r)$, we used the trivialization of the symplectic frame bundle given by the coordinate vector fields to construct a metaplectic-c prequantization. We could apply this same procedure to U; however, since U is not simply connected, it is not necessarily the case that the metaplectic-c prequantization so constructed would be isomorphic to the result of restricting $(Q_4, \Gamma_4, \delta_4)$ to U. Instead, we must show how the local change of variables from Cartesian to spherical coordinates can be lifted from Z to Q_4. This is the subject of the next section.

5.4.2 Change of variables over $\mathcal{C}$

Recall that the integral curve of ξ_K through the initial point $z_0 = (x_0, y_0)$ is

$$\psi_K^t(x_0, y_0) = \begin{pmatrix} \cos\sqrt{2\mathcal{E}}t & \frac{1}{\sqrt{2\mathcal{E}}}\sin\sqrt{2E}t \\ -\sqrt{2\mathcal{E}}\sin\sqrt{2\mathcal{E}}t & \cos\sqrt{2\mathcal{E}}t \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix},$$

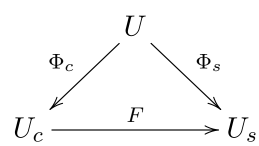
and its image $\mathcal{C}$ is a closed curve lying in $x_3x_4y_3y_4$-space. Since $\mathcal{C} \subset U$, the points in $\mathcal{C}$ can be rewritten in spherical coordinates. Upon converting, we find that any point in $\mathcal{C}$ satisfies $p_a = p_b = 0$ and $a = b = \frac{\pi}{2}$. Further, p_c is a constant value over $\mathcal{C}$ satisfying $p_c^2 = 2\mathcal{E}$. Since $\mathcal{C} \subset TS^3$, we also have $r = 1$ and $p_r = 0$. Therefore, in spherical coordinates, the orbit takes the form

$$\mathcal{C} = \left\{\left(\frac{\pi}{2}, \frac{\pi}{2}, c, 1, 0, 0, p_c, 0\right) : c \in \mathbb{R}/2\pi\mathbb{Z}\right\}.$$

Let $z(c)$ represent the point $\left(\frac{\pi}{2}, \frac{\pi}{2}, c, 1, 0, 0, p_c\right) \in \mathcal{C}$.

The change of coordinates from Cartesian to spherical must now be lifted to the symplectic frame bundle and the metaplectic-c prequantization for (Z, ω_4). The change of coordinates on $\mathrm{Sp}(Z, \omega_4)$ will take place over U, and that on Q_4 will take place over $\mathcal{C}$. We follow the same steps that we applied to the harmonic oscillator in Section 4.4.2.

On the neighborhood U, we have two different coordinate maps: the Cartesian map $\Phi_c : U \to \mathbb{R}^8$, and the spherical map $\Phi_s : U \to \mathbb{R}^4 \times (\mathbb{R}/\pi\mathbb{Z})^2 \times \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}$. The change of variables on U is simply the transition map $F = \Phi_s \circ \Phi_c^{-1}$. Let $\Phi_c(U) = U_c$ and $\Phi_s(U) = U_s$. Then each of the following maps is a diffeomorphism.



Let b_c be the section of $\mathrm{Sp}(Z, \omega_4)$ over U given by

$$b_c(z) : V_4 \to T_z Z \ \text{ such that } \ \widehat{v}_j \mapsto \left.\frac{\partial}{\partial x_j}\right|_z, \ \widehat{w}_j \mapsto \left.\frac{\partial}{\partial y_j}\right|_z, \quad \forall z \in U, \ j = 1, \ldots, 4.$$

That is, b_c is the section that defines the trivialization of $\mathrm{Sp}(Z, \omega_4)|_U$ with respect to Cartesian coordinates. Let the map $\widetilde{\Phi}_c : \mathrm{Sp}(Z, \omega)|_U \to U_c \times \mathrm{Sp}(V_4)$ be given by

$$\widetilde{\Phi}_c(b_c(z) \cdot g) = (\Phi_c(z), g), \quad \forall z \in U, \ \forall g \in \mathrm{Sp}(V_4).$$

Similarly, let b_s be the section of $\mathrm{Sp}(Z, \omega_4)|_U$ given by

$$b_s(z) : V_4 \to T_z Z \ \text{ such that } \ \widehat{v}_j \mapsto \left.\frac{\partial}{\partial a_j}\right|_z, \ \widehat{w}_j \mapsto \left.\frac{\partial}{\partial p_{a_j}}\right|_z, \quad \forall z \in U, \ j = 1, \ldots, 4,$$

and define the map $\widetilde{\Phi}_s : \mathrm{Sp}(Z, \omega)|_U \to U_s \times \mathrm{Sp}(V_4)$ by

$$\widetilde{\Phi}_s(b_s(z) \cdot g) = (\Phi_s(z), g), \quad \forall z \in U, \ \forall g \in \mathrm{Sp}(V_4).$$

To perform the change of coordinates on the level of the symplectic frame bundle, we must lift F to a map $\widetilde{F} : U_c \times \mathrm{Sp}(V_4) \to U_s \times \mathrm{Sp}(V_4)$ in such a way that the following diagram commutes.

$$\begin{array}{ccc} & \mathrm{Sp}(Z,\omega)|_U & \\ {\scriptstyle\widetilde{\Phi}_c}\swarrow & & \searrow{\scriptstyle\widetilde{\Phi}_s} \\ U_c \times \mathrm{Sp}(V_4) & \xrightarrow{\widetilde{F}} & U_s \times \mathrm{Sp}(V_4) \end{array}$$

In equations 5.4.1 and 5.4.2, we gave explicit formulas for x_j and y_j in terms of a_k and p_{a_k}. At each $z \in U$, let $G(z)$ be the 8×8 matrix consisting of the partial derivatives of the Cartesian coordinates with respect to the spherical ones:

$$G(z) = \begin{pmatrix} \left.\frac{\partial x_k}{\partial a_j}\right|_z & \left.\frac{\partial y_k}{\partial a_j}\right|_z \\ \left.\frac{\partial x_k}{\partial p_{a_j}}\right|_z & \left.\frac{\partial y_k}{\partial p_{a_j}}\right|_z \end{pmatrix}_{1\leq j,k\leq 4}.$$

Then

$$G(z) \begin{pmatrix} \left.\frac{\partial}{\partial x_k}\right|_z \\ \left.\frac{\partial}{\partial y_k}\right|_z \end{pmatrix}_{1\leq k\leq 4} = \begin{pmatrix} \left.\frac{\partial}{\partial a_j}\right|_z \\ \left.\frac{\partial}{\partial p_{a_j}}\right|_z \end{pmatrix}_{1\leq j\leq 4}, \quad \forall z \in U.$$

Since the Cartesian and spherical coordinate vectors are both symplectic bases for T_zZ, $G(z)$ is a symplectic matrix and can be treated as an element of $\mathrm{Sp}(V_4)$. By an identical argument to that in Section 4.4.2, the desired map $\widetilde{F}$ is given by

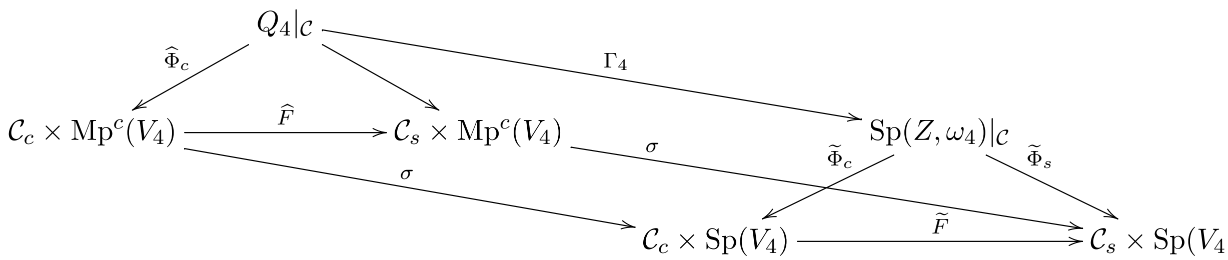
$$\widetilde{F}(\Phi_c(z), g) = (\Phi_s(z), G(z)g), \quad \forall z \in U, \ \forall g \in \mathrm{Sp}(V_4).$$

The final lift to the metaplectic-c prequantization will take place over $\mathcal{C}$. Let $\mathcal{C}_c$ and $\mathcal{C}_s$ be the images of $\mathcal{C}$ under Φ_c and Φ_s, respectively. Recall that points in $\mathcal{C}$ are denoted by $z(c)$ with $c \in \mathbb{R}/2\pi\mathbb{Z}$. We write $G(c)$ for $G(z(c))$. The components of $G(c)$ are single-valued with respect to c, so $G(c)$ is a closed path through $\mathrm{Sp}(V_4)$.

It is clear from the construction of Q_4 how to define a local trivialization with respect to the Cartesian coordinates. Let $\widehat{\Phi}_c : Q_4|_{\mathcal{C}} \to \mathcal{C}_c \times \mathrm{Mp}^c(V_4)$ be given by $\widehat{\Phi}_c(z(c), a) = (\Phi_c(z(c)), a)$ for all $z(c) \in \mathcal{C}$ and $a \in \mathrm{Mp}^c(V_4)$. Then $\widehat{\Phi}_c$ and $\widetilde{\Phi}_c$ make the following diagram commute.

$$\begin{array}{ccc} Q_4|_{\mathcal{C}} & \xrightarrow{\Gamma_4} & \mathrm{Sp}(Z,\omega_4)|_{\mathcal{C}} \\ \downarrow{\scriptstyle\widehat{\Phi}_c} & & \downarrow{\scriptstyle\widetilde{\Phi}_c} \\ \mathcal{C}_c \times \mathrm{Mp}^c(V_4) & \xrightarrow{\sigma} & \mathcal{C}_c \times \mathrm{Sp}(V_4) \end{array}$$

We require a local trivialization of Q_4 with respect to the spherical coordinates that has the analogous relationship to $\widetilde{\Phi}_s$. To find the appropriate map $\widehat{\Phi}_s : Q_4|_{\mathcal{C}} \to \mathcal{C}_s \times \mathrm{Mp}^c(V_4)$, we will construct a map $\widehat{F}$ that satisfies the diagram shown below, then define $\widehat{\Phi}_s = \widehat{F} \circ \widehat{\Phi}_c$.



As before, if $\widehat{F}$ exists, then it has the form

$$\widehat{F}(\Phi_c(z(c)), a) = (\Phi_s(z(c)), \widehat{G}(c)a), \quad \forall z(c) \in \mathcal{C}, \forall a \in \mathrm{Mp}^c(V_4),$$

where $\widehat{G}(c) \in \mathrm{Mp}^c(V_4)$ and $\sigma(\widehat{G}(c)) = G(c)$ for all c. To preserve the form of δ_4, we will lift the path $G(c)$ to a path in $\mathrm{Mp}(V_4)$, then check whether $\widehat{G}(c)$ is single-valued with respect to c.

If $\widehat{G}(c) \in \mathrm{Mp}(V_4)$, then its parameters take the form $(G(c), \mu(c))$, where $\mu(c)^2 \mathrm{Det}_{\mathbb{C}} C_{G(c)} = 1$. To determine $\mu(c)$, we must calculate $G(c)$. Using the expressions in equations 5.4.1 and 5.4.2, we compute the partial derivatives that form the matrix G, and evaluate at the point $z(c) = \left(\frac{\pi}{2}, \frac{\pi}{2}, c, 1, 0, 0, p_c, 0\right) \in \mathcal{C}_s$. Using the abbreviations $S(c) = \sin c$ and $C(c) = \cos c$, the result is

$$G(c) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -S(c) & C(c) & 0 & 0 & -p_c C(c) & -p_c S(c) \\ 0 & 0 & C(c) & S(c) & 0 & 0 & p_c S(c) & -p_c C(c) \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -S(c) & C(c) \\ 0 & 0 & 0 & 0 & 0 & 0 & C(c) & S(c) \end{pmatrix}.$$

Next, we evaluate $C_{G(c)} = \frac{1}{2}(G(c) - JG(c)J)$, using the matrix form for J noted in Section

5.1.1, and convert it to a 4×4 complex matrix. We find that

$$C_{G(c)} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -S(c) - \frac{i}{2}p_cC(c) & C(c) - \frac{i}{2}p_cS(c) \\ 0 & 0 & C(c) + \frac{i}{2}p_cS(c) & S(c) - \frac{i}{2}p_cC(c) \end{pmatrix},$$

which has complex determinant

$$\mathrm{Det}_{\mathbb{C}} C_{G(c)} = -1 - \frac{1}{4}p_c^2.$$

This value is real and constant over $\mathcal{C}$. Thus we can define $\widehat{G}(c)$ to be the element of $\mathrm{Mp}(V_4) \subset \mathrm{Mp}^c(V_4)$ with parameters $\left(G(c), i\left(1 + \frac{1}{4}p_c^2\right)^{-1/2}\right)$, for all $c \in \mathbb{R}/2\pi\mathbb{Z}$, whereupon $\widehat{G}(c)$ is the desired closed path through $\mathrm{Mp}(V_4)$.

Having determined $\widehat{G}(c)$, we now define the map $\widehat{F} : \mathcal{C}_c \times \mathrm{Mp}^c(V_4) \to \mathcal{C}_s \times \mathrm{Mp}^c(V_4)$ by

$$\widehat{F}(\Phi_c(z(c)), a) = (\Phi_s(z(c)), \widehat{G}(c)a) \quad \forall z(c) \in \mathcal{C}, \forall a \in \mathrm{Mp}^c(V_4).$$

The local trivialization of $Q_4|_{\mathcal{C}}$ that is compatible with spherical coordinates comes about by setting $\widehat{\Phi}_s = \widehat{F} \circ \widehat{\Phi}_c$. The one-form δ_4 on Q_4 induces a one-form δ_{4s} on $\mathcal{C}_s \times \mathrm{Mp}^c(V_4)$ that takes the form

$$\delta_{4s} = \frac{1}{i\hbar}\Pi^*\beta_4 + \frac{1}{2}\eta_*\vartheta_0,$$

where β_4 is written in spherical coordinates as in equation 5.4.3, and where ϑ_0 is now the trivial connection on $\mathcal{C}_s \times \mathrm{Mp}^c(V_4)$.

5.4.3 Restrictions to $\mathcal{C} \subset TS^3$

From now on, we no longer write the maps Φ_s, $\widetilde{\Phi}_s$ and $\widehat{\Phi}_s$ explicitly, but write elements of Z, $\mathrm{Sp}(Z, \omega_4)|_U$ and $Q_4|_{\mathcal{C}}$ with respect to spherical coordinates and the local spherical trivializations. Further, we treat TS^3 as a six-dimensional manifold with local coordinates (a, b, c, p_a, p_b, p_c) on $TS^3 \cap U$, and with symplectic form $\omega_3 = \sum_{j=1}^3 da_j \wedge dp_{a_j}$.

Recall from Section 5.3.2 that we defined the map $R(x, y) = \frac{1}{2}|x|^2$ on Z, and set $A = R^{-1}(\frac{1}{2})$.

We argued that the symplectic frame bundle $\mathrm{Sp}(TS^3, \omega_3)$ is isomorphic to $\mathrm{Sp}(TA/TA^{\perp})|_{TS^3}$, and that the metaplectic-c prequantization (Q, Γ, δ) for (TS^3, ω_3) is obtained by restricting $(Q_{4A}, \Gamma_{4A}, \delta_{4A})$ to $\mathrm{Sp}(TA/TA^{\perp})|_{TS^3}$.

In spherical coordinates, we have

$$R(z) = \frac{1}{2}r^2, \quad \forall z \in Z \cap U,$$

which has Hamiltonian vector field

$$\xi_R = r\frac{\partial}{\partial p_r}.$$

It is immediate that at each point $z \in A \cap U$,

$$T_zA = \mathrm{span}\left\{\frac{\partial}{\partial a}, \frac{\partial}{\partial b}, \frac{\partial}{\partial c}, \frac{\partial}{\partial p_a}, \frac{\partial}{\partial p_b}, \frac{\partial}{\partial p_c}, \frac{\partial}{\partial p_r}\right\}\bigg|_z \quad \text{and} \quad T_zA^{\perp} = \mathrm{span}\left\{\frac{\partial}{\partial p_r}\right\}\bigg|_z,$$

which implies that

$$T_zA/T_zA^{\perp} = \mathrm{span}\left\{\left[\frac{\partial}{\partial a}\right], \left[\frac{\partial}{\partial b}\right], \left[\frac{\partial}{\partial c}\right], \left[\frac{\partial}{\partial p_a}\right], \left[\frac{\partial}{\partial p_b}\right], \left[\frac{\partial}{\partial p_c}\right]\right\}\bigg|_z.$$

We see that over $A \cap U$, the local trivialization of $\mathrm{Sp}(Z, \omega_4)|_U$ with respect to spherical coordinates induces a local trivialization

$$\mathrm{Sp}(Z, \omega_4; A)|_{A\cap U} = A \cap U \times \mathrm{Sp}(V_4; W_4).$$

The group homomorphism $\nu : \mathrm{Sp}(V_4; W_4) \to \mathrm{Sp}(V_3)$ then induces a local trivialization

$$\mathrm{Sp}(TA/TA^{\perp})|_{A\cap U} = A \cap U \times \mathrm{Sp}(V_3).$$

Restricting further to $TS^3 \cap U \subset A \cap U$, we find that

$$\mathrm{Sp}(TS^3, \omega_3)|_{TS^3\cap U} = TS^3 \cap U \times \mathrm{Sp}(V_3),$$

with the local trivialization given by

$$\widehat{v}_j \mapsto \left.\frac{\partial}{\partial a_j}\right|_z, \quad \widehat{w}_j \mapsto \left.\frac{\partial}{\partial p_{a_j}}\right|_z, \quad \forall z \in TS^3 \cap U, \ j = 1, 2, 3.$$

On the level of metaplectic-c bundles, we can write

$$Q_4^A|_{\mathcal{C}} = \mathcal{C} \times \mathrm{Mp}^c(V_4; W_4)$$

by restricting $Q_4|_{\mathcal{C}}$ to $\mathrm{Sp}(Z, \omega; A)|_{\mathcal{C}}$. Then the group homomorphism $\hat{\nu} : \mathrm{Mp}^c(V_4; W_4) \to \mathrm{Mp}^c(V_3)$ induces the local trivialization

$$Q_{4A}|_{\mathcal{C}} = Q|_{\mathcal{C}} = \mathcal{C} \times \mathrm{Mp}^c(V_3).$$

The prequantization one-form on Q is

$$\delta = \frac{1}{i\hbar}\Pi^*\beta_3 + \frac{1}{2}\eta_*\vartheta_0,$$

where ϑ_0 is the trivial connection on $\mathcal{C} \times \mathrm{Mp}^c(V_3)$, and where

$$\beta_3 = -\sum_{j=1}^{3} p_j da_j = -p_c dc$$

over $\mathcal{C}$.

5.4.4 Quantized Energy Levels for (TS^3, ω_3, K)

In the local coordinates (a, b, c, p_a, p_b, p_c), the map $K : TS^3 \to \mathbb{R}$ is given by

$$K = \frac{1}{2}\left(p_a^2 + \frac{p_b^2}{\sin^2 a} + \frac{p_c^2}{\sin^2 a \sin^2 b}\right),$$

and its Hamiltonian vector field is

$$\xi_K = p_a \frac{\partial}{\partial a} + \frac{p_b}{\sin^2 a}\frac{\partial}{\partial b} + \frac{p_c}{\sin^2 a \sin^2 b}\frac{\partial}{\partial c} + \left(\frac{p_b^2}{\sin^3 a}\cos a + \frac{p_c^2 \cos a}{\sin^3 a \sin^2 b}\right)\frac{\partial}{\partial p_a} + \frac{p_c^2 \cos b}{\sin^2 a \sin^3 b}\frac{\partial}{\partial p_b}.$$

Recall that we fixed $\mathcal{E} > 0$ and defined $\mathcal{S} = K^{-1}(\mathcal{E})$, and that z_0 is a point in $\mathcal{S}$. The closed curve $\mathcal{C}$ is the orbit of ξ_K through z_0. We will show that the orbit of $\widetilde{\xi}_K$ through $(z_0, I) \in \mathrm{Sp}(T\mathcal{S}/T\mathcal{S}^{\perp})$ is closed, and then we will determine the values of $\mathcal{E}$ for which the orbit of $\widehat{\xi}_K$ through $(z_0, I) \in Q_{\mathcal{S}}$ is also closed.

The lift of the flow ψ_K^t to $\widetilde{\psi}_K^t$ on $\mathrm{Sp}(TS^3, \omega_3)$ is given by

$$\widetilde{\psi}_K^t(z, I) = (z, \psi_{K*}^t|_z), \quad \forall z \in TS^3 \cap U,$$

which implies that the lifted vector field $\widetilde{\xi}_K$ is

$$\widetilde{\xi}_K(z, I) = \left(\xi_K(z), \left.\frac{d}{dt}\right|_{t=0} \psi_{K*}^t|_z \right), \quad \forall z \in TS^3 \cap U.$$

As a 6×6 matrix, $\left.\frac{d}{dt}\right|_{t=0} \psi_{K*}^t|_z$ can be interpreted as an element of the Lie algebra $\mathfrak{sp}(V_3)$, and its components are

$$\left(\left.\frac{d}{dt}\right|_{t=0} \psi_*^t|_z \right)_{jk} = \left.\frac{\partial (\xi_K)_j}{\partial X_k}\right|_{X=z},$$

where X_k ranges over (a, b, c, p_a, p_b, p_c). In particular, when we evaluate this matrix of partial derivatives at $z(c) \in \mathcal{C}$, we obtain

$$\left.\frac{d}{dt}\right|_{t=0} \psi_{K*}^t|_{z(c)} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -p_c^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -p_c^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This value is constant over $\mathcal{C}$; we denote it by κ. Thus

$$\widetilde{\xi}_K(z(c), I) = (\xi_K(z(c)), \kappa), \quad \forall z(c) \in \mathcal{C},$$

which implies that the flow of $\widetilde{\xi}_K$ through $(z_0, I) \in \mathrm{Sp}(TS^3, \omega_3)$ is

$$\widetilde{\psi}^t_K(z_0, I) = (\psi^t_K(z_0), \exp(t\kappa)).$$

Let $\lambda = \sqrt{p_c^2} = \sqrt{2\mathcal{E}}$. A calculation shows that

$$\exp(t\kappa) = \begin{pmatrix} \cos\lambda t & 0 & 0 & \frac{1}{\lambda}\sin\lambda t & 0 & 0 \\ 0 & \cos\lambda t & 0 & 0 & \frac{1}{\lambda}\sin\lambda t & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -\lambda\sin\lambda t & 0 & 0 & \cos\lambda t & 0 & 0 \\ 0 & -\lambda\sin\lambda t & 0 & 0 & \cos\lambda t & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus the orbit through (x_0, I) is closed, with period $\frac{2\pi}{\lambda}$.

Over $\mathcal{C}$, ξ_K reduces to

$$\xi_K = p_c \frac{\partial}{\partial c}.$$

Therefore, for all $z(c) \in \mathcal{C}$, we have

$$T_{z(c)}\mathcal{S}^\perp = \operatorname{span}\left\{\frac{\partial}{\partial c}\right\}\bigg|_{z(c)}, \quad T_{z(c)}\mathcal{S} = \operatorname{span}\left\{\frac{\partial}{\partial a}, \frac{\partial}{\partial b}, \frac{\partial}{\partial c}, \frac{\partial}{\partial p_a}, \frac{\partial}{\partial p_b}\right\}\bigg|_{z(c)},$$

and

$$T_{z(c)}\mathcal{S}/T_{z(c)}\mathcal{S}^\perp = \operatorname{span}\left\{\left[\frac{\partial}{\partial a}\right], \left[\frac{\partial}{\partial b}\right], \left[\frac{\partial}{\partial p_a}\right], \left[\frac{\partial}{\partial p_b}\right]\right\}\bigg|_{z(c)}.$$

We identify $\mathrm{Sp}(T\mathcal{S}/T\mathcal{S}^\perp)|_{\mathcal{C}}$ with $\mathcal{C} \times \mathrm{Sp}(V_2)$ in the obvious way.

Upon descending to $\mathrm{Sp}(T\mathcal{S}/T\mathcal{S}^\perp)|_{\mathcal{C}}$, the induced flow takes the form

$$\widetilde{\psi}^t_K(z_0, I) = (\psi^t_K(z_0), \nu(\exp(t\kappa))),$$

where ν is the group homomorphism $\mathrm{Sp}(V_3;W_3) \xrightarrow{\nu} \mathrm{Sp}(V_2)$. We calculate that

$$\nu(\exp(t\kappa)) = \begin{pmatrix} \cos\lambda t & 0 & \frac{1}{\lambda}\sin\lambda t & 0 \\ 0 & \cos\lambda t & 0 & \frac{1}{\lambda}\sin\lambda t \\ -\lambda\sin\lambda t & 0 & \cos\lambda t & 0 \\ 0 & -\lambda\sin\lambda t & 0 & \cos\lambda t \end{pmatrix}.$$

The corresponding vector field on $\mathrm{Sp}(T\mathcal{S}/T\mathcal{S}^{\perp})|_{\mathcal{C}}$ is

$$\widetilde{\xi}_K(z(c),I) = (\xi_K(z(c)),\overline{\kappa}), \quad \forall z(c)\in\mathcal{C},$$

where

$$\overline{\kappa} = \frac{d}{dt}\bigg|_{t=0}\nu(\exp(t\kappa)) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\lambda^2 & 0 & 0 & 0 \\ 0 & -\lambda^2 & 0 & 0 \end{pmatrix} \in \mathfrak{sp}(V_2).$$

The local trivializations lift to the level of metaplectic-c bundles, so that $Q_{\mathcal{S}}|_{\mathcal{C}} = \mathcal{C}\times\mathrm{Mp}^c(V_2)$. The induced one-form $\delta_{\mathcal{S}}$ on $Q_{\mathcal{S}}$ takes the form

$$\delta_{\mathcal{S}} = -\frac{1}{i\hbar}p_c\,dc + \frac{1}{2}\eta_*\vartheta_0$$

over $\mathcal{C}$. Therefore the horizontal lift of $\widetilde{\xi}_K$ to $Q_{\mathcal{S}}|_{\mathcal{C}}$ is

$$\widehat{\xi}_K(z(c),I) = \left(\xi_K(z(c)),\overline{\kappa}\oplus\frac{\lambda^2}{i\hbar}\right), \quad \forall z(c)\in\mathcal{C}.$$

Since the $\mathfrak{mp}^c(V_2)$ component is constant, the orbit of $\widehat{\xi}_K$ through $(z_0,I)\in Q_{\mathcal{S}}$ is

$$\widehat{\psi}^t_K(z_0,I) = \left(\psi^t_K(z_0),\exp\left(t\overline{\kappa}\oplus\frac{\lambda^2 t}{i\hbar}\right)\right).$$

We can write the $\mathrm{Mp}^c(V_2)$ component as $\exp(t\kappa\oplus 0)e^{\lambda^2 t/i\hbar}$, where $\exp(t\overline{\kappa}\oplus 0)\in\mathrm{Mp}(V_2)\subset\mathrm{Mp}^c(V_2)$ and $e^{\lambda^2 t/i\hbar}\in U(1)\subset\mathrm{Mp}^c(V_2)$.

The parameters of $\exp(t\overline{\kappa}\oplus 0)$ have the form $(\exp t\overline{\kappa},\mu(t))$ where $\mu(t)^2\mathrm{Det}_{\mathbb{C}}C_{\exp(t\overline{\kappa})} = 1$. As a 2×2 complex matrix,

$$C_{\exp(t\overline{\kappa})} = \begin{pmatrix} \cos\lambda t + \frac{i}{2}\left(\lambda + \frac{1}{\lambda}\right)\sin\lambda t & 0 \\ 0 & \cos\lambda t + \frac{i}{2}\left(\lambda + \frac{1}{\lambda}\right)\sin\lambda t \end{pmatrix},$$

which has determinant

$$\mathrm{Det}_{\mathbb{C}} C_{\exp(t\overline{\kappa})} = \left(\cos\lambda t + \frac{i}{2}\left(\lambda + \frac{1}{\lambda}\right)\sin\lambda t\right)^2.$$

This value circles the origin twice as t ranges from 0 to $2\pi/\lambda$. Therefore the parameters of $\exp(t\overline{\kappa} \oplus 0)$ are

$$\left(\exp(t\overline{\kappa}), \left(\cos\lambda t + \frac{i}{2}\left(\lambda + \frac{1}{\lambda}\right)\sin\lambda t\right)^{-1}\right),$$

which describes a closed orbit in $\mathrm{Mp}(V_2)$ with period $2\pi/\lambda$.

Thus the orbit in $Q_{\mathcal{S}}$ is closed if and only if the $U(1)$ term is also a closed orbit with period $2\pi/\lambda$. We require $e^{\lambda^2 t/i\hbar} = 1$ when $t = 2\pi/\lambda$, or equivalently, $\frac{\lambda^2}{i\hbar}\frac{2\pi}{\lambda} = -2\pi i n$ for some $n \in \mathbb{Z}$. Rearranging and recalling that $\lambda^2 = p_c^2 = 2\mathcal{E}$ results in $\mathcal{E} = \frac{1}{2}n^2\hbar^2$ for some $n \in \mathbb{Z}$.

We now apply the argument made in Section 5.3.4. We are only concerned with the strictly positive quantized energies of K, because those correspond to quantized energies of the Delaunay Hamiltonian D on T^+S^3. Thus we can ignore the energy level corresponding to $n = 0$, and we can also dismiss $n < 0$ as redundant. This leaves us with

$$\mathcal{E}_n = \frac{1}{2}n^2\hbar^2, \quad n \in \mathbb{N}.$$

The positive value $\mathcal{E}_n$ is a quantized energy level of K if and only if $f(\mathcal{E}_n)$ is a quantized energy level of D, where the diffeomorphism f is given in equation 5.3.4, and the quantized energy levels of D are precisely the negative quantized energy levels of the hydrogen atom. At last, we find that the negative quantized energy levels of the hydrogen atom are

$$E_n = f(\mathcal{E}_n) = -\frac{m_e k^2}{2n^2\hbar^2}, \quad n \in \mathbb{N},$$

which is exactly the quantum mechanical prediction for our model of the hydrogen atom.

Chapter 6

Generalization of the Quantized Energy Condition

The quantized energy level definition presented in Chapter 4 has a natural generalization from a single function to a family of k Poisson-commuting functions. In this chapter, we will present the generalized definition, and prove the corresponding generalized version of the dynamical invariance theorem. When the family of functions is of maximal size, we will show that the quantization condition reduces to a Bohr-Sommerfeld condition.

A comment concerning notation: in this chapter, superscripts j, l range over elements of the Poisson-commuting family, while subscripts a, c range over components with respect to local coordinates.

6.1 The quantization condition

Let (M, ω) be a $2n$-dimensional manifold, and let $H^1, \ldots, H^k$ be a family of Poisson-commuting functions, where $1 \leq k \leq n$. Then $H = (H^1, \ldots, H^k)$ is a map from M to $\mathbb{R}^k$. Let $E = (E^1, \ldots, E^k)$ be a regular value of H, and let $S = H^{-1}(E)$.

Let the Hamiltonian vector field for the function H^j be denoted by ξ^j. Note that if a regular value E of H exists, then the Hamiltonian vector fields $\xi^1, \ldots, \xi^k$ are linearly independent everywhere on S. Further, for any $s \in S$, $T_sS^{\perp} = \operatorname{span}\left\{\xi^1(s), \ldots, \xi^k(s)\right\} \subset T_sS$, which shows that T_sS is a coisotropic subspace of T_sM.

Let (V, Ω) be a $2n$-dimensional model symplectic vector space, and let $W \subset V$ be a coisotropic subspace of codimension k. Then $W/W^{\perp}$ acquires a symplectic structure from Ω. As in Section 4.2.1, we let $\mathrm{Sp}(V;W)$ be the subgroup of $\mathrm{Sp}(V)$ that preserves W, and $\mathrm{Mp}^c(V;W)$ be the preimage of $\mathrm{Sp}(V;W)$ in $\mathrm{Mp}^c(V)$. Robinson and Rawnsley constructed a lift of the natural group homomorphism $\mathrm{Sp}(V;W) \xrightarrow{\nu} \mathrm{Sp}(W/W^{\perp})$ to the level of metaplectic-c groups for any coisotropic subspace W [15]. Therefore we have the same commuting diagram as we did in Section 4.2.1.

$$\begin{array}{ccc}
\mathrm{Mp}^c(V) \supset \mathrm{Mp}^c(V;W) & \xrightarrow{\hat{\nu}} & \mathrm{Mp}^c(W/W^{\perp}) \\
\downarrow \sigma & & \downarrow \sigma \\
\mathrm{Sp}(V) \supset \mathrm{Sp}(V;W) & \xrightarrow{\nu} & \mathrm{Sp}(W/W^{\perp})
\end{array}$$

Assume that (M, ω) admits a metaplectic-c prequantization (P, Σ, γ). By exactly the same process as in Section 4.2.2, we construct the following three-level structures.

$$\begin{array}{ccccc}
(P, \gamma) & \xleftarrow{\text{incl.}} & (P^S, \gamma^S) & \xrightarrow{\hat{\nu}} & (P_S, \gamma_S) \\
\downarrow \Sigma & & \downarrow & & \downarrow \\
\mathrm{Sp}(M, \omega) & \xleftarrow{\text{incl.}} & \mathrm{Sp}(M, \omega; S) & \xrightarrow{\nu} & \mathrm{Sp}(TS/TS^{\perp}) \\
\downarrow \rho & & \downarrow & & \downarrow \\
(M, \omega) & \xleftarrow{\text{incl.}} & S & \xrightarrow{=} & S
\end{array}$$

The second column is obtained by restriction, and the third column is obtained by taking associated bundles. The Hamiltonian vector fields $\xi^1, \ldots, \xi^k$ lift to $\widetilde{\xi}^1, \ldots, \widetilde{\xi}^k$ on the level of symplectic frame bundles, and $\widehat{\xi}^1, \ldots, \widehat{\xi}^k$ on the level of metaplectic-c bundles.

With these constructions established, we can now state the generalized version of our quantized energy condition.

Definition 6.1.1. Let $E = (E^1, \ldots, E^k) \in \mathbb{R}^k$ be a regular value for the family $H = (H^1, \ldots, H^k)$ of Poisson commuting functions on M. Suppose that the one-form γ_S has trivial holonomy over any closed path $u(t)$ in $\mathrm{Sp}(TS/TS^{\perp})$ such that every tangent vector $\dot{u}(t)$ is in span $\left\{\widetilde{\xi}^1, \ldots, \widetilde{\xi}^k\right\}$. Then the value E is a *quantized energy level* for the system (M, ω, H).

In Section 6.2, we state and prove a generalized version of the dynamical invariance theorem. The proof follows essentially the same steps as in Section 4.3.2, and as such, we omit some of the details of the calculations. Then, in Section 6.3, we examine the special case of a completely integrable system. When $k = n$, we will see that the quantization condition reduces to a Bohr-Sommerfeld condition.

6.2 Generalized dynamical invariance

Let $H = (H^1, \dots, H^k)$ and $L = (L^1, \dots, L^k)$ be two families of linearly independent Poisson commuting functions on M. Suppose $E = (E^1, \dots, E^k)$ and $F = (F^1, \dots, F^k)$ are regular values of H and L, respectively, such that $H^{-1}(E) = L^{-1}(F)$. We will show that the quantization condition is identical for H and for L.

Let $\xi^1, \dots, \xi^k$ be the Hamiltonian vector fields corresponding to $H^1, \dots, H^k$, and let them have flows $\phi^{1t}, \dots, \phi^{kt}$. Similarly, let $\eta^1, \dots, \eta^k$ be the Hamiltonian vector fields corresponding to $L^1, \dots, L^k$, and let these vector fields have flows $\rho^{1t}, \dots, \rho^{kt}$.

Let $S = H^{-1}(E) = L^{-1}(F)$. Then S is a $2n - k$-dimensional submanifold of M. For all $s \in S$, $T_sS^\perp$ is a k-dimensional subspace of T_sS, and

$$T_sS^\perp = \operatorname{span}\left\{\xi^1(s), \dots, \xi^k(s)\right\} = \operatorname{span}\left\{\eta^1(s), \dots, \eta^k(s)\right\}.$$

The above relationship implies that there is a function $C : S \to \mathbb{R}^{k \times k}$ such that

$$\eta^j(s) = \sum_{l=1}^{k} C^{jl}(s)\xi^l(s), \quad \forall s \in S.$$

For any $1 \le j \le k$, $\xi^j(s)$ and $\eta^j(s)$ are elements of T_sS, which is naturally identified with $T_\zeta T_sS$ for any $\zeta \in T_sS$. Since $T_\zeta T_sS \subset T_\zeta TS$, we can think of $\xi^j(s)$ or $\eta^j(s)$ as an element of $T_\zeta TS$. The following lemma is stated in terms of that identification.

Lemma 6.2.1. For all $s \in S$ and all $\zeta \in T_sS$,

$$\left.\frac{d}{dt}\right|_{t=0} (\rho^{jt}_*|_s\zeta) = \sum_{l=1}^{k} \left(C^{jl}(s) \left.\frac{d}{dt}\right|_{t=0} \left(\phi^{lt}_*|_s\zeta\right) + \left(\zeta C^{jl}\right)\xi^l(s) \right),$$

for any $1 \le j \le k$.

Proof. We work on the $2n-k$-dimensional manifold S. Let $U \subset S$ be a coordinate neighborhood for $s \in S$, and let $Z = (Z_1, \ldots, Z_{2n-k})$ be local coordinates on U. With respect to these coordinates, we write $\phi^{jt} = (\phi^{jt}_1, \ldots, \phi^{jt}_{2n-k})$ near s, and likewise for ρ^{jt}.

Since $\eta^j = \sum_{l=1}^k C^{jl}\xi^l$ on S, we have $\eta^j_b = \sum_{l=1}^k C^{jl}\xi^l_b$ on U for each $1 \le b \le 2n-k$, and so

$$\frac{d}{dt}\bigg|_{t=0} \rho^{jt}_b = \sum_{l=1}^k C^{jl} \, \frac{d}{dt}\bigg|_{t=0} \phi^{lt}_b$$

on U.

The pushforward $\phi^{jt}_*|_s$ is a $2n-k \times 2n-k$ matrix with entries given by $\left(\phi^{jt}_*|_s\right)_{ac} = \frac{\partial \phi^{jt}_a}{\partial Z_c}\Big|_{Z=s}$. For any $\zeta \in T_sS$, $\left(\phi^{jt}_*|_s\zeta\right)_a = \zeta\phi^{jt}_a$. The same relationship holds for $\rho^{tj}_*|_s\zeta$. Taking the time derivative yields

$$\frac{d}{dt}\bigg|_{t=0} \left(\rho^{jt}_*|_s\zeta\right)_a = \sum_{l=1}^k \left(C^{jl}(s) \, \frac{d}{dt}\bigg|_{t=0} \left(\phi^{lt}_*|_s\zeta\right)_a + \left(\zeta C^{jl}\right) \xi^l_a(s) \right), \tag{6.2.1}$$

for $1 \le j \le k$ and $1 \le a \le 2n-k$.

Given the coordinates Z on U, we get natural coordinates for $TS|_U$, and thence coordinates for $T_\zeta TS$. In terms of these,

$$\begin{aligned}
\frac{d}{dt}\bigg|_{t=0} \left(\rho^{jt}_*|_s\zeta\right) &= \left(\eta^j_1(s), \ldots, \eta^j_{2n-k}(s), \, \frac{d}{dt}\bigg|_{t=0} \left(\rho^{jt}_*|_s\zeta\right)_1, \ldots, \, \frac{d}{dt}\bigg|_{t=0} \left(\rho^{jt}_*|_s\zeta\right)_{2n-k}\right), \\
\frac{d}{dt}\bigg|_{t=0} \left(\phi^{jt}_*|_s\zeta\right) &= \left(\xi^j_1(s), \ldots, \xi^j_{2n-k}(s), \, \frac{d}{dt}\bigg|_{t=0} \left(\phi^{jt}_*|_s\zeta\right)_1, \ldots, \, \frac{d}{dt}\bigg|_{t=0} \left(\phi^{jt}_*|_s\zeta\right)_{2n-k}\right), \\
\xi^l(s) &= \left(0, \ldots, 0, \xi^l(s)_1, \ldots, \xi^l(s)_{2n-k}\right).
\end{aligned}$$

If we apply Equation (6.2.1) in the first line and compare with the second two lines, recalling that $\eta^j(s) = \sum_{l=1}^k C^{jl}(s)\xi^l(s)$, we conclude that

$$\frac{d}{dt}\bigg|_{t=0} \left(\rho^{jt}_*|_s\zeta\right) = \sum_{l=1}^k \left(C^{jl}(s) \, \frac{d}{dt}\bigg|_{t=0} \left(\phi^{lt}_*|_s\zeta\right) + \left(\zeta C^{jl}\right) \xi^l(s) \right).$$

□

Now we use Lemma 6.2.1 to establish a relationship between the vector fields $\widetilde{\eta}^1, \ldots \widetilde{\eta}^k$ and $\widetilde{\xi}^1, \ldots, \widetilde{\xi}^k$ on $\mathrm{Sp}(TS/TS^\perp)$.

Lemma 6.2.2. For all $s \in S$ and all $b' \in \mathrm{Sp}(TS/TS^\perp)_s$,

$$\widetilde{\eta}^j(b') = \sum_{l=1}^{k} C^{jl}(s)\widetilde{\xi}^l(b'), \quad 1 \le j \le k.$$

Proof. For V, choose a symplectic basis $\widehat{x}_1, \ldots, \widehat{x}_n, \widehat{y}_1, \ldots, \widehat{y}_n$. Let

$$W = \mathrm{span}\left\{\widehat{x}_1, \ldots, \widehat{x}_n, \widehat{y}_{k+1}, \ldots, \widehat{y}_n\right\}.$$

Consequently,

$$W^\perp = \mathrm{span}\left\{\widehat{x}_1, \ldots, \widehat{x}_k\right\}, \quad W/W^\perp = \mathrm{span}\left\{[\widehat{x}_{k+1}], \ldots, [\widehat{x}_n], [\widehat{y}_{k+1}], \ldots, [\widehat{y}_n]\right\}.$$

For convenience, let

$$(\widehat{z}_1, \ldots, \widehat{z}_n) = (\widehat{x}_1, \ldots, \widehat{x}_n, \widehat{y}_{k+1}, \ldots, \widehat{y}_n, \widehat{y}_1, \ldots, \widehat{y}_k)$$

so that

$$W = \mathrm{span}\left\{\widehat{z}_1, \ldots, \widehat{z}_{2n-k}\right\}, \quad W^\perp = \mathrm{span}\left\{\widehat{z}_1, \ldots, \widehat{z}_k\right\}, \quad W/W^\perp = \mathrm{span}\left\{[\widehat{z}_{k+1}], \ldots, [\widehat{z}_{2n-k}]\right\}.$$

For any $s \in S$ and any $b \in \mathrm{Sp}(M,\omega)_s$, identify b with the basis $(\zeta_1, \ldots, \zeta_{2n}) \in (T_sM)^{2n}$ where $b\widehat{z}_c = \zeta_c$ for $1 \le c \le 2n$. Note that $\zeta_c \in T_sS$ for all $1 \le c \le 2n-k$ and $\zeta_c \in T_sS^\perp$ for all $1 \le c \le k$. Similarly, identify $b' \in \mathrm{Sp}(TS/TS^\perp)_s$ with $([\zeta_{k+1}], \ldots, [\zeta_{2n-k}]) \in \left(T_sS/T_sS^\perp\right)^{2n-2k}$, where $b'[\widehat{z}_c] = [\zeta_c]$ for $k+1 \le c \le 2n-k$.

The lifted flow $\widetilde{\rho}^{jt}$ on $\mathrm{Sp}(M,\omega)$ acts on b by

$$\widetilde{\rho}^{jt}(b) = \rho_*^{jt}|_s \circ b = \left(\rho_*^{jt}|_s\zeta_1, \ldots \rho_*^{jt}|_s\zeta_{2n}\right).$$

This element b descends to the element $b' \in \mathrm{Sp}(TS/TS^\perp)$ that is represented by $([\zeta_{k+1}], \ldots, [\zeta_{2n-k}]) \in \left(T_sS/T_sS^\perp\right)^{2n-2k}$. The induced flow $\widetilde{\rho}^{jt}$ on $\mathrm{Sp}(TS/TS^\perp)$ acts on b' by

$$\widetilde{\rho}^{jt}(b') = \left([\rho_*^{jt}|_s\zeta_{k+1}], \ldots, [\rho_*^{jt}|_s\zeta_{2n-k}]\right),$$

which is a path through $(TS/TS^{\perp})^{2n-2k}$. If we take the time derivative, we get

$$\widetilde{\eta}^j(b') = \frac{d}{dt}\bigg|_{t=0} \widetilde{\rho}^{jt}(b') = \left(\frac{d}{dt}\bigg|_{t=0} [\rho_*^{jt}|_s\zeta_{k+1}], \ldots, \frac{d}{dt}\bigg|_{t=0} [\rho_*^{jt}|_s\zeta_{2n-k}] \right)$$
$$\in T_{[\zeta_{k+1}]}(TS/TS^{\perp}) \times \ldots \times T_{[\zeta_{2n-k}]}(TS/TS^{\perp}).$$

The pushforward of the projection map $TS \to TS/TS^{\perp}$, based at ζ_c, is a linear surjection $T_{\zeta_c}TS \to T_{[\zeta_c]}(TS/TS^{\perp})$, with kernel equal to $T_{\zeta_c}T_sS^{\perp}$. Identifying this tangent space with the vector space $T_sS^{\perp}$ yields a natural identification between $T_{[\zeta_c]}(TS/TS^{\perp})$ and $T_{\zeta_c}TS/T_sS^{\perp}$. Using this, we get

$$\widetilde{\eta}^j(b') = \left[\left(\frac{d}{dt}\bigg|_{t=0} \rho_*^{jt}|_s\zeta_{k+1} \right), \ldots, \left[\frac{d}{dt}\bigg|_{t=0} \rho_*^{jt}|_s\zeta_{2n-k} \right] \right]$$
$$\in T_{\zeta_{k+1}}TS/T_sS^{\perp} \times \ldots \times T_{\zeta_{2n-k}}TS/T_sS^{\perp}.$$

Now, since $\zeta_c \in T_sS$ for all $k+1 \leq c \leq 2n-k$, Lemma 6.2.1 applies and we have

$$\frac{d}{dt}\bigg|_{t=0} \rho_*^{jt}|_s\zeta_c = \sum_{l=1}^{k} \left(C^{jl}(s)\, \frac{d}{dt}\bigg|_{t=0} \phi_*^{lt}|_s\zeta_c + (\zeta_c C^{jl})\xi^l(s) \right)$$

for all c. Recall that $\xi^l(s) \in T_sS^{\perp}$ for all $1 \leq l \leq k$. Therefore, upon taking equivalence classes with respect to the quotient by $T_sS^{\perp}$, the terms of the form $(\zeta_c C^{jl})\xi^l(s)$ all vanish and we are left with

$$\left[\frac{d}{dt}\bigg|_{t=0} \rho_*^{jt}|_s\zeta_c \right] = \left[\sum_{l=1}^{k} C^{jl}(s)\, \frac{d}{dt}\bigg|_{t=0} \phi_*^{jt}|_s\zeta_c \right], \quad k+1 \leq c \leq 2n-k.$$

It follows that

$$\widetilde{\eta}^j(b') = \sum_{l=1}^{k} C^{jl}(s)\widetilde{\xi}^l(b'),$$

as desired. □

We have established that each of the Hamiltonian vector fields $\widetilde{\eta}^1, \ldots, \widetilde{\eta}^k$ on $\mathrm{Sp}(TS/TS^{\perp})$ is a linear combination of the vector fields $\widetilde{\xi}^1, \ldots, \widetilde{\xi}^k$. Therefore, if there is a closed curve in $\mathrm{Sp}(TS/TS^{\perp})$ whose tangent at every point is in the subspace spanned by $\{\widetilde{\eta}^1, \ldots, \widetilde{\eta}^k\}$, then that tangent is also in the subspace spanned by $\left\{\widetilde{\xi}^1, \ldots, \widetilde{\xi}^k\right\}$, and vice versa. From our definition of a quantized energy level, the following is immediate.

Theorem 6.2.3. If $H = (H^1, \ldots, H^k)$ and $L = (L^1, \ldots, L^k)$ are two families of Poisson-commuting functions on M such that $H^{-1}(E) = L^{-1}(F)$ for some regular values E of H and F of L, then E is a quantized energy level for (M, ω, H) if and only if F is a quantized energy level for (M, ω, L).

6.3 Completely integrable systems and Bohr-Sommerfeld conditions

In this section, we consider the special case in which $k = n$. As we will show, when the size of the Poisson-commuting family is maximal, the quantization condition simplifies to a Bohr-Sommerfeld condition. First, we briefly review Bohr-Sommerfeld conditions in the context of Kostant-Souriau quantization. Our summary is based on the more detailed treatments available in [11, 18, 19, 24].

6.3.1 Kostant-Souriau quantization and Bohr-Sommerfeld conditions

A completely integrable system on the $2n$-dimensional manifold (M, ω) is a family of n Poisson-commuting functions $H = (H^1, \ldots, H^n)$ that are linearly independent almost everywhere. Excluding the singular points, the Hamiltonian vector fields $\xi^1, \ldots, \xi^n$ form a real polarization: that is, an involutive, Lagrangian subbundle of the tangent bundle TM. A leaf of the polarization is a Lagrangian submanifold of M.

Assume that (M, ω) admits a Kostant-Souriau prequantization line bundle (L, ∇). A leaf of a real polarization F is called *Bohr-Sommerfeld* provided there exists a global, nonvanishing section of L over S that is horizontal in the directions of F, with respect to the connection ∇. Equivalently, a leaf S is Bohr-Sommerfeld if the connection has trivial holonomy over every closed curve in S. Since the connection is flat over a Lagrangian submanifold, homotopic curves have equal holonomy. Therefore it suffices to check the holonomy of ∇ over the generators of $\pi_1(S)$.

Now assume that (M, ω) also admits a metaplectic structure, which implies that we can construct the half-form bundle $\wedge^{1/2} F$. Then we modify the Bohr-Sommerfeld condition so that the Bohr-Sommerfeld leaves are those over which there is a horizontal section of $L \otimes \wedge^{1/2} F$. This corrected Bohr-Sommerfeld condition is necessary in order to obtain the quantum mechanical energy levels for the harmonic oscillator [19]. In the next sections, we will show that our quantization condition reduces to a Bohr-Sommerfeld condition when $k = n$, and specifically to one that replicates the half-form correction for the harmonic oscillator.

6.3.2 Quantized energy condition for $k = n$

Let $H = (H^1, \ldots, H^n)$ be a family of n Poisson-commuting functions on M. As usual, let $E = (E^1, \ldots, E^n)$ be a regular value of H, and let $S = H^{-1}(E)$. The crucial observation is that for all $s \in S$, $T_s S^\perp = T_s S = \text{span}\left\{\xi^1(s), \ldots, \xi^n(s)\right\}$, which implies that $\text{Sp}(TS/TS^\perp)$ has trivial fiber and can be identified with S. The bundle (P_S, γ_S) is a principal circle bundle with connection one-form over $\text{Sp}(TS/TS^\perp)$, and can now be thought of as a principal circle bundle with connection one-form over S. In particular, $d\gamma_S = 0$ since $S \subset M$ is Lagrangian.

Per our definition, E is a quantized energy level of (M, ω, H) if γ_S has trivial holonomy over all closed paths in $\text{Sp}(TS/TS^\perp)$ whose tangents are in the span of the Hamiltonian vector fields $\xi^1, \ldots, \xi^n$. But now $\text{Sp}(TS/TS^\perp)$ can be identified with S, and $\xi^1, \ldots, \xi^n$ span all of TS. Thus E is a quantized energy level of (M, ω, H) if γ_S has trivial holonomy over all closed paths in S. This is a Bohr-Sommerfeld condition, as described in the previous section.

Our primary examples in previous chapters have been the harmonic oscillator and the hydrogen atom. In both cases, the Hamiltonian energy function can be made part of a completely integrable system. In the remainder of this chapter, we revisit each system and apply the generalized quantization condition to a regular level set of the completely integrable system.

6.3.3 The n-dimensional harmonic oscillator

Let $M = \mathbb{R}^{2n}$, with Cartesian coordinates $(p_1, \ldots, p_n, q_1, \ldots, q_n)$ and symplectic polar coordinates $(s_1, \ldots, s_n, \theta_1, \ldots, \theta_n)$. We use all of the same definitions that were established in Sections 4.4.1 and 4.4.2. In particular, the metaplectic-c prequantization for (M, ω) is (P, Σ, γ), where $P = M \times \text{Mp}^c(V)$, $\Sigma : P \to \text{Sp}(M, \omega)$ is defined with respect to the global trivialization of

$\mathrm{Sp}(M,\omega)$ given by the symplectic frame $\left(\frac{\partial}{\partial p_1},\ldots,\frac{\partial}{\partial p_n},\frac{\partial}{\partial q_1},\ldots,\frac{\partial}{\partial q_n}\right)$, and

$$\gamma = \frac{1}{i\hbar}\Pi^*\beta + \frac{1}{2}\eta_*\vartheta_0,$$

where ϑ_0 is the trivial connection on P and

$$\beta = \frac{1}{2}\sum_{j=1}^{n}(p_j dq_j - q_j dp_j)$$

on M.

Let $H = (H^1,\ldots,H^n)$, where

$$H^j = \frac{1}{2}(p_j^2 + q_j^2) = s_j$$

for each $1 \le j \le n$. In Cartesian coordinates, the corresponding Hamiltonian vector fields are

$$\xi^j = q_j\frac{\partial}{\partial p_j} - p_j\frac{\partial}{\partial q_j}.$$

These vector fields clearly Poisson commute, and they are linearly independent unless $q_j = p_j = 0$ for some j. Thus a regular value of the family takes the form $E = (E_1,\ldots,E_n) \in \mathbb{R}^k$, where $E_j > 0$ for all j. Assume such a regular value E has been fixed, and consider the regular level set $S = H^{-1}(E)$. Then S is an n-torus.

We know from Section 4.4.2 that we can change coordinates from Cartesian to symplectic polar over the open subset of $U \subset M$ where $s_j > 0$ for all j. In terms of these coordinates, we have the Hamiltonian vector fields $\xi^j = -\frac{\partial}{\partial\theta_j}$ for all j. Let ϕ^{jt} be the flow for the vector field ξ^j. It is clear that all orbits in the level set S are closed, with period 2π.

Points in S take the form $(E_1,\ldots,E_n,\theta_1,\ldots,\theta_n)$. If we fix a starting point, then the orbits of the Hamiltonian vector fields ξ^j through that point span the fundamental group $\pi_1(S)$. Therefore we need to show that γ_S has trivial holonomy over each of these orbits.

Our procedure is exactly the same as that in Sections 4.4.2 and 4.4.3: we locally change variables from Cartesian to symplectic polar, lift that change of variables to the level of metaplectic-c structures, and construct local trivializations for all of the relevant bundles over an orbit $\mathcal{C}$. The only difference is that the curve $\mathcal{C}$ is now the orbit for ξ^j. We omit the details of the calculations when they are repetitions of those given in Section 4.4.

Without loss of generality, fix the starting point $m_0 = (E_1, \ldots, E_n, 0, \ldots, 0) \in S$. Let the orbit for ξ^j through m_0 be $\mathcal{C}$. Then a point in $\mathcal{C}$ takes the form

$$m(\tau) = (E_1, \ldots, E_n, 0, \ldots, \tau, \ldots, 0),$$

where $\tau \in \mathbb{R}/2\pi\mathbb{Z}$ is the jth angle coordinate.

For all $m \in U$, we construct the matrix of partial derivatives $G(m)$ that corresponds to the change of variables, just as in Section 4.4.2, and then we restrict it to lie over $\mathcal{C}$. Let $G(\tau) = G(m(\tau))$. In order to lift the change of variables to the level of metaplectic-c structures, we calculate

$$\mathrm{Det}_{\mathbb{C}} C_{G(\tau)} = \frac{1}{2}\left(\sqrt{2E_j} + \frac{1}{\sqrt{2E_j}}\right) e^{i\tau},$$

which we denote by $Ke^{i\tau}$ where K is a positive real constant. Then a lift of $G(\tau)$ to $\mathrm{Mp}(V)$ has parameters

$$\widehat{G}(\tau) \mapsto \left(G(\tau), \frac{1}{\sqrt{K}} e^{-i\tau/2}\right).$$

The matrix $G(\tau)$ is single-valued with respect to τ, but $e^{-i\tau/2}$ is not. As τ ranges from 0 to 2π, $G(\tau)$ ranges from $(I, 1)$ to $(I, -1)$. We use $\widehat{G}(\tau)$ to perform the change of variables over the subset $\dot{\mathcal{C}}$, then manually correct for the change in sign of the μ parameter when we close the loop.

Let $(\widehat{x}_1, \ldots, \widehat{x}_n, \widehat{y}_1, \ldots, \widehat{y}_n)$ be the usual symplectic basis for V. Let $W = \mathrm{span}\,\{\widehat{y}_1, \ldots, \widehat{y}_n\}$, so $W^\perp = W$ and $W/W^\perp = \{0\}$. Using the local trivializations over U given by $\widehat{x}_j \mapsto \frac{\partial}{\partial s_j}$ and $\widehat{y}_j \mapsto \frac{\partial}{\partial \theta_j}$, we obtain the identifications $\mathrm{Sp}(M,\omega)|_U = U \times \mathrm{Sp}(V)$, $\mathrm{Sp}(M,\omega;S)|_U = U \times \mathrm{Sp}(V;W)$, and $\mathrm{Sp}(TS/TS^\perp)|_U = U \times \{I\} = U$.

Over $\dot{\mathcal{C}}$, the lifted change of variables gives us the identifications $P|_{\dot{\mathcal{C}}} = \dot{\mathcal{C}} \times \mathrm{Mp}^c(V)$, $P^S|_{\dot{\mathcal{C}}} = \dot{\mathcal{C}} \times \mathrm{Mp}^c(V;W)$, and $P_S|_{\dot{\mathcal{C}}} = \dot{\mathcal{C}} \times U(1)$. The one-form γ_S takes the form

$$\gamma_S|_{\dot{\mathcal{C}}} = \frac{1}{i\hbar}\sum_{j=1}^{n} E_j \, d\theta_j + \frac{1}{2}\eta_* \vartheta_0.$$

Since $\mathrm{Sp}(TS/TS^{\perp})|_U$ is identified with U, the lift of $\xi^j(m(\tau))$ to $\mathrm{Sp}(TS/TS^{\perp})$ is simply

$$\widetilde{\xi^j}(m(\tau), I) = \xi^j(m(\tau)).$$

Then the horizontal lift to P_S with respect to γ_S is

$$\widehat{\xi^j}(m(\tau), I) = \left(\xi^j(m(\tau)), -\frac{1}{i\hbar}E_j\right).$$

Since the Lie algebra component is constant over the orbit, we obtain the lifted flow by exponentiating:

$$\widehat{\phi^{jt}}(m_0, I) = \left(\phi^{jt}(m_0), e^{-E_j t/i\hbar}\right).$$

On S, ϕ^{jt} has period 2π. For the $U(1)$ component, we introduce a negative sign upon closing the loop from $\dot{\mathcal{C}}$ to $\mathcal{C}$. Therefore the quantization condition is

$$\frac{2\pi i E_j}{\hbar} = 2\pi i \left(N_j + \frac{1}{2}\right)$$

for some $N_j \in \mathbb{N} \cup \{0\}$, which implies that

$$E_j = \hbar \left(N_j + \frac{1}{2}\right), \quad N_j \in \mathbb{N} \cup \{0\}.$$

Note that the Hamiltonian energy function for the n-dimensional harmonic oscillator is just $H^1 + \ldots + H^n$. Using the generalized dynamical invariance property, we see immediately that if we describe S as a level set of the completely integrable system $(H^1 + \ldots + H^n, H^2, \ldots, H^n)$, the quantized energy levels for $H^1 + \cdots + H^n$ are $E_1 + \ldots + E_n = \hbar\left(N_1 + \frac{1}{2} + \ldots + N_n + \frac{1}{2}\right) = \hbar\left(N + \frac{n}{2}\right)$, for some $N \in \mathbb{N} \cup \{0\}$. This result agrees with the analysis from Section 4.4 using just a single function, and also with the quantum mechanical calculation.

6.3.4 The 2-dimensional hydrogen atom

For simplicity, we restrict our attention to the two-dimensional version of the hydrogen atom (equivalently, the Kepler problem). Our initial definitions are special cases of those that appear in Section 5.2.

Let $M = \dot{\mathbb{R}}^2 \times \mathbb{R}^2$ with Cartesian coordinates (q_1, q_2, p_1, p_2) and symplectic form $\omega = \sum_{j=1}^{2} dq_j \wedge dp_j$. As before, the energy function for the hydrogen atom is

$$H = \frac{1}{2m_e}|p|^2 - \frac{k}{|q|},$$

where $k, m_e > 0$. Let (P, Σ, γ) be the metaplectic-c prequantization for M, where $P = M \times \mathrm{Mp}^c(V_4)$, $\Sigma : P \to \mathrm{Sp}(M, \omega)$ is defined with respect to the global trivialization of $\mathrm{Sp}(M, \omega)$ given by the Cartesian coordinate vector fields $\left(\frac{\partial}{\partial q_1}, \frac{\partial}{\partial q_2}, \frac{\partial}{\partial p_1}, \frac{\partial}{\partial p_2}\right)$, and

$$\gamma = \frac{1}{i\hbar}\Pi^*\beta + \frac{1}{2}\eta_*\vartheta_0,$$

where ϑ_0 is the trivial connection on P and

$$\beta = \sum_{j=1}^{2} (q_j dp_j + d(q_j p_j))$$

on M.

Let (r, θ) be polar coordinates for $\dot{\mathbb{R}}^2$, with conjugate momenta (p_r, p_θ) on $\mathbb{R}^2$. The change of coordinates is given by

$$\begin{aligned} r =& \sqrt{q_1^2 + q_2^2}, \\ \theta =& \tan^{-1}\left(\frac{q_2}{q_1}\right), \\ p_r =& \frac{1}{r}\left(q_1 p_1 + q_2 p_2\right), \\ p_\theta =& q_1 p_2 - q_2 p_1, \end{aligned}$$

and the inverse transformation is

$$\begin{aligned} q_1 =& r\cos\theta, \\ q_2 =& r\sin\theta, \\ p_1 =& p_r\cos\theta - \frac{1}{r}p_\theta\sin\theta, \\ p_2 =& p_r\sin\theta + \frac{1}{r}p_\theta\cos\theta. \end{aligned}$$

Note that polar coordinates are defined everywhere on M, so we could have defined $\Sigma : P \to \mathrm{Sp}(M,\omega)$ in terms of the global trivialization for $\mathrm{Sp}(M,\omega)$ given by the symplectic frame $\left(\frac{\partial}{\partial}, \frac{\partial}{\partial\theta}, \frac{\partial}{\partial p_r}, \frac{\partial}{\partial p_\theta}\right)$, and let γ be as given above, with β written in polar coordinates. However, since (M,ω) is not simply connected, it admits multiple nonisomorphic metaplectic-c prequantizations. It is not obvious that converting from Cartesian to polar has no effect on the form of γ. For the sake of comparison with our results in Chapter 5, we begin with the metaplectic-c prequantization that is defined in terms of Cartesian coordinates, and convert.

At any point $m \in M$, the matrix of partial derivatives representing the transformation from Cartesian to polar coordinates is

$$G(m) = \begin{pmatrix} \frac{\partial q_1}{\partial r} & \frac{\partial q_2}{\partial r} & \frac{\partial p_1}{\partial r} & \frac{\partial p_2}{\partial r} \\ \frac{\partial q_1}{\partial \theta} & \frac{\partial q_2}{\partial \theta} & \frac{\partial p_1}{\partial \theta} & \frac{\partial p_2}{\partial \theta} \\ \frac{\partial q_1}{\partial p_r} & \frac{\partial q_2}{\partial p_r} & \frac{\partial p_1}{\partial p_r}, & \frac{\partial p_2}{\partial p_r} \\ \frac{\partial q_1}{\partial p_\theta} & \frac{\partial q_2}{\partial p_\theta} & \frac{\partial p_1}{\partial p_\theta} & \frac{\partial p_2}{\partial p_\theta} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta & \frac{1}{r^2}p_\theta \sin\theta & -\frac{1}{r^2}p_\theta\cos\theta \\ -r\sin\theta & r\cos\theta & -p_r\sin\theta - \frac{1}{r}p_\theta\cos\theta & p_r\cos\theta - \frac{1}{r}p_\theta\sin\theta \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\frac{1}{r}\sin\theta & \frac{1}{r}\cos\theta \end{pmatrix}.$$

To determine a lift to $\mathrm{Mp}(V_4)$, we calculate $C_{G(m)}$, convert to a complex matrix, and take the determinant. The result is

$$\mathrm{Det}_{\mathbb{C}} C_{G(m)} = r + \frac{1}{r} + \frac{1}{4r^3}p_\theta^2.$$

While this is not necessarily constant, it is real and positive everywhere on M. Therefore, over any closed orbit in M, we can consistently take $\sqrt{\mathrm{Det}_{\mathbb{C}} C_{G(m)}}$. Thus a lift of $G(m)$ to $\mathrm{Mp}(V_4)$ is given by

$$\widehat{G}(m) = \left(G(m), \frac{1}{\sqrt{\mathrm{Det}_{\mathbb{C}} C_{G(m)}}}\right),$$

which is single-valued over any closed orbit in M. This demonstrates that we can, in fact, convert to polar coordinates without affecting the form of γ.

From now on, we use the global trivializations of P and $\mathrm{Sp}(M,\omega)$ given by the symplectic frame $\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial p_r}, \frac{\partial}{\partial p_\theta}\right)$. The one-form γ is given by

$$\gamma = \frac{1}{i\hbar}\Pi^* \beta + \frac{1}{2}\vartheta_0,$$

where

$$\beta = 2r dp_r + p_r dr - p_\theta d\theta$$

on M.

In terms of the polar coordinates,

$$H = \frac{1}{2m_e}\left(p_r^2 + \frac{1}{r^2}p_\theta^2\right) - \frac{k}{r},$$

and $\omega = dp_r \wedge dr + dp_\theta \wedge d\theta$. A calculation establishes that

$$\xi_H = \left(\frac{k}{r^2} - \frac{1}{m_e r^3}p_\theta^2\right)\frac{\partial}{\partial p_r} - \frac{1}{m_e}p_r\frac{\partial}{\partial r} - \frac{1}{m_e r^2}p_\theta\frac{\partial}{\partial \theta}.$$

Now consider p_θ. Physically, p_θ is the angular momentum of the system. Let $E < 0$ be a fixed negative energy value. As discussed in Section 5.2.1, the angular momentum can take values in $\left[0, \sqrt{-\frac{m_e k^2}{2E}}\right]$. The value $p_\theta = 0$ represents a degenerate elliptical orbit in which the electron collapses into the proton in finite time. The value $p_\theta = \sqrt{-\frac{m_e k^2}{2E}}$ represents a circular orbit.

Observe that

$$\xi_{p_\theta} = -\frac{\partial}{\partial \theta},$$

from which we see that $\{H, p_\theta\} = 0$. Therefore the system (H, p_θ) Poisson commutes. Further, ξ_H and ξ_{p_θ} are linearly independent unless $p_r = 0$ and $\frac{k}{r^2} - \frac{1}{m_e r^3}p_\theta^2 = 0$. These two conditions imply that $r = -\frac{k}{2E}$ and $p_\theta^2 = -\frac{m_e k^2}{2E}$: that is, ξ_H and ξ_{p_θ} are linearly independent everywhere except on the circular orbit.

Let $(E, L) \in \mathbb{R}^2$ be such that $E < 0$ and $0 < L < \sqrt{-\frac{m_e k^2}{2E}}$. Then (E, L) is a regular value of the family (H, p_θ). Let S be the corresponding level set. Then, from Section 5.2.1, all orbits of ξ_H in S are closed with period $\frac{2\pi}{\Lambda}$, where $\Lambda = \sqrt{-\frac{8E^3}{m_e k^2}}$. All orbits of ξ_{p_θ} in S are closed with period 2π.

Given a starting point $m_0 \in S$, the orbits of ξ_H and ξ_{p_θ} generate $\pi_1(S)$. Since S is a Lagrangian submanifold of M, we have $TS^\perp = TS$, and so $\mathrm{Sp}(TS/TS^\perp)$ is naturally identified with S. We must therefore evaluate the holonomy of γ_S over the orbits ξ_H and ξ_{p_θ} through m_0. Let ξ_H have flow ϕ_H^t, and let p_θ have flow $\phi_{p_\theta}^t$ on M.

First, we consider ξ_{p_θ}. Let $m(\tau)$ be a point on the orbit of ξ_{p_θ} through m_0, where $\tau \in [0, 2\pi]$ and $m(0) = m_0$. On $\mathrm{Sp}(TS/TS^\perp)$, we simply have

$$\widetilde{\xi}_{p_\theta}(m(\tau), I) = \xi_{p_\theta}(m(\tau)).$$

Using the expression for β in polar coordinates, we calculate that the horizontal lift to P_S is

$$\widehat{\xi}_{p_\theta}(m(\tau), I) = \left(\xi_{p_\theta}(m(\tau)), -\frac{L}{i\hbar}\right).$$

Upon exponentiating, we obtain the integral curve through (m_0, I):

$$\widehat{\phi}^t_{p_\theta}(m_0, I) = \left(\phi^t_{p_\theta}, e^{-Lt/i\hbar}\right).$$

Since the orbit in S has period 2π, one sees immediately that the quantization condition for the angular momentum is

$$L = N\hbar, \quad N \in \mathbb{Z}.$$

This is consistent with the Bohr model of the hydrogen atom, in which angular momentum is assumed to be quantized in units of $\hbar$.

Now we apply the same process to ξ_H. Let $m(\tau)$ be a point on the orbit of ξ_H through m_0, where now $\tau \in \left[0, \frac{2\pi}{\Lambda}\right]$ and $m(0) = m_0$. Since

$$\xi_H \lrcorner \beta = \frac{2k}{r} - \frac{1}{m_e}\left(p_r^2 + \frac{1}{r^2}p_\theta^2\right) = -2E,$$

we see that the lift to P_S is

$$\widehat{\xi}_H(m(\tau), I) = \left(\xi_H(m(\tau)), \frac{2E}{i\hbar}\right).$$

Then the integral curve through m_0 is

$$\widehat{\phi}^t_H(m_0, I) = \left(\phi^t_H(m_0), e^{2Et/i\hbar}\right).$$

Using the fact that the period of the orbit in S is $\frac{2\pi}{\Lambda}$ where $\Lambda = \sqrt{-\frac{8E^3}{m_e k^2}}$, we find that the quantization condition for the energy is

$$E = -\frac{m_e k^2}{2\hbar^2 N^2}, \quad N \in \mathbb{N}.$$

This agrees with our result from Chapter 5.

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