

BALANCED PRESENTATIONS OF THE TRIVIAL GROUP AND 4-DIMENSIONAL  
GEOMETRY

by

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A thesis submitted in conformity with the requirements  
for the degree of Doctor of Philosophy  
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# Abstract

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Doctor of Philosophy

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2016

We construct a sequence of balanced presentations of the trivial group with two generators and two relators with the following property: The minimal number of relations required to demonstrate that a generator represents the trivial element grows faster than the tower of exponentials of any fixed height of the length of the finite presentation.

We prove that 1) There exist infinitely many non-trivial codimension one “thick” knots in  $\mathbb{R}^5$ ; 2) For each closed four-dimensional smooth manifold  $M$  and for each sufficiently small positive  $\epsilon$  the set of isometry classes of Riemannian metrics with volume equal to 1 and injectivity radius greater than  $\epsilon$  is disconnected; and 3) For each closed four-dimensional  $PL$ -manifold  $M$  and any  $m$  there exist arbitrarily large values of  $N$  such that some two triangulations of  $M$  with  $< N$  simplices cannot be connected by any sequence of  $< \exp_m(N)$  bistellar transformations, where  $\exp_m(N) = \exp(\exp(\dots \exp(N)))$  ( $m$  times).

We construct families of trivial 2-knots  $K_i$  in  $\mathbb{R}^4$  such that the maximal complexity of 2-knots in any isotopy connecting  $K_i$  with the standard unknot grows faster than a tower of exponentials of any fixed height of the complexity of  $K_i$ . Here we can either construct  $K_i$  as smooth embeddings and measure their complexity as the ropelength (a.k.a the crumpledness) or construct  $PL$ -knots  $K_i$ , consider isotopies through  $PL$  knots, and measure the complexity of a  $PL$ -knot as the minimal number of flat 2-simplices in its triangulation.

For any  $m$  we produce an exponential number of balanced presentations of the trivial group with four generators and four relations of length  $N$  such that the minimal number of Andrews-Curtis transformations needed to connect any two of the presentations is at least  $\exp_m(N)$ .

## Acknowledgements

I am grateful to my supervisor Alexander Nabutovsky, who optimistically took me as a student so late in my studies, for his enormous help as both an advisor and a collaborator. I am also thankful for the financial assistance he provided.

I want to thank the faculty, postdocs and students of the Department of Mathematics for inspiring me to do research and sharing their knowledge. I especially value the involvement of Vitali Kapovitch and Kasra Rafi in geometry seminars, and conversations with my fellow students John Yang, Zhifei Zhu, Dan Fusca and Leonid Monin about mathematics. Additionally, I found talking to mathematicians from other departments, especially Slava Krushkal, motivating and useful.

I want to thank the administrative staff, particularly Jemima Merisca for all her help and, if it were possible, Ida Bulat, who welcomed me to the University of Toronto.

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## Introduction

In this thesis we are primarily interested in balanced presentations of the trivial group and their applications to 4-dimensional geometry. A finite presentation is called balanced if the number of generators is equal to the number of relations. We want to construct a sequence of balanced presentations of the trivial group with a constant number of generators (say, two) with the following property: the minimal number of factors in a product of conjugates of relators expressing a generator (called the area of the generator) grows very fast compared to the length of relations.

To understand geometric applications of such presentations recall a well known fact that for  $n \geq 4$ , it is possible to realize any finitely presented group as the fundamental group of some  $n$ -dimensional compact manifold. One can use a finite presentation of the group to construct such a manifold as a smooth hypersurface in  $\mathbb{R}^{n+1}$ . This manifold depends on the finite presentation of the group both topologically and geometrically (we can take the induced Riemannian metric from  $\mathbb{R}^{n+1}$ ). The fundamental group of the resultant manifold is naturally (from the geometric point of view) given by the presentation: the closed curves on the manifold can be “easily” deformed to a product of the generators of the presentation. This gives us a way to relate the properties of a presentation with the geometry of the constructed manifold. If one can obtain a smooth sphere, the most basic compact manifold, with a desired geometric property, one can then use this sphere to modify the geometry of other Riemannian manifolds by taking the geometric connected sum. Since the fundamental group of the sphere is trivial, one is often interested in presentations of the trivial group.

The area of the words representing the identity in a finite presentation can be related to how difficult it is to contract closed curves in the corresponding manifold. One can use the unsolvability of the word problem in groups to obtain a sequence of finite presentations of the trivial group such that the area of the generators grows faster than any computable function (as a function of the length of the relations in the sequence). Unfortunately, the triviality of the fundamental group is not enough in dimensions greater than 3 to guarantee that the compact manifold is a sphere. Indeed, finite presentations of groups that were obtained from Turing machines in the course of proving the algorithmic unsolvability of the triviality problem for groups always have more relations than generators. Therefore, these presentations are not balanced, which leads to non-vanishing of the second homology group of the manifolds constructed from the presentations in the most basic way. However, in dimensions greater than 4 there is enough freedom to first modify these finite presentations and then ensure that the manifolds corresponding to the chosen presentations are spheres that will have the desired geometric properties. And it was done before (cf. [Nab96a]). In dimension 4 we are interested in balanced presentations of the trivial group, because the corresponding compact manifolds have trivial second homology group, and therefore they are homotopically spheres and thus topological spheres by the Freedman-Poincaré theorem. Moreover, the presentations studied in this thesis satisfy the Andrews-Curtis conjecture and, therefore, the resulting spheres have the standard smooth structure.

In Chapter 1 we construct a sequence of presentations of the trivial group with two generators and two relations, where the area of generators grows faster than any tower of exponentials of fixed height. This growth is not as fast as some computable functions but is sufficient for most geometric applications. To find a sequence with a non-computable growth one would need to answer negatively to the following open question of Magnus: are presentations of the trivial group algorithmically unrecognizable among balanced presentations? But we are not aware of any ideas that might lead to a solution of this problem. Instead of using the areas of the generators one can use a slightly different measure of complexity:

how many Andrews-Curtis transformations are required to transform the given balanced presentation of the trivial group to the trivial presentation. Andrews-Curtis transformations are a set of simple modifications one can do to a presentation that preserve the presented group. Furthermore, these transformations leave the topology and the smooth structure of the associated manifolds unchanged. This is why it is interesting to know if any two balanced presentations of the trivial group can be connected by a sequence of Andrews-Curtis transformations. The assertion that this is always possible is called the Andrews-Curtis conjecture. The sequence of presentations constructed in Chapter 1 shows that the number of Andrews-Curtis transformations can grow faster than any tower of exponentials of fixed length as a function of the length of relations. Nevertheless, our presentations do satisfy the Andrews-Curtis conjecture and therefore can be used to construct spheres with the standard smooth structure but complicated geometry. Our construction is based on the Baumslag-Gersten group, a group with two generators and one relation that has words of very large area relative to their lengths. We add one more relation that equates one of the generators to a word of large area and thus kill the group. The fact that the used word has large area in the Baumslag-Gersten group suggests but does not imply that it has large area in the obtained presentation of the trivial group. Moreover, in the trivial group most techniques of estimating area break down. We developed a version of small cancellation theory over HNN extensions that is applicable to some presentations of the trivial group and used it make the necessary area estimates. This theory is similar to the usual small cancellation theory over HNN extensions but treats the HNN extension as an “effective pseudogroup”. To make this theory work we needed some ideas of Ol’shanskii ([Ol’91]). Chapter 1 is based on [Lis].

In Chapter 2 we exploit the result of Chapter 1 to obtain geometric applications. Alternatively, one can use [Bri15], where the results similar to ours from Chapter 1 were obtained independently and using different techniques. As described before, from these presentations we construct a sequence of spheres where to contract some closed curve one would need to make the curve very long in the process. Furthermore, these spheres when normalized to have volume 1, will have relatively small sectional curvatures, because they come from presentations with relatively short relations. This leads to several applications. For example, for a manifold  $M$  of non-zero Euler characteristic the space of isomorphism classes of Riemannian metrics with a two sided bound on sectional curvatures has infinitely many deep minimas (allowing slightly singular metrics) of the diameter functional. For example, that means there are other “optimal” (in the sense of minimizing the diameter) Riemannian metrics on the 4-sphere besides the round one. We require a non-zero Euler characteristic to obtain a uniform lower bound on volume. We do not know how to get the result without this assumption. Another application is a construction of triangulations of the standard sphere with the following property. If one wants to transform any of these triangulations to the boundary of 5-simplex this will require a lot (relative to the number of simplices in the triangulations) of bistellar transformations. Bistellar transformations are simple modifications of a triangulation that leave PL-structure unchanged, any two triangulations of the same PL manifold can be connected by a sequence of bistellar transformations. Chapter 3 is based on a joint work with A. Nabutovsky ([LN]).

In Chapter 3 we find a sequence of finite presentations of  $\mathbb{Z}$  such that if we kill a group generator (call it  $a$ ) by adding the relation  $a = 1$ , we get a sequence of balanced presentations of the trivial group of low complexity. Furthermore, before adding the relation  $a = 1$  we have a letter in the presentations of  $\mathbb{Z}$  representing identity, area of which is very large. We use this sequence to construct 2-unknots in  $\mathbb{R}^4$  that are “difficult” to untie, meaning that any isotopy connecting our unknot with the standard

unknot has to go through some very complicated embeddings. Chapter 3 is based on a joint work with A. Nabutovsky ([LN15]).

In Chapter 5 we find exponentially many (as a function of the length of the relations) presentations of the trivial group with 4 generators and 4 relations, which are far (in the sense of number of Andrews-Curtis transformations required) from each other. Our constructions is a mixture of the groups from Chapter 1 and from [Bri15]. Both of these constructions are based on the Baumslag-Gersten group. Therefore, to see that our presentations are indeed very different from each other we first need to understand automorphisms of the Baumslag-Gersten group. We classify them in Chapter 4 (which is based on [Lis15]). The results of Chapter 4 are not new (see [Bru80]) but are proven here in a slightly different way (using van Kampen diagrams) and in this regard are more applicable to the “effective” setting of Chapter 5. To construct these presentations we combine the small cancellation theory from Chapter 1, some ideas of Bridson ([Bri15]) and an adaptation of an idea of Collins to the setting of “effective pseudogroups”. In the future we hope to apply these results to obtain exponentially many triangulations of the PL-sphere, which are distant from each other (in the sense of the number of bistellar transformations).

# Chapter 1

# Balanced Presentations of the Trivial Group

## 1.1 Introduction

The purpose of this paper is to construct a sequence of balanced finite presentations of the trivial group with two generators and two relators, one of which is the same for all finite presentations in the sequence, with the following property: the minimal number of Tietze transformations required to bring these finite presentations to the empty presentation of the trivial group grows faster than the tower of exponentials of any fixed height of the length of the variable relation (or, equivalently, the length of these finite presentations). The minimal area of the van Kampen diagram required to demonstrate that either of the generators is trivial also grows faster than the tower of exponentials of any fixed height.

Balanced finite presentations can be realized as “obvious” finite presentations of the (trivial) fundamental group of 4-dimensional spheres and discs. This fact leads to numerous geometric applications of results presented in this chapter that will be discussed in Chapter 5. However, they also provide the following group-theoretic implications.

Recall, the Magnus problem (cf. [MK14]) asks whether or not the triviality problem for balanced group presentations is algorithmically solvable. Equivalently, it asks whether or not the size of van Kampen diagrams required to show that all generators in a balanced finite presentation of the trivial group are trivial cannot be majorized by any computable function of the complexity of the finite presentation. Although we are not able to prove that every computable function can serve as a lower bound for the size of such van Kampen diagrams for all sufficiently large values of the complexity of the finite presentations, we establish that it already grows very fast for balanced finite presentations with two generators and two relators.

Another implication of this work is that one can have balanced finite presentations of the trivial group satisfying the (balanced) Andrews-Curtis conjecture where one needs an enormous number of elementary Tietze moves to transform a given finite presentation to the trivial finite presentation of the trivial group. This fact is relevant in light of recent work ([Mia99], [HR03]), where it was verified by means of explicit computations that certain specific balanced finite presentations of the trivial group cannot be transformed into the trivial one by means of not too many elementary Tietze operations. Our result casts some doubt on whether these results can be considered as a strong empirical evidence that

these balanced finite presentations are counterexamples to the Andrews-Curtis conjecture. Also, they can potentially help to exclude approaches to proving this conjecture that would result in estimates that are not very rapidly growing.

In order to prove our results we start from the Baumslag-Gersten group, which has Dehn function that is not bounded by any tower of exponents of a finite length ([Ger92], [Ger91], [Pla04]). This group has a finite presentation  $\langle x, y | x^{x^t} = x^2 \rangle$ , and there exist words  $w_n$  of length  $O(2^n)$  representing the identity such that the area of any van Kampen diagram grows as a tower of exponents of height  $n$ . The proof of a lower bound for the Dehn function given by Gersten uses the fact that this group can be obtained from a cyclic group by a sequence of two HNN-extensions and, therefore, is aspherical. Therefore, the universal covering of its presentation complex is contractible, has trivial second homology group, and each filling of each trivial word is unique on 2-chain level (see survey papers [Sap11], [Bri02], [Sho07] for discussions of the Dehn functions and alternative proofs for the lower bounds). A natural idea is to kill this group using a new relation,  $w_n = t$ . However, as the resulting group is trivial, we cannot use covering spaces of the realization complex, and direct combinatorial proofs of desired lower bound for the areas of van Kampen diagrams for generators of these finite presentations seem elusive.

I learned about this problem from my Ph.D. advisor Alexander Nabutovsky. He unsuccessfully tried to prove that when one kills the Baumslag-Gersten group by adding an extra relation such as  $t = w_n$ , one obtains a sequence of desired “complicated” balanced finite presentations of the trivial group. He discussed this problem with several mathematicians in 1994-2000 but no solution was found. This problem was also mentioned in [Nab06a] at the end of Section 1.1.

After the results of this chapter appeared as a preprint on the arXiv, Martin Bridson informed us that he had announced similar results in talks starting in 2003, including his 2006 ICM talk ([Bri06] p. 977). He has now posted a preprint with his results on the arXiv [Bri15]. His constructions and methods are different from ours and, in particular, they do not produce examples of rank 2.

Our approach is to kill the Baumslag-Gersten group using a longer (variable) second relator, so as to be able to use a version of the small cancellation theory to prove the desired property of van Kampen diagrams. We use a combination of the small cancellation theory for HNN-extension developed by G. Sacerdote and P. Schupp in [SS74] (see also [LS01]) and ideas of A. Olshanskii related with the concept of “contiguity subdiagrams” ([OI’91],[OI’93]). Finally, we modify this theory to introduce a concept of equivalence between words based not on their equality in a group but its quantitative version, namely, the equality that can be established by means of a van Kampen diagram of area that does not exceed a specified number  $N$ . This chapter is self-contained: it does not directly use any results from the above-mentioned papers.

This chapter deals with one particular case of the words  $w_n$ . It is, of course, possible to apply the same techniques to many other words  $w_n$ . We guess, that by using another small cancellation condition one can prove similar results about a wider class of words  $w_n$ , including the most simple  $w_n$ . We are working on this right now, but so far we were unable to formulate a complete version of such a result. Further, this chapter discusses only one particular property of such presentations, namely that they present an “effectively” nontrivial group. It is possible to define precisely the “effective group” given by a presentation and study its other (geometric) properties besides the non-triviality. For example, in Chapter 5 we define an exponentially many balanced presentations of the trivial group, which are pairwise “effectively” distinct.

## 1.2 Main Results

In this section we construct a sequence of balanced group presentations of the trivial group with the following properties. Loosely speaking, the presentations are simple but to transform them to the empty presentation requires to go through increasingly more complex presentations. To make this notion precise we use elementary Tietze transformations (cf. [BHP68]). We will slightly abuse notation by using triangular brackets for both a presentation and the group presented by it.

**Definition 1.2.1.** Let  $\mu = \langle x_1, \dots, x_r | a_1, \dots, a_p \rangle$ .

Elementary Tietze transformations:

$Op_1$   $\mu$  is replaced by  $\langle x_1, \dots, x_r | a_1, \dots, a_{i-1}, a'x_j^\epsilon x_j^{-\epsilon} a'', a_{i+1}, \dots, a_p \rangle$ , where  $a_i \equiv a' a''$  and  $\epsilon = \pm 1$ .

$Op_1^{-1}$  The inverse of  $Op_1$  - it deletes  $x_j^\epsilon x_j^{-\epsilon}$  in one of the relators.

$Op_2$   $\mu$  is replaced by  $\langle x_1, \dots, x_r | a_1, \dots, a_{i-1}, a'_i, a_{i+1}, \dots, a_p \rangle$ , where the word  $a'_i$  is a cyclic permutation of the word  $a_i$ .

$Op_3$   $\mu$  is replaced by  $\langle x_1, \dots, x_r | a_1, \dots, a_{i-1}, a_i^{-1}, a_{i+1}, \dots, a_p \rangle$ .

$Op_4$   $\mu$  is replaced by  $\langle x_1, \dots, x_r | a_1, \dots, a_{i-1}, a_i a_j, a_{i+1}, \dots, a_p \rangle$ , where  $i \neq j$ .

$Op_5$   $\mu$  is replaced by the presentation  $\langle x_1, \dots, x_r, x_{r+1} | a_1, \dots, a_p, x_{r+1} a \rangle$ , where  $a$  is a word in the letters  $x_1^{\pm 1}, \dots, x_r^{\pm 1}$ .

$Op_5^{-1}$  The inverse of  $Op_5$ .

It is well-known (and can be found, for example, in [BHP68]) that any presentation of the trivial group could be reduced to the empty presentation by a sequence of Tietze transformations if one can also add empty relators to the presentation. This set of transformations is similar to Andrews-Curtis transformations, the difference being we treat relators as words, while Andrews-Curtis transformations treat them as elements of the free group. But otherwise they lead to the same presentations of the trivial group. The stable Andrews-Curtis conjecture says that we do not need to add empty relators to transform balanced presentations of the trivial group to the empty presentation.

We introduce some notation. Denote by  $E_n$  the tower of exponents of height  $n$ , i.e.  $E_i$  are recursively defined by  $E_0 = 1$ ,  $E_{n+1} = 2^{E_n}$ . As usual,  $x^y$  denotes  $y^{-1}xy$ , where  $x, y$  can be words or group elements. Let  $l(w)$  be the length of the word  $w$ . If  $w$  represents the identity element, denote by  $Area_\mu(w)$  the minimal number of 2-cells in a van Kampen diagram over the presentation  $\mu$  with boundary cycle labeled by  $w$ . Now we can state the main theorem:

**Theorem 1.2.2.** *There exist presentations of the trivial group  $\mu_i = \langle x, y, t | x^y x^{-2}, x^t y^{-1}, a_i \rangle$  for  $i \in \{5, 6, 7, \dots\}$ , where  $l(a_i) < 100 \cdot 2^i$ , but the minimal number of elementary Tietze transformations required to bring  $\mu_i$  to the empty presentation is at least  $E_{i-2}$ .*

First, we give an outline of the proof. Notice that  $\mu_i$  without the last relator  $a_i$  (denoted  $\mu_0 = \langle x, y, t | x^y x^{-2}, x^t y^{-1} \rangle$ ) presents the Baumslag-Gersten group  $G = \langle x, y, t | y^{-1}xy = x^2, y = t^{-1}xt \rangle$  (the base group for this HNN extension is called Baumslag-Solitar). It is known that the Dehn function for  $G$  is  $E_{\log_2(n)}$  (see [Pla04], [Ger91]). In particular they produce words  $w_n$  representing the identity element of length less than  $16 \cdot 2^n$  but of area greater than  $E_n$ . If we add the relation  $t = w_n$  to  $G$ , then the

group collapses to the trivial group. Actually  $a_n$  will be slightly more complicated than just  $t^{-1}w_n$ , we will define it precisely later, but in any case it will kill  $t$  by equating  $t$  to a word of large area. Then we will prove the following theorem:

**Theorem 1.2.3.** *The presentations of the trivial group  $\mu_i = \langle x, y, t | x^y x^{-2}, x^t y^{-1}, a_i \rangle$  for  $i \in \{5, 6, 7, \dots\}$ , where  $l(a_i) < 100 \cdot 2^i$ , have the property that  $\text{Area}_{\mu_i}(x) > E_{i-1}$ .*

Theorem 1.2.2 follows easily from Theorem 1.2.3, and we will supply all the details in the end of this paper. Now, we give a brief outline of the proof of Theorem 1.2.3.

What makes the theorem difficult is that though  $\text{Area}_{\mu_0}(w_n)$  is large, this fact a priori doesn't give us any bounds on  $\text{Area}_{\mu_n}(w_n)$ . Furthermore, standard techniques for proving lower bounds for the area do not work for presentations of the trivial group. Unlike an HNN-extension  $G$  the trivial group has no structure, no normal form theorem, to use to prove that different 2-cells in a van Kampen diagram cannot cancel each other to form a smaller van Kampen diagram. Therefore we will proceed as follows.

The authors of [SS74] have developed the small cancellation theory over HNN extensions: given a group  $H$ , an HNN extension with the stable letter  $t$ , and a new group formed from  $H$  by adding new relations, one can check if the new relations satisfy small cancellation condition (over  $H$ ) (see also the exposition of this theory in [LS01]). Recall, the usual small cancellation condition is satisfied if the cyclically reduced relators don't have large common pieces. However, the same element of  $H$  can be represented by different freely reduced words, therefore to measure pieces over  $H$  a different metric is used. The length is the number of occurrences of the stable letter of a normal form of the element (i.e. the minimal number of  $t, t^{-1}$  among all words representing the element). Note that such a small cancellation condition is not applicable to our case:  $a_i \equiv t^{-1}w_n =_H t^{-1}$ , meaning  $a_i$  has length 1. What we are going to do is to limit the number of applications of the relation  $x^t y^{-1}$ ,  $a_i$  then will not be equal to  $t^{-1}$  given this restriction. After we define everything appropriately this "effective" small cancellation theory will tell us that, for example,  $x$  is not trivial in  $\mu_i$  given the restriction on the number of applications of the relation  $x^t y^{-1}$ , and we will be done.

The approach taken in [SS74] is difficult to modify to obtain an "effective" theory. Therefore, we decided to use the techniques of contiguity subdiagrams used for stratified small cancellation theory [Ol'91], small cancellation theory over hyperbolic groups [Ol'93], relatively hyperbolic groups [Osi10]. Still, our exposition is self-contained and we are not going to use any concrete results from either [SS74] or [Ol'91]. We will first define  $a_i$  so that the reader has a motivating example, then develop a small cancellation theory over HNN extensions with a limited number of applications of relations, and then prove Theorem 1.2.3.

Let  $a_n = t^{-1}u_n$ , where  $u_n$  is defined inductively as follows. Let

$$u_{n,0} = [y^{-E_n} x y^{E_n}, x^3] [y^{-E_n} x y^{E_n}, x^5] [y^{-E_n} x y^{E_n}, x^7].$$

Suppose  $u_{n,m}$  is defined, then let  $u_{n,m+1}$  be the word obtained from  $u_{n,m}$  by replacing subwords  $y^{\pm E_{n-m}}$  with  $t^{-1} y^{-E_{n-m-1}} x^{\pm 1} y^{E_{n-m-1}} t$ . Finally, let  $u_n = u_{n,n}$ .

*Remark 1.2.4.* Since in the inductive step we replace subwords by equivalent (in  $G$ ) subwords,  $u_n =_G u_{n,n} =_G u_{n,n-1} =_G \dots =_G u_{n,0}$ . Each commutator in  $u_{n,0}$  is the identity element in  $G$ , which makes  $u_n =_G 1$ . Note that if we take  $w_{n,0} = [y^{-E_n} x y^{E_n}, x^1]$  and apply the same inductive procedure we will get  $w_n$ , the word from [Pla04] of area at least  $E_n$  (see Figure 1.1). The reason for us to make  $u_n$  slightly different is to satisfy a small cancellation condition.  $x$  to the powers  $\pm 3, \pm 5, \pm 7$  act as a unique signature

among cyclic permutations of  $u_n$ . It will be clear later why we avoid powers of 2. We can make an estimate  $l(a_n) \leq 100 \cdot 2^n$ .

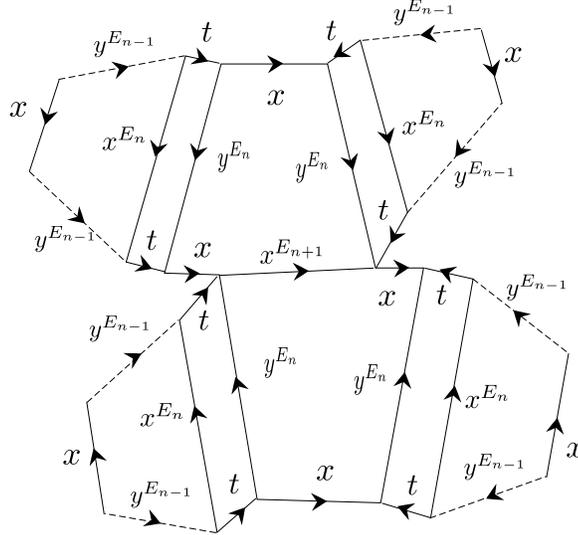


Figure 1.1: Van Kampen diagram for  $w_{n,1}$ . Parts of the boundary to be replaced by shorter paths to get  $w_{n,2}$  are marked by dashed lines.

### 1.3 Proofs

Now we develop small cancellation theory over HNN extensions with the limited number of operations with the stable letter. We will be using the diagram approach to HNN extensions first used in [MS73], see also an exposition in [Sap11], and in [Sho07] for dual diagrams. The main instrument of this approach is the  $t$ -band. Given a presentation of an HNN extension  $H$  with the stable letter  $t$ , a  $t$ -band is a collection of  $t$ -cells in a van Kampen diagram over this presentation such that the cells are adjacent to each other along  $t$ -edges.

Let  $H = \langle K, t | t^{-1}a_1t = b_1, \dots, t^{-1}a_k t = b_k \rangle$  (where  $a_i, b_i \in K$ ),  $A$  be the subgroup of  $K$  generated by  $\{a_i\}$  and  $B$  be the the subgroup of  $K$  generated by  $\{b_i\}$ . Recall the following fact, called Britton’s Lemma ([LS01], [Sap11]). If  $w = g_0 t^{\delta_1} g_1 t^{\delta_2} g_2 \dots g_{m-1} t^{\delta_m} g_m$  ( $g_i$  are words in the letters of the base group,  $\delta_i = \pm 1$ ) is a word representing 1 in  $H$ , then for some  $i$  either  $\delta_i = -1$ ,  $\delta_{i+1} = 1$  and  $g_i \in A$  or  $\delta_i = 1$ ,  $\delta_{i+1} = -1$  and  $g_i \in B$ . In particular that means we can rewrite  $w$  with one less  $t$  letter and one less  $t^{-1}$ . This is called a  $t$ -reduction or a “pinch”. If  $t$ -reduction is impossible the word is called reduced. Britton’s Lemma, therefore states that reduced words are not trivial, unless they are trivial already in  $K$ .

It is not hard to see why Britton’s Lemma is true using  $t$ -bands. Consider a van Kampen diaram for (freely reduced)  $w$ . If the diagram does not have any  $t$ -cells we are done, otherwise there are  $t$ -bands in the diagram. They can either be circular or begin and end on the boundary. Consider an innermost circular  $t$ -band. Its inner edge is a word representing the trivial element in  $K$ , therefore by assumption ( $A$  and  $B$  being isomorphic through conjugation by  $t$ ) the outer edge is also trivial in  $K$ . Thus we can replace the innermost circular  $t$ -band by a subdiagram not containing  $t$ -cells. In this way we can get rid of all circular  $t$ -bands. Now, consider a  $t$ -band originating on the boundary, a “semi-circular”  $t$ -band.

Take an innermost semi-circular  $t$ -band, it shows a  $t$ -reduction (see Figure 1.2), and we are done. See [LS01], [Sap11] for more details.

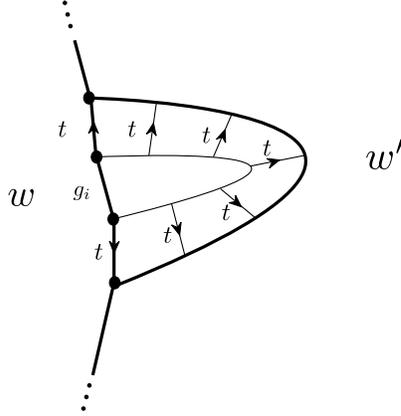


Figure 1.2: Van Kampen diagram for  $w = w'$ ,  $w'$  being  $w$  after one  $t$ -reduction is performed. Note,  $g_i$  is equal (in the base group) to the inner edge of the  $t$ -band, because there are no other  $t$ -cells on this diagram, thus showing  $g_i \in A$ .

*Remark 1.3.1.* Once Britton's Lemma is proven, one can replace circular  $t$ -bands by subdiagrams without  $t$ -cells in one steps. The outer edge of a circular  $t$ -band is an element in  $K$  trivial in  $H$ , and therefore trivial in  $K$ . We will also use another fact explained in the previous paragraph: if a part of the boundary of a van Kampen diagram is labeled by a reduced word, then it is impossible for a  $t$ -band to both originate and end on this part.

**Definition 1.3.2.** Let  $H = \langle K, t | t^{-1}a_1t = b_1, \dots, t^{-1}a_kt = b_k \rangle$  (where  $a_i, b_i \in K$ ) be a presentation of an HNN extension of the base group  $K$ . Let  $A$  be the subgroup of  $K$  generated by  $\{a_i\}$  and  $B$  be the the subgroup of  $K$  generated by  $\{b_i\}$ . Let  $N$  be a natural number. We call a word  $w = g_0t^{\delta_1}g_1t^{\delta_2}g_2\dots g_{m-1}t^{\delta_m}g_m$  ( $g_i$  are words in the letters of the base group,  $\delta_i = \pm 1$ )  $N$ -reduced if the following holds:

1. If  $\delta_i = -1$  and  $\delta_{i+1} = 1$  then either  $g_i$  is not in  $A$ , or  $g_i \in A$  but for any  $g$  (a word in  $K$  satisfying  $t^{-1}g_it = g$ , e.g. if  $g_i = a_3a_1$  then  $g$  could be  $b_3b_1$ ) any van Kampen diagram for  $t^{-1}g_itg^{-1}$  contains more than  $N - 1$   $t$ -cells.
2. If  $\delta_i = 1$  and  $\delta_{i+1} = -1$  then either  $g_i$  is not in  $B$ , or  $g_i \in B$  but for any  $g$  (a word in  $K$  satisfying  $tg_it^{-1} = g$ ) any van Kampen diagram for  $tg_it^{-1}g^{-1}$  contains more than  $N - 1$   $t$ -cells.

*Remark 1.3.3.* That is, pinches in  $N$ -reduced words are allowed only if the reduction requires long enough  $t$ -bands;  $\infty$ -reduced is the usual notion of reduced for HNN extensions. Note that this definition is independent of the particular words  $g_i$  as long as they represent the same elements of  $K$ . Therefore we can think of reduced words as sequences  $g_0, t^{\delta_1}, g_1, \dots, g_{m-1}, t^{\delta_m}, g_m$ , where  $g_i$  are elements of the base group. However, two different reduced sequences can represent the same element of  $H$ : unlike normal forms, where coset representatives are fixed (see [LS01] for details), reduced forms have this ambiguity. We write  $w_1 \equiv w_2$  if they are the same as sequences of letters  $t^{\pm 1}$  and elements of the base group. Equivalently, there is a van Kampen diagram for  $w_1w_2^{-1}$  without  $t$ -cells and to spell  $w_1$  one does not traverse a  $t$ -edge on the boundary twice, similarly for  $w_2^{-1}$ . The last condition is to make sure, for example,  $tt^{-1}t \neq t$ .

Similarly, we introduce cyclically  $N$ -reduced:

**Definition 1.3.4.** Let  $\mu_H$  be a presentation of an HNN extension  $H$ . Let  $N$  be a natural number. We call a word  $w = g_0 t^{\delta_1} g_1 \dots g_{m-1} t^{\delta_m} g_m$  ( $g_i$  are in the base group,  $\delta_i = \pm 1$ ) cyclically  $N$ -reduced if all cyclic permutations of  $g_0, t^{\delta_1}, g_1, \dots, g_{m-1}, t^{\delta_m}, g_m$  are  $N$ -reduced.

We illustrate these definitions on the Baumslag-Gersten group:

**Lemma 1.3.5.** Let  $v_1 = t^{-1}g_1t$ ,  $v_2 = tg_2t^{-1}$ , where  $g_1, g_2$  are words in the letters  $x, y, x^{-1}, y^{-1}$  and  $g_1 = x^i$ ,  $g_2 = y^i$  in the Baumslag-Solitar group. Then  $v_1, v_2$  are  $i$ -reduced (relative to  $\mu_0$ ).

*Proof.* We prove this lemma for  $v_1$ , for  $v_2$  it's completely analogous. Consider a van Kampen diagram for  $t^{-1}g_1tg^{-1}$ , where  $g = y^i$  (see Figure 1.3). Since there are only two letters  $t$  on the boundary of the diagram, they must be connected by a  $t$ -band, say of length  $k$ . Because  $G$  is an HNN extension of the Baumslag-Solitar group, the latter embeds in  $G$ . Therefore  $i = k$ .  $\square$

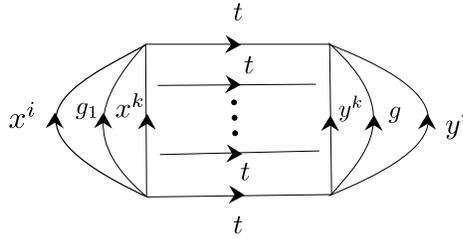


Figure 1.3:

*Remark 1.3.6.* Notice that

$$u_{n,1} = [t^{-1}y^{-E_{n-1}}x^{-1}y^{E_{n-1}}tx^{-1}y^{-E_{n-1}}x^1y^{E_{n-1}}t, x^3] \dots$$

is  $E_n$ -reduced by the preceding lemma. (Here we omitted the other two commutators involving  $x^5$  and  $x^7$  instead of  $x^3$ , which look the same.) Similarly,  $u_{n,2}$  is  $E_{n-1}$ -reduced,  $u_{n,3}$  is  $E_{n-2}$ -reduced, etc... Also,  $u_{n,1}$  is cyclically  $E_n$ -reduced, and so is  $t^{-1}u_{n,1}$  and  $(t^{-1}u_{n,1})^{-1}$ . Since our theory will only work with cyclically  $N$ -reduced words, it is a problem that  $u_n, a_n$  are not cyclically  $N$ -reduced for large  $N$ . We will deal with this in the proof of Theorem 1.2.3, but for now we prove results about  $u_{n,1}$ .

We will state the small cancellation condition we need. First we will define a piece.

**Definition 1.3.7.** Let  $H = \langle K, t | t^{-1}a_1t = b_1, \dots, t^{-1}a_kt = b_k \rangle$  (where  $a_i, b_i \in K$ ) be a presentation of an HNN extension of the base group  $K$ . Let  $A$  be the subgroup of  $K$  generated by  $\{a_i\}$  and  $B$  be the subgroup of  $K$  generated by  $\{b_i\}$ . Let  $N$  be a natural number. Let  $R$  be a set of cyclically  $N$ -reduced words in  $H$ , such that  $R$  is closed under inversion and cyclic permutation of the reduced sequences (see Remark 1.3.3). If  $R$  is such a set, we call it  $N$ -symmetrized. Let  $r \equiv pb$ ,  $r' \equiv p'b'$  be in  $R$ . We call  $p$  an  $N$ -piece if for some  $v_1, v_2 \in K$  the following holds:

1. both  $p$  and  $p'$  start and end with  $t^{\pm 1}$ ,
2.  $p = v_1p'v_2$ , furthermore there exists a van Kampen diagram for  $p = v_1p'v_2$ , where all  $t$ -bands are of length less than  $N$ ,

3.  $b \neq v_2^{-1}b'v_1^{-1}$  (see Remark 1.3.3 for the definition of  $\equiv$ ).

Someone familiar with the small cancellation theory over (relatively) hyperbolic groups would notice that a more natural requirement in (3) would be  $b \neq_H v_2^{-1}b'v_1^{-1}$ . We delay the explanation until Theorem 1.3.10.

Denote by  $l_t(w)$  ( $t$ -length of  $w$ ) the number of occurrences of the letters  $t, t^{-1}$  in the word  $w$ . We will measure the length of pieces with  $l_t$ . Two  $N$ -reduced words have the same  $l_t$  if the words are equal in  $H$  through diagrams with  $t$ -bands of length less than  $N$ . We define a metric small cancellation condition:

**Definition 1.3.8.** Let  $R$  be an  $N$ -symmetrized set (as in the previous definition). We say condition  $C'(\lambda, N)$  is satisfied for  $R$  if  $r \in R$ ,  $r \equiv pb$ , where  $p$  is an  $N$ -piece, implies  $l_t(p) < \lambda l_t(r)$ .

*Remark 1.3.9.* We would like to clarify the difference between our  $C'(\lambda, N)$  and the usual  $C'(\lambda)$  (see, for example, p. 291 in [LS01]). This remark can be skipped if desired, as it will not be used later in this chapter.

We claim that by changing condition (3) of Definition 1.3.7 to  $b \neq_H v_2^{-1}b'v_1^{-1}$  we would make  $C'(\lambda, \infty)$  equivalent to  $C'(\lambda)$  from p. 291 of [LS01]. Their definition of a symmetrized set is as follows. It is a set of elements of  $H$  (not reduced words or reduced sequences as in our definition) which are cyclically reduced and closed under conjugation (cyclically reducing after conjugation) and inverses. So, we have to explain what we mean by saying that these conditions are equivalent. Let  $R$  be an  $\infty$ -symmetrized set (with condition (3) modified as above). Let  $R' = \{w | w =_H w' \in R\}$ . Let  $\bar{R}'$  be the closure of  $R'$  under cyclic permutations and inversions. The projection of this set to  $H$ , denoted  $[\bar{R}']$ , is a symmetrized set as defined in [LS01]. Our precise claim is that  $[\bar{R}']$  is  $C'(\lambda)$ . Their definition of a piece is the following. Let  $r_1 = pb_1, r_2 = pb_2 \in [\bar{R}'] \subset H$ , such that here is no cancellation in  $pb_1$  and  $pb_2$  (meaning that if  $\tilde{p}, \tilde{b}_1$  are reduced sequences representing  $p, b_1$ , then  $\tilde{p}\tilde{b}_1$  is already a reduced sequence) and  $r_1 \neq r_2$ . Then  $p$  is a piece.

The claim follows from the following. Suppose  $r_i, p, b_i, \tilde{p}, \tilde{b}_i$  are defined as above. Then there exist  $\tilde{r}'_1, \tilde{r}'_2 \in R$  such that  $r_i = [h_i^{-1}\tilde{r}'_i h_i]$ . See Figure 1.4 for the definition of  $\tilde{p}'_i, \tilde{b}'_i, k_i, l_i$ . Since  $\tilde{p}'_i \tilde{b}'_i$  is a cyclic permutation of  $\tilde{r}'_i$ ,  $\tilde{p}'_i \tilde{b}'_i$  is in  $R$ . Furthermore, since  $\tilde{p}'_i =_H k_i \tilde{p} l_i$ , we have  $\tilde{p}'_i =_H v_1 \tilde{p}'_2 v_2$ , where  $v_1 = k_1 k_2^{-1}, v_2 = l_2^{-1} l_1$ . If in addition,  $r_1 \neq r_2$ , then  $\tilde{b}'_1 \neq_H v_2^{-1} \tilde{b}'_2 v_1^{-1}$  (suppose  $\tilde{b}'_1 =_H v_2^{-1} \tilde{b}'_2 v_1^{-1}$ , then  $l_1 \tilde{b}'_1 k_1 =_H l_2 \tilde{b}'_2 k_2$  or  $\tilde{b}'_1 =_H \tilde{b}'_2$ , which implies  $r_1 = r_2$ ). It follows that  $\tilde{p}'_1, \tilde{p}'_2$  are pieces (our definition) of the same  $t$ -length as  $p$ .

Now we can state and prove the only small cancellation result we need:

**Theorem 1.3.10.** *Let  $H$  be a presentation of an HNN extension with the stable letter  $t$  and associated subgroups  $A, B$  ( $t^{-1}At = B$ ). Let  $r_1, \dots, r_m, w$  be cyclically  $N$ -reduced words, let  $R$  be the smallest  $N$ -symmetrized set containing  $r_1, \dots, r_m, w \neq 0$  in  $H$  and  $l_t(w) = 0$ . Let  $\mu = \langle H | r_1, \dots, r_m \rangle$ . Then if  $R$  satisfies  $C'(\frac{1}{6}, N)$  there is no van Kampen diagram over  $\mu$  with boundary cycle  $w$ , which contains less than  $N$   $t$ -cells ( $t$ -cells from  $H$ ).*

*Proof.* By  $r$ -cells we will call cells corresponding to the relators  $r_1, \dots, r_m$ .

Suppose for the sake of contradiction that we have such a diagram  $D$  which has less than  $N$   $t$ -cells. We can assume this diagram is reduced. We also want it to be  $r$ -reduced in the following sense. We choose  $D$  to have the minimal number of  $r$ -cells among all diagrams with less than  $N$   $t$ -cells.

Now consider  $t$ -bands in the diagram  $D$ . Since the boundary does not have any letters  $t$ ,  $t$ -bands have to either be rings or originate and end at the  $r$ -cells. We call two  $t$ -bands consecutive if they begin

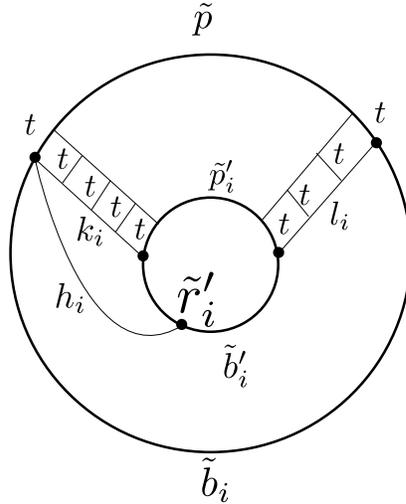


Figure 1.4: An annular van Kampen diagram corresponding to  $\tilde{p}\tilde{b}_i =_H h^{-1}\tilde{r}'_i h$ . Two  $t$ -bands are marked on this diagram. Note that these  $t$ -bands have to go from the outer boundary to the inner because there is no cancellation between  $\tilde{p}$  and  $\tilde{b}_i$  by assumption.

and end on consecutive letters  $t^{\pm 1}$  of the same  $r$ -cell (by this we mean they can be separated by letters other than  $t^{\pm 1}$ ) and the subdiagram bounded by this two  $t$ -bands on the sides and two  $r$ -cells at the ends contains only cells from  $H$ .

The subdiagram containing a maximal sequence of  $t$ -bands and the cells in between, such that neighbouring bands are consecutive, will be called a  $t$ -cable (see Figure 1.5). We claim that the ends of a  $t$ -cable are  $N$ -pieces. All requirements of Definition 1.3.7 are straightforward to check, we only note that requirement (3) follows from the fact that  $D$  is  $r$ -reduced. If (3) is not satisfied, we can replace the subdiagram bounded by  $b^{-1}v_1 b'v_2$  (in the notation of the definition 1.3.7) by a diagram containing only cells from the base of the HNN-extension  $H$ .

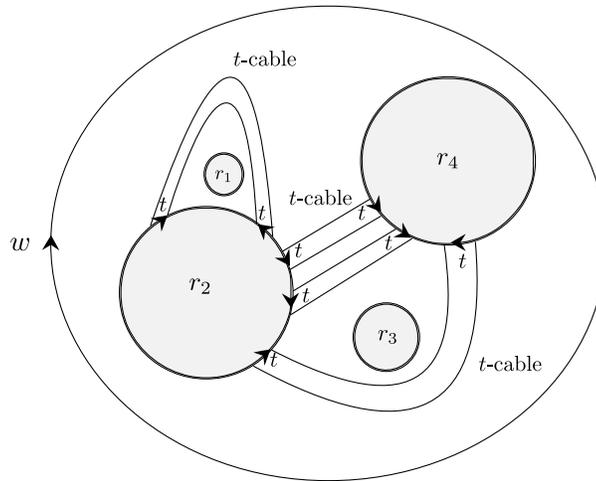


Figure 1.5: An example of  $D$ .  $r$ -cells are marked with grey. There are four  $t$ -bands and three  $t$ -cables in this example. Note that this example is not realistic, because  $r_1, r_3$  have no letters  $t$ .

We define a dual diagram to  $D$ . Associate to each  $r$ -cell a point. Connect them by edges corresponding

to the  $t$ -cables. Take an innermost connected component of this graph and call it  $D^*$ . We claim that  $D^*$  does not have any faces with less than 3 edges. The fact that  $D^*$  is an innermost component implies that the faces of  $D^*$  are faces of the whole dual to  $D$ .

A face with 1 edge ( $L_1$  on Figure 1.6) would mean that there is a  $t$ -cable with both ends on the same cell, such that there are no other  $t$ -cables in between that  $t$ -cable and the relator (call that subdiagram  $D'$ ). That implies that there are no letters  $t^{\pm 1}$  on the boundary of that relator between the pieces (marked as  $g_1$  on the figure) and there are no other  $r$ -cells in  $D'$  (other  $r$ -cells can not be connected to the  $r$ -cell because there are no  $t$ -cables to connect them with, and they can not be disconnected because we took the innermost component of the dual graph), but that contradicts that all cyclic permutations of  $r_1, \dots, r_m$  are  $N$ -reduced (see Remark 1.3.1 for a similar argument).

A face with 2 edges (marked  $L_2, L_3$  on Figure 1.6) would arise from two  $t$ -cables between two (possibly one) cells such that there are no other  $r$ -cells in between. But that would contradict that  $t$ -cables are maximal sequences of consecutive  $t$ -bands.

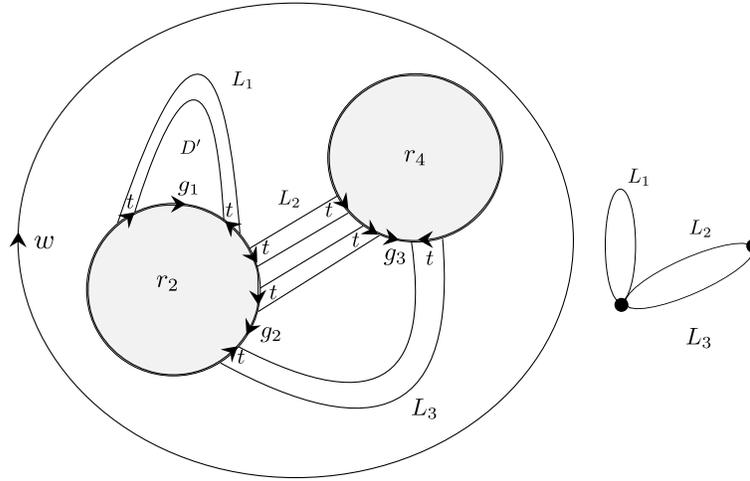


Figure 1.6: An (impossible) example of  $D$  and  $D^*$  next to it.  $L_1$  can not be a  $t$ -cable and  $L_2, L_3$  should be one  $t$ -cable.

$D^*$  is a planar connected graph, therefore we have the Euler's Formula

$$V - E + F = 1.$$

From the fact that each face of  $D^*$  has at least 3 edges we have  $F \leq \frac{2E}{3}$ . Euler's Formula then becomes

$$1 = V - E + F \leq V - \frac{E}{3}.$$

Since the ends of a  $t$ -cable correspond to  $N$ -pieces, from  $C'(\frac{1}{6}, N)$  we know that each vertex has degree at least 6 (if we count a looping edge twice). Therefore we have  $E \geq \frac{6V}{2} = 3V$ , and the inequality becomes

$$1 \leq V - \frac{E}{3} \leq V - V = 0.$$

We have reached a contradiction.

□

*Remark 1.3.11.* One can strengthen the above theorem, by obtaining an analogue of Greendlinger’s Lemma: not only the  $t$ -length of  $w$  should be non-zero, but  $w$  has to contain a part of  $r_i$  of  $t$ -length greater than  $\frac{1}{2}l_t(r_i)$  for some  $i$ . See Theorem 11.2 from [LS01] for such a result in the usual small cancellation theory over HNN-extensions. The “effective” theorem can be obtained in exactly the same way – by considering the geometry of  $D^*$ . We will not include this result as we will not use it later.

*Remark 1.3.12.* In Remark 1.3.9 we discussed why making condition (3) in Definition 1.3.7  $b \neq_H v_2^{-1}b'v_1^{-1}$  instead of  $b \neq v_2^{-1}b'v_1^{-1}$  would make our theory more similar to the one presented in [LS01]. In the previous theorem we see why we need the stronger condition. We used it to remove pairs or  $r$ -cells without adding any new  $t$ -cells. And this is a reason why it was inconvenient to use the preexisting small cancellation theory in our setting. We can afford this weaker condition, because  $R$  can be chosen to be much smaller than  $\bar{R}'$  (see Remark 1.3.9 for notation). This is demonstrated by Lemma 1.3.14, where we prove our set is  $C'(\frac{1}{6}, N)$  with the weaker condition (3).

To apply the preceding theorem we will define a  $C'(\lambda, N)$   $N$ -symmetrized set. We will first need the following lemma.

**Lemma 1.3.13.** *In the Baumslag-Solitar group  $\langle x, y | y^{-1}xy = x^2 \rangle$  (the base group of  $G$ ) if  $y^i x^m y^j = x^k$ , then  $i = -j$  and  $\frac{m}{k} = 2^j$ .*

*Proof.* Since the right part of the given equation does not contain any letter  $y^{\pm 1}$ ,  $i = -j$ . If  $i < 0$ ,  $y^i x^m y^j = x^{2^i m} y^i y^j = x^{2^i m}$ , and the result follows. If  $i > 0$ , then by conjugating both sides of the equation we have  $x^m = y^{-i} x^k y^{-j} = x^{2^{-i} k}$  as in the previous case. □

Now we can prove

**Lemma 1.3.14.** *The set of all cyclic permutations of  $t^{-1}u_{n,1}$  and  $(t^{-1}u_{n,1})^{-1}$  is an  $E_n$ -symmetrized set satisfying condition  $C'(\frac{1}{6}, E_n)$  over  $G = \langle x, y, t | y^{-1}xy = x^2, y = t^{-1}xt \rangle$ .*

*Proof.* Recall,  $u_{n,1} = [t^{-1}y^{-E_{n-1}}x^{-1}y^{E_{n-1}}t x t^{-1}y^{-E_{n-1}}x^1 y^{E_{n-1}}t, x^3] \dots$

The set is  $E_n$ -symmetrized by Remark 1.3.6. If  $p = t^{\delta_1} g_1 t^{\delta_2} g_2 \dots g_{m-1} t^{\delta_m}$  is a piece corresponding to  $p' = t^{\delta'_1} g'_1 t^{\delta'_2} g'_2 \dots g'_{m-1} t^{\delta'_m}$  ( $g_i$  are in Baumslag-Solitar,  $\delta_i = \pm 1$ ), there is a van Kampen diagram for  $p = v_1 p' v_2$  (from the definition of the piece). The  $t$ -bands on the diagram have to go from  $p$  to  $p'$ , because  $p$  and  $p'$  are  $E_n$ -reduced while the length of all  $t$ -bands is less than  $E_n$  (i.e. they can’t start and end on the same  $p$  or  $p'$ ). See Figure 1.7 for an example.

There is a limited number of possibilities for  $g_i$ , namely, it could be  $x^{\pm 1}, x^{\pm 3}, x^{\pm 5}, x^{\pm 7}$  or  $x^{\pm E_n}$  ( $y^{-E_{n-1}}x^{\pm 1}y^{E_{n-1}} = x^{\pm E_n}$  in Baumslag-Solitar). Assume  $g_i$  is a small power of  $x$ , then  $g_i$  occurs in  $t^{-1}u_{n,1}$  or  $u_{n,1}^{-1}t$  as  $tg_i t^{-1}$ . Therefore, the sides of the  $t$ -bands neighbouring  $g_i$  are labeled by powers of  $y$  rather than powers of  $x$ :  $y^k g_i y^j =_H g'_i$ . By Britton’s lemma (or by removing all circular  $t$ -bands, see Remark 1.3.1) this equality holds in Baumslag-Solitar. Therefore we can apply Lemma 1.3.13 to obtain  $k = j$ , and since none of the integers 1, 3, 5, 7 are divisible by 2 we have  $g_i = g'_i$  in the Baumslag-Solitar group, which in turn implies  $k = j = 0$ .

Every piece of  $t$ -length greater than 2 has a  $g_i$  a small power of  $x$ , which implies that for every piece of  $t$ -length greater than 2 all  $t$ -bands have length 0 and  $v_1 = 1 = v_2$ .

Assume  $p$  contains any of  $x^{\pm 3}, x^{\pm 5}, x^{\pm 7}$ . As before all  $t$ -bands have length 0 and  $v_1 = 1 = v_2$  in this case. Such a power of  $x$  is located uniquely within  $t^{-1}u_{n,1}$  and  $(t^{-1}u_{n,1})^{-1}$ . Therefore if both  $p$

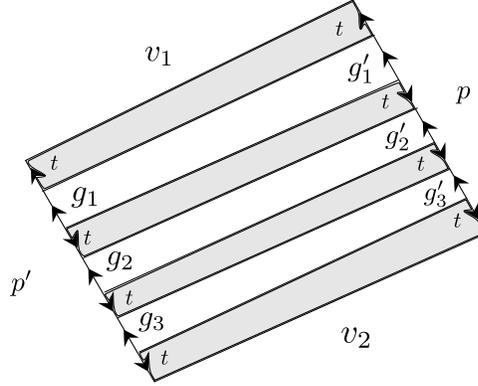


Figure 1.7:

and  $p'$  are parts of two cyclic permutations of the same relator, then condition (3) of Definition 1.3.7 is not satisfied. This implies that the only possibility is for  $p$  and  $p'$  to be from cyclic permutations of the relator and its inverse. We compare the relator, say, around  $x^5$ :

$$\dots, t, x, t^{-1}, x^{E_n}, t, x^5, t^{-1}, x^{-E_n}, t, x^{-1}, t^{-1}, \dots$$

to the inverse of the relator around  $x^5$

$$\dots, t, x^{-1}, t^{-1}, x^{E_n}, t, x^5, t^{-1}, x^{-E_n}, t, x, t^{-1}, \dots$$

Therefore, we see that if  $p$  contains  $x^5$  then  $l_t(p) \leq 4$ . We can proceed similarly for pieces containing  $x^{\pm 3}, x^{-5}, x^{\pm 7}$ . If  $p$  does not contain any of the  $x^{\pm 3}, x^{\pm 5}, x^{\pm 7}$ , then its length is also at most 4 which is the length between these powers of  $x$  except for  $x^{-7}$  to  $x^3$ , where there is an extra  $t^{-1}$  letter, which makes the pieces even shorter. The total length is  $4 * 6 + 1$ , so  $C'(\frac{1}{6}, E_n)$  holds.  $\square$

Now we can prove Theorem 1.2.3.

*Proof.* Suppose for the sake of contradiction  $x = \prod_{i=1}^N g_i u_i^{\pm 1} g_i^{-1}$ , where the equality is in the free group,  $N \leq E_{n-1}$  and  $u_i$  are the relators from  $\mu_n$ . We want to rewrite this equality using  $t^{-1}u_{n,1}$  instead of  $t^{-1}u_n$ .

We need to apply the relations of  $\mu_0$   $2E_{n-1}$  times to convert  $t^{-1}y^{-E_{n-2}}xy^{E_{n-2}}t$  to  $y^{E_{n-1}}$ . Therefore, we need  $24 \cdot 2E_{n-1}$  applications of the relations to convert  $u_{n,2}$  to  $u_{n,1}$ . Similarly, we need  $48 \cdot 2E_{n-2}$  applications to convert  $u_{n,3}$  to  $u_{n,2}$ , etc. Since  $E_{n-2} + 2E_{n-3} + 4E_{n-4} + \dots$  do not add up to more than  $E_{n-1}$ , we see that we do not need more than  $96 \cdot E_{n-1}$  applications of the relations to convert  $t^{-1}u_n$  to  $t^{-1}u_{n,1}$ .

There are at most  $N$  such relators in the product. Therefore, we need at most  $96 \cdot E_{n-1} \cdot E_{n-1}$  relations of  $\mu_0$  to convert all of them. Since for  $n > 5$ ,  $E_n > 96 \cdot E_{n-1} \cdot E_{n-1}$ , we have  $x = \prod_{i=1}^{N'} g'_i (u'_i)^{\pm 1} (g'_i)^{-1}$ , where the equality is in the free group,  $N' \leq E_n$  and  $u'_i$  are either from  $G$  or are  $t^{-1}u_{n,1}$ . The theorem now follows from Lemma 1.3.14 and Theorem 1.3.10.  $\square$

We prove our main result – Theorem 1.2.2.

*Proof.* Consider a sequence of presentations  $\mu_i = \mu_i^{(0)}, \mu_i^{(1)}, \mu_i^{(2)}, \dots$ , where the last presentation is the empty presentation, and each step is an elementary Tietze transformation. For some  $n$   $\mu_i^{(n+1)}$  will be obtained from  $\mu_i^{(n)}$  by applying  $Op_5^{-1}$  to kill the generator  $x$ . Therefore  $Area_{\mu_i^{(n)}}(x) = 1$ , while  $Area_{\mu_i^{(0)}}(x) > E_{n-1}$  by Theorem 1.2.3. Clearly,  $Op_1^{\pm 1}, Op_2, Op_3$  do not change the area of the word  $x$ . We show that  $Op_5^{\pm 1}$  also do not change the area.

Let  $\mu_i^{(k+1)} = \langle x_1, \dots, x_r, x_{r+1} | a_1, \dots, a_p, x_{r+1}a \rangle$  (where  $a$  is a word in the letters  $x_1^{\pm 1}, \dots, x_r^{\pm 1}$ ) be obtained from  $\mu_i^{(k)} = \langle x_1, \dots, x_r | a_1, \dots, a_p \rangle$  by applying  $Op_5$ . Suppose  $x = \prod_{i=1}^N g_i u_i^{\pm 1} g_i^{-1}$ , where  $u_i$  are relators of  $\mu_i^{(k)}$ , then  $u_i$  are also relators of  $\mu_i^{(k+1)}$ , implying  $Area_{\mu_i^{(k)}}(x) \geq Area_{\mu_i^{(k+1)}}(x)$ . On the other hand, suppose  $x = \prod_{i=1}^N g_i u_i^{\pm 1} g_i^{-1}$ , where  $u_i$  are relators of  $\mu_i^{(k+1)}$ . Let  $\tilde{g}_i, \tilde{u}_i$  be  $g_i, u_i$  with the letter  $x_{r+1}$  replaced by  $a^{-1}$ . The equality in the free group is preserved:  $x = \prod_{i=1}^N \tilde{g}_i \tilde{u}_i^{\pm 1} \tilde{g}_i^{-1}$ . We can remove the relators corresponding to  $x_{r+1}a$  (replaced with  $a^{-1}a$ ) from the last equation while preserving the equality, thus proving  $Area_{\mu_i^{(k)}}(x) \leq Area_{\mu_i^{(k+1)}}(x)$ . Therefore  $Op_5^{\pm 1}$  preserve the area.

The remaining transformation,  $Op_4$ , can reduce the area by a factor that cannot exceed 2 for the following reason. Suppose  $\mu_i^{(k+1)} = \langle x_1, \dots, x_r | a_1, \dots, a_{i-1}, a_i a_j, a_{i+1}, \dots, a_p \rangle$  is obtained from  $\mu_i^{(k)} = \langle x_1, \dots, x_r | a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_p \rangle$  by applying  $Op_4$ . If it takes  $K$  relators to show that  $x = 1$  in  $\mu_i^{(k+1)}$ , then in the worst case scenario all of them are  $a_i a_j$ , and we will need  $2K$  relators in  $\mu_i^{(k)}$  to show that  $x = 1$ . Therefore,  $Area_{\mu_i^{(k)}}(x) \geq \frac{Area_{\mu_i^{(k+1)}}(x)}{2}$ . So, we have  $n > E_{n-2}$ . □

Finally, we will prove the claim from the introduction.

**Corollary 1.3.15.** *There exist presentations of the trivial group  $\mu'_i$  with 2 generators and 2 relations, such that the length of the relators is less than  $300 \cdot 2^i$ , but the area of a generator of  $\mu'_i$  is greater than  $E_{i-2}$ .*

*Proof.* We obtain  $\mu'_i$  from  $\mu_i$  by eliminating  $y$  using the equation  $y = x^t$ . The number of Tietze transformations needed to do that grows linearly with the length of the relator  $a_i$ . The length of the relators does not increase by more than 3, so we have the length estimate. From the proof of Theorem 1.2.2 we can see that the area of the generators can not decrease by more than 2 with each elementary transformation, while Theorem 1.2.3 states that  $Area_{\mu_i}(x) > E_{i-1}$ . Therefore we have the estimate  $Area_{\mu'_i}(x) > E_{i-2}$ . □

# Chapter 2

## Four-dimensional Geometry

### 2.1 Main results

The goal of this paper is to extend results of [Nab95], [Nab96a], [Nab96b] to the four-dimensional situation.

**Theorem 2.1.1.** *Let  $M$  be any closed four-dimensional Riemannian manifold. Let  $I_\epsilon(M)$  denote the space of isometry classes of Riemannian metrics on  $M$  with volume equal to 1 and injectivity radius greater than  $\epsilon$ . (This space is endowed with the Gromov-Hausdorff metric  $d_{GH}$ .) Then for all sufficiently small  $\epsilon > 0$ ,  $I_\epsilon(M)$  is disconnected, and, moreover, can be represented at the union of two non-empty subsets  $A_1, A_2$  such that for any  $\mu_1 \in A_1, \mu_2 \in A_2$   $d_{GH}(\mu_1, \mu_2) > \frac{\epsilon}{10}$ .*

*Furthermore, let for each  $m$ ,  $\exp_m(x)$  denote  $\exp(\exp(\dots(\exp x)))$  ( $m$  times). Then for each  $m$  for all sufficiently small  $\epsilon$  there exist  $\mu, \nu \in I_\epsilon(M)$  with the following property. Let  $\mu_1 = \mu, \mu_2, \dots, \mu_N = \nu$  be a sequence of isometry classes of Riemannian metrics on  $M$  of volume one such that for each  $i$   $d_{GH}(\mu_i, \mu_{i+1}) \leq \frac{\epsilon}{10}$ . Then  $\inf_i \text{inj}(\mu_i) \leq \frac{1}{\exp_m(\frac{1}{\epsilon})}$ .*

A (stronger) analog of this theorem for  $n > 4$  as well as for a class of closed four-dimensional manifolds representable as the connected sum of any closed 4-manifold and several copies of  $S^2 \times S^2$  can be found in [Nab96a] (Theorem 1 and section 5.A). (More precisely, “several” means 14. The minimal number of copies of  $S^2 \times S^2$  required for the method of [Nab96a] to work is equal to the number of relators in a sequence of finite presentations, where the triviality problem is algorithmically unsolvable; cf. [Sta07], [Sta04]).

The next theorem is a four-dimensional analog of Theorem 11 from [Nab96a]. For each smooth manifold  $M$  define  $Al_1(M)$  as the space of  $C^1$ -smooth Alexandrov spaces of curvature  $-1 \leq K \leq 1$ , diffeomorphic to  $M$  (cf. [BN93] for a definition of Alexandrov spaces with two-sided bounds on sectional curvature). A result of I. Nikolaev ([Nik91]) implies that all of them are Gromov-Hausdorff limits of sequences of smooth Riemannian structures on  $M$ . The classical Gromov-Cheeger compactness theorem implies that all elements of  $Al_1(M)$  are  $C^{1,\alpha}$ -smooth Riemannian structures on  $M$  for any  $\alpha \in (0, 1)$ . We can consider diameter as a functional on  $Al_1(M)$ .

**Theorem 2.1.2.** *Let  $M$  be a closed 4-dimensional manifold such that either its Euler characteristic is not equal to zero, or its simplicial volume is not equal to zero. Then the diameter regarded as a functional*

on  $Al_1(M)$  has infinitely many local minima. The set of values of the diameter at its local minima on  $Al_1(M)$  is unbounded.

The assumptions about  $M$  imply a uniform positive lower bound for the volume of all elements of  $Al_1(M)$ . Now the Gromov-Cheeger theorem implies the compactness of sublevel sets of  $diam$ ,  $diam^{-1}((0, x])$ , on  $Al_1(M)$  for all values of  $x$ . Now we see that it is sufficient to prove that there exists an unbounded sequence of values of  $x$  such that the set of all smooth Riemannian structures on  $M$  with  $-1 \leq K \leq 1$  and  $diam \leq x$  is disconnected, and, moreover, can be represented as a union of two non-empty subsets with disjoint closures. After noticing that the classical Cheeger inequality implies that for all such smooth Riemannian structures the injectivity radius will be bounded below by an explicit positive function of  $x$  (that behaves as  $const \exp(-3x)$ ) we see that this theorem is similar to the previous one, and, in fact, has a very similar proof.

Theorem 11 in [Nab96a] should not be confused with a much deeper and significantly more difficult main theorem in [NW00] (see also [Nab10b], [NW03] and [Wei05]) that does not have the assumption that a smooth manifold  $M$  of dimension greater than four has either a non-zero Euler characteristic or a non-zero simplicial volume, and, therefore, one lacks an a priori uniform positive lower bound for the volumes of the considered metrics. At the moment we are not able to prove a four-dimensional analog of the main theorem of [NW00].

In order to state the next theorem define *crumpledness* (a.k.a *ropelength*) of an embedded closed manifold  $X^n$  in a complete Riemannian manifold  $Y^{n+k}$  as  $\kappa(X^n) = \frac{vol^{\frac{1}{n}}(X^n)}{r(X^n)}$ , where  $r(X^n)$  denotes the injectivity radius of the normal exponential map of  $X^n$ . Informally speaking,  $r(X^n)$  can be interpreted as the largest radius of a nonself-intersecting tube around  $X^n$ . This functional was defined in [Nab95] for hypersurfaces and named “crumpledness”, but in later papers on “thick” knots in  $\mathbb{R}^3$  it had been given a new name “ropelength”, as it can be interpreted as the length of a similar knot such that the maximal radius of a nonself-intersecting tube around this knot is equal to one (i.e. it is the length of a similar knot tied on “thick” rope of radius one). One of the ideas of [Nab95] was that one can similarly consider higher-dimensional “thick” knots. Two knots (=embeddings of  $S^n$  in  $\mathbb{R}^{n+k}$ ) belong to the same  $x$ -thick knot type if they both are in the same path component of the sublevel set  $\kappa^{-1}((0, x])$  of  $\kappa$ .

To state our main result about “thick” knots it is convenient to first introduce the space of non-parametrized  $C^{1,1}$ -smooth embeddings  $E_n = Emb(S^n, \mathbb{R}^{n+1})/Diff(S^n)$  of  $S^n$  into  $\mathbb{R}^{n+1}$ , and then define  $Knot_{n,1}$  as the quotient of  $E_n$  with respect to the action of the group generated by isometries and homotheties of the ambient Euclidean space  $\mathbb{R}^{n+1}$ . The choice of smoothness is motivated by the facts that 1)  $r(\Sigma^n) > 0$  for every  $C^{1,1}$ -smooth closed hypersurface; 2)  $r$  is an upper semi-continuous functional on  $E_n$  and, therefore,  $Knot_{n,1}$  (Theorem 5.1.1 of [Nab95]); and 3) The sublevel sets  $\kappa^{-1}((0, x])$  in  $Knot_{n,1}$  are compact (see [Nab95], proof of Theorem 5.2.1). These facts are true for all dimensions  $n$ . Now for each  $x$  we can consider “thick” knot  $x$ -types as subsets of either  $E_n$  or  $Knot_{n,1}$ . A knot  $x_1$ -type and  $x_2$ -type are *distinct* if they do not intersect in  $E_n$  (or, equivalently, in  $Knot_{n,1}$ ). (Assuming that, say,  $x_1 \leq x_2$ , this is equivalent to the  $x_1$ -knot not being a subset of the  $x_2$ -knot.) Our next results imply that there exist non-trivial types of “thick” four-dimensional knot types of codimension one.

**Theorem 2.1.3.** *There exists an infinite sequence of distinct  $x_i$ -knot types in  $E_4$  (correspondingly,  $Knot_{4,1}$ ), where  $x_i$  is an unbounded increasing sequence. Moreover, there exists an unbounded increasing sequence of  $x_i$ , which are the values of  $\kappa$  at its local minima  $k_i$  on  $E_4$  (or, equivalently,  $Knot_{4,1}$ ). Further, for each  $m$  one can find such a sequence of numbers  $\{x_i\}$  and knots  $k_i$  with the additional property that*

any isotopy between  $k_i$  and the standard 4-sphere of radius 1 in  $\mathbb{R}^5$  must pass through hypersurfaces, where the value of  $\kappa$  is greater than  $\exp_m(x_i)$ .

*Remark 2.1.4.* The second assertion of the theorem is stronger than the first assertion, as each local minimum of  $\kappa$  with value  $x$  gives rise to an  $x$ -knot type that consists of one knot, if the local minimum is strict, and a connected set of knots in  $\kappa^{-1}(\{x\})$  otherwise. On the other hand, the second assertion immediately follows from the first assertion and the compactness of sublevel sets of  $\kappa$  (see Theorem 5.1.1 in [Nab95]).

*Remark 2.1.5.* The local minima of  $\kappa$  were called self-clenching hypersurfaces in [Nab95]. The idea behind this metaphor is that one can imagine that this hypersurface is made of very thin material that bends but cannot be stretched. If it also cannot be squeezed, then the “thick” hypersurface is tightly folded in  $\mathbb{R}^5$ . It can move (other than a rigid body movement) only if the local minimum is not strict, and only by “sliding movements”, so that at each moment of time it is still a local minimum of  $\kappa$  (i.e. it cannot be unfolded into a less crumpled shape).

*Remark 2.1.6.* Two very interesting questions are whether or not there exist non-trivial “thick” knots of codimension one in  $\mathbb{R}^3$  and  $\mathbb{R}^4$ . The second of these questions is related to the smooth Schoenflies conjecture, that asserts that each smooth embedding of  $S^3$  into  $\mathbb{R}^4$  is isotopic to the standard round sphere of radius one. (This fact is known for all other dimensions, cf. [Mil15]). Note, that if the smooth Schoenflies conjecture turns out to be false one can still ask whether or not there are non-trivial “thick” knot types in the component of  $Knot_{3,1}$  that consists of 3-spheres in  $\mathbb{R}^4$  that are isotopic to the round sphere. It seems almost “self-evident” that there are no non-trivial “thick” knots  $S^1 \subset \mathbb{R}^2$ , but we do not know a proof of this fact and are not aware of any publications in this direction.

To state our third result for every closed four-dimensional  $PL$ -manifold  $M$  consider the set of all simplicial isomorphism classes of simplicial complexes  $PL$ -homeomorphic to  $M$ . For brevity, we call them *triangulations* of  $M$ . The discrete set  $T(M)$  of all triangulations of  $M$  can be turned into a metric space using *bistellar transformations*. Bistellar transformations are operations that transform one triangulation into another as follows. Let  $T_1$  be a triangulation of  $M$ . Assume that it contains a simplicial subcomplex  $K$  that consists of  $k$ ,  $1 \leq k \leq 5$ , 4-dimensional simplices (together with their faces) and is simplicially isomorphic to a subcomplex  $C$  of the boundary of a 5-dimensional simplex  $\partial\Delta^5$ . To perform the corresponding bistellar transformation one first removes these  $k$  simplices (and all their faces) and then attaches the closure of the complement  $\partial\Delta^5 \setminus C$  to the boundary of  $K$  (which is simplicially isomorphic to the boundary of  $\partial\Delta^5 \setminus C$ ). Since we exchange one  $PL$ -disc (triangulated with  $k$  4-simplices) for another (triangulated with  $6 - k$  4-simplices), we obtain a triangulation  $T_2$  of the same manifold. Moreover, endow  $T_1$  and  $T_2$  with length metrics such that each simplex is a flat regular simplex with side length one. In this case it is easy to see that  $T_1$  and  $T_2$  will be bi-Lipschitz homeomorphic, and the Lipschitz constants of the homeomorphism and its inverse will not exceed an absolute constant that can be explicitly evaluated. U. Pachner proved that every two triangulations of the same closed  $PL$ -manifold can be connected by a finite sequence of bistellar transformations ([Pac91]). Now one can define the distance  $d_{Bist}(T_1, T_2)$  on  $T(M)$  as the minimal number of bistellar transformations required to transform  $T_1$  into  $T_2$ .

**Theorem 2.1.7.** *For each 4-dimensional closed  $PL$ -manifold  $M$  and each positive integer value of  $m$  there exist arbitrarily large values of  $N$  and two triangulations  $T_1, T_2$  with  $\leq N$  simplices such that*

$d_{Bist}(T_1, T_2) > \exp_m(N)$ . (In other words,  $T_1$  and  $T_2$  cannot be connected by any sequence of less than  $\exp_m(N)$  bistellar transformations).

A stronger version of this theorem was proven in [Nab96b] for all manifolds of dimension greater than four as well as all four-dimensional manifolds that can be represented as a connected sum of  $k$  copies of  $S^2 \times S^2$ , where the value of  $k$  can be chosen as 14 using [Sta07], [Sta04]. Note that results of [Nab96b] and, especially, Theorem 2.1.7 for  $M = S^4$  have potential implications for four-dimensional Euclidean Quantum Gravity (see [Nab06b] and references there).

## 2.2 Proofs

### 2.2.1 Balanced finite presentations of the trivial group

Recall, in Chapter 1 we constructed a sequence of finite balanced presentations of the trivial group. These finite presentations have two generators and two relations. They can be described as the Baumslag-Gersten group  $B = \langle x, t | x^{x^t} = x^2 \rangle$  with an added variable second relation. (Here  $a^b$  denotes  $aba^{-1}$ .) To describe this relation note that there exists a sequence of words  $v_n$  in  $B$  of length  $O(n)$  representing  $x^{E(\lceil \log_2 n \rceil)}$ , where  $E(m)$  denotes  $2^{2^{\dots^2}}$  ( $m$  times). Clearly,  $v_n$  commutes with all powers of  $x$ . Words  $w_n = [v_n, x]$  represent the trivial element but one needs to apply the only relation at least  $E(\lceil \log_2 n \rceil - \text{const})$  times to establish this fact. (Thus, the Dehn function of the one relator group  $B$  grows faster than any tower of exponentials of a fixed height of  $n$ , cf. [Ger91], [Pla04].) The extra relation added to  $B$  is  $[v_n, x^3][v_n, x^5][v_n, x^7] = t$ , where  $[a, b]$  denotes the commutator  $aba^{-1}b^{-1}$ . The most important property of this sequence of groups is that any representation of either  $x$  or  $t$  as a product of conjugates of the two relators and their inverses will require at least  $E(\lceil \log_2 n \rceil - 2)$  factors. Also, note that these finite presentations satisfy the Andrews-Curtis conjecture. The importance of the last observation is in the fact that when one constructs a presentation complex  $K$  of such a finite presentation (that is, a 2-complex with one 0-dimensional cell, two 1-dimensional cells corresponding to the generators and two 2-dimensional cells corresponding to the relators), embeds it in  $\mathbb{R}^5$ , takes the boundary of a small neighborhood of the embedding, and smoothes it out, one obtains not merely a smooth homotopy 4-sphere that must be homeomorphic to  $S^4$  by virtue of Freedman's proof of the 4-dimensional Poincaré conjecture, but a manifold that is diffeomorphic to  $S^4$ . This fact can be demonstrated without the 4-dimensional Poincaré conjecture using instead the fact that the operations in the Andrews-Curtis conjecture correspond to certain diffeomorphisms of the underlying manifold ("handle slidings"). A sequence of these diffeomorphisms corresponding to handle slides will eventually result in the standard sphere that corresponds to the representation 2-complex of the trivial finite presentation of the trivial group (cf. [BHP68]).

It is convenient to define these hypersurfaces more carefully to have better control over their geometry. First we triangulate the presentation complex  $K$  by subdividing 1-cells into three intervals and 2-cells into  $3l$  triangles in the obvious way (by adding a vertex in the centre of the cell), where  $l$  is the length of the relator. This still is not a triangulation, because a pair of triangles can have an edge identified while a separate from the edge vertex (the centre) is always identified. To correct that we subdivide the 2-cells further by making a small  $l$ -gon around the centre and adding the necessary edges. The total number of triangles for our triangulation of  $K$  is  $4n + \text{const}$ . We linearly embed  $K$  into  $\mathbb{R}^5$ . We distinguish the following 7 vertices of  $K$ : 5 coming from the generators and 2 the centers of the 2-cells,

and make sure that our embedding is such that the distance between any two of them is between 1 and 2. Then we thicken this embedded complex up and smooth the boundary out to obtain a manifold diffeomorphic to  $S^4$ , we call it  $S^4(v_n)$ . After rescaling, it can be interpreted as an element of  $I_{\epsilon_n}(S^4)$  (for an appropriate  $\epsilon_n$ ) or  $Al_1(S^4)$ . We can also interpret them as elements of  $Knot_{4,1}$  or  $E_4$ . Finally, we can add simplices around  $K$  to obtain a hypersphere triangulated into flat simplices (instead of a smooth hypersphere). It is easy to see that the number of simplices will grow linearly with the length of the word  $v_n$ : first we thicken the 1-skeleton of  $K$  with *const*  $n$  simplexes, then each triangle of the relations will add *const* simplices (to make it full dimensional and to “cut through” simplices around  $K$ ). Similarly, in the smooth case,  $\frac{vol^{\frac{1}{4}}}{inj}$ ,  $\frac{vol^{\frac{1}{4}}}{r}$  and  $\sup |K|diam^2$  (were  $K$  is the sectional curvature and the supremum is taken over all points and planes) will be bounded above by an exponential function of *const*  $n$  for some *const* (in fact, one can ensure much better bounds, but we do not need this).

These hypersurfaces in  $\mathbb{R}^5$  constructed using the balanced presentations of the trivial group introduced in Chapter 1 will be used in the proofs of all our results. But note that in this construction one can alternatively use another family of balanced presentations of the trivial group with similar properties that were independently discovered by Martin Bridson. I was not aware of his work until the results of this chapter were almost finished. But after the results of Chapter 1 appeared on the arXiv Bridson e-mailed me and to tell that he found such finite presentations in 2003. Although they were mentioned in his ICM-2006 talk ([Bri06], p. 977), he has never published or posted any details of his construction on the Internet. Two weeks after his preprint [Bri15] has also appeared on the arXiv. Note that in his ICM-2006 talk Bridson expresses a hope that such finite presentations of the trivial group can be used to extend results of Nabutovsky and Weinberger on the sublevel sets of diameter on moduli spaces to dimension 4 (which is something that we were not yet able to accomplish). So, his work [Bri15] was also partially motivated by potential applications that are similar in spirit to our results in this paper. Even earlier, in the 90s, Nabutovsky attempted to prove the results of this chapter using balanced finite presentations of the trivial group obtained from  $B$  in the most obvious way, namely, by adding the second relation  $w_n = t$ . Yet he was not able to verify that these balanced presentations have the desired properties.

### 2.2.2 The filling length

Following [Nab96a] we are going to use the following characteristic of simply-connected closed Riemannian manifolds that measures how “difficult” is to contract closed curves. We define it as the supremum over all closed curves  $\gamma$  of the ratio  $\frac{fl(\gamma)}{length(\gamma)}$ , where the filling length  $fl(\gamma)$  denotes the infimum over all homotopies  $H = (\gamma_t)_{t \in [0,1]}$ ,  $\gamma_0 = \gamma$ , contracting  $\gamma$  to a point (=constant curve)  $\gamma_1$  of the maximal length  $\sup_t length(\gamma_t)$  of the closed curves arising during the homotopy  $H$ . We are going to denote this quantity by  $Fl$  and regard it as a functional on a considered space of (isometry classes) of Riemannian metrics.

To see that  $Fl < +\infty$  first note that all sufficiently short curves  $\gamma$  can be contracted to a point without length increase (and, therefore,  $fl(\gamma) = length(\gamma)$ ). On the other hand for long curves  $\gamma$  we can choose any point  $z$ , and connecting  $z$  with a sequence of sufficiently close points on  $\gamma$  by minimal geodesics reduce the contraction of  $\gamma$  to consecutive contractions of triangles formed by a very short arc of  $\gamma$  and two minimal geodesics between  $z$  and two very close points on  $\gamma$ . Perimeters of these triangles are bounded by  $2d + \epsilon$ , where  $d$  is the diameter of the Riemannian manifold and  $\epsilon$  is arbitrarily small. This easily implies that  $\frac{fl(\gamma)}{length(\gamma)} \rightarrow 1$  as  $length(\gamma) \rightarrow \infty$ . This fact was first noticed by M.

Gromov ([Gro98]). (The existence of the supremum for closed curves of length  $\leq 2d + \epsilon$  follows from the compactness of the set of Lipschitz curves of length  $\leq 2d + \epsilon$  parametrized by the arclength.) Gromov also introduced the term “filling length” and the notation  $fl$  (with a slightly different meaning than what we use here).

Note that  $Fl$  can also be defined for all simply-connected length spaces such that for some positive  $\epsilon$  all closed curves of length  $\leq \epsilon$  can be contracted to a point without length increase. So, in particular, we can consider  $Fl$  as a functional on the spaces of triangulations of closed manifolds after we endow each simplex of the maximal dimension by the metric of a regular flat simplex with side length 1. (Actually, it is easy to see that  $Fl$  will not depend on the choice of the side length here.)

Our observation is that if  $S^4(v_n)$  is a (smooth or PL) sphere constructed starting from the word  $v_n$  in the Baumslag-Gersten group as explained above (using either the idea from Chapter 1 or the idea from [Bri15]) then:

**Proposition 2.2.1.** *The value of  $Fl_n = Fl(S^4(v_n))$  grows faster than any finite tower of exponentials of  $n$ .*

*Proof.* Indeed, if not, then we can prove that the area of van Kampen diagrams for generators of  $S^4(v_n)$  will also be bounded by towers of exponentials of  $n$  of a fixed height. In the proof below we use the same notation  $const$  for different constants that can be, in principle, evaluated.

The idea is that one can choose a way to represent each closed curve  $\gamma$  of length  $\leq x$  by a word of length  $\leq const x$  so that if another curve  $\alpha$  is  $const$ -close to  $\gamma$ , then the corresponding words can be connected by a sequence of at most  $const x$  relations. In order to achieve this we first project  $\gamma$  to the embedding of the presentation complex of the balanced presentation in  $\mathbb{R}^5$ . Recall, that  $S^4(v_n)$  is the smoothed-out boundary of a small tubular neighborhood of the presentation complex  $K$ , so this step will increase the length by at most  $const$  factor (we might need to increase the length because  $\gamma$  might be making shortcuts through the thickness around  $K$ ). Denote the projection of  $\gamma$  to the embedded presentation complex by  $\tilde{\gamma}$ .

Now we want to associate with a curve  $\tilde{\gamma}$  a word  $\hat{\gamma}$  on the generators of the group presentation of length  $const x$ , where  $x$  is the length of  $\tilde{\gamma}$ . Note that if  $D$  is a Riemannian 2-disc one can choose a way to replace each arc with endpoints on  $\partial D$  by a shortest arc of  $\partial D$  with the same endpoints. Consider now one of the two 2-cells in the presentation complex and a connected smooth arc  $A$  in its interior with end points on its boundary. The boundary of the 2-cell has some self-intersections that appeared as the result of taking the quotient map, when the cell was attached to the 1-skeleton. Yet we can canonically lift  $A$  to the 2-disc with the same Riemannian metric in the interior and with nonself-intersecting boundary, then extend  $A$  to the boundary, replace it by a shortest arc of the boundary with the same endpoints. Finally, project this arc back to the 1-skeleton of the embedded representation complex.

Now, take each component of the intersection of  $\tilde{\gamma}$  with 2-cells and replace it by an arc in the boundary as explained above. This will result in a path in 1-skeleton. If it has a part that stops halfway through a generator and then goes back, then we either stretch it (if it passed the midpoint) or retract it to the vertex. The result is a word, denoted  $\hat{\gamma}$ , of length increased by at most  $const$  factor relative to  $\tilde{\gamma}$  for the following reason. Recall that the discs  $D$  are triangulated into flat simplices. There is a very small polygon around the centre of the disc, the boundary of  $D$  is broken into intervals of length at most 2 which are connected with the vertices of the central polygon by straight lines of length at least 1. Therefore, a piece of the boundary of length less than a half of the length of the boundary differs from the geodesic by at most a  $const$  factor.

There are two sources of ambiguity in the definition of  $\hat{\gamma}$ . First, when we chose the shortest arc (they could be of equal size). Second, when we chose to either extend or contract the the path to fill the 1-cell. If there are two close curves  $\tilde{\gamma}_1, \tilde{\gamma}_2$  then  $\hat{\gamma}_1$  is the result of applying relations of the presentation (coming from the first ambiguity) or a “relation” of the free group (the cancellation of a letter with its inverse, coming from the second ambiguity) to  $\hat{\gamma}_2$ . We use this fact in the following.

If we have a homotopy  $H_t$ , we can obtain a word homotopy  $\hat{H}_t$  such that the word for each  $t$  is of length at most  $n \text{ const}$  the length of  $H_t$ . Furthermore, every word in that homotopy can be obtained from  $\hat{H}_0$  by a finite application of relations or adding/canceling letters with their inverses where all intermediate steps result in a word not longer than the maximal length in the homotopy. One can see that by taking sup of all  $t$  such that  $\hat{H}_t$  can be obtained from  $\hat{H}_0$  in such a way, then one can show that the  $\hat{H}_{\text{sup}}$  can also be obtained from  $H_0$  and also  $\hat{H}_{\text{sup}+\epsilon}$  can.

Because the number of words of bounded length is finite one can estimate that the number of applications of relations needed to go from  $\hat{H}_0$  to  $\hat{H}_1$  is at most  $\exp(\text{length})$ . This completes the proof of the proposition.  $\square$

Let  $M_0$  be any closed simply connected 4-dimensional Riemannian manifold. We can form a Riemannian connected sum of  $M_0$  with the spheres  $S^4(v_n)$  in an obvious way and observe that  $Fl$  for resulting Riemannian manifolds grows faster than any tower of exponentials of  $n$  of a fixed height.

### 2.2.3 Proof of theorems

It is now easy to prove the main theorems for simply connected manifolds. In order to prove Theorem 2.1.7 recall that each bistellar transformation leads to a bi-Lipschitz homeomorphism of the underlying simplicial complexes regarded as metric spaces, where each face of dimension four is given the metric of the regular 4-simplex with the side length 1. The Lipschitz constants for the map and its inverse do not exceed an absolute constant  $\text{const}$ . Now note that in this situation  $Fl$  cannot change by more than the factor  $\text{const}^2$ . The value of  $Fl$  for the boundary  $\partial\Delta^5$  of the regular 4-simplex (endowed with the standard metric) is 1. Therefore, the value of  $Fl$  on each triangulation of  $S^4$  that can be connected with  $\partial\Delta^5$  by at most  $M$  bistellar transformation is at most  $\text{const}^{2M}$ . This fact immediately implies the assertion of the theorem.

The proofs of the first three theorems are similar. The idea is to prove that if the assertion does not hold, then  $Fl_n$  is bounded above by a tower of exponentials of  $n$  of a fixed height, and this would contradict the assertion of Proposition 2.2.1.

One can follow [Nab96a] to finish the proof of Theorem 2.1.1. One starts from the observation that if two Riemannian structures in  $I_\epsilon(M)$  are  $\frac{\epsilon}{8.5}$ -close (in the Gromov-Hausdorff metric), then the values of  $Fl$  can differ by a factor that does not exceed 1000000 (Lemma 2 in [Nab96a]). (The idea is that if  $M_1$  and  $M_2$  are close Riemannian manifolds and one can contract any closed curve in  $M_2$  through not too long curves, one can try to contract any closed curve  $\gamma$  in  $M_1$  by 1) discretizing it, moving points to the closest points in  $M_2$  and connecting them by minimal geodesics, thereby obtaining a closed curve  $\gamma_2$  that can be regarded as a “transfer” of  $\gamma$  to  $M_2$ ; 2) Contracting  $\gamma_2$  through not too long curves in  $M_2$ ; 3) Discretizing this homotopy and transferring closed curves in the discretization back to  $M_1$ ; 4) Connecting the transfers of the nearest closed curves by homotopies in  $M_1$ , thus, obtaining a homotopy contracting  $\gamma$  in  $M_1$ .)

The second observation used in [Nab96a] is that one can use the well-known proof of the fact that

$I_\epsilon(M)$  is precompact to give an explicit upper bound of the form  $\exp(\frac{const}{\epsilon^9})$  (in the four-dimensional case) for the cardinality of an  $\epsilon/20$ -net in  $I_\epsilon(M)$ . This estimate can then be used to conclude that any two Riemannian structures in the same connected component of  $I_\epsilon(M)$  can be connected by a sequence of  $\epsilon/9$ -long “jumps”, so that the number of jumps does not exceed  $\exp(\frac{const}{\epsilon^9})$  (see the proof of Lemma 3 in [Nab96a]). Combining this estimate with the previous observation we see that the ratio of values of  $Fl$  at any two elements of the same connected component of  $I_\epsilon(M)$  is bounded by a double exponential of a power of  $\frac{1}{\epsilon}$  (and, thus, by a triple exponential function of  $\frac{1}{\epsilon}$  for all sufficiently small values of  $\epsilon$ ). This estimate can be generalized to a stronger equivalence relation on  $I_\epsilon(M)$  than being in the same connected component, namely, the transitive closure of the relation “to be  $\frac{\epsilon}{9}$ -close in the Gromov-Hausdorff metric”.

A comparison of these triply exponential upper bounds with lower bounds for  $Fl$  that grow faster than any tower of exponential of a fixed height of  $n$  yields the assertion of Theorem 2.1.1.

As it had been noticed, Theorem 2.1.2 would follow from the disconnectedness of sublevel sets of the diameter  $diam^{-1}((0, x])$  on  $Al_1(M)$ , and the injectivity radius is bounded below by  $\exp(-const \ x)$  on these sets. Now one can use the same argument as in the proof of Theorem 2.1.1.

To prove Theorem 2.1.3 we can rescale the hypersurface to have the value of the volume equal to 1. Now note that the definition of  $\kappa$  implies that  $\kappa \geq |k|$ , where  $k$  denotes any of the principal curvatures of the hypersurface. This implies the obvious upper bound for the absolute values of its sectional curvatures, when it is regarded as a Riemannian manifold. It is not difficult to establish an upper bound for the diameter of the hypersurface in the intrinsic metric (which immediately follows from Theorem 1.1 in [Top08]). Now the Cheeger inequality implies an explicit lower bound for the injectivity radius of the hypersurface that behaves as  $\exp(-const \ \kappa^{const})$ , and we can prove the disconnectedness of sublevel sets of  $\kappa$  for an unbounded sequence of values of  $x$  exactly as we proved the disconnectedness of  $I_\epsilon(M)$ .

The proofs of Theorems 2.1.1, 2.1.2 and 2.1.7 in the case of a nonsimply connected manifold can be based on the same ideas. We form a Riemannian connected sum of  $M$  endowed with some Riemannian metric with  $S^4(v_n)$ . Now  $Fl$  is not defined, but we can look at how much the length of the closed curves corresponding to the generators of the balanced presentations must be increased before they can be contracted to a point. But will also need an additional property: these generators are “separated” from the complexity of  $M$ . By choosing a set of generators of the fundamental group of  $M$  represented by geodesics, call them  $g_1, \dots, g_k$ , we can distinguish between the two parts of the complexity of  $M$ : 1) the asymptotic complexity of a Dehn function of its fundamental group and 2) how difficult is to homotope curves to a concatenation of  $g_1, \dots, g_k$ , which does not depend on the length of curves (see also Section 3.4 for a discussion of  $Fl$  for a non-simply connected manifolds). We can make  $n$  large enough so that  $S^4(v_n)$  is much more complicated than the second type of complexity of  $M$ . We claim that any homotopy  $H_t$  in  $M \# S^4(v_n)$  contracting a generator of our presentation of the trivial group (e.g.  $x$ ) to a word on  $g_1, \dots, g_k$  has to go through very long curves. This claim is enough to prove the theorems. As in the simply connected case we have a sequence of short jumps (or bistellar transformations) between  $M \# S^4(v_n)$  and  $M$ . With a small number of jumps the property of  $x$  not being “effectively” contractible to words on  $g_1, \dots, g_k$  will be preserved (on each step there will be no “effective” homotopy between the “transfer” of  $x$  and  $\tilde{g}_1, \dots, \tilde{g}_k$ , where  $\tilde{g}_i$  are “transfers” of  $g_i$  and therefore effectively homotopic to each other). Since in  $M$  everything is effectively (relatively to the high complexity of  $S^4(v_n)$ ) contracts to words on  $g_1, \dots, g_k$ , but the transfer of  $x$  is not, we have a contradiction to the small number of steps.

The claim follows from the following. Suppose there is such a homotopy  $H_t$ . Choose a small round

sphere  $S^3$  connecting the two summands of  $M \# S^4(v_n)$ . We can “pause” the effects of the homotopy  $H_t$  in the first summand (to the left of  $S^3$ ), by gradually slowing it down as it approaches the separating sphere. We can do that while not increasing the lengths compared to the original  $H_t$  by much. Then it shows a contraction of  $x$  inside of  $S^4(v_n)$ .

## Chapter 3

# Complexity of unknotting trivial 2-knots

### 3.1 Main result

Let  $k$  be a PL-unknot in  $R^3$  with  $N$  crossings on one of its plane projection. The results of [Dyn06] imply that  $k$  can be isotoped to the standard unknot through PL-unknots with at most  $2(N + 1)^2$  crossings. (See also [Lac15] for further results in this direction.) On the other hand it was proven in [NW96] that for each  $n \geq 3$  and each *computable* function  $f$  there exists a trivial knot  $k : S^n \rightarrow \mathbb{R}^{n+2}$  triangulated into  $N$  (flat)  $n$ -simplices such that any isotopy between  $k$  and the trivial unknot that passes through PL-knots must pass through a knot that cannot be triangulated into less than  $f(N)$  simplices.

Alternatively, one can consider smooth embeddings of  $S^n$  into  $\mathbb{R}^{n+2}$  (or  $S^{n+2}$ ) and measure the complexity of knots as their ropelength (also known as crumpledness - see [Nab95]) that was defined as  $\frac{\text{vol}(k)^{\frac{1}{n}}}{r(k)}$ , where  $\text{vol}(k)$  is the volume of  $k$ , and  $r(k)$  denotes the injectivity radius of the normal exponential map for  $k$ . In other words,  $r(k)$  is the supremum of all  $x$  such that any two normals to  $k$  of length  $\leq x$  do not intersect. Informally speaking, one can think of  $r(k)$  as the maximal radius of a nonself-intersecting tube centered at  $k$ . For this measure of complexity it will still be true that if  $n = 1$ , then there exists a polynomial upper bound for the complexity of knots in an optimal isotopy connecting an unknot with the standard unknot, and for  $n \geq 3$  the worst case complexity of knots in the optimal isotopies grows faster than any computable function.

It is natural to conjecture that the results of [NW96] for  $n > 2$  will also hold in the case  $n = 2$ . Here we will prove that the complexity of untying of a trivial 2-knot can grow faster than a tower of exponentials of any fixed height of the complexity of the unknot. Let  $\text{exp}_k(x) = 2^{2^{\dots 2^x}}$  ( $k$  times).

**Theorem 3.1.1.** *For each positive  $k$  and arbitrarily large  $N$  there exists a trivial 2-knot with complexity  $x \geq N$  in  $\mathbb{R}^4$  or  $S^4$  such that any isotopy between this knot and the standard 2-sphere passes through 2-knots of complexity  $\geq \text{exp}_k(x)$ . (In other words, one needs to increase the complexity more than any tower of exponentials of a fixed height of the initial complexity before the 2-knots can be untied.) Here “complexity” means either the number of flat 2-simplices in a triangulation of the original PL-knot and each of the intermediate 2-knots (and in this case intermediate unknots also must be PL), or, if the original knot and intermediate knots are smooth, the complexity of a knot can be defined as its ropelength,*

$\frac{\sqrt{\text{Area}}}{r}$ , where  $r$  is the injectivity radius of the normal exponential map in the ambient  $\mathbb{R}^4$ .

In order to prove this theorem we first construct a sequence of finite presentations of  $\mathbb{Z}$ . These finite presentations have certain algebraic properties that help to realize them as “visible” finite presentations of 2-knots of complexity comparable with the total length of the corresponding finite presentations. Moreover, these finite presentations have the following additional property: In each of them there exists a trivial element of length  $n$  comparable with the total length of the presentation,  $Cn$ , such that one needs to apply the relations at least  $2^{2^{\dots 2^n}}$  (*const n times*) in order to demonstrate that this element is, indeed, trivial.

Then we prove that the finiteness/effective compactness of the set of trivial 2-knots of bounded complexity (modulo the group of transformations of  $\mathbb{R}^4$  generated by dilations and translations) implies that if all trivial 2-knots could be “untied” without a very large increase of complexity, then we would be able to contract any null-homotopic closed curve in the complement to the original 2-knot to a point through closed curves that are not much longer than the original one. But the algebraic property of the finite presentations of 2-knot groups explained in the previous paragraph implies that this is not the case.

## 3.2 Finite presentations of the infinite cyclic group

Recall that the group that has the finite presentation with two generators and one relator  $\langle x_1, x_2 | x_1^{x_2} = x_1^2 \rangle$  is called the Baumslag-Solitar group. (Here and below we use the standard notation  $x^y$  for  $yx y^{-1}$ .) Note that for each  $m$   $x_1^{x_2^m} = x_1^{2^m}$ , and therefore the commutator  $w_m = [x_1^{x_2^m}, x_1] = e$ . However, one needs to apply the relation  $\sim 2^m$  times in order to demonstrate this fact in the most obvious way. In fact, it is well-known that there is no essentially shorter way to write  $w_m$  as a product of the conjugates of the relator and its inverse. In other words, the Dehn function of the Baumslag-Solitar group is (at least) exponential. A proof of this fact can be found in [Ger92] (and a sketch of another simpler proof using van Kampen diagrams can be found in [Sap11]). The idea of the proof in the paper of Gersten is that the Baumslag-Solitar group is an HNN-extension and, therefore, the realization complex of the finite presentation will be aspherical. (Recall that the realization complex of a finite presentation has one 1-dimensional cell corresponding to each generator of the group and one 2-cell for each relator of the group.) So, its universal covering will be contractible, and, in particular, will have the trivial second homology group. Therefore, there will be a unique way to fill each null homologous 1-chain in the universal covering by a 2-chain. Each way to represent  $w_m$  as the product of conjugates of the relator and its inverse corresponds to a filling of the lift of the loop corresponding to  $w_m$  in the realization complex to its universal covering. Therefore, it must have the same number of 2-cells counted with multiplicities as the filling corresponding to the obvious presentation of  $w_m$  as a product of conjugates of the relator and its inverse. It remains only to check that 2-cells in the universal covering that correspond to the obvious representation of  $w_m$  as the product of conjugates of the relator and its inverse do not cancel, which can be done using the theory of normal forms in HNN-extensions. Of course, the same proof implies that for each  $N$  any representation of  $w_m^N$  as a product of conjugates of the relator and its inverse has at least  $\sim N2^m$  terms.

One can iterate the idea used in the construction of the Baumslag-Solitar groups and consider the

following sequence of finite presentation of groups (see [Bri02]). For each  $n = 1, 2, \dots$

$$G_n = \langle x_1, \dots, x_n \mid x_1^{x_2} = x_1^2, x_2^{x_3} = x_2^2, \dots, x_{n-1}^{x_n} = x_{n-1}^2 \rangle.$$

This finite presentation has  $n$  generators and  $n - 1$  relators of total length  $5(n - 1)$ . These groups are sometimes called Gromov groups. One can prove that the Dehn function of  $G_n$  grows as  $2^{2^{\dots 2^k}}$ , where the height of the tower of exponentials is  $n$ , using the approach of [Ger92] (the universal cover is contractible because  $G_n$  is obtained as a sequence of HNN-extension where each extension has the associate subgroups  $\mathbb{Z}$ , alternatively one can use [Ger98] or  $x_i$ -bands as in [Bri02]). In particular, one can consider  $g_k \in G_n$  defined as  $x_1^{x_2^{\dots x_n^k}}$ . It is easy to see that  $g_{n,k} = x_1^{2^{\dots 2^k}}$  ( $(n - 1)$  times). Therefore,  $v_{n,k} = g_{n,k} x_1 g_{n,k}^{-1} x_1^{-1}$  will be trivial. One needs to apply relations more than  $2^{2^{\dots 2^{O(k)}}}$  ( $(n - 1)$ ) times to demonstrate that  $v_{n,k}$  is trivial, when one proceeds in the obvious way. As above, one can use the asphericity of the representation 2-complex to conclude that the filling in its universal covering is unique on the 2-chain level. Below we will choose  $k = 1$  to obtain  $2^{2^{\dots 2^2}}$  ( $(n - 1)$ ) times lower bound will for the area of  $v_{n,1}$  with the same proof.

Now consider the following finite presentations  $P_n = \langle G_n, t \mid t v_n t^{-1} = v_n x_n \rangle = \langle x_1, \dots, x_n, t \mid x_1^{x_2} = x_1^2, \dots, x_{n-1}^{x_n} = x_{n-1}^2, t v_n t^{-1} = v_n x_n \rangle$ , where  $v_n = v_{n,1} = [x_1^{x_2^{\dots x_n^1}}, x_1]$  are words considered in the previous paragraph. It is easy to see that  $P_n$  is the finite presentation of  $\mathbb{Z} = \langle t \rangle$ , as  $v_n = e$  in the corresponding Gromov group of total length *const*  $2^n$ . The following theorem is the key technical fact in our paper. It asserts that any way to demonstrate that, say,  $x_n = e$  in  $P_n$  would involve at least  $2^{2^{\dots 2^2}}$  ( $O(n)$  times) applications of the relations. Equivalently, each representation of  $x_n$  as the product of conjugates of the relators and their inverses must involve at least this number of terms. We are stating and proving this fact using the language of van Kampen diagrams (cf. [LS01]).

**Theorem 3.2.1.** *Each van Kampen diagram with the boundary  $x_n$  in  $P_n$  contains at least  $2^{2^{\dots 2^2}}$  ( $O(n)$  times) cells.*

*Proof.* Consider a minimal van Kampen diagram with  $x_n$  on the outer boundary. It must contain cells corresponding to the last relation. As there are no copies of  $t$  on the boundary, these cells must form annuli ( $t$ -annuli) (see Figure 3.1). Consider one of the innermost  $t$ -annuli (that does not have any  $t$ -cells inside). Its inner boundary must be a non-zero power of either  $v_n$  or  $v_n x_n$ . The second option is impossible, as this would imply that the power of  $v_n x_n$  is trivial in  $G_n$ , which is false. So, we have a non-zero power of  $v_n$  on the innermost boundary. The part of the van Kampen diagram inside this innermost boundary is a van Kampen diagram in  $G_n$ . But we already established that any such diagram must have size of at least  $2^{2^{\dots 2^2}}$  ( $O(n)$  times).  $\square$

### 3.3 Construction of a 2-knot

It is known (see [Ker65]) how to realize groups as  $\pi_1$  of the complement of a knot if the group satisfies certain conditions. In this section we make this procedure effective. Note that if one adds one more relator to the finite presentation  $P_n$ , namely,  $t$ , then one obtains a finite presentation of the trivial group. Denote the resulting finite presentation of the trivial group by  $Q_n$ . The finite presentation  $Q_n$  can be transformed to the trivial finite presentation of the trivial group by performing  $O(n)$  elementary

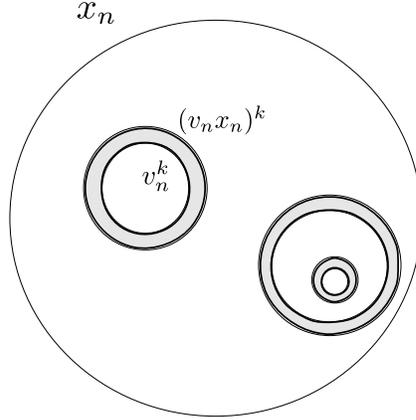


Figure 3.1: A Van Kampen diagram for  $x_n$ , with some  $t$ -annuli marked by grey.

operations of the following types: 1) Replacing a relator by its inverse; 2) Replacing a relator by its product with another relator; and 3) Replacing a relator  $r$  by  $grg^{-1}$  or  $g^{-1}rg$ , where  $g$  is a generator. Indeed, one can use  $O(n)$  operations involving only the relators  $tv_n t^{-1} x_n^{-1} v_n^{-1}$  and  $t$  to replace these two relators by  $x_n$  and  $t$ , and then  $O(1)$  operations to transform each of the first  $n - 1$  relators to  $x_i$  for an appropriate  $i$ . (The trivial finite presentation here is the finite presentation, where the set of relators coincides with the set of generators). Note the  $Q_n$ , the considered trivial finite presentation and all intermediate finite presentations are balanced, that is the number of generators is equal to the number of relators.

For each balanced finite presentation  $P$  of the trivial group we can construct a smooth 4-manifold in  $\mathbb{R}^5$  by starting from the connected sum of several copies of  $S^1 \times S^3$ , where  $S^1$  in each copy of  $S^1 \times S^3$  corresponds to one of the generators, and then performing surgeries killing the relators. More precisely, we realize each relator by a simple closed curve  $\gamma$ , remove the tubular neighbourhood of  $\gamma$ , glue in a copy of  $D^2 \times S^2$  so that its boundary is glued to the boundary of the removed tubular neighbourhood of  $\gamma$ , so that the boundary of  $D^2$  is glued to a curve isotopic to  $\gamma$ , and smooth out the boundary. Alternatively, we could start from the 2-complex with one 0-cell, 1-cells corresponding to the generators and 2-cells corresponding to relators of  $P$  (i.e. the realization complex of  $P$ ), embed it into  $\mathbb{R}^5$ , take the boundary of an open neighbourhood of  $P$  and smooth-out the corners. The resulting smooth 4-manifold will have the fundamental group with the obvious finite presentation  $P$  (in particular, it will be isomorphic to the trivial group), and the trivial second homology group. So, it will be a homotopy 4-sphere. Denote it by  $M^4(P)$ . But it is easy to see that  $M^4(P)$  will be diffeomorphic to  $S^4$ . The reason is that if two balanced finite presentations  $P$  and  $Q$  are related by one elementary operation of any of the three types introduced in the previous paragraph, the manifolds  $M^4(P)$  and  $M^4(Q)$  are diffeomorphic via a diffeomorphism that can be described as a “handle slide”. Each elementary operation with finite presentation corresponds to an isotopy of a curve bounding a 2-disc forming an axis of a 2-handle. The isotopy of the boundary of a 2-disc can be extended to an isotopy of the 2-disc, the whole 2-handle and the whole 4-manifold. After finitely many operations we will end up with the 4-manifold  $M(T_n)$  constructed from a finite presentation  $T_n$  with the same generator as  $Q_n$  and relators killing all the generators, which is diffeomorphic to the standard  $S^4$  by means of an obvious diffeomorphism.

Thus, each handle slide as well as the whole sequence of handle slides used to construct diffeomorphisms between  $M^4(P_n)$  and  $M^4(T_n)$  can be regarded as an isotopy  $M_t^4(Q_n)$ ,  $t \in [0, 1]$ ,  $M_0^4(Q_n) =$

$M^4(Q_n)$ ,  $M_1^4(Q_n) = M^4(T_n)$ . This isotopy can then be extended to an obvious isotopy between  $M^4(P_n)$  and the round sphere  $S^4$  of radius one.

Before moving further we are going to give the following definition:

**Definition 3.3.1.** Let  $f$  and  $g$  be two positive valued functions defined on a closed unbounded subset  $D$  of  $[0, \infty)$ . We say that they have *similar growth* if there exist  $k$  such that  $f(x) < \exp_k(g(\{\exp_k(x)\}_D))$  and  $g(x) < \exp_k(f(\{\exp_k(x)\}_D))$ . Where  $\{y\}_D$  means  $\min\{x \in D, x \geq y\}$ . Increasing functions that do *not* have similar growth with  $f(x) = x$  (restricted to their domain) are called *rapidly growing* functions. An increasing function that is not rapidly growing is called *reasonably growing*.

Now note that our explanation of why  $M(Q_n)$  is diffeomorphic to  $S^4$  can be used to construct an explicit diffeomorphism such that its Lipschitz constant regarded as a function of  $n$  is bounded by a reasonably growing function of  $n$ .

Next consider the last relator,  $t$ , and represent it by a simple curve in  $M(Q_n)$  that we will also denote  $t$ . It corresponds to a 2-handle  $H_t$  in  $M(Q_n)$ . The 2-disc  $D$  filling  $t$  forms a generator of this handle, which is diffeomorphic to  $S^2 \times D$ . Consider the 2-sphere  $S_n = S^2 \times c$ , where  $c$  is a point inside  $D$ . We claim that the fundamental group of the complement  $M(Q_n) \setminus S_n$  is isomorphic to  $\mathbb{Z}$ , and, in fact, it is that this group has “apparent” finite presentation  $P_n$ . Indeed,  $M(Q_n) \setminus S_n$  can be deformed to  $M(P_n)$  minus a tubular neighbourhood of  $t$ . Yet the deleted tubular neighbourhood of  $t$  decomposes into a 3-cell and a 4-cell. Therefore, if one attaches the deleted tubular neighbourhood back then the “apparent” finite presentation of the fundamental group remains unchanged. Thus,  $P_n$  is an “apparent” finite presentation of  $M(Q_n) \setminus S_n$ .

The meaning of “apparent” finite presentation here is that each loop in  $M(Q_n) \setminus S_n$  can be homotoped to a bouquet of loops representing the generators of  $P_n$  with an insignificant length increase. (Here and below a length increase is regarded as insignificant if it is measured by a reasonably growing function of  $n$ .)

Consider a 2-knot  $S_n$  in  $M(Q_n)$ . Now consider a diffeomorphism between  $M(Q_n)$  and  $M(T_n)$  that can be obtained as a sequence of handle slides corresponding to elementary operations transforming  $Q_n$  into  $T_n$ . Take the composition of this diffeomorphism with a diffeomorphism between  $M(T_n)$  and the standard round sphere  $S^4$ . Denote the resulting diffeomorphism between  $M(Q_n)$  and the round  $S^4$  by  $\phi_n$ . Note that it is easy ensure that the Lipschitz constants of  $\phi_n$  and its inverse were bounded by reasonably growing functions of  $n$ . Indeed, it is sufficient to verify that this will hold for diffeomorphisms corresponding to the individual handle slides. Now each handle slide can be regarded as an isotopy extension that extends an isotopy of a simple curve bounding a 2-disc forming an axis of a 2-handle. One can discretize this isotopy of closed curve into small isotopes where the closed curves at the beginning and the end of the isotopy are normal variations of each other inside tubes of radius bounded by the injectivity radii of the normal exponential map of the curves. Further, one can ensure that the reciprocals of these injectivity radii of closed curves during the handle slide isotopies are uniformly bounded by a reasonably growing function of  $n$ . Now it is obvious that Lipschitz constant of diffeomorphisms corresponding to the individual steps of the discretized isotopy are bounded by a reasonably growing function of  $n$ . Now, it remains to check that the number of small steps in the discretization is also bounded by a reasonably growing function of  $n$ . In other words, the isotopies do not need to be too long. In order to see this we can just analyze the isotopies corresponding to each of the elementary operations. (Alternatively, one can use an argument that provides an explicit upper bound for the number of points in a minimal  $\epsilon$ -net

in the space of Lipschitz curves of bounded length with injectivity radius of the normal exponential map bounded below by a positive parameter.)

The desired family of 2-knots are  $\phi_n(S_n)$  in the standard round  $S^4$ . One can also perform a stereographic projection from a point on  $S^4$  far from  $\phi_n(S_n)$  and obtain desired 2-knots in  $\mathbb{R}^4$ .

### 3.4 Filling functions

Recall a concept from [Gro98] that we used in Chapter 2: For each Riemannian manifold, or, more generally, length space  $X$  if finite diameter we define its *filling length* function  $filllength_X(x)$  as follows. For each positive  $x$  let  $C_x$  denote the set of all contractible closed curves on  $X$  that have length  $\leq x$ . For each  $\gamma \in C_x$  let  $H(\gamma)$  be the set of all homotopies contracting  $\gamma$  to a point. We consider elements of  $H(\gamma)$  as one-parametric families of closed curves starting at  $\gamma$  and ending at a point. For each  $h \in H(\gamma)$  let  $fill(h)$  denotes the maximal length of a closed curve in  $h$ . Then define  $filllength_X(\gamma)$  as the  $\inf_{h \in H(\gamma)} fill(h)$ . Finally, we define *filling*  $Fl(X)$  of  $X$  as the supremum of  $filllength_X(\gamma)/length(\gamma)$  over all closed *contractible* curves  $\gamma$ . In order for this supremum to be finite it is helpful if all sufficiently short closed curves can be contracted to a point without length increase as it happens, for example, for Riemannian manifolds. Further, note that if  $X$  is, in addition, simply connected, then  $Fl$  can be majorized in terms of the value  $Fl_{\leq 2diam(X)}(X)$  of this supremum on curves of length  $\leq 2diam(X)$ .

The choice of  $2diam(X)$  here is due to the well-known fact [Gro98] that each closed curve  $\gamma$  in a Riemannian manifold (possibly with boundary)  $X$  can be homotoped with almost no increase of length to a join of closed curves of length  $\leq 2diam(X)$  and contracted through closed curves of length  $\leq length(\gamma) + constdiam(X)$ . Indeed, one can choose any point  $z \in X$ , a finite set of points  $x_1, \dots, x_N$  on  $\gamma$  such that  $x_i$  and  $x_{i+1}$  are close to each other, and to reduce contracting  $\gamma$  to contracting geodesic triangles  $zx_i x_{i+1}$ . We see that if  $X$  is simply-connected, then the length of  $\gamma$  will increase by at most  $4diam(X) + Fl_{\leq (2diam(X)+\epsilon)}(X)(2diam(X) + \epsilon)$  for the described contracting homotopy, and then one can pass to the limit as  $\epsilon \rightarrow 0$ .

In this chapter we are going to apply this concept to spaces that are not simply connected, but have fundamental groups isomorphic to  $\mathbb{Z}$ . These spaces will be complements to trivial 2-knots, and it is easy to see that the generators of  $\mathbb{Z}$  can be represented by based loops of length that does not exceed twice the diameter of these spaces plus an arbitrarily small  $\epsilon$ . Indeed, one can take a very small circle around the embedded  $S^2$  and connect it with the base point by two minimizing geodesics traveled in the opposite directions. Denote the resulting curve by  $\tau$ . Now proceed as in the simply connected case with the only difference that distances between points  $x_i$  and  $x_{i+1}$  on  $\gamma$  in the metric of  $\gamma$  are now chosen as  $\frac{1}{2}diam(X)$  rather than a very small  $\epsilon$ . In this way we control the number of the triangles  $zx_i x_{i+1}$  in terms of  $\frac{length(\gamma)}{diam(X)}$ . Each of these triangles has length  $< 3diam(X)$  and is homotopic to a point or  $\tau$  which is possibly iterated several times and maybe also traveled in the opposite direction. Once we homotope  $\gamma$  into a collection of integer iterates of  $\tau$  (where the exponents must sum to zero), we will be able to cancel them and contract the resulting curve without increasing its length. Therefore, in order to prove the finiteness of  $Fl$  it is sufficient to demonstrate the existence of the supremum of the maximal length of loops in an “optimal” homotopy contracting a loop of length  $\leq 3diam(X)$  to an integer power of  $\tau$ . (Here the supremum is taken over all loops of length  $\leq 3diam(X)$ ; the word “optimal” means that we are taking the infimum over all such homotopies; the power of  $\tau$  is uniquely determined by the initial curve and is locally constant). The existence of this supremum becomes evident when we combine

the following two facts: First, note that the Arzela-Ascoli theorem implies the compactness of the set of closed curves in  $X$  of bounded length parametrized proportionally to the arclength. Second, assume the existence of  $\delta > 0$  such that each closed loop of length  $\leq \delta$  can be contracted to a point without length increase. Now let  $\gamma_1, \gamma_2$  be two loops such that for each  $t$   $d(\gamma_1(t), \gamma_2(t)) \leq \delta$ . Then  $\gamma_1, \gamma_2$  will be homotopic to the same power of  $\tau$ , and given a homotopy between  $\gamma_1$  and this power of  $\tau$ , we can extend it to a homotopy for  $\gamma_2$  by merely adding a homotopy between  $\gamma_2$  and  $\gamma_1$  that does not increase length by much. This implies the second fact that the maximal length of loops in an “optimal” isotopy cannot significantly increase under (controllably) small perturbations of loops.

Now we plan to use these filling functions similarly to how it was done in [Nab96a]. The main idea is that they behave in a similar way to their algebraic counterparts that measure how difficult it is to see that all trivial words in a “visible” finite presentation of the fundamental group of  $X$  are, indeed, trivial (or, more concretely, the maximum over all trivial words of a given length of the minimal area of a van Kampen diagram for the considered word). More specifically, we would like to consider complements to  $S_n$  in  $M(Q_n)$ ,  $\phi_n(S_n)$  in the standard  $S^4$  and to establish that 1) the values of  $Fl$  for these complements are similarly growing functions of  $n$ ; 2)  $Fl$  for the complements of  $S_n$  in  $M(Q_n)$  is a not reasonably growing function of  $n$ , as its growth is more or less the same as the growth of the area of van Kampen diagrams required to demonstrate that the groups of these 2-knots are trivial (see Theorem 3.2.1); and 3) If knots  $\phi_n(S_n)$  in the standard  $S^4$  can be untied through 2-knots of not too high complexity, then the second of these two filling functions is reasonably growing. Taken together these three facts establish that the constructed 2-knots can be untied only through 2-knots of a very high complexity. One technical difficulty that arises here is the following: As the considered complements to 2-sphere are not compact, it is not clear that the values of  $Fl$  for the considered complements are finite. More specifically, one can have contractible curves in, say,  $M(Q_n) \setminus S_n$  that include many very short arcs that go around  $S_n$  in opposite directions. Also, the proof of the existence of  $Fl$  given above used the existence of  $\delta$  such that all closed curves of length  $\leq \delta$  are contractible (and even contractible by a length non-increasing homotopy).

Therefore, we are going to remove not only the 2-knots but also their open tubular neighbourhoods with radii given as the inverse values of a reasonably growing function of  $n$ . More specifically, we proceed as follows.

In order to establish that  $\phi_n(S_n)$  cannot be untied through 2-knots of a not too high complexity we proceed by contradiction. We assume that  $\phi_n(S_n)$  in the standard  $S^4$  can be isotoped to the standard unknot through 2-knots of complexity that is a reasonably growing function of  $n$ . Consider the smooth case, when the complexity is defined as  $\frac{\sqrt{Area}}{r}$ , where  $r$  is the injectivity radius of the normal exponential map. (The proof in the simplicial case is quite similar.) First, we observe that there is a reasonably growing function  $f(n)$  that majorizes  $\frac{\sqrt{Area}}{r}$  not only for  $\phi_n(S_n)$  in the standard  $S^4$  and  $S_n$  in  $M(Q_n) \setminus S_n$  but also for all 2-knots in an isotopy connecting  $\phi_n(S_n)$  with an unknot in the round  $S^4$ . (This fact follows from our assumption that  $\phi_n(S_n)$  can be untied through knots of not very high complexity.) Without any loss of generality we can normalize all metrics and assume that the areas of  $S_n, \phi_n(S_n)$ , and all knots in an isotopy of  $\phi_n(S_n)$  to a round 2-sphere in a round 4-sphere are between one and  $g(n)$ , and also  $\frac{1}{F(n)}$  is a lower bound for the injectivity radii of all these 2-knots, where  $F$  and  $g$  are some reasonably growing functions of  $n$ . For each of these 2-knots  $k$  let  $N(k)$  denote its open tubular neighbourhood of radius  $\frac{1}{10F(n)}$ . We modify our idea and consider the complements to  $N(S_n)$  in  $M(Q_n)$  and  $N(\phi_n(S_n))$  in a round  $S^4$  rather than to  $S_n$  and  $\phi_n(S_n)$ . In this way we obtain compact metric spaces

and immediately see that their  $Fl$  are finite. Moreover, the boundaries of these spaces are hypersurfaces (diffeomorphic to  $S^2 \times S^1$ ) with principal curvatures bounded by a reasonably growing function of  $n$ . The same will hold also for the complements of  $N(k_t)$  in the round  $S^4$ , where  $k_t$  denote the 2-knots in the considered isotopy between  $\phi_n(S_n)$  and a round 2-sphere. Now we are going to prove the following three lemmas:

**Lemma 3.4.1.** *The functions  $Fl(M(Q_n) \setminus N(S_n))$  and  $Fl(S^4 \setminus N(\phi_n(S_n)))$  are similarly growing functions of  $n$ , where  $S^4$  denotes the standard round sphere.*

*Proof.* The assertion of the lemma immediately follows from the fact that Lipschitz constants of  $\phi_n$  and its inverse are bounded by reasonably growing functions of  $n$ . (We explained this fact at the end of the previous section.) This easily implies that  $M(Q_n) \setminus N(S_n)$  and  $S^4 \setminus N(\phi_n(S_n))$  are also bi-Lipschitz homeomorphic with both Lipschitz constants bounded by reasonably growing functions.  $\square$

Now we are going to prove that:

**Lemma 3.4.2.**  *$Fl(M(Q_n) \setminus N(S_n))$  is not reasonably growing.*

*Proof.* The idea of our proof of this lemma is that  $Fl(M(Q_n) \setminus N(S_n))$  behaves essentially as the Dehn function(s) for the family of finite presentations  $P_n$ . More precisely, assume that  $Fl(M(Q_n) \setminus N(S_n))$  is reasonably growing. Then we are going to prove that there exist van Kampen diagrams for  $x_n$  in  $P_n$  with  $C(n)$  cells, where  $C(n)$  is a reasonably growing function. This will contradict Theorem 3.2.1.

Let us represent  $x_n$  by a sufficiently short loop in  $M(Q_n) \setminus N(S_n)$  and contract it to a point via loops of length  $\leq l(n)$ , where  $l$  is a reasonably growing function of  $n$ . Each of these intermediate loops can be first homotoped to a loop in the 2-skeleton of  $Fl(M(Q_n) \setminus N(S_n))$  that can be regarded as the 2-dimensional Dehn complex corresponding to the finite presentation  $P_n$ , and afterwards almost canonically represented as the product by  $\leq const l(n)$  of loops representing the generators of  $P_n$ . “Almost canonically” means that the only ambiguities appear as the result of having different ways to represent the same loop  $\gamma$  by a small number of short words that correspond to relators of  $P_n$ . The number of these small words is proportional to the length of  $\gamma$  and is, therefore, bounded by a reasonably growing function of  $n$ . These ambiguities will correspond to the discontinuities in our presentations of loops by words, and together will provide a representation of  $x_n$  as the product of conjugates of words corresponding to these ambiguities. As the length of words is also bounded by a reasonably growing function of  $n$ , so will be the number of cells in the corresponding van Kampen diagram. This completes the proof. Note that this argument is very similar to an analogous argument in the proof of Proposition 2.2.1. The only difference is that before projecting curves in  $Fl(M(Q_n) \setminus N(S_n))$  to the presentation complex of  $P_n$  we first need to push them out from the partially removed handle  $H_t$ . It can be done without much length increase, because we removed an open neighbourhood of the centre of the disc killing  $t$ . Therefore the length would not increase by more than the ratio of the radius of the disc killing  $t$  to the radius of the removed disc.  $\square$

Finally, we are going to prove that:

**Lemma 3.4.3.** *If the constructed 2-knots in  $S^4$  can be untied through 2-knots of complexity bounded by a reasonably growing function, then  $Fl(S^4 \setminus N(\phi_n(S_n)))$  is a reasonably growing function.*

It is clear that Lemmas 3.4.1, 3.4.2 and 3.4.3 together immediately yield the contradiction that implies that the constructed family of 2-knots satisfies the conditions of our main theorem.

*Proof.* To prove Lemma 3.4.3 we assume that the constructed trivial knots can be untied through knots  $k_t$ ,  $t \in [0, 1]$  of complexity bounded by a reasonably growing function. As we are considering the smooth case, we can assume that the areas of the knots during the homotopy are between 1 and  $g(n)$ , where  $g$  is a reasonably growing function, and the injectivity radius of the normal exponential map during a contracting isotopy is bounded below by  $\frac{1}{F(n)}$ , where  $F$  is also a reasonably growing function of  $n$ .

Now our goal is to demonstrate that any closed curve  $\gamma$  of length  $l \leq \frac{1}{100F(n)}$  in  $S^4 \setminus N(k_t)$  can be contracted with an increase of length bounded by a constant factor (say, 2). If  $\gamma$  is not in  $l$ -neighbourhood of  $\partial N(k_t)$ , then it is a convex metric ball of radius  $\frac{l}{2}$  in  $S^4$  and can be contracted within this ball without any length increase. Otherwise, it can be moved along outer normals to the  $(\frac{1}{10F(n)} + 2l)$ -neighbourhood of  $k_t$ . The length of  $\gamma$  during this homotopy less than by a factor of 2 for the following reasons. If it moved to a distance  $d$  the vectors parallel to the normals do not change in length. We consider what happens to the vectors in the tangent space at the points of the curve. The vectors perpendicular to the normal at the point but parallel to  $k_t$  (we can use Fermi coordinates to define what it means, see for example [Gra12]) change at most by the factor  $\frac{\frac{1}{F(n)} + \frac{1}{10F(n)} + d}{\frac{1}{F(n)} + \frac{1}{10F(n)}}$ . The vectors perpendicular to the normal and to  $k_t$  change by the factor not exceeding  $\frac{\frac{1}{10F(n)} + d}{\frac{1}{10F(n)}}$ . Since  $d \leq 2l < \frac{2}{100F(n)}$ , we see that all of the factors do not exceed 2. Finally, any curve of length less than  $2l$  in  $(\frac{1}{10F(n)} + 2l)$ -neighbourhood of  $k_t$ , can be contracted inside the complement of  $N(k_t)$  without length increase, since it is inside a convex metric ball of radius  $l$ .

This argument applies to the complement of  $\frac{1}{10F(n)}$ -neighbourhood of any 2-knot  $k$  in  $S^4$  such that its area and the injectivity radius of the normal exponential map satisfy the same bounds as the bounds for  $k_t$ . Now we can adapt the argument from [Nab96a] to prove that if two such 2-knots  $k^1$  and  $k^2$  with the injectivity radius of the normal exponential map greater than  $\frac{1}{F(n)}$  and volume between 1 and  $g(n)$  are  $\frac{1}{1000F(n)}$ -close, then the values of  $Fl$  for the complements of  $N(k^i)$ ,  $i = 1, 2$ , differ from each other by not more than a constant factor (say,  $10^6$ ). The idea that in order to contract a contractible closed curve  $\gamma_1$  in, say, the complement to  $N(k^1)$ , one can transfer the curve to (the close in Gromov-Hausdorff metric) complement of  $N(k^2)$  without a significant length increase, contract the resulting curve  $\gamma_2$  there, discretize the contracting homotopy, transfer it back to the complement of  $N(k^2)$  and “fill” the discretized homotopy. In order for this program to work one need to be able to contract without a significant length increase “short” closed curves of length that does not exceed the distance between the complements to  $N(k^i)$  times an appropriate constant. Note that in order to see that  $\gamma_2$  is contractible in the complement of  $N(k^2)$  we can construct a contracting homotopy by similarly “transferring” a homotopy contracting  $\gamma_1$  in the complement to  $N(k^1)$ . See [Nab96a] for detailed descriptions of such transfers. This argument works, for example, if the distance between  $\partial N(k^i)$ ,  $i = 1, 2$ , does not exceed  $\frac{1}{10000F(n)}$ .

Now note that the isotopy between the standard unknot and given unknot can be replaced by a sequence of “jumps” of “length”  $\leq \frac{1}{100000F(n)}$  where the number of jumps is bounded by a reasonably growing function of  $n$ . Here “length” means the Hausdorff distance between the considered knots. The number of these jumps is bounded by twice the number of pairwise disjoint metric balls of radius  $\frac{1}{100000F(n)}$ -net in the considered space of hypersurfaces in  $S^4$  satisfying the same bounds for the volume and lower bounds  $(\frac{1}{10F(n)})$  for the injectivity radius of the normal exponential map as  $\partial N(k_t)$  for 2-knots  $k_t$ . Indeed, we can replace any subsequence of “jumps” where the distance between the beginning and the end does not exceed  $\frac{1}{100000F(n)}$  by just one jump.

Now note that sizes of  $\frac{const}{F(n)}$ -nets in the space of hypersurfaces in  $S^4$  with areas between 1 and a reasonably growing function, and the injectivity radius of the normal exponential map bounded below

by the inverse of a reasonably growing function is also bounded by a reasonably growing function. Such a bound will follow from any proof of the precompactness of the corresponding space of hypersurfaces of bounded complexity and volume (cf. [Nab95]). One possible idea is to represent these hypersurfaces as zero sets of appropriate  $C^2$ -functions that vary in the same way along each normal segment to the hypersurface from  $-1$  and  $1$ . It is easy to majorize  $C^2$ -norms of these functions in terms of the available data, and to use a standard effective proof of the Arzela-Ascoli theorem (see [Nab95] for the details of this argument).

Alternatively, to estimate the maximal number of required jumps one can use precompactness of the space of compact manifolds with boundary with a uniform bound on the injectivity radius of the normal exponential map of the boundary, on the diameter, and the curvature of the manifold (see [Won08]). Note that the injectivity radius of the normal exponential map of  $\partial N(k_t)$  is  $\frac{1}{F(n)} - \frac{1}{10F(n)} = \frac{9}{10F(n)}$  (see, for example, [Gra12]).

Since the change of  $Fl$  of the complements to the neighbourhoods of 2-knots under such jumps does not exceed an explicit constant factor (cf. [Nab96a] or Chapter 2), we can start at the standard unknot, “jump” back to the given 2-knot and observe that  $Fl$  for the complement to its open neighbourhood of radius  $\frac{1}{10F(n)}$  does not exceed  $const \# \text{ jumps}$ , which is bounded by a reasonably growing function of  $n$ .

Note that one can do a similar argument for 2-knots in  $\mathbb{R}^4$  instead of  $S^4$ , but in order to have the desired compactness one needs to transfer all 2-knots during isotopies to a neighbourhood of the origin (by appropriate translations).  $\square$

Again, one can easily adapt this proof for the PL -case.

## Chapter 4

# Automorphisms of the Baumslag-Gersten group

### 4.1 Introduction

The Baumslag-Gersten group introduced in [Bau69] enjoys the interesting property of having the fast growing Dehn function (see [Ger92], [Ger91], [Pla04]), while it can be presented by the number of relators one less than the number of generators. This property was found to be useful in constructing Riemannian metrics which are non-trivial minimums of some functionals of Riemannian structures on  $S^n$  (and other 4-dimensional manifolds) (see Chapter 2) by constructing balanced presentations of the trivial groups (Chapter 1), which are far from the trivial presentation. The following description of the outer automorphisms of the Baumslag-Gersten group will help us to find exponentially growing number of such presentations in Chapter 5.

Recall, we can present the Baumslag-Gersten group by  $G = \langle x, y, t \mid x^y = x^2, x^t = y \rangle$ , where  $a^b$  denotes  $b^{-1}ab$ . From this presentation we can see that this group is an HNN extension of the Baumslag-Solitar group, presented by  $H = \langle x, y \mid x^y = x^2 \rangle$ . The latter is an HNN extension of the group  $\langle x \rangle$ . We state our main result.

**Theorem 4.1.1.** *Out(G) is isomorphic to the dyadic rationals -  $\langle \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots \rangle$ , the subgroup of the rationals with addition, which is not finitely generated.*

To prove this theorem we classify all homomorphisms of  $G$  into itself, then prove  $Out(G)$  is isomorphic to  $C_H(x)$ , the centralizer of  $x$  in  $H$ , which we then describe as a subgroup of rational numbers. We use the techniques of Collins [Col78], mainly the Collins' Lemma ([Col78], [LS01], [Sho07]). We present this tool in the next section and prove the main results in the last. These and similar results for a class of groups that includes  $G$  were obtained in [Bru80].

### 4.2 HNN extensions and Collins' lemma

Recall, we can represent elements of HNN-extensions by reduced sequences of the stable letter, its inverse and elements of the base group. We denote by  $|w|$  the number of occurrences of the stable letter and its inverse in the reduced form of  $w$ . We can also cyclically permute these sequences. If all the permutations

are also reduced we call this element cyclically reduced. Any element is conjugate to a cyclically reduced element. Here is a form of Collins lemma we will use (see also [LS01], [Sho07]). The statement is cumbersome but the proof is simple.

**Lemma 4.2.1.** *Let  $u, v$  be cyclically reduced conjugate elements in some HNN-extension (given by  $t^{-1}Ct = B$ , where  $C, B$  are subgroups of the base group).  $u = g^{-1}vg$ , where  $g = a_1t^{\sigma_1}a_2t^{\sigma_2}\dots a_nt^{\sigma_n}$ . Then  $|u| = |v|$ . Furthermore, if  $|u| = |v| = 0$ , then there is a finite chain of words  $v_1, u_1 \dots v_n, u_n$ , where  $u_i, v_i \in C \cup B$  such that either  $u_i = t^{-1}v_it$  or  $v_i = t^{-1}u_it$  and  $v$  is conjugate to  $u_1$ ,  $u$  is conjugate to  $v_n$ ,  $v_i$  is conjugate to  $u_{i+1}$ , where all conjugations are in the base group (in fact by  $a_i$ ).*

*Proof.* Consider a van Kampen's diagram for  $u = g^{-1}vg$ . Note that  $t$ -bands on the diagram can not start and end on the same  $u$  or  $v$  because they are cyclically reduced. Therefore there are two possibilities for the way the  $t$ -bands go on the diagram (Figure 4.1) corresponding to  $|u| = |v| = 0$  and  $|u| = |v| \neq 0$ . The second part of the statement follows from Figure 4.1 too.  $\square$

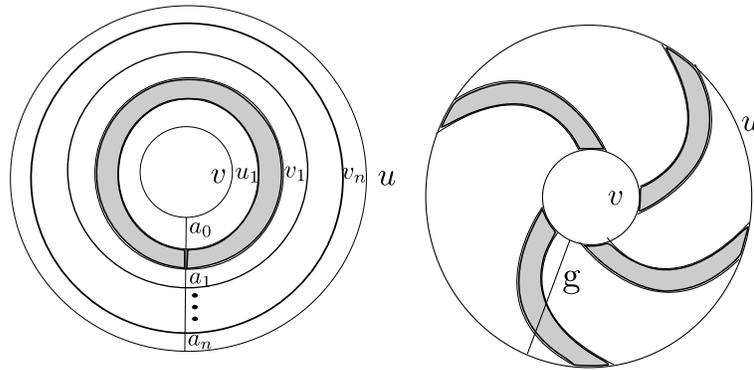


Figure 4.1: Two possible van Kampen diagrams for  $u = g^{-1}vg$ . The inside circle is  $v$ , the outside is  $u$ , the vertical line is  $g$ .  $t$ -bands are marked by grey.

### 4.3 Main Results

We prove several lemmas that we will apply in order to a homomorphism  $G \rightarrow G$ .

**Lemma 4.3.1.** *Let  $F : G \rightarrow G$  be a homomorphism, then there exists an inner automorphism  $A$  such that  $(A \circ F)(x)$  is in  $\langle x, y \rangle$ .*

*Proof.* We can choose  $A$  such that  $(A \circ F)(x)$  (and therefore  $((A \circ F)(x))^2$ ) is cyclically reduced, then it follows from Lemma 4.2.1 applied to  $(A \circ F)(x) = g^{-1}((A \circ F)(x))^2g$  that  $|(A \circ F)(x)| = 0$ .  $\square$

**Lemma 4.3.2.** *Let  $F : G \rightarrow G$  be a homomorphism, such that  $F(x) \in \langle x, y \rangle$ , then there exists an inner automorphism  $A$  such that  $(A \circ F)(x) = x^i$ .*

*Proof.* Consider  $(F(y))^{-1}F(x)F(y) = (F(x))^2$  and apply Collins' lemma to it (if there are  $t$  letters in  $F(y)$  apply it to the second HNN extension (stable letter  $t$ ), otherwise apply to the first one (stable letter  $y$ )). In the latter case  $F(x)$  might not be cyclically  $y$  reduced, but its conjugate is. In either case  $v_1$  or  $u_1$  will be  $x^i$  (see Figure 4.2). In the degenerate case of  $F(y) = 1$  we have  $F(x) = 1 = x^0$ .  $\square$

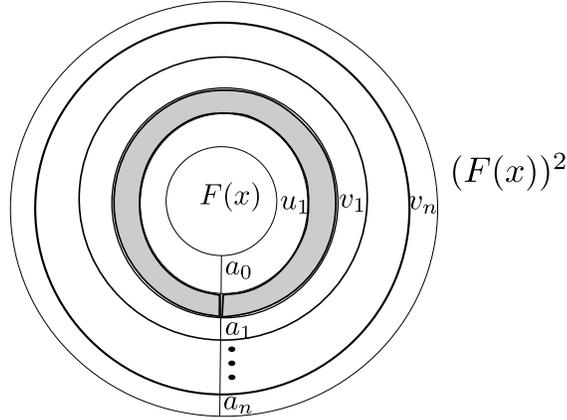


Figure 4.2: A van Kampen diagram for  $(F(y))^{-1}F(x)F(y) = (F(x))^2$ . The inside circle is  $F(x)$ , the outside is  $(F(x))^2$ , the vertical line is  $F(y)$ .  $t$ -bands are marked by grey. One of the edges of the  $t$ -band is  $x^i$ .

**Lemma 4.3.3.** *Let  $F : G \rightarrow G$  be a homomorphism, such that  $F(x) = x^i \neq 1$ , then  $F(y) \in \langle x, y \rangle$ .*

*Proof.* Consider  $(F(y))^{-1}x^iF(y) = x^{2i}$  and apply Collins' lemma to it. The outside of the first  $t$ -ring is  $y^j$  and it has to be Baumslag-Solitar conjugate to the inside of the second  $t$ -ring (see Figure 4.3), which is either  $x^m$  or  $y^m$ . The first case is impossible, and the second case is only possible if  $y^j$  is conjugated by  $y^k$  (in the Baumslag-Solitar group anything else conjugates  $y^j$  outside of  $\langle y \rangle$ ). Then the first and the second  $t$ -rings can be reduced. There could not be only one  $t$ -ring because the outside of the diagram is  $x^{2i}$ . Therefore  $F(y)$  has no letters  $t$  (in reduced form). □

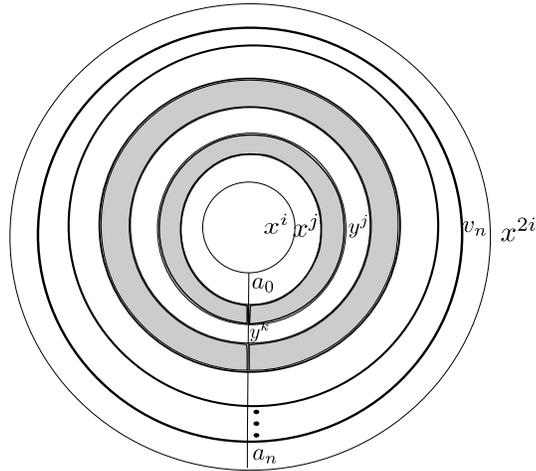


Figure 4.3: A van Kampen diagram for  $(F(y))^{-1}F(x)F(y) = (F(x))^2$ . The inside circle is  $F(x)$ , the outside is  $(F(x))^2$ , the vertical line is  $F(y)$ .  $t$ -bands are marked by grey. One of the edges of the  $t$ -band is  $x^i$ .

**Lemma 4.3.4.** *Let  $F : G \rightarrow G$  be a homomorphism, such that  $F(x) = x^i \neq 1$  and  $F(y) \in \langle x, y \rangle$ . Then  $F(t)$  has one letter  $t$ , and there is an inner automorphism  $A$  such that  $(A \circ F)(x) = x$  and  $(A \circ F)(y) = y$ .*

*Proof.*  $F(y) = y^{n+1}x^jy^{-n}$  for  $n > 0$  because of the equality  $(F(y))^{-1}x^iF(y) = x^{2i}$ . If  $F(t)$  had no  $t$  letters we would have that  $x^i$  is conjugate to  $y^{n+1}x^jy^{-n}$  in Baumslag-Solitar, which is impossible: the sum of exponents of  $y$  can not be changed by conjugation. As in the proof of Lemma 4.3.3  $F(t)$  can have at most one letter  $t$  (in reduced form). Let  $F(t) = g_1tg_2$ , where  $g_1, g_2$  are in the Baumslag-Solitar. Again by Collins' lemma  $g_1^{-1}x^i g_1 = x^m$  and  $g_2^{-1}y^m g_2 = F(y)$ . The left hand side of the last equation has sum of exponents of  $y$  equal to 1, therefore  $m = 1$ , which implies there exists an inner automorphism  $A'$  such that  $(A' \circ F)(x) = x^m = x$ . Let us redefine  $n, j$  such that  $(A' \circ F)(y) = y^{n+1}x^jy^{-n}$ .

Let  $A''$  be the conjugation by  $y^n x^j y^{-n}$ , then  $(y^n x^{-j} y^{-n})x(y^n x^j y^{-n}) = x$  and  $(y^n x^{-j} y^{-n})(y^{n+1}x^jy^{-n})(y^n x^j y^{-n}) = (y^n x^{-j})yx^j(x^j y^{-n}) = y^n y y^{-n} = y$ . We get the conclusion of the lemma by setting  $A = A'' \circ A'$ .  $\square$

**Proposition 4.3.5.** *Let  $F : G \rightarrow G$  be a homomorphism. If  $F(x) = 1$ , then  $F(y) = 1$  and  $F(t)$  can be any element. Otherwise,  $F$  is an automorphism, and there exists an inner automorphism  $A$  such that  $(A \circ F)(x) = x$ ,  $(A \circ F)(y) = y$ , and  $(A \circ F)(t) = gt$ , where  $g \in C_H(x)$ .*

*Proof.* If  $F(x) \neq 1$  we apply Lemmas 4.3.1, 4.3.2, 4.3.3, 4.3.4 in succession to obtain  $A$ , such that  $(A \circ F)(x) = x$ ,  $(A \circ F)(y) = y$ , and  $(A \circ F)(t) = g_1tg_2$ , where  $g_1, g_2$  are in Baumslag-Solitar. Since  $g_2^{-1}y^k g_2 = y$ , we have  $g_2 = y^m$  and  $k = 1$ . Therefore  $g_1^{-1}x^k g_1 = g_1^{-1}xg_1 = x$ , i.e.  $g_1 \in C_H(x)$ . Since  $g_1tg_2 = g_1ty^m = g_1x^m t$ , we set  $g = g_1x^m \in C_H(x)$ .  $\square$

*Remark 4.3.6.* It follows that  $G$  is both Hopfian and co-Hopfian. Now we can describe  $Out(G)$ .

**Proposition 4.3.7.**  *$Out(G)$  is isomorphic to  $C_H(x)$ , the centralizer of  $x$  in the Baumslag-Solitar group. The isomorphism  $\phi : C_H(x) \rightarrow Out(G)$  is defined by  $\phi(g) = [F]$ , where  $F : x \mapsto x, y \mapsto y, t \mapsto gt$  is an automorphism.*

*Proof.* This map is one-to-one because  $C_H(\langle x, y \rangle) = 1$ . This map is onto because of Proposition 4.3.5. The multiplication is clearly preserved.  $\square$

*Remark 4.3.8.*  $C_H(x) = \{y^n x^j y^{-n}\}$ , all elements of  $H$  that have total power of  $y$  being 0. This subgroup is generated by  $y^i x y^{-i}$ ,  $i > 0$ . We can construct a map  $\psi : C_H(x) \rightarrow \mathbf{Q}$ , defined by  $y^i x y^{-i} \mapsto \frac{1}{2^i}$ . This map is an injective homomorphism with respect to addition:  $\psi(y^n x^j y^{-n}) = \frac{j}{2^n}$ . The image of this map is not finitely generated, because all of its finitely generated subgroups are cyclic, but the image itself is not cyclic.

The main theorem, Theorem 4.1.1, follows from Proposition 4.3.7 and Remark 4.3.8.

## Chapter 5

# The Growth of the Number of Balanced Presentations of the Trivial Group

### 5.1 Introduction

In Chapter 1 we constructed a sequence of balanced presentations of the trivial group such that the minimal number of Tietze transformations required to bring these finite presentations to the empty presentation grows faster than the tower of exponentials of any fixed height of the length of the finite presentation. In this chapter we want to construct an exponentially growing number of such presentations pairwise distant (in the above sense) from each other.

In Chapter 2 we used the fact that balanced finite presentations can be realized as “obvious” finite presentations of the (trivial) fundamental group of 4-dimensional spheres and discs to derive some geometric applications. In particular, there are triangulations of the sphere which require a lot (compared to the number of simplices) of bistellar transformations to become the boundary of the standard 5-simplex. Possible applications of this chapter include a proof of existence of exponentially many triangulations of the sphere that can not be obtained from each other through application of the number of bistellar transformations smaller than a tower of exponentials.

The outline of the construction is the following. We take the presentations used in [Bri15] (also based on the Baumslag-Gersten group) and modify them to obtain exponentially many different presentations. They have very similar properties to the ones obtained in Chapter 1. One can think of them presenting “effectively” non-trivial “groups” (“effective pseudogroups”). This notion can be made precise using ideas from [Nab10a], but we won’t do that here. One can ask the question of when two such effective pseudogroups are isomorphic. In this chapter we essentially prove that the effective pseudogroups coming from the constructed presentations are not isomorphic and therefore are far from each other (in the sense of the Tietze transformations). We do that by modifying the construction of [Bri15] with our words from Chapter 1 satisfying an “effective” small cancellation condition to obtain control over finer properties of the presentations. However, note that we do not need a definition of pseudogroups and do not use them. We prove that they are not isomorphic by using the “effectivised” idea of Collins (used in Chapter 4

to classify automorphisms of the Baumslag-Gersten group) and our quantitative understanding of the Baumslag-Gersten group accumulated in Chapter 1. We do not at the moment know how to do (if at all possible) the same to the simpler presentations of Chapter 1.

In the next section we describe the construction. In the third section we prove some quantitative results about Baumslag-Gersten group. In the fourth section we essentially prove that any two different presentations from our construction are not isomorphic as effective semigroups. In the last section we state and prove the main theorem.

## 5.2 Notation and the Construction

Denote by  $exp_n(x)$  the tower of exponents of height  $n$ , i.e.  $exp_n$  are recursively defined by  $exp_0(x) = x$ ,  $exp_{n+1} = 2^{exp_n}$ . Let  $E_n = exp_n(1)$ . As usual,  $x^y$  denotes  $y^{-1}xy$ , where  $x, y$  can be words or group elements. Let  $l(w)$  be the length of the word  $w$ . If  $w$  represents the identity element, denote by  $Area_\mu(w)$  the minimal number of 2-cells in a van Kampen diagram over the presentation  $\mu$  with boundary cycle labeled by  $w$ . For a presentation  $\mu$  denote by  $l(\mu)$  its total length, i.e. the sum of lengths of the relators.

Let  $G = \langle x, y, t \mid x^y = x^2, x^t = y \rangle$ ,  $G$  is called the Baumslag-Gersten group, an HNN-extension of the group  $K = \langle x, y \mid x^y = x^2 \rangle$  (called the Baumslag-Solitar group).

We are going to use the same word of large area as in Chapter 1. We repeat the definition for the reader's convenience.

Let  $w_n$  be defined inductively as follows. Let

$$w_{n,0} = [y^{-E_n}xy^{E_n}, x^3][y^{-E_n}xy^{E_n}, x^5][y^{-E_n}xy^{E_n}, x^7].$$

Suppose  $w_{n,m}$  is defined, then let  $w_{n,m+1}$  be the word obtained from  $w_{n,m}$  by replacing subwords  $y^{\pm E_{n-m}}$  with  $t^{-1}y^{-E_{n-m-1}}x^{\pm 1}y^{E_{n-m-1}}t$ . Finally, let  $w_n = w_{n,n}$ .

*Remark 5.2.1.* We can make an estimate  $l(w_n) \leq 100 \cdot 2^n$ . Recall from Chapter 1 that  $w_{n,1}$  is  $E_n$ -reduced, i.e. any  $t$ -band in a digram for  $w_{n,1}$  is of length at least  $E_n$ . Also, the area of  $w_n$  is at least  $E_n$ .

Before defining the presentations we prove a lemma.

**Lemma 5.2.2.** *Let  $v, v'$  be two different words in the alphabet  $\{y, yx\}$ , then  $v \neq v'$  in  $G$ .*

*Proof.* One can see that by choosing  $x$  as the coset representative for  $x\langle x^2 \rangle$  and applying the theory of normal forms for an HNN-extension  $K$ . Alternatively, one can see that  $v = y^i x^j$  in  $K$ , where  $j$  can be represented in the binary notation as follows. The length of the number is equal to  $i$ . There is a digit 1 for each  $x$  in the word  $v$ , the digit is placed in the  $n$ -th position if there are  $n - 1$  letters  $y$  to the right of that  $x$ . The rest of the digits are 0. For example,  $xyyxyxyxy = y^5 x^j$ , where  $j = 10110_2$ . Clearly, this number is unique for a word  $v$ . To finish the proof we notice that if  $v = y^i x^j = v' = y^m x^n$ , then  $i = m$  because  $y$  is the stable letter of  $K$ , and therefore  $j = n$ . Finally, if  $v \neq v'$  in  $K$ , then  $v \neq v'$  in  $G$ , because  $G$  is an HNN-extension of  $K$ .  $\square$

Define  $H_v = \langle x, y, t, s \mid x^y = x^2, x^t = y, s^{-1}vw_n^{-1}v^{-1}w_n s = t \rangle$ , and a second copy of this presentation  $\hat{H}_v = \langle \hat{x}, \hat{y}, \hat{t}, \hat{s} \mid \hat{x}^{\hat{y}} = \hat{x}^2, \hat{x}^{\hat{t}} = \hat{y}, \hat{s}^{-1}\hat{v}\hat{w}_n^{-1}\hat{v}^{-1}\hat{w}_n \hat{s} = \hat{t} \rangle$ . Note, these are presentations of the infinite cyclic group.

Let  $\mu_v = H_v \underset{s=\hat{x}, x=\hat{s}}{*} \hat{H}_v$ .

*Remark 5.2.3.* In [Bri15] it was proven that, in particular,  $Area_{\mu_v}(x) \geq Area_G(vw_n^{-1}v^{-1}w_n)$ . We are going to study maps between such presentations for different words  $v$  of the type defined in the previous lemma. The exponential number of such words as a function of length will give us the exponential number of such presentations.

We introduce more notation. For  $w \in \mu_v$  denote by  $\hat{l}(w)$  the minimal  $m$  such that  $w = a_1 \dots a_m$  (equality in the free group), where neighbouring  $a_k$  are from different factors of  $\mu_v$ . For  $w \in H_v$  denote by  $l_s(w)$  the number of letters  $s, s^{-1}$  in  $w$ . Similarly, for  $w \in \hat{H}_v$  denote by  $\hat{l}_s(w)$  the number of letters  $\hat{s}, \hat{s}^{-1}$  in  $w$ , and for  $w \in G$  denote by  $l_t(w)$  the number of letters  $t, t^{-1}$  in  $w$ .

### 5.3 Quantitative Results about the Baumslag-Gersten Group

We recommend skipping this section for now. It is better to return to the lemmas proved here when they are required.

Recall, that  $K$  denotes the Baumslag-Solitar group,  $G$  is the Baumslag-Gersten group, and  $w_n$  were defined in the previous section.

**Lemma 5.3.1.** *Let  $\tilde{w}$  be a non-empty word in  $\{a, a^{-1}, b\}$  that does not contain more than 1 letter  $b$  consecutively. Let  $w$  be a word obtained from  $\tilde{w}$  by replacing  $a$  with  $w_n$ ,  $a^{-1}$  with  $w_n^{-1}$ , and  $b$  with an element  $B$  of  $K$ , possibly different for any particular instance of  $b$ , satisfying the following conditions. If  $b$  is between two letters  $a$ , then  $x^{-7}B \neq y^i$ , if  $b$  is between  $a$  and  $a^{-1}$ , then  $x^{-7}Bx^7 \neq y^i$ , if  $b$  is between  $a^{-1}$  and  $a$ , then  $B \neq y^i$ , and if  $b$  is between two letters  $a^{-1}$ , then  $Bx^7 \neq y^i$  (not equal in  $K$ ). Then if  $w = 1$  in  $G$ ,  $Area_G(w) \geq E_n$ .*

*Proof.* Consider a van Kampen diagram for  $w$ . Let us call letters  $t, t^{-1}$  on the boundary of this diagram “outer” if they come from  $w_{n,1}$ . Recall that  $w_{n,1}$  is the product of three commutators of the form

$$[t^{-1}y^{-E_{n-1}}x^{-1}y^{E_{n-1}}txt^{-1}y^{-E_{n-1}}x^1y^{E_{n-1}}t, x^3],$$

where in the other two commutators  $x^3$  is replaced with  $x^5$  and  $x^7$ . First, notice that any  $t$ -band in the diagram originating on an outer letter has to end on an outer letter. That follows from counting the number of letters  $t$  and subtracting from that the number of letters  $t^{-1}$  between the two ends of a  $t$ -band. This difference has to be equal to 0. Pick an outer letter and a  $t$ -band corresponding to it. Pick another outer letter between the ends of this  $t$ -band, there is the corresponding inner  $t$ -band. Continue until we find a  $t$ -band between neighbouring outer letters. We claim that this  $t$ -band has length  $E_n$ . The only possible neighbouring outer letters are  $t^{-1}At$  ( $A =_G x^{\pm E_n}$ ),  $tx^i t^{-1}$  (not a pinch),  $tx^{-7}Bx^7 t^{-1}$  (coming from  $w_n B w_n^{-1}$ ),  $tBt^{-1}$  (coming from  $w_n^{-1} B w_n$ ),  $tx^{-7}Bt^{-1}$  (coming from  $w_n B w_n$ ),  $tBx^7 t^{-1}$  (coming from  $w_n^{-1} B w_n^{-1}$ ). The last four pairs are not pinches because of the requirements on  $B$ . The claim and the lemma follow. □

**Lemma 5.3.2.** *Let  $Area_G(x^i u_1 x^j u_2) < E_n$ , where  $u_1, u_2$  are powers of  $vw_n^{-1}v^{-1}w_n$  or  $t$ ,  $i \neq 0$  and  $v$  is as in Lemma 5.2.2. Then  $u_1, u_2 = 1$  as words. Similarly, if  $u_1 \neq 0$ , then  $Area_G(u_1) \geq E_n$ .*

*Proof.* We look at cases. If one of the words is a power of  $t$  then the other has to be the inverse of this power, and since  $i \neq 0, j \neq 0$ , which implies  $v_1 = v_2 = 0$ . If one of the words is a power of  $vw_n^{-1}v^{-1}w_n$ ,

then so is the other one and  $i = j$ . We want to apply Lemma 5.3.1 and therefore want to check the conditions in its statement. Between  $w_n^{-1}$  and  $w_n$  we can have  $v^{-1} = x^{-m}y^{-k} \neq y^p$ . Between  $w_n$  and  $w_n^{-1}$  we can have  $v, x^i, x^i v, v^{-1}x^i$ , or  $v^{-1}x^i v$ . We need to check that if we conjugate any of them by  $x^7$  we won't get a power of  $y$ . It is true for the second and the fifth element because  $i \neq 0$ . For the rest we will use the following fact, if  $v = y^k x^m$ , then  $k > 0$  and  $m$  is even. Checking for the first element:  $x^{-7} v x^7 = x^{-7} y^k x^m x^7 = y^k x^{\text{odd}} \neq y^p$ . Similarly for the third:  $x^{-7} x^i v x^7 = x^{-7} x^i y^k x^m x^7 = y^k x^{\text{odd}} \neq y^p$ . And the fourth:  $x^{-7} v^{-1} x^i x^7 = x^{-7} x^{-m} y^{-k} x^i x^7 = x^{\text{odd}} y^{-k} \neq y^p$ . Therefore, the lemma follows from Lemma 5.3.1.  $\square$

**Lemma 5.3.3.** *Let  $\text{Area}_G(gu_1g^{-1}u_2) < E_{n-1}$ , where  $g$  is a word in  $G$ ,  $u_1 = A^i, u_2 = B^j$ , where  $A, B$  are  $vw_n^{-1}v^{-1}w_n$  or  $t$ ,  $i, j$  are less than  $E_{n-1}$ , and  $v$  is as in Lemma 5.2.2. Then  $A = B$  and  $i = \pm j$ . Furthermore, if  $A = B = t$ , then  $g =_G t^k$  through area less than  $E_{n-1}$ .*

*Proof.* When  $A$  or  $B$  is  $t$  it is clear. Suppose  $A = B = vw_n^{-1}v^{-1}w_n$ . Let  $u'_1$  be  $u_1$  after all  $w_n$  are replaced with  $w_{n,1}$ . Similarly define  $u'_2$ . Then  $\text{Area}_G(gu'_1g^{-1}u'_2) \leq E_n$  (see the proof of Theorem 1.2.3 for a complete calculation). Consider an annular van Kampen diagram of area less than  $E_n$  for this conjugation. The  $t$ -bands on this diagram can not originate and end on the same boundary component (see the proof of Lemma 5.3.2). Therefore the number of letters  $t$  is the same for both boundary components and  $i = \pm j$ .  $\square$

Notice the decrease of the upper bound ( $E_{n-1}$  versus  $E_n$  in Lemma 5.3.2) in the previous lemma. It is there for technical reasons, and we believe can be eliminated with some extra work. Similarly, the factor of 2 in the next lemma is probably unnecessary.

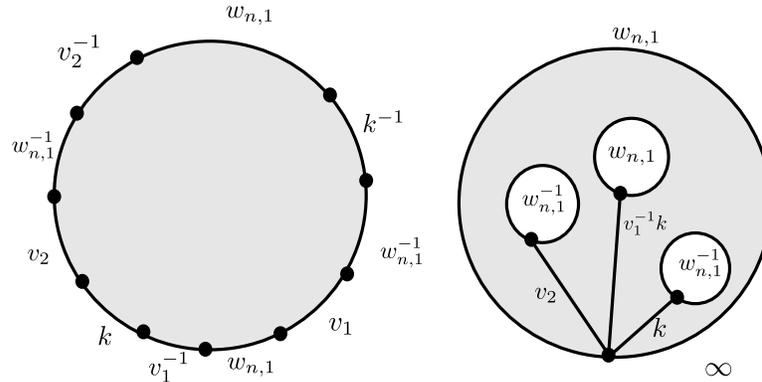


Figure 5.1: Two Van Kampen diagrams for  $k^{-1}(v_1w_{n,1}^{-1}v_1^{-1}w_{n,1})k = v_2w_{n,1}^{-1}v_2^{-1}w_{n,1}$ . The one on the right was obtained from the one on the left by gluing parts of the boundary together. One can think of the right diagram as spherical by placing  $\infty$  outside of the outer  $w_{n,1}$ .

**Lemma 5.3.4.** *Let  $v_1 \neq v_2$  be as in Lemma 5.2.2, then  $\text{Area}_G(k^{-1}(v_1w_{n,1}^{-1}v_1^{-1}w_{n,1})k(v_2w_{n,1}^{-1}v_2^{-1}w_{n,1})^{-1}) \geq \frac{1}{2}E_n$ , where  $k$  is a word in  $G$ .*

*Proof.* We can apply less than  $100E_{n-1}$  relations to convert  $w_n$  to  $w_{n,1}$ , obtaining a van Kampen diagram for  $k^{-1}(v_1w_{n,1}^{-1}v_1^{-1}w_{n,1})k = v_2w_{n,1}^{-1}v_2^{-1}w_{n,1}$  of area less than  $E_n$ . We want to apply our “effective” small cancellation theory to this equality. We can view it as a van Kampen diagram with boundary  $w_{n,1}$  over  $\langle G|w_{n,1} \rangle$ , because we have the equality in the free group:  $w_{n,1} = (v_2^{-1})^{-1}w_{n,1}(v_2^{-1}) \cdot (v_1^{-1}k)^{-1}w_{n,1}^{-1}(v_1^{-1}k)$ .

$(k)^{-1}w_{n,1}(k)$ . Since the boundary of this diagram is also  $w_{n,1}$  it can be viewed as a spherical van Kampen diagram over  $\langle G|w_{n,1} \rangle$  (see Figure 5.1), call it  $D$ . We want to apply Theorem 1.3.10 to  $D$ . There are three differences. First, the diagram is spherical, not planar. That is not an issue because we just replace the Euler characteristic of the disk (1) with that of the sphere (2) in the proof of Theorem 1.3.10 to get the same result. Second, the relation is  $w_{n,1}$  instead of  $t^{-1}w_{n,1}$ . This is not a problem, since we still have the condition (6,3). We don't have the metric condition  $C'(\frac{1}{6})$  (our pieces can be *exactly* of length  $\frac{1}{6}$ ), but it is only needed if the boundary is of non-zero  $t$ -length, in our case the boundary is just empty. We are left with the third issue: the diagram is not what we called  $r$ -reduced in Theorem 1.3.10, that is there is a large  $t$ -cable between one of  $w_{n,1}$  and  $w_{n,1}^{-1}$ . In Lemma 1.3.14 we proved that all large  $t$ -cables have only  $t$ -bands of length 0. Consider two cases. Case 1: there is a large  $t$ -cable involving the first  $w_{n,1}^{-1}$  cell (conjugated by  $v_2$ ) and case 2: a large  $t$ -cable is on the other  $w_{n,1}^{-1}$  cell. In case 1 we have two options:  $pv_2^{-1}k^{-1}v_1p^{-1}$  (see Figure 5.2 for the definition of  $p$ ) is a loop on the diagram, or  $v_2$  is. Note, loops in  $D$  represent 1 in  $G$  because  $\langle G|w_{n,1} \rangle = G$  as groups. By assumption,  $v_2 \neq_G 1$ , therefore  $pv_2^{-1}k^{-1}v_1p^{-1} =_G 1$ , or  $kv_2 =_G v_1$ . After canceling this pair of  $w_n$ -cells we obtain a spherical diagram  $D'$  that has one pair of  $w_n$ -cells. In  $D$  these cells were connected by a curve spelling  $k$ . After the cancellation the curve spells  $k' =_G k$ . Note,  $k$  might not be equal to  $k'$  effectively. For example they might differ by one of the removed cells. In  $D'$  the two remaining cells have to cancel each other, therefore  $k'$  is a loop and thus  $k =_G k' =_G 1$ . We know from before  $kv_2 =_G v_1$ , and so  $v_2 =_G v_1$ . Case 2 is dealt with similarly.

□

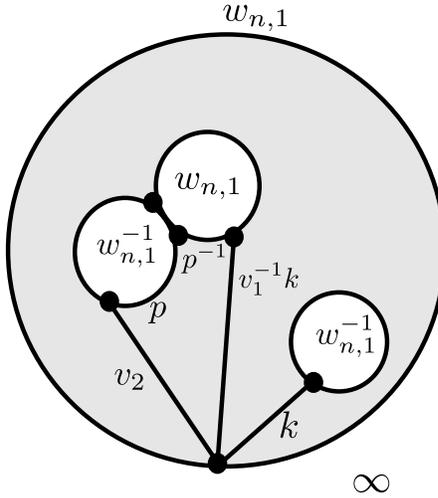


Figure 5.2: Two cells are touching along a  $t$ -cable of 0 thickness. Since this  $t$ -cable is long enough the cells touch in the same position.

By combining Lemma 5.3.2 and a result from [Bri15] we can get the following lemma. We reproduce the proof here for completeness.

**Lemma 5.3.5.** *Let  $w$  be a non-empty freely reduced word in  $\langle s, x \rangle$ . We can think of  $w$  as a word in  $H_v$ . Then if  $w =_{H_v} 1$ ,  $\text{Area}_{H_v}(w) \geq E_n$ .*

*Proof.* Suppose the area is less than  $E_n$ . First we prove, that circular  $s$ -bands are impossible. Find an innermost circular  $s$ -band. Then we see that either a power of  $t$  or a power of  $vw_n^{-1}v^{-1}w_n$  is 1 in

$G$  through area  $< E_n$ . This is impossible by Lemma 5.3.1. We checked the conditions required for an application of Lemma 5.3.1 in the proof of Lemma 5.3.2. Semi-circular  $s$ -bands are impossible because  $1 \neq_G x^i \neq_G t^j$ . Therefore  $w =_G 1$ , which is impossible, a contradiction.  $\square$

**Lemma 5.3.6.** *Let  $r_1, r_2$  be non-empty freely reduced words in  $\langle s, x \rangle$ . If  $gr_1g^{-1} =_{H_v} r_2$  (via area  $< N < E_n$ ) for some  $g \in H_v$ , then either  $g$  is equal to a word in  $\langle s, x \rangle$  in  $H_v$  via area  $< N$  or  $r_1 = x^i$ ,  $r_2 = x^j$  for some  $i, j$ .*

*Proof.* Consider a diagram for  $gr_1g^{-1} =_{H_v} r_2$ . As in the proof of Lemma 5.3.5 there are no circular  $s$ -bands, so the subdiagrams between  $s$ -bands are over  $G$ . Suppose there is a letter  $s^{\pm 1}$  in  $r_1$  (otherwise  $r_1, r_2$  are powers of  $x$ ). Then the  $s$ -band originating on it has to go to  $r_2$  for the same reasons explained in the proof of Lemma 5.3.5. The subdiagrams between such  $s$ -bands are in  $G$ , and we can use Lemma 5.3.2 to show that these  $s$ -bands have length 0. That shows  $g$  is equal to a word in  $\langle s, x \rangle$  in  $H_v$  via area  $< N$ .  $\square$

## 5.4 Effective Isomorphisms

**Definition 5.4.1.** For a presentations  $\mu$ , denote by  $[\mu]$  the free group on the letters of the presentation.

**Definition 5.4.2.** For presentations  $\mu, \mu'$ , we say a group map  $F : [\mu] \rightarrow [\mu']$  is  $(L, N)$  (we assume  $N > L$ ) if for a letter  $a \in \mu$ ,  $F(a)$  has length  $< L$ , and for a relator  $u$  of  $\mu$ ,  $Area_{\mu'}(F(u)) < N$ . If in addition  $Area_{\mu'}(F(a)) > M$  (possibly  $F(a) \neq_{\mu'} 1$ ), then we say  $F$  is  $(L, N, M)$ .

*Remark 5.4.3.* One should think of an  $(L, N)$  map  $F$  as an “effective” group homomorphism.

Let  $v_1, v_2$  be words like in Lemma 5.2.2. Let  $\mu_1 = \mu_{v_1}, \mu_2 = \mu_{v_2}, l = \max\{l(\mu_1), l(\mu_2)\}$ . In this section we will prove several lemmas about an effective homomorphism from  $\mu_1$  to  $\mu_2$ . We start with one and then simplify it until in the end we are able to prove that it exists only if  $v_1 = v_2$ . Each modification of the homomorphism either would be trivial on the level of pseudogroups, or a composition with a simple automorphism of  $\mu_2$ : a conjugation or the transposition of the two factors of  $\mu_2$ .

The next lemma is an analogue of Lemma 4.3.1, but instead of using Collins’ Lemma for an HNN-extension we have to use its analogue for a free product with amalgamation, furthermore, in this setting we need an effective version of such a result, as we can’t simply work with reduced words.

**Lemma 5.4.4.** *If  $F : [\mu_1] \rightarrow [\mu_2]$  is  $(L, N, M)$  (where  $lN^3 < E_n$ ), then there exists  $F' : [\mu_1] \rightarrow [\mu_2]$  of type  $(3N^3, lN^3, M - lN^3)$  such that  $F'(x) \in \langle x, y, t, s \rangle$ .*

*Proof.* Diagrams over any free product with amalgamation consist of subdiagrams of cells from one or the other factor (regions), bounded by the mixed cells (the cells responsible for amalgamation). In the case of  $\mu_2$  we have regions made up from  $H_{v_2}$  cells and regions from  $\hat{H}_{v_2}$  bounded by the cells corresponding to  $s = \hat{x}, x = \hat{s}$ . Therefore the pure regions are bounded by words in  $\langle x, s \rangle$  (or  $\langle \hat{x}, \hat{s} \rangle$ ) and pieces of the boundary of the diagram.

Consider a diagram for  $F(x)^{F(y)} = (F(x))^2$  of area less than  $N$ . Our goal is to find  $F'$  such that  $\hat{l}(F'(x)) = 1$ . Suppose  $\hat{l}(F(x)) \geq 2$ , then  $\hat{l}(F^2(x)) \geq 4$ . Consider the regions of the diagram in  $H_{v_2}$ . In total they have twice as many boundary pieces on  $F^2(x)$  than on  $F(x)$ . Therefore there either exists a “no-hat” region with boundary on  $F^2(x)$  only (case 1), or a region with more than one boundary piece on  $F^2(x)$  (case 2), see Figure 5.3. In case 1 we have the equality of a no-hat piece of the boundary of the

diagram to a word in  $\langle \hat{x}, \hat{s} \rangle$ , in case 2 we have the equality of a mixed piece (the part of the boundary between some two no-hat pieces) to a word in  $\langle x, s \rangle$ . In both cases we can find a word  $w =_{\mu_2} k^{-1}F(x)k$  such that  $\text{Area}_{\mu_2}(w^{-1}k^{-1}F(x)k) < N$ , the length of  $k$  is less than  $L$ , the length of  $w$  is less than  $L + N$  ( $N$  is a bound for the length of the words in  $\langle \hat{x}, \hat{s} \rangle$  or  $\langle x, s \rangle$  – the boundaries between pure regions), and  $\hat{l}(w) < \hat{l}(F(x))$ . Thus, we define  $F^{(1)}$  of type  $(L + N + L, N + lN, M - N)$  to be  $F^{(1)}(x) = w$ , for  $a \neq x$  define  $F^{(1)}(a) = k^{-1}F(a)k$ . We have  $\hat{l}(F^{(1)}(x)) < \hat{l}(F(x))$ .

We can continue in this way defining  $F^{(2)}, F^{(3)}$ , etc. Until  $\hat{l}(F^{(m)}(x)) = 1$  for some  $m \leq \hat{l}(F(x)) \leq L$ . We can estimate  $F^{(m)}$  to be  $(L + N + 4N + 7N + 10N + \dots + (3L + 1)N = L + LN + 3N(L + 1)L/2 < 3NL^2, lNL^2)$  or  $(3N^3, lN^3)$ . Define  $F'$  to be either  $F^{(m)}$  or  $\hat{F} \circ F^{(m)}$ , where  $\hat{F}$  is the automorphism on  $[\mu_2]$  interchanging  $H_{v_2}$  with  $\hat{H}_{v_2}$ .  $\square$

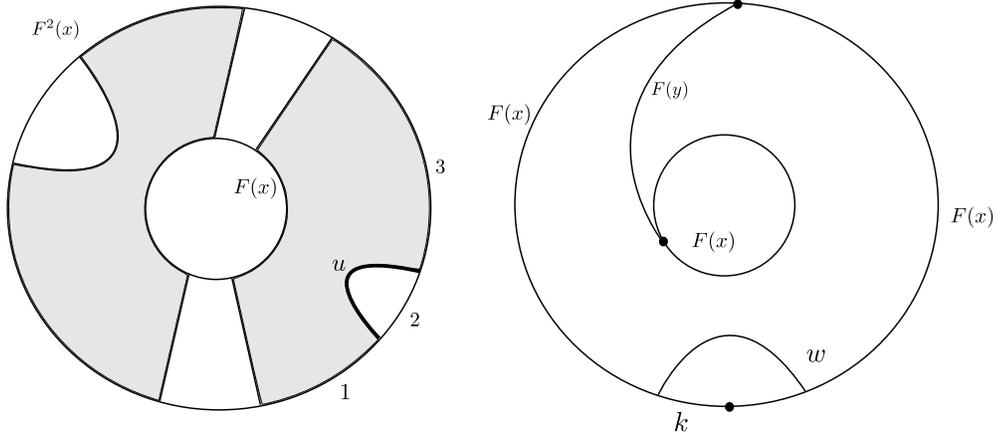


Figure 5.3: Two annular van Kampen diagrams for  $F(x)^{F(y)} = (F(x))^2$ . The left diagram is an example of  $\hat{l}(F(x)) = 4$ . If the white regions are the no-hat regions, and the grey ones are hat regions, then it represents case 1: the piece of the boundary marked by 2 can be replaced by  $u$ , thus reducing  $\hat{l}(F(x))$ . If, on the other hand, the grey regions are no hat regions, then we can see case 2: one of the two grey regions has two pieces on the outer boundary (piece 1 and piece 2), they are connected by a piece  $u$  that can be used to replace piece 2. On the right diagram we see what happens if this reduction goes over the point of concatenation of two  $F(x)$  – we can conjugate by  $k$  to recover  $F(x)$  from  $w$ .

**Lemma 5.4.5.** *If  $F : [\mu_1] \rightarrow [\mu_2]$  is  $(L, N, M)$  (where  $lN^3 < E_n$ ) is such that  $F(x)$  is a word in  $H_{v_2}$ , then there exists  $F' : [\mu_1] \rightarrow [\mu_2]$  such that  $F'(x), F'(y)$  are in  $H_v$ , where  $F'$  is  $(N^3, lN^3, M - lN^3)$ .*

*Proof.* Consider a diagram for  $F(x)^{F(y)} = (F(x))^2$  of area less than  $N$ . Our goal is (as in the previous lemma) to decrease  $\hat{l}(F(y))$  to 1. Observe that by applying Lemma 5.3.5 we see that there are no topologically trivial regions on the diagram. Therefore, this diagram looks like concentric circles of alternating type (hat, no-hat), see Figure 5.4. Since both components of the boundary are in  $H_{v_2}$ , if there is more than one circle (i.e.  $\hat{l}(F(y)) > 1$ ), there has to be more than two ( $\hat{l}(F(y)) > 2$ ). This gives us a subdiagram (the second annulus) to apply Lemma 5.3.6. Then if  $g$  from the conclusion of the lemma is a word in  $\langle s, x \rangle$  we can obtain  $F^{(1)}$  having  $\hat{l}(F^{(1)}(y)) = \hat{l}(F(y)) - 2$ , or if the conclusion is  $r_1 = r_2 = x^j$  we can define  $F^{(1)}$  (by conjugating by  $g_1 \in H_{v_2}$ ) such that  $F^{(1)}(x) = x^j$ . In both cases  $F^{(1)}$  is  $(L + N + L, N + lN, M - N)$ .

Now we apply this process to  $F^{(1)}$ , but we notice that if  $F^{(1)}(x) = x^j$  the conclusion of the Lemma 5.3.6 can not be of the second case, because  $x^j$  is not conjugate to  $s^k$  in  $H_{v_2}$  (the conjugation corre-

sponding to the first annulus). Therefore, an application of Lemma 5.3.6 will always decrease  $\hat{l}(F(y))$  by 2 except possibly once. Similarly to the previous lemma we obtain  $F'$  of type  $(N^3, lN^3, M - lN^3)$ .  $\square$

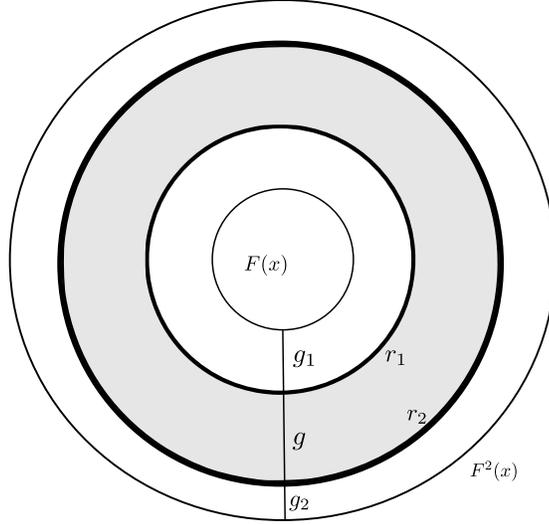


Figure 5.4: A possible diagram for  $F(x)^{F(y)} = (F(x))^2$  with  $\hat{l}(F(y)) = 3$ . Here  $g_1 g g_2 = F(y)$ . We apply Lemma 5.3.6 to  $g r_2 g^{-1} = r_1$ .

**Lemma 5.4.6.** *If  $F : [\mu_1] \rightarrow [\mu_2]$  is  $(L, N, M)$  ( $lN^3 < E_n$ ) is such that  $F(x), F(y)$  are words in  $H_{v_2}$ , then there exists  $F' : [\mu_1] \rightarrow [\mu_2]$  such that  $F'(x), F'(y), F'(t)$  are in  $H_{v_2}$ , where  $F'$  is  $(N^3, lN^3, M - lN^3)$ .*

*Proof.* The proof is completely analogous to the proof of Lemma 5.4.5, the only difference is we consider the diagram for  $F(x)^{F(t)} = F(y)$ . Note that the only thing we used about a diagram for  $F(x)^{F(y)} = (F(x))^2$  in Lemma 5.4.5 is that its boundary is in  $H_{v_2}$ . As in the proof of Lemma 5.4.5 we might need to conjugate  $F$  to deal with the possible conclusion of Lemma 5.3.6 not resulting in the decrease of  $\hat{l}(F(t))$ , but we conjugate by something in  $H_{v_2}$  thus not interfering with  $F(x), F(y)$  being in  $H_{v_2}$ .  $\square$

**Lemma 5.4.7.** *If  $F : [\mu_1] \rightarrow [\mu_2]$  is  $(L, N, M)$  ( $lN^3 < E_n$ ) is such that  $F(x), F(y), F(t)$  are words in  $H_{v_2}$ , then there exists  $F' : [\mu_1] \rightarrow [\mu_2]$  such that  $F'(x), F'(y), F'(t), F'(s)$  are in  $H_{v_2}$ , where  $F'$  is  $(N^3, lN^3, M - lN^3)$ .*

*Proof.* We proceed as in Lemma 5.4.6 by considering a diagram for  $(v w_n^{-1} v^{-1} w_n)^{F(s)} = F(t)$ .  $\square$

**Lemma 5.4.8.** *If  $F : [\mu_1] \rightarrow [\mu_2]$  is  $(L, N, M)$  ( $lN^3 < E_n$ ) is such that  $F(x), F(y), F(t), F(s)$  are words in  $H_{v_2}$ , then there exists  $F' : [\mu_1] \rightarrow [\mu_2]$  of type  $(lN^3, lN^3, M - lN^3)$  such that  $F'(x)$  is in  $G$ , while  $F'(y), F'(t), F'(s)$  are in  $H_{v_2}$ .*

*Proof.* We mimic Lemma 5.4.4, with  $s$ -bands instead of regions. Consider a diagram for  $F(x)^{F(y)} = (F(x))^2$  of area less than  $N$ . If  $l_s(F(x)) > 0$  then there is an  $s$ -band attached to the  $(F(x))^2$  part of the boundary. One difference is that the length of  $F(x)$  can grow faster – by the length of  $v w_n^{-1} v^{-1} w_n$  which is less than  $l$ .  $\square$

**Lemma 5.4.9.** *If  $F : [\mu_1] \rightarrow [\mu_2]$  is  $(L, N, M)$  ( $N < E_{n-1}$ ) is such that  $F(y), F(t), F(s)$  are words in  $H_{v_2}$  and  $F(x)$  is in  $G$ , then  $F(y) \in G$ .*

*Proof.* Consider the diagram for  $F(x)^{F(y)} = (F(x))^2$ . Suppose  $l_s(F(y)) > 0$ , then by applying Lemma 5.3.3 to the concentric subdiagrams between  $s$ -bands we see that  $F(x)$  is conjugate to  $A^i$  while  $F^2(x)$  is conjugate to  $B^i$  (Lemma 5.3.3 makes sure it's the same  $i$ ), where  $A, B$  are either  $t$  or  $(vw_n^{-1}v^{-1}w_n)$ . Then we have that  $A^{2i}$  is conjugate to  $B^i$ , so that  $A = B$  and again by Lemma 5.3.3 that's impossible ( $i \neq 0$  because  $M > N$ ). □

**Lemma 5.4.10.** *If  $F : [\mu_1] \rightarrow [\mu_2]$  is  $(L, N, M)$  ( $lN^3 < E_n$ ) is such that  $F(t), F(s)$  are words in  $H_{v_2}$ ,  $F(x), F(y)$  are in  $G$  and  $\text{Area}(F(x)) > N$ , then there exists  $F' : [\mu_1] \rightarrow [\mu_2]$  such that  $F'(x) = x^i$  and  $F'(y) \in \langle x, y \rangle$ , where  $i \neq 0$ , while  $F'(t), F'(s)$  stay in  $H_{v_2}$ ,  $F'$  is  $(3N^3, lN^3, M - lN^3)$ .*

*Proof.* We proceed similarly to Lemma 5.4.5 reducing  $F(y)$  until its  $l_t$  is equal to 0. The first step is to define  $F^{(1)}$  such that  $F^{(1)}(x) = x^i$  ( $i \neq 0$  follows from  $M > lN^3$ ), how to do it is described in the proof of Lemma 4.3.2. Then we proceed as in the proof of Lemma 4.3.3 to make sure  $F^{(m)}(y) \in \langle x, y \rangle$  for some  $m \leq L$ . □

**Lemma 5.4.11.** *If  $F : [\mu_1] \rightarrow [\mu_2]$  is  $(L, N, M)$  ( $N < E_n$ ) is such that  $F(t), F(s)$  are words in  $H_{v_2}$ , and  $F(x) = x^i, F(y) \in K$ , then  $F(t) \in G$ .*

*Proof.* Consider the diagram for  $F(x)^{F(t)} = F(y)$ . If there is an  $s$ -band it means  $x^i$  is conjugate to  $t^j$  or 1 in  $G$ , which is impossible since  $i \neq 0$  (because  $M > 0$ ). □

**Lemma 5.4.12.** *If  $F : [\mu_1] \rightarrow [\mu_2]$  ( $2lN^3 < E_n$ ) is  $(L, N, M)$  is such that  $F(s)$  is a word in  $H_{v_2}$ ,  $F(t) \in G$ ,  $F(x) = x^i$  and  $F(y) \in K$ , then there exists  $F' : [\mu_1] \rightarrow [\mu_2]$  such that  $F'(x) = x, F'(y) = y, F'(t) = gt, F'(s) \in H_{v_2}$ , where  $g$  is in  $K$  and commutes with  $x$ ,  $F'$  is  $(6N^3, 2lN^3, M - 2lN^3)$ .*

*Proof.* First, consider an equality in  $K$   $(x^i)^{F(y)} = x^{2i}$ . As in the proof of Lemma 4.3.4, we note that the total  $y$  power of  $F(y)$  (counted with signs) is 1. In the same lemma it is also described how to reduce  $l_t$  of  $F(t)$  to 1 (by considering a diagram for  $F(x)^{F(t)} = F(y)$ ) and noticing that  $x^i$  is then conjugate to  $x$ . Let  $F'(t) = g_1tg_2$ . From the same diagram we see that  $F(y)^{g_1} =_K y$  and therefore we can remove pinches one by one in  $F(y)$  until it is  $y$ . Furthermore, we see that  $g_i =_K y^m$  and therefore we can make  $F'(t) = gt$ , where  $g$  commutes with  $x$ . □

**Lemma 5.4.13.** *If  $F : [\mu_1] \rightarrow [\mu_2]$  of type  $(L, N, M)$  is such that  $F(s)$  is a word in  $H_{v_2}$ ,  $F(x) = x, F(y) = y$  and  $F(t) = gt$ , where  $g \in K$  commutes with  $x$ , then there exists  $F' : [\mu_1] \rightarrow [\mu_2]$  such that  $F'(x) = x, F'(y) = y, F'(t) = t$ , and  $F'(s) = hsk$  for  $h, k \in G$ , where  $F'$  is  $(N^3, lN^3, M - lN^3)$ .*

*Proof.* Consider a diagram for  $F(v_1^{-1}w_nv_1)^{F(s)} = F(t)$  of area less than  $N$ . In  $G$   $F(v_1^{-1}w_nv_1)$  can not be conjugate to  $F(t)$ , because of different total power of  $t$  (counted with signs). Therefore  $F(s)$  has at least one letter  $s$  or  $s^{-1}$ . Consider the nearest to  $F(t)$  circular  $s$ -band on the diagram. Since  $F(t) = gt$  is not conjugate to 1 or  $t^i$  for  $i \neq 1$ , we have that this  $s$  band has length 1, and  $gt$  is conjugate to  $t$  in  $G$ . Consider the subdiagram corresponding to this conjugation. There is one  $t$  band on that subdiagram, giving  $x^i gy^i = 1$  in  $G$ , which is possible only for  $i$  being 0 and  $g$  being 1, because  $g$  commuting with  $x$

means it has the total  $y$  power (counted with signs) 0. Now, we want to reduce  $l_s((F(s))$  to 1. Notice that by Lemma 5.3.3 whenever two  $s$ -bands face each other with their  $t$  side, we can cancel them. Furthermore, it is not possible to have just two  $s$ -bands facing each other with their 1 side because we know  $F(t) \neq_G 1$ . Now, the lemma follows by the standard argument.  $\square$

**Proposition 5.4.14.** *If  $F : \mu_1 \rightarrow \mu_2$  of type  $(N, N, M)$  is such that  $M > 2(lN)^{21}$  and  $2(lN)^{21} < E_{n-1}$ , then  $v_1 = v_2$ .*

*Proof.* Apply Lemmas 5.4.4, 5.4.5, 5.4.6, 5.4.7, 5.4.8, 5.4.9, 5.4.10, 5.4.11, 5.4.12, 5.4.13 successively to obtain  $F'$  of type  $(2(lN)^{7 \cdot 3}, 2(lN)^{7 \cdot 3}, M - 2(lN)^{7 \cdot 3})$  such that  $F'(x) = x, F'(y) = y, F'(t) = t$ , and  $F'(s) = hsk$  for  $h, k \in G$ .

Consider a diagram for  $F'(v_1^{-1}w_nv_1)^{F'(s)} = F'(t)$  of area less than  $E_n$ , or equivalently  $(v_1^{-1}w_nv_1)^{hsk} = t$ . The diagram has an  $s$ -band of length 1, from which we see that  $k^{-1}(v_1w_n^{-1}v_1^{-1}w_n)k = v_2w_n^{-1}v_2^{-1}w_n$  through area  $< E_n$ , which is impossible by Lemma 5.3.4 unless  $v_1 = v_2$ .  $\square$

## 5.5 Main Results

We recall the definition of Tietze transformations from Chapter 1.

**Definition 5.5.1.** Let  $\mu = \langle x_1, \dots, x_r | a_1, \dots, a_p \rangle$ .

Elementary Tietze transformations:

$Op_1$   $\mu$  is replaced by  $\langle x_1, \dots, x_r | a_1, \dots, a_{i-1}, a'x_j^\epsilon x_j^{-\epsilon} a'', a_{i+1}, \dots, a_p \rangle$ , where  $a_i \equiv a' a''$  and  $\epsilon = \pm 1$ .

$Op_1^{-1}$  The inverse of  $Op_1$  - it deletes  $x_j^\epsilon x_j^{-\epsilon}$  in one of the relators.

$Op_2$   $\mu$  is replaced by  $\langle x_1, \dots, x_r | a_1, \dots, a_{i-1}, a'_i, a_{i+1}, \dots, a_p \rangle$ , where the word  $a'_i$  is a cyclic permutation of the word  $a_i$ .

$Op_3$   $\mu$  is replaced by  $\langle x_1, \dots, x_r | a_1, \dots, a_{i-1}, a_i^{-1}, a_{i+1}, \dots, a_p \rangle$ .

$Op_4$   $\mu$  is replaced by  $\langle x_1, \dots, x_r | a_1, \dots, a_{i-1}, a_i a_j, a_{i+1}, \dots, a_p \rangle$ , where  $i \neq j$ .

$Op_5$   $\mu$  is replaced by the presentation  $\langle x_1, \dots, x_r, x_{r+1} | a_1, \dots, a_p, x_{r+1} \rangle$ .

$Op_5^{-1}$  The inverse of  $Op_5$ .

All presentations of the trivial group can be transformed from one to another using these operations if we also allow adding the empty relation.

**Definition 5.5.2.** For presentations  $\mu, \mu'$ , denote by  $T(\mu, \mu')$  the minimal number of Tietze transformations needed to go from one to another.

We describe a connection between the number of Tietze moves and effective isomorphisms in the next proposition.

**Proposition 5.5.3.** *If for some presentations  $\mu, \mu'$  of the trivial group we have  $T(\mu, \mu') \leq N$ , then there exists  $F : [\mu] \rightarrow [\mu']$  of type  $(2, 1 + 2^N)$ . Furthermore, if the letters in  $\mu$  have area greater than  $M$  then  $F'$  is  $(1, 1 + 2^N, 2^{-N}M)$ .*

*Proof.* Suppose  $\mu$  differs from  $\mu'$  by one operation. There is the obvious map  $F : [\mu] \rightarrow [\mu']$  (if a letter was removed, send it to 1). In the case of  $Op_1^{\pm 1}, Op_2, Op_3, Op_5^{\pm 1}$ ,  $F$  is clearly  $(2, 2)$  (recall, the identity map  $[\mu] \rightarrow [\mu]$  is also  $(2, 2)$  because the inequalities are strict). In the case of  $Op_4$ ,  $F$  is  $(2, 3)$ . Therefore, if  $T(\mu, \mu') < N$  then  $F$  is  $(2, 1 + 2^N)$ . Similarly, if we have a bound for the area of a generator  $M$ , then after one transformation it might decrease by at most 2, so that it becomes  $\frac{M}{2}$ . Therefore  $F$  is of type  $(2, 1 + 2^N, 2^{-N}M)$ .  $\square$

We need the following definition to state the main result.

**Definition 5.5.4.** If  $S$  is a set of balanced presentations of the trivial group of rank  $k$  and of length at most  $l$  such that for any  $\mu \neq \mu'$  in  $S$ ,  $T(\mu, \mu') > \exp_m(l)$ , we call  $S$   $(l, m, k)$ -disconnected. Let  $M_{l,m,k}$  denote the maximal size of  $(l, m, k)$ -disconnected sets. We are going to show that for any  $m$ ,  $M_{l,m,4}$  grows at least exponentially.

**Theorem 5.5.5.** *For each  $m$  there exists a constant  $\text{const}(m)$  such that if  $l > \text{const}(m)$ , then  $M_{l,m,4} > 1.18^l$ .*

*Proof.* Choose minimal  $n$  such that  $E_{n-1} > \exp_{m+1}(41l+1) > 2(\exp_{m+1}(l))^{42}$ , or equivalently  $E_{n-m-2} > 42l + 1$ . We can choose  $n$  to be less than  $\log_2(l) - 20$  for large enough  $l$ .

Let  $S_{n,l} = \{\mu_v | l(\mu_v) \leq l\}$ , where  $v$  is any word of the form described in Lemma 5.2.2. Note that  $n$  was already used in the definition of  $\mu_v$ . Suppose  $T(\mu, \mu') \leq \exp_m(l)$ , then by Proposition 5.5.3 there exists  $F : [\mu] \rightarrow [\mu']$  of type  $(2, 1 + \exp_{m+1}(l), \frac{E_n}{\exp_{m+1}(l)})$  (we used here the fact that the identity map on  $[\mu]$  is  $(2, 2, E_n)$ ). To apply Proposition 5.4.14 to  $F$  we need to check that  $2(l(1 + \exp_{m+1}(l)))^{21} < E_{n-1}$  and  $\frac{E_n}{\exp_{m+1}(l)} > 2(l(1 + \exp_{m+1}(l)))^{21}$ . The first inequality follows because  $2(l(1 + \exp_{m+1}(l)))^{21} < 2(\exp_{m+1}(l))^{42} < E_{n-1}$  by the choice of  $n$ , the second one holds because  $E_n > (E_{n-1})^2$ . Therefore  $\mu = \mu'$  and  $S_{n,l}$  is  $(l, m, 4)$ -disconnected.

Now we want to estimate the size of  $S_{n,l}$ . Recall that  $l(\mu_v) < 2l(v) + 200 \cdot 2^n + 20$ . Using our estimate for  $n$  we obtain  $l(\mu_v) < 2l(v) + 0.01l$ . Therefore to make sure  $\mu_v \in S_{n,l}$ , it is enough to make  $l(v) < \frac{0.99l}{2}$ . There are at least  $2^{0.99l/4} < 1.18^l$  such  $v$ .  $\square$

We conclude with one application, for which we need the following definition.

**Definition 5.5.6.** Let  $\Gamma_{l,m,k}$  be a graph, the vertices of which are the balanced presentations of the trivial group of rank  $k$  and of length at most  $l$ . Any two presentation are connected by an edge if they require at most  $\exp_m(l)$  Tietze transformations to go from one to another.

**Theorem 5.5.7.** *For each  $m$  there exists a constant  $\text{const}(m)$  such that if  $l > \text{const}(m)$ , then the number of connected components of  $\Gamma_{l,m,4}$  is at least  $1.18^l$ .*

*Proof.* We can make a crude estimate that the number of vertices of  $\Gamma_{l,m}$  is less than  $ll^l$ . Therefore, if two vertices  $\mu, \mu'$  are connected by some path, then  $T(\mu, \mu') < ll^l \exp_m(l)$ . Noting that  $\exp_{m+1}(l) > ll^l \exp_m(l)$  for large enough  $l$ , we see that this theorem follows from  $M_{l,m+1,4} > 1.18^l$  (Theorem 5.5.5).  $\square$

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