Convexity and Cohomology of the Based Loop Group

by

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Abstract

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Let K be a compact, connected, simply connected Lie group and define ΩK to be the loops on K. Let $\Omega_{\text{alg}}K$ be those loops which are the restriction of algebraic maps $\mathbb{C}^\times \to K_\mathbb{C}$. Herein we establish two distinct but related results. In the first, we demonstrate the module structure for various generalized abelian equivariant cohomology theories as applied to equivariantly stratified spaces. This result is applied to the algebraic based loop group for the cases of equivariant singular cohomology, K-theory, and complex cobordism cohomology. Subsequently, we examine the image of the based loop group under the non-abelian moment map. We show that both the Kirwan and Duistermaat convexity theorems hold in this infinite dimensional setting.

For my grandmother Ayako (Irene) Tabata, who passed away while I was writing this thesis, and never got to see me finish.

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Chapter 1

Introduction and Outline

If X is a pointed topological space, its based loop space ΩX consists of the continuous maps from the circle S^1 to X. Used to study homotopy groups, define spectra for cohomology theories, and being essential to Bott periodicity, the based loop space frequently finds itself amidst the homotopy theorist's toolbox.

Additional structures on X effect additional structures on ΩX, and a plethora of interesting properties manifest as a result. For example, if X is a Lie group, then ΩX also has the structure of a Lie group, albeit as the more generalized notion of a Fréchet manifold. If that group is furthermore compact, ΩX does its best to imitate a compact group despite being non-compact, admitting a locally bijective exponential map, finite singular cohomology, and playing nicely with its symplectic structure. While this imitation is doomed to be imperfect, the relative simplicity of its study drives our interest in researching its characteristics.

Changing the regularity of the maps permitted between S^1 and X establishes a hierarchy of related spaces, many of which find themselves of interest in different mathematical fields. Restricting to maps of algebraic varieties is of interest to geometric representation theorists, while loops preserving differential structure is of interest to differential geometers. Loops of Sobolev class H^1 find application in the study of partial differential equations and harmonic analysis.

Our focus in this thesis will lie primarily with the smooth loops, though we will not hesitate to exploit the aforementioned algebraic and Sobolev H^1 loops to assist our study. In fact, Chapter 4 is based on the paper [CH15], whose origins are rooted in the study of the algebraic based loops. The notation and terminology has been translated from the algebraic language to the smooth language in thematic consonance, though the original presentation remains undoctored *ibid.*

The outline of this treatise is as follows: In Chapter 2 we begin by introducing an in-depth treatment of the mathematical foundation underlying the study of the based loop group of a compact, connected, simply connected Lie group. This is followed by a short summary of generalized equivariant cohomology theories. For this latter component our goal is only to establish a sufficient infrastructure to convince the reader that the subsequent vocabulary is not derivative of our imagination and that our work is well-founded.

Chapter 3 offers the reader a brief literature review, including a discussion of papers dedicated to the study of cohomology on the based loop group, followed by a treatment of the equivariant cohomology on stratified spaces in general. This is succeeded by a survey of finite dimensional convexity results and their analogous generalizations to the based loop group.

Chapter 4 summarizes the results of the paper [CH15]. Here we study the equivariant singular cohomology, K-theory, and complex cobordism cohomology of a space which admits an equivariant stratification. The corresponding cohomology of each stratum is computed then reassembled to yield the cohomology of the entire space. The chapter is concluded with an explicit demonstration of the results applied to the algebraic based loop group.

This is followed by Chapter 5, which is based on the paper [Hol15]. Herein three classical finite-dimensional convexity theorems are presented and generalized. Specifically, the Atiyah-Guillemin-Sternberg, Duistermaat, and Kirwan convexity theorems are generalized to apply in the case of algebraic, smooth, and Sobolev class H^1 loops on a Lie group. The proof uses the stratification of the algebraic based loop group into Schubert varieties to demonstrate convexity on each stratum, which then allows for the argument to be made for the entire algebraic based loop group. Topological arguments permit the extension of the results to the smooth and Sobolev H^1 loops.

Chapter 2

Mathematical Foundation

Other than an appealing alliteration, the tools and ideas used for studying cohomology and convexity are quite disparate. The application of these tools further requires an intimate knowledge of the based loop group, which itself lies at the intersection of so many fields as to inherit a surfeit of structure. As this dissertation draws extensive inspiration from the underlying ideas and strategies established in the seminal literature. We have therefore chosen to prioritize the mathematical introduction ahead of our discussion the existing literature. With jargon in hand, we may explore the details of those results with the necessary depth to contextualize our work in the field.

This section assumes familiarity with the fields of symplectic geometry, Kähler geometry, and Lie groups and algebras.

2.1 The Based Loop Group

The collection of based loop groups over compact Lie groups is our space of primary interest. Such spaces find applications in string theory, simplified models of quantum field theory, integrable systems of partial differential equations, group valued moment maps, and represent well-behaved infinite dimensional manifolds which exhibit many of the characteristics of their finite dimensional counterparts. A close relative of the based loops are the free loops, whose behaviour and structure will be paramount to our study of the based loops, and in particular serve as the intermediary between the differential and algebraic geometric paradigms of our study.

2.1.1 Definition

Let K be a compact, connected, simply connected Lie group, and choose a maximal torus $T \subseteq K$. Let $\mathfrak{k}$ and $\mathfrak{t}$ denote the Lie algebras of K and T respectively. We will use a subscript $\mathbb{C}$ to indicate the appropriate complexification, so that $\mathfrak{k}_\mathbb{C}$ and $\mathfrak{t}_\mathbb{C}$ denote the complexified algebras, while to $K_\mathbb{C}, T_\mathbb{C}$ we assign the corresponding Lie groups. The existence of these complexified groups is a result of the compact and connected hypotheses [Bum13].

Definition 2.1.1. Let S^1 denote the unit circle, though of as a Lie group. We define the *free loop space over* K, denoted LK, to be the Sobolev class H^1 maps $S^1 \to K$:

$$LK = H^1(S^1, K).$$

The based loop group is the subgroup of LK which fixes the identity elements,

$$\Omega K = \{\gamma \in LK : \gamma(e_{S^1}) = e_K\}.$$

As the argument of our maps will be essential to our conversation, one is confronted with a choice as to represent S^1 additively or multiplicatively. The relationship between the two presentations arises from embedding S^1 in $\mathbb{C}$ as follows:

$$\text{if } z = e^{i\theta} \text{ and } w = e^{i\phi} \quad \text{then} \quad zw = e^{i(\theta+\phi)}.$$

Wherever convenient, we will represent S^1 multiplicatively. However, there are times when it is necessary or more advantageous to convert to an additive notation, such as when dealing with derivatives which appear in the symplectic form, discussed in Section 2.1.2. We will specify this as necessary.

The compactness of S^1 ensures that LK is well defined, and moreover is a Hilbert manifold. Pointwise multiplication of loops endows LK with an analytic group structure, taking the constant loop $e(\theta) = e_K$ as the group identity, so we can realize LK as a Hilbert Lie group.

Proposition 2.1.2. *There is an action of LK on ΩK which realizes ΩK as a homogeneous LK-space.*

Proof. Define the map $\Omega K \times LK \to \Omega K$ by $(\beta, \gamma) \mapsto \gamma \cdot \beta = \gamma(1)^{-1}\beta(z)\gamma(z)$. It is straightforward to check that this defines a right LK-action. The stabilizer at the identity is

$$\begin{aligned}
\mathrm{Stab}_{LK}(e) &= \{\gamma \in LK : e_{\Omega K} \cdot \gamma = e_{\Omega K}\} \\
&= \left\{\gamma \in LK : \gamma(1)^{-1}\gamma(z) = e_G, \forall z \in S^1\right\} \\
&= \left\{\gamma \in LK : \gamma(1) = \gamma(z), \forall z \in S^1\right\} \\
&\cong K
\end{aligned}$$

where the last congruence is under the identification of K with the constant loops. This implies that LK/K is a homogeneous LK-space.

Consider the map $\phi : LK/K \to \Omega K$ by $[\gamma(z)] \mapsto \gamma(1)^{-1}\gamma(z)$. This is a well-defined map, since if $[\gamma] = [\lambda]$ then there is some constant loop $g \in K$ such that $\gamma = g\lambda$. Under the image of ϕ we then have

$$\phi(\gamma) = \phi(g\lambda) = (g\lambda)(1)^{-1}g\lambda(z) = \lambda(1)^{-1}g^{-1}g\lambda(z) = \lambda(1)^{-1}\lambda(z) = \phi(\lambda).$$

Bijectivity comes from defining a set theoretic inverse: the map $\psi : \Omega K \to LK/K$ sending $\beta \to [\beta]$. For $\beta \in \Omega K$ and $[\gamma] \in LK/K$ we have

$$\begin{aligned} (\phi \circ \psi)(\beta) &= \phi([\beta]) = \beta(1)^{-1}\beta(z) = \beta && \text{since } \beta(1) = e \\ (\psi \circ \phi)([\gamma]) &= \psi(\gamma(1)^{-1}\gamma(z)) = [\gamma(1)^{-1}\gamma(z)] = [\gamma] && \text{since } \gamma(1) \in K. \end{aligned}$$

Finally we show that ϕ is LK-equivariant. Recall that LK acts on LK/K by right-translation, so that if $\gamma, \lambda \in LK$ then

$$\phi(\gamma \cdot \lambda) = \lambda(1)^{-1} \underbrace{\gamma(1)^{-1}\gamma(\theta)}_{\phi(\gamma)} \lambda(\theta) = \phi(\gamma) \cdot \lambda.$$

Thus $LK/K \cong \Omega K$ as LK-sets, and we conclude that ΩK is a homogeneous LK space. □

The Lie algebra of LK is $L\mathfrak{k} = H^1(S^1, \mathfrak{k})$. The exponential map $\exp : \mathfrak{k} \to K$ defines a pointwise exponential map $L\mathfrak{k} \to LK$ which is bijective in a sufficiently small neighbourhood of $0 \in L\mathfrak{k}$. Of note is that $L\mathfrak{k}$ admits a central extension $\widehat{L\mathfrak{k}}$ which exponentiates to give a central extension of the group $\widehat{LK}$. These are examples of Kac-Moody Lie algebras/groups, though we will not dwell on this particular subject.

One can use the roots of $\mathfrak{k}_\mathbb{C}$ to deduce the appropriate root system for $L\mathfrak{k}_\mathbb{C}$. Choose a faithful unitary representation of K so that $K_\mathbb{C}$ is realized as a subgroup of $\mathrm{SL}_n\mathbb{C}$. Let Φ denote the roots of $\mathfrak{k}_\mathbb{C}$, so that the root decomposition is expressed as

$$\mathfrak{k}_\mathbb{C} = \mathfrak{t}_\mathbb{C} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{k}_\alpha.$$

Using the Fourier decomposition that comes from realizing $\mathfrak{k}_\mathbb{C} \subseteq \mathfrak{sl}_n\mathbb{C}$ gives

$$L\mathfrak{k}_\mathbb{C} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{k}_\mathbb{C} z^k = \bigoplus_{k \in \mathbb{Z}} \mathfrak{t}_\mathbb{C} z^k \oplus \bigoplus_{(k,\alpha) \in \mathbb{Z} \times \Phi} \mathfrak{k}_\alpha z^k.$$

The root system for $L\mathfrak{k}_\mathbb{C}$ may then be identified with $\mathbb{Z} \times \Phi$, which we will refer to as the *affine root system*. The new root spaces $L\mathfrak{k}_{n,\alpha}$ consists of the algebraic morphisms $\mathbb{C}^\times \to \mathfrak{k}_\alpha$ of homogeneous degree n.

If $W = N(T)/T$ is the Weyl group for $\mathfrak{k}_\mathbb{C}$ it is well known that W permutes the roots in Φ. There is a corresponding *affine Weyl group* $\mathcal{W}$ which permutes the affine root system, and may be given as the normalizer of a torus $\mathcal{W} = N(S^1 \times T)/(S^1 \times T)$, where here $S^1 \times T$ is thought of as sitting in $S^1 \ltimes LK$. Alternatively, if $\{\alpha_1, \ldots, \alpha_\ell\}$ are a choice of simple roots of $\mathfrak{k}_\mathbb{C}$, let s_i be the simple reflections about the hyperplanes $\ker \alpha_i$, known to generate W. If $\tilde{\alpha}$ is the highest root of $K_\mathbb{C}$, let s_0 be the reflection about the affine hyperplane $\{x \in \mathfrak{t}_\mathbb{C} : \tilde{\alpha}(x) = -1\}$. The affine Weyl group is then generated by $\{s_0, s_1, \ldots, s_\ell\}$.

One may then generalize the notion of positive and simple roots to those of positive and simple *affine roots*, together with chambers generalizing to alcoves and the like. For our purpose, the most important point is that the simple affine roots are precisely

$$\Sigma = \{(0,\alpha_1),\ldots,(0,\alpha_\ell),(-1,\tilde{\alpha})\}.$$

We will use these simple roots to construct parabolic subgroups in Section 2.1.6, though the following proposition gives an alternative characterization.

Proposition 2.1.3 ([PS86, Proposition 5.1.2])**.** *If $X_*(T_\mathbb{C})$ denotes the coweights of $K_\mathbb{C}$ with respect to its maximal torus $T_\mathbb{C}$, then $\mathcal{W} \cong X_*(T_\mathbb{C}) \rtimes W$.*

The corresponding action of $\mathcal{W}$ on $\mathbb{Z}\times\Phi$ is given by

$$(\lambda, w)\cdot(n,\alpha) = (n+\alpha(\lambda), w\alpha).$$

This discussion naturally restricts to the based loop group, whose Lie algebra is $\Omega\mathfrak{k} \cong L\mathfrak{k}/\mathfrak{k}$ consisting of those class H^1 loops $\xi : S^1 \to \mathfrak{k}$ such that $\xi(e) = 0$.

Our conversation thus far has focused on those loops of Sobolev class H^1 between S^1 and K, though we need not limit ourselves to this regularity. Other common choices include the analytic maps, smooth maps, and algebraic maps. In imposing such stringent conditions on our maps, the free loop space trades its status as a Hilbert manifold for that of a Banach manifold. In fact, we shall make great use of the algebraic maps in Section 2.1.4, but will otherwise not dwell upon the technical differences.

2.1.2 The Kähler Structure on ΩK

By compactness, $\mathfrak{k}$ admits an invariant inner product $\langle\cdot,\cdot\rangle$,

$$\langle \operatorname{Ad}_g \xi, \operatorname{Ad}_g \eta\rangle = \langle \xi,\eta\rangle, \qquad g\in G, \quad \xi,\eta\in\mathfrak{k}.$$

As K is connected, this is equivalent to

$$\langle \operatorname{ad}_\xi \eta, \zeta\rangle + \langle \eta, \operatorname{ad}_\xi \zeta\rangle = 0, \qquad \xi,\eta,\zeta\in\mathfrak{k}.$$

Define the following 2-cocycle on $L\mathfrak{k}$,

$$\omega_e(\xi,\eta) = \frac{1}{2\pi}\int_0^{2\pi} \big\langle \xi(\theta), \eta'(\theta)\big\rangle\, d\theta, \qquad \xi,\eta\in L\mathfrak{k}, \tag{2.1}$$

where we have written S^1 additively, to account for the fact that we are expressing these as real rather than complex derivatives.

Proposition 2.1.4. *Equation (2.1) defines a 2-cocycle, and so satisfies the cocycle condition*

$$\omega([\xi,\eta],\mu) + \omega([\nu,\mu],\xi) + \omega([\mu,\xi],\eta) = 0.$$

Proof. We make extensive use of the Leibniz identity for the Lie bracket:

$$\frac{\mathrm{d}}{\mathrm{d}\theta}[\xi(\theta), \eta(\theta)] = [\xi'(\theta), \eta(\theta)] + [\xi(\theta), \eta'(\theta)].$$

The invariance of $\langle \cdot, \cdot \rangle$ becomes important in this next step, as

$$\begin{aligned}
\omega([\xi, \eta], \mu) + \omega([\eta, \mu], \xi) + \omega([\mu, \xi], \eta) &= \frac{1}{2\pi} \int_0^{2\pi} \big\langle [\xi, \eta], \mu' \big\rangle + \big\langle [\eta, \mu], \xi' \big\rangle + \big\langle [\mu, \xi], \eta' \big\rangle \, d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \big\langle \xi, [\eta, \mu'] \big\rangle - \big\langle \xi, [\eta, \mu]' \big\rangle + \big\langle \xi, [\eta', \mu] \big\rangle \, d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \big\langle \xi, [\eta, \mu'] - [\eta, \mu]' + [\eta', \mu] \big\rangle \, d\theta \\
&= 0.
\end{aligned}$$

In the last inequality, we used the Leibniz property of the Lie bracket to deduce that the second argument of the inner product is zero. □

This cocycle restricts two a symplectic 2-form on ΩK as follows: As ΩK is a Lie group it is parallelizable, so its tangent spaces are modelled by the pushforward of the Lie algebra by an appropriate multiplication. For the sake of concreteness, let $L_\gamma : \Omega K \to \Omega K$ be the left-multiplication map for $\gamma \in \Omega K$, so that $T_\gamma \Omega K = d_e L_\gamma \Omega \mathfrak{k}$. The symplectic form is just the pullback of the cocycle ω by $L_{\gamma^{-1}}$, so that if $\xi, \eta \in T_\gamma \Omega K$ then

$$\begin{aligned}
\omega_\gamma(\xi, \eta) &= \left(L^*_{\gamma^{-1}} \omega_e \right)_\gamma (\xi, \eta) \\
&= \omega_e \left(d_e L_{\gamma^{-1}} \xi, d_e L_{\gamma^{-1}} \eta \right) \\
&= \frac{1}{2\pi} \int_0^{2\pi} \Big\langle \gamma(\theta)^{-1} \xi(\theta), \big(\gamma(\theta)^{-1} \eta(\theta) \big)' \Big\rangle \, d\theta.
\end{aligned}$$

Proposition 2.1.5. *The 2-cocycle as defined on ΩK is a symplectic form.*

Proof. It suffices to show that ω_e is a non-degenerate, skew-symmetric 2-form. Using the Leibniz formula on the inner product gives rise to an integration-by-parts formula, which when applied directly to (2.1) yields

$$\begin{aligned}
\omega(\xi, \eta) &= \frac{1}{2\pi} \int_0^{2\pi} \big\langle \xi(\theta), \eta'(\theta) \big\rangle \, d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \frac{\mathrm{d}}{\mathrm{d}\theta} \langle \xi(\theta), \eta(\theta) \rangle \, d\theta - \frac{1}{2\pi} \int_0^{2\pi} \big\langle \xi'(\theta), \eta(\theta) \big\rangle \, d\theta \\
&= \frac{1}{2\pi} \langle \xi(\theta), \eta(\theta) \rangle \Big|_0^{2\pi} - \frac{1}{2\pi} \int_0^{2\pi} \big\langle \eta(\theta), \xi'(\theta) \big\rangle \, d\theta \\
&= -\omega(\eta, \xi).
\end{aligned}$$

Here we have used the fact that $\xi(\theta), \eta(\theta)$ are loops so when evaluated from 0 to 2π their difference is zero. To show that this form is closed, we recall that for any two form ω we can write $d\omega$ as

$$\begin{aligned} d\omega(X,Y,Z) &= X\omega(Y,Z) - Y\omega(X,Z) + Z\omega(X,Y) \\ &\qquad - \omega([X,Y],Z) + \omega([X,Z],Y) - \omega([Y,Z],X) \\ &= X\omega(Y,Z) - Y\omega(X,Z) + Z\omega(X,Y) \end{aligned}$$

Where we have exploited cocyclicity of ω to annihilate the last three terms. To see that the first three terms also vanish, note that for fixed vector fields X, Y, the function

$$\omega(X,Y) : \Omega K \to \mathbb{R}, \qquad \beta \mapsto \omega_\beta(X_\beta, Y_\beta)$$

is constant with respect to the parameter β. This is seen immediately, since ω is defined via the pullback to the Lie algebra and so only depends on the generating vectors for X and Y. Hence the action of a vector field on $\omega(X,Y)$ will just yield zero, and we are able to conclude that $d\omega = 0$.

The previous two proofs made no explicit use of our restriction to the based loop group and in fact hold for LK in general. It is the property of non-degeneracy which requires the restriction to ΩK. It suffices to show that for any $\xi \in \Omega\mathfrak{k}$ there is an $\eta \in \Omega\mathfrak{k}$ such that $\omega(\xi,\eta) \neq 0$. Let $\xi = \eta'$ in which case we have that

$$\omega(\xi,\eta) = \frac{1}{2\pi}\int_0^{2\pi} \langle \xi, \eta' \rangle \, d\theta = \frac{1}{2\pi}\int_0^{2\pi} \langle \xi, \xi \rangle \, d\theta \geq 0.$$

Since $\langle \cdot, \cdot \rangle$ is an inner product, this is always non-negative and is zero precisely when $\xi = 0$. This occurs only if η is a constant loop, but the only constant loop in $\Omega\mathfrak{k}$ is the 0 loop. This is because any other constant loop would project onto a non-identity constant loop in ΩK, for which there are none. We conclude that that the form is non-degenerate. □

The identification of $\Omega K \cong LK/K$ through Proposition 2.1.2 caries through to the Lie algebra level so that $\Omega\mathfrak{k} \cong L\mathfrak{k}/\mathfrak{k}$. Similarly, the complexification carries through this to yield $\Omega\mathfrak{k}^{\mathbb{C}} \cong L\mathfrak{k}^{\mathbb{C}}/\mathfrak{k}^{\mathbb{C}}$. Each $\xi \in \Omega\mathfrak{k}^{\mathbb{C}}$ has a Fourier decomposition of the form

$$\xi(z) = \sum_{k \neq 0} \xi_k z^k, \qquad \xi_k \in \mathfrak{k}^{\mathbb{C}}, z = e^{i\theta},$$

and an element $\xi \in \Omega\mathfrak{k}^{\mathbb{C}}$ corresponds to an element of the real Lie algebra $\Omega\mathfrak{k}$ precisely when it is invariant under complex conjugation; that is, $\xi_k = \overline{\xi}_{-k}$. Hence we can endow $\Omega\mathfrak{k}$ with a complex structure by giving $\Omega\mathfrak{k}^{\mathbb{C}}$ a complex structure. Define $J_0 : \Omega\mathfrak{k}^{\mathbb{C}} \to \Omega\mathfrak{k}^{\mathbb{C}}$ by

$$J_0\xi(z) = i\sum_{k>0} \xi_k z^k - i\sum_{k<0} \xi_k z^k.$$

It is easy to see that

$$\begin{aligned} J_0^2 \xi(z) &= J_0 \left(i \sum_{k>0} \xi_k z^k - i \sum_{k<0} \xi_k z^k \right) \\ &= i^2 \sum_{k>0} \xi_k z^k + (-i)^2 \sum_{k<0} \xi_k z^k \\ &= - \sum_{k>0} \xi_k z^k - \sum_{k<0} \xi_k z^k \\ &= -\xi(z) \end{aligned}$$

so that J_0 is almost complex. Extending J_0 to an almost complex structure J on all of $T\Omega K$ by left-translation, we need only show that J is integrable. This follows by Birkhoff's theorem, and can be found in [PS86; Ser10].

Proposition 2.1.6. *The complex structure J is a symplectomorphism, and $g(\xi, \eta) = \omega(\xi, J\eta)$ defines a Riemannian metric on ΩK.*

Proof. That J is a symplectomorphism can be seen by a straightforward computation:

$$\begin{aligned} \omega(\xi, \eta) &= \frac{1}{2\pi} \int_0^{2\pi} \left\langle \sum_{k \neq 0} \xi_k z^k, \sum_{\ell \neq 0} \eta_\ell z^\ell i\ell \right\rangle d\theta \\ &= -\frac{i}{2\pi} \int_0^{2\pi} \sum_{k,\ell \neq 0} \left\langle \xi_k e^{ik\theta}, \ell \eta_\ell e^{il\theta} \right\rangle \\ &= -\frac{i}{2\pi} \sum_{k \neq 0} \int_0^{2\pi} \langle \xi_k, -k\eta_{-k} \rangle \, d\theta \\ &= i \sum_{k \neq 0} k \, \langle \xi_k, \eta_{-k} \rangle . \end{aligned}$$

On the other hand,

$$\begin{aligned} \omega(J\xi, J\eta) &= \frac{1}{2\pi} \int_0^{2\pi} \left\langle i \sum_{k \neq 0} \operatorname{sgn}(k) \xi_k e^{i\theta k}, i \sum_{\ell \neq 0} \operatorname{sgn}(\ell) \eta_\ell (i\ell) e^{i\ell\theta} \right\rangle d\theta \\ &= -\frac{i}{2\pi} \int_0^{2\pi} \sum_{k \neq 0} \operatorname{sgn}(k) \operatorname{sgn}(-k) k \, \langle \xi_k, \eta_{-k} \rangle \\ &= i \sum_{k \neq 0} k \, \langle \xi_k, \eta_{-k} \rangle \end{aligned}$$

and we conclude that $\omega(J\xi, J\eta) = \omega(\xi, \eta)$ as required.

To see that g is a Riemannian metric, we use the fact that $\xi \in \Omega\mathfrak{k}$ if and only if $\xi_k = \bar{\xi}_{-k}$:

$$\begin{aligned} g(\xi,\xi) = \omega(\xi, J\xi) &= i\left[\sum_{k>0} k\left\langle \xi_k, i\xi_{-k}\right\rangle + \sum_{k<0} k\left\langle \xi_k, -i\xi_{-k}\right\rangle\right] \\ &= i\left[-i\sum_{k>0} k\left\langle \xi_k, \xi_{-k}\right\rangle + i\sum_{k<0} k\left\langle \xi_k, \xi_{-k}\right\rangle\right] \\ &= 2\sum_{k>0} k\left\langle \xi_k, \xi_{-k}\right\rangle \\ &= 2\sum_{k>0} k\left\langle \xi_k, \overline{\xi_k}\right\rangle \end{aligned}$$

However, the inner product is a *real* inner product over $\mathfrak{k}$. In the complex setting, pairing a vector with its conjugate over a real inner product is still positive definite, hence it is clear that $g(\xi,\xi) > 0$ and is zero only if $\xi_k \equiv 0$ for all k; that is, when $\xi = 0$. □

2.1.3 Hamiltonian Actions and Moment Maps

Identify K with the subgroup of LK consisting of constant maps whose image lies in K. We can restrict the natural conjugation action of LK on itself to those constant maps

$$(k\cdot\gamma)(z) = k\gamma(z)k^{-1}.$$

On the other hand, a natural S^1-action may be induced by loop-rotation

$$(s\cdot\gamma)(z) = \gamma(sz)\gamma(s)^{-1}. \tag{2.2}$$

We are more interested in the corresponding actions on ΩK, which accounts for the presence of the $\gamma(s)^{-1}$ term in (2.2) which acts to re-base the loop. It is straightforward to verify that these independently define left-actions on ΩK. More importantly, the actions commute and hence can be combined into an $S^1 \times T$-action.

Proposition 2.1.7. *The S^1- and T-actions on ΩK commute, and hence define an $S^1 \times T$-action on ΩK.*

Proof. Let $t \in T$ and $s \in S^1$ so that

$$\begin{aligned} (t\cdot(s\cdot\gamma))(z) &= t\cdot\left(\gamma(sz)\gamma^{-1}(s)\right) \\ &= t\gamma(sz)\gamma^{-1}(s)t^{-1} \\ (s\cdot(t\cdot\gamma))(z) &= s\cdot\left(t\gamma(z)t^{-1}\right) \\ &= \lambda(zs)\lambda^{-1}(s) \qquad\qquad \lambda(z) = t\gamma(z)t^{-1} \\ &= t\gamma(zs)t^{-1}\left(t\gamma(s)t\right)^{-1} \\ &= t\gamma(zs)\gamma^{-1}(s)t^{-1} \end{aligned}$$

in which case we conclude that $t \cdot (s \cdot \gamma) = s \cdot (t \cdot \gamma)$ as required. □

Theorem 2.1.8. *The $S^1 \times K$-action on ΩK is a Hamiltonian group action whose moment map is $\mu : \Omega K \to \mathbb{R} \oplus \mathfrak{k}, \gamma \mapsto (E(\gamma), p(\gamma))$, where*

$$E(\gamma) = \frac{1}{4\pi}\int_0^{2\pi} \left\|\gamma'(\theta)\gamma(\theta)^{-1}\right\|^2 \mathrm{d}\theta, \qquad p(\gamma) = \frac{1}{2\pi}\int_0^{2\pi} \gamma'(\theta)\gamma(\theta)^{-1}\mathrm{d}\theta.$$

Here the norm is induced by our invariant inner product, and we have used that same inner product to identify the Lie algebra with its dual.

Proof. The choice of a left- versus right-trivialization of the tangent spaces $T_\gamma \Omega K$ is important for the map p, but not for E, due to the Ad-invariance of the inner product. As such we will actually use $\gamma^{-1}\gamma'$ for E and $\gamma'\gamma^{-1}$ for p. See Remark 2.1.9 for more details.

The more simple of the two cases is that of the energy map, since it corresponds to an S^1 action. Let $X \in \mathrm{Lie}(S^1) \cong \mathbb{R}$ for which we may compute its fundamental vector field as

$$\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(tX)\cdot\gamma(z) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left[\gamma(ze^{tX})\gamma(e^{tX})^{-1}\right] \\
&= \gamma'(ze^{tX})(+X)\gamma(e^{tX})^{-1} \\
&\qquad - \gamma(ze^{tX})\gamma(e^{tX})^{-1}\gamma'(e^{tX})\gamma(e^{tX})^{-1}(X)\,\Big|_{t=0} \\
&= \left[\gamma'(z) - \gamma(z)\gamma'(0)\right] X
\end{aligned}$$

where we have used the fact that $\gamma\gamma^{-1} = e$ to derive

$$\frac{\mathrm{d}\gamma^{-1}}{\mathrm{d}t} = -\gamma^{-1}\frac{\mathrm{d}\gamma}{\mathrm{d}t}\gamma^{-1}.$$

Our goal is to show that for any $X \in \mathfrak{k} \cong \mathbb{R}$ and induced fundamental vector field $X^\#$ we have

$$\iota_{X^\#}\omega = dE.$$

To do this, let $\lambda : [0, 1] \to \Omega K$ with $\lambda(0) = \gamma(\theta)$ and $\lambda'(0) = \gamma(\theta)Y$ for some $Y \in \Omega\mathfrak{k}$. For simplicity sake, we may denote $\lambda(t,\theta) = [\lambda(t)](\theta)$ in which case

$$\begin{aligned} dE_\gamma(\gamma Y) = dE_{\lambda(0)}(\lambda'(0)) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (E \circ \lambda) \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \frac{1}{4\pi}\int_0^{2\pi} \left\langle \lambda(t,\theta)^{-1}\partial_\theta\lambda(t,\theta), \lambda(t,\theta)^{-1}\partial_\theta\lambda(t,\theta)\right\rangle d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \left\langle \partial_t\lambda^{-1}\partial_\theta\lambda + \partial_{t\theta}\lambda, \lambda^{-1}\partial_\theta\lambda\right\rangle\big|_{t=0} d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \left\langle -\gamma^{-1}(\gamma Y)\gamma^{-1}\gamma' + \gamma^{-1}(\gamma Y)', \gamma^{-1}\gamma'\right\rangle d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \left\langle -Y\gamma^{-1}\gamma' + \gamma^{-1}\gamma' Y + Y', \gamma^{-1}\gamma'\right\rangle d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \left\langle [\gamma^{-1}\gamma', Y] + Y', \gamma^{-1}\gamma'\right\rangle d\theta. \end{aligned} \tag{2.3}$$

By the ad-invariance of the inner-product we have

$$\left\langle [\gamma^{-1}\gamma', Y], \gamma^{-1}\gamma'\right\rangle = \left\langle [\gamma^{-1}\gamma', \gamma^{-1}\gamma'], Y\right\rangle = 0.$$

so that one of the terms in (2.3) vanishes. Furthermore,

$$\frac{1}{2\pi}\int_0^{2\pi} \left\langle Y', \gamma'(0)\right\rangle d\theta = \left\langle Y(\theta), \gamma'(0)\right\rangle \Big|_{\theta=0}^{2\pi} = 0 \tag{2.4}$$

since $Y(0) = Y(2\pi) = 0$. Thus we may add the term given in (2.4) to that of (2.3) which then yields

$$\begin{aligned} dE_\gamma(\gamma Y) &= \frac{1}{2\pi}\int_0^{2\pi} \left\langle Y', \gamma^{-1}\gamma'\right\rangle d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \left\langle Y', \gamma^{-1}\gamma' - \gamma'(0)\right\rangle d\theta \\ &= \omega(Y, \gamma^{-1}\gamma' - \gamma'(0)) \\ &= (\iota_{X^\#}\omega)_\gamma(\gamma Y) \end{aligned}$$

where in the last step, we recall that the inner product translates the vector back to the Lie algebra in order to perform the calculation.

For the momentum calculation, we proceed in a similar fashion to that of the energy functional and begin by computing the fundamental vector field for the T-action. If $X \in \mathfrak{t}$ and

$\gamma \in \Omega K$ then

$$\begin{aligned} X_\gamma^\# &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \exp(tX)\gamma(\theta) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \exp(tX)\gamma(\theta)\exp(-tX) \\ &= X\gamma(\theta) - \gamma(\theta)X \\ &= [X, \gamma(\theta)]. \end{aligned}$$

We would like to show that

$$\iota_{X^\#}\omega = d\left\langle p(\cdot), X\right\rangle,$$

and here we respect the right-trivialization of the momentum map. If $\lambda : [0,1] \to \Omega K$ satisfies $\lambda(0) = \gamma$ and $\partial_t\lambda(0) = \gamma Y$ then one derives

$$\begin{aligned} \left[d_\gamma \left\langle p(\gamma), X\right\rangle\right](\gamma Y) &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \frac{1}{2\pi}\int_0^{2\pi} \left\langle X, (\partial_\theta\lambda)\lambda^{-1}\right\rangle \mathrm{d}\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \left\langle X, (\partial_\theta\partial_t\lambda)\lambda^{-1} + (\partial_\theta\lambda)(\partial_t\lambda^{-1})\right\rangle_{t=0} \mathrm{d}\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \left\langle X, (\gamma Y)'\gamma^{-1} - \gamma'\gamma^{-1}\gamma Y\gamma^{-1})\right\rangle \mathrm{d}\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \left\langle X, \gamma' Y\gamma^{-1} + \gamma Y'\gamma^{-1} - \gamma' Y\gamma^{-1}\right\rangle \mathrm{d}\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \left\langle X, \gamma Y'\gamma^{-1}\right\rangle \mathrm{d}\theta \end{aligned}$$

On the other hand, we have

$$\omega\gamma([X,\gamma(\theta)], \gamma Y) = \frac{1}{2\pi}\int_0^{2\pi} \left\langle \gamma^{-1}[X,\gamma(\theta)], (\gamma^{-1}\gamma Y)'\right\rangle \mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} \left\langle \gamma^{-1}X\gamma - X, Y'\right\rangle \mathrm{d}\theta$$

Note that

$$\frac{1}{2\pi}\int_0^{2\pi} \left\langle X, Y'\right\rangle \mathrm{d}\theta = \left\langle X, Y\right\rangle_0^{2\pi} = 0$$

since $Y(0) = Y(2\pi) = 0$, while

$$\frac{1}{2\pi}\int_0^{2\pi} \left\langle \gamma^{-1}X\gamma, Y'\right\rangle \mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} \left\langle X, \gamma Y'\gamma^{-1}\right\rangle \mathrm{d}\theta$$

by Ad-invariance of the inner product. Both facts combined yield

$$\left[d_\gamma \left\langle p(\gamma), X\right\rangle\right](\gamma Y) = \frac{1}{2\pi}\int_0^{2\pi} \left\langle X, \gamma Y'\gamma^{-1}\right\rangle \mathrm{d}\theta = \omega_\gamma\left([X,\gamma], \gamma Y\right)$$

as required. □

Remark 2.1.9. In verifying that p is indeed the momentum map for the corresponding K action, it was essential that we used the right-trivialization of the tangent spaces. Indeed, using the left-trivialization results in a computational quagmire that ultimately fails to demonstrate the desired result, together with several incorrect consequences regarding the image of one-parameter subgroups under the $E \times p$ map. This is an easy fact to overlook as it does not affect the overall moment map image. We hope this comment obviates hours of frustration for the reader who finds him/herself unable to prove ostensibly straightforward facts.

2.1.4 Algebraic Loops

Critical to our study will be the *algebraic based loop group* $\Omega_{\text{alg}}K$, which arises as the dense subgroup of ΩK whose maps are the restriction of algebraic maps $\mathbb{C}^\times \to K_\mathbb{C}$. This group exhibits many of the properties of a finite dimensional reductive algebraic group, which we will exploit in turn to derive many of our results. The most important property is that of the Bruhat decomposition, discussed further in Section 2.1.6.

Let $L_{\text{alg}}K_\mathbb{C} = \{f : \mathbb{C}^\times \to K^\mathbb{C} : f \text{ algebraic}\}$ and $L^+_{\text{alg}}K_\mathbb{C}$ be those maps which extend holomorphically to the interior of the unit disk. In the language of algebraic geometry, set $\mathcal{O} = \mathbb{C}[z]$ and $\mathcal{K} = \mathbb{C}[z, z^{-1}]$ so that $L_{\text{alg}}K_\mathbb{C}$ may be identified with $K_\mathbb{C}(\mathcal{K})$ and $L^+_{\text{alg}}K_\mathbb{C}$ with $K_\mathbb{C}(\mathcal{O})$, the $\mathcal{K}$- and $\mathcal{O}$-valued points of $K_\mathbb{C}$ respectively.

In the literature one often sees the definitions $\mathcal{O} = \mathbb{C}[[z]]$ and $\mathcal{K} = \mathbb{C}((z))$, indicating the use of the ring of formal power series and Laurent series respectively. The rings $\mathbb{C}[z]$ and $\mathbb{C}[z, z^{-1}]$ are the "finite" analogs of these rings. Our interest will be limited to studying the corresponding quotient $K_\mathbb{C}(\mathcal{K})/K_\mathbb{C}(\mathcal{O})$, in which case the distinction between power series and polynomials vanishes [Lus83].

In much of our discussion of the algebraic loops, it will be useful to have an explicit characterization in terms of Fourier series. Choosing a faithful unitary representation realizes K as a subgroup of $\text{SU}(n)$, so that $\Omega_{\text{alg}}K \subseteq \Omega_{\text{alg}}\text{SU}(n)$ in turn. Maps in the latter space admit a description by Fourier series:

$$\Omega_{\text{alg}}\text{SU}(n) = \left\{ \sum A_k z^k : A_k \neq 0 \text{ for finitely many } k \right\}.$$

Note that the A_k are $n \times n$ matrices, but need not be elements of $\text{SU}(n)$.

We can use this representation to filter $\Omega_{\text{alg}}K$ over $\mathbb{N}$. For any fixed $m \in \mathbb{N}$ define the space

$$\Omega_{\text{alg},m}K = \left\{ \gamma \in \Omega_{\text{alg}}K : \gamma(z) = \sum_{k=-m}^{m} A_k z^k \right\}. \tag{2.5}$$

It is easy to check that the $\Omega_{\text{alg},m}K$ are invariant under the $S^1 \times K$-action, owing to the fact that S^1 preserves the basis element z^k and the K action acts only on the A_k. As such, we have an equivariant filtration of $\Omega_{\text{alg}}K$:

$$K = \Omega_{\text{alg},0}K \subseteq \Omega_{\text{alg},1}K \subseteq \Omega_{\text{alg},2}K \subseteq \cdots \subseteq \Omega_{\text{alg}}K.$$

In general these $\Omega_{\mathrm{alg},m}K$ are singular varieties and so do not interact nicely with our smooth structures. However, they will serve as a conduit for a smooth embedding, to be discussed in Section 2.1.5.

2.1.5 The Grassmannian Model

Herein we wish to realize $L_{\mathrm{alg}}K_{\mathbb{C}}$ as operators on a Hilbert space; an objective which will serve several purposes. The first will be an identification of the algebraic based loop group with the affine Grassmannian discussed in Section 2.1.6. That identification alone will endow $\Omega_{\mathrm{alg}}K$ with various filtrations and decompositions which we will be invaluable to our study of the group's convexity and cohomology. Secondly, this model will make obvious an embedding of our loops into an infinite dimensional Grassmannian which shares many of the attributes of its finite dimensional analogs.

Our treatment will focus on the algebraic loops, though a similar procedure holds for the general Sobolev class H^1 loops. For an in depth treatment, we refer the reader to [PS86].

For the moment, let K be a centerfree compact group with Lie algebra $\mathfrak{k}$. Define $\mathcal{H}^{\mathfrak{k}} = L^2(S^1, \mathfrak{k}_{\mathbb{C}})$ to be those loops which are square integrable with respect to the Hermitian inner product $\langle x, y\rangle_{\mathfrak{k}_{\mathbb{C}}}$ induced by the Killing form on $\mathfrak{k}_{\mathbb{C}}$. The group $L_{\mathrm{alg}}K_{\mathbb{C}}$ acts naturally on $\mathcal{H}^{\mathfrak{k}}$ via the adjoint action induced pointwise

$$(\gamma \cdot f)(z) = \mathrm{Ad}_{\gamma(z)} f(z), \qquad \gamma \in L_{\mathrm{alg}}K_{\mathbb{C}}, f \in \mathcal{H}^{\mathfrak{k}}.$$

If the complex dimension of $\mathfrak{k}_{\mathbb{C}}$ is n, choose a basis $\{\epsilon_i\}_{i=1}^n$ for $\mathfrak{k}_{\mathbb{C}}$ so that

$$\left\{\epsilon_i z^k : i \in \{1, \ldots, n\}, k \in \mathbb{Z}\right\}$$

is a basis for $\mathcal{H}^{\mathfrak{k}}$. We polarize $\mathcal{H}^{\mathfrak{k}} = \mathcal{H}_+ \oplus \mathcal{H}_-$ by taking $\mathcal{H}_+ = \mathrm{span}_{k \geq 0}\left\{\epsilon_i z^k\right\}$ and setting $\mathcal{H}_-$ to be its orthogonal complement in the L^2 inner product. The *Grassmannian* of $\mathcal{H}^{\mathfrak{k}}$ is

$$\mathrm{Gr}(\mathcal{H}^{\mathfrak{k}}) = \left\{W \subseteq \mathcal{H}^{\mathfrak{k}} : {\scriptstyle \mathrm{pr}_+ : W \to \mathcal{H}_+ \text{ Fredholm} \atop \mathrm{pr}_- : W \to \mathcal{H}_- \text{ Hilbert-Schmidt}}\right\},$$

where $\mathrm{pr}_\pm : W \to \mathcal{H}_\pm$ is the orthogonal projection map. The requirement that the projections be Fredholm and Hilbert-Schmidt manifest

Proposition 2.1.10 ([PS86, Theorem 7.1.2])**.** *The space* $\mathrm{Gr}(\mathcal{H}^{\mathfrak{k}})$ *is an infinite dimensional Hilbert manifold, modelled on the space of Hilbert-Schmidt operators* $\mathrm{HS}(\mathcal{H}_+, \mathcal{H}_-)$*.*

One can describe the coordinate chart about any given point as follows: if $W \in \mathrm{Gr}(\mathcal{H}^{\mathfrak{k}})$ then the coordinate chart centred at W is $U_W := \mathrm{HS}(W, W^\perp)$, the space of Hilbert-Schmidt maps $W \to W^\perp$. Additionally, $\mathrm{Gr}(\mathcal{H}^{\mathfrak{k}})$ has a Kähler form at $T_{\mathcal{H}_+}\mathrm{Gr}(\mathcal{H}^{\mathfrak{k}})$ given by

$$\omega_{\mathrm{HS}}(X, Y) = -i\mathrm{Tr}(X^*Y - Y^*X), \qquad X, Y \in \mathrm{HS}(\mathcal{H}_+, \mathcal{H}_-). \tag{2.6}$$

These points will be invaluable in the proof of Proposition 2.1.14.

Analogous to (2.5) we define the following subspace:

$$\mathrm{Gr}_{0,m}(\mathcal{H}^{\mathfrak{k}}) = \left\{ W \in \mathrm{Gr}(\mathcal{H}^{\mathfrak{k}}) : z^m \mathcal{H}_+ \subseteq W \subseteq z^{-m}\mathcal{H}_+ \right\}.$$

As each $\mathrm{Gr}_{0,m}(\mathcal{H}^{\mathfrak{k}})$ is homogeneous beyond the $z^{\pm m}$ basis component, one can quickly check that $\mathrm{Gr}_{0,m}(\mathcal{H}^{\mathfrak{k}})$ is isomorphic to the finite dimensional Grassmannian of complex nm-dimensional subspaces of $\mathbb{C}^{2nm}$ via the identification $W \mapsto W/z^m\mathcal{H}_+$. For the sake of notation, we will explicitly indicate when we are conceptualizing the finite dimensional Grassmannian by writing $\mathcal{G}_m := \mathrm{Gr}_{\mathbb{C}}(nm, 2nm)$.

The *algebraic Grassmannian* $\mathrm{Gr}_0(\mathcal{H}^{\mathfrak{k}})$ is the dense subspace of $\mathrm{Gr}(\mathcal{H}^{\mathfrak{k}})$ formed by the union

$$\mathrm{Gr}_0(\mathcal{H}^{\mathfrak{k}}) := \bigcup_k \mathrm{Gr}_{0,m}(\mathcal{H}^{\mathfrak{k}})$$

and as such has a natural $\mathbb{N}$-indexed filtration given by the $\mathcal{G}_m$. Additionally, we can define a $\mathbb{C}^\times \times K_{\mathbb{C}}$ action on $\mathrm{Gr}_0(\mathcal{H}^{\mathfrak{k}})$ as follows: Let $W \in \mathrm{Gr}_0(\mathcal{H}^{\mathfrak{k}})$,

1. For $a \in K_{\mathbb{C}}, a \cdot W = \{\mathrm{Ad}_a f(z) : f \in W\}$,
2. For $s \in \mathbb{C}^\times, s \cdot W = \{f(sz) : f \in W\}$.

It is easy to check that both are group actions, and in fact we recognize that the $K_{\mathbb{C}}$-action is just the $L_{\mathrm{alg}}K_{\mathbb{C}}$ action on $\mathcal{H}^{\mathfrak{k}}$ restricted to the constant maps.

With the Plücker embedding, finite dimensional Grassmannians are closed submanifolds of projective space and hence are Kähler themselves. The Fubini-Study form on the Grassmannian is the pullback of the Fubini-Study form on projective space, and in local coordinates is given by $\omega'_{FS} = i\partial\bar{\partial}\log(\det(Z^*Z))$. The following lemma ensures that the symplectic structures of the Grassmannian filtration are all mutually compatible with the one inherited as a submanifold of $\mathrm{Gr}_0(\mathcal{H}^{\mathfrak{k}})$.

Lemma 2.1.11 ([Har+06, Proposition 2.3])**.** *Let $M = \binom{2nm}{nm}$ and $(\mathbb{P}^M, \omega_{FS})$ be projective space with the Fubini-Study form. If $(\mathrm{Gr}_0(\mathcal{H}^{\mathfrak{k}}), \omega_{HS})$ is the algebraic Grassmannian with the Hilbert-Schmidt form, and $p : \mathcal{G}_m \hookrightarrow \mathbb{P}^{M-1}$ is the Plücker embedding, then the restriction of ω_{HS} to $\mathcal{G}_m$ is precisely $\omega'_{FS} = p^*\omega_{FS}$.*

As alluded to earlier, these Grassmannians will serve as hosts for our embedding. The set of interest, the image of the embedding, is given in the following definition:

Definition 2.1.12. Define the set $\mathrm{Gr}_0^{\mathfrak{k}}$ to be the collection of subspaces $W \in \mathrm{Gr}_0(\mathcal{H}^{\mathfrak{k}})$ satisfying

1. $zW \subseteq W$,
2. $zW = \overline{W}^{\perp}$,

3. If W_{sm} consists of the smooth maps, then W_{sm} is involutive under the Lie bracket on $\mathfrak{k}$,

where by zW we mean multiplication by z, or shifted upwards in the corresponding Fourier expansion.

Intersecting with the filtration of $\mathrm{Gr}_0(\mathcal{H}^{\mathfrak{k}})$ yields a filtration of $\mathrm{Gr}_0^{\mathfrak{k}}$ by the components $\mathrm{Gr}_{0,m}^{\mathfrak{k}} := \mathcal{G}_m \cap \mathrm{Gr}_0^{\mathfrak{k}}$.

Proposition 2.1.13 ([PS86, Theorem 8.6.1])**.** *The action of $L_{alg}K_{\mathbb{C}}$ on $\mathcal{H}^{\mathfrak{k}}$ extends to an action on $\mathrm{Gr}_0(\mathcal{H}^{\mathfrak{k}})$ according to*

$$\gamma \cdot W = \{\gamma \cdot f : f \in W\}, \qquad \gamma \in L_{alg}K_{\mathbb{C}}, W \in \mathrm{Gr}_0(\mathcal{H}^{\mathfrak{k}}).$$

which preserves $\mathrm{Gr}_0^{\mathfrak{k}}$.

While we have relegated this proof to the realm of citation, the conditions used to define $\mathrm{Gr}_0^{\mathfrak{k}}$ deserve some explanation. Imposing that $zW \subseteq W$ implies that zW has codimension n in W, so its orthogonal complement has dimension n. Choosing a basis $\{w_i(z)\}_{i=1}^n$ for the complement and arranging the basis elements to form an $(n \times n)$-matrix gives an element of $L_{\text{alg}}K_{\mathbb{C}}$. The details of this argument for Sobolev class H^1 loops can be found in [PS86, Theorem 8.3.2], and for piecewise smooth curves we refer the reader to [HJS14, Lemma 3.4].

The space $\overline{W}^{\perp}$ is the orthogonal complement of W with respect to the Killing form, and the group action above must preserve the Killing form. Furthermore, $zW = \overline{W}^{\perp}$ further ensures that the complement of zW in W looks like $W \cap \overline{W}$, and so is the complexification of a real n-dimensional space; in fact, $W \cap \overline{W} \cong \mathfrak{k}_{\mathbb{C}}$.

Proposition 2.1.14. *The map $\phi : \Omega_{alg}K \to \mathrm{Gr}_0(\mathcal{H}^{\mathfrak{k}})$ given by $\gamma \mapsto \gamma \cdot \mathcal{H}_+$ defines an $S^1 \times K$ equivariant symplectic embedding whose image is $\mathrm{Gr}_0^{\mathfrak{k}}$, and moreover $\phi(\Omega_{alg,m}K) = \mathrm{Gr}_{0,m}^{\mathfrak{k}}$, preserving the filtration.*

Proof. We shall proceed by first examining the result in $\mathrm{SU}(n)$, then argue the general case afterwards. Let $X, Y \in T_e\Omega_{\text{alg}}\mathrm{SU}(n)$, written as a Fourier series

$$X(\theta) = \sum_k A_k e^{ik\theta}, \qquad Y(\theta) = \sum_\ell B_\ell e^{i\ell\theta}.$$

Applying the symplectic form, we get

$$\begin{aligned}
\omega_{\Omega K}(X, Y) &= \frac{1}{2\pi}\int_0^{2\pi} \sum_{k,\ell} \left\langle A_k e^{ik\theta}, i\ell B_\ell e^{i\ell\theta} \right\rangle d\theta \\
&= \frac{i}{2\pi}\sum_{k,\ell} \ell \mathrm{Tr}(A_k^* B_\ell) \int_0^{2\pi} e^{i(k-\ell)\theta} d\theta \\
&= i \sum_k k \mathrm{Tr}(A_k^* B_k).
\end{aligned}$$

On the other hand, the coordinate chart about $W \in \mathrm{Gr}_0(\mathcal{H})$ is defined as

$$U_W = \left\{ V \oplus TV : T : W \to W^{\perp} \text{ Hilbert-Schmidt } \right\}$$

which allows us to uniquely identify V with the map T. If $W = \mathcal{H}_+$ then $U_{\mathcal{H}_+}$ equivalently consists of those spaces V such that $\pi_V : V \to \mathcal{H}_+$ is an isomorphism, in which case T can be explicitly defined as

$$T = \mathrm{pr}_-|_W \circ \pi_W^{-1} : \mathcal{H}_+ \to W \to \mathcal{H}_-.$$

In coordinates around the identity, $\phi(\gamma) = \mathrm{pr}_- \circ \pi^{-1}_{\gamma\mathcal{H}_+}$, and so $d\phi_e(X(z)) = \mathrm{pr}_- \circ L_{X(z)}$, where $L_{X(z)}$ is left-multiplication by $X(z)$.

Let $f_X = d\phi_e(X) = \mathrm{pr}_- \circ L_X$ and $f_Y = d\phi_e(Y) = \mathrm{pr}_- \circ L_Y$, so that

$$\begin{aligned}
\omega_{HS}(f_X, f_Y) &= -i\mathrm{Tr}\left(f_X^* f_Y - f_Y^* f_X\right) \\
&= -i \sum_{\substack{i\in\{1,\ldots,n\} \\ j\geq 0}} \left\langle \epsilon_i z^j, (f_X^* f_Y - f_Y^* f_X)\epsilon_i z^j \right\rangle_{L^2} \\
&= -i \sum_{\substack{i\in\{1,\ldots,n\} \\ j\geq 0}} \left[\left\langle f_X \epsilon_i z^j, f_Y \epsilon_i z^j \right\rangle - \left\langle f_Y \epsilon_i z^j, f_X \epsilon_i z^j \right\rangle_{L^2} \right].
\end{aligned}$$

Examining just half of this equation for now, we see that

$$\begin{aligned}
\sum_{j\geq 0}\sum_{i=1}^{n} \left\langle f_X \epsilon_i z^j, f_Y \epsilon_i z^j \right\rangle_{L^2} &= \sum_{j\geq 0}\sum_{i=1}^{n}\sum_{r,s<-j} \left\langle (A_r\epsilon_i) z^{j+r}, (B_s\epsilon_i) z^{j+s} \right\rangle_{L^2} \\
&= \sum_{j\geq 0}\sum_{i=1}^{n}\sum_{r<-j} \langle A_r\epsilon_i, B_r\epsilon_i \rangle_{\mathfrak{k}_{\mathbb{C}}} \\
&= \sum_{j\geq 0}\sum_{r<-j} \mathrm{Tr}(A_r B_r^*) = \sum_{k<0} k\mathrm{Tr}(A_k B_k^*).
\end{aligned}$$

On the other hand, we know that $A_{-k} = \overline{A_k}$ and hence

$$\sum_{k<0} k\mathrm{Tr}(A_k B_k^*) = -\sum_{k>0} k\mathrm{Tr}(\overline{A_k B_k}^*) = -\sum_{k>0} k\mathrm{Tr}(A_k^* B_k)$$

where in the last inequality we have used transpose- and cyclic-invariance of the trace. By symmetry, we thus have

$$\omega_{HS}(f_X, f_Y) = i\sum_{k\geq 0} \left[k\mathrm{Tr}(A_k^* B_k) - k\mathrm{Tr}(A_{-k}^* B_{-k}) \right] = i\sum_{k} k\mathrm{Tr}(A_k^* B_k)$$

which is the same result we got from $\omega(X, Y)$, as required.

For a general compact group, assume temporarily that the group is centerfree. Since $\langle\cdot,\cdot\rangle_{\mathfrak{k}_{\mathbb{C}}}$ is the inner product induced by the Killing form on $\mathfrak{k}$ (itself related to the Killing form on $\mathfrak{k}_{\mathbb{C}}$

by conjugating the second argument), we have the $\langle \cdot, \cdot \rangle_{\mathfrak{k}_\mathbb{C}}$ is Ad-invariant. The action of $\Omega_{\mathrm{alg}} K$ on $\mathrm{Gr}_0(\mathcal{H}^{\mathfrak{k}})$ thus realizes $\Omega_{\mathrm{alg}} K$ as a submanifold of $\Omega_{\mathrm{alg}}\mathrm{SU}(\mathfrak{k}_\mathbb{C})$, and since K is centerfree this is actually an embedding. Choosing an appropriate basis for $\mathfrak{k}_\mathbb{C}$, we identify $\mathrm{SU}(\mathfrak{k}_\mathbb{C})$ with $\mathrm{SU}(n)$ and $\mathcal{H}^{\mathfrak{k}}$ with $\mathbb{C}^n$. The result then follows from commutativity of the embeddings:

$$\begin{array}{ccc} \Omega_{\mathrm{alg}}\mathrm{SU}(n) & \hookrightarrow & \mathrm{Gr}_0(\mathcal{H}^{\mathbb{C}^n}) \\ \uparrow & & \uparrow \\ \Omega_{\mathrm{alg}} K & \hookrightarrow & \mathrm{Gr}_0(\mathcal{H}^{\mathfrak{k}}). \end{array}$$

Once again, we are not concerned with the centerfree requirement, since when K is not centerfree we have that $\Omega_{\mathrm{alg}} K$ is a disjoint union of the connected components of $\Omega\,\mathrm{Ad}(K)$.

Equivariance follows quickly once we recognize that if $a \in A_\mathbb{C}$ then $\mathrm{Ad}_a \mathcal{H}_+ = \mathcal{H}_+$. As $(a \cdot \gamma)(z) = a\gamma(z)a^{-1}$ one has

$$\begin{aligned} \phi(a \cdot \gamma) &= \mathrm{Ad}_{a\gamma a^{-1}} \mathcal{H}_+ = \mathrm{Ad}_a \circ \mathrm{Ad}_\gamma \circ \mathrm{Ad}_{a^{-1}} \mathcal{H}_+ \\ &= \mathrm{Ad}_a \circ \mathrm{Ad}_\gamma \mathcal{H}_+ \\ &= a \cdot \phi(\gamma). \end{aligned}$$

For $s \in S^1$, $(s \cdot \gamma)(z) = \gamma(sz)\gamma(s)^{-1}$. The map $f(z) \mapsto f(s^{-1}z)$ is bijective on $\mathcal{H}_+$, so that

$$\phi(s \cdot \gamma) = \mathrm{Ad}_{\gamma(sz)\gamma(s^{-1})} \mathcal{H}_+ = \mathrm{Ad}_{\gamma(sz)} \mathcal{H}_+ = s \cdot \phi(\gamma).$$

Both actions commute, so ϕ is equivariant as required. Finally, the $\mathbb{C}^\times \times K_\mathbb{C}$-action commutes with z-multiplication, leaving $z^m\mathcal{H}_+$ invariant and preserving the filtration. □

Proposition 2.1.13 immediately tells us that $\mathrm{Gr}_0^{\mathfrak{k}}$ is an $L_{\mathrm{alg}} K_\mathbb{C}$-homogeneous space. Moreover, the stabilizer of $\mathcal{H}_+$ is $L^+_{\mathrm{alg}} K_\mathbb{C}$, implying that $\mathrm{Gr}_0^{\mathfrak{k}} \cong L_{\mathrm{alg}} K_\mathbb{C}/L^+_{\mathrm{alg}} K_\mathbb{C}$. Proposition 2.1.14 implies that $\mathrm{Gr}_0^{\mathfrak{k}} \cong \Omega_{\mathrm{alg}} K$, yielding the identification

$$\Omega_{\mathrm{alg}} K \cong L_{\mathrm{alg}} K_\mathbb{C}/L^+_{\mathrm{alg}} K_\mathbb{C}.$$

2.1.6 Bruhat Decomposition

Definition 2.1.15. Consider the evaluation map $\mathrm{ev}_0 : L^+_{\mathrm{alg}} K_\mathbb{C} \to K_\mathbb{C}$ given by $\mathrm{ev}_0(\gamma) = \gamma(0)$. The *Iwahori subgroup* $\mathcal{B}$ is the preimage of the Borel under ev_0; that is, $\mathcal{B} = \mathrm{ev}_0^{-1}(B)$.

In the analog between the finite- and infinite-dimensional regimes, the Iwahori subgroup occupies the role normally played by the Borel. It is the minimal parabolic subgroup of $L_{\mathrm{alg}} K_\mathbb{C}$, so that $L_{\mathrm{alg}} K_\mathbb{C}/\mathcal{B}$ is the *complete affine flag variety.* In contrast, it should be clear that $\mathcal{B} \subseteq L^+_{\mathrm{alg}} K_\mathbb{C}$, so that $L^+_{\mathrm{alg}} K_\mathbb{C}$ is also parabolic. In fact, $L^+_{\mathrm{alg}} K_\mathbb{C}$ is a maximal parabolic subgroup, so that $\Omega_{\mathrm{alg}} K \cong L_{\mathrm{alg}} K_\mathbb{C}/L^+_{\mathrm{alg}} K_\mathbb{C}$ is known as the *affine Grassmannian.*

The Bruhat decomposition, which confers a cellular decomposition to both the group and its flag varieties, also carries over into the infinite dimensional regime. Let $I, J \subseteq \Sigma$ be subsets of the affine simple roots, with the corresponding parabolic subgroups P_I, P_J respectively. Let $\mathcal{W}_I, \mathcal{W}_J$ be the Weyl groups generated by the reflections in I and J, and define the double coset $\mathcal{W}_{IJ} = \mathcal{W}_I \backslash \mathcal{W} / \mathcal{W}_J$. The Bruhat decomposition is the expression of $L_{\text{alg}} K_{\mathbb{C}}$ as the disjoint union

$$L_{\text{alg}} K_{\mathbb{C}} = \bigsqcup_{w \in \mathcal{W}_{IJ}} P_I w P_J.$$

The closure of a particular cell, indexed by say w, has an appealing description as the union of all cells which precede w in the Bruhat order; namely

$$\overline{P_I w P_J} = \bigsqcup_{x \leq w} P_I x P_J, \qquad x \in \mathcal{W}_{IJ}. \tag{2.7}$$

A general discussion of Bruhat decompositions with an emphasis on loop groups may be found in [Mit88].

With a myriad of choices for our various parabolic subgroups, we have a diverse array of decompositions to choose from. We saw above that $\Omega_{\text{alg}} K \cong L_{\text{alg}} K_{\mathbb{C}} / L^+_{\text{alg}} K_{\mathbb{C}}$. Setting $P_J = L^+_{\text{alg}} K_{\mathbb{C}}$ we get $\mathcal{W}_J = W$ so that $\mathcal{W}/W \cong X_*(T)$ and we can decompose the algebraic based loop group as

$$\Omega_{\text{alg}} K = \bigsqcup_{w \in W_I \backslash X_*(T)} P_I w.$$

Our two most important choices for P_I correspond to $P_I = L^+_{\text{alg}} K_{\mathbb{C}}$ and $P_I = \mathcal{B}$, wherein the latter case $\mathcal{W}_I$ is trivial. Our decompositions are thus

$$\Omega_{\text{alg}} K = \bigsqcup_{\lambda \in X_*(T)} \mathcal{B}\lambda = \bigsqcup_{\lambda \in X_*(T)_+} L^+_{\text{alg}} K_{\mathbb{C}} \lambda. \tag{2.8}$$

In the latter decomposition, we have chosen to use the dominant coweights as representatives of the W-cosets in $X_*(T)$. The former decomposition is known as the *Bruhat decomposition*, with $\mathcal{B}\lambda$ referred to as the *Bruhat cells* and indexed by the coweights $X_*(T)$. In the later decomposition, the cells $L^+_{\text{alg}} K_{\mathbb{C}} \lambda$ are known as *Bruhat manifolds* and indexed by the *dominant* coweights $X_*(T)_+$.

The cell closure relation given in (2.7) descends through the quotient, allowing us to write

$$\overline{\mathcal{B}\lambda} = \bigsqcup_{\eta \leq \lambda} \mathcal{B}\eta, \qquad \overline{L^+_{\text{alg}} K_{\mathbb{C}} \lambda} = \bigsqcup_{\eta \leq \lambda} \overline{L^+_{\text{alg}} K_{\mathbb{C}} \eta}. \tag{2.9}$$

We refer to $\overline{\mathcal{B}\lambda}$ as Schubert varieties, in analog of the finite dimensional case.

In general, the Schubert varieties $\overline{\mathcal{B}\lambda}$ are singular, but come with the advantage that for any coweight λ there is a sufficiently large m such that $\overline{\mathcal{B}\lambda} \hookrightarrow \Omega_{\text{alg},m} K$ is an embedding of varieties [AP83; Mar10]. Proposition 2.1.14 showed that $\Omega_{\text{alg},m} K$ is symplectomorphic to $\text{Gr}^{\mathfrak{k}}_{0,m}$

which itself is a subvariety of a smooth Kähler manifold $\mathcal{G}_m$. Composing these maps gives an embedding of $\overline{\mathcal{B}\lambda}$ into a smooth Kähler manifold:

$$\overline{\mathcal{B}\lambda} \hookrightarrow \Omega_{\mathrm{alg},m}K \xrightarrow{\cong} \mathrm{Gr}^{\mathfrak{k}}_{0,m} \hookrightarrow \mathcal{G}_m.$$

2.1.7 Determinant Bundle

If $k < n$ and $\mathrm{Gr}(k,n)$ represents the Grassmannian of complex k-planes in $\mathbb{C}^n$, then $\mathrm{Gr}(k,n)$ has a tautological rank k vector bundle, $\gamma_{k,n} \to \mathrm{Gr}(k,n)$ where

$$\gamma_{k,n} = \{(W,w) : W \in \mathrm{Gr}(k,n), w \in W\}.$$

Since this is a rank k vector bundle, we may define a line bundle $\det^* \to \mathrm{Gr}(k,n)$ by taking the k-th exterior power over each fibre. More explicitly, if $\{w_1,\ldots,w_k\}$ is a basis for W, a typical fibre element over W will be of the form $(W, \alpha w_1 \wedge \cdots \wedge w_k)$, which we write as $(W,[\alpha,w])$. Naturally, choosing another basis should not change the structure of the fibre. If $\{w'_1,\ldots,w'_k\}$ is another basis and C is the change of basis matrix from w to w', then

$$w'_1 \wedge \cdots \wedge w'_k = (\det C) w_1 \wedge \cdots \wedge w_k,$$

so we identify the elements $(W,[\alpha,w']) \sim (W,[\alpha(\det C),w])$.

We are more interested in the dual bundle $\det \to \mathrm{Gr}(k,n)$. It is easy to check that $\det \to \mathrm{Gr}(k,n)$ is the pullback of $\mathcal{O}(1) \to \mathbb{P}^{\binom{n}{k}-1}$ by the Plücker embedding $p : \mathrm{Gr}(k,n) \to \mathbb{P}^{\binom{n}{k}-1}$. If $\nabla_{\mathcal{O}(1)}$ is the Chern connection on $\mathcal{O}(1)$, then $\nabla = p^*\nabla_{\mathcal{O}(1)}$ is a Chern connection on det, and moreover the curvature satisfies

$$F_{p^*\nabla_{\mathcal{O}(1)}} = p^* F_{\nabla_{\mathcal{O}(1)}} = -2i\pi p^*\omega_{FS} = -2i\pi\omega'_{FS},$$

which shows that $(\det,\nabla)$ is a positive prequantum line bundle. When applied to our $\mathcal{G}_m$, we shall denote the bundle by $\det_m \to \mathcal{G}_m$. If we combine this with the embedding of $\overline{\mathcal{B}\lambda} \hookrightarrow \mathcal{G}_m$ we get

$$\begin{array}{ccccc} & & \det_m & \longrightarrow & \mathcal{O}(1) \\ & & \downarrow & & \downarrow \\ \overline{\mathcal{B}\lambda} & \hookrightarrow & \mathcal{G}_m & \hookrightarrow & \mathbb{P}^{\binom{2nm}{nm}}. \end{array}$$

2.2 Generalized Equivariant Cohomology

2.2.1 General Overview

In the interest of clarity, we will begin with a brief overview of the pertinent parts of generalized equivariant cohomology theory. Let T denote a fixed compact torus, and define a T-space to be a compactly generated weak Hausdorff topological space X endowed with a continuous action of

T. Taking such spaces as the objects and letting morphisms consist of T-equivariant continuous maps defines a category $\mathcal{C}_T$. We will let $(\mathcal{C}_T)_*$ denote the category of *based* T-spaces. This is the natural environment in which to define generalized equivariant cohomology, though many of our arguments will be more transparent if cohomology is induced in the language of spectra. More precisely, we will examine the homotopy category of T-equivariant spectra.

Fix a complete T-universe

$$\mathcal{U} = \bigoplus_{V \in RO(T)} V^{\mathbb{N}},$$

where $RO(T)$ is the real representation ring of T; namely, $\mathcal{U}$ is a real orthogonal T-representation of countably infinite dimension, such that $\mathcal{U}$ contains countably infinitely many copies of each finite-dimensional T-representation. Recall that T-spectra indexed on $\mathcal{U}$ form a category $TS^{\mathcal{U}}$ [HPS97, Definition 9.4.1], and to any pointed T-space one can define a spectrum by taking the sequence of spaces induced by compounding suspension. This induces a functor $\Sigma^\infty : (\mathcal{C}_T)_* \to TS^{\mathcal{U}}$, assigning to $X \in (\mathcal{C}_T)_*$ the spectrum satisfying $(\Sigma^\infty X)_n = \Sigma^n X$. Based T-spaces thus yield T-spectra, and we will sometimes make no distinction between a based T-space X and its suspension spectrum $\Sigma^\infty(X)$.

The functor Σ^∞ is just one of a family of suspension functors $(\mathcal{C}_T)_* \to TS^{\mathcal{U}}$ indexed by finite-dimensional real T-representations. Suspensions of spheres are invaluable in defining non-equivariant cohomology, and so too will suspensions of representation spheres play a key role in defining equivariant cohomology. More precisely, let V be a representation and denote by S^V its one-point compactification, realized as a based space with point at infinity. The action of T on V extends to an action on S^V that fixes the base point, so that S^V is in fact a based T-space. Smashing against these spheres generalizes the usual suspension process, defining a functor $\Sigma^V : (\mathcal{C}_T)_* \to (\mathcal{C}_T)_*$ by

$$\Sigma^V(X) := S^V \wedge X.$$

If $V \subseteq W$ is an inclusion of finite dimensional T-representations, we define the relative suspension of a based T-space X to be

$$(\Sigma_V^\infty(X))(W) := \Sigma^{V^\perp}(X),$$

where $V^\perp$ is the orthogonal complement of V in W. If V does not include into W, we take $(\Sigma_V^\infty(X))(W)$ to be a point. The spaces $\{(\Sigma_V^\infty(X))(W)\}_W$ constitute a T-prespectrum and therefore determine a T-spectrum $\Sigma_V^\infty(X)$ via the process of spectrification. Furthermore, $X \mapsto \Sigma_V^\infty(X)$ defines a functor $\Sigma_V^\infty : (\mathcal{C}_T)_* \to TS^{\mathcal{U}}$, which one may use to define desuspensions of representation spheres:

$$S^{-V} := \Sigma_V^\infty(S^0).$$

If W is another finite-dimensional T-representation, we set

$$S^{W-V} := S^W \wedge S^{-V}.$$

This gives us a T-spectrum S^α for each α in the representation ring $RO(T;\mathcal{U})$(see [May96]).

Cohomology via spectra:

We have now developed the machinery necessary to explain how generalized T-equivariant cohomology theories arise from T-spectra. Denote by $\overline{h}TS^{\mathcal{U}}$ the stable homotopy category of T-spectra obtained by inverting the weak equivalences in $TS^{\mathcal{U}}$. Fix a T-spectrum E, and define a functor $\tilde{E}^0_T : \overline{h}TS^{\mathcal{U}} \to \mathbb{Z}$-mod by associating to each T-spectrum F the abelian group $[F,E] := \mathrm{Hom}(F,E)$ of morphisms in $\overline{h}TS^{\mathcal{U}}$. One may extend $\tilde{E}^0_T$ to an $RO(T;\mathcal{U})$-graded functor by setting

$$\tilde{E}^\alpha_T(F) := [S^{-\alpha} \wedge F, E], \quad \alpha \in RO(T;\mathcal{U}). \tag{2.10}$$

We will be primarily interested in the underlying $\mathbb{Z}$-graded functor. More explicitly, if $n \in \mathbb{Z}$, then $\tilde{E}^n_T : \overline{h}TS^{\mathcal{U}} \to \mathbb{Z}$-mod is defined via (2.10) by setting α equal to the appropriately signed $|n|$-dimensional trivial T-representation. The resulting $\mathbb{Z}$-graded functor $\tilde{E}^*_T$ then restricts to a reduced generalized T-equivariant cohomology theory on $(\mathcal{C}_T)_*$, with the associated unreduced theory E^*_T on $\mathcal{C}_T$ given by

$$E^*_T(X) := \tilde{E}^*_T(X_+).$$

Here X_+ is the T-space formed by taking a disjoint union of X and an additional base point.

If E is additionally a commutative T-ring spectrum [May96, Chapter XII], then E^*_T take values in the category $\mathrm{CRing}_{\mathbb{Z}}$ of $\mathbb{Z}$-graded commutative rings. We then have the following definition of a generalized T-equivariant cohomology theory suitable for our purposes.

Definition 2.2.1. A generalized T-equivariant cohomology theory is a $\mathbb{Z}$-graded functor $E^*_T : \mathcal{C}_T \to \mathrm{CRing}_{\mathbb{Z}}$ resulting from a commutative ring T-spectrum E as indicated above.

2.2.2 The Generalized Thom-Gysin Sequence

Our goal here is to provide a detailed look at the Thom-Gysin sequence in generalized equivariant cohomology, and in particular show that the Gysin map is an $E^*_T(\mathrm{pt})$-module map. In order to do this, we will need to require that our theories satisfy additional properties, and in addition we will need to define Thom spaces for vector bundles.

There are two common ways of thinking about the Thom spaces which arise in the Thom isomorphism. In either case, let $p : \xi \to X$ be a rank-n (T-equivariant) complex vector bundle:

1. Use a partition of unity to give ξ a metric, and let $D(\xi) \to X$ be the unit disk-bundle associated to ξ. Similarly, take $S(\xi) \to X$ to be the associated sphere bundle, and define the Thom space $T(\xi) = D(\xi)/S(\xi)$.

2. Let x_0 be the complement of the image of the zero section $z : X \to \xi$. Define the Thom space to be the topological pair $(E(\xi), E_0(\xi))$.

To see that these constructions are homotopy equivalent, one can consider the following commutative diagram and apply the short 5-lemma:

$$\begin{array}{ccccc} S(\xi) & \longrightarrow & D(\xi) & \longrightarrow & (T(\xi), *) \\ \simeq \downarrow & & \simeq \downarrow & & \phi \downarrow \\ E_0(\xi) & \longrightarrow & E(\xi) & \longrightarrow & (E(\xi), E_0(\xi)) \\ & & p \downarrow & & \\ & & B & & \end{array}$$

The induced map ϕ is guaranteed to be a homotopy equivalence. For consistency of notation, we will henceforth denote the Thom space of a vector bundle $\xi \to X$ by $\mathrm{Th}(\xi)$.

Complex Orientability

To even guarantee the existence of Thom and Euler classes means ensuring that our theory plays well with complex vector bundles. The corresponding property of the theory is that it be complex oriented [CGK02]. There are several possible definitions but we present below the one which makes the existence of Thom classes manifest:

Definition 2.2.2. Let E_T^* be a multiplicative generalized equivariant cohomology theory. We say that E_T^* is *complex oriented* if whenever $\xi \to X$ is a complex vector bundle of dimension n, there exists a class $u_T(\xi) \in \tilde{E}_T^{2n}(\mathrm{Th}(\xi))$, called the *Thom class*, such that

1. The image Thom class $u_T(\xi)$ under the map $\tilde{E}_T^{2n}(\mathrm{Th}(\xi)) \to E_T^0(\mathrm{pt})$ is the identity element.
2. The Thom class $u_T(\xi)$ is natural under pullbacks; that is, if $f : Y \to X$ then $u_T(f^*\xi) = f^* u_T(\xi)$.
3. The thom class $u_T(\xi)$ is multiplicative over the Whitney sum, so if $W \to X$ is also an n-vector bundle, $u_T(\xi \oplus W) = u_T(\xi) \smile u_T(W)$.

Associated to the Thom class is the Euler class, defined as follows: Let $\xi \to X$ be a vector bundle of dimension n, and take $z : X_+ \to \mathrm{Th}(\xi)$ to be the zero section of the natural projection. Define the *Euler class* $e_T(\xi) \in E_T^{2n}(X)$ as

$$e_T(\xi) := z^*(u_T(\xi)) \in \tilde{E}_T^{2n}(X_+) = E_T^{2n}(X).$$

In the case where our cohomology theory is defined using a (ring) spectrum, such spheres act in lieu of the usual sphere spectrum of the non-equivariant regime. Even when the cohomology theory is geometrically motivated, such representation spheres lend themselves to suspension-type isomorphisms. To be more precise, the theory E_T^* is said to be a *complex stable ring theory* (see [CGK02]) if for all complex T-representations V, there exists a class $\alpha_V \in \tilde{E}_T^{\dim_{\mathbb{R}}(V)}(S^V)$ with the property that multiplication by α_V defines an isomorphism $\tilde{E}_T^*(X) \to \tilde{E}_T^*(S^V \wedge X)$ for all T-spaces X. In this case, setting $X = S^0$ implies that $\tilde{E}_T^*(S^V)$ is freely generated by α_V as a module over $E_T^*(\mathrm{pt})$.

The Gysin Sequence

Recall that for a topological T-space X with T-invariant subspace Y, there is a long exact sequence in relative cohomology, induced by the inclusions

$$(Y, \emptyset) \hookrightarrow (X, \emptyset) \hookrightarrow (X, Y),$$

which together with a connecting morphism yields

$$\longrightarrow E_T^*(X, Y) \longrightarrow E_T^*(X) \longrightarrow E_T^*(Y) \longrightarrow . \tag{2.11}$$

Our motivation is to inductively build the cohomology of X from its components in a "Meyer-Vietoris" like fashion; namely, if $A \subseteq X$ is a closed subspace we would like to describe $E_T^*(X)$ in terms of $E_T^*(A)$ and $E_T^*(X \setminus A)$. By setting $Y = X \setminus A$, all that needs to be done is to relate $E_T^*(X, X \setminus A)$ to $E_T^*(A)$ in (2.11), and this will be done with the Thom isomorphism.

Theorem 2.2.3 (Thom Isomorphism). *If $p : \xi \to X$ is an orientable rank-n complex vector bundle, then there exists a class $\tau \in E_T^{2n}(\mathrm{Th}(\xi))$ such that $\Phi : E_T^*(X) \to E_T^{*+2n}(\mathrm{Th}(\xi))$ given by $\Phi(\eta) = p^*(\eta) \smile \tau$ is an isomorphism, where $\smile$ is the multiplicative structure on $E_T^*(X)$.*

The Thom isomorphism thus gives us a strategy for translating relative cohomological data into non-relative cohomology. The next step is to realize the pair $(X, X \setminus A)$ as a vector bundle, which will be aided by the use of the Excision Theorem.

Theorem 2.2.4 (Excision Theorem). *If $U \subseteq V \subseteq X$ are topological spaces such that $\bar{U} \subseteq V^\circ$, then there is an isomorphism*

$$E_T^*(X, V) \cong E_T^*(X \setminus U, V \setminus U).$$

Let T be a tubular neighbourhood of A in X, so that $T = \iota(\nu_X A)$ where $\iota : \nu_X A \hookrightarrow X$ is an embedding of the normal bundle $\nu_X A$ into X. The rank of $\nu_X A$ is the codimension of A in X, which we set to be d. Notice that $X \setminus T$ is closed, $X \setminus A$ is open, and $X \setminus T \subseteq X \setminus A$, so certainly the subspaces $X \setminus T \subseteq X \setminus A \subseteq X$ satisfy the hypothesis of the Excision Theorem. Excising $X \setminus T$ from $(X, X \setminus A)$ we get

$$X \setminus (X \setminus T) = X \cap T = T, \qquad (X \setminus A) \setminus (X \setminus T) = X \setminus A \cap T = T \setminus A.$$

Notice that A is isomorphic to the zero section $\iota \circ z : A \to \nu_X A \to T$, so applying the Excision theorem and then using the Thom isomorphism we get

$$E_T^*(X, X \setminus A) \cong E_T^*(T, T \setminus A) \cong E_T^*(\nu_X A, (\nu_X A)_0) \cong E_T^{*-n}(A),$$

where $(\nu_X A)_0$ is the complement of the zero section.

Substituting this equivalence into (2.11) we have the equivariant Thom-Gysin sequence

$$\longrightarrow E_T^{*-n}(A) \xrightarrow{g} E_T^*(X) \longrightarrow E_T^*(X \setminus A) \longrightarrow . \tag{2.12}$$

The Module Structure

Given a commutative T-ring spectrum E and a T-space X, the trivial map $\phi : X \to \mathrm{pt}$ in $\mathcal{C}_T$ yields a morphism $E_T^*(\phi) : E_T^*(\mathrm{pt}) \to E_T^*(X)$ of $\mathbb{Z}$-graded commutative rings. This map renders $E_T^*(X)$ a module over the ring $E_T^*(\mathrm{pt})$ by defining

$$r \cdot \eta = E_T^*(\phi)(r) \smile \eta, \qquad r \in E_T^*(\mathrm{pt}), \eta \in E_T^*(X).$$

The next question is whether the Gysin map $g : E_T^{*-n}(A) \to E_T^*(X)$ given in (2.12) is an $E_T^*(\mathrm{pt})$-module morphism.

Proposition 2.2.5. *Let $f : X \to Y$ be a morphism of commutative rings. Let R be another ring with ring maps $\phi_X : R \to X, \phi_Y : R \to Y$. Define an R-module structure on X and Y via*

$$r \cdot x = \phi_X(r)x, \qquad r \cdot y = \phi_Y(r)y.$$

If the following diagram commutes

$$\begin{array}{ccc} R & = & R \\ {\scriptstyle \phi_X}\downarrow & & \downarrow{\scriptstyle \phi_Y} \\ X & \xrightarrow{f} & Y \end{array}$$

then f is an R-module map.

Proof. Additivity is immediate since f is a ring map, so we only need to show that the scalar multiplication in R is preserved. The fact that the diagram commutes means that $f(\phi_X(r)) = \phi_Y(r)$. Now let $x \in X, r \in R$, and notice that

$$\begin{aligned} f(r \cdot x) &= f(\phi_X(r)x) \\ &= f(\phi_X(r))f(x) && \text{since } f \text{ is an ring map} \\ &= \phi_Y(r)f(x) && \text{since the diagram commutes} \\ &= r \cdot f(x), \end{aligned}$$

as required. ☐

One is then immediately able to deduce that all maps in (2.12) are $E_T^*(\mathrm{pt})$-module morphisms since the following diagram commutes:

$$\begin{array}{ccccc} \mathrm{pt} & = & \mathrm{pt} & = & \mathrm{pt} \\ \uparrow & & \uparrow & & \uparrow \\ T(\nu_X A) & \longleftarrow & X & \longleftarrow & X \setminus A. \end{array}$$

Applying the functor $E_T^*(\cdot)$ to pass to equivariant cohomology, this diagram becomes

$$\begin{array}{ccccc} E_T^*(\mathrm{pt}) & = & E_T^*(\mathrm{pt}) & = & E_T^*(\mathrm{pt}) \\ \downarrow & & \downarrow & & \downarrow \\ E_T^*(T(\nu_X A)) & \longrightarrow & E_T^*(X) & \longrightarrow & E_T^*(X \setminus A). \end{array} \tag{2.13}$$

Now $\tilde{p} : T(\xi) \to B(\xi)$ is an $E_T^*(\mathrm{pt})$-module map since

$$\begin{array}{ccc} \mathrm{pt} & = & \mathrm{pt} \\ \uparrow & & \uparrow \\ T(\xi) & \xrightarrow[\tilde{p}]{} & B. \end{array}$$

As the Thom isomorphism $E_T^{*-n}(A) \to E_T^*(T(\nu_X A))$ is just multiplication by the Thom class, we then have

$$\begin{aligned} \Phi(r \cdot \eta) &= \Phi(\phi_A(r) \smile \eta) = p^*(\phi_A(r) \smile \eta) \smile \tau \\ &= p^*(\phi_A(r)) \smile p^*(\eta) \smile \tau && \text{functoriality of cup product} \\ &= \phi_{T(\nu_X A)}(r) \smile (p^*(\eta) \smile \tau) \\ &= r \cdot \Phi(\eta). \end{aligned}$$

This implies that the Thom isomorphism is an $E_T^*(\mathrm{pt})$-module map, so combining this with (2.13) tells us that all the maps in (2.12) are $E_T^*(\mathrm{pt})$-maps as required.

2.2.3 Important Examples

Despite having discussed generalized equivariant cohomology theories in the abstract, we emphasize three important generalized T-equivariant cohomology theories: equivariant singular cohomology H_T^*, complex equivariant K-theory K_T^*, and equivariant complex cobordism MU_T^*.

Singular Equivariant Cohomology

We denote by $ET \to ET/T = BT$ the universal principal T-bundle, characterized by the property that ET is a contractible space on which T acts freely. If X is a T-space, then the product $X \times ET$ carries a T-action and we may form the Borel mixing space

$$X_T := (X \times ET)/T.$$

We then define the ordinary T-equivariant cohomology of X (with integer coefficients) to be

$$H_T^*(X) := H^*(X_T; \mathbb{Z}),$$

the integral cohomology of X_T. Of course, H_T^* arises from the Eilenberg-MacLane T-spectrum [May96, Chapter XIII].

There is a natural ring isomorphism between the base ring $H_T^*(\mathrm{pt})$ and $\mathrm{Sym}_{\mathbb{Z}}(X^*(T))$, the symmetric algebra of the weight lattice $X^*(T)$ of T. Indeed, a weight $\mu : T \to S^1$ yields an associated line bundle

$$L(\mu) := \frac{ET \times \mathbb{C}}{(\alpha, z) \sim (t\alpha, \mu(t)z)} \to BT,$$

where $t \in T$ and $(\alpha, z) \in ET \times \mathbb{C}$. The ring isomorphism then associates to $\mu \in X^*(T)$ the first Chern class $c_1(L(\mu)) \in H^2(BT;\mathbb{Z}) = H_T^2(\mathrm{pt})$.

Equivariant K-Theory

Our treatment follows that given in [Seg68]. Recall that for a compact T-space X, $K_T^0(X)$ is defined to be the Grothendieck group of the category of T-equivariant complex vector bundles over X. The operation of taking the tensor product of equivariant vector bundles renders $K_T^0(X)$ a commutative ring. One extends the definition of K_T^0 to a definition of $K_T^n(X)$ for X locally compact and n any integer. By virtue of Bott periodicity, there are natural $\mathbb{Z}$-module isomorphisms $K_T^n(X) \cong K_T^{n+2}(X)$, $n \in \mathbb{Z}$. In particular, if $n \in \mathbb{Z}$, then $K_T^{2n}(\mathrm{pt})$ is naturally isomorphic to (the underlying abelian group of) the representation ring $R(T)$ of T. Note that $R(T)$ is freely generated over $\mathbb{Z}$ by $\{e^\mu : \mu \in X^*(T)\}$, where $e^\mu \in R(T)$ denotes the class of the one-dimensional complex T-representation of weight μ. Furthermore, $K_T^{2n+1}(\mathrm{pt}) = K_T^{-1}(\mathrm{pt}) = 0$. Hence, we shall identify $K_T^*(\mathrm{pt})$ as a $\mathbb{Z}$-graded abelian group with $R(T)^{\oplus 2\mathbb{Z}}$. If we multiply elements in the grading components of the latter as elements of $R(T)$, then this becomes an isomorphism of $\mathbb{Z}$-graded commutative rings.

It will later be necessary to discuss the T-equivariant K-theory of spaces that are not locally compact. To encompass this larger class of spaces, we will define T-equivariant K-theory via its ring T-spectrum [May96, Chapter XIV].

Equivariant Complex Cobordism

Our discussion of the equivariant complex cobordism follows that of [May96; Sin01]. It is important to note that as an equivariant cohomology theory whose construction hinges entirely upon spectra, equivariant complex cobordism cohomology does not agree with its natural geometric interpretation of classifying cobordant spaces. More specifically, one would envision that such a cohomology theory would describe the cobordisms of G-spaces (at least in the coefficient ring), but the failure of transversality in the equivariant context obstructs such an equivalence [May96].

If V is a complex representation of T, let $|V|$ denote its dimension as a complex vector space. As in Section 2.2, fix a complete T-universe $\mathcal{U}$ and let $BU^T(n)$ denote the Grassmannian of complex linear n-planes in $\mathcal{U}$. This Grassmannian comes equipped with a tautological line bundle $\xi_n^T \to BU^T(n)$, which is well known to serve as a model for the universal complex n-

plane bundle. If V is a finite-dimensional complex T-representation, let $\xi_V^T = \xi_{|V|}^T$. One then forms $\mathrm{Th}(U)$, an $R(T)$-indexed pre-spectrum whose V-th entry is $\mathrm{Th}(\xi_V^T)$. The spectrification of $\mathrm{Th}(U)$ yields the spectrum MU_T.

Chapter 3

Literature Review

To place our work in context, we present here a short summary of the more common results related to our work. In particular, we will discuss the attempts and techniques and computing the cohomology of the based loop group as well as various convexity results. As necessary, we will allude to various finite dimensional theorems to draw an analogy between what is classically known, and what has been generalized to the based loop group.

3.1 Cohomological Results

3.1.1 Non-Equivariant Cohomology

The ordinary cohomology of ΩK over $\mathbb{Q}$ was known to Serre [Ser51], who showed therein that K looks rationally like a product of odd-dimensional circles, making the computation of $H_*(\Omega K; \mathbb{Q})$ very simple. More explicitly, there exist integers $(n_1, \ldots, n_k) \in \mathbb{Z}^k$ and a map

$$f : S^{2n_1-1} \times S^{2n_2-1} \times \cdots \times S^{2n_k-1} \to K$$

which induces an isomorphism on $H^*(\,\cdot\,; \mathbb{Q})$. With appropriate manipulation of definitions, a relative of this map will induce an isomorphism at the level of loop spaces. It is worth mentioning as well that Serre is responsible for introducing the path fibration to aid in studying the cohomology of ΩK. In particular, let PK be the collection of continuous maps $[0, 1] \to K$ based at the identity, and consider the fibering

$$\Omega K \to PK \xrightarrow{p} K$$

with $p(f) = f(1)$, mapping a path to its endpoint. As PK is contractible, this opens the homology ΩK to analysis by means of spectral sequences. Serre's original study was for loop spaces in general, not necessarily those of Lie groups.

The first study of the cohomology of ΩK with coefficients in $\mathbb{Z}$ can be traced to Bott [Bot56]. A general point of K is any point contained in a unique maximal torus of K. Fix a general point $p \in K$ and let S be the collection of geodesics in K between p and the identity. Bott showed that the $H_*(\Omega K; \mathbb{Z})$ could be written as the free module generated by such geodesics, though the isomorphism is not explicitly rendered. Computation of the Betti numbers shows that the homology is concentrated in even degree. In a companion paper co-authored with Samelson [BS58], these results were extended by giving the explicit isomorphism between the homology and corresponding free module.

Bott concludes the seminal triumvirate with a description of the Pontryagin ring of $H_*(\Omega K; \mathbb{Z})$. As a quick review, recall that an H-space X is any space with a continuous map $\mu : X \times X \to X$ and an identity e such that $\mu(e, x) = \mu(x, e) = x$ for all $x \in X$. Using the Künneth formula, the induced map on cohomology $\mu_* : H_*(X) \times H_*(X) \to H_*(X)$ is called the *Pontryagin product*, and gives $H_*(X)$ a ring structure. Combined with the diagonal map $X \to X \times X$, which gives $H^*(X)$ its ring structure, $H_*(X)$ can be viewed as a Hopf algebra. Concatenation of loops endows any loop space ΩX with an H-space structure, but when X is a Lie group, pointwise multiplication also makes $H_*(X)$ an H-space. Interestingly, in such instances loop concatenation is homotopic to pointwise multiplication, and the Pontryagin products are equivalent.

Now let $s : S^1 \to K$ be a group morphism and K_s be the centralizer of the image of s in K. If $\Omega_0 K$ is the connected component of the identity, one can define a map on the coset space

$$g^s : K/K_s \to \Omega_0 K, \qquad [xK_s](t) = xs(t)x^{-1}s(t)^{-1}.$$

Given special conditions on s, the author shows that the image of $g^s_* : H_*(K/K_s) \to H_*(\Omega_0 K)$ generates the Pontryagin ring. This in turn is used to deduce the Hopf algebra structure of $H^*(\Omega K)$ and $H_*(\Omega K)$, and in particular Bott gives the algorithm explicitly in the case of $K = \mathrm{SU}(n), \mathrm{SO}(n), \mathrm{Spin}(n)$, and the exception Lie group G_2. It is worth noting though that the dependence on choice of a generating circle makes this construction non-canonical.

Of course, we have been remiss in our cavalier disregard for the differences between homology and cohomology. This may be remedied by appealing to the theory of (quasi-)Hopf algebras, as follows:

Theorem 3.1.1 ([MT91, Theorem 2.17])**.** *If a quasi-Hopf algebra A, which is free of finite type, has as simple system of positive generators: $A = \Delta(x_1, x_2, \ldots)$, then A^* is a commutative Hopf algebra and $A^* = \Lambda(x_1^*, x_2^*, \ldots)$ for x_i^* dual to x_i with respect to the basis $\{x_{i_1} \cdots x_{i_r} : i_1 < \cdots < i_r\}$ of A.*

Here $\Delta(x_1, x_2, \ldots)$ is the graded ($\mathbb{Z}$-)algebra with generators $\{x_{i_1} \cdots x_{i_r} : i_1 < i_2 < \cdots\}$ while Λ is the corresponding exterior algebra. Our previous results indicate that the homology groups $H_*(\Omega K)$ are free and of finite type with simple generators, so the cohomology groups are obtained by taking appropriate duals. The book [MT91] in general gives an overview of the results above, along with a more detailed classification of $H^*(\Omega K)$ for the exceptional Lie groups.

Of related interest is that one may use the Bruhat decomposition of $\Omega_{\mathrm{alg}} K$ to facilitate the computation of $H^*(\Omega K; \mathbb{Z})$. It is known that $\Omega_{\text{alg}} K$ is homotopy equivalent to ΩK [PS86, Proposition 8.6.6], so they agree in cohomology. In Chapter 2 we saw that $\Omega_{\text{alg}} K$ admits a decomposition into Schubert varieties. These varieties may be used to determine the product structure on $H^*(\Omega_{\text{alg}} K)$ in direct analog with the finite dimensional case. We refer the reader to [Mag] for more detail.

Many other papers exist which examine the cohomology $H^*(\Omega K)$. For example, [Hil85] examines the cohomology with $\mathbb{Z}/p$ coefficients. The role played by the affine Grassmannian in algebraic geometry means it has received a fair amount of attention in that field as well, see for example [Gin95; Gin98]. When applied to the loop spaces of symmetric spaces $\Omega K/K'$, we refer the reader to [Koc95].

3.1.2 Equivariant Cohomology

A great deal of information regarding Kac-Moody groups, generalized flag varieties, and their cohomology may be found in [Kum02], though the aforementioned treatise operates in far greater generality than necessary for our discussion. A treatment of the equivariant cohomology purely in terms of the Schubert varieties may be found in [Mag]. In a collection of two papers [HJS13; HJS14], Harada, Jeffrey, and Selick deduce the module and product structure of $K_T^*(\Omega \mathrm{SU}(2))$.

Generalizing the GKM method [GKM98] for computing equivariant cohomology, the paper [HHH05] generalizes the main result by allowing for non-abelian groups G and generalized cohomology theories E_G^*. It is this paper in particular which is most closely related to our work on equivariant cohomology. In particular, using an equivariantly stratified space the authors are able to reduce the computation of E_G^* to combinatorial data. As an example, their technique is applied to $H_T^*(\Omega_{\text{alg}} \mathrm{SU}(2); \mathbb{Z})$ and $K_T^*(\Omega_{\text{alg}} \mathrm{SU}(2))$.

3.2 Convexity Results

Herein we recall a spattering of convexity results for symplectic and Kähler manifolds, including both classical finite dimensional results, and those relating expressly to the based loop group. One might contend that the inclusion of the classical results is unnecessary, but we include them as a basis for contextualizing how our work acts to further generalize these theorems.

3.2.1 The Classical Theorems

Let G be a compact group and (M, ω) a compact symplectic manifold, acted upon by G in a Hamiltonian fashion with moment map $\mu : M \to \mathfrak{g}^*$. Fix a maximal torus $T \subseteq G$ and let $\mathfrak{t}_+^*$ be the positive Weyl chamber.

Our first result was proved independently by Atiyah [Ati82] and Guillemin and Sternberg [GS82].

Theorem 3.2.1 (Atiyah-Guillemin-Sternberg, [Ati82; GS82])**.** *Suppose that* (M, ω) *is a compact symplectic manifold endowed with a Hamiltonian torus action of* T*. If* $\mu : M \to \mathfrak{t}^*$ *is the moment map of this action, then* $\mu(M)$ *is convex, and moreover is the convex hull of the images of the fixed points of the* T*-action.*

This theorem is typically stated with several other consequences, including connectedness of preimages under the moment map. However, we will not worry about this connectedness property, except when we mention how it was generalized in the case of the based loop group. The proof offered by Atiyah involves embedding the closures of T-orbits in Kähler manifolds. This technique is the foundation for the approach of several other convexity arguments, as well as our own in Chapter 4. We refer the reader to [Aud91] for more details about this theorem and its consequences.

It was quickly realized that the moment map images for many complex manifolds agreed with that computed from their corresponding real locus. This was proven more formally by Duistermaat [Dui83]:

Theorem 3.2.2 (Duistermaat, [Dui83])**.** *Suppose that* (M, ω) *is a compact symplectic manifold and* $\tau : M \to M$ *is an anti-symplectic involution. If* T *acts on* M *in a Hamiltonian fashion and* τ *is compatible with this action in the sense that*

$$\tau(t \cdot m) = t^{-1}\tau(m), \qquad t \in T, m \in M.$$

then $\mu(M^\tau) = \mu(M)$*, where* M^τ *are the* τ*-fixed points of* M*.*

Once again we have omitted a part of the result which gives an explicit description of a perfect Morse function on M. The τ-fixed point set M^τ is often called the *real locus* of M, wherein one thinks of τ as describing complex conjugation, so that M^τ is the corresponding real analog of M. This terminology finds its origins with $\mathbb{CP}^n$ endowed with the Fubini-Study form and the natural $\mathrm{U}(n+1)$-action on $\mathbb{CP}^n$, where τ actually is complex conjugation, and $\mu(\mathbb{CP}^n) = \mu(\mathbb{RP}^n)$.

Finally, when the group action is not given by an abelian group, one can show that the image of the moment map is not necessarily convex. Kirwan [Kir84a] showed that to guarantee convexity, one should intersect the image of the moment map with the positive Weyl chamber.

Theorem 3.2.3 (Kirwan, [Kir84a])**.** *If* (M, ω) *is a compact symplectic manifold endowed with a Hamiltonian* G*-action and moment map* $\mu : M \to \mathfrak{g}^*$*, then* $\mu(M) \cap \mathfrak{t}^*_+$ *is convex.*

Of course, these theorems hold if compactness of M is dropped and replaced with suitable substitute, such as a proper moment map μ. As this section is designed solely to draw analogy with the infinite dimensional results, we will not dwell on the possible substitute conditions one can impose.

3.2.2 Convexity of ΩK

One again we will take K to be compact, connected, and simply connected with a fixed maximal torus T. Recall from Chapter 2 that there is a Hamiltonian $S^1 \times K$-action on ΩK which restricts to a Hamiltonian torus actions of $S^1 \times T$. Let μ be the corresponding moment map.

It is known that the behaviour of infinite-dimensional manifolds under moment maps is not nearly as tidy as its classical counterparts. One of the reasons that the based loop group is of interest is that despite the bad reputation typically afforded infinite dimensional symplectic manifolds, ΩK acts reasonably well. Extending the Atiyah-Guillemin-Sternberg Theorem, Atiyah and Pressley [AP83] proved the following:

Theorem 3.2.4 (Atiyah-Pressley [AP83])**.** *The image of ΩK under the moment map for the $S^1 \times T$ action is convex. Moreover, the image is prescribed as the convex hull of the $S^1 \times T$ fixed points, which are exactly the group homomorphisms $\mathrm{Hom}(S^1, T)$.*

We would like to expound upon this result in modest detail, as it forms the inspiration for our own convexity result elucidated in Chapter 5. The authors focus on the case where $K = \mathrm{SU}(n)$, arguing that for more general K one can embed $K \hookrightarrow \mathrm{SU}(n)$. We have mentioned previously that the algebraic based loops admits a decomposition into cells C_λ index by coweights $X_*(T_\mathbb{C})$. The closures of these cells $\overline{C_\lambda}$ are generally singular varieties, but can be embedded into finite dimensional Grassmannians. Using the results of [Ati82], one can show that $\mu(\overline{C_\lambda})$ are convex. As $X_*(T_\mathbb{C})$ is a directed system, convexity of $\mu(\Omega_{\mathrm{alg}} K)$ follows quickly. As the $\Omega_{\mathrm{alg}} K$ is dense in ΩK and the image $\mu(\Omega_{\mathrm{alg}} K)$ is closed, continuity of the moment map immediately implies that $\mu(\Omega K) = \mu(\Omega_{\mathrm{alg}} K)$, giving the desired result. In Chapter 5 we will comment on the obstacles encountered when attempting to generalize this process to our results, as well as the techniques used to overcome those hurdles.

The interested and knowledgeable reader might note that the technique used in [Ati82] to demonstrate convexity hinged upon showing that the preimage of level sets is convex. This result is still true in the case of ΩK, as was demonstrated for regular level sets in [Har+06] and later for all level sets in [Mar10]. For those interested in connectivity and convexity properties of general Hilbert manifolds, we refer the reader to [Smi14].

Jeffrey and Mare [JM10] were able to generalize the Duistermaat Convexity Theorem for a special class of involution. In particular, assume that the Lie group K comes endowed with an involutive auto-morphism $\sigma : K \to K$ such that for all $t \in T$, $\sigma(t) = t^{-1}$. Define $\tau : \Omega K \to \Omega K$ by $[\tau(\gamma)](z) = \sigma(\gamma(\bar{z}))$, which can be shown to be an anti-symplectic involution on ΩK.

Theorem 3.2.5 (Jeffrey-Mare)**.** *If ΩK^τ is the real locus consisting of τ-fixed points, then $\mu(\Omega K^\tau) = \mu(\Omega K)$.*

It is straightforward to show that $\mathrm{Hom}(S^1, T) \subseteq \Omega K^\tau$, and the authors were able to show that $\mu(\Omega K^\tau)$ was convex using Terng convexity for isoparametric manifolds [Ter93]. These two results suffice to demonstrate equality. The authors posed the problem to determine whether the result also held for $\Omega_{\mathrm{alg}} K$, which we will demonstrate in the affirmative in Chapter 5.

We presented three theorems in Section 3.2.1 and only two analogies in this section thus far. Notably, we do yet have an appropriate generalization of the Kirwan convexity theorem. This is one of the results of this thesis, and will be demonstrated in Section 5.

Chapter 4

Generalized Equivariant Cohomology of Stratified Spaces

Generalized equivariant cohomology theories have received considerable attention in the modern research literature. Particular emphasis has been placed on cohomology computations in the presence of well-behaved equivariant stratifications. Atiyah and Bott [AB83] gave an inductive procedure for computing the equivariant cohomology of a manifold in terms of the cohomologies of the strata in an equivariant stratification. Kirwan [Kir84b] then applied related ideas to a Morse-type stratification arising from the norm-square of a moment map. A paper by Harada, Henriques, and Holm [HHH05] subsequently broadened this Atiyah-Bott-Kirwan framework to include generalized equivariant cohomology calculations via infinite stratifications. This work was partly motivated by a desire to develop a GKM-type theory for the partial flag varieties of Kac-Moody groups.

This work represents a joint venture with Peter Crooks, wherein our initial objective was to compute the equivariant singular cohomology of the algebraic based loop group $H^*_{S^1\times T}(\Omega_{\text{alg}}K)$. Once completed, it was realized that the details of the proof depended neither upon the particular choice of singular cohomology, nor upon the particular structure of $\Omega_{\text{alg}}K$. This spawned an immediate inquiry into the most general possible result that could be obtained. The abstract nature of that generalization obfuscates the original motivation, and as such the work below may appear to be non sequitur. However, we will return to $\Omega_{\text{alg}}K$ as an application of our theorem, putting the result into appropriate context.

As mentioned above and in Chapter 3, the equivariant cohomology of stratified spaces in general was considered in [HHH05], who gave a combinatorial description of the corresponding ring structure. While much of our work was inspired by that paper, there are some important distinctions to be made. In [HHH05], the authors first work in the context of a topological group G and a fairly general stratified G-space X. Among other things, they provide some conditions on the stratification which explicitly determine the generalized G-equivariant cohomology of X in terms of the cohomologies of the strata. By contrast, herein we narrow our scope and deal with stratifications only in the context of a compact torus T and a smooth complex projective $T_{\mathbb{C}}$-variety X. Further demanding that our strata admit finitely many T-fixed points will allow us to more concretely, if non-canonically, present the module structure of our associated cohomology theory. We try to emphasize that the task of computing the generalized T-equivariant cohomology of X, or direct limits thereof, is especially simple.

This chapter has two principal objectives: The first is to provide a straightforward, self-contained account of how to perform generalized torus-equivariant cohomology computations with a finite equivariant stratification of a smooth complex projective variety. While this is readily deducible from existing work, we believe it might serve as a convenient reference for other authors. More importantly, however, it provides the context for the second of our objectives– a computation of the generalized torus-equivariant cohomology of a direct limit of smooth projective varieties with finitely many T-fixed points. More specifically, we will prove the following theorem.

Theorem 4.0.1. *Suppose that T is a compact torus with complexification $T_{\mathbb{C}}$, and let E_T^* be one of $H_T^*(\cdot;\mathbb{Z})$, K_T^*, and MU_T^*. Let $X_0 \subseteq X_1 \subseteq X_2 \subseteq \ldots$ be a sequence of equivariant closed embeddings of smooth complex projective $T_{\mathbb{C}}$-varieties, each with finitely many T-fixed points. If we define X to be the direct limit of the varieties X_n in their classical topologies, then*

$$E_T^*(X) \cong \prod_{x \in X^T} E_T^*(\mathrm{pt})$$

as $E_T^*(\mathrm{pt})$-modules.

Before miring the reader in the technical subtleties of our proof, the high level reasoning is as follows: Let X be a space with a finite equivariant stratification, defined rigorously below. Imitating [AB83], one can use a generalized Thom-Gysin sequence to reconstruct the cohomology of X in terms of its strata. If furthermore the T-fixed points of X are known to be finite, the module is free and of finite rank. If X now admits a filtration into components, each of which has an equivariant stratification and finitely many T-fixed points, then one can realize the cohomology of X as a direct limit over the filter elements explicitly.

4.1 Finite Stratifications

Throughout this section let T be a compact torus with complexification $T_{\mathbb{C}}$, and assume that E_T^* is a complex oriented generalized equivariant cohomology theory.

Definition 4.1.1. Let X be a smooth complex projective variety on which $T_{\mathbb{C}}$ acts algebraically. A *T-equivariant stratification of X* consists of a finite partially ordered set B and a collection $\{X_\beta\}_{\beta \in B}$ of pairwise disjoint, smooth, T-invariant, locally closed subvarieties of X satisfying

(i) $X = \bigcup_{\beta \in B} X_\beta$, and

(ii) $\overline{X_\beta} = \bigcup_{\gamma \leq \beta} X_\gamma$ for all $\beta \in B$.

Fix a smooth complex projective $T_\mathbb{C}$-variety X and let $\{X_\beta\}_{\beta \in B}$ be a given equivariant stratification. For each fixed $\beta \in B$, let $N_\beta \to X_\beta$ denote the normal bundle of X_β in X and let $d(\beta)$ denote its rank. The bundle N_β has a T-equivariant Thom class $u_T(\beta) \in \tilde{E}_T^{2d(\beta)}(\mathrm{Th}(N_\beta))$ and an associated Euler class $e_T(\beta) \in E_T^{2d(\beta)}(X_\beta)$.

Theorem 4.1.2. *Assume that for each $\beta \in B$, $E_T^*(X_\beta)$ is a free module over $E_T^*(\mathrm{pt})$, and that $e_T(\beta)$ is not a zero-divisor in $E_T^*(X_\beta)$. There is an isomorphism*

$$E_T^*(X) \cong \bigoplus_{\beta \in B} E_T^*(X_\beta)$$

of $E_T^*(\mathrm{pt})$-modules.

Proof. Following [AB83], we define a subset $J \subseteq B$ to be *open* if whenever $\beta \in J$ and $\gamma \in B$ satisfy $\beta \leq \gamma$, we have $\gamma \in J$. This definition has the desirable property that if $J \subseteq B$ is open, then

$$X_J := \bigcup_{\beta \in J} X_\beta$$

is an open subset of X.

Choose a maximal element $\beta_1 \in B$ and set $J_1 := \{\beta_1\}$, an open subset of B. We inductively define subsets $J_k \subseteq B$, $k \in \{2, \ldots, |B|\}$, by the condition that $J_k = \{\beta_1, \ldots, \beta_k\}$ with β_k a maximal element of $B \setminus J_{k-1}$. By construction, J_k is open for all k.

We have graded $E_T^*(\mathrm{pt})$-module isomorphisms

$$E_T^*(X_{J_k}, X_{J_{k-1}}) \cong E_T^*(\mathrm{Th}(N_{\beta_k})) \cong E_T^{*-2d(\beta_k)}(X_{\beta_k}), \tag{4.1}$$

the second being the Thom Isomorphism (see [May96], Theorem 9.2). Using (4.1), the long exact sequence of the pair $(X_{J_k}, X_{J_{k-1}})$ takes the form

$$\ldots \longrightarrow E_T^{i-2d(\beta_k)}(X_{\beta_k}) \xrightarrow{\phi} E_T^i(X_{J_k}) \longrightarrow E_T^i(X_{J_{k-1}}) E_T^{i-2d(\beta_k)+1}(X_{\beta_k}) \longrightarrow \ldots. \tag{4.2}$$

If $E_T^i(X_{J_k}) \to E_T^i(\beta_k)$ is the restriction map, the composition

$$E_T^{i-2d(\beta_k)}(X_{\beta_k}) \xrightarrow{\phi} E_T^i(X_{J_k}) \longrightarrow E_T^i(X_{\beta_k})$$

is equivalent to multiplication by the equivariant Euler class $e_T(\beta_k)$. As $e_T(\beta_k)$ is not a zero divisor, the composition is injective, forcing ϕ to be injective. Hence (4.2) degenerates to the short exact sequence

$$0 \longrightarrow E_T^{*-2d(\beta_k)}(X_{\beta_k}) \longrightarrow E_T^*(X_{J_k}) \longrightarrow E_T^*(X_{J_{k-1}}) \longrightarrow 0 \tag{4.3}$$

of $E_T^*(\mathrm{pt})$-modules. Using (4.3) and induction, we will prove that

$$E_T^*(X_{J_k}) \cong \bigoplus_{\ell \leq k} E_T^*(X_{\beta_\ell}) \tag{4.4}$$

for all $k \in \{2, \ldots, |B|\}$, from which the theorem will follow.

In the base case $k = 2$, our short exact sequence is

$$0 \longrightarrow E_T^{*-2d(\beta_2)}(X_{\beta_2}) \longrightarrow E_T^*(X_{J_2}) \longrightarrow E_T^*(X_{\beta_1}) \longrightarrow 0.$$

This sequence splits by virtue of the fact that $E_T^*(X_{\beta_1})$ is a free $E_T^*(\mathrm{pt})$-module, hence

$$E_T^*(X_{J_2}) \cong E_T^*(X_{\beta_1}) \oplus E_T^*(X_{\beta_2}).$$

Assume now that (4.4) holds for some $k \leq |B|-1$ and replace k with $k+1$ in (4.3) to obtain the sequence

$$0 \longrightarrow E_T^{*-2d(\beta_{k+1})}(X_{\beta_{k+1}}) \longrightarrow E_T^*(X_{J_{k+1}}) \longrightarrow E_T^*(X_{J_k}) \longrightarrow 0. \tag{4.5}$$

By assumption, $E_T^*(X_{J_k})$ is free, so (4.5) splits, and (4.4) holds if we replace k with $k+1$, completing the induction. □

Remark 4.1.3. The isomorphism in Theorem 4.1.2 does not respect the $\mathbb{Z}$-gradings of $E_T^*(X)$ and $\bigoplus_{\beta \in B} E_T^*(X_\beta)$. To compensate for the degree-shift of $2d(\beta)$ appearing in (4.3), one can identify $E_T^*(X_\beta)$ as an $E_T^*(\mathrm{pt})$-module with the principal ideal $\langle e_T(\beta) \rangle$ generated by $e_T(\beta)$. This gives us an isomorphism

$$E_T^*(X) \cong \bigoplus_{\beta \in B} \langle e_T(\beta) \rangle \tag{4.6}$$

on the level of both $E_T^*(\mathrm{pt})$-modules and $\mathbb{Z}$-graded abelian groups.

4.2 The Case of Finitely Many Fixed Points

The approach outlined in Section 4.1 can be combined with a suitable Białynicki-Birula stratification to yield the E_T^*-module structure of a smooth complex projective $T_{\mathbb{C}}$-variety with finitely many T-fixed points. More explicitly, we will prove the following theorem:

Theorem 4.2.1. *Suppose that E_T^* is one of H_T^*, K_T^*, and MU_T^*. If X is a smooth complex projective $T_{\mathbb{C}}$-variety with finitely many T-fixed points, then $E_T^*(X)$ is a free $E_T^*(\mathrm{pt})$-module of rank $|X^T|$.*

For the duration of this section, we will assume that everything is as given in the statement of Theorem 4.2.1.

Lemma 4.2.2. *There exists a coweight $\lambda : \mathbb{C}^* \to T_{\mathbb{C}}$ with the property that the fixed points of the resulting $\mathbb{C}^*$-action on X are precisely the T-fixed points.*

Proof. Choose a coweight λ such that for every $w \in X^T$ and weight $\mu : T_{\mathbb{C}} \to \mathbb{C}^*$ of the isotropy representation T_wX, the pairing $\langle \lambda, \mu \rangle$ is non-zero. This coweight yields an algebraic action of $\mathbb{C}^*$ on X, and we suppose that Y is an irreducible component of $X^{\mathbb{C}^*}$. Note that Y is a smooth closed $T_{\mathbb{C}}$-invariant subvariety of X. By the Borel Fixed Point Theorem, Y has a T-fixed point y. Since T_yY is precisely the trivial weight space of the $\mathbb{C}^*$-representation on T_yX, our choice of λ implies that $T_yY = \{0\}$. It follows that $Y = \{y\}$, giving the inclusion $X^{\mathbb{C}^*} \subseteq X^T$. □

Now, select $\lambda : \mathbb{C}^* \to T_{\mathbb{C}}$ as in Lemma 4.2.2. Given $w \in X^{\mathbb{C}^*} = X^T$, one has the smooth locally closed subvariety

$$X_w := \left\{ x \in X : \lim_{t \to 0} (\lambda(t) \cdot x) = w \right\}. \tag{4.7}$$

The X_w constitute a Białynicki-Birula stratification [BB73], a T-equivariant stratification of X. Furthermore, X_w is T-equivariantly homeomorphic to the T-submodule $(T_wX)^+$ of T_wX spanned by the weight vectors whose weights have strictly positive pairing with λ. In particular, X_w equivariantly retracts onto its T-fixed point $\{w\}$ and we have a ring isomorphism $r_w : E_T^*(X_w) \xrightarrow{\cong} E_T^*(\{w\})$. If $e_T(w) \in E_T^*(X_w)$ denotes the T-equivariant Euler class of the normal bundle of X_w in X, then $r_w(e_T(w))$ is the T-equivariant Euler class of the quotient representation $T_w(X)/T_wX_w \to \{w\}$.

Lemma 4.2.3. *Let V be a finite-dimensional complex T-representation such that $V^T = \{0\}$, viewed as a T-equivariant vector bundle over a point. If E_T^* is H_T^*, K_T^*, or MU_T^*, then the T-equivariant Euler class $e_T(V) \in E_T^*(\mathrm{pt})$ is not a zero divisor.*

Proof. Note that $E_T^*(\mathrm{pt})$ is an integral domain for each of the above three theories. By virtue of the Whitney sum formula, it therefore suffices to prove that $e_T(V)$ is non-zero when V is one-dimensional.

Let $\mu \in X^*(T)$ be the (non-zero) weight of V. If $E_T^* = H_T^*$, then $e_T(V)$ is the ordinary Euler class of the associated bundle $ET \times_T V \to BT$. Under the usual ring isomorphism $H^*(BT; \mathbb{Z}) \cong \mathrm{Sym}_{\mathbb{Z}}(X^*(T))$, this Euler class corresponds to the weight μ.

When $E_T^* = K_T^*$, the equivariant Euler class of a complex T-representation is given by the alternating sum of its exterior powers in $K_T^*(\mathrm{pt})$[May96, Chapter XIV, Theorem 3.2]. Hence, $e_T(V) = 1 - [V] \in K_T^2(\mathrm{pt})$, which is identified with $1 - e^{\mu}$ under the isomorphism $K_T^2(\mathrm{pt}) \cong R(T)$. We thus see that $e_T(V) \neq 0$.

In the case of MU_T^*, we simply appeal to Theorem 1.2 of [Sin01]. □

Since the T-fixed points in X are isolated, zero is not a weight of the representation T_wX/T_wX_w. By Lemma 4.2.3, we conclude that $r_w(e_T(w))$ is not a zero-divisor in $E_T^*(\{w\})$, meaning that $e_T(w)$ is not a zero divisor. An application of Theorem 4.1.2 then yields an

$E_T^*(\mathrm{pt})$-module isomorphism

$$E_T^*(X) \cong \bigoplus_{w \in X^T} E_T^*(X_w).$$

In particular, $E_T^*(X)$ is free of rank $|X^T|$, proving Theorem 4.2.1.

Theorem 4.2.1 will prove essential in extending our results to the case of direct limits of projective varieties. To realize the extension, we will require the following lemma.

Proposition 4.2.4. *If Y is a smooth closed $T_{\mathbb{C}}$-invariant subvariety of X, then*

(i) $E_T^*(X,Y)$ is a free $E_T^*(\mathrm{pt})$-module of finite rank vanishing in odd grading degrees, and

(ii) the restriction map $E_T^*(X) \to E_T^*(Y)$ is surjective.

Proof. To prove (i), we will appeal to some general properties of model categories. Indeed, T-spaces form a model category in which the weak equivalences are the T-homotopy equivalences and the cofibrations are the morphisms with the T-homotopy extension property. Accordingly, we will begin by proving the following claim by induction: If $w_1, \ldots, w_n \in Y^T$ and X_{w_i} are the associated Białynicki-Birula strata, then the inclusion

$$Y \to Y \cup \bigcup_{i=1}^{n} X_{w_i}$$

is an acyclic cofibration; that is, a cofibration that is also a weak equivalence.

For the base case, let $Y_{w_1} \subseteq Y$ denote the Białynicki-Birula stratum of Y associated with $w_1 \in Y^T$. One has the pushout square

$$\begin{array}{ccc} Y_{w_1} & \longrightarrow & X_{w_1} \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y \cup X_{w_1} \end{array}$$

of inclusions. Note that $Y_{w_1} \to X_{w_1}$ is an acyclic cofibration. Since the pushout of an acyclic cofibration is itself an acyclic cofibration, it follows that $Y \to Y \cup X_{w_1}$ is an acyclic cofibration. Now, assume that our claim holds for $\leq n$ points in Y^T. Given $w_1, \ldots, w_{n+1} \in Y^T$, we consider the pushout square

$$\begin{array}{ccc} Y & \xrightarrow{i_1} & Y \cup \bigcup_{i=1}^{n} X_{w_i} \\ \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle j_2} \\ Y \cup X_{w_{n+1}} & \xrightarrow{j_1} & Y \cup \bigcup_{i=1}^{n+1} X_{w_i} \end{array}$$

of inclusions. Noting that i_1 is an acyclic cofibration, the same is true of j_1. The inclusion $Y \to Y \cup \bigcup_{i=1}^{n+1} X_{w_i}$ is then a composition of the acyclic cofibrations i_2 and j_1, and so is itself an acyclic cofibration. This completes the induction. Setting

$$Z := \bigcup_{w \in Y^T} X_w,$$

it follows that $Y \to Z$ is an acyclic cofibration. In particular, $E_T^*(Z, Y) = 0$, and it just remains to prove that $E_T^*(X, Z)$ is free of finite rank and vanishes in odd degrees.

Recall that if $w \in X^T$, then X_w is T-equivariantly homeomorphic to a finite-dimensional complex T-representation V_w. Choose an enumeration $\{w_1, \ldots, w_m\}$ of $X^T \setminus Y^T$ with the property that for all $k \in \{1, \ldots, m\}$, the quotient of $Z \cup \bigcup_{j=1}^k X_{w_j}$ by $Z \cup \bigcup_{j=1}^{k-1} X_{w_j}$ is T-equivariantly homeomorphic to the one-point compactification $S^{V_{w_k}}$. For simplicity of notation, write $V_k = Z \cup \bigcup_{j=1}^k X_{w_j}$ with $V_0 = Z$. Using induction, we will prove that $E_T^*(V_k, Z)$ is free of finite rank for all $k \in \{1, \ldots, m\}$, and that it vanishes in odd grading degrees.

Since $V_0 \cap X_{w_1} = \emptyset$, the inclusion $V_0 \hookrightarrow V_1$ is a cofibration, hence

$$E_T^*(V_1, V_0) \cong \tilde{E}_T^*(V_1/V_0) \cong \tilde{E}_T^*\left(S^{V_{w_k}}\right)$$

is free of finite rank, and vanishes in odd grading degrees. Now, assume that $E_T^*(V_k, V_0)$ vanishes in odd degrees and is free of finite rank. Since the inclusion $V_k \hookrightarrow V_{k+1}$ is a cofibration, we find that

$$E_T^*(V_{k+1}, V_k) \cong \tilde{E}_T^*(V_{k+1}/V_k) \cong \tilde{E}_T^*\left(S^{V_{w_{k+1}}}\right)$$

is also free of finite rank and vanishes in odd degrees. Therefore, the long exact sequence of the pairs

$$(V_{k+1}, V_k), \quad (V_{k+1}, V_0), \quad (V_k, V_0)$$

splits to give the short exact sequence

$$0 \to E_T^*(V_{k+1}, V_k) \to E_T^*(V_{k+1}, V_0) \to E_T^*(V_k, V_0) \to 0.$$

Since $E_T^*(V_{k+1}, V_k)$ and $E_T^*(V_k, V_0)$ are free of finite rank, the same is true of $E_T^*(V_{k+1}, V_0)$. We have therefore proved (i).

For (ii), we consider the long exact sequence of the pair (X, Y). Indeed, (i) is then seen to imply that $E_T^n(X) \to E_T^n(Y)$ is surjective for even n. Furthermore, the isomorphism (4.6) establishes that both $E_T^*(X)$ and $E_T^*(Y)$ vanish in odd grading degrees. The proof is therefore complete. □

4.3 Direct Limits of Projective Varieties

We now provide a generalization of our findings in Section 4.2, replacing projective varieties with direct limits thereof. As before, T denotes a compact torus with complexification $T_{\mathbb{C}}$, and

E_T^* is one of H_T^*, K_T^*, and MU_T^*. Suppose that

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq \ldots \subseteq X_n \subseteq \ldots$$

is a sequence of equivariant closed embeddings of smooth complex projective $T_{\mathbb{C}}$-varieties with $(X_n)^T$ finite for each $n \geq 0$. Let X be the topological direct limit of the X_n in their analytic topologies, and endow X with the induced direct limit topology. Note that X then carries a continuous action of T. The following theorem then generalizes Theorem 4.2.1:

Theorem 4.3.1. *Under the conditions stated above, there is an $E_T^*(\mathrm{pt})$-module isomorphism*

$$E_T^*(X) \cong \prod_{x \in X^T} E_T^*(\mathrm{pt}).$$

Proof. By Proposition 4.2.4, each restriction map $E_T^*(X_{n+1}) \to E_T^*(X_n)$ is surjective. Hence, the inverse system $\{E_T^*(X_n)\}_n$ of $E_T^*(\mathrm{pt})$-modules has vanishing Milnor $\varprojlim^1$. It follows that the canonical map $E_T^*(X) \to \varprojlim_n E_T^*(X_n)$ is an isomorphism[HPS97].

It will therefore suffice to prove that $\{E_T^*(X_n)\}_n$ and $\{\bigoplus_{x \in (X_n)^T} E_T^*(\mathrm{pt})\}_n$ are isomorphic as inverse systems of $E_T^*(\mathrm{pt})$-modules, where the maps in the latter system are precisely the projection maps resulting from the inclusions $(X_n)^T \subseteq (X_{n+1})^T$. We will do this by inductively constructing $E_T^*(\mathrm{pt})$-module isomorphisms

$$\psi_n : E_T^*(X_n) \to \bigoplus_{x \in (X_n)^T} E_T^*(\mathrm{pt})$$

making the diagrams

$$\begin{array}{cccc} D_n := & E_T^*(X_{n+1}) & \xrightarrow{\psi_{n+1}} & \bigoplus_{x \in (X_{n+1})^T} E_T^*(\mathrm{pt}) \\ & \downarrow & & \downarrow \\ & E_T^*(X_n) & \xrightarrow{\psi_n} & \bigoplus_{x \in (X_n)^T} E_T^*(\mathrm{pt}) \end{array}$$

commute.

By Theorem 4.2.1, we have an $E_T^*(\mathrm{pt})$-module isomorphism

$$\psi_0 : E_T^*(X_0) \to \bigoplus_{x \in (X_0)^T} E_T^*(\mathrm{pt}).$$

Assume now that we have constructed isomorphisms $\psi_k : E_T^*(X_k) \to \bigoplus_{x \in (X_k)^T} E_T^*(\mathrm{pt})$ for all $k \leq n$ so that the diagrams $D_0, \ldots, D_{n-1}$ commute. Since the restriction $\pi_n : E_T^*(X_{n+1}) \to E_T^*(X_n)$ is surjective, the long exact sequence of the pair (X_{n+1}, X_n) degenerates to a short exact sequence

$$0 \to E_T^*(X_{n+1}, X_n) \to E_T^*(X_{n+1}) \xrightarrow{\pi_n} E_T^*(X_n) \to 0 \tag{4.8}$$

of $E_T^*(\mathrm{pt})$-modules. Theorem 4.2.1 implies that $E_T^*(X_n)$ is free, so that (4.8) admits a splitting $\varphi_n : E_T^*(X_{n+1}) \to E_T^*(X_{n+1}, X_n)$. Also, Proposition 4.2.4 implies that $E_T^*(X_{n+1}, X_n)$ is free of rank $|(X_{n+1})^T \setminus (X_n)^T|$, therefore we may choose an $E_T^*(\mathrm{pt})$-module isomorphism

$$\theta_n : E_T^*(X_{n+1}, X_n) \xrightarrow{\cong} \bigoplus_{x \in (X_{n+1})^T \setminus (X_n)^T} E_T^*(\mathrm{pt}).$$

The composite map

$$E_T^*(X_{n+1}) \xrightarrow{(\pi_n, \varphi_n)} E_T^*(X_n) \oplus E_T^*(X_{n+1}, X_n) \xrightarrow{\psi_n \oplus \theta_n} \bigoplus_{x \in (X_{n+1})^T} E_T^*(\mathrm{pt})$$

is then an $E_T^*(\mathrm{pt})$-module isomorphism that we shall call ψ_{n+1}. By construction, D_n commutes for this choice of ψ_{n+1}, completing the proof. □

4.4 The Algebraic Based Loop Group

As promised, we return to a concrete application of Theorem 4.3.1 to the algebraic based loop group.

Let K be a compact, connected, simply connected Lie group with complexification $K_{\mathbb{C}}$ and algebraic based loop group $\Omega_{\mathrm{alg}} K$. Choose a faithful irreducible embedding $K_{\mathbb{C}} \hookrightarrow \mathrm{SL}_n(\mathbb{C})$, so that loops can be realized as Fourier series with coefficients in $\mathrm{M}_n(\mathbb{C})$. In Chapter 2 we saw that $\Omega_{\mathrm{alg}} K \cong L_{\mathrm{alg}} K_{\mathbb{C}} / L^+_{\mathrm{alg}} K_{\mathbb{C}}$. Let $z^{-k} L^+_{\mathrm{alg}} K_{\mathbb{C}}$ represent the loops with pole order k, so that

$$L^+_{\mathrm{alg}} K_{\mathbb{C}} \subseteq z^{-1} L^+_{\mathrm{alg}} K_{\mathbb{C}} \subseteq z^{-2} L^+_{\mathrm{alg}} K_{\mathbb{C}} \subseteq \cdots \subseteq L_{\mathrm{alg}} K_{\mathbb{C}}.$$

The spaces $z^{-k} L^+_{\mathrm{alg}} K_{\mathbb{C}}$ are invariant under the action of $L^+_{\mathrm{alg}} K_{\mathbb{C}}$ and hence we can pass to the quotient. Defining $\mathcal{G}r_k = L_{\mathrm{alg}} K_{\mathbb{C}} / (z^{-k} L^+_{\mathrm{alg}} K_{\mathbb{C}})$ gives the filtration:

$$1 \subseteq \mathcal{G}r_1 \subseteq \mathcal{G}r_2 \subseteq \cdots \subseteq \Omega_{\mathrm{alg}} K_{\mathbb{C}}.$$

The $\mathcal{G}r_k$ are very similar to the $\Omega_{\mathrm{alg},k} K$ filtration, and a similar argument shows that $\mathcal{G}r_k$ is invariant under the $S^1 \times T$ action, making the above filtration equivariant.

With Theorem 4.3.1 in mind, it remains only to prove that $(\mathcal{G}r_k)^T$ is finite for all $k \geq 0$. To this end, let $\lambda \in X_*(T_{\mathbb{C}})$ be a coweight, and consider the point in $L_{\mathrm{alg}} K_{\mathbb{C}}$ given by the composition

$$\mathbb{C}^* \xrightarrow{\lambda} T_{\mathbb{C}} \hookrightarrow K_{\mathbb{C}}, \tag{4.9}$$

where $T_{\mathbb{C}} \hookrightarrow G$ is the inclusion. Let $z^\lambda \in \Omega_{\mathrm{alg}} K$ denote the class of (4.9) in the algebraic loops. We recall from Theorem 3.2.4 that the $S^1 \times T$ fixed points of $\Omega_{\mathrm{alg}} K$ are precisely the elements $X_*(T_{\mathbb{C}})$, which leads to the following theorem:

Lemma 4.4.1. *For $n \geq 0$,*

$$(\mathcal{G}r_n)^{S^1 \times T} = \{z^{w\lambda} : w \in W,\ \lambda \in X_*(T_{\mathbb{C}})_+, \langle \lambda, \alpha_0 \rangle \geq -n\},$$

where $\alpha_0 \in X^*(T_{\mathbb{C}})$ is the lowest weight of $\mathrm{SL}_n(\mathbb{C})$ in its standard representation. In particular, $(\mathcal{G}r_n)^T$ is finite.

Proof. Since $\mathcal{G}r_n$ is $K_{\mathbb{C}}$-invariant, one has an induced action of W on $(\mathcal{G}r_n)^{T_{\mathbb{C}}}$. Because the actions of $K_{\mathbb{C}}$ and $\mathbb{C}^*$ commute, the W-action leaves $(\mathcal{G}r_n)^{\mathbb{C}^* \times T_{\mathbb{C}}} = (\mathcal{G}r_n)^{S^1 \times T}$ invariant. Hence, if $\mu \in X_*(T_{\mathbb{C}})$ is W-conjugate to $\lambda \in X_*(T_{\mathbb{C}})_+$, then $z^\mu \in (\mathcal{G}r_n)^T$ if and only if $z^\lambda \in (\mathcal{G}r_n)^{S^1 \times T}$. Our task is therefore to prove that if $\lambda \in X_*(T_{\mathbb{C}})_+$, then $z^\lambda \in (\mathcal{G}r_n)^{S^1 \times T}$ if and only if $\langle \lambda, \alpha_0 \rangle \geq -n$.

Suppose that $\lambda \in X_*(T_{\mathbb{C}})_+$, and let $v \in \mathbb{C}^n$ be a vector of weight $\xi \in X^*(T_{\mathbb{C}})$. Note that for all $t \in \mathbb{C}^*$,

$$\lambda(t) \cdot v = \xi(\lambda(t))v = z^{\langle \lambda, \xi \rangle} v.$$

Hence, if we regard λ as a point in $G(\mathcal{K})$, then

$$\lambda \cdot v = z^{\langle \lambda, \xi \rangle} v \in \mathbb{C}^n \otimes z^{\langle \lambda, \xi \rangle} \mathbb{C}[t].$$

Since $\mathbb{C}^n$ has a basis of weight vectors, it follows that the pole-order of λ is the minimum of $\langle \lambda, \xi \rangle$, where ξ ranges over the weights of the standard representation. Noting that α_0 is the lowest weight of standard representation, this naturally occurs at $\langle \lambda, w_0 \alpha \rangle$. Therefore, $\lambda \in G(\mathcal{K})_n$ if and only if $\langle \lambda, \alpha_0 \rangle \geq -n$. □

We may thus apply Theorem 4.3.1 to compute the module structure of $E_T^*(\Omega_{\mathrm{alg}} K)$ for $E_T^* = H_T^*$, K_T^*, or MU_T^*. Indeed, we have

$$E_T^*(\Omega_{\mathrm{alg}} K) \cong \prod_{\lambda \in X_*(T_{\mathbb{C}})} E_T^*(\mathrm{pt})$$

as $E_T^*(\mathrm{pt})$-modules.

4.5 Further Work

Our entire discussion has been predicated on the study of the module structure of our cohomology theories. A natural question is whether one can derive the algebra structure from the same argument. Unfortunately, the morphisms induced by the equivariant Thom-Gysin sequence fail to be ring maps, and the direct limits in Section 4.3 are therefore restricted to the category of modules. Determining the product structure therefore remains an open question, and one that seems reasonable using the equivariant stratification approach.

With specific regard to the based loop group, we have computed the module structure of $E_T^*(\Omega_{\mathrm{alg}} K)$. The question remains as to whether this can be extended to the entire loop group, and whether or not $E_T^*(\Omega_{\text{alg}}K) = E_T^*(\Omega K)$.

The result is known to be true in the case where E_T^* is equivariant singular cohomology H_T^*, for in that case one can use the homotopy equivalence $\Omega_{\text{alg}}K \simeq \Omega K$ to show homotopy equivalence of the Borel mixing spaces $ET \times_T \Omega_{\text{alg}}K \simeq ET \times_T \Omega K$, from which

$$H_T^*(\Omega_{\text{alg}}K) = H^*(ET \times_T \Omega_{\text{alg}}K) \cong H^*(ET \times_T \Omega K) = H_T^*(\Omega K).$$

However, the Borel mixing space approach does not generalize to K-theory, nor to complex cobordism.

One avenue of attack would be to demonstrate the equivariant homotopy equivalence of $\Omega_{\text{alg}}K$ and ΩK. Work in this direction can be found in [HJS14], who demonstrated the SU(2)-equivariant homotopy equivalence of $\Omega_{\text{alg}}\text{SU}(2)$ and $\Omega\text{SU}(2)$. The aforementioned paper has established the basic framework for how such a proof could be approached in the case of a general compact, connected, simply connected Lie group K, and so awaits generalization. If this result could be demonstrated, the above work would immediately yield the generalized equivariant cohomology of ΩK.

Chapter 5

Convexity of the Based Loop Group

Generalizing upon the finite dimensional results of the Atiyah-Guillemin-Sternberg Theorem [Ati82; GS84], Atiyah and Pressley [AP83] showed that by restricting the $S^1 \times K$ action to that of a maximal torus $S^1 \times T$, the image $\mu(\Omega K)$ is an unbounded convex region, generated as the convex hull of infinitely many discrete points. Generalizing the work of Duistermaat [Dui83], Jeffrey and Mare [JM10] showed that if ΩK is additionally endowed with an anti-symplectic involution $\tau : \Omega K \to \Omega K$, then the image of ΩK under the moment map coincides with that of its real locus ΩK^τ.

The purpose of this chapter is two-fold: The first is to describe the image of ΩK for the non-abelian moment map, while the second is to give a non-abelian version of Duistermaat convexity. In both cases we use the Bruhat decomposition of $\Omega_{\text{alg}} K$ to show the result for the Schubert varieties before collating for the result on the entirety of the algebraic loops. Density arguments then yield the result of ΩK.

5.1 Non-abelian Convexity

Herein we examine the image of the moment map $\mu : \Omega K \to \mathbb{R} \oplus \mathfrak{k}$ for the $S^1 \times K$-action. It is well known that when the group is non-abelian, the image of the moment is not necessarily convex. Rather, Kirwan [Kir84a] demonstrated that the correct analog lies in considering that portion of the moment map image which resides within the positive Weyl chamber $\mathfrak{t}^*_+$. Several authors have generalized the aforementioned work, though our particular interest lies with the "highest weight polytope" approach of Brion [Bri87], who showed that $K_\mathbb{C}$-invariant subvarieties have convex image in $\mathfrak{t}^*_+$. Guillemin and Sjamaar [GS06] extended Brion's work to those varieties invariant under just the action of the Borel subgroup B, and it is this result that we exploit here. The following is the main result of that paper, paraphrased to omit the structure of the highest weight polytope whose particular structure is not necessary for our result.

Theorem 5.1.1 ([GS06, Theorem 2.1])**.** *Let M be a compact, complex manifold and $L \to M$ a positive Hermitian line bundle with Hermitian connection ∇ and curvature form ω. Let K*

be a compact Lie group with complexification $K_{\mathbb{C}}$ and $B \subseteq K_{\mathbb{C}}$ a Borel subgroup. Assume that K acts on L by line bundle automorphisms, which preserve the complex structure on M and Hermitian structure on L. If X is a B-invariant irreducible closed analytic subvariety of M, then $\Delta(X) = \mu(X) \cap \mathfrak{t}_+^*$ is a convex polytope, where $\mu : M \to \mathfrak{k}^*$ is the moment map corresponding to ω.

Let $\mu : \Omega K \to \mathbb{R} \oplus \mathfrak{k}$ be the moment map in Theorem 2.1.8 and define $\Delta(X) = \mu(X) \cap \mathfrak{t}_+^*$. By applying Theorem 5.1.1 to the Schubert varieties $\mathcal{B}\lambda$ of $\Omega_{\text{alg}} K$, we deduce the following theorem:

Theorem 5.1.2. *The image set $\Delta(\Omega_{alg} K)$ is convex.*

Proof. Fix some $\lambda \in X_*(T)$ and consider $X = \overline{\mathcal{B}\lambda} \subseteq \Omega_{\text{alg}} K$. Since the Iwahori subgroup $\mathcal{B}$ is the preimage of the Borel subgroup under the evaluation map, $\overline{\mathcal{B}\lambda}$ is clearly B-invariant. In Section 2.1.7 we showed that for sufficiently large $m \in \mathbb{N}$, there is an embedding $\overline{\mathcal{B}\lambda} \hookrightarrow \Omega_{\text{alg},m} K \hookrightarrow \mathcal{G}_m$, and a positive prequantum line bundle $\det_m \to \mathcal{G}_m$, whose curvature form is the Kähler form ω'_{FS}.

To define a $\mathbb{C}^\times \times K_{\mathbb{C}}$ linearization on $\det_m$, we will first define one on $\det_m^*$. Let $W \in \mathcal{G}_m$ and choose a homogeneous basis $\{w_i\}$ for W. Elements of $\det_m^*$ look like $(W, [\alpha, w])$ where $w = w_1 \wedge \cdots \wedge w_n$ and $\alpha \in \mathbb{C}$. Let $s \in \mathbb{C}^\times$ and let s act on each w_i by loop rotation, so that $s \cdot w_i = s^{k_i} w_i$, for some $k_i \in \mathbb{Z}$. This action is diagonal in this basis, and the induced action of $\mathbb{C}^\times$ on $\Lambda^{\text{top}} W$ is given by

$$s \cdot (W, [\alpha, w_1 \wedge \cdots \wedge w_n]) = (s \cdot W, [\alpha, w_1(sz) \wedge \cdots \wedge w_n(sz)]) = (s \cdot W, [\alpha s^k, w]),$$

where $k = k_1 + \cdots + k_n$. Similarly, if $k \in K_{\mathbb{C}}$ then $k \cdot w_i = k w_i$ is certainly linear. Moreover, since $K_{\mathbb{C}}$ is connected and semisimple, $\det(k) = 1$:

$$k \cdot (W, [\alpha, w]) = (k \cdot W, [\alpha, (kw_1) \wedge \cdots \wedge (kw_n)]) = (k \cdot W, [\alpha, w]).$$

These actions commute and hence define a $\mathbb{C}^\times \times K_{\mathbb{C}}$ action on $\det_m^*$ which commutes with the projection map.

The $K_{\mathbb{C}}$ action trivially preserves the Hermitian structure on $\det_m$. The $\mathbb{C}^\times$ action is also Hermitian, since

$$\begin{aligned} \langle s \cdot [\alpha_1, w], s \cdot [\alpha_2, w] \rangle &= \left\langle [s^k \alpha_1, w], [s^k \alpha_2, w] \right\rangle \\ &= s^k s^{-k} \langle [\alpha_1, w], [\alpha_2, w] \rangle \\ &= \langle [\alpha_1, w], [\alpha_2, w] \rangle . \end{aligned}$$

By passing to the dual, one derives an action of $\mathbb{C}^\times \times K_{\mathbb{C}}$ on $\det_m$. Since the Hermitian structure on $\det_m$ is just the dual of that on $\det_m^*$, one also concludes that the Hermitian structure on $\det_m$ is $\mathbb{C}^\times \times K_{\mathbb{C}}$-invariant. Furthermore, the $\mathbb{C}^\times \times K_{\mathbb{C}}$-action acts holomorphically on $\mathcal{G}_m$. Since $S^1 \times K$ includes into $\mathbb{C}^\times \times K_\mathbb{C}$ as its compact real form, we conclude that the corresponding real group action is Hermitian and preserves the complex structure. Theorem 5.1.1 thus implies that $\Delta(\overline{\mathcal{B}\lambda})$ is convex.

To extend from each Schubert variety to the whole algebraic based loop group, choose any two points $x_1, x_2 \in \Delta(\Omega_{\text{alg}}K)$ and take $p_i \in \Omega_{\text{alg}}K$ such that $\mu(p_i) = x_i$. There exist $\lambda_1, \lambda_2 \in X_*(T)$ such that $p_i \in \mathcal{B}\lambda_i$, and since $X_*(T)$ is directed, a λ such that both $\mathcal{B}\lambda_1 \subseteq \mathcal{B}\lambda$ and $\mathcal{B}\lambda_2 \subseteq \mathcal{B}\lambda$. Consequently, both $p_1, p_2 \in \overline{\mathcal{B}\lambda}$ whose image is convex, as required. □

Corollary 5.1.3. *The image set $\Delta(\Omega K)$ is also convex.*

Proof. The algebraic based loops are dense in ΩK [AP83]. Since the moment map is continuous, $\mu(\Omega_{\text{alg}}K)$ is dense in $\mu(\Omega K)$, and so $\overline{\Delta(\Omega_{\text{alg}}K)} = \Delta(\Omega K)$. Since the closure of a convex set is convex, the result follows. □

5.2 Duistermaat-type Convexity

We now turn our attention to the real locus of the based loop group; a discussion which necessitates introducing an anti-symplectic involution on ΩK. Assume that K is equipped with an involutive group automorphism $\sigma : K \to K$ such that if $T \subseteq K$ is a maximal torus, then $\sigma(t) = t^{-1}$ for every $t \in T$. Such an involution is guaranteed to exist by [Loo69]. The differential $d_e\sigma : \mathfrak{k} \to \mathfrak{k}$ is thus involutive as well and induces the eigenspace decomposition $\mathfrak{k} = \mathfrak{p} \oplus \mathfrak{q}$, where $\mathfrak{p}$ and $\mathfrak{q}$ are the ± 1-eigenspaces respectively. Extend $d_e\sigma$ to $\mathfrak{k}_\mathbb{C}$ in an anti-holomorphic fashion by setting

$$\hat{\sigma} : \mathfrak{k}_\mathbb{C} \to \mathfrak{k}_\mathbb{C}, \qquad X + iY \mapsto d_e\sigma(X) - i d_e\sigma(Y). \tag{5.1}$$

We define an involutive group automorphism $\tau : \Omega K \to \Omega K$ by $(\tau\gamma)(z) = \sigma(\gamma(\bar{z}))$ for $\gamma \in \Omega K$.

One can easily check that τ leaves $\Omega_{\text{alg}}K$ invariant, and so also defines an involution there. It is straightforward to check that τ also preserves the filtration $\Omega_{\text{alg},m}K$. We will often conflate τ with the corresponding $\mathbb{Z}_2$ action it induces on ΩK.

The τ fixed points of ΩK are denoted ΩK^τ, and are often referred to as the *real locus* of ΩK. The nomenclature is derivative of the finite dimensional regime, where our manifold is a complex variety and τ is complex conjugation. In [JM10], Jeffrey and Mare showed that $\mu_T(\Omega K) = \mu_T(\Omega K^\tau)$; in analog of Duistermaat's convexity theorem in finite dimensions [Dui83]. In the former paper, the question of whether $\mu_T(\Omega_{\text{alg}}K) = \mu_T(\Omega_{\text{alg}}K^\tau)$ was proposed. Our main theorem in this section will answer the more general non-abelian case in the affirmative, and give the result for the full class of Sobolev H^1-loops as an immediate corollary.

Once again, our strategy will be to demonstrate the result on the Schubert varieties $\overline{\mathcal{B}\lambda}$. Guillemin and Sjamaar [GS06] generalize Duistermaat's convexity result in finite dimensions to the non-abelian regime for singular varieties. Goldberg [Gol09] in turn combined this result with that of [OS00] to derive the same result for singular B-invariant varieties.

Definition 5.2.1 ([GS06]). Let U be a compact group with an involutive automorphism $\tilde{\sigma} : U \to U$. A *Hamiltonian* $(U, \tilde{\sigma})$*-manifold* is a quadruple (M, ω, τ, μ) such that (M, ω) is a smooth symplectic manifold, endowed with a Hamiltonian U-action for which $\mu : M \to \mathfrak{u}^*$ is the moment map. In addition, τ is an anti-symplectic involution, compatible with the U-action as follows:

1. $\mu(\tau(m)) = -(d_e\tilde{\sigma})^*(\mu(m))$,
2. $\tau(u \cdot m) = \tilde{\sigma}(u)\tau(m)$.

A $(U, \tilde{\sigma})$*-pair* $(X, Y) \subseteq M \times M^\tau$ is a pair such that

1. X is a U- and τ-stable irreducible closed complex subvariety of M,
2. $X_{\text{reg}} \cap M^\tau \neq \emptyset$, where X_{reg} are the regular points of X,
3. Y is the closure of a connected component of $X_{\text{reg}} \cap M^\tau$.

Theorem 5.2.2 ([Gol09, Theorem 1.7]). *Let (M, ω, τ, μ) be a compact, connected, $(U, \tilde{\sigma})$-manifold, equipped with a complex structure compatible with ω, relative to which τ is anti-holomorphic. Assume that the Borel subgroup $B \subseteq K_{\mathbb{C}}$ is preserved by the anti-holomorphic extension of $\tilde{\sigma}$, and take $\mathfrak{u} = \mathfrak{p} \oplus \mathfrak{q}$ to be the ± 1-eigenspace decomposition of $\mathfrak{u}$ with respect to $d_e\tilde{\sigma}$. Let $L \to M$ be a holomorphic line bundle with connection ∇ such that the curvature form $F_\nabla = -2\pi i\omega$, and assume τ lifts to an involutive, anti-holomorphic bundle map τ_L which preserves ∇. If (X, Y) is a $(U, \tilde{\sigma})$-pair such that X is B-invariant, τ-invariant, and X^τ is non-empty, then*

$$\Delta(Y) = \Delta(X) \cap \mathfrak{m}^* \tag{5.2}$$

where $\mathfrak{m}^* \subseteq \mathfrak{q}^*$ is a maximal abelian subspace of $\mathfrak{q}^*$.

Remark 5.2.3.

1. Note that when $\sigma(t) = t^{-1}$ as is our case, then $\mathfrak{t} \subseteq \mathfrak{q}$. Consequently, $\mathfrak{m} = \mathfrak{t}$, so that (5.2) simply becomes $\Delta(Y) = \Delta(X)$.
2. Equation (5.2) is stated for group actions that are not necessarily abelian, requiring the intersection of the image with the positive Weyl chamber. It is well known [GS05] that the moment polytope of the abelian action is intimately connected with the non-abelian one (being the union of the convex hull of the Weyl orbits of the $\lambda \in \mathfrak{t}^*$). Hence this theorem will be just as useful in the abelian case and gives a hint at further generalizations.

The Schubert varieties $\overline{\mathcal{B}\lambda}$ will play the role of X, while the Grassmannian $\mathcal{G}_m$ will play that of M. To begin, we convince ourselves that $\overline{\mathcal{B}\lambda}$ is a candidate for X, by showing that it is preserved by τ and that the embedding $\overline{\mathcal{B}\lambda} \hookrightarrow \mathcal{G}_m$ is $\mathbb{Z}_2$-equivariant. The following lemma is partially sketched in [Mit88, Theorem 5.9], and was communicated to us by Mare.

Proposition 5.2.4. *Each Schubert variety $\overline{\mathcal{B}\lambda}$ is invariant under τ.*

Proof. We first claim that the Borel subgroup B is invariant under the complexification of σ. Write $\mathfrak{k}_{\mathbb{C}}$ in its root decomposition as

$$\mathfrak{k}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{k}_\alpha^{\mathbb{C}}$$

where $\mathfrak{t}_{\mathbb{C}}$ is the complexification of $\mathfrak{t}$ and $\mathfrak{k}_\alpha^{\mathbb{C}}$ are the corresponding root spaces. Let $\hat\sigma : \mathfrak{k}_{\mathbb{C}} \to \mathfrak{k}_{\mathbb{C}}$ be the anti-holomorphic map defined in (5.1). It is sufficient to show that $\mathfrak{b}$ is invariant under $\hat\sigma$. Our choice of Borel corresponds to

$$\mathfrak{b} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha\in\Phi_+} k_\alpha^{\mathbb{C}},$$

and the definition of σ ensures that $\mathfrak{t}_{\mathbb{C}}$ is invariant under $\hat\sigma$, so we need only show that each $\mathfrak{k}_\alpha^{\mathbb{C}}$ is invariant under $\hat\sigma$. For this, let $X \in \mathfrak{k}_\alpha^{\mathbb{C}}$ so that $[H, X] = \alpha(H)X$ for all $H \in \mathfrak{t}_{\mathbb{C}}$. Note that it is in fact sufficient to just take $H \in \mathfrak{t}$, and that $\alpha|_{\mathfrak{t}} \in i\mathbb{R}$. Since $\hat\sigma$ is a Lie algebra automorphism, this yields

$$\begin{aligned}\hat\sigma[H, X] &= [\hat\sigma(H), \hat\sigma(X)] = -[H, \hat\sigma(X)]\\ &= \hat\sigma(\alpha(H)X) = \overline{\alpha(H)}\hat\sigma(X) = -\alpha(H)\hat\sigma(X)\end{aligned}$$

so that $[H, \hat\sigma(X)] = \alpha(H)\hat\sigma(X)$ showing that $\mathfrak{k}_\alpha^{\mathbb{C}}$ is invariant under $\hat\sigma$.

Invariance of $\mathcal{B}$ under τ follows a similar procedure, wherein we demonstrate that Lie($\mathcal{B}$) is invariant under $d\tau$. We have that

$$\text{Lie}(\mathcal{B}) = \left\{ \gamma(z) = \sum_{k=0}^m A_k z^k : A_0 \in \mathfrak{b}, m \in \mathbb{N} \right\},$$

so that if $\gamma \in \text{Lie}(\mathcal{B})$ then

$$d\tau(\gamma) = d\tau\left(\sum_{k=0}^m A_k z^k\right) = \sum_{k=0}^\infty \hat\sigma(A_k \bar z^k) = \sum_{k=0}^\infty \hat\sigma(A_k) z^k.$$

Clearly $\hat\sigma(A_0) \in \mathfrak{b}$ by our previous argument, showing that $\mathcal{B}$ is invariant under τ. It is easy to check that each group morphism $\lambda : S^1 \to T$ is fixed by τ, and hence $\mathcal{B}\lambda$ is τ invariant.

Finally, the closure relation on the Schubert varieties is given by

$$\overline{\mathcal{B}\lambda} = \bigsqcup_{\eta\le\lambda} \mathcal{B}\eta,$$

and as each $\mathcal{B}\eta$ is invariant under τ, so too is $\overline{\mathcal{B}\lambda}$. □

In light of the previous proposition, it makes sense to write $\overline{\mathcal{B}\lambda}^\tau$ and to consider its image under the moment map. We next extend τ to $Gr_0(\mathcal{H}^\mathfrak{k})$ in order to utilize the Grassmannian model of the algebraic based loops. Define $\hat{\tau} : \mathcal{H}^\mathfrak{k} \to \mathcal{H}^\mathfrak{k}$ by $(\hat{\tau} f)(z) = \hat{\sigma}(f(\bar{z}))$, which extends to a map

$$\hat{\tau} : Gr_0(\mathcal{H}^\mathfrak{k}) \to Gr_0(\mathcal{H}^\mathfrak{k}), \qquad \hat{\tau} W = \{\hat{\sigma}(f(\bar{z})) : f \in W\}.$$

The map $\hat{\tau}$ is well defined on $Gr_0(\mathcal{H}^\mathfrak{k})$ since $\hat{\sigma}$ acts by Lie algebra automorphism, and is involutive since $\hat{\sigma}$ is the derivative of an involutive map.

Proposition 5.2.5. *The action of τ on $Gr_0(\mathcal{H}^\mathfrak{k})$ preserves $Gr_0^\mathfrak{k}$ and its corresponding filtration. Moreover, if $\phi : \Omega_{alg} K \to Gr_0(\mathcal{H}^k)$ is the symplectic embedding given in Proposition 2.1.14, then ϕ is $\mathbb{Z}_2$-equivariant.*

Proof. We begin by showing that $Gr_0^\mathfrak{k}$ is preserved. Multiplication by z commutes with $\hat{\tau}$, since for any $W \in Gr_0(\mathcal{H}^\mathfrak{k})$ we have

$$\hat{\tau}(zW) = \{\hat{\sigma}(\bar{z} f(\bar{z})) : f \in W\} = \{z\hat{\sigma}(f(\bar{z})) : f \in W\} = z \cdot \hat{\tau} W,$$

keeping in mind the anti-holomorphic nature of $\hat{\sigma}$. Consequently, τ leaves $z^m \mathcal{H}_+$ invariant, preserving the filtration. Furthermore, $\hat{\tau}$ is an isometry of the Killing form on $\mathfrak{k}_\mathbb{C}$ and so commutes with the map $W \mapsto \overline{W}^\perp$. Now if $W \in Gr_0^\mathfrak{k}$ then $zW \subseteq W$ implies that

$$z(\hat{\tau} W) = \hat{\tau}(zW) \subseteq \hat{\tau} W,$$

while $zW = \overline{W}^\perp$ implies that

$$z(\hat{\tau} W) = \hat{\tau}(zW) = \hat{\tau}(\overline{W}^\perp) = (\overline{\hat{\tau} W})^\perp.$$

The involutivity of W_{sm} follows immediately from the fact that $\hat{\sigma}$ is a Lie algebra automorphism, so $Gr_0^\mathfrak{k}$ is preserved by τ.

All that remains to be shown is the compatibility of τ with $\hat{\tau}$ through the embedding ϕ. Certainly $\hat{\tau}\mathcal{H}_+ = \mathcal{H}_+$, and for any $k \in K$ one has $\hat{\sigma} \circ \mathrm{Ad}_k = \mathrm{Ad}_{\sigma(k)} \circ \hat{\sigma}$, so that

$$\begin{aligned} \hat{\tau}\phi(\gamma) &= \left\{\hat{\sigma}(\mathrm{Ad}_{\gamma(\bar{z})} f(\bar{z})) : f \in \mathcal{H}_+\right\} \\ &= \left\{\mathrm{Ad}_{\sigma(\gamma(\bar{z}))} \hat{\sigma}(f(\bar{z})) : f \in \mathcal{H}_+\right\} \\ &= \{\mathrm{Ad}_{\tau\gamma} f : f \in \mathcal{H}_+\} \\ &= \phi(\tau\gamma). \end{aligned}$$

□

Fix a Schubert variety $\overline{\mathcal{B}\lambda}$ and choose m sufficiently large so that $\overline{\mathcal{B}\lambda}$ embeds into $\mathcal{G}_m$. Define $\tilde{\sigma} : S^1 \times K \to S^1 \times K$ by $\tilde{\sigma}(s, k) = (s^{-1}, \sigma(k))$. Whenever $(s, t) \in S^1 \times T$ one has $\tilde{\sigma}(s, t) = (s^{-1}, t^{-1})$. This implies that $\mathbb{R} \oplus \mathfrak{t}$ is contained in the (-1)-eigenspace of $d_e\tilde{\sigma}$, $\mathbb{R} \oplus \mathfrak{q}$, and hence $\mathfrak{m} = \mathbb{R} \oplus \mathfrak{t}$. In the context of this particular choice of $\tilde{\sigma}$, Equation (5.2) simply reads

$\Delta(Y) = \Delta(X)$.

Lemma 5.2.6. *$(\mathcal{G}_m, \omega'_{FS}, \tau, \mu)$ is a compact, connected, $(S^1 \times K, \tilde{\sigma})$-manifold.*

Proof. The majority of the desired properties are immediately true: $S^1 \times K$ is a compact group with involution $\tilde{\sigma}$, $(\mathcal{G}_m, \omega_{\mathrm{FS}})$ is a symplectic manifold with moment map μ and anti-symplectic involution τ. Thus all that needs to be checked are the compatibility conditions. We begin by showing condition (2); namely, if $(s,k) \in S^1 \times K$ then $\tau((s,k)\cdot W) = (s^{-1}, \sigma(k)) \cdot \tau(W)$.

We will check each action separately. Let $W \in Gr_0(\mathcal{H}^{\mathfrak{k}})$, $s \in S^1$, and $k \in K$, so that

$$\begin{aligned}
\tau(s \cdot W) &= \{\hat{\sigma}\left[f(s\bar{z})\right] : f \in W\} = \left\{\hat{\sigma}\left[f(\overline{s^{-1}z})\right] : f \in W\right\} \\
&= s^{-1}\left\{\hat{\sigma}[f(\bar{z})] : f \in W\right\} \\
&= s^{-1} \cdot \tau(W). \\
\tau(k \cdot W) &= \{\hat{\sigma}\left[\mathrm{Ad}_k f(\bar{z})\right] : f \in W\} = \left\{\mathrm{Ad}_{\sigma(k)} \hat{\sigma}[f(\bar{z})] : f \in W\right\} \\
&= \sigma(k) \cdot \{\hat{\sigma}[f(\bar{z})] : f \in W\} \\
&= \sigma(k) \cdot \tau(W).
\end{aligned}$$

On the other hand, since $S^1 \times K$ is connected, [OS00, Lemma 2.2] implies that condition (1) is true up to an appropriate shifting of μ. □

We are well acquainted with the fact that $\mathcal{G}_m$ is in fact Kähler, so it certainly has a complex structure which is compatible with ω'_{FS}. To see that τ is anti-holomorphic with respect to this structure, we introduce the following result:

Lemma 5.2.7. *If (M, ω, J, g) is a Kähler manifold and $\tau : M \to M$ is an involutive isometry, then τ is antiholomorphic if and only if τ is antisymplectic.*

Proof. Assume first that τ is anti-holomorphic. In particular, $d\tau \circ J = -J \circ d\tau$. To see that ω is anti-symplectic, we have

$$\begin{aligned}
(\tau^*\omega)(X,Y) &= \omega(d\tau X, d\tau Y) = g(J d\tau X, d\tau Y) \\
&= -g(d\tau J X, d\tau Y) \\
&= -g(JX, Y) \\
&= -\omega(X,Y).
\end{aligned}$$

Conversely, assume that τ is anti-symplectic. Note that since τ is involutive it is necessarily bijective, so that $d\tau$ is an isomorphism. Proceeding in a similar fashion as that above, we have

$$(\tau^* g)(X,Y) = g(d\tau X, d\tau Y) = \omega(d\tau X, J d\tau Y)$$

Since τ is an isometry, $\tau^* g = g$, so we also have

$$(\tau^* g)(X, Y) = g(X, Y) = \omega(X, JY) = -\omega(d\tau X, d\tau JY)$$

where in the last equality we have used the fact that τ is anti-symplectic. By multilinearity, we thus have

$$0 = \omega(d\tau X, Jd\tau Y + d\tau JY).$$

Since X was arbitrary and $d\tau$ is surjective, non-degeneracy of ω immediately implies that $(J \circ d\tau + d\tau \circ J)Y = 0$, and since Y was arbitrary, τ is anti-holomorphic. □

Note that we did not need involutivity to show that anti-holomorphic implied anti-symplectic. Similarly, we only needed that τ was surjective in order to show the converse direction.

Lemma 5.2.8. *The involution $\tau : \mathcal{G}_m \to \mathcal{G}_m$ lifts to an involutive anti-holomorphic bundle map $\tau_{\det_m} : \det_m \to \det_m$. Moreover, this map preserves the connection; that is, $\tau^*_{\det_m} \nabla = \nabla$.*

Proof. Define $\tau_{\det_m} : \det_m \to \det_m$ as follows: Fix $W \in \mathcal{G}_m$ and a basis $\{w_1, \ldots, w_{nm}\}$ of W, so that the fibre over W is given by $(W, [\alpha, w])$. Fibrewise, we define

$$\tau_{\det_m} : (\det_m)_W \to (\det_m)_{\tau W}, \qquad (W, [\alpha, w]) \mapsto (\tau W, [\overline{\alpha}, \tau(w)]) .$$

Certainly if w is a basis then $\tau(w)$ is a basis, convincing us that $\tau w_1 \wedge \cdots \wedge \tau w_k$ is indeed non-zero line element.

To see that the map is well defined, let w and w' be two different bases and C be the change of variable matrix sending w to w'; that is, $w'_i = C_{ij} w_j$. Since τ is anti-holomorphic

$$\tau w'_i = \tau \left[C_{ij} w_j \right] = \overline{C_{ij}} \tau w_j$$

implying that the change of basis between τw and $\tau w'$ is given by $\overline{C_{ij}}$. We identify $[\alpha, w']$ and $[\alpha \cdot \det C, w]$, so

$$\tau \left[\alpha w'_1 \wedge \cdots \wedge w'_k \right] = \overline{\alpha}(\tau w'_1) \wedge \cdots \wedge (\tau w'_k) = \overline{\alpha} \det \overline{C} (\tau w_1 \wedge \cdots \wedge \tau w_k).$$

On the other hand, we have

$$\tau \left[\alpha w_1 \wedge \cdots \wedge w_k \right] = \overline{\alpha \det C} (\tau w_1) \wedge \cdots \wedge (\tau w_k),$$

and these are quite naturally equal since $\overline{\det C} = \det \overline{C}$.

The map $\tau_{\det_m}$ is clearly involutive, and it is anti-holomorphic since in any trivializing neighbourhood U, $\tau : U \times \mathbb{C} \to U \times \mathbb{C}$ is separately anti-holomorphic on U and conjugate-linear on $\mathbb{C}$. By Hartogs' theorem, τ is thus anti-holomorphic.

All that remains to be shown is that τ preserves the connection. Since τ is an involution, it preserves the real part of the Hermitian metric h. Since it is additionally anti-holomorphic, by Lemma 4.4 of [OS00] we have that τ preserves the connection. □

Proposition 5.2.9. *If $\lambda \in X_*(T)_+$ is a dominant coweight, then*

$$\Delta(\overline{\mathcal{B}\lambda}) = \Delta(\mathcal{B}\lambda^\tau).$$

Proof. Choose m sufficiently large so that $\overline{\mathcal{B}\lambda}$ embeds into $\mathcal{G}_m$. We have already seen that $\overline{\mathcal{B}\lambda}$ is a closed irreducible subvariety of $\mathcal{G}_m$ invariant under B. Lemmas 5.2.6 through 5.2.8 show that $\mathcal{G}_m$ satisfies the hypotheses of Theorem 5.2.2, and Lemma 5.2.4 shows that $\overline{\mathcal{B}\lambda}$ is τ-stable. Furthermore, $\lambda \in \overline{\mathcal{B}\lambda}$ is a smooth point of X which is also in $\mathcal{G}_m^\tau$, showing that $X_{\text{sm}} \cap M^\tau \neq \emptyset$.

Applying Theorem 5.2.2, if Y is the closure of any connected component of $X_{\text{sm}} \cap M^\tau$ we will have $\Delta(X) = \Delta(\overline{\mathcal{B}\lambda}) = \Delta(Y)$. Our challenge is thus to exploit this fact to yield $\Delta(\overline{\mathcal{B}\lambda}^\tau)$ on the right-hand-side.

Since λ is dominant, the smooth locus of $\overline{\mathcal{B}\lambda}$ is precisely $\mathcal{B}\lambda$ [MOV05], giving $X_{\text{reg}} \cap M^\tau = \mathcal{B}\lambda^\tau$. This set is closed and consists of only finitely many connected components, say $\mathcal{B}\lambda^\tau = \bigsqcup_k C_k$ where each C_k is also closed in $\mathcal{G}_m$. By Theorem 5.2.2 we have $\Delta(\overline{\mathcal{B}\lambda}) = \Delta(C_k)$ and hence

$$\Delta(\mathcal{B}\lambda^\tau) = \Delta\left(\sqcup_k C_k\right) = \sqcup_k \Delta(C_k) = \sqcup_k \Delta\left(\overline{\mathcal{B}\lambda}\right) = \Delta\left(\overline{\mathcal{B}\lambda}\right). \qquad \square$$

Theorem 5.2.10. *The moment map image of $\Omega_{\text{alg}}K$ coincides with its real-locus; that is, $\Delta(\Omega_{alg}K) = \Delta(\Omega_{alg}K^\tau)$.*

Proof. We can write $\Omega_{\text{alg}}K$ as a disjoint union of the cells $\mathcal{B}\lambda$ as in (2.8), so that

$$\Omega_{\text{alg}}K^\tau = \bigsqcup_{\lambda \in X_*(T)} \mathcal{B}\lambda^\tau.$$

Let $c \in \mu_A(\Omega_{\text{alg}}K)$ and choose any $\gamma \in \Omega_{\text{alg}}K$ such that $\mu_A(\gamma) = c$. There exists a unique $\mathcal{B}\lambda'$ such that $\gamma \in \mathcal{B}\lambda'$, and since the collection of dominant coweights is cofinal in the Bruhat order, there exists a λ such that $\gamma \in \mathcal{B}\lambda' \subseteq \overline{\mathcal{B}\lambda}$. By Proposition 5.2.9,

$$c = \Delta(\gamma) \in \Delta(\overline{\mathcal{B}\lambda}) = \Delta(\mathcal{B}\lambda^\tau) \subseteq \Delta(\Omega_{\text{alg}}K^\tau),$$

showing that $\Delta(\Omega_{\text{alg}}K) \subseteq \Delta(\Omega_{\text{alg}}K^\tau)$. The other inclusion is trivial, and the result then follows. □

One can apply Theorem 5.2.10 to immediately deduce several useful corollaries, the first of which is that we can weaken the regularity conditions on loops:

Corollary 5.2.11. *Let $A \subseteq K$ be any torus, and $\mu_A = (\text{id} \times \text{pr}_{\mathfrak{a}}) \circ \mu : \Omega K \to \mathbb{R} \oplus \mathfrak{a}$ be the corresponding moment map for the $S^1 \times A$ action, where $\text{pr}_{\mathfrak{a}} : \mathfrak{k} \to \mathfrak{a}$ is the projection map. It follows that*

$$\mu_A(\Omega_{alg}K) = \mu_A(\Omega_{alg}K^\tau).$$

Proof. Consider the case where $A = T$ is a maximal torus of K. It is known that the image of μ_T may be derived from that of Δ by examining the union of the convex hulls of the Weyl orbits of elements in the image of Δ [GS05, Theorem 1.2.2]. Applying this to Theorem 5.2.10 the result follows for μ_T. If A is any other torus, it is contained in a maximal torus T, and as projections of convex set are still convex, the result follows in general. □

Corollary 5.2.12. *If $A \subseteq K$ is any torus, then*

$$\mu_A(\Omega K) = \mu_A(\Omega K^\tau).$$

This result was stated and proved in [JM10], wherein the authors showed that $\mu_T(\Omega K^\tau)$ is convexity using a convexity theorem of Terng [Ter93]. Additionally, the fixed points of the $S^1 \times T$ action on ΩK are the group morphisms $\mathrm{Hom}(S^1, T)$, and it is straightforward to see that these are also fixed under the τ action, yielding the desired result. However, once one has the result for the algebraic based loops, the full class of Sobolev H^1 loops quickly follows:

Proof. The inclusion $\mu_A(\Omega K^\tau) \subseteq \mu_A(\Omega K)$ is trivial. On the other hand, by [AP83] we know that $\Delta(\Omega K) = \Delta(\Omega_{\mathrm{alg}} K)$, so by Corollary 5.2.11

$$\mu_A(\Omega K) = \mu_A(\Omega_{\mathrm{alg}} K^\tau) \subseteq \mu_A(\Omega K^\tau).$$ □

Corollary 5.2.13. *The closures of the moment-map image for ΩK and its real-locus coincide; that is, $\overline{\Delta(\Omega K)} = \overline{\Delta(\Omega K^\tau)}$.*

Proof. It was mentioned in Chapter 2 that $\Omega_{\mathrm{alg}} K$ is dense in ΩK, so that $\overline{\Omega_{\mathrm{alg}} K} = \Omega K$. By continuity of the moment map,

$$\overline{\Delta(\overline{\Omega_{\mathrm{alg}} K})} = \overline{\Delta(\Omega_{\mathrm{alg}} K)} \tag{5.3}$$

showing that $\overline{\Delta(\Omega K)} = \overline{\Delta(\Omega_{\mathrm{alg}} K)}$. By Theorem 5.2.10 we thus have

$$\overline{\Delta(\Omega K)} = \overline{\Delta(\Omega_{\mathrm{alg}} K)} = \overline{\Delta(\Omega_{\mathrm{alg}} K^\tau)} \subseteq \overline{\Delta(\Omega K^\tau)}.$$

The reverse inclusion is trivial, and the result follows. □

We conjecture that Corollary 5.2.13 holds without the closures; that is, $\Delta(\Omega K) = \Delta(\Omega K^\tau)$. This would follow immediately if it could be shown that $\Delta(\Omega_{\mathrm{alg}} K)$ is closed, for then Equation (5.3) would would simply read $\Delta(\Omega K) = \Delta(\Omega_{\mathrm{alg}} K)$. The remainder of the proof would follow, *mutatis mutandis.*

5.3 Further Work

As mentioned after the proof of Corollary 5.2.13, we conjecture that $\Delta(\Omega K) = \Delta(\Omega K^\tau)$, for which it would be sufficient to show that $\Delta(\Omega_{\mathrm{alg}} K)$ is closed. We suspect that the key to demonstrating closure lies in the details of the proofs of Theorems 5.1.1 and 5.2.2. Both proofs expand upon the highest weight polytope approach of Brion using geometric invariant theory.

Recall that we required the involution $\sigma : K \to K$ to restrict to inversion on a maximal torus T of K. A problem originally posed in [JM10] is to consider the problem when the involution $\sigma : K \to K$ is allowed to be arbitrary. That same paper demonstrates that in general, the loop space and its real locus will have different images, but one might wonder if a there is a compromise hypothesis that would secure the veracity of the claim. We have not yet examined the consequences of adapting our approach to those weakened hypotheses.

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