THE EQUIVARIANT K-THEORY OF COMMUTING 2-TUPLES IN SU(2)

by

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Abstract

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In this thesis, we study the space of commuting n-tuples in SU(2), $Hom(\mathbb{Z}^n, SU(2))$. We describe this space geometrically via providing an explicit G-CW complex structure, an equivariant analog of familiar CW- complexes. For the n=2 case, this geometric description allows us to compute various cohomology theories of this space, in particular the G-equivariant K-Theory $K^*_G(Hom(\mathbb{Z}^2, SU(2)))$, both as an $R(SU(2))$-module and as an $R(SU(2))$-algebra. This space is of particular interest as $\phi^{-1}(e)$ in a quasi-Hamiltonian system $M \xrightarrow{\phi} G$ consisting of the G-space $SU(2) \times SU(2)$, together with a moment map ϕ given by the commutator map. Finite dimensional quasi-Hamiltonian spaces have a bijective correspondence with certain infinite dimensional Hamiltonian spaces, and we additionally compute relevant components of this larger picture in addition to $\phi^{-1}(e) = Hom(\mathbb{Z}^2, SU(2))$ for this example.

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Contents

Chapter 1

Introduction

1.1 Overview

In this thesis, our principle objective is the study of the commuting n-tuples in SU(2), $\{(g_1, \cdots, g_2) \in (SU(2))^n \mid g_i g_j = g_j g_i \; \forall i, j\}$. Since a homomorphism is determined by the mappings of the generators, this is the space $Hom(\mathbb{Z}^n, SU(2))$. This is an SU(2)-space where SU(2) acts on $Hom(\mathbb{Z}^n, SU(2))$ by conjugation on each component. Alternatively, this space is homomorphisms from the fundamental group of $(S^1)^n$ into G which can be identified with the space of flat G-connections on $(S^1)^n$ modulo based gauge transformations. Our main goal is to understand this space on various cohomology theories, in particular SU(2)-equivariant K-theory.

A standard methodology to compute homology and cohomology groups associated to topological spaces is to establish a CW-complex structure on the spaces. The appropriate equivariant analog of the familiar CW-complexes is the notion of a G-CW complex developed independently by [Mat] and [Ill1]. From the skeleta filtration X^n of a G-CW space X, we can consider the corresponding long exact sequences on various cohomology theories corresponding to $X^n \hookrightarrow X^{n+1} \to X^{n+1}/X^n$, much as we would for CW-complexes. In the specific case of equivariant K-theory, which is a $\mathbb{Z}_2$ graded cohomology theory due to Bott Periodicity, this yields six term cyclical exact sequences. As quotients X^{n+1}/X^n can be well descibed, this provides a methodology to compute the unknown terms in the sequence. Thus by establishing an explicit G-CW complex for a space, this geometric description gives us considerable power in computing equivariant cohomology theories.

We apply this methodology to a range of G-spaces, but our principal computation is to establish a G-CW complex structure for G=SU(2) on $Hom(\mathbb{Z}^n, G)$. From this, in the case of n=2, we go on to use the long exact sequences corresponding to the skeletal filtration to compute $X = Hom(\mathbb{Z}^2, SU(2))$ on various cohomology theories such as $K_G^*(X)$, ordinary $H^*(X, \mathbb{Z})$, $K_T^*(X)$, $H_G^*(X)$, and $H_T^*(X)$. The former two, as modules, have previously been computed by [AG] and [BJS] respectively for $X = Hom(\mathbb{Z}^2, G)$.

Beyond providing an alternate and pleasingly geometric computation that motivates future work using this G-CW approach, our results extend the literature in three ways. Firstly, our geometric G-CW decompositions allows us to also establish the multiplicative structure, thus computing $K_G^*(Hom(\mathbb{Z}^2, G))$ as an R(G)-algebra, not just as an R(G)-module. Additionally, our methods allow us to write this on other cohomology theories in particular $H_G(Hom(\mathbb{Z}^2, G))$. Finally, we equip the commuting n-tuples in SU(2) with a G-CW structure that generalizes the 2-tuples case, from which the various cohomologies theories mentioned above could be computed analogously. However, we lack a closed form solution for the general n case and note that computations quickly become cumbersome. As such, only the n=2 case has been explicitly written out in this thesis.

One motivation for this study of $Hom(\mathbb{Z}^2, G)$ is the quasi-Hamiltonian systems introduced by [AMM]. Indeed, the commutator map from $G \times G \to G$ provides an example of such a quasi-Hamiltonian system with $Hom(\mathbb{Z}^2, G)/G$ as the symplectic quotient where G acts regularly on $Hom(\mathbb{Z}^2, G)$. $Hom(\mathbb{Z}^2, G)/G$ has orbifold singularities and we can identify it with based gauge equivalence classes of the moduli space of flat connections on a closed 2-manifold Σ of genus 1 [Jef]. There are a range of other spaces related to the larger pictures of quasi-Hamiltonian systems and their bijective correspondence with particular Hamiltonian LG systems, and for this example we additionally compute $K_G(\Omega G)$, $K_G(G)$ and $K_G(G \times G)$, and finally relate them to larger questions in quasi-Hamiltonian systems.

1.2 Outline of the Thesis

This thesis is organized by chapter as follows:

- In Chapter 2 we outline various results in the literature relevant to this thesis. In particularly, we dicuss known computations on equivariant K-theory and cohomology for several relevant spaces and where our results match or extend them.

- In Chapter 3 we provide an introduction into the objects and tools that we will use in this thesis. In particular, we define G-CW complexes in Section 3.1 and discuss a few important facts. In Section 3.2 we introduce Equivariant K-theory, following [Seg]; in particular, we cite various propositions that will be critical for our computations on equivariant K-theory. In Section 3.3 we provide a brief overview of equivariant cohomology. Finally, in Section 3.4 we discuss various facts about the Lie group SU(2). All of the G-spaces under consideration in this thesis are for G= SU(2) and details about how one diagonalizes elements, for instance, in this space are critical.

- In Chapter 4 we establish G-CW complexes on all the various spaces we are interested in computing in this thesis. In particular, we study G, $G \times G$, $Hom(\mathbb{Z}^n, G)$, and ΩG. Of critical importance is describing the attaching maps in our G-CW structure sufficiently clearly so that we can compute equivariant K-theory from them.

- In Chapter 5, we compute the equivariant K-theory of our range of spaces as R(G)-modules. Note that while we have a G-CW structure on commuting n-tuples, we only go on to compute equivariant K-theory in the n=2 case. In Section 5.3 and 5.4 we then study $Hom(\mathbb{Z}^2, G)$ on various other cohomology theories. Additionally, we make explicit the R(G)-algebra structure for $K^*_G(Hom(\mathbb{Z}^2, G))$ in Section 5.5. We study ΩG in Section 5.6 and finally, in 5.7, we discuss some potential for future work.

- In Chapter 6, we discuss the relationship between $Hom(\mathbb{Z}^2, G)$ and the other spaces we compute here and the quasi-Hamiltonian systems that originally motivated this study. We conduct a review of the literature on analogs of Kirwan surjectivity, and conclude the thesis with suggestions for futher work along these themes.

1.3 Conventions

Henceforth in this thesis, unless otherwise specified when wishing to make a more general statement, we will be considering G=SU(2), acting on itself by conjugation. T will denote the maximal torus, as will be described in Section 3.4.

Chapter 2

Historical Results and Context

in this chapter we summarize prior computations on various cohomology theories in the literature that overlap with our result and describe where we match the literature.

$K_G^*(G)$ was first computed in 2000 by [BZ] for simply connected Lie groups G acting on themselves by conjugation as the algebra of Grothendieck differentials on the representation ring. We will match, as expected, this result in the case of G=SU(2) in Section 5.1.

The space $Hom(\pi, G)$ for π a finitely generated discrete group, and G a Lie group, was studied by [AC]. In particular, for π a free abelsian group of rank equal to n, cohomology groups were explicitly computed in the n=2 and n=3 cases for $G = SU(2)$. [BJS] further study $Hom(\mathbb{Z}^n, SU(2))$, computing the integral cohomology groups for all positive n, in agreement with [AC2], an errata to the original [AC] in the n=3 case. Our results in this thesis match the n=2 case on integral cohomology as noted in Section 5.4.

[AG] provide a computation for $K^*_{SU(2)}(Hom(\mathbb{Z}^2, SU(2)))$ as an R(G)-module. This result uses the Atiyah-Hirzebruch-Segal Spectral Sequence and Segal's localization to reduce the problem to computing $H^p_W(T^2, \mathcal{R}_T)$, which is computed using a W-CW decomposition of T^2. We reproduce this result in Section 5.2 through computing $K^*_{SU(2)}(Hom(\mathbb{Z}^2, SU(2))$ via providing a G-CW structure on $(Hom(\mathbb{Z}^2, SU(2)))$.

[HJS1] and [HJS2] are twin papers that compute $K^*_G(\Omega G)$ for $G = SU(2)$ acting on itself by conjugation as a module and as an algebra respectively. We will reinterpret the last few steps in the module computation by making explicit a G-CW structure on $\Omega SU(2)$ in Section 5.6.

Chapter 3

Preliminaries

This section contains a brief overview of G-CW complexes, equivariant K-theory and a few facts on SU(2) that will be used in our computations. It is by no means intended to be exhaustive on these subjects.

3.1 G-CW Complexes

An equivariant analog of normal CW-Complexes, referred to as G-CW complexes in this thesis, was developed independently by [Mat] and [Ill1]. Let X by a G-space for a compact Lie group G; that is, X is a Hausdorff topological space with a left action of G on X. Let H_k denote a closed subgroup of G for each k. Following [May],

Definition 1. *A G-CW complex X is a union of G-spaces X^n where X^0 is a disjoint union of orbits G/H_k, referred to as 0-cells, and X^{n+1} is determined inductively by attaching (n+1)-cells $(G/H_k) \times D^{n+1}$ to X^n via attaching G-maps $\sigma_k : (G/H_k) \times S^n \to X^n$*

That is, X^{n+1} is determined by the pushout of the following diagram:

$$\begin{array}{ccc} \coprod_k (G/H_k) \times S^n & \xrightarrow{\coprod_k \sigma_k} & X^n \\ \downarrow & & \downarrow \\ \coprod_k (G/H_k) \times D^{n+1} & \longrightarrow & X^{n+1}. \end{array} \tag{3.1}$$

A point of the form $(eH_k, t) \in (G/H_k) \times D^{n+1}$ is fixed under the H_k action, and so maps under σ_k into the fixed point set of H_k, denoted $(X^n)^{H_k}$. Thus, an attaching G-map σ_k is determined by its restriction $S^n \to (X^n)^{H_k}$.

By convention, a $(G/H) \times D^n$ cell will be referred to as an n-cell, be said to have degree n, and have dimension $(Dim(G/H) + n)$. The space X^n, formed by attaching cells of degree up to and including n, is referred to as the n-skeleton of X. A G-CW subcomplex Y of a G-CW complex X is a subcollection of cells from the G-CW complex for X that is itself a G-CW complex.

Given two G-CW complexes X and Y, one can form a $(G \times G)$-CW complex on $X \times Y$ by taking all products of cells $(G/H \times D^n) \times (G/K \times D^m) \cong ((G \times G)/(H \times K)) \times D^{n+m}$. If a G-CW complex for each $(G \times G)/(H \times K)$ can be determined (in a noncanonical way) then a G-CW complex for $X \times Y$ can be determined.

The following let's us interpret the quotients of skeleta from our G-CW complex as a suspension. We will henceforth write $T(G/H_k) \times D^n)$ to refer to the Thom space of the product G-bundle with base G/T with the usual action and trivial action on D^n.

Proposition 1. *For a G-CW skeleton X^n on a G-space X,*

$$X^n/X^{n-1} \cong \bigvee_k T((G/H_k) \times D^n) \cong \bigvee_k (S^n((G/H_k) \coprod *)) \cong \bigvee_k (S^n(G/H_k) \vee S^n)$$

with k indexing all n-cells of our G-CW complex.

The first identification follows from the observation that an attaching map for a $G/H_k \times D^n$ cell glues $G/H_k \times S^n$ into the (n-1)-skeleton resulting in the quotient of the trivial disk bundle by the trivial sphere bundle, hence $T(G/H_k \times D^n)$. The second identification follows from the more general fact that for a vector bundle ξ, the fibrewise direct sum $T(\xi \oplus \mathbb{R}^n) \simeq S^n \wedge T(\xi)$ applied to the case of ξ being the 0 dimensional bundle over base G/H_k which has Thom space $T(\xi) = G/H_k \coprod pt$.

3.2 Equivariant K-theory

Let G be a compact Lie group and X a compact G-space. Then [Seg] defines a $\mathbb{Z}/2$-graded algebra $K^*_G(X)$ generated by equivalence classes of complex G-vector bundles over X. We follow Segal's approach outlining the basic facts.

Equivariant K-theory generalizes from two simple cases. Firstly, when $X = \{pt\}$,

$$K_G^q(pt) \cong \begin{cases} R(G) & \text{if } q \text{ even;} \\ 0 & \text{if } q \text{ odd.} \end{cases}.$$

where R(G) is the representation ring of G. Secondly, when G is trivial, $K_G^*(X) = K^*(X)$, ordinary K-theory. The map $X \to pt$ induces a map $K_G^1(pt) = R(G) \to K_G^1(X)$ which makes equivariant K-theory an R(G)-algebra.

Further, there are two extreme cases on the action of G on X. Firstly, if G acts trivially on X, then any vector bundle can be thought of as an equivariant vector bundle, giving us a map $K^*(X) \to K_G^*(X)$. Together with the ring homomorphism $K_G(pt) = R(G) \to K_G(X)$ induced by the map $X \to pt$, we get a ring homomorphism $\mu : R(G) \otimes K(X) \to K_G(X)$.

Proposition 2. *If G acts trivially on X, the map μ described above is an isomorphism.*

Secondly, for G acting freely:

Proposition 3. *If G acts freely on X then the quotient map $\pi : X \to X/G$ induces an isomorphsim*

$$\pi^* : K(X/G) \to K_G(X)$$

Proposition 4. *For H a closed subgroup of G, and an H-space X, the map $i : H \hookrightarrow G$ induces an isomorphism $i^* : K_G^*(G \times_H X) \to K_H^*(X)$*

In particular, we have

$$K_G^q(G/H) \cong K_H^q(pt) \cong \begin{cases} R(H) & \text{if } q \text{ even;} \\ 0 & \text{if } q \text{ odd.} \end{cases}.$$

The aforementioned $\mathbb{Z}/2$ grading is a consequence of the critical property of Bott Periodicity:

Proposition 5. *The Thom homomorphism $\phi^* : K_G(X) \to \tilde{K}_G(T(E))$ is an isomorphism for any G-vector bundle E on a compact G-space X.*

A consequence of Bott Periodicity is that the typical long exact sequence becomes a cyclical six-term sequence that is critical for many of our computations:

Proposition 6 (Exact Sequences). *Let A be a closed G-subspace of a compact G-space X, then the following six-term sequence is exact.*

$$\begin{array}{ccccc} \tilde{K}^0_G(A) & \longleftarrow & \tilde{K}^0_G(X) & \longleftarrow & \tilde{K}^0_G(X/A) \\ \downarrow & & & & \uparrow \\ \tilde{K}^1_G(X/A) & \longrightarrow & \tilde{K}^1_G(X) & \longrightarrow & \tilde{K}^1_G(A) \end{array} \tag{3.2}$$

3.3 Equivariant cohomology

While we are largely interested in equivariant K-theory, we will also record various results on equivariant cohomology as well. We thus recall some essential facts regarding equivarariant cohomology in this section.

Let X be a G-space for a topological group G. We can construct the homotopy quotient $EG \times_G X = (EG \times X)/G$ where $E_G \to BG := EG/G$ is a universal principal G-bundle. The homotopy quotient is thus homotopy equivalent (as EG is contractible) to X, but the action of G on the homotopy quotient is free. Further, our definition will be well defined as the classifying space BG is unique up to homotopy. We then define the equivariant integral cohomology ring as

$$H^*_G(X;\mathbb{Z}) := H^*(EG \times_G X;\mathbb{Z}).$$

We will stop making the $\mathbb{Z}$ explicit from now on.

If G acts freely on X, we just have $H^*_G(X) = H^*(X/G)$ as $EG \times_G X$ is thus a bundle over X/G with contractible fibre. For X a point, we get that $EG \times_G \{pt\} = BG$. Much like equivariant K-theory, the map $X \to pt$ thus induces an $H^*_G(pt)$-module structure on $H^*_G(X)$.

Now let G be a compact Lie group, and T a maximal torus in G. Finally, let W be the corresponding Weyl group. Then, the splitting principle gives us that

$$H^*_G(pt) = H^*_T(pt)^W.$$

We will consider our specific case of $G = SU(2)$ and $T = S^1$ in Section 5.4.

3.4 On SU(2)

$G = SU(2)$ consists of all 2x2 unitary matrices with determinant 1. As a manifold, $SU(2)$ is homeomorphic to S^3. All maximal tori are conjugate to $T = \{\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \mid \theta \in [0, 2\pi)\}$ which topologically is homeomorphic to S^1 and any two maximal tori intersect at $\pm I$. Two elements in SU(2) commute precisely if they lie on a common maximal torus.

Elements in $G \backslash \{\pm I\}$ can be uniquely diagonalized up to right multiplication by elements in T as $g = hth^{-1}$ with $t \in T^+ := \{t \in T \mid 0 < \theta < \pi\}$. G has G-projections $G \to G/T$, which is topologically homeomorphic to S^2, and to $G \to T^+$. Let $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, then the Weyl group is given by $W = \{T, wT\}$ and has a W action on T that takes $t \mapsto t^{-1}$ and on G/T corresponding to the antipodal action on S^2.

For G=SU(2) and T the maximal torus, $R(T) \cong \mathbb{Z}[b, b^{-1}]$, the representation ring of T with b the weight 1, one dimensional representation of T and $R(G) \cong \mathbb{Z}[v]$, the representation ring of G with v the standard two dimensional representation of G. The nontrivial action of the Weyl group is given by $b \mapsto b^{-1}$ and the restriction of v to T is given by $b \oplus b^{-1}$. $R(T)$, as a group, is $R(G) \oplus R(G)$

Chapter 4

G-CW Complex Calculations

In this chapter we identify G-CW structures on several G-spaces, in particular for $Hom(\mathbb{Z}^n, G)$. We begin with the more elementary examples to develop our intuition. Then in 4.2 and 4.3 we discuss commuting 2-tuples and n-tuples respectively before concluding with ΩG

A list of the G-CW complexes computed in this section is tabulated below, excluding $Hom(\mathbb{Z}^n, G)$:

Table 4.1: G-CW Complexes

Space	Cells
G	$\coprod_2 (G/G) \times D^0$, $(G/T) \times D^1$
$(G/T) \times (G/T)$	$\coprod_2 (G/T) \times D^0$, $(G/*) \times D^1$
$G \times G$	$\coprod_4 (G/G) \times D^0$, $\coprod_4 (G/T) \times D^1$, $\coprod_2 (G/T) \times D^2$, $(G/*) \times D^3$
$G \vee G$	$\coprod_3 (G/G) \times D^0$, $\coprod_2 (G/T) \times D^1$
$Hom(\mathbb{Z}^2, G)$	$\coprod_4 (G/G) \times D^0$, $\coprod_4 (G/T) \times D^1$, $\coprod_2 G/T \times D^2$
ΩG	$(G/G) \times D^0$, $(G/T) \times D^2$, $(G/T) \times D^{2(2r-1)}$
$((G \times G) \backslash Hom(\mathbb{Z}^2, G))^+$	$G/G \times D^0$, $G/(\mathbb{Z}/2) \times D^3$

4.1 G-CW structures for various G-spaces

Example 1. $G = SU(2)$

X^0 consists of two $(G/G) \times D^0$ 0-cells thought of as the North and South poles of SU(2), which is topologically S^3, and are fixed points under the conjugation action.

$X = X^1$ consists of a $(G/T) \times D^1$ cell attached to X^0 via the attaching map that collapses each of the two connected components of $G/T \times S^0$ to the two 0-cells respectively. Topologically, one can think of this as the construction of S^3 by taking $S^2 \times D^1$ and collapsing the ends $S^2 \times \partial D^1$ to two points. Identifying D^1 with T^+, described earlier, we have a G-homeomorphism from $(G/T) \times D^1$ to $G \backslash \{\pm e\}$ given by $(gT, t) \mapsto gtg^{-1}$ where G acts on $(G/T) \times D^1$ by left multiplication on the first factor and trivially on the second. Note that we have two choices for this diagonalization, depending on whether we assert that $t \in T^+$ or $t \in T^-$. We choose the former.

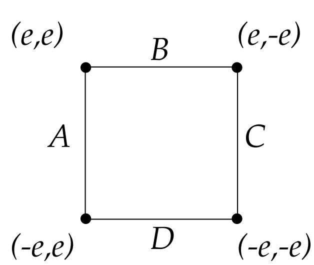
Example 2. *$G \times G$ (as a $(G \times G)$-CW Complex)*

For $G \times G$, we begin by considering the collection of cells formed by taking formal products of cells in G. This results in four $(G/G) \times (G/G) \times D^0 \times D^0$ 0-cells, four $(G/T) \times (G/G) \times D^1 \times D^0$ 1-cells and one $(G/T) \times (G/T) \times D^1 \times D^1$ 2-cell. The zero and one degree cells in the resulting $(G \times G)$-CW complex can be immediately rewritten as four $(G/G) \times D^0$ cells and four $(G/T) \times D^1$ cells to form a G-CW complex. The only nontrivial $G \times G$ cell to rewrite as a G cell, to be done shortly, is the 2-cell.

The 1-skeleton for $G \times G$, denoted $(G \times G)^1$, consists of a hollow square where the four vertices represent the four 0-cells consisting of pairs $(\pm e, \pm e)$, and the four segments represent the four 1-cells, the pairs with $\pm e$ in one factor and elements from $G/T \times D^1$ in the other.

In the following diagram, a non vertex point on the square represents a copy of G/T. The space then consists of four copies of the suspension of G/T, topologically S^3, labeled A,B,C,D, and wedged together as per the diagram. Note that this can be visualized as $S^2 \times S^1$ that has been 'pinched' at the four vertices.

Diagram 1. $(G \times G)^1$



Attaching G-maps for the one skeleton are determined as per the diagram where for a specific $(G/T) \times D^1$ 1-cell, $(G/T) \times \partial D^1$ attaches to the two adjacent vertices induced by the G-map from $G/T \to *$. For instance, the 1-cell consisting of pairs $\{(e, g), \forall g \in G\}$ is attached to the vertices (e, e) and $(e, -e)$. Note that much like for G we make the choice of diagonalization with elements $t \in T^+$ at all points in Diagram 1.

Note that X^1 is G-homotopy equivalent to $(\bigvee_4 S(G/T) \bigvee S^1$ with trivial action on the last factor.

Example 3. $G/T \times G/T$

We construct the following G-CW structure. It has a 0-cell $G/T \times D^0$ and a 2-cell $G/T \times D^2$ with an attaching map that glues $\partial D^2 = S^1$ to the point D^0 on the second factors and identity on the first. We have thus constructed the space $G/T \times S^2$. Topologically this is the same as $G/T \times G/T$ which is topologically just $S^2 \times S^2$ but they have different G-actions on the second factor.

Consider a generic point $(gT, hT) \in G/T \times G/T$ which we can rewrite as $(gT, g(g^{-1}h)T)$. We construct a G-homeomorphism from $G/T \times G/T \to G/T \times D^2$ via $(gT, g(g^{-1}h)T) \mapsto (gT, \mathbf{x})$ where $\mathbf{x} \in S^2$ is the antipode of the coset $g^{-1}hT \in G/T$ where we forget the G-action and thus have a point in S^2. As such the D^0 in $G/T \times D^0$ is thought of as one of the poles of S^2 and points in $G/T \times D^0$ map diagonally into $G/T \times G/T$. Stereoprographic projection from D^2 into the sphere minus a pole determines the nondiagonal points.

Example 4. $G \times G$ *(now as a G-CW Complex)*

The 0 and 1 skeletons of $G \times G$, as a G-CW complex where established in Example 2. Thought of as a $(G \times G)$-CW complex, we have this top $(G/T \times G/T) \times D^2$ 2-cell where the attaching map $\sigma : (G/T) \times (G/T) \times \partial D^2 \to X^1$ consists of collapsing to a point at each of the four vertices, the projection map $\pi_1 : (G/T) \times (G/T) \to G(T)$ onto the first factor along the A and C segments, and the projection $\pi_2 : (G/T) \times (G/T) \to G(T)$ onto the second factor along the B and D segments.

Now, however, we can rewrite this top cells as a G-CW complex, following Example 3, and as such in addition to the four 0-cells and four 1-cells described previously, we also have one$(G/T \times D^0) \times D^2 = G/T \times D^2$ 2-cell and one $(G/T \times D^2) \times D^2 = G/T \times D^4$ 4-cell. Here, the attaching map for the 2-cell (recall we thought of these as the diagonal elements) is to attach $G/T \times \partial D^2$ where the $\partial D^2 = S^1$ wraps around the boundary square of Diagram 1 while the G/T maps by the identity map, except for the corners where it collapses to a point. For the 4-cell, $\partial D^4 = (\partial D^2 \times D^2) \bigcup (D^2 \times \partial D^2)$. As such the $G/T \times S^1 \times D^2$ component is attached by collapsing to $G/T \times pt \times D^2$ and identifying this with the $G/T \times D^2$ cell thought of as the diagonal elements. The nondiagonal elements represented by points $G/T \times D^2 \times S^1$ are identified along the boundary circle as described above, namely collapsing to a point at the four vertices, but projecting to the first factor along the A and C segments and projecting to the second factor along the B and D segments.

Example 5. $G \vee G$

For $G \vee G$, with the wedge attaching the North pole of the first copy to the South pole of the second copy, X^0 consists of three $(G/G) \times D^0$ 0-cells, labelled S_1, $N_1 = S_2, N_2$, and two $(G/T) \times D^1$ 1-cells attached as in the previous case with one of the 1-cells attached to S_1 and $N_1 = S_2$ and the other to $N_1 = S_2$ and N_2.

Example 6. *Non-commuting 2-tuples*

In this example we reinterpret work in Proposition 6.11 of [AG] in our G-CW language, and in a way that makes for slightly easier computations.

Following [AG], $Y := (G \times G) \backslash Hom(\mathbb{Z}^2, G))$. Consider the restriction of the commutator map $\phi : G \times G \to G$ to the noncommuting 2 tuples, $\phi_| : Y \to G \backslash \{1\}$. As per [AC], this is a locally trivial G-fibre bundle with fibre

$$F := \phi^{-1}(-1) = \{(x_1, x_2,) \in G \times G \mid [x_1, x_2] = -1\}$$

which has a G-equivariant homotopy equivalence to

$$Y \simeq_G F \times \mathcal{SU}_2 \cong_G F \times D^3$$

Following from [ACG], $F \cong G/Z(G) \cong G/(\mathbb{Z}/2)$ where G acts on $G/(\mathbb{Z}/2)$ by left translation.

Now consider Y^+ which identifies $F \times \partial D^3$ with a point. We can thus equip Y^+ with a G-CW structure via a single point, $G/G \times D^0$ as the zero skeleton, and then a 3-cell $G/(\mathbb{Z}/2) \times D^3$ identified by the trivial map on both factors $G/(\mathbb{Z}/2) \times \partial D^3 \to G/G \times D^0$.

4.2 G-CW structure for $Hom(\mathbb{Z}^2, G)$

We build our intuition first through an explicit description of commuting n-tuples in the n=2 case (which is the only case we will go on to compute the equivariant K-theory of) before describing the general n-tuple case in the next section. Consider commuting 2-tuples in G,

$$Hom(\mathbb{Z}^2, G) = \{(g_i, g_j) \in G \times G \mid g_i g_j g_i^{-1} = g_j\}$$

Elements in G commute if they are on the same maximal torus; all maximal tori are conjugate to each other and intersect at the two poles. Let the zero skeleton consist of pairs $(\pm e, \pm e)$, and the one skeleton consist of pairs $(\pm e, g)$ or $(g, \pm e)$ for $g \in G$. Thus $Hom(\mathbb{Z}^2, G)^1$ = $(G \times G)^1$ and is as in Diagram 1.

For a pair $(g_1, g_2) \in Hom(\mathbb{Z}^2, G) \backslash (Hom(\mathbb{Z}^2, G))^1$, we have two options for how to diagonalize g_1 where the action of the Weyl group alternates between these presentations by acting on G/T via the antipodal map and on the maximal torus T via taking $t \to t^{-1}$. We can either write $g_1 = h_1 t_1 h_1^{-1}$ for $h_1 \in G/T$ and $t_1 \in T^+$ or we can write $g_1 = (w \cdot h_1) t_1^{-1} (w \cdot h_1)^{-1}$ for $w \cdot h_1 \in G/T$ being the antipode of h_1 and $w \cdot t_1 = t_1^{-1} \in T^-$. Aiming to determine a $(G/T) \times D^n$ cell, the former of these two choices for g_1 fixes our choice for an element in G/T as h_1 and so for g_2 we must now use $g_2 = h_1 t_2 h_1^{-1}$ where t_2 may be in either T^- or T^+. That is,

$$Hom(\mathbb{Z}^2, G) \backslash (Hom(\mathbb{Z}^2, G))^1 \cong (G/T \times (T \backslash \{\pm e\})^2)/W \cong ((G/T) \times \coprod_4 D^2)/W$$

Applying W,

$$Hom(\mathbb{Z}^2, G) \backslash (Hom(\mathbb{Z}^2, G))^1 \cong (G/T) \times \coprod_2 D^2$$

We thus get two $(G/T) \times D^2$ 2-cells where the first copy refers to pairs where t_1, t_2 are both in T^+ and the second copy refers to pairs where $t_1 \in T^+$ but $t_2 \in T^{-1}$. We can think of the two copies as representing pairs that are on the same side of the maximal torus, and on opposite sides, respectively.

To explicitly define our G-homeomorphism

$$\sigma : (G/T) \times \coprod_{2} D^2 \to Hom(\mathbb{Z}^2, G) \backslash (Hom(\mathbb{Z}^2, G))^1$$

we first identify $((G/T) \times \coprod_2 D^2)$ with $((G/T) \times T^+ \times T^+) \coprod((G/T) \times T^+ \times T^-)$ where $t \in T^-$ if $t^{-1} \in T^+$. Our maps should be compatible with our choice of writing elements in the one skeleton in terms of T^+. For the first connected component, elements are already written this way so our G-map takes $(gT, t_1, t_2) \to (gt_1g^{-1}, gt_2g^{-1})$. For the second connected component, the third factor is written in terms of T^- and so our G-map becomes $(gT, t_1, t_2) \to (gt_1g^{-1}, (w \cdot g)t_2^{-1}(w \cdot g)^{-1})$. As such, our attaching map for the first connected component $(G/T) \times \partial D^2 \to (Hom(\mathbb{Z}^2, G))^1$ is induced by the identity map on G/T and the identification of ∂D^2 with the boundary square in Diagram 1, and for the second connected component is induced by the same map along the boundary square but has the identity map on G/T along the A and C segments of ∂D^2 and the antipodal map on G/T along the segments B and D.

Therefore our G-CW complex for $Hom(\mathbb{Z}^2, G)$ consists of the same four $(G/G) \times D^0$ 0-cells and four $(G/T) \times D^1$ 1-cells as $(G \times G)^1$, along with two $(G/T) \times D^2$ 2-cells, attached as above.

4.3 G-CW structure for $Hom(\mathbb{Z}^n, G)$

Now consider the generalization to commuting n-tuples in G,

$$Hom(\mathbb{Z}^n, G) = \{(g_1, \cdots, g_n) \in G^n \mid g_i g_j g_i^{-1} = g_j \ \forall i, j\} \tag{4.1}$$

which for G=SU(2) can be described as

$$Hom(\mathbb{Z}^n, G) = \{(g_1, \cdots, g_n) \in G^n \mid \text{all } g_i \text{ lie in a common maximal torus}\}. \tag{4.2}$$

We define a filtration on $Hom(\mathbb{Z}^n, G)$ via

$$(Hom(\mathbb{Z}^n, G))^k = \{(g_1, \cdots, g_n) \in Hom(\mathbb{Z}^n, G) \mid \ g_i \notin \{\pm e\} \text{ for at most k of the } g_i\} \tag{4.3}$$

which satisfies

$$(Hom(\mathbb{Z}^n, G))^n = Hom(\mathbb{Z}^n, G). \tag{4.4}$$

We equip each $(Hom(\mathbb{Z}^n, G))^k$ with a G-CW structure inductively as follows. For convenience, define

$$F_n(m) := \text{the number of m-faces of an n-hypercube} = 2^{n-m}\binom{n}{m}. \tag{4.5}$$

The zero skeleton consists of $F_n(0) = 2^n$ points:

$$(Hom(\mathbb{Z}^n, G))^0 = \{(g_1, \cdots, g_n) \in Hom(\mathbb{Z}^n, G) \mid \; g_i \in \{\pm e\} \forall i\} \tag{4.6}$$

and thus can be given a G-CW structure by considering 2^n discrete $G/G \times D^0$ cells. Now consider

$$(Hom(\mathbb{Z}^n, G))^k \backslash (Hom(\mathbb{Z}^n, G))^{k-1} =$$

$$= \{(g_1, \cdots, g_n) \in Hom(\mathbb{Z}^n, G) \mid \; g_i \notin \{\pm e\} \text{ for exactly k of the } g_i\}. \tag{4.7}$$

In each n-tuple, we have n-k of the coordinates $g_i \in \{\pm e\}$ and each such choice can be thought of as defining a "k-face" where we are generalizing the notion of the k-skeletons of an n-hypercube. There are $F_n(k)$ such k-faces, and taking interiors as we are leaves each face disjoint. The points in each face then consist of k nontrivial coordinates, and so for notational convenience we will shorten our n-tuple with k nontrivial coordinates to a k-tuple. That is,

$$(Hom(\mathbb{Z}^n, G))^k \backslash (Hom(\mathbb{Z}^n, G))^{k-1} \cong \coprod_{F_n(k)} \{(g_1, \cdots, g_k) \in Hom(\mathbb{Z}^k, G) \mid \; g_i \notin \{\pm e\} \forall i\}. \tag{4.8}$$

Recall that $(g_1, \cdots, g_k) \in Hom(\mathbb{Z}^k, G)$ precisely when all g_i lie in a common maximal torus. Recall further that for an element $(g_1, \cdots, g_k) \in Hom(\mathbb{Z}^k, G)$, we have a choice on how to diagonalize g_1 where the action of the Weyl group alternates between these presentations by acting on G/T via the antipodal map and on the maximal torus T via taking $t \to t^{-1}$. We can either write $g_1 = h_1 t_1 h_1^{-1}$ for $h_1 \in G/T$ and $t_1 \in T^+$ or we can write $g_1 = (w \cdot h_1) t_1^{-1} (w \cdot h_1)^{-1}$ for $w \cdot h_1 \in G/T$ being the antipode of h_1 and $w \cdot t_1 = t_1^{-1} \in T^-$. Preferring to diagonalize all coordinates with $t_i \in T^+$, the $h_1 \in G/T$ specified by the diagonalization for g_1 either is used, or its antipode is used, for all subsequence coordinates. That is, either $g_i = h_1 t_i h_1^{-1}$ or $g_i = (w \cdot h_1) t_i (w \cdot h_1)^{-1}$ for $2 \leq i \leq k$.

Insisting that all $g_i \notin \{\pm e\}$, this breaks $(Hom(\mathbb{Z}^n, G))^k \backslash (Hom(\mathbb{Z}^n, G))^{k-1}$ into disjoint sections, characterized by whether each of the 2nd through kth nontrivial coordinates are located on the same side of the maximal torus as g_1 or on the opposite side. For instance, in the commuting 2-tuples, there are two top cells corresponding to nontrivial pairs on the same side of the maximal torus, and pairs on the opposite sides of the maximal torus. Thus elements in $Hom(\mathbb{Z}^k, G)$ are uniquely determined by the following three pieces of data: an element of G/T, an n-tuple $(t_1, \cdots, t_n) \in {T^+}^n$, and the knowledge of which of the latter k-1 coordinates we ought to apply the antipodal map to, the latter having 2^{k-1} possibilities and corresponding to the 2^{k-1} cells for each k-face. A particular $(G/T) \times D^k$ cell records the first two pieces of data in the first and second factor respectively, while the third is brought by the attaching map to be discussed shortly.

Algebraically, we have

$$(Hom(\mathbb{Z}^n, G))^k \backslash (Hom(\mathbb{Z}^n, G))^{k-1} \cong \coprod_{F_n(k)} (G/T \times (T \backslash \{\pm e\})^k)/W \cong \coprod_{F_n(k)} ((G/T) \times \coprod_{2^k} I^k)/W. \tag{4.9}$$

Applying W,

$$\begin{aligned}(Hom(\mathbb{Z}^n, G))^k \backslash (Hom(\mathbb{Z}^n, G))^{k-1} &\cong \coprod_{F_n(k)} ((G/T) \times \coprod_{2^{k-1}} I^k) \cong \\ &\cong \coprod_{F_n(k) 2^{k-1}} ((G/T) \times I^k) \cong \coprod_{2^{n-1}\binom{n}{k}} ((G/T) \times I^k).\end{aligned} \tag{4.10}$$

This can be thought of as having 2^{k-1} $(G/T) \times D^k$ k-cells corresponding to each of our $F_n(k)$ k-faces. For example, in $Hom(\mathbb{Z}^2, G)$, we have two such 2-cells where the first copy refers to pairs where t_1, t_2 are both in T^+ and the second copy refers to pairs where $t_1 \in T^+$ but $t_2 \in T^-$. We can think of the two copies as representing pairs that are on the same side of the maximal torus, and on opposite sides, respectively. The W action identifies these two cells with two corresponding cells having the first coordinate being instead given by $t_1 \in T^-$.

Putting this together, we state the cells in the G-CW structure:

$$0\ skeleton:\ 2^n \text{ cells of the form } (G/G) \times D^0 \tag{4.11}$$

$$k\ skeleton,\ 1 \leq k \leq n:\ 2^{n-1}\binom{n}{k} \text{ cells of the form } (G/T) \times D^k \tag{4.12}$$

While the cells have been listed, we still need to specify the attaching maps. The 1-cells are attached analogously to the attaching map for the (single) 1-cell in our G-CW complex for G. That is, we have a G-map from $(G/T) \times \partial D^1 \to \{\pm e\}$ that identifies S^0 with the points $\{\pm e\}$ along with the map $G/T \to \{*\}$. The difference now is we identify the right hand factor in each $(G/T) \times D^1$ with one of the $F_n(1)$ edges in an n-hypercube and $(G/T) \times \partial D^1$ is attached to the corresponding vertices.

The analogy with the right hand factors and an n-hypercube continue, where we continue to attach the boundaries of the $2^{k-1}\ \ G/T \times D^k$ cells corresponding to a each of the various $F_n(k)$ k-faces in the n-hypercube into the k-1 skeleton surrounding each such k-face where the map on the right hand factor sends ∂D^k to various D^{k-1} in the way expected for cubes. The trick, however, comes in the first factor and we need to be careful about the map from $G/T \to G/T$ even if the maps on the second factor follow precisely as one expects from the skeleta of an n-hypercube.

Explicitly defining our G-homeomorphism

$$\sigma : (G/T) \times D^k \to (Hom(\mathbb{Z}^n, G))^k \backslash (Hom(\mathbb{Z}^n, G))^{k-1} \tag{4.13}$$

we send $(gT, t_1, \cdots, t_k) \mapsto (g_1, \cdots, g_k)$ where $g_1 = gt_1g^{-1}$ and for $2 \leq i \leq k$ either $g_k = gt_kg^{-1}$ or $(w \cdot g)t_k(w \cdot g)^{-1})$, depending on which cell we are identifying. Each of the 2^{k-1} cells for a particular k-face corresponded with a particular choice of whether or not to apply the antipode in coordinates 2 through k, and this data is now recorded in the G-map.

We can make the data encapsulated by the G-homeomorphism precise by defining an index $\Lambda^n = \mathbb{Z}_2^n$ with a 1 in the ith coordinate corresponding to applying the antipode at that coordinate and a 0 means not doing this. Given our above preference, the first coordinate is thus always a 0. A cell corresponding to index (0,1,1), for instance, would consist of those 3-tuples without trivial coordinates who had the first element on one side of the maximal torus, and the other two on the other side. Then a cell consists of a $(G/T) \times D^k$ together with a G-map σ_λ where

$$\sigma_\lambda(gT, t_1, \cdots, t_k) = ((w^{\lambda_1} \cdot g)t_1(w^{\lambda_1} \cdot g)^{-1}, \cdots, (w^{\lambda_k} \cdot g)t_k(w^{\lambda_k} \cdot g)^{-1}). \tag{4.14}$$

For the attaching G-maps from $(G/T) \times \partial D^k \to (Hom(\mathbb{Z}^n, G))^{k-1}$ we begin with a $(G/T) \times D^k$ k-cell identified by choice of index $\lambda \in \Lambda^k$ that has a 0 in the first coordinate. We identify a particular (k-1)-face of $(G/T) \times \partial D^k$ analogous to the (k-1)-face of ∂D^k. Choosing a face is equivalent to ignoring a particular coordinate in our index so the face can be thought of as being indexed by an element $\tilde{\lambda} \in \Lambda^{k-1}$ whose first coordinate is not necessarily a 0. On the corresponding (k-1)-face in $(Hom(\mathbb{Z}^n, G)^{k-1}$ we have 2^{k-2} $(G/T) \times D^{k-1}$ cells, each identified with a particular choice of index $\delta \in \Lambda^{k-1}$ that has a 0 in the first coordinate.

We thus have two possibilities for an attaching map: $(gT, t_1, \cdots, t_{k-1}) \mapsto (gT, t_1, \cdots, t_{k-1})$ or $(gT, t_1, \cdots, t_{k-1}) \mapsto ((w \cdot g)T, t_1, \cdots, t_{k-1})$. These possibilities correspond to whether the induced index $\tilde{\lambda}$ has a 0 in the first coordinate, in which case there is a cell from the k-1 skeleton with a corresponding index and we don't need to apply w, or whether it has a 1, in which case we do apply w. Notice that there will always be two k-cells whose boundary attaches to particular (k-1)-cell, given the two fold choice in the first coordinate.

4.4 G-CW structure for ΩG

The G-space ΩG has a filtration $\Omega_{poly,r} G$ that is studied in detail in [HJS1]'s computation of the module and product structure of the G-equivariant K-theory of ΩG. For our purposes, a key fact is that this filtration, henceforth labeled F_{2r} for consistency with [HJS1], has a convenient geometric description as quotients F_{2r}/F_{2r-2} are G-homeomorphic to $Thom(\tau^{2r-1})$ where the G-bundle τ is G-isomorphic to the tangent bundle on $\mathbb{P}^1$.

As $G/T \cong \mathbb{P}^1$, we get the following G-CW complex where the G action on the second components are the trivial SU(2) action on $(D^2)^r \cong \mathbb{C}^r$.

$$F_0 \cong * \cong (G/G) \times D^0$$

$$F_2/F_0 \cong Thom(\tau) \cong G/T \times D^2$$

$$F_4/F_2 \cong Thom(\tau^3) \cong (G/T) \times D^6$$

$$\cdots$$

$$F_{2r}/F_{2r-2} \cong Thom(\tau^{2r-1}) \cong (G/T) \times D^{2(2r-1)}$$

The attaching map $(G/T) \times \partial D^2 \to (G/G) \times D^0$ is induced via the map from $(G/T) \to *$ and the usual construction of S^2 by gluing the boundary of D^2 to the point F_0. For the attaching G-map $(G/T) \times \partial D^6 \to (G/G) \times D^2$, consider D^6 as $D^2 \times D^2 \times D^2 \in \mathbb{C}^3$. A G-map from $D^2 \times D^2 \times D^2 \to D^2$ is given by $(z_1, z_2, z_3) \mapsto z_1 + z_2 + z_3$ where the left hand disks are of radius 1 and the right hand disk is of radius 3. The attaching G-map is this map on the second component and the identity on G/T on the first. For the general term, $(z_1, z_2, ..., z_{2r-1}) \mapsto (z_1, z_2, ..., z_{2r-4}, z_{2r-3} + z_{2r-2} + z_{2r-1})$.

Chapter 5

Equivariant K-theory computations

This chapter computes $\tilde{K}^*_G(X)$ of the various G-CW complexes determined in Chapter 4. In Sections 5.3 and 5.4 we additionally compute $Hom(\mathbb{Z}^2, G)$ on other cohomology theories. In Section 5.5 we discuss the R(G) algebra structure for $K^*_G(Hom(\mathbb{Z}^2, G))$. Finally, we discuss some possible generalizations in section 5.7. The module structure for the equivariant K-theory of the various spaces computed in this Chapter are tabulated below.

Table 5.1: $\tilde{K}^*_G(X)$ computations

Space	$\tilde{K}^0_G(X)$	$\tilde{K}^1_G(X)$
G	0	R(G)
$(G/T)\times(G/T)$	$R(G)\oplus R(G)\oplus R(G)$	0
$G\times G$	R(G)	$R(G)\oplus R(G)$
$Hom(\mathbb{Z}^2, G)$	$R(G)\oplus(R(G)\oplus R(G))/<(v,-2)>$	$R(G)\oplus R(G)$
ΩG	$\prod_{k=1}^{\infty} R(G)$	0

5.1 $\tilde{K}^*_G(G),\ \tilde{K}^*_G(G\times G),\ \ etc.$

Example 1. $\tilde{K}^q_G(G)$

The spaces involved are as follows:

$$X^0 \cong S^0$$

$$S(X^0) \cong S^1 \cong S(*) \vee S^1$$

$$X^1/X^0 \cong Thom(G/T \times D^1) \cong S(G/T) \vee S^1$$

The map of spaces $i : X^1/X^0 \hookrightarrow S(X^0)$ is induced by the map $G/T \to *$ in the first factor and identity in the second.

We now compute $\tilde{K}^*_G$:

$$\tilde{K}^0_G(X^0) = \tilde{K}^0_G(2pts) = R(G) = \tilde{K}^1_G(S(X^0))$$
$$\tilde{K}^1_G(X^0) = \tilde{K}^1_G(2pts) = 0 = \tilde{K}^0_G(S(X^0))$$
$$\tilde{K}^0_G(X^1/X^0) = \tilde{K}^0_G(S(G/T)) \oplus \tilde{K}^0_G(S^1) = \tilde{K}^1_G(G/T) \oplus \tilde{K}^1_G(S^0) = 0$$
$$\tilde{K}^1_G(X^1/X^0) = \tilde{K}^1_G(S(G/T)) \oplus \tilde{K}^1_G(S^1) = \tilde{K}^0_G(G/T) \oplus \tilde{K}^0_G(S^0) = R(G) \oplus R(G)$$

We thus get the following six term exact sequence

$$\begin{array}{ccccc} R(G) & \longleftarrow & \tilde{K}^0_G(X) & \longleftarrow & 0 \\ \Big\downarrow{\scriptstyle i^*} & & & & \Big\uparrow \\ R(G) \oplus R(G) & \longrightarrow & \tilde{K}^1_G(X) & \longrightarrow & 0 \end{array} \tag{5.1}$$

$i^* : R(G) \to R(G) \oplus R(G)$, induced by the map of spaces i, takes $v \mapsto (0, v)$. Therefore:

$$\tilde{K}^0_G(X) = ker(i^*) = 0$$
$$\tilde{K}^1_G(X) = coker(i^*) = R(G)$$

Note that this corresponds with our expected answer from ordinary reduced cohomology with integer coefficients which, as G is topologically just S^3, has a single copy of $\mathbb{Z}$ in degree three (which appears here in degree one due to the Z_2 grading in equivariant K-theory). Indeed, one can write out the corresponding exact sequence for ordinary cohomology and obtain the same result. This result also corresponds to [BZ]'s result for $K^q_G(G)$ in the case of $G = SU(2)$.

Example 2. $\tilde{K}^q_G(G/T \times G/T)$

Recall that our G-CW complex for $(G/T) \times (G/T)$ yields the following spaces

$$X^0 \cong G/T \times D^0$$
$$X^2/X^0 \cong Thom(G/T \times D^2) \cong S^2(G/T) \vee S^2$$

by Prop 1 of Section 3.1.

Computing $\tilde{K}^*_G$ gives us

$$\tilde{K}^q_G(X^0) \cong \tilde{K}^q_G(G/T) \cong \begin{cases} R(G) & \text{if } q \text{ even;} \\ 0 & \text{if } q \text{ odd} \end{cases}$$

and

$$\tilde{K}^q_G(X^2/X^0) \cong \tilde{K}^q_G(S^2(G/T) \vee S^2) \cong \tilde{K}^{q+2}_G(G/T) \oplus \tilde{K}^{q+2}(S^0) \cong \begin{cases} R(G) \oplus R(G) & \text{if } q \text{ even;} \\ 0 & \text{if } q \text{ odd} \end{cases}.$$

We thus get the following six term exact sequence corresponding to $X^0 \hookrightarrow X^2 \to X^2/X^0$

$$\begin{CD} R(G) @<<< \tilde{K}^0_G(X^2) @<<< R(G) \oplus R(G) \\ @VVV @. @AAA \\ 0 @>>> \tilde{K}^1_G(X^2) @>>> 0 \end{CD} \tag{5.2}$$

The sequence splits, and thus we get

$$\tilde{K}^0_G(G/T \times G/T) = R(G) \oplus R(G) \oplus R(G) \tag{5.3}$$

$$\tilde{K}^1_G(G/T \times G/T) = 0 \tag{5.4}$$

Note that computing the analogous exact sequence on ordinary reduced cohomology with integer coefficients for $(G/T) \times (G/T)$, which is topologically $S^2 \times S^2$, gives two copies of $\mathbb{Z}$ in degree 2 and one copy of $\mathbb{Z}$ in degree 4 as expected.

Example 3. $\tilde{K}^q_G(G \times G)$

Recall that our one skeleton $X^1 \cong (G \times G)^1$ is as in Diagram 1, and is thus G-homotopic to $(\bigvee_4 S(G/T)) \bigvee S^1$ where G acts trivially on the S^1. Denoting $X = G \times G$, we not that as per Theorem 2,

$$X/X^1 \cong Thom((G/T) \times (G/T) \times D^2) \cong S^2((G/T) \times (G/T)) \vee S^2$$

We now compute $\tilde{K}^q_G(X^1)$, $\tilde{K}^q_G(X/X^1)$,

$$\tilde{K}^0_G(X^1) = (\bigoplus_4 \tilde{K}^0_G(S(G/T))) \bigoplus \tilde{K}^0_G(S^1) = (\bigoplus_4 \tilde{K}^1_G(G/T)) \bigoplus \tilde{K}^1_G(S^0) = 0$$

$$\tilde{K}^1_G(X^1) = (\bigoplus_4 \tilde{K}^1_G(S(G/T))) \bigoplus \tilde{K}^1_G(S^1) = (\bigoplus_4 \tilde{K}^0_G(G/T)) \bigoplus \tilde{K}^0_G(S^0) = \bigoplus_5 R(G)$$

$$\tilde{K}^0_G(X/X^1) = \tilde{K}^0_G(S^2((G/T)\times(G/T))\vee S^2) = \tilde{K}^0_G((G/T)\times(G/T)) \bigoplus \tilde{K}^0_G(S^0) = \bigoplus_4 R(G)$$

$$\tilde{K}^1_G(X/X^1) = \tilde{K}^1_G(S^2((G/T) \times (G/T)) \vee S^2) = \tilde{K}^1_G((G/T) \times (G/T)) \bigoplus \tilde{K}^1_G(S^0) = 0$$

We thus get the following six term exact sequence on equivariant K-theory corresponding to the exact sequence of spaces $X^1 \hookrightarrow X^2 \twoheadrightarrow X^2/X^1$

$$\begin{array}{ccccc} 0 & \longleftarrow & \tilde{K}^0_G(G\times G) & \longleftarrow & \oplus_4 R(G) \\ \downarrow & & & & \uparrow i^* \\ 0 & \longrightarrow & \tilde{K}^1_G(G\times G) & \longrightarrow & \oplus_5 R(G) \end{array} \tag{5.5}$$

To determine i^*, note that corresponding to to the identity map from $S^1 \to S^1$ we get an identity between the right most $R(G)$ factors. Recall that the attaching map for the $(G/T) \times (G/T) \times D^2$ cell is determined by maps $(G/T) \times (G/T) \to G/T$ that are projections onto the first factor along the A and C segments of Diagram 1, and projections onto the second factor along the B and D segments of Diagram 1. In $\tilde{K}^q_G((G/T) \times (G/T))$, there were two copies of $R(G)$, one for each of the two copies of G/T, and a final copy of R(G) coming from the $G/T \times D^2$ cell representing the diagonal elements. If we denote $x_1, \cdots, x_5$ as generators of the summands in $\oplus_5 R(G)$ and $y_1, \cdots, y_4$ as generators of the summands in $\oplus_4 R(G)$ then the map $i^* : \oplus_5 R(G) \to \oplus_4 R(G)$ is described in these bases is given by

$$\begin{pmatrix} Id & 0 & Id & 0 & 0 \\ 0 & Id & 0 & Id & 0 \\ Id & Id & Id & Id & 0 \\ 0 & 0 & 0 & 0 & Id \end{pmatrix}$$

Thus,

$$\tilde{K}^0_G(G \times G) = coker(i^*) = R(G)$$
$$\tilde{K}^1_G(G \times G) = ker(i^*) = R(G) \oplus R(G)$$

Note that doing the analogous computation on reduced ordinary cohomology gives the same result as that expected when considering the space topologically as $S^3 \times S^3$; that is, two copies of $\mathbb{Z}$ in degree 3 and one in degree 6.

Example 4. $\tilde{K}^q_G(((G \times G)\backslash Hom(\mathbb{Z}^2, G))^+)$

Proof. Using our G-CW complex for $((G \times G)\backslash Hom(\mathbb{Z}^2, G))^+$, we compute:

$$K^q_G(((G \times G)\backslash Hom(\mathbb{Z}^2, G))^+) \cong K^q_G(Thom(G/(\mathbb{Z}/2) \times D^3))$$

$$\cong K^q_G(S^3(G/(\mathbb{Z}/2) \sqcup *)) \cong K^{q+1}_G(G/(\mathbb{Z}/2) \sqcup *)$$

Recall that for normal subgroups H of G,

$$K^q_G(G/H) \cong K^q_H(pt) \cong \begin{cases} R(H) & \text{if } q \text{ even;} \\ 0 & \text{if } q \text{ odd.} \end{cases}$$

Thus,

$$\tilde{K}^q_G((G \times G)\backslash Hom(\mathbb{Z}^2, G))^+) \cong \begin{cases} 0 & \text{if } q \text{ even;} \\ R(\mathbb{Z}/2) & \text{if } q \text{ odd.} \end{cases}$$

□

This computation lets us state a result regarding $i^* : K^q_G(G \times G) \to K^q_G(Hom(\mathbb{Z}^2, G)$, which is equivalent to Proposition 6.11 of [AG].

Theorem 1. *Let* $i : Hom(\mathbb{Z}^2, G) \to G^2$ *be the inclusion map. Then*

$$i^* : K^1_G(G \times G) \xrightarrow{\cong} K^1_G(Hom(\mathbb{Z}^2, G))$$

is an isomorphism and there is a short exact sequence of R(G)-modules

$$0 \to K^0_G(G \times G) \to K^0_G(Hom(\mathbb{Z}^2, G)) \to R(\mathbb{Z}/2) \to 0$$

Proof. Consider the six-term exact sequence corresponding to the pair $(G\times G, Hom(\mathbb{Z}^2, G)$. The result follows from the above computation for the noncommuting 2-tuples together with the isomorphism

$$K^q_G(G \times G, Hom(\mathbb{Z}^2, G)) \cong \tilde{K}^q_G((G \times G)\backslash Hom(\mathbb{Z}^2, G))^+)$$

□

5.2 Module structure of $K^*_G(Hom(\mathbb{Z}^2, G))$

From the G-CW structure described in section 4.2, we will compute the module structure for $K^*_G(Hom(\mathbb{Z}^2, G))$ and $K^*_T(Hom(\mathbb{Z}^2, G))$ and verify it works out as expected for ordinary cohomology $H^*(Hom(\mathbb{Z}^2, G))$. We begin the computation on reduced G-equivariant K-theory.

First note that the that $K^*_T(G/T)$ was computed in [HJS1] to be

$$K^*_T(G/T) = R(T)[L]/ < L^2 - (b + b^{-1})L + 1 > \tag{5.6}$$

and so for the module structure we will use $\tilde{K}^0_T(G/T) = R(T)$.

Recall that our one skeleton $X^1 \cong (Hom(\mathbb{Z}^2, G))^1$ is the same as $(G \times G)^1$ and hence we restate our previous computation.

$$\tilde{K}^0_G(X^1) = 0 \tag{5.7}$$

$$\tilde{K}^1_G(X^1) = \bigoplus_5 R(G) \tag{5.8}$$

Further, we have $(G/T) \times D^2$ 2-cells, and so Proposition 1 of 3.1 gives

$$X^2/X^1 \cong \bigvee_2 Thom((G/T) \times D^2) \cong \bigvee_2((S^2(G/T)) \vee S^2) \tag{5.9}$$

and so

$$\tilde{K}^0_G(X^2/X^1) = \tilde{K}^0_G(\bigvee_2((S^2(G/T)) \vee S^2)) = \bigoplus_2(\tilde{K}^0_G(G/T) \bigoplus \tilde{K}^0_G(S^0)) = \bigoplus_4 R(G) \tag{5.10}$$

$$\tilde{K}^1_G(X^2/X^1) = \tilde{K}^1_G(\bigvee_2((S^2(G/T)) \vee S^2)) = \bigoplus_2(\tilde{K}^1_G(G/T) \bigoplus \tilde{K}^1_G(S^0)) = 0. \tag{5.11}$$

We thus get the following six term exact sequence on equivariant K-theory corresponding to the exact sequence of spaces $X^1 \hookrightarrow X^2 \twoheadrightarrow X^2/X^1$

$$\begin{array}{ccccc} 0 & \longleftarrow & \tilde{K}^0_G(Hom(\mathbb{Z}^2, G)) & \longleftarrow & \oplus_4 R(G) \\ \downarrow & & & & \uparrow{\scriptstyle i^*} \\ 0 & \longrightarrow & \tilde{K}^1_G(Hom(\mathbb{Z}^2, G)) & \longrightarrow & \oplus_5 R(G) \end{array} \tag{5.12}$$

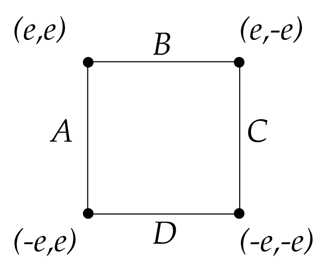
And hence

$$\tilde{K}^0_G(Hom(\mathbb{Z}^2, G)) = coker(i^*) \tag{5.13}$$

$$\tilde{K}^1_G(Hom(\mathbb{Z}^2, G)) = ker(i^*) \tag{5.14}$$

To determine i^*, first recall the diagram

Diagram 2. $(Hom(\mathbb{Z}^2, G))^1$



For each of our two 2-cells, $\partial D^2 \cong S^1$ is identified with the boundary square in this diagram, and hence we get the diagonal map from $R(G) \to R(G) \oplus R(G)$ on the right most factors. Recall that the attaching map for the first of the two $(G/T) \times D^2$ cells is induced by the identity map on G/T along all four segments while the attaching map for the second of two $(G/T) \times D^2$ cells is induced by the identity on G/T for the A and C segments, but by the antipodal map a on G/T for the B and D segments. If we denote $x_1, \cdots, x_5$ as generators of the summands in $\oplus_5 R(G)$ and $y_1, \cdots, y_4$ as generators of the summands in $\oplus_4 R(G)$ then the map $i^* : \oplus_5 R(G) \to \oplus_4 R(G)$ is described in these bases is given by

$$\begin{pmatrix} Id & Id & Id & Id & 0 \\ Id & a^* & Id & a^* & 0 \\ 0 & 0 & 0 & 0 & Id \\ 0 & 0 & 0 & 0 & Id \end{pmatrix}$$

We aim to compute the kernel and cokernel of this matrix. After reducing this matrix, the only nonobvious component to consider is the image of $(a^* - Id) : \tilde{K}^0_G(G/T) \to \tilde{K}^0_G(G/T)$, which we will study first on unreduced $(a^* - Id) : K^0_G(G/T) \to K^0_G(G/T)$.

Recall that $K^0_G(G/T) = K^0_T(pt)$ which, as an $R(T)$-module, is just $R(T) = \mathbb{Z}[b, b^{-1}]$; that is, Laurent polynomials in the variable b. Identifying $b^{-1} = v - b$, we can express a generic element as $p + qb$ where $p, q \in \mathbb{Z}[v]$.

Consider the antipodal map $a : G/T \to G/T$ given by $a(gT) = (w \cdot g)T$ where w is the nontrivial element of the Weyl group. As the induced action of w, the nontrivial element of the Weyl group, on K-theory is to interchange b and b^{-1}, a induces the map $a^* : \mathbb{Z}[b, b^{-1}] \to \mathbb{Z}[b, b^{-1}]$ given by $(a^*)(p + qb) = p + qb^{-1}$. Hence, $(a^* - Id)(p + qb) = q(b^{-1} - b)$.

The cokernel of this map is as an R(T)-module is thus:

$$coker((a^* - Id) : K^0_G(G/T) \to K^0_G(G/T)) = R(T)/ < b^{-1} - b > . \quad (5.15)$$

However, we wish to write this as an R(G)-module as we are doing G-equivariant K-Theory. Recall that we were describing a generic element in $R(T)$ as $p + qb$ where $p, q \in \mathbb{Z}[v]$. We can thus describe R(T) as an R(G)-module as $R(G) \oplus R(G)$ using the $\{1, b\}$ basis. In particular, we can denote b^{-1} as $1v + (-1)b$ in this basis. Since as an R(T) module, $a^*(p+qb) = p+qb^{-1} = p+vq-qb$, writing it in this way as an R(G)-module we get that $a^* : R(G) \oplus R(G) \to R(G) \oplus R(G)$ thus takes $a^*(p, q) = (p+vq, -q)$. Hence, as an R(G)-module, $(a^* - Id)(p, q) = (vq, -2q)$. The cokernel of this map as an R(G)-module is thus:

$$coker((a^* - Id) : K^0_G(G/T) \to K^0_G(G/T)) = (R(G) \oplus R(G))/ < (v, -2) > . \quad (5.16)$$

Finally, combining with the other components of the matrix thus gives

$$K^0_G(Hom(\mathbb{Z}^2, G)) = coker(i^*) = R(G) \oplus (R(G) \oplus R(G))/ < (v, -2) > \tag{5.17}$$

$$K^1_G(Hom(\mathbb{Z}^2, G)) = ker(i^*) = R(G) \oplus R(G). \tag{5.18}$$

Note that this result matches that of [AG].

5.3 Module Structure of $K^*_T(Hom(\mathbb{Z}^2, G))$

We can repeat the computation of Section 5.2 tersely on T-equivariant K-theory knowing that, in [HJS2], the R(T)-algebra $K^*_G(G/T)$ is computed to be $R(T)[L]/ < L^2 - (b^{-1} - b)L + 1 >$. Thus on reduced as R(T)-modules we get $K^0_T(G/T) = R(T)$ and $K^1_T(G/T) = 0$.

$$\tilde{K}^0_T(X^2/X^1) = \tilde{K}^0_T(\bigvee_2((S^2(G/T)) \vee S^2)) = \bigoplus_2(\tilde{K}^0_T(G/T) \bigoplus \tilde{K}^0_G(S^0)) = \bigoplus_4 R(T) \tag{5.19}$$

$$\tilde{K}^1_T(X^2/X^1) = \tilde{K}^1_T(\bigvee_2((S^2(G/T)) \vee S^2)) = \bigoplus_2(\tilde{K}^1_T(G/T) \bigoplus \tilde{K}^1_G(S^0)) = 0. \tag{5.20}$$

We thus get the following six term exact sequence on equivariant K-theory corresponding to the exact sequence of spaces $X^1 \hookrightarrow X^2 \twoheadrightarrow X^2/X^1$

$$\begin{array}{ccccc} 0 & \longleftarrow & \tilde{K}^0_T(Hom(\mathbb{Z}^2, G)) & \longleftarrow & \oplus_4 R(T) \\ \downarrow & & & & \uparrow i^* \\ 0 & \longrightarrow & \tilde{K}^1_T(Hom(\mathbb{Z}^2, G)) & \longrightarrow & \oplus_5 R(T) \end{array} \tag{5.21}$$

with the same matrix for i^* only now written in terms of basis elements for $\oplus_5 R(T)$ and $\oplus_4 R(T)$ as opposed to $\oplus_5 R(G)$ and $\oplus_4 R(G)$. The same key map, $(a^* - Id)$ appears upon reducing and since as R(T)-modules $\tilde{K}^0_T(G/T) = K_G(G/T)$ we have already computed the cokernel above. Combining everything, we get

$$K^0_T(Hom(\mathbb{Z}^2, G)) = coker(i^*) = R(T) \oplus R(T) \oplus R(T)/ < b^{-1} - b > \tag{5.22}$$

$$K^1_T(Hom(\mathbb{Z}^2, G)) = ker(i^*) = R(T) \oplus R(T). \tag{5.23}$$

Note that we thus get a torsion term in T-equivariant K-Theory whose corresponding term on G-equivariant K-theory is not a torsion term. For an alternative way to view this, we can consider $(\sum_{i=0}^{\infty} a_i v^i, \sum_{i=0}^{\infty} b_i v^i) \in R(G) \oplus R(G)$ with at most a finite number of nonzero $a_i,\ b_i \in \mathbb{Z}$. Under the identification that $(v, -2) = (0, 0)$, we can thus identify $(\sum_{i=0}^{\infty} a_i v^i, \sum_{i=0}^{\infty} b_i v^i) = (a_0, \sum_{i=0}^{\infty}(b_i + 2a_{i+1})v^i)$ so this nontrivial summand in our cokernel can be identified with $\mathbb{Z} \oplus \mathbb{Z}[v]$. Hence this is not a free R(G)-module. Yet another representation of a generic element is

$$(\sum_{i=0}^{\infty} a_i v^i, \sum_{i=0}^{\infty} b_i v^i) = (a_0 + \sum_{i=1}^{\infty}(1/2b_{i-1} + a_i)v^i, 0)$$

Hence if we invert the $2 \in R(G)$, this becomes a free module. This is analogous to the 2-torsion in degree 4 that occurs in the integral cohomology calculation that disappears upon inverting 2.

We note that in this case $K^*_G(Hom(\mathbb{Z}^2, G)) = K^*_T(Hom(\mathbb{Z}^2, G))^W$. Indeed, $R(T)^W = R(G)$. Further, T-equivariantly we have a torsion term consisting of $R(T)/ < b^{-1} - b >$ which is polynomials in just the variable b. Hence, upon taking Weyl Invariance, only the constant polynomials survive and so $(R(T)/ < b^{-1} - b >)^W = \mathbb{Z}$. This is the same copy of the integers that appears in the G-equivariant computation, hence the equality after taking Weyl Invariance.

5.4 $Hom(\mathbb{Z}^2, G)$ on other cohomology theories

Let's repeat the computation on various other cohomology theories. In the case of $H^*(Hom(\mathbb{Z}^2, G); \mathbb{Z})$ we verify that the results match the expected results from [BJS] and [AC]. Secondly we do the case of $H^*_T(Hom(\mathbb{Z}^2, G); \mathbb{Z})$ and finally $H^*_G(Hom(\mathbb{Z}^2, G); \mathbb{Z})$ neither of which has not been recorded in the literature.

5.4.1 $H^*(Hom(\mathbb{Z}^2, G); \mathbb{Z})$

We begin on ordinary integral cohomology and verify that it works as expected. Following the computation from Section 5.2, recall that our 1 skeleton for $Hom(\mathbb{Z}^2, G)$

consists of $X^1 = (\bigvee_4 SG/T) \vee S^1$ which is topologically $(\bigvee_4 (S^3) \vee S^1$ and our quotient has the form $X^2/X^1 = \bigvee_2 (S^2 G/T) \vee S^2)$ which is topologically $\bigvee_2 (S^4 \vee S^2)$. The long exact sequence on for cohomology for $X^1 \hookrightarrow X^2 \to X^2/X^1$ thus has all zeros in even degree for X^1 terms and zeros in odd degree for X^2/X^1 terms. We are left with the following exact sequences of the form:

$$0 \leftarrow \tilde{H}^{2q}(X^2) \leftarrow \tilde{H}^{2q}(\bigvee_2 (S^4) \vee S^2)) \overset{i^*}{\leftarrow} \tilde{H}^{2q-1}((\bigvee_4 S^3) \vee S^1) \leftarrow \tilde{H}^{2q-1}(X^2) \leftarrow 0$$

which by desuspending can be rewritten for $q \geq 1$

$$0 \leftarrow \tilde{H}^{2q}(X^2) \leftarrow \tilde{H}^{2q-2}(\bigvee_2 (S^2) \vee S^0)) \overset{i^*}{\leftarrow} \tilde{H}^{2q-2}((\bigvee_4 S^2) \vee S^0) \leftarrow \tilde{H}^{2q-1}(X^2) \leftarrow 0$$

and so we aim to compute the kernel and cokernel of i^* as before.

For the rightmost factors S^0 vectors in the wedge, this induces on integral cohomology the diagonal map $H^0(pt) = \mathbb{Z} \to H^0(pt) \oplus H^0(pt) = \mathbb{Z} \oplus \mathbb{Z}$. This results in a single factor of $\mathbb{Z}$ coming from the cokernel of this map in degree 2.

For the left hand factors, first let us consider the antipodal map $a : G/T \to G/T$ thought of now topologically as $a : S^2 \to S^2$. This now induces the map $a^* : H^2(S^2; \mathbb{Z}) = \mathbb{Z} \to H^2(S^2; \mathbb{Z}) = \mathbb{Z}$ that is just multiplication by -1 on integral cohomology. The $a^* - Id$ map we saw previously is thus multiplication by -2. As such, denoting $x_1, \cdots, x_4$ as generators for each factor in $H^2(\bigvee_4 G/T) = \mathbb{Z}^4$ and denoting y_1, y_2 as generators for each factor in $H^2(\bigvee_2 S^2) = \mathbb{Z}^2$ we thus get the map from $H^2(\bigvee_2 S^2) \to H^2(\bigvee_4 G/T)$ given by the matrix in this basis:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

As this matrix has kernel $\mathbb{Z}^2$ and cokernel $\mathbb{Z}_2$, putting all of this together gives us

$$\tilde{H}^q(Hom(\mathbb{Z}^2, G)) = \begin{cases} \mathbb{Z} & q = 2 \\ \mathbb{Z} \oplus \mathbb{Z} & q = 3 \\ \mathbb{Z}/2 & q = 4 \\ 0 & \text{else} \end{cases}$$

Note that this result matches that of [BJS] and [AC].

5.4.2 $H_T^*(Hom(\mathbb{Z}^2, G); \mathbb{Z})$

Now let us consider $H_T^*(Hom(\mathbb{Z}^2, G))$, following from the introduction to equivariant cohomology in Section 3.3.

Following the presentation of [HJS2], let b denote the weight 1 one-dimensional representation of T, as before. Let $\bar{b} = c_1^T(b) \in H_T^2(pt; \mathbb{Z}) = H^2(BT; \mathbb{Z})$ be the equivariant Chern class of b. We thus get $H_T^*(pt; \mathbb{Z}) = \mathbb{Z}[\bar{b}]$. The nontrivial element of the Weyl group W acts on $\mathbb{Z}[\bar{b}]$ via $w(\bar{b}) = -\bar{b}$.

[HJS2] compute that

$$H_T^*(G/T; \mathbb{Z}) = \mathbb{Z}[\bar{b}][L]/\langle L^2 - \bar{b}^2 \rangle.$$

where $\bar{L}$ is the first chern class of the bundle L described in [HJS2]; for our purposes here it sufficies to note that it is a complex line bundle that is Weyl invariant.

In particular, we observe that as a $\mathbb{Z}$ module we get that

$$H_T^{2q+1}(G/T; \mathbb{Z}) = 0$$

and

$$H_T^{2q}(G/T; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$$

with the factors generated by $\bar{b}^q$ and $\bar{L}\bar{b}^{q-1}$.

Now consider the antipodal map between G/T and G/T which is given by multiplying by the nontrivial element of the Weyl group. This clearly induces the trivial map on odd degrees. Further, $a^* : H_G^{4q}(G/T) \to H_G^{4q}(G/T)$ is given by

$$a^*(p\bar{b}^{2q} + q\bar{L}\bar{b}^{2q-1}) = pw \cdot \bar{b}^{2q} + qw \cdot (\bar{L}\bar{b}^{2q-1}) = p\bar{b}^{2q} - q\bar{L}\bar{b}^{2q-1}$$

for $p, q \in \mathbb{Z}$, as the Weyl group acts trivially on $\bar{L}$ and even powers of $\bar{b}$, and by multiplication by -1 on odd powers of $\bar{b}$. Repeating this story in degree $4q+2$ we get

$$a^*(p\bar{b}^{2q+1} + q\bar{L}\bar{b}^{2q}) = pw \cdot \bar{b}^{2q+1} + qw \cdot (\bar{L}\bar{b}^{2q}) = -p\bar{b}^{2q+1} + q\bar{L}\bar{b}^{2q}.$$

Repeating these computations for the map we will really care about, $a^* - Id$ where Id represents the identity map, gives us

$$(a^* - Id)(p\bar{b}^{2q} + q\bar{L}\bar{b}^{2q-1}) = -2q\bar{L}\bar{b}^{2q-1}$$

in degree 4q and

$$a^*(p\bar{b}^{2q+1} + q\bar{L}\bar{b}^{2q}) = -p\bar{b}^{2q+1}$$

in degree 4q-2. In either even degree case, we thus get a cokernel with a $\mathbb{Z}$ summand and a $\mathbb{Z}_2$ summand, only with the order of the factors differing between the two cases.

Now let us consider $Hom(\mathbb{Z}^2, G)$. Following the computation from Section 5.2, recall that our 1 skeleton consists of $X^1 = (\bigvee_4 SG/T) \vee S^1$ and our quotient has the form $X^2/X^1 = \bigvee_2 (S^2G/T) \vee S^2)$. The long exact sequence on T-equivariant cohomology for $X^1 \hookrightarrow X^2 \to X^2/X^1$ thus has all zeros in even degree for X^1 terms and zeros in odd degree for X^2/X^1 terms. We are left with

$$0 \leftarrow \tilde{H}^{2q}_T(X^2) \leftarrow \tilde{H}^{2q}_T(\bigvee_2 (S^2G/T) \vee S^2)) \xleftarrow{i^*} \tilde{H}^{2q-1}_T((\bigvee_4 SG/T) \vee S^1) \leftarrow \tilde{H}^{2q-1}_T(X^2) \leftarrow 0$$

which by desuspending can be rewritten for $q \geq 1$ as

$$0 \leftarrow \tilde{H}^{2q}_T(X^2) \leftarrow \tilde{H}^{2q-2}_T(\bigvee_2 (G/T) \vee S^0)) \xleftarrow{i^*} \tilde{H}^{2q-2}_T((\bigvee_4 G/T) \vee S^0) \leftarrow \tilde{H}^{2q-1}_T(X^2) \leftarrow 0.$$

We thus need to compute the kernel and cokernel of i^*. For the right hand S^0 factors in the wedge, we get the diagonal map $H^{2q-2}_T(pt) = \mathbb{Z} \to H^{2q-2}_T(pt) \oplus H^{2q-2}_T(pt) = (\mathbb{Z} \oplus \mathbb{Z}) \oplus (\mathbb{Z} \oplus \mathbb{Z})$. This has trivial kernel, but picks up a factor of $(\mathbb{Z} \oplus \mathbb{Z})$ coming from the cokernel in all even degrees $2q$ for $q \geq 2$.

For the left factors in the wedge, for $q \geq 1$ let $x_1, \cdots x_4$ be generators for each of the summands in $\bigoplus_4 \tilde{H}_T^{2q-2}(G/T)$ and let y_1, y_2 be generators for the summands in $\tilde{H}_T^{2q-2}(G/T)$. The attaching maps work as before giving us in this basis the matrix

$$\begin{pmatrix} Id & Id & Id & Id \\ Id & a^* & Id & a^* \end{pmatrix}$$

Recall that we had previously studied the map $a^* - Id$ which occurs upon reducing the matrix and observed that the cokernel of this was $\mathbb{Z} \oplus \mathbb{Z}_2$ and a kernel of two copies of $H_T^{2q-2}(G/T); \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$ in degree $2q-2$ for $q \geq 2$. Putting all of this together gives us

$$\tilde{H}_T^0(Hom(\mathbb{Z}^2, G)) = 0$$

$$\tilde{H}_T^1(Hom(\mathbb{Z}^2, G)) = 0$$

$$\tilde{H}_T^2(Hom(\mathbb{Z}^2, G)) = \mathbb{Z}$$

$$\tilde{H}_T^{2q-1}(Hom(\mathbb{Z}^2, G)) = \oplus_4 \mathbb{Z} \text{ for } q \geq 2$$

$$\tilde{H}_T^{2q}(Hom(\mathbb{Z}^2, G)) = (\mathbb{Z}_2 \oplus \mathbb{Z}) \oplus (\mathbb{Z} \oplus \mathbb{Z}) \text{ for } q \geq 2.$$

5.4.3 $H_G^*(Hom(\mathbb{Z}^2, G); \mathbb{Z})$

Finally, let us consider the case of $H_G^*(Hom(\mathbb{Z}^2, G))$. Following [HJS2] again, we denote $\bar{t} := \bar{b}^2$ which lives in degree 4. Then

$$H_G^*(pt; \mathbb{Z}) = H(BG; \mathbb{Z}) = (H^*(BT; \mathbb{Z})^W = \mathbb{Z}[\bar{t}].$$

The arguments of Section 3 of [HJS2] while stated rationally all work integrally as well. In particular we have that

$$H_G^*(G/T; \mathbb{Z}) = \mathbb{Z}[\bar{t}][\bar{L}]/\langle \bar{L}^2 - \bar{t} \rangle$$

where $\bar{L}$ is as before.

In particular, we observe that as a $\mathbb{Z}$ module we get that

$$H_G^{2q+1}(G/T; \mathbb{Z}) = 0$$

and

$$H^{2q}_G(G/T;\mathbb{Z}) = \mathbb{Z}$$

with the factors generated by $\bar{t}^q$ in degrees 4q and generated by $\bar{L}\bar{t}^{q-1}$ in degrees 4q-2 .

Since $w(\bar{b}) = -\bar{b}$, $w(\bar{t}) = \bar{t}$. As such, all of $H^*_G(G/T;\mathbb{Z})$ is invariant under the action of w. The core map that we will have to study is $a^* - Id : H^*_G(G/T;\mathbb{Z}) \to H^*_G(G/T;\mathbb{Z})$ is thus just the zero map. Repeating what we did for $H^*_T(Hom(\mathbb{Z}^2, G))$, we get segments of the long exact sequence corresponding to $X^1 \hookrightarrow X^2 \to X^2/X^1$ that look like

$$0 \leftarrow \tilde{H}^{2q}_G(X^2) \leftarrow \tilde{H}^{2q}_G(\bigvee_2 (S^2 G/T) \vee S^2)) \overset{i^*}{\leftarrow} \tilde{H}^{2q-1}_G((\bigvee_4 SG/T) \vee S^1) \leftarrow \tilde{H}^{2q-1}_G(X^2) \leftarrow 0$$

which by desuspending can be rewritten for $q \geq 1$ as

$$0 \leftarrow \tilde{H}^{2q}_G(X^2) \leftarrow \tilde{H}^{2q-2}_G(\bigvee_2 (G/T) \vee S^0)) \overset{i^*}{\leftarrow} \tilde{H}^{2q-2}_G((\bigvee_4 G/T) \vee S^0) \leftarrow \tilde{H}^{2q-1}_G(X^2) \leftarrow 0.$$

As before, for the right hand S^0 factors in the wedge, we get the diagonal map $H^{2q-2}_G(pt) = \mathbb{Z} \to H^{2q-2}_G(pt) \oplus H^{2q-2}_T(pt) = \mathbb{Z} \oplus \mathbb{Z}$. This has trivial kernel, but picks up a factor of $\mathbb{Z}$ coming from the cokernel in all even degrees $2q$ for $q \geq 1$.

For the left factors in the wedge, for $q \geq 1$ let $x_1, \cdots x_4$ be generators for each of the summands in $\bigoplus_4 \tilde{H}^{2q-2}_G(G/T)$ and let y_1, y_2 be generators for the summands in $\tilde{H}^{2q-2}_G(G/T)$. The attaching maps work as before giving us in this basis the matrix

$$\begin{pmatrix} Id & Id & Id & Id \\ Id & a^* & Id & a^* \end{pmatrix}$$

which reduces to

$$\begin{pmatrix} Id & Id & Id & Id \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

because of our earlier analysis that the antipodal map acts via the identity. Since as a $\mathbb{Z}$-module each $\tilde{H}^{2q}_G(G/T)$ is a single factor of $\mathbb{Z}$ in all even degrees, we get three copies of the integers in the kernel, and 1 in cokernel in degree $2q$ for $q \geq 1$.

Putting this all together we get:

$$\tilde{H}^0_G(Hom(\mathbb{Z}^2, G)) = 0$$
$$\tilde{H}^1_G(Hom(\mathbb{Z}^2, G)) = 0$$
$$\tilde{H}^2_G(Hom(\mathbb{Z}^2, G)) = \mathbb{Z}$$
$$\tilde{H}^{2q-1}_G(Hom(\mathbb{Z}^2, G)) = \oplus_3\mathbb{Z} \text{ for } q \geq 2$$
$$\tilde{H}^{2q}_G(Hom(\mathbb{Z}^2, G)) = \oplus_2\mathbb{Z} \text{ for } q \geq 2.$$

5.5 Algebra structure for $K^*_G(Hom(\mathbb{Z}^2, G))$

Recall that we had described both the quotient of skeleta, X^2/X^1, and the one skeleton X^1, as wedges of suspensions:

$$X^2/X^1 \cong \bigvee_2((S^2(G/T)) \vee S^2) \tag{5.24}$$

$$X^1 \cong (\bigvee_4 S(G/T)) \vee S^1 \tag{5.25}$$

As such, the multiplicative structure on any of our cohomology theories for either of these spaces is automatically trivial. Thus, if we consider the aforementioned six term exact sequence

$$\begin{array}{ccccc} 0 & \longleftarrow & \tilde{K}^0_T(Hom(\mathbb{Z}^2, G)) & \longleftarrow & \oplus_4 R(T) \\ \downarrow & & & & \uparrow{\scriptstyle i^*} \\ 0 & \longrightarrow & \tilde{K}^1_T(Hom(\mathbb{Z}^2, G)) & \longrightarrow & \oplus_5 R(T) \end{array} \tag{5.26}$$

these maps are all ring homomorphisms and thus the trivial multiplicative structure carries forward onto $K^*_T(Hom(\mathbb{Z}^2, G))$. Denoting a generator for each of the four summand in the module structure by x_i, we thus get

$$\tilde{K}^*_T(Hom(\mathbb{Z}^2, G)) = R(T)[x_1, x_2, x_3, x_4, x_5]/\langle\{x_ix_j = 0\ \forall i, j\}, (b^{-1} - b)x_2\rangle. \tag{5.27}$$

Repeating the computation G-equivariantly, we get

$$\tilde{K}^*_G(Hom(\mathbb{Z}^2, G)) = R(T)[x_1, x_2, x_3, x_4, x_5]/\langle\{x_ix_j = 0\ \forall i, j\}, vx_2 - 2x_3\rangle. \tag{5.28}$$

5.6 Module Structure of $\tilde{K}^*_G(\Omega G)$

Recall that following the work of [HJS1], we had established a G-CW complex for $\Omega_{poly}G$ consisting of a point in degree zero and a $(G/T) \times D^{2(2r-1)}$ cell in each even degree. Computing $\tilde{K}^q_G$,

$$\tilde{K}^q_G(F_0) \cong \tilde{K}^q_G(pt) \cong 0 \; \forall q$$

$$\tilde{K}^q_G(F_{2r}/F_{2(r-1)}) \cong \tilde{K}^q_G(Thom((G/T) \times D^{2(2r-1)})) \cong$$
$$\cong \tilde{K}^q_G((G/T) \coprod *) \cong \begin{cases} R(G) \oplus R(G) & \text{if } q \text{ even;} \\ 0 & \text{if } q \text{ odd.} \end{cases}$$

Thus our exact sequence

$$\begin{array}{ccccc} \tilde{K}^0_G(F_{2(r-1)}) & \longleftarrow & \tilde{K}^0_G(F_{2r}) & \longleftarrow & \tilde{K}^0_G(F_{2r}/F_{2(2r-1)}) \\ \downarrow & & & & \uparrow \\ \tilde{K}^1_G(F_{2r}/F_{2(2r-1)}) & \longrightarrow & \tilde{K}^1_G(F_{2r}) & \longrightarrow & \tilde{K}^1_G(F_{2(r-1)}) \end{array}$$

becomes

$$\begin{array}{ccccc} \tilde{K}^0_G(F_{2(r-1)}) & \longleftarrow & \tilde{K}^0_G(F_{2r}) & \longleftarrow & R(G) \oplus R(G) \\ \downarrow & & & & \uparrow \\ 0 & \longrightarrow & \tilde{K}^1_G(F_{2r}) & \longrightarrow & \tilde{K}^1_G(F_{2(r-1)}) \end{array}$$

where, by induction with $\tilde{K}^*_G(F_0)$ as the basis case, $\tilde{K}^*_G(F_{2(r-1)})$ is 0 in odd degree and a free R(G)-module in even degree and hence the sequence in the top row splits,

$$\tilde{K}^q_G(F_{2r}) \cong \begin{cases} \tilde{K}^q_G(F_{2(r-1)}) \oplus R(G) \oplus R(G) & \text{if } q \text{ even;} \\ 0 & \text{if } q \text{ odd.} \end{cases}$$

By induction we get

$$\tilde{K}^q_G(F_{2r}) \cong \begin{cases} \prod_{k=1}^{2r} R(G) & \text{if } q \text{ even;} \\ 0 & \text{if } q \text{ odd.} \end{cases}$$

Finally,

$$\tilde{K}^q_G(\Omega G) \cong \tilde{K}^q_G(\Omega_{poly} G) \cong \varprojlim \tilde{K}^q_G(\Omega_{poly,r} G) \cong \begin{cases} \prod_{k=1}^{\infty} R(G) & \text{if } q \text{ even;} \\ 0 & \text{if } q \text{ odd.} \end{cases}$$

5.7 Future Work

There are two immediate generalizations of our present computation of the commuting 2-tuples.

- Commuting n-tuples in SU(2). The G-CW complex is written down in Chapter 3, from which K_G chould follow. Heuristically, the important information encoded by our G-CW complex is a various maps between G/T and G/T; either the identity map or the antipodal map. Hence, analogously to the n=2 case, one will get matrices whose components are either the identity map or the antipodal map (or the zero map). A closed form solution may not be possible, and even n=3 is cumbersome, but at least this should be able to be programmed for a computer to produce the computation for a given value of n.

- Commuting m-tuples (or just 2-tuples) in SU(n). For SU(n) the maximal torus is a torus of dimension n-1 opposed to only being dimension 1 as it was in our SU(2) case. This adds some complexity, but nonetheless constructing a G-CW complex on $Hom(\mathbb{Z}^m, SU(n))$ from which one can compute various cohomology theories seems promising.

Chapter 6

Quasi-Hamiltonian Systems

In this chapter we outline the key ideas regarding quasi-Hamiltonian Systems. In section 6.1 we will introduce the concept and note that the spaces we have been computing throughout this thesis are motivated, in part, by the spaces in a particular example of a quasi-Hamiltonian system. In Section 6.2 we outline the literature results on Kirwan surjectivity. In Section 6.3 we discussed some potential for future work.

6.1 Quasi-Hamiltonian Systems

Alekseev, Malkin, and Meinrenken [AMM] introduced quasi-Hamiltonian G-spaces which are certain finite-dimensional G-spaces equipped with a G-valued quasi-Hamiltonian moment map $\Phi : M \to G$. Following their presentation, for a compact Lie group G, we make a choice of an invariant positive definite inner product $\langle \cdot, \cdot \rangle$ on the lie algebra $\mathcal{G}$. Let $\theta, \bar{\theta} \in \Omega^1(G, \mathcal{G})$ denote the left- and right-invariant Maurer-Cartan forms, respectively. Let $\chi = \frac{1}{12}\langle \theta, [\theta, \bar{\theta}] \rangle$ denote the canonical bi-invariant 3-form on G. Finally, let ν_ξ denote the genderating vector field on M for an element $\xi \in \mathcal{G}$. Then a quasi-Hamiltonian G-space is a G-manifold M with an invariant 2-form $\omega \in \Omega(M)^G$ and a G-map $\phi \in C^\infty(M, G)^G$ such that

1. $d\omega = -\phi^* \chi$
2. $i(\nu_\xi)\omega = \frac{1}{2}\phi^* \langle \theta + \bar{\theta}, \xi \rangle$
3. For each $x \in M$, $ker \omega_x = \{\nu_\xi, \ \xi \in ker(Ad_{\phi(x)} + 1)\}$

These finite dimensional quasi-Hamiltonian G-spaces are in bijective correspondence with certain infinite dimensional Hamiltonian L_sG-spaces $\mathcal{M}$ that fit into the following diagram where $\mathcal{M}$ is a pullback of the bottom right square and the columns are fibrations with fibre ΩG. $\Phi^{-1}(e)$ is referred to as the symplectic quotient $M//G$.

$$\begin{array}{ccccc} & & \Omega G & \xrightarrow{=} & \Omega G \\ & & \downarrow & & \downarrow \\ \tilde{\Phi}^{-1}(0) & \hookrightarrow & \mathcal{M} & \xrightarrow{\tilde{\Phi}} & L\mathcal{G}^* \\ \downarrow = & & \downarrow \pi & & \downarrow Hol \\ \Phi^{-1}(e) & \hookrightarrow & M & \xrightarrow{\Phi} & G \end{array} \tag{6.1}$$

Consider the following specific example of a quasi-Hamiltonian G-space. Let $G = SU(2)$ which acts on itself by conjugation. The space $M = (G \times G)^h$ with the diagonal action and equipped with the G-map $\Phi : (G \times G)^h \to G$ given by taking the commutator map $\Phi(a_1, \ldots, a_h, b_1, \ldots, b_h) = \prod_{i=1}^{h}[a_i, b_i]$ provides an example of a quasi-Hamiltonian G-space. The corresponding symplectic quotient $\mathcal{M}//L_sG$ is the moduli space of flat connections on the trivial principal G-bundle over a compact, connected 2-manifold Σ of genus h with boundary $\partial\Sigma = S^1$ (See [AMM]). In particular, we will consider the h=1 case which has kernel $\Phi^{-1}(e) = Hom(\mathbb{Z}^2, G)$. For this specific case we relabel parts of the diagram:

$$\begin{array}{ccccc} & & \Omega G & \xrightarrow{=} & \Omega G \\ & & \downarrow & & \downarrow \\ Hom(\mathbb{Z}^2, G) & \hookrightarrow & \mathcal{M} & \xrightarrow{\tilde{\Phi}} & L\mathcal{G}^* \\ \downarrow = & & \downarrow \pi & & \downarrow Hol \\ Hom(\mathbb{Z}^2, G) & \hookrightarrow & G \times G & \xrightarrow{\Phi} & G \end{array} \tag{6.2}$$

While this chapter is near the end of this thesis, a key motivation for our work is to understand this diagram on equivariant K-theory. In so doing, while the principle computation is the equivariant K-theory of the symplectic quotient $Hom(\mathbb{Z}^2, G)$, we have also reinterpreted the work of [HJS1] to compute ΩG and the unsurprising G and $G \times G$.

6.2 Kirwan Surjectivity

We collect in this section an overview of the original Kirwan surjectivity and its various analogs.

Kirwan [Kir] showed that for a compact Lie group G and a finite Hamiltonian

G-space M with proper moment map $\mu : M \to \mathcal{G}^*$, the function $\|\mu\|^2 : M \to \mathbb{R}$ is a G-equivariantly perfect Morse function on M. It follows that the inclusion $\mu^{-1}(0) \hookrightarrow M$ induces a surjection on rational G-equivariant cohomology

$$H_G^*(M;\mathbb{Q}) \twoheadrightarrow H_G^*(\mu^{-1}(0);\mathbb{Q}). \tag{6.3}$$

When 0 is a regular value of μ, $\mu^{-1}(0)/G$ is a symplectic orbifold called the symplectic quotient $M//G$. G then acts locally freely on $\mu^{-1}(0)$ and so we have a surjection called the Kirwan map

$$\kappa : H_G^*(M;\mathbb{Q}) \twoheadrightarrow H^*(M//G;\mathbb{Q}). \tag{6.4}$$

Harada and Landweber [HL] extend this result to G-equivariant K-theory

$$K_G^*(M) \twoheadrightarrow K_G^*(\mu^{-1}(0)) \cong K^*(\mathcal{M}//G) \tag{6.5}$$

where the latter isomorphism holds when G acts freely.

There is an infinite-dimensional analogue of the above situation which results in analogous surjectivity theorems. The compact group G is replaced with the Banach Lie group $L_sG := \{\gamma : S^1 \to G \mid \gamma \; is \; of \; Sobolev \; class \; s\}$ for $s > 3/2$. A Hamiltonian L_sG-space consists of an infinite-dimensional symplectic Banach manifold $\mathcal{M}$ together with an L_sG-action and moment map $\tilde{\Phi} : \mathcal{M} \to L_s\mathcal{G}^*$.

Bott, Tolman, and Weitsman [BTW] proved that the inclusion $\tilde{\Phi}^{-1}(0) \hookrightarrow \mathcal{M}$ induces a surjection on rational G-equivariant cohomology for 0 a regular value:

$$H_G^*(\mathcal{M};\mathbb{Q}) \twoheadrightarrow H_G^*(\tilde{\Phi}^{-1}(0);\mathbb{Q}) \cong H^*(\mathcal{M}//L_sG;\mathbb{Q}). \tag{6.6}$$

Harada and Selick [HS] proved the G-equivariant K-theory analogue

$$K_G^*(\mathcal{M}) \twoheadrightarrow K_G^*(\tilde{\Phi}^{-1}(0)) \cong K^*(\mathcal{M}//L_sG). \tag{6.7}$$

where the latter isomorphism holds when G acts freely.

The third setting to be considered is that of the quasi-Hamiltonian G-spaces introduced by Alekseev, Malkin, and Meinrenken [AMM] which are certain finite-dimensional G-spaces equipped with a G-valued quasi-Hamiltonian moment map $\Phi : M \to G$. These finite dimensional quasi-Hamiltonian G-spaces are in bijective correspondence with certain infinite dimensional Hamiltonian L_sG-spaces $\mathcal{M}$ that fit into the following diagram where $\mathcal{M}$ is a pullback of the bottom right square and the columns are fibrations with fibre ΩG.

$$\begin{array}{ccccc} & & \Omega G & \xrightarrow{=} & \Omega G \\ & & \downarrow & & \downarrow \\ \tilde{\Phi}^{-1}(0) & \hookrightarrow & \mathcal{M} & \xrightarrow{\tilde{\Phi}} & L\mathcal{G}^* \\ \downarrow{=} & & \downarrow{\pi} & & \downarrow{Hol} \\ \Phi^{-1}(e) & \hookrightarrow & M & \xrightarrow{\Phi} & G \end{array} \tag{6.8}$$

Indeed, this correspondence is instrumental in [BTW] and [HS]'s proofs of surjectivity in the Hamiltonian L_sG-space situation for rational G-equivariant cohomology and G-equivariant K-theory respectively.

In the quasi-Hamiltonian case, the inclusion $\Phi^{-1}(e) \hookrightarrow M$ fails to necessarily induce surjections on either rational G-equivariant cohomology or G-equivariant K-theory. Nonetheless, [BTW] prove a similar result in a way that depends on the surjectivity for Hamiltonian L_sG-spaces. Loosely, they prove that $H^*_G(\Phi^{-1}(e))$ is generated as a ring by the image of $H^*_G(M) \to H^*_G(\Phi^{-1}(e))$ together with classes that originate in the fibre.

6.3 Relating to future work

One of our motivations was to understand this example in the larger context of quasi-Hamiltonian G-spaces. We have computed the equivariant K-theory of the symplectic quotient, the base $G \times G$, and the fibre ΩG for our example, but we have not computed this for the infinite dimensional space $\mathcal{M}$. We note a few facts regarding this space. Firstly, note that the the space is defined as a pullback. If we restrict the base to just commuting pair in $G \times G$ then we get the map $G \times G|_{Hom(\mathbb{Z}^2,G)} \to G$ via the commutator map, and the pull back for this restriction really is a product as all such pairs map to e. However, $\mathcal{M}$ itself is not a product topologically which is unfortunate as it prevents us from easily writing down a G-CW complex from our knowledge of each factor. Note however that it is indistinguishable on rational cohomology from a product because the commutator map factors through the smash product which is topologically $S^3 \wedge S^3 \cong S^6 \to S^3$ and $\pi_6(S^3) = \mathbb{Z}_{12}$.

As such, an immediate future goal is to attempt to resolve this difficulty and be able to describe $\mathcal{M}$ in our example more fully, or perhaps at least the low dimensional skeleta of it. [BTW] and [HS] provide a filtration of this space where it is proven that the surjectivity of the inclusion $\phi^{-1}(0) \hookrightarrow \mathcal{M}$ induces a surjection on rational cohomology and equivariant K-theory respectively. This surjection is true for the inclusion into each space in the filtration, and thus even understanding the first of these filtration components would be an advantage.

One immediate generalization from our present situation is to student the example of a quasi-Hamiltonian system where $M = SU(2)^{2n}$ with moment map a product of commutators as before. This has the symplectic quotient the moduli space of flat connections on a closed 2-manifold Σ of genus n. We have done the n=1 case. Note that this does not have commuting n-tuples in SU(2) as the symplectic quotient. The G-CW complex for M is simple enough and of course ΩG remains as the fibre, but work is needed to establish the G-CW structure on the symplectic quotient in this case.

An important long term goal is to extend this body of Kirwan surjectivity results into the quasi-Hamiltonian setting on G-equivariant K-theory. For our specific example, $\phi^{-1}(e) \hookrightarrow M$ does not induce a surjection on equivariant K-theory. However, we conjecture that a K-theoretic analogue of [BTW]'s result in rational equivariant cohomology for the quasi-Hamiltonian case holds true. Specifically, from [HS] the inclusion $\tilde{\Phi}^{-1}(0) \hookrightarrow \mathcal{M}$ induces a surjection on $K_G^*()$. As $\tilde{\Phi}^{-1}(0) = \Phi^{-1}(e)$, some set of generators of $K_G^*(\mathcal{M})$ are needed to generate $K_G^*(\Phi^{-1}(e))$. On cohomology, the result of [BTW] uses cochains and the Leray-Hirsch theorem, unavailable to us on equivariant K-theory. Nonetheless, it is conjectured that an anolog can be stated on equivariant K theory. Here, one source is the image $\pi^*(K_G^*(M)) \subset K_G^*(\mathcal{M})$. Unlike the Hamiltonian G-space and Hamiltonian L_sG-space situations, this source alone is insufficient to ensure surjectivity. Additionally, it is conjectured that one needs generators originating in an appropriate sense from the fibre ΩG.

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