Width, Ricci Curvature, and Bisecting Surfaces

by

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Abstract

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In this thesis we studied width-volume inequalities, bisecting surfaces in three spheres, and the planar case of Larry Guth's sponge problem. Our main result is a width-volume inequality for conformally non-negatively Ricci curved manifolds. We obtain several estimates on the size of minimal hypersurfaces in such manifolds. Concerning geometric subdivision and 3-spheres, we give a positive answer to a question of Papasoglu . Regarding the sponge problem, we show that any open bounded Jordan measurable set in the plane of small area admits an expanding embedding in to a strip of unit height. We also prove that a generalization of the planar sponge problem is NP-complete. This thesis is partially based on joint work with Ye. Liokumovich [17] and Z. Zhu [18].

This thesis is dedicated to the memory of:
Christopher Glynn, Drew Adey, Greg Adey, Pam McKeever (née Adey), Matt Collinson,
Ida Bulat, and Charles Brunner.

Contents

Chapter 1

Introduction

1.1 Preface

This thesis is concerned with the quantitative geometry of Riemannnian manifolds. To use Misha Kapovich's apt phrase, it is "geometric geometry" and studies geometric quantities like lengths and areas. The three themes of the work are: width-volume inequalities, bisecting surfaces, and Larry Guth's sponge problem.

The main results of this thesis were obtained in two life changing collaborations. I am thankful to Regina Rotman for suggesting and encouraging these projects among her students. For further acknowledgements, of which there are many, see Section 1.5. The first collaboration was with Yevgeny Liokumovich and resulted in Chapter 2 concerning width-volume inequalities and Ricci curvature. The second collaboration was with Zhifei Zhu and resulted in Chapter 3 concerning bisecting surfaces. The remaining material in this thesis was obtained while working on the sponge problem. Robert Young listened patiently while I, somewhat quixotically, worked on the problem. In Chapter 4, I give a computational hardness result for the problem and a solution to a related problem in the plane.

Chpt. 2 Width-Volume Inequalities describes joint work with Yevgeny Liokumovich on the relationship between width-volume inequalities and Ricci curvature [17].

Chpt. 3 Homological Filling and Bisection Area grew out of joint work with Zhifei Zhu on knot theoretic metric obstructions to small width [18]. This chapter contains our result about coarsely subdividing generic Riemannian 3-spheres. This work answered a question posed by Papasoglu [47].

Chpt. 4 Sponges and Width details work on the sponge problem. I answered a related problem in the plane, and showed that the problem of detemining whether there is an expanding embedding between two planar domains is NP-complete.

1.2 Prelude

When I first started at the University of Toronto, Regina Rotman asked me to read Frankel and Katz's alluringly titled "The Morse Landscape of a Riemannian Disk" [16] about Riemannian 2-disks with bumpy metrics. Some fellow geometry students, including Dominic Dotterrer, organized a seminar where we read Guth and Gromov's "Generalizations of the Kolmogorov-Barzdin Embedding Estimates" [28] on thickly embedding complexes in euclidean space. These two articles lead to me take an interest in the field of quantitative geometry and lead directly to the work in this thesis. Chapter 2 arose out of trying to better understand "width". The former article lead to the question we answered in Chapter 3. The measure of complexity used in the latter article is strongly related to the work in Chapter 4.

Here is a single example which contains the germ of the projects in this thesis. We are going to look at a result of Barzdin and Komologorov from the 70s. Motivated by speculation about the structure of the brain, Barzdin and Kolmogorov investigated thick embeddings of graphs in to $\mathbb{R}^3$. Their model for the brain was that neurons are balls and axons are tubes that go between the neurons. To measure the complexity of a graph, or a brain, they thought to study the radius of the smallest ball in euclidean space containing a thickened copy of the graph. This radius is a quantitative measure of complexity which is related to width and sponges, both of which we discuss below. Consider a graph topologically as a simplicial complex where each edge is homeomorphic to an interval. An embedding $f : \Gamma \to \mathbb{R}^3$ is 1-thick if $d(f(\sigma_i), f(\sigma_j)) \geq 1$ for any non-adjacent simplices σ_i and σ_j.

Theorem 1.2.1 (Barzdin-Kolmogorov [4])**.** *There are constants $0 < c(d) < C(d)$ such that: If Γ is a graph of degree d with N vertices then Γ admits a 1-thick embedding in to $B^3(R)$ for $R \leq C(d) \cdot N^{1/2}$. Moreover, there are graphs of degree d with N vertices which require a ball of radius at least $c(d) \cdot N^{1/2}$ to contain any 1-thick embedding in to $\mathbb{R}^3$.*

To get a feel for the result, we are going to show that there are graphs which require large balls. We are going to use some special graphs which will not appear in the body of the thesis, but whose general properties inspired the results contained in this work. We say that a graph is a λ-expander if: for any subset of the vertices $S \subset V$ satisfying $|S| \leq |V|/2$ one has that there are at least $\lambda|S|$ edges from S to its complement $V \setminus S$. For our purposes, an expander is a sparse graph which require many edges to be crossed by any partition of its vertex set. It is a surprising fact that d-regular expanders exist for $\lambda > 0$. We will blackbox the existence problem and suppose, without proof, that there are "λ-expanders".

Suppose that a d-regular $1/10$-expander with N vertices has a 1-thick embedding in to $B^3(R)$. We estimate R. Generically there will be a plane Π such that half the vertices of Γ lie above the plane, and half below. Cutting the graph according to this plane would give a partition of the graph in to two parts with an equal number of vertices. Thus, by expansion, there are at least $(1/10) \cdot (1/2) \cdot N$, or roughly N, edges meeting Π. Since the embedding is 1-thick, the intersection $\Pi \cap B^2(R)$ contains roughly N unit disks. It follows that $\text{area}(\Pi \cap B^3(R)) \gtrsim N$. Therefore, $R \gtrsim N^{1/2}$. Put plainly, an expander graph requires a large ball to contain the image of any 1-thick embedding in to euclidean space because expanders require that we cut many edges to partition their vertex sets in to two parts.

This result, and the idea behind it, got me interested in all sorts of things. Expander graphs can be characterized by spectral estimates, and that lead to thinking about spectra and width in Chapter 2. The notion of using the size of a bisecting surface to describe complexity lead to Chapter 3. The idea of measuring the complexity of a thing by folding in to a ball lead to Chapter 4.

1.3 Overview

We now give a detailed chapter by chapter overview of the thesis.

1.3.1 Width-Volume Inequalities

The width of a manifold measures the complexity of slicing the manifold into hypersurfaces. Let (M^n, g) be a compact oriented Riemannian manifold. Using tools from geometric measure theory, one can naturally metrize $\mathcal{Z}_k(M)$ the space of k-cycles in M. A continuous $(n-k)$-dimensional family $z : X^{n-k} \to \mathcal{Z}_k(M)$ of k-cycles sweeps out M if z assembles to the fundamental class of M under a certain canonical gluing operation. The precise statement of the gluing operation is somewhat technical but one can imagine an $(n-k)$-dimensional family of k-cycles gluing together to form a n-chain in M.

Definition 1.3.1. The k-width of (M, g) is $\mathrm{W}_k(M) = \inf_z \sup_p \mathrm{vol}_k(z_p)$ where z ranges over all sweep outs of M by k-cycles and the supremum runs over all cycles in the family.

In this work we only deal with $\mathrm{W}_{n-1}(M)$ which we abbreviate to $\mathrm{W}(M)$. An important example of a sweep out by $(n-1)$-cycles is obtained by taking the level sets of a smooth Morse function $f : M \to \mathbb{R}$. One has that $f^{-1}(t)$ will be empty for $|t|$ sufficiently large. Thus, the family of $(n-1)$-cycles given by $z_t = f^{-1}(t)$ will start and end at $0 \in \mathcal{Z}_{n-1}(M)$. That is to say: the level sets of a Morse function produce a loop in the space of $(n-1)$-cycles. To get a sense of the gluing operation note that $f^{-1}([t-\epsilon, t+\epsilon])$ will be a n-chain of small volume with boundary $z_{t-\epsilon}$ and $z_{t+\epsilon}$ when ϵ is sufficiently small. Gluing the levels sets together, using the intermediate n-chains, gives a n-cycle. Since f is Morse function, this n-cycle represents the fundamental class of M.

If a family of $(n-1)$-cycles assembles to the fundamental class of M then the family represents a non-trivial loop in $\mathcal{Z}_{n-1}(M)$. By applying a high dimensional analogue of curve shortening to such a non-trivial loop we may find minimal hypersurfaces in M effectively. This technique was introduduced by Pitts [48]. This is the approach of Chapter 2. This approach to controlling the size of minimal surfaces using width is only useful if we can estimate width.

It is desirable to have estimates for width since it is difficult to explicitly compute width. Consider, for a moment, the fact that (at present) the exact value of the width of the unit cube remains unknown [30]. Consequently, we seek to estimate width in terms of quantities which are easier to compute such as volume. The main inspiration for the results in Chapters 2 and 4 of this thesis was the following width estimate:

Theorem 1.3.1 (Guth's Width-Volume Inequality)**.** *If $U \subset \mathbb{R}^n$ is open and bounded then:*

$$\mathrm{W}_k(U) \leq C(n)\,\mathrm{vol}(U)^{\frac{k}{n}}$$

for a universal constant $C(n)$ depending only on dimension.

We call this the "euclidean width-volume inequality" since it holds for subspaces of euclidean space. In general, we may ask about the width of various kinds of spaces. Note that the width-volume inequality does not hold for all Riemannian manifolds. This was remarked by Guth in [32] and follows from work by Burago and Ivanov [6]. The problem of characterizing exactly when such an inequality holds remains wide open. The first positive result for a large class of Riemannian manifolds was the following non-euclidean width-volume inequality:

Theorem 1.3.2 (Balacheff-Sabourau [3])**.** *If* (Σ_k, g) *is a closed Riemannian surface of genus* k *then:*

$$\mathrm{W}_1(\Sigma_k) \leq 10^{18}\sqrt{(k+1)\,\mathrm{vol}_2(\Sigma_k)}$$

The dependence on genus and area is asymptotically optimal. Originally, arithmetic hyperbolic surfaces were used by Brooks to show the tightness of these dependencies [5]. One can also show tightness of the bound by using expanders to construct surfaces which are difficult to subdivide. By using the uniformization theory for orientable surfaces, we were able to improve the constant in Balacheff and Sabourau's result from 10^{18} to 220. The main result of Chapter 2 however, from our current perspective, is the following generalization of the width-volume inequality:

Theorem 1.3.3 (Glynn-Adey & Liokumovich)**.** *If* (M^n, g) *is conformally non-negatively Ricci curved then:*

$$\mathrm{W}(M) \leq C(n)\,\mathrm{vol}(M)^{\frac{n-1}{n}}$$

We will derive several useful consequences of this below. We will effectively construct minimal hypersurfaces in manifolds of dimension $3 \leq d \leq 7$. In particular, we get explicit upper bounds for minimal hypersurfaces who exists was proven by Neves and Marques [39].

In this thesis we work with the metaphor that width is a non-linear analogue of eigenvalue estimates for the Laplacian. This metaphor was first suggested by Gromov in [21]. One heuristic connection between width estimates and spectra is explained by Cheeger's inequality for Riemannian manifolds.

Theorem 1.3.4 (Cheeger [9])**.** *For a compact Riemannian manifold* (M, g) *define:*

$$h(M) = \inf_{\Sigma} \frac{\mathrm{area}(\Sigma)}{\min\{\mathrm{vol}(A), \mathrm{vol}(B)\}}$$

where the infimum is taken over all hypersurfaces Σ *in* M *which subdivide* M *in to two disjoint submanifolds* A *and* B*. Let* λ_1 *be the smallest positive eigenvalue of the Laplacian of* M*. One has* $\lambda_1 \geq h^2/4$*.*

The width of a manifold controls the size of the largest cycle in an optimal sweep out of M by small $(n-1)$-cycles. To compute $\mathrm{W}_{n-1}(M)$ one considers continuous families of cycles. To estimate the smallest positive eigenvalue the Laplacian of M one may evaluate the Cheeger constant of M. The Cheeger constant is an infimum taken over surfaces splitting M in to two parts. Width, on the other hand, is properly considered a parametric version of this slicing. One looks for families of $(n-1)$-cycles in M with the hope of capturing more of the global geometry of M. Thus, width is a parametric or non-linear version of the Cheeger constant $h(M)$. The parametric aspect of estimating width makes for a more robust measure of geometric complexity than the spectrum.

1.3.2 Subdividing disks

Before discussing the result in Chapter 3, we will introduce some of the history of the problem. Motivated by problems from geometric group theory, Gromov asked the following question in [26]:

Question 1.3.1. *We say a Riemannian 2-disk is small if* $\mathrm{length}(\partial D^2) \leq 1$ *and* $d(p, \partial D^2) \leq 1$ *for all* $p \in D^2$*. Is there a universal constant* C *such that the following holds? Every small 2-disk admits a homotopy of curves contracting its boundary circle to a point through curves of length at most* C*.*

At the time, Gromov was concerned with the diameter and area of van Kampen diagrams for groups. It was asked with the hope that a positive answer could reduce the growth rate of a bound on Dehn's function, a combinatorially defined isoperimetric profile for finitely presented groups.

S. Frankel and M. Katz [16] answered Gromov's question negatively by using a combinatorial construction. The key feature of their construction is the observation that the complete binary tree T_n has large "combinatorial width": any continuous map $T_n \to \mathbb{R}$ from the complete binary tree of height n to the reals must have a fiber containing $O(n)$ points. The metric on the disk theyconstructed was negatively curved andconcentrated around a large binary tree in such a way that any curve meeting the tree many times must be long. Their observation was that any contraction of the boundary to a point would meet many edges of the tree. The use of combinatorics of trees to provide lower bounds on width estimates for spheres was further developed by Liokumovich [37]. Continuing in this vein, the work of Liokumovich, Nabutovsky, and Rotman [38] answered questions raised by Frankel and Katz and related to Gromov's question on disks. Motivated by their work, Papasoglu asked in [47]:

Question 1.3.2. *Let M be a Riemannian manifold homeomorphic to a 3-disk satisfying: (i)* $\operatorname{diam}(M) = d$*, (ii)* $\operatorname{area}(\partial M) = A$*, (iii) and* $\operatorname{vol}_3(M) = V$*. Is it true that there is a homotopy $S_t : \partial M \times [0,1] \to M$ such that: $S_0 = id_{\partial M}$ and S_1 is a point and* $\operatorname{vol}_2(S_t) \leq f_1(A, d, V)$ *for some function f_1?*

Question 1.3.3. *Let M be as above. Is it true that there is a relative 2-disk D splitting M in to two regions of volume at least $V/4$ such that* $\operatorname{area}(D) \leq f_2(A, d, V)$ *for some function f_2?*

In work with Zhifei Zhu [18], we answered Papasoglu's questions negatively. Our construction involved linking a pair of tori in the disk. The kinds of obstructions that links can create in high dimensional metrics remains to be explored further. Independently, an elegant expander-based counterexample was given by Papasoglu and Swenson in [46]. In the same work with Zhu, we also provided a positive result related to Papasoglu's Question 1.3.3 about subdividing disks. This work is contained in Chapter 3. Before we can summarize our work, we will need to introduce some definitions.

We wish to partition a three sphere into two parts both of which contain at least a $1/4 - \epsilon$ fraction of the total volume. Any embedded surface which does so will be called a subdividing surface.

Definition 1.3.2. Given a Riemannian 3-sphere M with volume V, let

$$\operatorname{SA}_\epsilon(M) = \inf_{\Sigma \subset M} \left\{ \operatorname{vol}_2(H) : M \setminus \Sigma = R_1 \sqcup R_2, \operatorname{vol}_3(R_i) > \left(\frac{1}{4} - \epsilon\right) V \text{ for } i = 1, 2 \right\}$$

be the subdivision area of M. The infimum is taken over all embedded subdividing surfaces. We define

$$\operatorname{HF}_1(\ell) = \sup_{||z||_1 \leq \ell} \left(\inf_{\partial c = z} \operatorname{vol}_2(c) \right)$$

to be the first homological filling function. In the definition of $\operatorname{HF}_1(\ell)$ the supremum is taken over all 1-cycles z satisfying $\operatorname{vol}_1(z) \leq \ell$ and the infimum computes the size of the smallest 2-cycle c filling $z = \partial c$.

The homological filling functions are a natural generalization of the isoperimetric profile to high co-dimension contexts. Whereas the isoperimetric profile quantifies the difficulty of filling a given amount of surface area by a volume, the k-th homological filling function quantifies the amount $(k+1)$-volume needed to fill a k-dimensional cycle. The homological filling functions were used by Nabutovsky and Rotman [45] to give the first curvature-free upper bound for the smallest area of a minimal hypersurface in a k-connected manifold. The proof of the their result for minimal surfaces motivated the work in Chapter 3. And now we come back to disk subdivision. In Chapter 3 we show the following:

Theorem 1.3.5 (Glynn-Adey & Zhu)**.** *For any Riemannian 3-sphere* (M, g) *with* M *diffeomorphic to* S^3 *we have:*

$$\mathrm{SA}_\epsilon(M) \leq 3\,\mathrm{HF}_1(2d)$$

where d *is the diameter of* (M, g)*.*

This result says that any 3-sphere can be coarsely bisected by an embedded surface of controlled size. It is interesting to note that the diameter term also has a homological interpretation. We can interpret the diameter as the supremal length of a filling of pairs of points. In this sense, $d = \mathrm{HF}_0(2)$.

1.3.3 Sponges and Width

In Chapter 4 we will study Guth's sponge problem. This problem originated in Guth's thesis [32]. The precise statement of the problem is as follows:

Question 1.3.4 (Guth's Sponge Problem)**.** *Are there dimensional constants* $\epsilon = \epsilon(d) > 0$ *such that: If* U *is an open bounded set in the* $\mathbb{R}^d$ *of Lebesgue measure at most* ϵ *then there exists an embedding* $U \to B^d(1)$ *which increases the length of all curves.*

The intuition for the sponge problem is that a set of small measure which is large and diffuse should resemble a physical sponge. A physical sponge has little volume itself, but encloses lot of volume in pockets which can be squeezed out. The problem asks if it is possible to "squeeze all the water out of a sponge" by an expanding embedding. The problem however remains open, even in the plane.

An embedding which increases the length of all curves is a difficult thing to construct. The algorithmic difficulty of constructing expanding embeddings between planar regions in general is illustrated by the following result:

Theorem 1.3.6 (Glynn-Adey)**.** *Given* U *and* V *open bounded sets in the plane, it is NP-complete to determine if there is an expanding embedding from* U *to* V*.*

At present, the sponge problem in full generality seems out of reach. However, I was able to provide a positive solution to a related planar sponge problem. The result is sufficient to prove a width-volume inequality for Jordan measurable sets in the plane.

Theorem 1.3.7 (Glynn-Adey)**.** *If* U *is an open bounded Jordan measurable set in the* $\mathbb{R}^2$ *of Lebesgue measure at most one then there exists an embedding* $U \to [0, 10] \times \mathbb{R}$ *which increases the length of all curves.*

The sponge problem is important since it would offer a new conceptually different proof of the euclidean width-volume inequality. One can use an expanding embedding to place any open set in to the ball, sweep out the ball, and pull back the sweep-out to the original set. Since a size of sweep-out is monotonic with respect to expanding embeddings, the sweep-out you obtain is no bigger than the sweep-out of the ball. A scaling argument then yields the euclidean width-volume inequality. In addition to offering a new proof of the euclidean width-volume inequality one could use a positive answer to the sponge problem in geometric divide and conquer algorithms.

1.4 Open Questions

In this brief section we list and describe some interesting avenues for future work that arose during the completion of this thesis. We describe the questions and the current state of affairs concerning them.

Question 1.4.1. *Is there a universal constant C such that:*

$$\mathrm{W}(M) \leq C \cdot \mathrm{HF}_1(C \cdot \mathrm{diam}(M))$$

for any Riemannian 3-manifold such that $H_1(M) = 0$?

The result in Chpt. 3 suggest that there should be a bound of this form, at least for $M = S^3$, but we do not have a proof. One can subdivide an arbitrary S^3 by a surface satisfying the proposed bound, but it is not clear how to extend a single cut continuously to obtain a sweep-out. A positive constructive answer to this question would provide an effective realization of the results in Nabutovsky and Rotman [45]. That is, we could use the result to produce minimal hypersurfaces in k-connected manifolds.

Question 1.4.2. *What controls HF_1 and, in particular, is there non-trivial bound by Ricci curvature and volume?*

The homological filling function ought to be amenable to techniques from the calculus of variations. Presently the question of controlling the homological filling function by Ricci curvature is being investigated by Zhifei Zhu. It is desirable to have these bounds in order to obtain estimates for sizes of minimal surfaces.

Question 1.4.3. *Does the sponge problem admit a positive solution for simply connected planar domains?*

It seems that restricting the topology of the open sets might make the problem easier. All simply connected planar sets should have enough structure to admit a simple constructive technique for producing an expanding embedding in to the ball.

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[1] Stefan constantly quoting the verse below lead to learn topology at an early age:

Top, Top, Topology!
Holes are found by homology!
Orientations
Cause complications;
So use $\mathbb{Z}_2$ on all occasions.
— Seven Years of Manifold [52]

Chapter 2

Width and Ricci Curvature

Introduction In this chapter, we establish the main results of the thesis. We show that a width-volume inequality holds for every conformally non-negatively Ricci curved manifold. We use this width-volume inequality to obtain several results about minimal hypersurfaces and stationary integral varifolds in such manifolds

2.1 Statement of Results

In [48] Pitts proved existence of a smooth closed embedded minimal hypersurface in any closed Riemannian manifold M of dimension n, for $3 \leq n \leq 5$. This result was extended to manifolds of dimension $n \leq 7$ by Schoen and Simon [51]. The main result of this chapter is a bound on the volume of this hypersurface for certain conformal classes of Riemannian metrics.

Theorem 2.1.1. *Suppose M_0 is a closed Riemannian manifold of dimension n, for $3 \leq n \leq 7$. If M is in the conformal class of M_0 then M contains a smooth closed embedded minimal hypersurface Σ with volume bounded above by $C(M_0)\operatorname{Vol}(M)^{\frac{n-1}{n}}$. When $\operatorname{Ricci}(M_0) \geq 0$ the constant $C(M_0)$ is an absolute constant that depends only on n. In general, for M_0 with $\operatorname{Ricci}(M_0) \geq -(n-1)a^2$ we can take $C(M_0) = C(n)\max\{1, a\operatorname{Vol}(M_0)^{\frac{1}{n}}\}$.*

If $n > 7$ the same upper bound will hold for the $(n-1)$-volume of a closed minimal hypersurface with singularities of dimension at most $n-8$.

Theorem 2.1.1 follows from a bound on the width of M. One can find background and many results about widths of manifolds in [25, App.1F], [30], [3], [39]. Informally, the width $\operatorname{W}(M)$ of a manifold M is the smallest number such that every sweep-out of M by hypersurfaces contains a hypersurface of volume at least $\operatorname{W}(M)$. We give a precise definition of width in Section 2.4.

To state our bound on the width of manifolds it will be convenient to define a conformal invariant called the min-conformal volume. This invariant was recently introduced in a work of Hassannezhad [33].

Definition 2.1.1. Let M be a compact Riemannian manifold. Define the min-conformal volume of M to be: $\operatorname{MCV}(M) = \inf\{\operatorname{Vol}(M')\}$, where the infimum is taken over all manifolds M' in the conformal class of M with $\operatorname{Ricci}(M') \geq -(n-1)$.

Theorem 2.1.2. *Let M be a closed Riemannian manifold of dimension n then*

$$\mathrm{W}(M) \leq C(n) \max\{1, \mathrm{MCV}(M)^{\frac{1}{n}}\} \operatorname{Vol}(M)^{\frac{n-1}{n}}$$

Corollary. *Let M be a closed Riemannian manifold of dimension n, $a \geq 0$ and suppose that* $\operatorname{Ricci}(M) \geq -(n-1)a^2$. *Then*

$$\mathrm{W}(M) \leq C(n) \max\{1, \ a \operatorname{Vol}(M)^{\frac{1}{n}}\} \operatorname{Vol}(M)^{\frac{n-1}{n}}$$

More generally, Theorem 2.1.2 holds if we replace the min-conformal volume in the estimate by the infimum of $\{d \operatorname{Vol}(M')\}$, where d is any positive integer, such that M admits a degree d conformal branched covering onto a manifold M' with $\operatorname{Ricci}(M') \geq -(n-1)$.

The conformal invariant MCV is somewhat reminiscent of (but different from) the conformal volume studied by Li and Yau in [35]. Consider the following:

Example 2.1.1. $\mathrm{MCV}(M)$ can be computed explicitly when M is a Riemannian surface of genus g. By the uniformization theorem M is conformally equivalent to a surface M_0 of constant curvature 1 (if $g = 0$), 0 (if $g = 1$) or -1 (if $g \geq 2$). When the genus is 0 or 1 it follows that $\mathrm{MCV}(M) = 0$. When $g \geq 2$ and the Gaussian curvature of a surface M' satisfies $K \geq -1$ then by Gauss-Bonnet theorem $\operatorname{Area}(M') \geq 4\pi(g-1)$ with equality holding exactly when $K = -1$ everywhere. We conclude that $\mathrm{MCV}(M) = 4\pi(g-1)$

Theorem 2.1.2 then implies that for any surface M of genus g we have $\mathrm{W}(M) \leq C\sqrt{(g+1)\operatorname{Area}(M)}$. This result was previously obtained by Balacheff and Sabourau in [3] with constant $C = 10^8$. Using a slightly modified version of our proof and invoking the Riemann-Roch theorem we can get a somewhat better constant for orientable surfaces.

Theorem 2.1.3. *Any closed orientable Riemannian manifold S_g of dimension 2 and genus g satisfies*

$$\mathrm{W}(S_g) \leq 220\sqrt{(g+1)\operatorname{Area}(S_g)}$$

In [5] Brooks constructed hyperbolic surfaces of large genus and Cheeger constant bounded away from zero. These surfaces have width $\mathrm{W}(M)$ bounded below by $c\sqrt{g \operatorname{Area}(M)}$ for some constant $c > 0$. Hence, the inequality in Theorem 2.1.2 is optimal up to the value of the constant $C(n)$. Upper bounds on the higher parametric versions of width $\mathrm{W}^k(M)$ for all Riemannian surfaces were recently obtained by Liokumovich [37].

It follows from the works of Almgren [2], Pitts [48], and Schoen and Simon [51] that estimates on width yield upper bounds on the volume of smooth embedded minimal hypersurfaces in manifolds of dimension less than or equal to 7. In higher dimensions, we obtain bounds on the volume of stationary integral varifolds, which are smooth hypersurfaces everywhere except possibly for a set of Hausdorff dimension at most $n-8$.

It is possible to obtain more minimal hypersurfaces if one considers parametric families of sweep-outs. In Section 2.4 we define families of hypersurfaces that correspond to cohomology classes of mod 2 $(n-1)$-cycles on M. To each such p-dimensional family we assign the corresponding min-max quantity $\mathrm{W}^p(M)$. Let S^n be the round unit n-sphere. In [21, 4.2.B] Gromov showed that there are constants $0 < c(n) < C(n)$ so that $\mathrm{W}^p(S^n)$ satisfies:

$$c(n)p^{\frac{1}{n}} \operatorname{Vol}(S^n)^{\frac{n-1}{n}} \leq \mathrm{W}^p(S^n) \leq C(n)p^{\frac{1}{n}} \operatorname{Vol}(S^n)^{\frac{n-1}{n}}$$

Guth [31] derived similar bounds for min-max quantities corresponding to the Steenrod algebra generated by the fundamental class λ. Marques and Neves [39], building on the work of Gromov and Guth, proved existence of infinitely many minimal hypersurfaces on a manifold M of dimension n, for $3 \le n \le 7$, under the assumption that M has positive Ricci curvature.

In Section 2.9 we show that if M has non-negative Ricci curvature then $\mathrm{W}^p(M) \le C(n)p^{\frac{1}{n}} \operatorname{Vol}(M)^{\frac{n-1}{n}}$. We use this bound to derive an effective version of the theorem of Marques and Neves. Let $\operatorname{sys}_{n-1}(M)$ be the infimum of volumes of smooth closed embedded minimal hypersurfaces in M.

Theorem 2.1.4. *Suppose M is a closed Riemannian manifold of dimension n, $3 \le n \le 7$, and positive Ricci curvature. For every k there exists k smooth closed embedded minimal hypersurfaces of volume bounded above by $C(n)k^{\frac{1}{n-1}} \operatorname{Vol}(M) \left(\operatorname{sys}_{n-1}(M)\right)^{-\frac{1}{n-1}}$, where $C(n)$ depends only on n.*

2.2 Previous Work on Ricci and Width

The main estimates of this chapter were motivated by similar estimates on the spectrum of the Laplace operator on Riemannian manifolds.

Let M be a closed Riemannian manifold in the conformal class of M_0. In [34] Korevaar constructed a decomposition of M into annuli (and other regions) which measures the 'volume concentration' of the metric M with respect to the base metric of M_0. This annular decomposition is then used to estimate Rayleigh quotients, thus bounding the spectrum of the Laplacian of M. Korevaar's method was further developed by Grigor'yan-Yau in [20] and Grigor'yan-Netrusov-Yau in [19] to obtain upper bounds on the eigenvalues of elliptic operators on various metric spaces. In [22] Gromov used a different approach (based on Kato's inequality) to obtain upper bounds for the spectrum of the Laplacian on Kähler manifolds [22].

In [33] Hassannezhad, combining methods of [10] and [19], obtained upper bounds for eigenvalues of the Laplacian in terms of the conformal invariant MCV (see Definition 2.1.1) and the volume of the manifold.

We do not use a discretization technique, but instead work at all times with our original manifold. This suggests that our techniques are more general. Our techniques however give an explicit geometric construction of the sweep-out withhout using any spectral estimates.

As explained in F. Coda Marques and A. Neves [39], estimates on width give, via min-max constructions, explicit upper bounds on the area of minimal hypersurfaces in manifolds of small dimension. In high dimensions, we obtain bounds on the area of stationary integral varifolds. Estimating the width of a manifold, or the area of minimal surfaces in it, is a highly non-linear problem. This paper implicity linearizes the problem, by reducing the problem of constructing a sweep-out to that of estimating the spectrum of the Laplacian, which is a linear problem. For further insights into this linearization approach to width estimates see [49, Sec §8] and [27, Sec §5.2].

As suggested by Gromov in [21] the problem of bounding width $\mathrm{W}(M)$ and its parametric version W^k can be thought of as a nonlinear analogue of finding the spectrum of the Laplacian on M. In this paper we were guided by this analogy.

Recall for a moment, Guth's Width-Volume Inequality. In dimension $n \ge 3$ one can find a family of parallel hyperplanes in $\mathbb{R}^n$ yielding the desired sweep-out. This follows from the work of Falconer [11] on the (n, k)-Besicovitch conjecture. In dimension two, however, it may happen that any slicing of U by parallel lines contains an arbitrarily large segment. To surpass problems of this kind Guth developed a method of sweeping out regions by 'bending planes' around the skeleton of the unit lattice in $\mathbb{R}^n$. This method was developed further in [31] to bound higher parametric versions of width. We do a similar bending construction in Chapter 4.

It follows from the work of Burago and Ivanov [6] that on any manifold of dimension greater than two there exists a Riemannian metric of small volume and arbitrarily large $(n-1)$-width. Our results show that this does not happen for certain conformal classes of manifolds. In particular, this does not happen in the presence of curvature bound.

In [44] Nabutovsky and Rotman showed that any closed Riemannian manifold possesses a stationary 1-cycle of mass bounded by $C(n)\operatorname{Vol}(M)^{\frac{1}{n}}$. One wonders if this result can be generalized to the case of minimal surfaces on Riemannian manifolds.

Some results in this direction were obtained by A. Nabutovsky and R. Rotman in [45] where they bounded volumes of minimal surfaces on Riemannian manifolds in terms of homological filling functions of M. The k-th homological filling function $\mathrm{HF}_k : \mathbb{R}_+ \to \mathbb{R}_+$ is defined as the smallest number $\mathrm{HF}_k(x)$, such that every k-cycle of mass at most x can be filled by a $(k+1)$-chain of mass at most $\mathrm{HF}_k(x)+\epsilon$.

Theorem 2.2.1 (Nabutovsky-Rotman [44])**.** *Let M be a closed Riemannian manifold of dimension n, for $3 \leq n \leq 7$, such that the first $n-1$ homology groups are trivial, $H_1(M) = ... = H_{n-1}(M) = 0$. There exists a smooth, closed, embedded minimal hypersurface of volume bounded by*

$$C(n)\,\mathrm{HF}_{n-1}(C(n)\,\mathrm{HF}_{n-2}(\cdots \mathrm{HF}_2(C(n)\operatorname{Vol}(M)^{\frac{1}{n}})\cdots))$$

.

Their proof uses a combination of Almgren-Pitts min-max method and other techniques. In particular, a bound on the width of M in terms of homological filling functions does not follow from their argument. It would be interesting to know whether such a bound exists. It is also interesting to know whether homological filling functions can be controlled in terms of Ricci curvature of M.

Other important results are contained in a paper of Marques and Neves [40] where, among other things, they prove a *sharp* upper bound on $W(M)$ when M is a Riemannian 3-sphere with Ricci > 0 and scalar curvature $R \geq 6$.

Remark. In [50] Stephane Sabourau independently obtained upper bounds on the width and volume of the smallest minimal hypersurface on Riemannian manifolds with Ricci ≥ 0.

2.3 Plan of the Chapter

The structure of the proof of Theorem 2.1.2 is as follows: To construct a sweep-out of M, we subdivide M repeatedly, using an isoperimetric inequality adapted to our context. Once we have subdivided M into a collection of small volume open subsets, we construct a sweep-out of each small volume piece using the fact that at small scales M is locally Euclidean. We then assemble these local sweep-outs into a global sweep-out of M.

In Section 2.4 we define what it means for a family of $(n-1)$-cycles to sweep-out M. We define the width $\mathrm{W}(M)$ and its higher parametric version $\mathrm{W}^k(M)$. We also prove Proposition 1, which gives us control of the width of M in terms of widths of its open subsets.

In Section 2.5 we use an idea of Colbois and Maerten from [10] together with the length-area method to prove an isoperimetric inequality (Theorem 2.5.1) which allows us to partition any open set in M in two parts with both parts satisfying a lower volume bound. The subdividing surface satisfies an upper bound on area which depends on the volume of the open set. The lower bound on volume of the two parts ensures that after repeatedly subdividing, using Theorem 2.5.1, we will have many subsets of small volume.

In Section 2.6 we estimate the width of small volume submanifold $M' \subset M$ in terms of $(n-1)$-volume of its boundary. The proof proceeds by covering M' with small balls, which are $(1+\epsilon_0)$-bilipschitz diffeomorphic to balls in Euclidean space. We construct a sequence of nested open subsets U_i of M' with volumes tending to 0, such that the difference $U_i \setminus U_{i+1}$ is contained in a small ball. Since the ball is almost Euclidean, we can sweep out $U_i \setminus U_{i+1}$ by cycles of controlled volume. We then use Proposition 1 to assemble a sweep-out of M'.

In Section 2.7 we prove Theorem 2.1.2 by inductively constructing sweep-outs of larger and larger subsets of M. The result of Section 2.6 serves as the base of the induction. To do so, we carry out the necessary estimates to apply Theorem 2.5.1 and Proposition 2 to subdivide our manifold as needed.

In Section 2.8 we prove Theorem 2.1.3. We also describe how to obtain a version of Theorem 2.1.2 for manifolds, which admit a conformal mapping into some nice space M_0.

In Section 2.9 we show that a manifold with non-negative Ricci curvature can be covered by balls of small n-volume, small $(n-1)$-volume of the boundary, and such that the cover has controlled multiplicity. We use this decomposition to bound the volume of k-parametric sweep-outs of M and, consequently, volumes of stationary integral varifolds or minimal hypersurfaces in M.

2.4 Width of Riemannian manifolds

Let G be an abelian group. We denote the space of flat G-chains in M by $F_k(M;G)$ and the space of flat G-cycles by $Z_k(M;G)$. The space of integral flat chains was defined in [13]. For flat chains with coefficients in an abelian group G see [12, (4.2.26)]. The deformation theorem of Federer and Fleming states that a flat chain of finite mass and boundary mass can be approximated by a piecewise linear polyhedral chain (see [12, (4.2.20),$(4.2.20)^\nu$]). The deformation theorem will be used throughout this paper. Often we will abuse notation and use the same letter for a flat G-chain and a polyhedral chain approximating it. We will denote the mass of a k-chain c by $\mathrm{Vol}_k(c)$.

In [1] F. Almgren constructed an isomorphism

$$F_A : \pi_k(Z_{n-1}(M;G);0) \to H_{n+k}(M;G)$$

For $k=1$ the map F can be described as follows. Let $c_t \in Z_{n-1}(M;G)$, $t \in S^1$, be a continuous family of cycles. Pick a fine subdivision $t_0, ..., t_m$ of S^1 and let C_i be a (nearly) volume minimizing n-chain filling $c_i - c_{i-1}$ (for $i \in \mathbb{Z}_m$). Then $C = \sum_{i\in\mathbb{Z}_m} C_i$ is an n-cycle. It turns out that homology class of C is independent of the choice of the subdivision and filling chains C_i as long as the subdivision is fine enough and the mass of chains C_i is close to the mass of a minimal filling.

If M is a manifold with boundary we may also consider the space of flat cycles relative to the boundary of M. Let q be a quotient map $q : F_k(M;G) \to F_k(M, \partial M;G) = F_k(M;G)/F_k(\partial M;G)$. The boundary map on $F_k(M;G)$ descends to a boundary map ∂ on the quotient. This allows us to define the space of relative cycles $Z_k(M, \partial M;G)$. Cycles in this space can be represented by $(n-1)$-chains with boundary in ∂M. Almgren's map then defines an isomorphism $\pi_1(Z_{n-1}(M, \partial M;G), \{0\}) \cong H_{n-1}(M, \partial M;G)$.

For simplicity from now on we assume everywhere that group $G = \mathbb{Z}_2$. Henceforth we drop the reference to the group G from our notation. $\mathbb{Z}_2$ coefficients will suffice for all applications to volumes of minimal surfaces that we obtain in this paper. When manifold M is orientable the bound in Theorem 2.1.2 holds for sweep-outs with integer coefficients. The proof is essentially the same with some minor modifications to account for orientation of cycles.

Definition 2.4.1. We define the following two notions:

1. For a closed manifold M a map $f : S^1 \to Z_{n-1}(M)$ is called a *sweep-out* of M if it is not contractible, i.e. $F_A([f]) \neq 0$. Similarly, if M has a boundary we call $f : S^1 \to Z_{n-1}(M, \partial M)$ a sweep-out if the image of $[f]$ under Almgren's isomorphism is non-zero.

2. The *width of* M is
$$\mathrm{W}(M) = \inf_{\{f\}} \sup_t \mathrm{Vol}_{n-1}(f(t))$$
where the infimum is taken over the set of all sweep-outs of M.

For manifolds with boundary it will be convenient to consider a particular type of sweep-outs that start on a trivial cycle and end on ∂M. We will call them ∂-sweep-outs.

Definition 2.4.2. 1. Let M be a manifold with boundary. A *∂-sweep-out of* M is a map $f : [0, 1] \to Z_k(M)$, such that:

   (a) $f(0)$ is a trivial k-cycle and $f(1) = \partial M$

   (b) Let $q \circ f : [0, 1] \to Z_{n-1}(M, \partial M)$ be the composition of f with the quotient map q. When we identify $q \circ f(0)$ and $q \circ f(1)$ we obtain a sweep-out of M.

2. The *∂-width of* M is:
$$\mathrm{W}^{\partial}(M) = \inf_{\{f\}} \sup_t \mathrm{Vol}_{n-1}(f(t))$$
where the infimum is taken over the set of all ∂-sweep-outs of M.

From the definition we have inequalities $\mathrm{W}(M) \leq \mathrm{W}^{\partial}(M)$ and $\mathrm{W}^{\partial}(M) \geq \mathrm{Vol}_{n-1}(\partial M)$.

Definition 2.4.2 is motivated by the following proposition.

Proposition 1. *Let $U_0 \subset ... \subset U_{m-1} = M$ be a sequence of nested open subsets of M, and let A_i denote the closure of $U_i \setminus U_{i-1}$ for $1 \leq i \leq m-1$ and A_0 denote the closure of U_0. Then* $\mathrm{W}^{\partial}(M) \leq \sup\{W^{\partial}(A_0), W^{\partial}(A_1) + \mathrm{Vol}_{n-1}(\partial U_0), ..., W^{\partial}(A_{m-1}) + \mathrm{Vol}_{n-1}(\partial U_{m-2})\}$.

Proof. By the definition of ∂-width for each i there exists a map $c_i : [0, 1] \to Z_{n-1}(M)$ that starts on a trivial cycle, ends on ∂A_i and is bounded in volume by $\mathrm{W}^{\partial}(A_i) + \epsilon$. By definition of A_i, $\partial A_i \subset \partial U_i \cup \partial U_{i-1}$ and $\partial U_i + c_{i+1}(1) = \partial U_{i+1}$.

We define a sweep-out $c : [0, 1] \to Z_{n-1}(M)$ as follows. For $0 \leq t \leq \frac{1}{m}$ we set $c(t) = c_0(t/m)$ and for $\frac{i}{m} \leq t \leq \frac{i+1}{m}$, $i = 1, ..., m-1$ we set $c(t) = c_i(m(t - \frac{i}{m})) + \partial U_{i-1}$.

Let F_A be the Almgren's isomorphism. We can represent the homology class $F_A(c)$ by a sum of n-chains $\sum_{i=0}^{m-1} C_i$, such that $\partial C_i = c(\frac{i}{m}) - c(\frac{i-1}{m})$. Moreover, since each c_i is a ∂-sweep-out of A_i we may assume that C_i represents a non-trivial homology class in $H_n(M, \overline{M \setminus A_i}) \cong H_n(A_i, \partial A_i)$.

We claim that the sum $\sum_{i=0}^{k} C_i$ represents a non-trivial homology class in $H_n(M, M \setminus U_k)$. Indeed, assume this to hold for $\sum_{i=0}^{k-1} C_i$. Let V_1 denote a small tubular neighbourhood of the set U_{k-1} inside

U_k. Let V_2 be a small tubular neighbourhood of $U_k \setminus U_{k-1}$ inside U_k. Let $V_3 = V_1 \cap V_2 \subset U_k$. The pair $(V_3, V_3 \cap \partial U_k)$ is homotopy equivalent to $(\partial U_{k-1}, \partial U_{k-1} \cap \partial U_k)$. From the Mayer-Vietoris sequence we have an isomorphism $H_n(\overline{U_k}, \partial U_k) \to^{\partial} H_{n-1}(\partial U_{k-1}, \partial U_{k-1} \cap \partial U_k)$. This map sends $[\sum_{i=0}^{k-1} C_i + C_k]$ to the fundamental class $[\overline{\partial U_{k-1} \setminus \partial U_k}]$.

We conclude that $c(t)$ is a ∂-sweep-out of M. □

In the last section of this paper we will obtain upper bounds on the k-parametric sweep-outs $\mathrm{W}^k(M)$ of M. By Almgren's isomorphism theorem we have $\pi_m(Z_{n-1}(M; Z); 0) = 0$ for $m > 1$ and $\pi_1(Z_{n-1}(M; \mathbb{Z}_2); 0) \cong \mathbb{Z}_2$. Hence, the connected component Z^0_{n-1} of $Z_{n-1}(M; \mathbb{Z}_2)$ that contains the 0-cycle is weakly homotopy equivalent to the Eilenberg-MacLane space $K(\mathbb{Z}_2, 1) \simeq \mathbb{RP}^\infty$.

Let K be a k-dimensional polyhderal complex and $\sigma : K \to Z^0_{n-1}(M)$ be continuous and assume that $\sigma(x)$ has finite mass for all x. Following [39] we define k-parametric width W^k as follows.

Definition 2.4.3. We introduce the following parametric version of Definition 2.4.1:

1. For a closed manifold M we say that σ is a k-parametric sweep-out of M if $\sigma(K)$ represents the non-zero class in $H_k(Z^0_{n-1}, \mathbb{Z}_2) \cong \mathbb{Z}_2$.

2. Define the k-parametric width to be $\mathrm{W}^k(M) = \inf_\sigma \sup_{t \in K} \mathrm{Vol}_{n-1}(f(t))$, where the infimum is taken over the set of all k-parametric sweep-outs σ.

It follows from the definition that $\mathrm{W}^1(M) = \mathrm{W}(M)$ and $\mathrm{W}^k(M) \leq \mathrm{W}^{k+1}(M)$.

Using Almgren-Pitts min-max theory it is possible to obtain minimal hypersurfaces from sweep-outs of M. In [39] Marques and Neves proved the following results.

Theorem 2.4.1 (Marques-Neves)**.** *Let M be a closed Riemannian manifold of dimension n, $3 \leq n \leq 7$.*

1. *There exists a smooth, closed, embedded minimal hypersurface in M of volume $\leq \mathrm{W}(M)$.*

2. *If $\mathrm{W}^k(M) = \mathrm{W}^{k+1}(M)$ then there exists infinitely many smooth, closed, embedded minimal hypersurfaces in M of volume $\leq \mathrm{W}^k(M)$.*

3. *Suppose M is a manifold of positive Ricci curvature and there exists only finitely many minimal hypersurfaces of volume $\leq \mathrm{W}^k(M)$. Then there exists a smooth, closed, embedded minimal hypersurface Σ_k and $a_k \in \mathbb{N}$, such that $a_k \mathrm{Vol}_{n-1}(\Sigma_k) = \mathrm{W}^k(M)$.*

Remark. In the proof of these results Marques and Neves impose an additional technical condition on W^k. Namely, they require that the infimum in the definition of W^k is taken over only those maps $f : K \to Z_{n-1}(M)$ that have no concentration of mass. This is defined as follows. Using the notation of [12] let $\|c\|$ denote the Radon measure associated with the flat chain c. Then a map f is said to have no concentration of mass if

$$\lim_{r \to 0} \sup\{\|f(x)\|(B(a)) : \; x \in K, a \in M\} = 0$$

All estimates on W^k in our paper come from explicit constructions of families of flat cycles (in fact, polyhedral cycles), which have no concentration of mass. Therefore we can safely combine our estimates with the conclusions of Theorem 2.4.1.

2.5 Isoperimetric inequality

Let (M, g_0) be a closed Riemannian n-manifold with Ricci $\geq -(n-1)$. Let $g = \phi^2 g_0$ be a Riemannian metric on (M, g_0) in the conformal class of g_0. Here $\phi : M_0 \to \mathbb{R}_+$ is a smooth function on (M, g_0).

Notation 2.5.1. We write M_0 for (M, g_0) and M for (M, g).

Below we use the convention that geometric structures measured with respect to g_0 have a superscript zero in their notation. Geometric structures measured with respect to g have no superscript.

Notation 2.5.2. Let $\mathrm{Vol}_k(U)$, $d(x, y)$, dV, $B(x, r)$ and ∇ denote the k-volume function, distance function, volume element, closed metric ball of radius r about x, and gradient with respect to g. Let $\mathrm{Vol}^0_k(U)$, $d^0(x, y)$, dV^0, $B^0(x, r)$, ∇^0 denote the corresponding quantities with respect to g_0.

Let W be a subset of U and let $N^0_l(W)$ denote the set $\{x \in U | d^0(W, x) \leq l\}$.

Lemma 2.5.1. *There exists a set $W \subset U$ and $l \in (0, \frac{1}{2}]$, such that*

1. $\mathrm{Vol}_n(U)/25^n \leq \mathrm{Vol}_n(W) \leq 2\,\mathrm{Vol}_n(U)/25^n$
2. $\mathrm{Vol}_n(N^0_l(W)) \leq (1 - \frac{1}{25^n})\,\mathrm{Vol}_n(U)$
3. $\mathrm{Vol}^0_n(N^0_l(W) \setminus W) \leq l^n \max\{2\,\mathrm{Vol}^0_n(U), c(n)\}$

Proof. The argument is essentially the same as the proof of Lemma 2.2 in the work of Colbois and Maerten [10]. Let r be the smallest radius with the property that $\mathrm{Vol}(B^0(a, r) \cap U) = \frac{\mathrm{Vol}(U)}{25^n}$ for some $a \in M$.

We consider two cases. If $r \leq 1$ we define $W = B^0(a, r) \cap U$ and $l = \frac{r}{2}$.

We observe, using curvature comparison for the space M_0, that the l-neighbourhood of $B^0(a, r)$ can be covered by at most 24.4^n balls of radius r. Indeed, let $\{B^0(x_i, r/2)\}_{i=1}^N$ be a maximal collection of disjoint balls with centers in $B^0(a, \frac{3r}{2})$. Since the collection is maximal, the union $\bigcup B(x_i, r)$ covers $B^0(a, \frac{3r}{2})$. Using the Bishop-Gromov comparison theorem we can estimate the number N. Let $\mathrm{Vol}^0_n(B(x_j, \frac{r}{2})) = \min_i\{\mathrm{Vol}^0_n(B(x_i, \frac{r}{2}))\}$.

$$N \leq \frac{\mathrm{Vol}^0_n(B^0(a, \frac{3r}{2}))}{\mathrm{Vol}^0_n(B(x_j, \frac{r}{2}))} \leq \frac{\mathrm{Vol}^0_n(B(x_j, \frac{5r}{2}))}{\mathrm{Vol}^0_n(B(x_j, \frac{r}{2}))} \leq \frac{V(\frac{5r}{2})}{V(\frac{r}{2})}$$

where $V(r)$ denotes the volume of a ball of radius r in n-dimensional hyperbolic space. When $r \in (0, 1]$ this quantity is maximized for $r = 1$. We conclude that $B^0(a, \frac{3r}{2})$ can be covered by

$$N \leq \frac{\int_0^{\frac{5}{2}} \sinh^{n-1}(s)ds}{\int_0^{\frac{1}{2}} \sinh^{n-1}(s)ds} \leq (2e^{\frac{5}{2}})^n \leq 24.4^n$$

balls, such that each of them has $\mathrm{Vol}_n(B^0(x_i, r) \cap U)$ at most $\frac{\mathrm{Vol}_n(U)}{25^n}$. This proves inequalities (1) and (2) for the case $r \leq 1$.

Volume of a unit ball in hyperbolic n-space satisfies $V(1) \leq \omega_n e^{n-1}$, where ω_n denotes the volume of a unit n-ball in Euclidean space. Hence, $\mathrm{Vol}^0_n(B^0(a, \frac{3r}{2}) \setminus B^0(a, r)) \leq 25^n e^{n-1} \omega_n r^n = c(n)$. This proves (3) for the case $r \leq 1$.

Suppose $r > 1$. Let k be the smallest number, such that there exists a collection of k balls of radius 1 $\{B^0(x_i, 1)\}_{i=1}^k$ with $\mathrm{Vol}(\bigcup B^0(x_i, 1) \cap U) \geq \frac{\mathrm{Vol}_n(U)}{25^n}$. Let $\{B^0(x_i, 1)\}_{i=1}^k$ be a collection of k

balls with the property that if $\{B^0(y_i,1)\}_{i=1}^k$ is any other collection of k balls then $\mathrm{Vol}(\bigcup B^0(x_i,1)\cap U)\geq \mathrm{Vol}(\bigcup B^0(y_i,1)\cap U)$. We set $W=\bigcup B^0(x_i,1)\cap U$. Note that by our definition of k we have $\mathrm{Vol}_n(W)<\frac{2\,\mathrm{Vol}_n(U)}{25^n}$.

Consider 1/2-neighbourhood of W and note that it can be covered by at most $(24.4)^n$ sets B_j, where each B_j is a union of k balls $B^0(y_i,1)$ of radius 1. By definition of W we have $\mathrm{Vol}(B_j)\leq \mathrm{Vol}(W)$, so $\mathrm{Vol}_n(N_l^0(W))\leq \frac{24.4^n+1}{25^n}\mathrm{Vol}(U)$. Finally, we observe that $\frac{\mathrm{Vol}^0(N_l^0(W))}{1/2}\leq 2\,\mathrm{Vol}^0(U)$. □

Theorem 2.5.1. *There exists a constant $c(n)$ such that the following holds: Let $U\subseteq M$ be an open subset. There exists an $(n-1)$-submanifold $\Sigma\subset U$ subdividing U into two open sets U_1 and U_2 such that* $\mathrm{Vol}_n(U_i)\geq (\frac{1}{25^n})\,\mathrm{Vol}_n(U)$ *and* $\mathrm{Vol}_{n-1}(\Sigma)\leq c(n)\max\{1,\mathrm{Vol}_n^0(U)^{\frac{1}{n}}\}\,\mathrm{Vol}_n(U)^{\frac{n-1}{n}}$.

Proof. We use the length-area method (see [25, p. 4]) to find a small volume hypersurface in $N_l^0(W)\backslash W$, where W and l are as in Lemma 2.5.1.

Let $f(x)=d^0(W,x)|_U:U\to\mathbb{R}^+$ be the d^0 distance form x to W restricted to the set U. By Rademacher's theorem, f is differentiable almost everywhere. By applying the co-area formula we have:

$$\begin{aligned}\int_0^l \mathrm{Vol}_{n-1}(f^{-1}(t))dt &= \int_{f^{-1}(0,l)}||\nabla f||dV\\ \text{(Hölder's inequality)}\quad &\leq \left(\int_{f^{-1}(0,l)}||\nabla f||^n dV\right)^{\frac{1}{n}}\left(\mathrm{Vol}_n(f^{-1}(0,l)\right)^{\frac{n-1}{n}}\\ &= \left(\mathrm{Vol}_n^0(f^{-1}(0,l))\right)^{\frac{1}{n}}\left(\mathrm{Vol}_n(f^{-1}(0,l))\right)^{\frac{n-1}{n}}\end{aligned}$$

The last equality holds since $||\nabla f||^n dV=||\nabla^0 f||^n dV^0$ is a conformal invariant. By Lemma 2.5.1 we have $\mathrm{Vol}_n^0(f^{-1}(0,l))^{\frac{1}{n}}\leq c(n)l\max\{\mathrm{Vol}_n^0(U)^{\frac{1}{n}},1\}$. For the second factor we apply the bound $\mathrm{Vol}_n(f^{-1}(0,l))\leq \mathrm{Vol}_n(U)$. It follows that

$$\min_{r<t<2r}\mathrm{Vol}_{n-1}(f^{-1}(t))\leq c(n)\max\{\mathrm{Vol}(U)^{\frac{1}{n}},1\}\,\mathrm{Vol}_n(U)^{\frac{n-1}{n}}$$

Thus for some regular value of t the level set $f^{-1}(t)$ with area no larger than average, is the desired submanifold Σ. We take $U_1=f^{-1}([0,t))$ and $U_2=f^{-1}((t,\infty))$.

Since $W\subseteq U_1$ by Lemma 2.5.1 we have $\mathrm{Vol}(U_1)\geq \frac{\mathrm{Vol}_n(U)}{25^n}$. On the other hand, $U_1\subseteq N_l^0(W)$ of volume at most $1-\frac{\mathrm{Vol}_n(U)}{25^n}$ so $\mathrm{Vol}(U_2)\geq \frac{\mathrm{Vol}_n(U)}{25^n}$. □

2.6 The width of small submanifolds

In this section we will show that if a submanifold M' of a Riemannian manifold M has small enough volume then its ∂-width can be bounded from above in terms of $\mathrm{Vol}_{n-1}(\partial M')$. First we show this for a submanifold that is contained in a very small ball.

Definition 2.6.1. For a closed Riemannian manifold M and $\epsilon_0\in(0,1)$ define $\epsilon(M,\epsilon_0)$ to be the largest radius r such that: for every $x\in M$ we have that $B(x,r)$ is $(1+\epsilon_0)$-bilipschitz diffeomorphic to the Euclidean ball of radius r.

Lemma 2.6.1. *If $M'\subset M$ is contained in a ball of radius $\epsilon(M,\epsilon_0)$ then* $\mathrm{W}^{\partial}(M')\leq(1+\epsilon_0)\,\mathrm{Vol}_{n-1}(\partial M')$.

Proof. A 2-dimensional version of the lemma appeared in [36]. Let $U \subset \mathbb{R}^n$ be the image of M' under $(1+\epsilon_0)$-bilipschitz diffeomorphism F. An argument similar to that in [41, §6] shows that for a generic line $l \in \mathbb{R}^n$ the projection of ∂U onto l is a Morse function. Let p denote such a projection map and assume that $p(U) = [0, c]$.

Define $f : [0, c] \to Z_{n-1}(U, \mathbb{Z})$ by setting

$$f(t) = \partial(p^{-1}([0, t]) \cap U)$$

Open subsets of hyperplanes in $\mathbb{R}^n$ are volume minimizing regions. Therefore we have $\mathrm{Vol}_{n-1}(f(t)) \leq \mathrm{Vol}(U)$ for all t. Composing f with F^{-1} we obtain the desired sweep-out.

□

We extend the result of the lemma to submanifolds of small volume.

Proposition 2. *There exist a constant $C_1(n) > 0$, such that for every closed Riemannian n-manifold M, $\epsilon_0 > 0$ and every embedded submanifold $M' \subset M$ of dimension n and volume $\mathrm{Vol}_n(M') \leq \epsilon(M, \epsilon_0)^n / C_1$ the following bound holds:*

$$\mathrm{W}^{\partial}(M') \leq 3(1+\epsilon_0)\,\mathrm{Vol}_{n-1}(\partial M')$$

The proof of Proposition 2 somewhat resembles a high dimensional analog of the Birkhoff curve shortening process. We cover M' by a finite collection of small balls B_i such that balls of 1/4 of the radius still cover M'. Since M' has very small volume it will not contain any of the balls B_i. Hence, we can cut away the part of $\partial M'$ that is contained in B_i and replace it with a minimal surface that does not intersect $(1/4)B_i$. As a result we obtain a new submanifold $M'' \subset M'$ that does not intersect $(1/4)B_i$. Moreover, we can do this in such a way that volume of the boundary does not increase. The difference $M' \setminus M''$ is contained in a small ball, so we can sweep it out by Lemma 2.6.1. After finitely many iterations we obtain a submanifold that is entirely contained in one of the small balls. We then apply Proposition 1 to assemble a sweep-out of M' from sweep-outs in small balls.

In the proof of Proposition 2 we will need the following isoperimetric inequality:

Theorem 2.6.1 (Federer–Fleming)**.** *There exists a constant $C_2(n) > 1$, such that every k-cycle A in a closed unit ball in $B \subset \mathbb{R}^n$ can be filled by a $(k+1)$-chain D in B, such that: (i) $\mathrm{Vol}(D) \leq C_2(n)\,\mathrm{Vol}(A)^{\frac{k+1}{k}}$, and (ii) D is contained in the $C_2(n)\,\mathrm{Vol}(A)^{\frac{1}{k}}$-neighbourhood of A.*

To show Proposition 2 we first need to prove the following lemma.

Definition 2.6.2. A k-chain A will be called δ-minimizing if $\mathrm{Vol}(A) - \delta \leq \inf\{A' \in C_k(M, \mathbb{Z}) : \partial A' = \partial M\}$.

Lemma 2.6.2. *There is a constant $C_3(n)$ such that the following holds: Let B be a ball of radius $r_0 \leq \epsilon(\epsilon_0, M)$ and $A \subset \partial B$ be an $(n-1)$-chain satisfying $\mathrm{Vol}(A) \leq C_3(n)\,\mathrm{Vol}(\partial B)$. For every $\delta > 0$ there exists δ-minimal filling D of ∂A in $B(x, r_0)$, such that $D \cap B(x, r_0/2) = \emptyset$. We may take $C_3(n) \leq \omega_{n-1}^{-1}(10 C_2(n))^{-n}$*

The proof of Lemma 2.6.2 is a variation of an argument in [25, §4.2-3]. See also [29, Lemma 6].

Proof. Fix $\delta' < \delta r_0 / 100 C_2(n)$. Let D_1 be some δ'-minimal filling of ∂A in B. We claim that D_1 is contained in a $r_0/4$-neighbourhood of ∂A except for a subset of volume at most δ'.

Since B is 2-bilipschitz homeomorphic to a Euclidean ball, we may apply the Federer-Fleming isoperimetric inequality (with a worse constant) inside B. We obtain that every $(n-2)$-cycle S can be filled in B by an $(n-1)$-chain of mass at most $2C_2(n)\operatorname{Vol}(S)^{\frac{n-1}{n-2}}$.

Let $A(r) = \operatorname{Vol}_{n-2}(\{x \in D_1 : d(x, \partial A) = r\})$ and $V(r) = \operatorname{Vol}_{n-1}(\{x \in D_1 : d(x, \partial A) > r\})$. The co-area inequality implies that $|V'(r)| \geq A(r)$.

It follows by the δ'-minimality of D_1 that every open subset $U \subset D_1$ not meeting ∂A must have volume at most:

$$\operatorname{Vol}_{n-1}(U) \leq 2C_2(n)\operatorname{Vol}_{n-2}(\partial U)^{\frac{n-1}{n-2}} + \delta'$$

In particular, we have: $V(r) \leq 2C_2(n)A(r)^{\frac{n-1}{n-2}} + \delta'$. Applying the co-area inequality again we obtain:

$$\frac{d}{dr}\left([V(r) - \delta']^{\frac{1}{n-1}}\right) \leq \frac{-1}{(n-1)(2C_2(n))^{\frac{n-1}{n-2}}}$$

Hence, $V(r) \leq \delta'$ for some

$$\begin{aligned} r &\leq (n-1)(2C_2(n))^{\frac{n-1}{n-2}}\operatorname{Vol}(D_1)^{\frac{1}{n-1}} \\ &\leq (n-1)(2C_2(n))^{\frac{n-1}{n-2}}\operatorname{Vol}(A)^{\frac{1}{n-1}} \\ &\leq 2(n-1)(2C_2(n))^{\frac{n-1}{n-2}}\left(C_3(n)n\omega_n r_0^{n-1}\right)^{\frac{1}{n-1}} \leq r_0/4 \end{aligned}$$

We will now cut off the piece of D_1 that lies outside of $(r_0/4)$-neighbourhood of ∂D_1. Again, by the co-area inequality we have that: $A(r') \leq \frac{8}{r_0}\delta'$ for some $(1/4)r_0 \leq r' \leq (3/8)r_0$. The Federer-Fleming isoperimetric inequality gives a filling of $\{d(x, \partial D_1) = r'\}$ by an $(n-1)$-chain D_2 satisfying:

$$\operatorname{Vol}(D_2) \leq 2C_2(n)\left(\frac{8}{r_0}\delta'\right)^{\frac{n-1}{n-2}} \leq \delta/2$$

Moreover, the filling has the property that the distance from $\{d(x, \partial D_1) = r'\}$ to every point of D_2 is at most $2C_2(n)\left(\frac{8}{r_0}\delta'\right)^{\frac{1}{n-1}} \leq r_0/8$. This gives the desired filling. □

Now we prove Proposition 2. We will construct a decomposition of M into open sets and then apply Proposition 1.

Proof. Set $C_1(n) = 4^n\omega_{n-1}(10C_3(n))^n$. Let $\epsilon = \epsilon(M, \epsilon_0)$ and assume that $M' \subset M$ has volume bounded by $1/C_1(n)\epsilon^n$. Let $B_i = B(x_i, \epsilon)$ for $i = 1, ..., N$, be a collection of balls such that M' is contained in the interior of $\bigcup B(x_i, \epsilon/4)$. Fix $\delta > 0$. We will construct a collection of open subsets $U_1 \subset ... \subset U_N$, with the following properties:

1. $U_N = M'$.

2. $\operatorname{Vol}(\partial U_i) \leq \operatorname{Vol}(\partial U_{i+1}) + \delta/2^i$.

3. $U_i \cap \bigcup_{j=i+1}^N B(x_j, \epsilon/4)$ is empty.

Assume that $U_{i+1}, ..., U_N$ have been defined. If $U_{i+1} \cap B(x_i, \epsilon/4)$ is empty we set $U_i = U_{i+1}$. Otherwise, to construct U_i we proceed as follows.

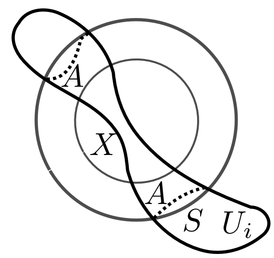

Figure 2.1: Proof of Proposition 2

By the co-area inequality we can find $S(x_i, r') = \partial B(x_i, r')$, with $\frac{3}{4}\epsilon < r' < \epsilon$, such that $S = U_{i+1} \cap S(r', x_i)$ satisfies $\mathrm{Vol}_{n-1}(S) \leq 4\,\mathrm{Vol}_n(U_{i+1} \cap B(x_i, \epsilon))^{1-1/n}$. By Lemma 2.6.2 there exists an $(n-1)$-chain $A \subset B(x_i, r')$ with $\partial A = \partial S$ which is $(\delta/2^i)$-minimizing and A does not intersect $B(x_i, \epsilon/4)$. Let X denote the union of the connected components of $U_{i+1} \setminus A$ that intersect $B(x_i, \epsilon/4)$. We define $U_i = U_{i+1} \setminus X$. Note that the volume decreased and by $\delta/2^i$-minimality of A and the volume of the boundary could not have increased by more than $\delta/2^i$.

By Lemma 2.6.1 we have $\mathrm{W}^{\partial}(X) \leq 2(1+\epsilon_0)\,\mathrm{Vol}_{n-1}(\partial M') + \delta$. By Proposition 1 we have $\mathrm{W}^{\partial}(M) \leq 3(1+\epsilon_0)\,\mathrm{Vol}_{n-1}(\partial M') + 2\delta$. Since δ can be chosen arbitrarily small this concludes the proof of Proposition 2.

□

2.7 Proof of the width inequality

In this section we prove Theorem 2.1.2.

Theorem 2.7.1. *Let M_0 be a manifold with* Ricci $\geq -(n-1)$ *and let M be in the conformal class of M_0. Let $M' \subseteq M$ be an n-dimensional submanifold. There exists a constant $C(n)$ that depends on the dimension, such that:*

$$\mathrm{W}^{\partial}(M') \leq C(n) \max\{1, \mathrm{Vol}_n^0(M')^{\frac{1}{n}}\}\, \mathrm{Vol}_n(M')^{\frac{n-1}{n}} + 3\,\mathrm{Vol}_{n-1}(\partial M')$$

Theorem 2.1.2 follows as a special case.

Proof. Pick the constant $C(n) = 4 \cdot 25^n c(n)$, where $c(n)$ is the constant in Theorem 2.5.1.

Let $\epsilon > 0$ be small enough that every submanifold of volume at most $25^n \epsilon$ satisfies conclusions of Theorem 2. Suppose that $M' \subseteq M$, and pick k so that: $k\epsilon < \mathrm{Vol}(M') \leq (k+1)\epsilon$ and $k > 25^n$. We proceed by induction on k.

Assume the desired sweep-out exists for every open subset of volume at most $k\epsilon$. By Lemma 2.5.1 we can find an $(n-1)$-submanifold Σ subdividing M' into M_1 and M_2 of volume at most $c(n) \max\{1, \mathrm{Vol}_n^0(M')^{\frac{1}{n}}\}\, \mathrm{Vol}_n(M')^{\frac{n-1}{n}}$, such that $\mathrm{Vol}_n(M_i) \leq (1 - 1/25^n)\,\mathrm{Vol}_n(M')$. Since $k > 25^n$ the inductive hypothesis is applicable to both halves M_i.

By inductive hypothesis we have

$$W^{\partial}(M_i) \leq 3(\mathrm{Vol}(\partial M' \cap M_i) + \mathrm{Vol}(\Sigma)) + C(n) \max\{1, \mathrm{Vol}_n(M)^{\frac{1}{n}}\}\, \mathrm{Vol}_n(U_i)^{\frac{n-1}{n}}$$

We apply Proposition 1 with $U_0 = M' \setminus M_2$ and $U_1 = M'$. We obtain

$$\mathrm{W}^{\partial}(M') \leq 3\,\mathrm{Vol}(\partial M') + 4\,\mathrm{Vol}(\Sigma) + C(n)\max\{1, \mathrm{Vol}_n^0(M')^{\frac{1}{n}}\} \max_{i=1,2}\{\mathrm{Vol}^n(M_i)^{\frac{n-1}{n}}\}$$

We use bounds $\mathrm{Vol}_n(M_i)^{\frac{n-1}{n}} \leq \frac{25^n-1}{25^n}\,\mathrm{Vol}_n(M')$ and

$$\mathrm{Vol}_{n-1}(\Sigma) \leq c(n)\max\{1, \mathrm{Vol}_n^0(M')^{\frac{1}{n}}\} Vol_n(M')^{\frac{n-1}{n}}$$

We compute that the resulting expression satisfies the desired bound.

□

Theorem 2.1.2 follows from Theorem 2.7.1 by taking the infimum of the total volume of M_0 over all manifolds M_0 that are conformally equivalent to M and have Ricci $\geq -(n-1)$.

2.8 The width of surfaces

In this section we prove a theorem of Balacheff and Sabourau [3] with an improved constant. Note that the result also follows as an immediate corollary of Theorem 2.1.2 with a worse constant. However, we observed that one can use a slightly modified version of our proof and invoke the Riemann-Roch theorem to get a somewhat better constant.

Below we prove a version of Theorem 2.1.2 which allows us to bound width of a manifold M if M admits a conformal map into some nice space M_0 with a small number of pre-images. We will then estimate the width of surfaces by applying uniformization theorem and the Riemann-Roch theorem. Our argument is parallel to the analogous arguments of Yang and Yau [54] and [34, §4] for eigenvalues of the Laplacian on Riemann surfaces.

Definition 2.8.1. Define $\tau = \tau(M_0)$ and $\nu = \nu(M_0)$ as follows: τ is the least number such that any annulus $B^0(x,2r) \setminus B^0(x,r)$ in M_0 can be covered by τ balls of radius r. We let $\nu(M_0)$ be the least constant such that $\mathrm{Vol}_n^0(B^0(x,r)) \leq \nu r^n$ for all $r > 0$ and all $x \in M_0$.

Theorem 2.8.1. *Let $\Phi : (M,g) \to (M_0, g_0)$ be a conformal map. Suppose the following holds: (i) Any point $x \in M_0$ has at most K pre-images, (ii) The set $\{x \in M, d\Phi(x) = 0\}$ is of measure 0. It follows that:*

$$\mathrm{W}(M) \leq \frac{8\nu^{\frac{1}{n}}K^{\frac{1}{n}}}{1 - (\frac{\tau+1}{\tau+2})^{\frac{n-1}{n}}}\,\mathrm{Vol}(M)^{\frac{n-1}{n}}$$

Proof. First, we prove an analog of our isoperimetric inequality, Theorem 2.5.1.

Let U be an open set in M. We show that there is an $(n-1)$-submanifold $\Sigma \subset U$ such that $U \setminus \Sigma = U_1 \sqcup U_2$ with $\mathrm{Vol}_n(U_i) \geq \frac{1}{\tau+2}\,\mathrm{Vol}_n(U)$ and $\mathrm{Vol}_{n-1}(\Sigma) \leq 2\nu^{\frac{1}{n}}K^{\frac{1}{n}}\,\mathrm{Vol}_n(U)^{\frac{n-1}{n}}$.

Let $p \in M$ and u and v be vectors in the tangent space T_pM. Since Φ is conformal we have

$$\langle \Phi_* u, \Phi_* v\rangle_{g_0} = \phi(x)\langle u, v\rangle_g$$

for some non-negative function ϕ. In a neighbourhood of a point $p \in M \setminus \{x \in M, d\Phi(x) = 0\}$ map Φ is a local diffeomorpism and

$$||\nabla(f \circ \Phi)|| = \phi^{1/2}||\nabla f|| \quad dV_g = \phi^{-n/2} dV_{g_0}$$

where $f : M_0 \to \mathbb{R}$ is a smooth function and dV_g, dV_{g_0} are volume elements.

The fact that the measure of the set $\{x \in M, d\Phi(x) = 0\}$ is zero guarantees that $\lim_{r\to 0} \mathrm{Vol}_n(\Phi^{-1}(B^0(a,r))) = 0$ for all $a \in M_0$. Let r be the smallest radius, such that there exists a ball $B(r,a)$ with $\mathrm{Vol}(\Phi^{-1}(B^0(a,r)) \cap U) = \mathrm{Vol}_n(U)/(\tau+2)$.

Let d^0 be the distance function on M_0 and define $f(x) = d^0(a,x)|_{\Phi(U)} : \Phi(U) \to \mathbb{R}^+$ to be the distance from $x \in \Phi(U) \subset M$ to a.

$$\begin{aligned}
\int_r^{2r} \mathrm{Vol}_{n-1}((f\circ\Phi)^{-1}(t))dt &= \int_{(f\circ\Phi)^{-1}(r,2r)} ||\nabla(f\circ\Phi)||dV_g \\
&\leq \left(\int_{(f\circ\Phi)^{-1}(r,2r)} ||\nabla(f\circ\Phi)||^n dV_g\right)^{\frac{1}{n}} \left(\mathrm{Vol}_n((f\circ\Phi)^{-1}(r,2r))\right)^{\frac{n-1}{n}} \\
&\leq K^{\frac{1}{n}} \left(\int_{f^{-1}(r,2r)} ||\nabla^0 f||^n dV_{g_0}\right)^{\frac{1}{n}} \left(\mathrm{Vol}_n((f\circ\Phi)^{-1}(r,2r))\right)^{\frac{n-1}{n}} \\
&\leq 2r\nu^{\frac{1}{n}} K^{\frac{1}{n}} \,\mathrm{Vol}(U)^{\frac{n-1}{n}}
\end{aligned}$$

It follows that the average of $\mathrm{Vol}_{n-1}((f\circ\Phi)^{-1}(t))$ is smaller than $2\nu^{\frac{1}{n}} K^{\frac{1}{n}}\,\mathrm{Vol}(U)^{\frac{n-1}{n}}$. We then take $\Sigma = (f\circ\Phi)^{-1}(t)$, with area at most average. This finishes the proof of the analog of Theorem 2.5.1.

The rest of the proof of Theorem 2.8.1 proceeds exactly as in Section 2.7 with $c(n)\max\{1, \mathrm{Vol}_n^0(U)^{\frac{1}{n}}\}$ replaced by $2r\nu^{\frac{1}{n}} K^{\frac{1}{n}}$. □

We now recover Theorem 2.1.3. Let S_g denote a genus g closed surface with a complete Riemannian metric. We write h for the metric on S_g.

The uniformization theorem for Riemannian surfaces guarantees that there is a metric ϕh of constant sectional curvature in the conformal class of h. If $g = 0$ or $g = 1$ then the result follows from Theorem 2.8.1 by taking M_0 to be S^2, $\mathbb{RP}^2$, T^2 or the Klein botle K with the standard metric. In all of these cases we have $\nu = \pi$ and $\tau = 6$ (see Remark 2.8 below).

Suppose now that the surface is orientable and the genuse $g > 1$. Take ϕh to have constant sectional curvature $\kappa = -1$. We now apply the Riemann-Roch theorem which gives a meromorphic function $\Phi : S_g \to S^2$ of degree at most $g+1$. Since Φ is a ramified conformal covering map, it has at most $g+1$ points in each fiber and there are finitely many points where $d\Phi = 0$. Applying Theorem 2.8.1 to Φ gives a width volume inequality for surfaces of genus $g > 1$, we obtain:

$$\mathrm{W}(S_g) \leq \frac{8\sqrt{\nu(S^2)}}{1-\sqrt{\frac{\tau(S^2)+1}{\tau(S^2)+2}}}\sqrt{(g+1)\,\mathrm{Area}(S_g)}$$

Remark. Clearly $\nu(S^2) = \nu(\mathbb{R}^2) = \pi$. It is well known that the smallest number of discs of radius 1 required to cover an annulus $B(2)\setminus B(1) \subset \mathbb{R}^2$ is 6. A similar covering also works on S^2 so $\tau(S^2) = \tau(\mathbb{R}^2) = 6$. With these values of τ and ν we compute $\frac{8\sqrt{\nu(S^2)}}{1-\sqrt{\frac{\tau(S^2)+1}{\tau(S^2)+2}}} \leq 220$ which improves the upper bound $C \leq 10^8$ from [3].

2.9 Volumes of hypersurfaces

In this section we prove Theorem 2.1.4.

Theorem 2.9.1. *If M is a manifold with non-negative Ricci curvature then* $\mathrm{W}^k(M) \leq C(n)k^{\frac{1}{n}} \operatorname{Vol}(M)^{\frac{n-1}{n}}$.

Note that Theorem 2.9.1 is consistent with the conjecture that the sequence of numbers $\mathrm{W}^k(M)$ obeys a Weyl type asymptotic formula (see [21] and the discussion in [39, §9]).

To prove Theorem 2.9.1 we will need to decompose M into open subsets of small sizes. Similar arguments for bounding W^k have been used by Gromov [21],[23] and Guth [31].

Lemma 2.9.1. *Let M be a closed Riemannian manifold with* $\operatorname{Ricci}(M) \geq 0$. *There exists a constant $C_4(n)$, such that for any p there exists $p' \leq p$ and a collection of open balls $\{U_i\}_{i=1}^{p'}$ with $\bigcup U_i = M$, $\operatorname{Vol}_n(U_i) \leq C_4(n)\frac{\operatorname{Vol}_n(M)}{p}$ and $\operatorname{Vol}_{n-1}(\partial U_i) \leq C_4(n)\left(\frac{\operatorname{Vol}_n(M)}{p}\right)^{\frac{n-1}{n}}$.*

Proof. It is a standard fact in comparison geometry that for any ball $B(x,r) \subset M$ we have $\operatorname{Vol}_n(B(x,3r)) \leq 3^n \operatorname{Vol}(B(x,r))$ and $\operatorname{Vol}_{n-1}(\partial B(x,3r)) \leq 3^{n-1} n\omega_n^{\frac{1}{n}} \operatorname{Vol}_n(B(x,r))^{\frac{n-1}{n}}$.

Both of these bounds can be deduced, for example, from the Bishop-Gromov inequality

$$\frac{\operatorname{Vol}_n(B(x,r-\epsilon))}{\omega_n(r-\epsilon)^n} \geq \frac{\operatorname{Vol}_n(B(x,r))}{\omega_n r^n}$$

where ω_n denotes the volume of a unit ball in Euclidean n-space.

To prove the second bound observe that $\operatorname{Vol}_n(B(x,r) \setminus B(r-\epsilon)) \leq \frac{n\epsilon}{r} \operatorname{Vol}_n(B(x,r)) + O(\epsilon^2)$. Since $\operatorname{Vol}_n(B(x,r)) \leq \omega_n r^n$ we can bound the volume of the annulus by $n\omega_n^{\frac{1}{n}}\epsilon(\operatorname{Vol}_n(B(r)))^{\frac{n-1}{n}} + O(\epsilon^2)$. Since $\operatorname{Vol}(3B_i) \leq 3^n \operatorname{Vol}(B_i)$ we obtain that for every $\epsilon > 0$ the volume of the annulus $B(x_i, 3r_i) \setminus B(x_i, 3r_i - \epsilon)$ is bounded by $3^{n-1}n\omega_n^{\frac{1}{n}}\epsilon \operatorname{Vol}(B(x_i,r_i))^{\frac{n-1}{n}} + O(\epsilon^2)$. Hence, there must exist a sphere $S(x,r')$ in the annulus, $3r-\epsilon \leq r' \leq 3r$, with $\operatorname{Vol}_{n-1}(S(x,r')) \leq 3^{n-1}n\omega_n^{\frac{1}{n}} \operatorname{Vol}_n(B(x,r))^{\frac{n-1}{n}} + O(\epsilon^2)$. By curvature comparison again the volume of a sphere can not suddenly jump up. Since ϵ was arbitrary we conclude $\operatorname{Vol}_{n-1}(\partial S(x,3r)) \leq 3^{n-1}n\omega_n^{\frac{1}{n}} \operatorname{Vol}_n(B(x,r))^{\frac{n-1}{n}}$.

Now we construct a covering of M by disjoint balls of volume $\frac{\operatorname{Vol}_n(M)}{p}$, such that balls of three times the radius cover M. This is also standard (see [24]). For each x choose $r_x > 0$ to be the radius of a ball $B(x,r_x)$, such that $\operatorname{Vol}_n(B(x,r_x)) = \frac{\operatorname{Vol}_n(M)}{p}$. By compactness there exists a finite subcollection of balls $B(x,r_x)$ that cover M. By the Vitali covering lemma we can further choose a subcollection of disjoint balls $B_1, \ldots, B_k$ with radii $r_1, \ldots, r_{p'}$, such that balls of three times the radius cover M. Note that we must have $p' \leq p$. Theorem now follows by taking $U_i = 3B_i$.

□

By Theorem 2.7.1 we have the following: for each open subset U of M there exists a family of cycles X_t, for $0 \leq t \leq 1$, sweeping-out U. Moreover, we have that X_0 is a trivial cycle, $X_1 = \partial U$ and $\operatorname{Vol}(X_t) \leq \operatorname{Vol}(\partial U) + C(n)\operatorname{Vol}(U)^{\frac{n-1}{n}}$. For each i we let X_t^i be the family of cycles with the above properties for the submanifold with boundary $U_i \setminus (\bigcup_{j=1}^{i-1} U_j)$. Let $V_i = \bigcup_{j=1}^{i} U_j$ for $1 \leq i \leq p'$ and $V_i = \emptyset$ otherwise. Define a family of mod 2 cycles Z_t for $0 \leq t \leq p'$ by setting $Z_t = \partial V_{i-1} + X_{t-[t]}^i$ for $i-1 \leq t \leq i$, here $[t]$ denotes the integer part of t. We identify the endpoints (which are trivial cycles) and rescale so that Z_t is parametrized by a unit circle.

Observe that for each t cycle Z_t can be decomposed into two $(n-1)$-cycles $Z_t = Z_t^1 + Z_t^2$ with $Z_t^1 \subset \bigcup \partial U_i$ and $\operatorname{Vol}(Z_t^2) \leq C_4(n)\left(\frac{\operatorname{Vol}(M)}{p}\right)^{\frac{n-1}{n}}$.

Following Gromov [21] and Guth [31] we will now define a p-cycle $F : \mathbb{RP}^p \to Z_{n-1}(M, \mathbb{Z}_2)$ which detects the cohomology element λ^p. Consider a truncated symmetric product $TP^p(S^1)$. Recall that the symmetric product is defined as a quotient of $(\mathbb{Z}_2 \times S^1)^p$ by the symmetric group S_p. The truncated symmetric product is then defined as a quotient of the symmetric product by an equivalence relation $1s_j + 1s_k = 0$ if $s_j = s_k$ and $j \neq k$. In [43] Mostovoy proved that $TP^p(S^1)$ is homeomorphic to $\mathbb{RP}^p$ (we learnt about this from [31]). We define $F(\sum_{i=1}^p a_i t_i) = \sum_{i=1}^p a_i Z_{t_i}$. Alternatively, we could define a map from $\mathbb{RP}^p$ to $Z_{n-1}(M, \mathbb{Z}_2)$ with the desired property using zeros of real polynomials of one variable with degree at most p as in [21, 4.2B] and [31, Section 5].

We claim that $\mathrm{Vol}(F(x)) \leq C(n) p^{\frac{1}{n}} \mathrm{Vol}(M)^{\frac{n-1}{n}}$. Indeed, we may decompose each of the p summands $Z_{t_i} = Z^1_{t_i} + Z^2_{t_i}$ with $Z^1_{t_i}$ contained in the union of the boundaries of U_i's and the volume of $Z^2_{t_i}$ bounded from above by a constant times $(\frac{\mathrm{Vol}(M)}{p})^{\frac{n-1}{n}}$. Since we are dealing with mod 2 cycles the sum of all $Z^1_{t_i}$ can not be greater than $\mathrm{Vol}(\bigcup \partial U_i) \leq C(n) p^{\frac{1}{n}} \mathrm{Vol}(M)^{\frac{n-1}{n}}$. We also have that the sum $\sum Z^2_{t_i} \leq C(n) p^{\frac{1}{n}} \mathrm{Vol}(M)^{\frac{n-1}{n}}$.

Finally, we show that $F^*(\lambda^p) = F^*(\lambda)^p \neq 0$. Observe that $S = \{1 \cdot t : t \in S^1\} \subset TP^p(S^1)$ represents a non-trivial homology class in $H_1(TP^p(S^1), \mathbb{Z}_2)$ and $F(S) = \{Z_t\}_{t \in S^1}$ is a sweep-out of M by Proposition 1. It follows that $F^*(\lambda) = 1$. This finishes the proof of Theorem 2.9.1.

We use this result to bound volumes of minimal hypersurfaces in a space with positive Ricci curvature. These minimal hypersurfaces arise from Almgren-Pitts min-max theory as supports of stationary almost minimizing integral varifolds. Pitts [48] and Schoen and Simon [51] proved that these hypersurfaces are smooth embedded subamanifolds when $n \leq 7$. In higher dimensions they may have singular sets of dimension at most $n - 8$.

Marques and Neves [39] showed that every manifold M of dimension n, for $3 \leq n \leq 7$, with positive Ricci curvature possesses infinitely many embedded minimal hypersurfaces. Theorem 2.1.4 is an effective version of their result. Note that for 2-dimensional surfaces an analogous result for periodic geodesics is false. Morse showed that for an ellipsoid of area 1 with distinct but very close semiaxes the length of the fourth shortest geodesic becomes uncontrollably large ([42]).

Proof of Theorem 2.1.4. Let V_k be the infimum of numbers such that there exists k distinct minimal hypersurfaces of volume less or equal to V_k. By [39, Prop. 4.8] and results in Section 2 of the same paper we may assume that each parametric width W^k can be written as a finite linear combination $\mathrm{W}^k = \sum a_j \mathrm{V}_j$, where a_j are integer coefficients. Moreover, when M has positive Ricci curvature (or, more generally whenever M has the property that any two embedded minimal hypersurfaces in M intersect) we have $\mathrm{W}^k = a_{j_k} \mathrm{V}_{j_k}$ for some positive integer a_{j_k}.

Let $C = C(n)$ be the constant from Theorem 2.9.1 and define $C' = 2^{\frac{1}{n-1}} C^{\frac{n}{n-1}}$. We proceed by contradiction. Suppose

$$\mathrm{V}_k > C' \,\mathrm{Vol}(M) \left(\mathrm{sys}_{n-1}(M)\right)^{-\frac{1}{n-1}} k^{\frac{1}{n-1}}$$

for some k. Let $A(N) = \{\mathrm{W}^i \leq N\}$. It follows from the proof of Theorem 6.1 in [39] that if $\mathrm{W}^i = \mathrm{W}^{i+1}$ for some i then there exists infinitely many hypersurfaces of volume at most W^i. Hence, we may assume that $\mathrm{W}^i < \mathrm{W}^{i+1}$ for all $i < k$. By Theorem 2.9.1 we have that the number of elements in the set $A(N)$ satisfies $\#A(N) \geq \frac{N^n}{C^n \,\mathrm{Vol}(M)^{n-1}} - 1$. In particular, we compute $\#A(\mathrm{V}_k) \geq 2\frac{kV_k}{V_1} - 1$. On the other hand, the set $\{a_i \mathrm{V}_i : a_i \in \mathbb{N}, a_i \mathrm{V}_i \leq \mathrm{V}_k\}$ has at most $\frac{k\,\mathrm{V}_k}{\mathrm{V}_1}$ elements, which is a contradiction.

Chapter 3

Homological Filling and Bisection Area

Introduction In this chapter we show that every 3-sphere can be coarsely bisected by a surface of area at most $3\,\mathrm{HF}_1(2d)$. This geometric bisection theorem holds without any curvature assumptions.

3.1 Background on Homological Filling Function

The proof techinque we use to show Theorem 3.1.1 is an adaptation of technique first developed by Gromov in [25, §1.2]. The version of the technique that we employ was used by A. Nabutovsky and R. Rotman in [45] to obtain the first curvature-free bounds on areas of minimal surfaces in Riemannian manifolds.

Question 3.1.1. *Let M be a Riemannian manifold homeormophic to a 3-disk satisfying: (i)* $\mathrm{diam}(M) = d$*, (ii)* $\mathrm{area}(\partial M) = A$*, (iii) and* $\mathrm{vol}_3(M) = V$*. Is it true that there is a homotopy $S_t : \partial M \times [0,1] \to M$ such that: $S_0 = id_{\partial M}$ and S_1 is a point and* $\mathrm{vol}_2(S_t) \leq f_1(A, d, V)$ *for some function f_1?*

Question 3.1.2. *Let M be as above. Is it true that there is a relative 2-disc D splitting M in to two regions of volume at least $V/4$ such that* $\mathrm{area}(D) \leq f_2(A, d, V)$ *for some function f_2?*

These questions were inspired by the work of Ye. Liokumovich, A. Nabutovsky, and R. Rotman in [38]. They proved that any Riemannian 2-sphere (M, g) can be swept-out by curves of length at most $200\,\mathrm{diam}(M)\max\left\{1, \log\frac{\sqrt{\mathrm{Area}(M)}}{\mathrm{diam}(M)}\right\}$ and showed that this bound is optimal up to a constant factor. Liokumovich, Nabutovsky, and Rotman's work was related to the work of S. Frankel and M. Katz [16]. For further refinements of that work, see Liokumovich [36] which constructs Riemannian 2-spheres which are hard to sweep out by 1-cycles.

In work with Zhu, we give negative answers to Question 3.1.1 and Question 3.1.2. We do, however, prove a positive result which majorizes the size of the disk in Question 3.1.2 by the homological filling function and diameter of M (Theorem 3.1.1).

Papasoglu's Question 3.1.1 is a natural extension of the following question asked by Gromov in 1992:

Question 3.1.3. *Consider all Riemannian metrics (D^2, g) on the 2-disk such that: (i) the length of the boundary is at most one and (ii) the diameter of the disk is at most one. Is there a universal constant C such that: For every such metric there is a free homotopy of curves which contracts the boundary to a point through curves of length at most C?*

Frankel and Katz answered Gromov's question negatively in [16]. They construct a metric on the disc with a "wall" whose base is shaped like a regular binary tree with many nodes. Combinatorial properties of the tree force any curve subdividing the nodes in to two equal parts to meet the edges of tree many times. This curve will have to "climb over the wall" many times. This combinatorial obstruction forces any contraction of the boundary of the disc to a point to contain a long curve. In our context we needed to produce a large surface in any contraction of the boundary of D^3 to a point. We did so by constructig a metric which is concentrated around two solid tori embedded in D^3 as a Hopf link. The fact that the tori are linked forces any sweep-out of the 3-disk to meet one component of link transversally and hence any sweep-out will contain a 2-cycle of large area.

In [18] we showed that any sweep-out of a 3-disk containing a pair of linked solid tori must contain an essential loop on the boundary of one of the tori. This essential loop will bound some disk in the solid torus. Our choice of the Riemannian metric on the solid tori forces any such disk to have a large area.

Remark. Our construction of the metric above was inspired by D. Burago and S. Ivanov's construction of a metric on the 3-torus (T^3, g) such that any homologically non-trivial 2-cycle in T^3 has large area [7]. L. Guth remarked that such a construction should provide an example of a sphere which is hard to bisect in [30].

We now describe the main result of this chapter.

Definition 3.1.1. Given a Riemannian 3-sphere M with diameter d and volume V, let

$$\mathrm{BA}(M) = \inf_{\Sigma \subset M} \{\mathrm{vol}_2(\Sigma) : M \setminus \Sigma = R_1 \sqcup R_2, \mathrm{vol}_3(R_i) > \frac{1}{6} V \text{ for } i = 1, 2\}$$

be the bisecting area of M. The infimum here is taken over all embedded surfaces $\Sigma \subset M$.

We define

$$\mathrm{HF}_1(\ell) = \sup_{||z||_1 \leq \ell} \left(\inf_{\partial c = z} \mathrm{vol}_2(c) \right)$$

to be the first homological filling function. The supremum here is taken over all 1-cycles z satisfying $\mathrm{vol}_1(z) \leq \ell$ and the infimum computes the size of the smallest 2-cycle c filling $z = \partial c$.

Theorem 3.1.1. *For any Riemannian 3-sphere (M, g) with M diffeomorphic to S^3 we have:*

$$\mathrm{BA}(M) \leq 3\,\mathrm{HF}_1(2d)$$

where d is the diameter of (M, g).

Given a Riemannian 3-sphere (M, g) with diameter d and volume V, consider the hypersurfaces Σ in M that subdivide M into two connected components, $M \setminus \Sigma = R_1 \sqcup R_2$, with both parts satisfying $\mathrm{vol}_3(R_i) > \frac{1}{6} V$. We claim that for any small $\delta > 0$, there exists such a subdividing surface Σ with area

$$\mathrm{vol}_2(\Sigma) \leq 3\,\mathrm{HF}_1(2d + \delta) + o(\delta^2)$$

By taking $\delta \to 0$, we obtain the result in Theorem 3.1.1. Our argument is derived from similar filling arguments found in Nabutovsky and Rotman's work on minimal hypersurfaces [45]. We prove the claim above by contradiction: we show that if there is no such surface then the fundamental class $[M] \in H_3(M;\mathbb{Z})$ is zero. During the proof we will try to construct a 4-chain which has the fundamental class of M as its boundary; the proof is similar to a standard coning argument.

Proof of Theorem 3.1.1. Suppose there is no subdividing hypersurface Σ satisfying the bounds above.

Let us choose a triangulation X^3 of M such that the length of each edge of the triangulation is at most $\delta > 0$, a small constant that we will eventually send to zero. We consider X^3 as a simplicial complex, though we have an identification of the vertices of X and points in M. For a simplicial complex Δ, we will use the notation $[v_0, v_1, \dots, v_k]$ to denote the k-simplex in Δ with vertex set $\{v_0, v_1, \dots, v_k\}$. We will use notation $\Delta^{(k)}$ for the k-skeleton of a complex. We now define the cone $\mathcal{K}$ to be the simplicial cone over X, our triangulation of M, with an additional distinguished cone point $[v_\star] \in \mathcal{K}^{(0)}$.

Let $C_*(\mathcal{K}) = C_*^{\text{simp}}(\mathcal{K};\mathbb{Z}_2)$ be the chain complex of simplicial chains in $\mathcal{K}$ with coefficients from $\mathbb{Z}_2$. Let $C_*(M) = C_*^{\text{sing}}(M;\mathbb{Z}_2)$ be the chain complex of singular chains in M with coefficients from $\mathbb{Z}_2$. We use $\mathbb{Z}_2$ coefficients in order to avoid orientation issues[1]. We will define our chain map $\Psi : C_*(\mathcal{K}) \to C_*(M)$ inductively skeleton by skeleton. Once we have defined Ψ on $\mathcal{K}^{(k)}$, the k-skeleton of $\mathcal{K}$, we extend it linearly to $C_k(\mathcal{K})$, the group of k-chains in $\mathcal{K}$. As we work with $\mathbb{Z}_2$ coefficients, we will disregard the order of the vertices.

We define Ψ as follows: For any vertex $[v] \neq [v_\star]$ we define $\Psi([v]_{\mathcal{K}}^{\text{simp}}) = [v]_M^{\text{sing}}$, using our identification of vertices of $\mathcal{K}$ with points in M. We send any simplex in $\mathcal{K}$ not meeting the coin point to the corresponding singular simplex in X induced from our triangulation. That is, for a simplex $[v_1, \dots, v_k] \in \mathcal{K}$ such that $\{v_1, \dots, v_k\} \cap \{v_\star\} = \emptyset$, we define $\Psi([v_1, \dots, v_k]) = [v_1, \dots, v_k]$.

It remains to define Ψ on those simplices in $\mathcal{K}$ which meet the cone point. For edges meeting the cone point we define $\Psi([v_\star, v_i])$ to be a minimal geodesic from $\Psi(v_\star)$ to $\Psi(v_i)$. We note that $\Psi(\partial[v_\star, v_i]) = \partial\Psi([v_\star, v_i]) = \Psi([v_\star]) + \Psi([v_i])$ by construction; the sign is irrelevant since we work in $\mathbb{Z}_2$. Observe that $\Psi([v_\star, v_i])$ has length at most d.

To define Ψ on the 2-faces that meet the cone point we proceed as follows: We have that $\partial[v_\star, v_i, v_j] = [v_\star, v_i] + [v_i, v_j] + [v_j, v_\star]$. This boundary has length at most $2d + \delta$. Thus there is a 2-chain c such that $\partial c = \partial[v_\star, v_i, v_j]$ and $\text{vol}_2(c) \leq \text{HF}_1(2d + \delta)$. We define $\Psi([v_\star, v_i, v_j]) = c$. The relation $\Psi(\partial[v_\star, v_i, v_j]) = \partial\Psi([v_\star, v_i, v_j])$ follows from the construction. We have now defined Ψ on the entire 2-skeleton of $\mathcal{K}$.

We note that the singular 2-cycles $\Psi(\partial[v_*, v_i, v_j, v_k])$ play a distinguished role in our construction; they are our candidate bisecting surfaces. We name them $z^2_{\alpha,\beta,\gamma,\star} = \Psi(\partial[\alpha, \beta, \gamma, v_\star])$ for $\alpha, \beta, \gamma \in \mathcal{K}^{(0)} \setminus \{[v_\star]\}$. In Lemma 3.1.1 we will show that, under our assumption about subdividing surfaces, each of the 2-cycles $z^2_{\alpha\beta\gamma\star}$ can be replaced by a 2-cycle $\hat{z}^2_{\alpha\beta\gamma\star}$ with the same support which, in addition, bounds a small volume 3-chain $\hat{\phi}^3_{\alpha\beta\gamma\star}$. Essentially $\hat{z}^2$ removes any intersections of the chain z^2 with itself so that we can meaningfully talk about the two halves of M bounded by the chain. In Lemma 3.1.1 we will also show that $\text{mass}(\hat{\phi}^3_{\alpha\beta\gamma\star}) \leq \frac{1}{6}V$. Let us assume, for now, that Lemma 3.1.1 holds.

We define Ψ on those 3-simplices in $\mathcal{K}$ which meet the cone point. We will define Ψ to be the 3-chain $\hat{\phi}^3 \in C_3(M)$ from Lemma 3.1.1 filling $z^2 = z^2_{[v_i],[v_j],[v_k],\star}$.

We now define Ψ on the four skeleton of $\mathcal{K}$ as follows: For a simplex $[v_\star, v_i, v_j, v_k, v_l]$ consider the 3-chain formed by taking all the small volume fillings and an original 3-simplex $\sigma^3_{ijkl} = [v_i, v_j, v_k, v_l]^{\text{sing}}_M$

[1]We thank Stefan Bilaniuk for this suggestion, see Footnote 1 on Page 8.

from the triangulation of M. That is, consider:

$$\hat{z}^3_{ijkl\star} = \hat{\phi}^3_{ijk\star} + \hat{\phi}^3_{jkl\star} + \hat{\phi}^3_{kli\star} + \hat{\phi}^3_{ijl\star} + \sigma^3$$

One can check that $\partial \hat{z}^3_{ikjl\star} = 0 \in C_2(M)$, since everything cancels in pairs. By construction we have $\mathrm{vol}_3(\hat{z}^3_{ijkl\star}) \leq \frac{4}{6}V + o(\delta^3)$. Thus, there is a point not in the support of $\hat{z}^3_{ijkl\star}$ and we obtain that $\hat{z}^3_{ijkl\star} = \partial\hat{\phi}^4_{ijkl\star}$ for some $\hat{\phi}^4_{ijkl\star} \in C_4(M;\mathbb{Z}_2)$ We define Ψ so that: $\Psi([v_\star, v_i, v_j, v_k, v_l]) = \hat{\phi}^4_{ijkl\star}$.

We have defined a chain map Ψ from $C_*(\mathcal{K})$ to $C_*(M)$. The image under Ψ of the sum of the 4-simplices of the $\mathcal{K}$ is a chain whose boundary is the fundamental class of M. To be precise, we have:

$$[M] = \left[\sum \sigma_{ijkl}\right] = \left[\sum \hat{z}^3_{ijkl\star}\right] = \left[\partial\left(\sum \hat{\phi}^4_{ijkl\star}\right)\right]$$

The second equality holds since $\sum \hat{\phi}^3_{ijkl\star} = 0$. We have now exhibited the fundamental class of M as a boundary, and so $[M] = 0 \in H_3(M;\mathbb{Z}_2)$. We obtain a contradiction since $H_3(M;\mathbb{Z}_2) = \mathbb{Z}_2 = \langle [M] \rangle$.

□

Lemma 3.1.1. *Suppose that (M,g) is a Riemmanian 3-sphere such that for all embedded surfaces $\Sigma \subset M$ we have the following: If $\mathrm{mass}_2(\Sigma) \leq 3\,\mathrm{HF}_1(2d)$ and $M \setminus \Sigma = R_1 \sqcup R_2$ then $\mathrm{mass}_3(R_i) \leq \frac{1}{6}V$ for $i = 1$ or $i = 2$. Given such an (M,g) we have that for any $z^2 = z^2_{\alpha\beta\gamma\star}$ as above there is $\hat{z}^2 \in Z_2(M,\mathbb{Z}_2)$ such that: (i) $\mathrm{supp}(z^2) = \mathrm{supp}(\hat{z}^2)$ and (ii) there is $\hat{\phi}^3 \in C_3(M;\mathbb{Z}_2)$ satisfying $\partial\hat{\phi}^3 = \hat{z}^2$ and $\mathrm{mass}_3(\hat{\phi}^3) \leq \frac{1}{6}V$.*

Proof. Let z^2 be as above. First note that z^2 is piecewise smooth by the Fleming Regularity Lemma [15, 13], since it is realized as a finite union of mass-minimizing surfaces in dimension three. We choose a sufficiently fine triangulation of the image of z^2 so that each simplex of the triangulation is a smoothly embedded surface f_i. We note that $z^2 = \sum \epsilon_i f_i$ where $f_i : \Delta^2 \to M$ is smooth and $\epsilon_i \in \mathbb{Z}$. By the mass_2-minimality of z^2 we have that $\epsilon_i = \pm 1$. Now we work with $\mathbb{Z}_2$-coefficients.

Consider $\hat{z}^2 = \sum f_i \in C_2(M;\mathbb{Z}_2)$. We may perturb the image of $\hat{z}^2$ so that it is in general position in M. That is, the image of $\hat{z}^2$ consists of: regular points, double arcs, triple points, and branch points [8, Chapter 4].) We have $\mathrm{mass}_2(\hat{z}^2) = \mathrm{mass}_2(z^2)$ because $|\epsilon_i| = 1$ and the cycles have the same support. By construction, $\partial\hat{z}^2 = 0$. Thus there is $\hat{\phi}^3$ such that $\hat{z}^2 = \partial\hat{\phi}^3$ since $H_2(M;\mathbb{Z}_2) = 0$.

We show that there must be a small volume filling of $\hat{z}^2$. Suppose that $\mathrm{mass}_3(\hat{\phi}^3) \geq \frac{1}{6}V$. Let $\hat{\psi}^3$ be a chain supported in $\overline{M \setminus \mathrm{supp}\,\hat{\phi}^3}$ satisfying $\partial\hat{\psi}^3 = \hat{z}^2$. We will prove that $\mathrm{mass}_3(\hat{\psi}^3) < \frac{1}{6}V$ by contradiction. We will use $\hat{z}^2$, $\hat{\phi}^3$ and $\hat{\psi}^3$ to construct a subdividing surface whose area is at most $3\,\mathrm{HF}_1(2d+\delta) + o(\delta^2)$. That is, we will show there is a surface Σ such that: $M \setminus \Sigma = R_1 \sqcup R_2$ with $\mathrm{vol}_3(R_i) \geq \frac{1}{6}V$ and $\mathrm{vol}_2(\Sigma) \leq 3\,\mathrm{HF}_1(2d+\delta) + o(\delta^2)$.

We now construct the surface Σ. We will first describe how to replace $\hat{z}^2$ with a union of closed embedded surfaces. Since $\hat{z}^2$ is a piecewise smooth 2-cycle in S^3 we may pick an open metric ball $B(p,\eta) \subset S^3 \setminus \mathrm{supp}(\hat{z}^2)$ such that $S^3 \setminus B(p,\eta)$ is homeomorphic to the closed unit ball in $\mathbb{R}^3$. Let ρ be this homeomorphism. Then ρ is $C = C(p,\eta,g)$-bilipschitz since its domain and target are both compact.

Consider image $\rho(\hat{z}^2)$ in $\mathbb{R}^3$. We want to replace this cycle with a union of closed surfaces. Let $U_\epsilon = \partial\{x \in \mathbb{R}^3 : d(x,\rho(\hat{z}^2) \leq \epsilon\}$ be the boundary of the ϵ-neighbourhood of $\rho(\hat{z}^2)$. By Ferry [14] we know that this is an embedded 2-manifold for an open dense set of $\epsilon \in \mathbb{R}^+$. We choose ϵ to be sufficiently small and we pick V_ϵ to be a connected component of U_ϵ which deformation retracts onto the image of the cycle $\rho(\hat{z}^2)$. Then $\hat{z}^2_\epsilon = \rho^{-1}(V_\epsilon)$ is a union of closed embedded surfaces. Since ρ is C-bilipschitz we can pick ϵ small enough so that $|\,\mathrm{mass}_2(\hat{z}^2) - \mathrm{mass}_2(\hat{z}^2_\epsilon)| \leq \epsilon$.

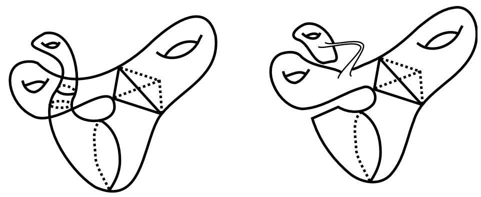

Figure 3.1: Construction of Σ: Take surfaces which are the boundary of a neighbourhood of $\hat{z}^2$. Connect the components by small tubes.

We now consider the surface Σ which is obtained in the following way: Add thin tubes to $\hat{z}^2_\epsilon$ so as to form one connected component. See Figure 3.1. We may do this while adding at most ϵ to both the surface area of $\hat{z}^2_\epsilon$, and volume of $\mathrm{supp}(\hat{\phi}^3)$. We let Σ be the boundary of $\mathrm{supp}(\hat{\phi}^3{}_\epsilon)$ union the thin tubes.

We have $\mathrm{vol}_2(\Sigma) \leq 3\,\mathrm{HF}_1(2d+\delta)+o(\delta^2)+2\epsilon$, and $M \setminus \Sigma = R_1 \sqcup R_2$. Note that $\mathrm{vol}_3(R_i) > \frac{1}{6}V - o(\epsilon^{\frac{3}{2}})$. Taking $\epsilon \to 0$ gives us a subdividing surface contradicting hypothesis about the subdivision area. $\square$

Chapter 4

Sponges

Introduction In this chapter, we establish two results concerning Larry Guth's sponge problem. We show that a generalized planar case of the problem is "computationally difficult". We also show that every small area open bounded set with nice boundary in the plane can be folded to fit in a strip.

While working on the euclidean width-volume inequality Larry Guth asked the following question:

Question 4.0.4 (Sponge Problem)**.** *Are there constants $\epsilon(n)$ such that: If $U \subset \mathbb{R}^n$ is a bounded open set with $\mu(U) < \epsilon(n)$ then there is an expanding embedding $U \to B^n(1) \subset \mathbb{R}^n$.*

Definition 4.0.2. A map $f : U \to \mathbb{R}^n$ is expanding if $||(\,\mathrm{d}f)_x v|| \geq ||v||$ for all $v \in T_x U$.

A positive answer to the Sponge Problem would provide a short and satisfying proof of the width-volume inquality: use an expanding embedding to map the domain U in to a ball of appropriate radius then sweep out the ball and pull back the sweep-out along the expanding embedding. Any expanding embedding could only increase the size of cycles in the sweep-out so one obtains an upper bound on the size of a sweep-out.

We impose

The sponge problem has a certain physically motivated reasonableness. A physical sponge has very small volume but is diffuse in space. A sponge with volume $\sim 1/1000$ L might hold or surround 1L of water. The sponge problem asks if we can squeeze out the 1L of water through an expanding embedding. That is, can such a general sponge be folded to fit in a sphere of radius 1cm? What about a sphere of radius 1km? These questions remain unanswered.

In Section 4.1 we address the complexity of the problem in general. We show that the decision problem: "Is there an expanding embedding $U \to V$?" for bounded planar sets U and V is NP-complete. In Section 4.2 we solve a version of the sponge problem by constructing an expanding embeddings $U \to \mathbb{R} \times [0, 5]$ for nice sets U. Our proof establishes a width-volume inequality for Jordan measurable planar domains.

4.1 Computability and Sponges

In this section we address the question: "How hard is the sponge problem generally?" We do so by showing that computationally difficult problems in combinatorics reduce to the problem of finding an expanding embedding between two planar regions. In order to be self-contained this section also contains a very brief introduction to computational complexity theory.

Problem 4.1.1 (Expanding embedding existence)**.** Given U and V, two nice subsets of $\mathbb{R}^2$, determine whether there exists an expanding embedding $f : U \to V$. See Definition 4.1.5 for the definition of nice.

We will show that determining whether there is an expanding embedding between two domains is at least as difficult as showing checking whether a planar trivalent graph possesses a Hamiltonian cycle.

Definition 4.1.1. A graph Γ has a Hamiltonian cycle if there is a cycle of edges which visits each vertex exactly once.

Problem 4.1.2 (Hamiltonian cycle)**.** Given a planar trivalent graph Γ, determine whether Γ has a Hamiltonian cycle.

It is well-known that Problem 4.1.2 is "algorithmically difficult" (NP-complete). To show that determining the existence of an expanding embedding is difficult, we give a polynomial time reduction from finding a Hamiltonian cycle in a graph to finding an expanding embedding. Thus, we show that finding an expanding embedding between two planar sets is at least as hard as finding a Hamiltonian cycle. Firstly, we introduce the relevant concepts for readers unfamiliar with computational complexity theory.

Definition 4.1.2. Fix a finite alphabet Σ. A word w from Σ is a finite ordered list of elements of Σ. We write $w = \sigma_1\sigma_2\ldots\sigma_n$ for a word. We write Σ^* for the set of all finite words from Σ. A decision problem is a map $D : \Sigma^* \to \{\text{Yes, No}\}$. We write $L(D) = \{w \in \Sigma^* : D(w) = \text{Yes}\}$ for the language determined by D.

Definition 4.1.3. A decision problem D is in P if: There is a deterministic turing machine $M(D)$ and constants C and $k \in \mathbb{N}$ such that: $M(D)$ halts on all $w \in \Sigma^*$ of length n in time at most $C \cdot n^k$ and M accepts $L(D)$. If the above holds we say that D is accepted by a deterministic turing machine in polynomial time. A decision problem D is in NP if: D is accepted by a non-deterministic turing machine in polynomial time.

Definition 4.1.4. A decision problem D is NP-complete if: D is in NP and every problem D' in NP admits a polynomial time reduction to D.

Remark. Decision problems which are NP-complete are generally considered "difficult to solve algorithmically". The exact sense of this difficulty is caught up in deep problems such as P vs NP.

Definition 4.1.5. A set $U \subset \mathbb{R}^2$ is nice if it is a union of finitely many open balls of rational radius centered at points with rational coordinates.

Remark. Any nice set is open, bounded, and can be specified with finitely much data.

Theorem 4.1.1. *Given $U, V \subset \mathbb{R}^2$ two bounded open subsets of $\mathbb{R}^2$, it is NP-complete to determine whether there exists an expanding embedding $f : U \to V$.*

Proof of Theorem 4.1.1. We construct U and V so that the problem of detecting a Hamiltonian cyle in a planar trivalent graph (Problem 4.1.2) reduces to the problem of determining whether there is an expanding embedding $U \to V$.

Fix a planar trivalent graph Γ. We construct V as follows: embed Γ in the plane. For each vertex of Γ place a disk of radius 10 in the plane. Connect the disks by tubes of unit width according to Γ. Thus, V is a vertex thickened copy of Γ in the plane. We construct U as an annulus of unit width with $|V(\Gamma)|$ balls of radius five along its circumference. Alternatively, U is a vertex thickened $|V(\Gamma)|$-cycle.

If there is an expanding embedding $U \to V$ then we can obtain a Hamiltonion cycle on Γ. We follow the meridian path in the annulus U, and obtain a path which visits each thick vertex of V exactly one. This completes the reduction. □

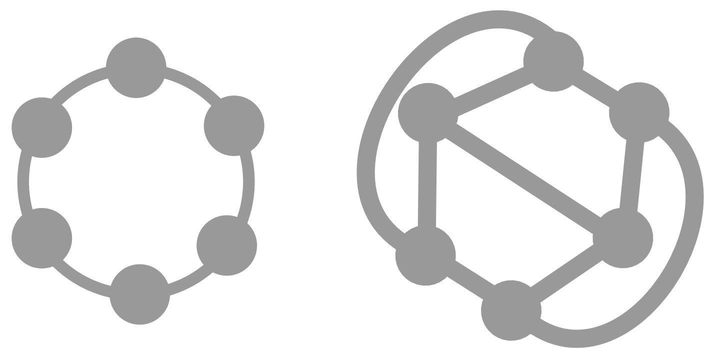

Figure 4.1: The construction used to prove Theorem 4.1.1.

4.2 The Sponge Problem for the Strip

In this section we show that any small area Jordan measurable set in the plane admits an expanding embedding to $\mathbb{R} \times [0, 5]$. This solves a problem analogous to the sponge problem. The result is sufficient to establish a width-volume inequality for Jordan measurable open bounded sets in the plane. We recall the notion of Jordan measurability.

Definition 4.2.1. An axis parallel rectangle in $\mathbb{R}^n$ is $R = [R_1^-, R_1^+] \times [R_2^-, R_2^+] \times \cdots \times [R_n^-, R_n^+]$ satisfying $R_i^- < R_i^+$ for $i = 1, 2, \ldots, n$. A rectangle in $\mathbb{R}^n$ is any subset of $\mathbb{R}^n$ congruent to an axis parallel rectangle.

Definition 4.2.2. We say that $E \subset \mathbb{R}^n$ is Jordan measurable if:

$$\sup_{R \subset E} \mu(R) = \inf_{E \subset S} \mu(S) = \mu(U)$$

where the supremum (resp. infimum) is taken over all *finite* unions of axes parallel rectangles contained in (resp. containing) U.

Remark. Definition 4.2.2 is strictly weaker than the now standard notion of Lebesgue measurability. Many Lebesgue measurable sets are not Jordan measurable, e.g. $\mathbb{Q} \subset \mathbb{R}$.

We may now state our result:

Theorem 4.2.1. *If U is an open bounded Jordan measurable subset of the plane satisfying $\mu(U) < 1$ then U admits an expanding embedding $U \to \mathbb{R} \times [0, 5]$.*

Remark. We note that the hypothesis "U is Jordan measurable" is probably not necessary, but serves as a geometric finiteness condition that we use to rule out pathelogical sets. Recall the classical characterization of Jordan measurability:

Theorem 4.2.2 (Lebesgue). *An open bounded subset U in $\mathbb{R}^n$ is Jordan measurable iff $\mu(\partial U) = 0$.*

Consider for a moment, an open dense subset U of $B^2(1000)$ with total measure $\mu(U) = 1/100$. We see U from afar as a disk of radius 1000 due to its density but the sponge problem would tell us that U should can be expanded to fit in to a ball of radius 1/10. To deal the issue of this type of example, we imposed the condition $\mu(\partial U) = 0$. If U is Jordan measurable then it (and its boundary) cannot be dense and we see it at any scale as a thin network of filaments in the disk. This concludes our remarks about Jordan measurability.

Our proof of Theorem 4.2.1 will rely on the following "plumbing" lemmas which describes how to fold a number of thin rectangles in to the strip.

Lemma 4.2.1. *For any $\ell, L_1, L_2 > 0$ there is an expanding embedding:*

$$\pi = \pi_{\ell,L_1,L_2} : [0,\ell] \times [0,1] \to [0, L_2 + 2\ell + 2] \times [0, L_1 + \ell + 1]$$

such that: (i) $\pi(x,0) = (x,0)$, (ii) $\pi(x,1) = (L_2 + 2\ell + 2 - x, 0)$, and (iii) the image of π does not intersect the rectangle $[\ell + 1, \ell + L_2 + 1] \times [0, L_1]$.

Remark. We note that the map $\pi = \pi_{\ell,L_1,L_2}$ folds the rectangle $[0,\ell] \times [0,1]$ in to a horseshoe with thickness ℓ, height $L_1 + \ell + 1$, and width $L_2 + 2\ell + 2$. One can see that the parameter L_1 controls the height of the horseshoe and L_2 controls its width. The important point is that we can choose a rectangle of arbitrary dimensions such that the image of π avoids this rectangle.

Proof of Lemma 4.2.1. We construct π out of two maps: a vertical stretch s, and a right angled bend b, both of which are defined below. First we stretch the rectangle to have height $2L_1 + L_2 + 2$. The final additive constant is to give us two strips of unit height on which to perform two right hand turns. We apply a right hand turn to the rectangles $[0,\ell] \times [L_1, L_1 + 1]$ and $[0,\ell] \times [L_1 + L_2 + 1, L_1 + L_2 + 2]$, both of which have unit height. We note the following facts about the maps we use to construct π:

- The vertical stretch $s = s_{L_1,L_2}(x,y) = (x, (2L_1 + L_2 + 2)y)$ is an expanding embedding for all $L_1, L_2 > 0$.
- The right angle bend $b = b_\ell(x,y) = \left((1+\ell) - (1+\ell-x)\cos\left(\frac{\pi}{2}y\right), (1+\ell-x)\sin\left(\frac{\pi}{2}y\right)\right)$ is an expanding embedding from $[0,\ell] \times [0,1]$ to $[0,\ell+1]^2$ which maps: $b(x,0) = (x,0)$ and $b(x,1) = (1+\ell, 1+\ell-x)$.

□

Lemma 4.2.2. *Let $S = \bigsqcup_{i=0}^{N} [x_{2i}, x_{2i+1}]$ be a finite union of disjoint closed intervals of total length ℓ. Label the intervals so that $x_0 < x_1 < \cdots < x_n < x_{2N+1}$. Let $\ell_i = x_{2i+1} - x_{2i}$ denote the length of the interval $[x_{2i-1}, x_{2i}]$. Let $L_i = x_{2i} - x_{2i-1}$ denote the space between adjacent intervals.*

For any $\Lambda > 0$ and $0 < \epsilon < \min\{L_i\}$ there is an expanding embedding $P : S \times [0,\epsilon] \to \mathbb{R} \times [0, \ell + \epsilon N]$ such that: $P(x,0) = (x,0)$ and if $x \in [x_{2i}, x_{2i+1}]$ then

$$P(x,1) = \left(\left[2\sum \ell_k + 2\sum L_k + \Lambda + 2N\epsilon\right] - \sum^{i-1} \ell_k - \sum^{i-1} L_k - i\epsilon - x, 0\right)$$

Remark. The claim above is significant because the strip has height independent of L_i, the spacing between the intervals. The height of the strip only depends on the original length of the intervals and a term which we can control by varying ϵ. The construction, in effect, "squeezes out the space between the intervals".

Proof of Lemma 4.2.2. One applies a scaling argument to change the height of the domain from ϵ to one, and then applies Lemma 4.2.1 to the image of each rectangle $[x_{2i}, x_{2i+1}] \times [0, \epsilon]$. By nesting the horsehoes we obtain a map. The proof is inductive, and uses Lemma 4.2.1 as a base case. Scaling back to height ϵ we obtain another map. We may stretch horizontally in order to introduce a Λ to vary the length of the image. This completes the construction. □

Proof of Theorem 4.2.1. We construct an expanding embedding from U to $\mathbb{R} \times [0, 5]$ using the plumbing construction. Since U is Jordan measurable we may assume that U in a finite union of axis parallel rectangles. If not, we may pass to superset of U which has these properties. We can find U', a finite union of rectangles, such that $U \subset U'$ and $\mu(U') < 1$. We may freely replace U with U' if needed. We now apply a slight rotational perturbation to U in order to ensure that $f(t) = \mathcal{H}^1(U \cap \{y = t\})$ is piecewise linear and continuous. The co-area formula applied to U gives us:

$$\int_0^1 \left(\sum_{k \in \mathbb{Z}} f(k+t) \right) dt = \mu(U) < 1$$

By Chebyshev's inequality, we may pick a $t \in [0, 1]$ such that $\sum_{k \in \mathbb{Z}} f(k+t) < 1$. We now use these small cuts to construct and expanding embedding in to the strip. To apply Lemma 4.2.2 we proceed as follows: For each k we may cover the set $U \cap \{y = k + t\}$ by a thin rectangles of height ϵ and total length $\ell < 1$. We apply Lemma 4.2.2 to these thin rectangles. To do so: we pick $\Lambda = 2 \operatorname{diam}(U)$ and ϵ small enough so that $\epsilon N < \ell$ where N is the number of disjoint intervals in the cut. This "unfolds" U along the cut. For each integer k we unfold along the cut $y = k + t$ as in Figure[1] 4.2. We unfold on the bottom of the cut, and then on the top of the cut, altering the orientation for each cut. Since U is bounded we only need to apply finitely many unfolding operations. The total height of the strip we map in to is at most $1 + 2\ell + 2\ell = 1 + 4\ell < 5$. This completes the construction.

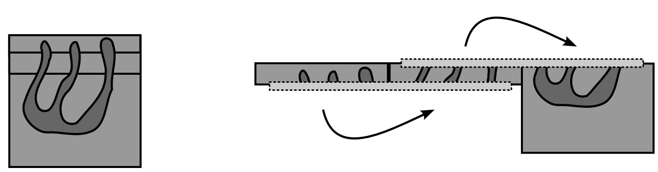

Figure 4.2: The proof of Theorem 4.2.1.

□

[1] "To a Mathematician, a 'proof by picture' is not a proof at all. For while it is true that 'a picture is worth a thousand words', often many of those words will be outright lies – or, at best, misrepresentations of the truth" — Tom Storer [53]

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