The Equivariant Geometry of Nilpotent Orbits and Associated Varieties

by

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Abstract

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Nilpotent orbits are highly structured algebraic varieties lying at the interface of algebraic geometry, Lie theory, symplectic geometry, and geometric representation theory. The interest in these objects has been long-standing, ranging from Kostant's foundational work in the 1950s and 1960s to Kronheimer's realization of nilpotent orbits as instanton moduli spaces. At the same time, nilpotent orbits are studied for the sake of understanding closely associated varieties, such as nilpotent Hessenberg varieties.

In this thesis, we study the equivariant algebraic geometry and topology of nilpotent orbits and related varieties. Our first group of results is principally concerned with presentations of T-equivariant cohomology rings. More specifically, we give concrete presentations of the T-equivariant cohomology rings of the regular and minimal nilpotent orbits, with the latter presentation providing an equivariant counterpart to existing work on the ordinary cohomology of the minimal nilpotent orbit. We also examine the family of Hessenberg varieties arising from the minimal nilpotent orbit, showing them to be GKM and obtaining presentations of their T-equivariant cohomology rings. In Lie type A, we explain how one would compute the Poincaré polynomials and irreducible components of these Hessenberg varieties.

Our second group of results includes a characterization of those semisimple real Lie algebras for which every complex nilpotent orbit contains a real one, building on Rothschild's criterion for such a Lie algebra to be quasi-split. We also consider the role of nilpotent orbits in quaternionic Kähler geometry by giving a new, self-contained proof of the LeBrun-Salamon Conjecture for equivariant contact structures on partial

flag varieties. This approach allows us to give an intrinsic description of the standard contact structure on the isotropic Grassmannian of 2-planes in $\mathbb{C}^{2n}$.

Contents

Chapter 1

Introduction

1.1 Broad Context

Conjugacy classes of matrices are concrete, well-studied objects that arise frequently in both classical and modern mathematics. In particular, there are many situations in which one specializes to the conjugacy classes of nilpotent matrices. Such classes enjoy a number of properties that distinguish them from other families of conjugacy classes. Indeed, over the complex numbers, Jordan canonical forms make it possible to index the conjugacy classes of nilpotent matrices with partitions. As a consequence, there are only finitely many conjugacy classes of nilpotent matrices.

Everything mentioned above has a Lie-theoretic interpretation, in which conjugacy classes of matrices generalize to adjoint orbits in a Lie algebra. Under this generalization, conjugacy classes of nilpotent matrices become so-called *nilpotent orbits*. These latter objects are studied at the interface of several sub-disciplines in mathematics. For instance, it is an ongoing algebro-geometric problem to describe the singularities arising in nilpotent orbit closures (see [43, 57]). In symplectic geometry, the Killing form establishes a correspondence between nilpotent orbits and conical (ie. dilation-invariant) coadjoint orbits. Combinatorialists have studied the natural poset structure on the collection of nilpotent orbits (see 2.51 for a definition of the partial order). For geometric representation theorists, nilpotent orbits arise in the context of Springer resolution (see [17]). Finally, some work has been done to compute algebraic topological invariants of nilpotent orbits (see [10, 18, 46]).

While they are independently interesting, nilpotent orbits give rise to other notable algebraic varieties. Specifically, each (non-zero) nilpotent orbit can be projectivized to yield a subvariety of projective space. The orbit's canonical symplectic structure then descends to a complex contact structure on the projectivization. In fact, the long-standing LeBrun-Salamon Conjecture [62] posits that every Fano contact variety with second Betti number 1 arises as the projectivization of a so-called minimal nilpotent orbit.

Nilpotent orbits are also closely related to the so-called nilpotent Hessenberg varieties [23]. The latter are closed, singular subvarieties of the flag variety occurring in combinatorial algebraic geometry [65,72,73], geometric representation theory [33], and equivariant topology [3].

1.2 Objective and Overview of Results

The principal objective of this thesis is to systematically study the equivariant geometry of nilpotent orbits and associated varieties (ex. projectivizations and Hessenberg varieties). We also include some related projects undertaken by the author during the doctoral program. Organized by chapter, the main results of this thesis are as follows.

- In Chapter 3, we give explicit presentations of the T-equivariant cohomology rings of the regular and minimal nilpotent orbits of a simple complex group (see Theorem 7). We thereby obtain an equivariant counterpart to Juteau's description [46] of the ordinary cohomology of the minimal nilpotent orbit. This chapter is based on the paper [20].

- Chapter 4 introduces the class of Hessenberg varieties associated with the minimal nilpotent orbit. We show them to be GKM varieties, and we give presentations of their ordinary and T-equivariant cohomology rings. In Lie type A, we develop an explicit combinatorial procedure for determining their Poincaré polynomials and irreducible components. This chapter is based on joint work with Hiraku Abe, some of which appears in the preprint [2].

- In Chapter 5, we characterize those semisimple real Lie algebras $\mathfrak{g}$ with the property that every nilpotent orbit in $\mathfrak{g}_{\mathbb{C}}$ contains a nilpotent orbit in $\mathfrak{g}$ (see Theorem 46). This result builds on Rothschild's criterion for $\mathfrak{g}$ to be quasi-split (see Proposition 5.1 of [61]). This chapter is based on the preprint [19].

- Chapter 6 gives a modern, self-contained, Lie-theoretic proof of Boothby's classification of homogeneous complex contact varieties [11,12] in a special case (see

Theorem 60). Our approach yields an intrinsic description of the canonical contact structure on the isotropic Grassmannian of 2-planes in $\mathbb{C}^{2n}$. This chapter is based on the joint work with Steven Rayan from [21].

Further details about the distribution of work in the joint projects can be found at the beginning of the respective chapters.

1.3 Structure of the Thesis

We now outline this thesis as a whole. Noting that each chapter begins with a summary of its contents, our outline will be broad in scope.

Chapter 2 is devoted to some of the notation, conventions, and background material one requires for subsequent chapters. This material includes, but is not limited to, partial flag varieties, equivariant cohomology, GKM theory, and nilpotent orbits. Chapter 3 then calculates the T-equivariant cohomology rings of the regular and minimal nilpotent orbits. The second calculation is the more complicated, and depends fundamentally on GKM theory and nilpotent orbit projectivizations.

Chapter 4 defines and studies the family of Hessenberg varieties associated with the minimal nilpotent orbit. While we establish several properties of these varieties, emphasis is placed on two main themes. Firstly, we use combinatorial and diagrammatic gadgets to find the Poincaré polynomials and irreducible components of these varieties in Lie Type A. Secondly, for each variety, we obtain two independent presentations of the T-equivariant cohomology ring. One of these is a consequence of showing our variety to be GKM, while the other comes from exhibiting the equivariant cohomology ring as a quotient of $H_T^*(G/B)$.

Chapter 5 explores relationships between real and complex nilpotent orbits by means of *nilpotent orbit complexification*. Using this framework, we classify the semisimple real Lie algebras $\mathfrak{g}$ with the property that every nilpotent orbit in $\mathfrak{g}_{\mathbb{C}}$ meets $\mathfrak{g}$. Any $\mathfrak{g}$ satisfying this property is necessarily quasi-split, and for such $\mathfrak{g}$ we characterize the complex nilpotent orbits meeting $\mathfrak{g}$. The chapter concludes with the result that distinct real nilpotent orbits lying in the same complex orbit are incomparable in the closure order. Our proof invokes the Kostant-Sekiguchi Correspondence and its properties.

In Chapter 6, our focus turns to the nilpotent orbits in quaternionic Kähler geometry. Specifically, we give a proof of the LeBrun-Salamon Conjecture for partial flag varieties with G-invariant complex contact structures. While this result is deducible from Boothby's work [11,12], our proof instead harnesses the equivariant geometry of partial flag varieties. Also, our approach is shown to imply an intrinsic description of the standard contact structure on the isotropic Grassmannian of 2-planes in $\mathbb{C}^{2n}$.

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Chapter 2

Preliminaries

2.1 Introduction and Structure

This chapter assembles some of the notation, conventions, and background on which subsequent chapters will depend. We begin by listing some of the fundamental conventions observed throughout this thesis. Section 2.3 subsequently introduces the main Lie-theoretic objects used in Chapters 3, 4, and 6. Next, Section 2.5 recalls the basics of equivariant cohomology. Using this as a foundation, Sections 2.6 and 2.7 review GKM theory and explain how it applies to compute the T-equivariant cohomology of a partial flag variety. Section 2.8 provides some of the basic background on nilpotent orbits in semisimple Lie algebras.

2.2 Conventions

In the interest of consistency, let us establish some of the fundamental notation and conventions to be used throughout this thesis.

- We will make extensive use of algebraic and geometric concepts for which, strictly speaking, a base field must be specified (ex. Lie algebras, algebraic varieties, etc.) However, parsimony is best served by our sometimes not mentioning the base field. In all such cases (except for the one mentioned in the next bullet point), this field is to be taken as $\mathbb{C}$.

- Whenever we invoke homology and cohomology (ordinary, equivariant, etc.) without specifying a coefficient ring, let us take this ring to be $\mathbb{Q}$.

- Whenever we introduce a group action without indicating it to be a left or right action, we will understand it to be a left action.

- Whenever we refer to a topological invariant of an algebraic variety (ex. ordinary cohomology), we will do so with the understanding that the variety possesses its classical topology. Similarly, all equivariant topological invariants (ex. equivariant cohomology) will be computed with respect to the classical topologies on both the varieties and groups in question.

- Throughout Chapters 2, 3, 4, and 6, the groups G, B, B_-, and T will be exactly as defined in 2.3.1.

- By *vector bundle*, we will mean an algebraic vector bundle $\mathcal{E}$ on a smooth variety X. If $x \in X$, then $\mathcal{E}_x$ will denote the fibre over x.

2.3 Lie-Theoretic Objects

2.3.1 Notation

Let G be a connected, simply-connected simple algebraic group with Lie algebra $\mathfrak{g}$. Denote by

$$\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g}) \text{ and } \mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$$

the adjoint representations of G and $\mathfrak{g}$, respectively. Fix a pair of positive and negative Borel subgroups, $B \subseteq G$ and $B_- \subseteq G$, respectively, and consider the maximal torus $T := B \cap B_-$. The inclusions $T \subseteq B \subseteq G$ give rise to inclusions of Lie algebras, which we denote by $\mathfrak{t} \subseteq \mathfrak{b} \subseteq \mathfrak{g}$. One then has the weight lattice $X^*(T)$, roots $\Delta \subseteq X^*(T)$, positive roots $\Delta_+ \subseteq \Delta$, negative roots $\Delta_- \subseteq \Delta$, and simple roots $\Pi \subseteq \Delta_+$. The weight lattice comes equipped with its usual bilinear form $\langle \cdot, \cdot \rangle : X^*(T) \otimes_{\mathbb{Z}} X^*(T) \to \mathbb{Z}$. Also, since G is simple, there is a highest root $\lambda \in \Delta_+$. Finally, let $W = N_G(T)/T$ denote the Weyl group, $s_\gamma \in W$ the reflection associated to $\gamma \in \Delta_+$, and $\ell : W \to \mathbb{Z}_{\geq 0}$ the length function resulting from our choice of simple roots.

Recall that a parabolic subgroup $P \subseteq G$ is called *standard* if $B \subseteq P$. Assuming this to be the case, set $S := \{\alpha \in \Pi : \mathfrak{g}_{-\alpha} \subseteq \mathfrak{p}\}$, where $\mathfrak{p}$ is the Lie algebra of P. The assignment of S to P defines a bijective correspondence between the standard parabolic subgroups and the subsets of Π. Given $S \subseteq \Pi$, we shall let $P_S \subseteq G$ denote the standard parabolic subgroup associated to S via this correspondence, and we shall let $\mathfrak{p}_S \subseteq \mathfrak{g}$ denote its

Lie algebra. Let W_S denote the Weyl group of P_S, and recall that W_S is the subgroup of W generated by the simple reflections s_α, $\alpha \in S$.

2.3.2 Specialization to Type A

While we will usually work in arbitrary Lie type, we will sometimes benefit from specializing to type A_{n-1}. In these cases only, the notation in 2.3.1 is to be interpreted as follows.

- $G = SL_n(\mathbb{C}) = \{g \in GL_n(\mathbb{C}) : \det(g) = 1\}$
- $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C}) = \{X \in \mathfrak{gl}_n(\mathbb{C}) : \mathrm{tr}(X) = 0\}$
- For $g \in SL_n(\mathbb{C})$ and $X \in \mathfrak{sl}_n(\mathbb{C})$, we have

$$\mathrm{Ad}(g)(X) = gXg^{-1}. \tag{2.1}$$

- B and B_- are the subgroups of upper and lower-triangular matrices in $SL_n(\mathbb{C})$, respectively.
- T is the subgroup of diagonal matrices in $SL_n(\mathbb{C})$, namely

$$T = \left\{ \begin{bmatrix} t_1 & 0 & 0 & \dots & 0 \\ 0 & t_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & t_n \end{bmatrix} : t_1 t_2 \cdots t_n = 1 \right\}. \tag{2.2}$$

- Abusing notation, $t_i \in X^*(T)$ shall denote the weight

$$T \to \mathbb{C}^*, \quad \begin{bmatrix} t_1 & 0 & 0 & \dots & 0 \\ 0 & t_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & t_n \end{bmatrix} \mapsto t_i.$$

  Note that $X^*(T)$ is then generated by $\{t_1, t_2, \dots, t_n\}$ with the relation $t_1 + t_2 + \dots + t_n = 0$.

- $\Delta = \{t_i - t_j : 1 \le i, j \le n,\ i \ne j\}$
  $\Delta_+ = \{t_i - t_j : 1 \le i < j \le n\}$

$\Pi = \{t_i - t_{i+1} : 1 \leq i \leq n-1\}$
$\lambda = t_1 - t_n.$

- We will regard W as the symmetric group S_n, so that the Weyl-action on $X^*(T)$ satisfies
$$wt_i = t_{w(i)}, \quad w \in S_n, \ i = 1, 2, \ldots, n. \tag{2.3}$$

- We will sometimes use the so-called *one-line notation* for permutations, in which one represents $w \in S_n$ by the list of values $w(1)\, w(2)\ \cdots\ w(n)$.

- If $1 \leq i < j \leq n$, then $(i\, j) \in S_n$ will denote the transposition switching i and j. Note that $(i\, j)$ is precisely s_γ for $\gamma = t_i - t_j$.

2.4 Partial Flag Varieties

The following section reviews some relevant aspects of partial flag varieties, objects of central importance to this thesis. To this end, recall that a *partial flag variety* (of G) is a projective variety equipped with a transitive algebraic G-action. Equivalently, a partial flag variety is a projective G-variety with the property of being equivariantly isomorphic to G/P_S for some subset $S \subseteq \Pi$. We shall refer to G/B, the partial flag variety with maximal dimension, as the *full flag variety*.

2.4.1 T-Fixed Points and Schubert Varieties

Recall that T acts on G/P_S with finitely many fixed points, indexed by W/W_S in the following way. Given $[w] \in W/W_S$, lift $[w]$ to $w \in W$, and then lift w to $g \in N_G(T)$. Let $x([w]) \in G/P_S$ denote the coset of g in G/P_S. This point is defined independently of our earlier choices, and the association

$$W/W_S \to (G/P_S)^T, \quad [w] \mapsto x([w]) \tag{2.4}$$

is a bijection.

The groups B and B_- act on G/P_S with respective orbit decompositions

$$G/P_S = \coprod_{[w] \in W/W_S} Bx([w]) \tag{2.5}$$

and

$$G/P_S = \coprod_{[w]\in W/W_S} B_-x([w]). \tag{2.6}$$

The orbit $Bx([w])$ is an affine space of dimension $\ell([w])$, where $\ell([w])$ denotes the length of a coset representative of $[w]$ having minimal length. Similarly, $B_-x([w])$ is an affine space of codimension $\ell([w])$ in G/P_S. One calls $Bx([w])$ (resp. $B_-x([w])$) a *Schubert cell* (resp. an *opposite Schubert cell*), alluding to the fact that (2.5) and (2.6) are cell decompositions of G/P_S.

The closures of Schubert and opposite Schubert cells give a geometric manifestation of the Bruhat order (see [25]) on W/W_S in the sense that

$$\overline{Bx([w])} = \coprod_{[v]\leq[w]} Bx([v]) \tag{2.7}$$

and

$$\overline{B_-x([w])} = \coprod_{[w]\leq[v]} B_-x([v]). \tag{2.8}$$

The subvariety $\overline{Bx([w])}$ is called a *Schubert variety*, while $\overline{B_-x([w])}$ is called an *opposite Schubert variety*. Henceforth, we shall denote the former by $X([w])$ and the latter by $X_-([w])$.

It will be important to note that the collections

$$\sigma([w]) := [X([w])] \in H^*(G/P_S;\mathbb{Z}), \quad [w] \in W/W_S \tag{2.9}$$

and

$$\sigma_-([w]) := [X_-([w])] \in H^{2\ell([w])}(G/P_S;\mathbb{Z}), \quad [w] \in W/W_S \tag{2.10}$$

are additive bases of $H^*(G/P_S;\mathbb{Z})$. The classes in (2.9) are called *Schubert classes*, while those in (2.10) are called *opposite Schubert classes*.

2.4.2 Examples in Type A

The partial flag varieties of $SL_n(\mathbb{C})$ are realizable in terms of flags of subspaces of $\mathbb{C}^n$. Indeed, for integers $1 \leq d_1 < d_2 < \ldots < d_k \leq n-1$, denote by $\mathcal{F}(d_1, d_2, \ldots, d_k; \mathbb{C}^n)$ the set of all flags $V_\bullet$ of the following form:

$$V_\bullet = (\{0\} \subseteq V_1 \subseteq V_2 \subseteq \ldots \subseteq V_k \subseteq \mathbb{C}^n),$$

where V_j is a d_j-dimensional subspace of $\mathbb{C}^n$. This set has a canonical projective variety structure, and the natural $SL_n(\mathbb{C})$-action then renders it a partial flag variety.

We note that the full flag variety is the one for which flags have maximal length, namely

$$SL_n(\mathbb{C})/B \cong \mathcal{F}(1, 2, \ldots, n-1; \mathbb{C}^n). \tag{2.11}$$

Equivalently, letting $E_k := \text{span}\{e_1, e_2, \ldots, e_k\}$ be the span of the first k standard basis vectors, one notices that B is precisely the $SL_n(\mathbb{C})$-stabilizer of the flag

$$E_\bullet := (\{0\} \subseteq E_1 \subseteq E_2 \subseteq \ldots \subseteq E_{n-1} \subseteq \mathbb{C}^n) \in \mathcal{F}(1, 2, \ldots, n-1; \mathbb{C}^n).$$

In the interest of concreteness, let us outline how our earlier discussion of T-fixed points and Schubert varieties applies to $\mathcal{F}(1, 2, \ldots, n-1; \mathbb{C}^n)$. Indeed, noting that $B = P_\emptyset$ and $W_\emptyset = \{e\}$, (2.4) implies that the T-fixed points are indexed by the full Weyl group $W = S_n$. Now, given $w \in S_n$ and $k \in \{1, 2, \ldots, n-1\}$, set $E_k^w := \text{span}\{e_{w(1)}, e_{w(2)}, \ldots, e_{w(k)}\}$ and

$$E_\bullet^w := (\{0\} \subseteq E_1^w \subseteq E_2^w \subseteq \ldots \subseteq E_{n-1}^w \subseteq \mathbb{C}^n) \in \mathcal{F}(1, 2, \ldots, n-1; \mathbb{C}^n).$$

The T-fixed points and Schubert varieties in $\mathcal{F}(1, 2, \ldots, n-1; \mathbb{C}^n)$ are then given by

$$\mathcal{F}(1, 2, \ldots, n-1; \mathbb{C}^n)^T = \{E_\bullet^w : w \in S_n\}$$

and

$$X(w) = \overline{BE_\bullet^w} = \left\{ V_\bullet \in \mathcal{F}(1, 2, \ldots, n-1; \mathbb{C}^n) : \begin{array}{c} \dim(V_j \cap E_k) \geq \dim(V_j \cap E_k^w) \\ 1 \leq j \leq n-1 \\ 1 \leq k \leq n-1 \end{array} \right\},$$

respectively. For a more extensive discussion of these descriptions, we refer the reader fo [31].

2.4.3 Equivariant Vector Bundles

Let $\mathcal{G}$ be an algebraic group and X a variety on which $\mathcal{G}$ acts algebraically. Recall that a vector bundle $\mathcal{E}$ on X is called $\mathcal{G}$-*equivariant* if

- $\mathcal{G}$ acts algebraically on $\mathcal{E}$ via a lift of the $\mathcal{G}$-action on X, and
- for all $x \in X$ and $g \in \mathcal{G}$, the action of g on $\mathcal{E}$ restricts to a vector space isomor-

phism $\mathcal{E}_x \xrightarrow{\cong} \mathcal{E}_{gx}$.

Given a $\mathcal{G} = $ G-equivariant vector bundle $\mathcal{E}$ on $X = G/P_S$, note that $\mathcal{E}_{[e]}$ is a P_S-representation (where $[e] \in G/P_S$ denotes the coset of the identity $e \in G$). Conversely, we may associate to each finite-dimensional P_S-representation $\varphi : P_S \to GL(V)$ a G-equivariant vector bundle over G/P_S. Explicitly, note that P_S acts freely on $G \times V$ via

$$p \cdot (g, v) := (gp^{-1}, \varphi(p)(v)), \quad p \in P_S, g \in G, v \in V. \tag{2.12}$$

The quotient variety, denoted $G \times_{P_S} V$, carries a residual G-action and has a natural map to G/P_S whose fibres are vector spaces. In this way, $G \times_{P_S} V$ is a G-equivariant vector bundle over G/P_S.

With $\mathcal{E}$ and V as above, one has

$$G \times_{P_S} \mathcal{E}_{[e]} \cong \mathcal{E} \tag{2.13}$$

as G-equivariant vector bundles and

$$(G \times_{P_S} V)_{[e]} \cong V \tag{2.14}$$

as P_S-representations. In other words, the associations $\mathcal{E} \mapsto \mathcal{E}_{[e]}$ and $V \mapsto G \times_{P_S} V$ are inverses in a bijective correspondence between the isomorphism classes of G-equivariant vector bundles on G/P_S and those of finite-dimensional P_S-representations.

Our correspondence restricts to give one between $\text{Hom}(P_S, \mathbb{C}^*)$ and the (isomorphism classes of) G-equivariant line bundles on G/P_S. This gives rise to an injection

$$\text{Hom}(P_S, \mathbb{C}^*) \to \text{Pic}(G/P_S), \tag{2.15}$$

where $\text{Pic}(G/P_S)$ denotes the Picard group of G/P_S. Noting that restriction to T defines an isomorphism $\text{Hom}(P_S, \mathbb{C}^*) \xrightarrow{\cong} X^*(T)^{W_S}$, we may present (2.15) as

$$X^*(T)^{W_S} \to \text{Pic}(G/P_S). \tag{2.16}$$

Now for each $\alpha \in X^*(T)^{W_S}$, consider

$$\mathcal{L}(\alpha) := G \times_{P_S} \mathbb{C}_\alpha, \tag{2.17}$$

where $\mathbb{C}_\alpha$ is the one-dimensional P_S-representation of T-weight α. The map (2.16) is then given by

$$X^*(T)^{W_S} \to \mathrm{Pic}(G/P_S), \quad \alpha \mapsto [\mathcal{L}(\alpha)]. \tag{2.18}$$

Proposition 1. *The map* (2.18) *is a group isomorphism.*

Proof. It is known that the structure sheaf of G/P_S has trivial first and second cohomology groups (see [66]). An examination of the exponential sequence in sheaf cohomology then implies that

$$\mathrm{Pic}(G/P_S) \to H^2(G/P;\mathbb{Z}), \quad [L] \mapsto c_1(L) \tag{2.19}$$

is an isomorphism. Hence, it will suffice to prove that the composition of (2.18) and (2.19) is an isomorphism. This composite map is given by

$$X^*(T)^{W_S} \to H^2(G/P_S;\mathbb{Z}), \quad \alpha \mapsto c_1(\mathcal{L}(\alpha)), \tag{2.20}$$

which is known to be an isomorphism (see [8]). □

2.5 Equivariant Cohomology

2.5.1 The General Setup

Equivariant cohomology will be our principal means of describing objects in the equivariant topological category. Here, we give a narrowly focused treatment of equivariant cohomology, emphasizing only those aspects directly pertinent to this thesis. For a more exhaustive study, we refer the reader to [14].

For $\mathcal{G}$ a topological group, there exists a contractible topological space $E\mathcal{G}$ on which $\mathcal{G}$ acts continuously and freely. The resulting principal $\mathcal{G}$-bundle

$$E\mathcal{G} \to E\mathcal{G}/\mathcal{G} =: B\mathcal{G} \tag{2.21}$$

is called a *universal principal $\mathcal{G}$-bundle*. This nomenclature reflects the universal property that every principal $\mathcal{G}$-bundle $E \to Y$ arises as a pullback of (2.21) along a suitable map $Y \to B\mathcal{G}$ (defined uniquely up to homotopy).

Given a topological space X carrying a continuous $\mathcal{G}$-action, let $\mathcal{G}$ act diagonally on the product $X \times E\mathcal{G}$. The quotient

$$(X \times E\mathcal{G})/\mathcal{G} =: X_{\mathcal{G}} \tag{2.22}$$

is called the *Borel mixing space* of X. The *$\mathcal{G}$-equivariant cohomology* of X is then defined by

$$H^*_{\mathcal{G}}(X) := H^*(X_{\mathcal{G}}). \tag{2.23}$$

Using the universal property of (2.21), one can show this $\mathbb{Z}$-graded commutative $\mathbb{Q}$-algebra to be well-defined up to isomorphism.

It will be prudent to mention a few elementary properties of $\mathcal{G}$-equivariant cohomology. To this end, suppose that X and Y have continuous $\mathcal{G}$-actions, and that $f : X \to Y$ is a $\mathcal{G}$-equivariant continuous map. Note that f induces a continuous map

$$f_{\mathcal{G}} : X_{\mathcal{G}} \to Y_{\mathcal{G}}, \tag{2.24}$$

which in turn gives a map on cohomology

$$f^*_{\mathcal{G}} : H^*_{\mathcal{G}}(Y) \to H^*_{\mathcal{G}}(X). \tag{2.25}$$

Let us consider the following special cases of this construction.

- **Module Structure:** The map from X to the one-point space, $X \to \mathrm{pt}$, induces a map of rings, $H^*_{\mathcal{G}}(\mathrm{pt}) \to H^*_{\mathcal{G}}(X)$. In this way, $H^*_{\mathcal{G}}(X)$ is canonically a module over $H^*_{\mathcal{G}}(\mathrm{pt})$.

- **Restriction:** The inclusion $X^{\mathcal{G}} \to X$ is $\mathcal{G}$-equivariant and therefore induces a map $H^*_{\mathcal{G}}(X) \to H^*_{\mathcal{G}}(X^{\mathcal{G}})$. We will often refer to this latter map as a *restriction map*.

- **Free Actions:** Let $\mathcal{G}$ act freely on X and consider the map

  $$(X \times E\mathcal{G})/\mathcal{G} \to X/\mathcal{G}$$

  defined by projection from the first component. This is a fibration with contractible fibre $E\mathcal{G}$, and it therefore induces an isomorphism on cohomology. Since the cohomology of the total space is precisely $H^*_{\mathcal{G}}(X)$, we have

  $$H^*_{\mathcal{G}}(X) \cong H^*(X/\mathcal{G}).$$

- **Homogeneous Spaces:** Let $\mathcal{K} \subseteq \mathcal{G}$ be a closed subgroup and consider the homogeneous $\mathcal{G}$-space $X := \mathcal{G}/\mathcal{K}$. The map

  $$E\mathcal{G}/\mathcal{K} \to ((\mathcal{G}/\mathcal{K}) \times E\mathcal{G})/\mathcal{G}, \quad [y] \mapsto [([e], y)] \tag{2.26}$$

is seen to be a well-defined homeomorphism, where $[e] \in \mathcal{G}/\mathcal{K}$ is the coset of the identity $e \in \mathcal{G}$. In particular, the domain and codomain of (2.26) have isomorphic cohomology rings. The cohomology of the latter is exactly $H^*_{\mathcal{G}}(\mathcal{G}/\mathcal{K})$. Also, as $E\mathcal{G}$ is a contractible space on which $\mathcal{K}$ acts freely, we may set $B\mathcal{K} := E\mathcal{G}/\mathcal{K}$. Hence, the cohomology of the codomain is $H^*(B\mathcal{K}) = H^*_{\mathcal{K}}(\mathrm{pt})$ and we have

$$H^*_{\mathcal{G}}(\mathcal{G}/\mathcal{K}) \cong H^*_{\mathcal{K}}(\mathrm{pt}). \tag{2.27}$$

- **Group Complexification:** Let K be a compact connected Lie group with complexification $K_{\mathbb{C}}$. Given a variety X with an algebraic $K_{\mathbb{C}}$-action, one has a fibration
$$K_{\mathbb{C}}/K \to (X \times EK_{\mathbb{C}})/K \to (X \times EK_{\mathbb{C}})/K_{\mathbb{C}}. \tag{2.28}$$
The fibre $K_{\mathbb{C}}/K$ is contractible and (2.28) induces an isomorphism on ordinary cohomology,
$$H^*((X \times EK_{\mathbb{C}})/K_{\mathbb{C}}) \xrightarrow{\cong} H^*(X \times EK_{\mathbb{C}})/K). \tag{2.29}$$
Note that the left-hand-side of (2.29) is precisely $H^*_{K_{\mathbb{C}}}(X)$. Also, as $EK_{\mathbb{C}}$ is a contractible space on which K acts freely, we may take $(X \times EK_{\mathbb{C}})/K$ to be the Borel mixing space of X with respect to the K-action. In this way, the right-hand-side of (2.29) is $H^*_K(X)$, giving us a canonical isomorphism
$$H^*_{K_{\mathbb{C}}}(X) \xrightarrow{\cong} H^*_K(X). \tag{2.30}$$

- **Equivariant Chern Classes:** If $\pi : \mathcal{E} \to X$ is a $\mathcal{G}$-equivariant vector bundle (as defined in 2.4.3), then $\pi_{\mathcal{G}} : \mathcal{E}_{\mathcal{G}} \to X_{\mathcal{G}}$ is naturally a vector bundle. This fact gives rise to $\mathcal{G}$-equivariant Chern classes $c^{\mathcal{G}}_n(\mathcal{E})$, namely
$$c^{\mathcal{G}}_n(\mathcal{E}) := c_n(\mathcal{E}_G) \in H^{2n}(X_{\mathcal{G}}) = H^{2n}_{\mathcal{G}}(X). \tag{2.31}$$

2.5.2 T-Equivariant Cohomology

We will be principally concerned with T-equivariant cohomology, to which the theory in 2.5.1 has interesting specializations.

- **The T-Equivariant Cohomology of a Point:** For $\alpha \in \mathbb{C}^*$, let $\mathbb{C}_\alpha$ denote the one-dimensional T-representation of weight α. Note that $\mathbb{C}_\alpha$ is equivalently a T-equivariant line bundle over pt, and as such has T-equivariant first Chern class

$c_1^T(\mathbb{C}_\alpha) \in H_T^2(\text{pt})$. The association $\alpha \mapsto c_1^T(\mathbb{C}_\alpha)$ determines a degree-doubling $\mathbb{Q}$-algebra isomorphism

$$\text{Sym}(X^*(T) \otimes_{\mathbb{Z}} \mathbb{Q}) \xrightarrow{\cong} H_T^*(\text{pt}). \tag{2.32}$$

Using (2.32), we will freely identify $H_T^*(\text{pt})$ and $\text{Sym}(X^*(T) \otimes_{\mathbb{Z}} \mathbb{Q})$.

Note that if $\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_r\}$, then $\text{Sym}(X^*(T) \otimes_{\mathbb{Z}} \mathbb{Q})$ is more concretely viewed as $\mathbb{Q}[\alpha_1, \alpha_2, \dots, \alpha_r]$. Hence, we will occasionally identify $H_T^*(\text{pt})$ with this polynomial algebra.

- **Equivariant Formality:** Let X be a variety on which T acts algebraically. One calls X *equivariantly formal* if the cohomology spectral sequence of the natural fibration

  $$X \xrightarrow{\iota} X_T \to BT \tag{2.33}$$

  collapses on its second page, so that

  $$H_T^*(X) \cong H^*(X) \otimes_{\mathbb{Q}} H_T^*(\text{pt})$$

  as $H_T^*(\text{pt})$-modules. The class of equivariantly formal spaces includes, for instance, all T-varieties with vanishing odd-degree cohomology.

  If X is equivariantly formal, then the map ι from (2.33) induces a surjection $\iota^* : H_T^*(X) \to H^*(X)$. Its kernel is $H_T^{>0}(\text{pt})H_T^*(X)$, where $H_T^{>0}(\text{pt})$ is the ideal of $H_T^*(\text{pt})$ generated by the positive-degree elements. In particular, $H^*(X)$ can be recovered from $H_T^*(X)$ via the ring isomorphism

  $$H^*(X) \cong \frac{H_T^*(X)}{H_T^{>0}(\text{pt})H_T^*(X)}.$$

- T**-Equivariant Schubert Classes:** It will be advantageous to note that the cohomology classes (2.9) and (2.10) have counterparts in T-equivariant cohomology. Given $[w] \in W/W_S$, let

  $$\sigma_T([w]) \in H_T^*(G/P_S) \tag{2.34}$$

  and

  $$\sigma_T^-([w]) \in H_T^{2\ell([w])}(G/P_S) \tag{2.35}$$

  denote the T-equivariant cohomology classes determined by $X([w])$ and $X_-([w])$, respectively. The former is called a T-*equivariant Schubert class*, while the latter is called a T-*equivariant opposite Schubert class*.

For details on the association of equivariant cohomology classes to T-invariant subvarieties, we refer the reader to Section 3.5 of [1].

2.6 GKM Theory

Having recalled the general theory of equivariant cohomology, let us discuss some techniques for computation. To this end, we begin with arguably the most powerful and celebrated such technique — one developed by Goresky, Kottwitz, and MacPherson (henceforth abbreviated GKM) in [35]. These authors developed a concrete framework for computing the T-equivariant cohomology rings of a large and encompassing class of varieties, the so-called *GKM varieties*.

Definition 2. A GKM variety is a projective variety X on which T acts algebraically and with the following properties:

(i) X^T is finite.

(ii) X has finitely many one-dimensional T-orbits.

(iii) X is equivariantly formal.

Prominent examples of varieties satisfying Definition 2 include the partial flag varieties G/P_S (see 2.7), the Schubert varieties $X([w])$, and smooth projective toric varieties. However, there have been some interest in broadening the class of spaces originally considered by GKM. In [37], Guillemin and Holm relax the requirement that X^T consist of isolated points. Also, Harada, Henriques, and Holm [39] substantially weaken the assumptions in Definition 2 to compute the equivariant cohomology rings of Kac-Moody flag varieties.

Let us now discuss some implications of Definition 2 for a GKM variety X. Accordingly, let $Y \subseteq X$ be the closure of a one-dimensional T-orbit in X. There exists a non-zero weight $\gamma \in X^*(T)$ for which Y is T-equivariantly isomorphic to $\mathbb{P}^1$ with the T-action

$$t \cdot [x_1 : x_2] = [\gamma(t)x_1 : x_2]. \tag{2.36}$$

Henceforth, we shall denote this latter T-variety by $(\mathbb{P}^1, \gamma)$.

Interestingly, the property $Y \cong (\mathbb{P}^1, \gamma)$ determines γ uniquely up to sign. This fact will feature prominently in our discussion of GKM graph edges, and we prove it via the following lemma.

Lemma 3. *If $\gamma_1, \gamma_2 \in X^*(T)$ are non-zero weights, then $(\mathbb{P}^1, \gamma_1)$ and $(\mathbb{P}^1, \gamma_2)$ are T-equivariantly isomorphic if and only if $\gamma_1 = \gamma_2$ or $\gamma_1 = -\gamma_2$.*

Proof. Assume that $(\mathbb{P}^1, \gamma_1)$ and $(\mathbb{P}^1, \gamma_2)$ are T-equivariantly isomorphic, and let $\varphi : (\mathbb{P}^1, \gamma_1) \to (\mathbb{P}^1, \gamma_2)$ be an isomorphism. Since $[1:0]$ and $[0:1]$ are the T-fixed points in each variety, we must have $\varphi([0:1]) = [0:1]$ or $\varphi([0:1]) = [1:0]$. In the former case, the differential of φ at $[0:1]$ gives an isomorphism

$$d_{[0:1]}\varphi : T_{[0:1]}\mathbb{P}^1 \xrightarrow{\cong} T_{[0:1]}\mathbb{P}^1 \tag{2.37}$$

of one-dimensional T-representations.

Now, let U denote the complement of $[1:0]$ in $(\mathbb{P}^1, \gamma_1)$, and let $\mathbb{C}_{\gamma_1}$ denote the one-dimensional T-representation of weight γ_1. The map

$$\mathbb{C}_{\gamma_1} \to U, \quad x_0 \mapsto [x_0 : 1]$$

is a T-equivariant isomorphism, and so induces a T-representation isomorphism

$$T_{[0:1]}\mathbb{P}^1 \cong T_0\mathbb{C}_{\gamma_1} \cong \mathbb{C}_{\gamma_1}.$$

In other words, the domain of (2.37) is acted upon by T with weight γ_1. Similarly, the weight of the codomain is γ_2, and we have $\gamma_1 = \gamma_2$.

In the case $\varphi([0:1]) = [1:0]$, the differential of φ at $[0:1]$ gives an isomorphism

$$d_{[0:1]}\varphi : T_{[0:1]}\mathbb{P}^1 \xrightarrow{\cong} T_{[1:0]}\mathbb{P}^1.$$

The weight of the domain is again γ_1, while consideration of the complement of $[0:1]$ in $(\mathbb{P}^1, \gamma_2)$ shows the weight of the codomain to be $-\gamma_2$. Hence, $\gamma_1 = -\gamma_2$.

To prove the converse, assume that $\gamma_1 = \gamma_2$ or $\gamma_1 = -\gamma_2$. Since the desired conclusion clearly holds in the former case, we shall assume that $\gamma_1 = -\gamma_2$. Now, consider the variety isomorphism

$$\psi : \mathbb{P}^1 \to \mathbb{P}^1, \quad [x_0 : x_1] \mapsto [x_1 : x_0].$$

If $t \in T$ and $[x_0 : x_1] \in \mathbb{P}^1$, then

$$\begin{aligned}\psi([\gamma_1(t)x_0 : x_1]) &= [x_1 : \gamma_1(t)x_0] \\ &= [\gamma_1(t)^{-1}x_1 : x_0] \\ &= [(-\gamma_1)(t)x_1 : x_0] \\ &= [\gamma_2(t)x_1 : x_0].\end{aligned}$$

So, in addition to being a variety isomorphism, ψ is a T-equivariant map $(\mathbb{P}^1, \gamma_1) \to (\mathbb{P}^1, \gamma_2)$. This completes the proof. □

Returning to the main discussion, let $Y \subseteq X$ be the closure of a one-dimensional T-orbit and let $\gamma \in X^*(T)$ be a non-zero weight for which $Y \cong (\mathbb{P}^1, \gamma)$. Since $(\mathbb{P}^1)^T = \{[1 : 0], [0 : 1]\}$, it follows that Y^T consists of two distinct T-fixed points in X. In this way, certain pairs of points in X^T are "connected" by the closures of one-dimensional T-orbits. Given distinct $x_1, x_2 \in X^T$, let us write $x_1 \to x_2$ when these points lie in the closure of a one-dimensional T-orbit. In this case, the closure Y_{x_1,x_2} is unique and acted upon by T with some non-zero weight $\gamma_{x_1,x_2} \in X^*(T)$, as in (2.36).

Perhaps surprisingly, the data of X^T and the weights γ_{x_1,x_2} for $x_1 \to x_2$ completely determine $H_T^*(X)$ as a subalgebra of $H_T^*(X^T)$.

Theorem 4 (The GKM Presentation, [35]). *Suppose that all objects are as defined above. The restriction map*

$$H_T^*(X) \to H_T^*(X^T) = \bigoplus_{x\in X^T} H_T^*(\{x\}) = \bigoplus_{x\in X^T} H_T^*(pt)$$

is injective, and its image is precisely

$$H_T^*(X) \cong \{(f_x)_{x\in X^T} \in \bigoplus_{x\in X^T} H_T^*(pt) : \gamma_{x_1,x_2} | (f_{x_1} - f_{x_2}) \textit{ whenever } x_1 \to x_2\}. \tag{2.38}$$

Remark. For purposes of interpreting (2.38), $f_x \in H_T^*(pt)$ is to be regarded as an element of $\mathrm{Sym}(X^*(T) \otimes_{\mathbb{Z}} \mathbb{Q})$. The condition $\gamma_{x_1,x_2} | (f_{x_1} - f_{x_2})$ then makes sense in the symmetric algebra.

The divisibility conditions appearing in (2.38) are conveniently encoded in an edge-labelled graph $\mathrm{GKM}(X)$, called the *GKM graph* of X. This graph has vertex set X^T, with $x_1 \neq x_2 \in X^T$ connected by an edge if and only if $x_1 \to x_2$ as defined above. In this case, the edge in question is given the label γ_{x_1,x_2}. While this label is defined only up to a choice of sign, the choice does not affect the right-hand-side of (2.38).

2.7 GKM Theory for Partial Flag Varieties

2.7.1 Some General Considerations

Equipped with their usual T-actions, the partial flag varieties of G are perhaps the most recognized examples of GKM varieties. In the interest of later sections, we now review the construction of the GKM graph of a partial flag variety G/P_S. For this, we will need to recall a classification of the closures of one-dimensional T-orbits in G/P_S (as presented in [32], for instance).

Given $\gamma \in \Delta_+$, let $SL_2(\mathbb{C})_\gamma \subseteq G$ denote the root subgroup with Lie algebra $\mathfrak{g}_{-\gamma} \oplus [\mathfrak{g}_{-\gamma}, \mathfrak{g}_\gamma] \oplus \mathfrak{g}_\gamma$. Consider the $SL_2(\mathbb{C})_\gamma$-orbit of $[e] \in G/P_S$, namely

$$Y_\gamma := SL_2(\mathbb{C})_\gamma[e] \subseteq G/P_S. \tag{2.39}$$

For an element $[w] \in W/W_S$, we set

$$Y_{[w],\gamma} := gY_\gamma, \tag{2.40}$$

where $g \in N_G(T)$ is a lift of $w \in W$. Note that $Y_{[w],\gamma}$ is a non-singleton if and only if $\gamma \notin \Delta_{S,+}$, the set of positive roots in the $\mathbb{Z}$-span of S. In this case, $Y_{[w],\gamma}$ is the closure of a one-dimensional T-orbit and

$$(Y_{[w],\gamma})^T = \{x([w]), x([ws_\gamma])\}. \tag{2.41}$$

Proposition 5. *If* $[w] \in W/W_S$ *and* $\gamma \in \Delta_+ \setminus \Delta_{S,+}$, *then* $Y_{[w],\gamma}$ *is* T*-equivariantly isomorphic to* $(\mathbb{P}^1, w\gamma)$.

Proof. Firstly, note that the $SL_2(\mathbb{C})_\gamma$-stabilizer of $[e] \in G/P_S$ is

$$B_\gamma := B \cap SL_2(\mathbb{C})_\gamma, \tag{2.42}$$

a Borel subgroup of $SL_2(\mathbb{C})_\gamma$. Hence, we have a T-equivariant isomorphism

$$Y_\gamma \cong SL_2(\mathbb{C})_\gamma/B_\gamma. \tag{2.43}$$

The latter variety is seen to equivariantly identify with $(\mathbb{P}^1, \gamma)$, so that $Y_\gamma \cong (\mathbb{P}^1, \gamma)$. In other words, there exists a variety isomorphism $\varphi : Y_\gamma \to \mathbb{P}^1$ satisfying

$$\varphi(tx) = [\gamma(t)x_0 : x_1] \tag{2.44}$$

for all $t \in T$ and $x \in Y_\gamma$, where $\varphi(x) := [x_0 : x_1]$.

Now, let $g \in N_G(T)$ represent w and recall that $Y_{[w],\gamma} = gY_\gamma$. With this in mind, consider the variety isomorphism $\psi : Y_{[w],\gamma} \to \mathbb{P}^1$ defined by $\psi(gx) = \varphi(x)$ for $x \in Y_\gamma$. To prove the proposition, it will suffice to establish that

$$\psi(t(gx)) = [(w\gamma)(t)x_0 : x_1] \tag{2.45}$$

for all $t \in T$ and $x \in Y_\gamma$ (where, again, $\psi(gx) = \varphi(x) = [x_0 : x_1]$).

We have

$$\begin{aligned}\psi(t(gx)) &= \varphi((g^{-1}tg)x) \\ &= [\gamma(g^{-1}tg)x_0 : x_1] \quad \text{(by (2.44))} \\ &= [(w\gamma)(t)x_0 : x_1],\end{aligned}$$

completing the verification of (2.45). □

It is known that the subvarieties $Y_{[w],\gamma}$, $[w] \in W/W_S$, $\gamma \in \Delta_+ \setminus \Delta_{S,+}$, constitute a complete list of the closures of one-dimensional T-orbits in G/P_S. In particular, we have the means to concretely describe the graph $\mathrm{GKM}(G/P_S)$.

- Using the description of $(G/P_S)^T$ given in (2.4), we will regard $\mathrm{GKM}(G/P_S)$ as having vertex set W/W_S.
- By (2.41), two vertices are connected by an edge if and only if they are of the form $[w]$ and $[ws_\gamma]$ for $w \in W$ and $\gamma \in \Delta_+ \setminus \Delta_{S,+}$. Noting Proposition 5, the edge in question is labelled with $w\gamma$.

Recalling Theorem 4, our description of $\mathrm{GKM}(G/P_S)$ is equivalent to the following GKM presentation of $H_T^*(G/P_S)$.

$$H_T^*(G/P_S) \cong \left\{ (f_{[w]}) \in \bigoplus_{[w] \in W/W_S} H_T^*(\mathrm{pt}) : \begin{array}{c} (w\gamma) | (f_{[w]} - f_{[ws_\gamma]}) \\ \forall w \in W \\ \forall \gamma \in \Delta_+ \setminus \Delta_{S,+} \end{array} \right\} \tag{2.46}$$

2.7.2 Example: The GKM Graph of a Full Flag Variety in Type A

Let us draw the GKM graph of $\mathcal{F}(1,2;\mathbb{C}^3)$, the full flag variety of $SL_3(\mathbb{C})$. To this end, note that $\mathcal{F}(1,2;\mathbb{C}^3) = SL_3(\mathbb{C})/B = SL_3(\mathbb{C})/P_S$ for $S = \emptyset$. In particular, W_S is the trivial group and $\Delta_{S,+} = \emptyset$. With these observations in mind, the framework from 2.7.1 applies to give the following description of $\mathrm{GKM}(\mathcal{F}(1,2;\mathbb{C}^3))$.

- The vertex set coincides with the Weyl group, which in this case is S_3. We will record these vertices using the one-line notation for elements of S_3.
- Two vertices $w, w' \in S_3$ are joined by an edge if and only if $w' = w(i\,j)$ for some $1 \leq i < j \leq 3$. In this case, the edge is given the label $w(t_i - t_j) = t_{w(i)} - t_{w(j)}$.

Setting $\gamma_{ij} := t_i - t_j$, the following is the GKM graph itself.

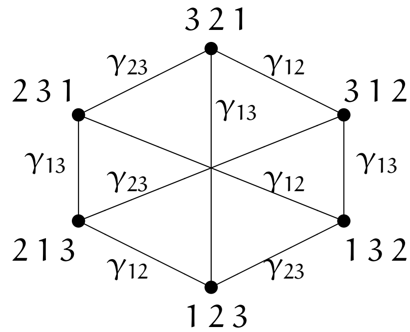

Figure 2.1: The GKM graph of $\mathcal{F}(1, 2; \mathbb{C}^3)$

2.8 Nilpotent Orbits in Semisimple Lie Algebras

The following section provides some of the relevant background and context for our study of nilpotent orbits in semisimple Lie algebras.

2.8.1 Basic Definitions and Features

Recall that $\xi \in \mathfrak{g}$ is called *nilpotent* if $\mathrm{ad}(\xi) : \mathfrak{g} \to \mathfrak{g}$ is nilpotent as a vector space endomorphism. More powerfully, if one fixes any faithful finite-dimensional $\mathfrak{g}$-module $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$, then ξ is nilpotent if and only if $\phi(\xi) : V \to V$ is a nilpotent endomorphism (see [13], Chapter I). In particular, if $\mathfrak{g}$ is presented as a Lie subalgebra of $\mathfrak{gl}_N(\mathbb{C})$, then the nilpotent elements of $\mathfrak{g}$ are precisely the nilpotent matrices lying in $\mathfrak{g}$.

The locus of all nilpotent $\xi \in \mathfrak{g}$ is called the *nilpotent cone* $\mathcal{N} \subseteq \mathfrak{g}$, namely

$$\mathcal{N} := \{\xi \in \mathfrak{g} : \xi \text{ is nilpotent}\}. \tag{2.47}$$

This closed subvariety of $\mathfrak{g}$ is known to be irreducible and singular (see (2.49) for more details on the latter property). More immediately, however, $\mathcal{N}$ is invariant under the adjoint representation of G. Equivalently, $\mathcal{N}$ is a union of adjoint G-orbits.

Definition 6. An adjoint G-orbit in $\mathcal{N}$ is called a *nilpotent orbit*.

We now mention a few of the salient properties of nilpotent orbits.

- **Locally Closed:** Being algebraic group orbits, nilpotent orbits are smooth, locally closed subvarieties of $\mathcal{N}$. However, only the trivial orbit $\{0\}$ is closed in $\mathcal{N}$.

- **Finiteness:** Applying the work of Kostant and others, one sees that there are only finitely many nilpotent orbits. Indeed, consider the inclusion $\mathbb{C}[\mathfrak{g}]^G \hookrightarrow \mathbb{C}[\mathfrak{g}]$ and the induced map of varieties

$$\chi : \mathfrak{g} \to \text{Spec}(\mathbb{C}[\mathfrak{g}]^G). \tag{2.48}$$

  The variety $\text{Spec}(\mathbb{C}[\mathfrak{g}]^G)$ identifies with affine space in such a way that $\mathcal{N}$ coincides with the fibre of χ over 0. The finiteness of nilpotent orbits then follows from Kostant's result that each fibre of χ is a finite union of adjoint orbits (see [53]).

- **Symplectic Structures:** Each nilpotent orbit $\mathcal{O} \subseteq \mathcal{N}$ is canonically a symplectic variety. Indeed, the Killing form is non-degenerate and induces an isomorphism between the adjoint and coadjoint representations. Through this isomorphism, $\mathcal{O}$ corresponds to a coadjoint orbit with its Kirillov-Kostant-Souriau symplectic structure (see [7], Section 7.5).

- **Instanton Moduli Spaces:** Nilpotent orbits arise as "instanton moduli spaces" in theoretical physics. More precisely let $K \subseteq G$ be a compact real form with Lie algebra $\mathfrak{k} \subseteq \mathfrak{g}$. One considers smooth curves $A : \mathbb{R} \to \text{Hom}(\mathfrak{su}(2), \mathfrak{k})$ into the vector space of $\mathbb{R}$-linear maps $\mathfrak{su}(2) \to \mathfrak{k}$. The space of such curves carries a natural action of K, and we shall let $C(\rho)$ denote the K-orbit of a Lie algebra morphism $\rho : \mathfrak{su}(2) \to \mathfrak{k}$. The instanton moduli space $\mathcal{M}(\rho)$ then consists of those smooth curves $A : \mathbb{R} \to \text{Hom}(\mathfrak{su}(2), \mathfrak{k})$ satisfying

$$\lim_{t\to\infty} A(t) \in C(\rho),$$

  as well as the anti-self-dual Yang-Mills equations on $SU(2) \times \mathbb{R}$ (see [28]).

  Now, ρ complexifies to give $\rho_{\mathbb{C}} : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{g}$. It is a celebrated result of Kronheimer's work [55] that $\mathcal{M}(\rho)$ is canonically diffeomorphic to the nilpotent G-orbit through $\rho_{\mathbb{C}}(e)$, where

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in \mathfrak{sl}_2(\mathbb{C})$$

is the standard nilpositive vector.

Conversely, suppose that $\mathcal{O} \subseteq \mathfrak{g}$ is a nilpotent orbit. Given any point $\xi \in \mathcal{O}$, the Jacobson-Morozov Theorem allows one to find a complex Lie algebra map $\phi : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{g}$ such that $\xi = \phi(e)$. By replacing ϕ with a G-conjugate if necessary, we may assume that $\phi(\mathfrak{su}(2)) \subseteq \mathfrak{k}$. It follows that $\phi = \rho_{\mathbb{C}}$, where $\rho : \mathfrak{su}(2) \to \mathfrak{k}$ is the restriction of ϕ to $\mathfrak{su}(2)$. Kronheimer's result then identifies $\mathcal{O}$ with $\mathcal{M}(\rho)$. Hence, our discussion demonstrates that every nilpotent G-orbit arises as an instanton moduli space.

- **Springer Theory:** Nilpotent orbits are ubiquitous in Springer Theory (see [17]). In more detail, one has the so-called *Springer resolution*

$$\mu : G \times_B \mathfrak{n} \to \mathcal{N}, \quad [(g, \xi)] \mapsto \mathrm{Ad}(g)(\xi), \tag{2.49}$$

where $\mathfrak{n} := \bigoplus_{\gamma \in \Delta_+} \mathfrak{g}_\gamma$. The fibres of μ are called *Springer fibres*. Notably, the isomorphism class of $\mu^{-1}(\xi)$ depends only on the nilpotent orbit containing ξ.

- **The Dilation Action:** Nilpotent orbits are distinguished from general adjoint orbits in that only the former are dilation-invariant. Indeed, let $\mathcal{O} \subseteq \mathfrak{g}$ be a nilpotent orbit. Given $\xi \in \mathcal{O}$ and $a \in \mathbb{C}^*$, we claim that $a\xi \in \mathcal{O}$. To this end, the Jacobson-Morozov Theorem allows us to find $h \in \mathfrak{g}$ for which $[h, \xi] = 2\xi$. Also, there exists $b \in \mathbb{C}$ for which $e^{2b} = a$.

$$\begin{aligned} \mathrm{Ad}(\exp(bh))(\xi) &= e^{\mathrm{ad}(bh)}(\xi) \\ &= \sum_{j=0}^{\infty} \frac{1}{j!} \left(\mathrm{ad}(bh)\right)^j (\xi) \\ &= \sum_{j=0}^{\infty} \frac{(2b)^j}{j!} \xi \\ &= e^{2b}\xi \\ &= a\xi, \end{aligned}$$

where $\exp : \mathfrak{g} \to G$ is the exponential map. Since $\mathrm{Ad}(\exp(bh))(\xi) \in \mathcal{O}$, this calculation establishes that $a\xi \in \mathcal{O}$.

Now, fix a non-zero nilpotent orbit $\mathcal{O}$. The dilation action of $\mathbb{C}^*$ on $\mathfrak{g}$ commutes with the adjoint representation of G, and both group actions respect the inclusion $\mathcal{O} \subseteq \mathfrak{g} \setminus \{0\}$. In particular, $\mathbb{P}(\mathfrak{g}) = (\mathfrak{g} \setminus \{0\})/\mathbb{C}^*$ carries a residual G-action for which

$$\mathbb{P}(\mathcal{O}) := \mathcal{O}/\mathbb{C}^* \tag{2.50}$$

is a G-orbit.

- **The Closure Order:** The set $\mathcal{N}/G$ of nilpotent orbits carries a canonical partial order, called the *closure order*. Given nilpotent orbits $\mathcal{O}_1, \mathcal{O}_2 \subseteq \mathcal{N}$, one defines

$$\mathcal{O}_1 \leq \mathcal{O}_2 \Longleftrightarrow \mathcal{O}_1 \subseteq \overline{\mathcal{O}_2}. \tag{2.51}$$

Strictly speaking, we should specify the topology in which the closure of $\mathcal{O}_2$ is taken. However, as $\mathcal{O}_2$ is a constructible subset of $\mathfrak{g}$, its Zariski and classical closures agree.

- **The Regular Nilpotent Orbit:** The poset $\mathcal{N}/G$ has a unique maximal element $\mathcal{O}_{\text{reg}}$, called the *regular nilpotent orbit*. As the nomenclature suggests, $\mathcal{O}_{\text{reg}}$ consists of those $\xi \in \mathfrak{g}$ which are simultaneously nilpotent and regular (meaning that the Lie algebra centralizer of ξ has dimension equal to the rank of $\mathfrak{g}$). Furthermore, $\mathcal{O}_{\text{reg}}$ is known to have a family of standard representatives. Letting $e_\alpha \in \mathfrak{g}_\alpha \setminus \{0\}$, $\alpha \in \Pi$, be a choice of non-zero vector in each simple root space, we have

$$\sum_{\alpha \in \Pi} e_\alpha \in \mathcal{O}_{\text{reg}} \tag{2.52}$$

(see [52], Lemma 5.2).

We note that $\mathcal{O}_{\text{reg}}$ is sometimes called the *principal nilpotent orbit*, reflecting a close connection to Kostant's notion of a principal three-dimensional subalgebra (TDS). By definition, a principal TDS is a Lie subalgebra of $\mathfrak{g}$ having the form $\mathfrak{a} = \text{span}_{\mathbb{C}}\{\eta, h, \xi\}$, where $\xi \in \mathcal{O}_{\text{reg}}$, $\eta \in \mathfrak{g}$, and $h \in \mathfrak{g}$ satisfy the following conditions:

$$[\xi, \eta] = h, \quad [h, \xi] = 2\xi, \quad [h, \eta] = -2\eta.$$

In particular, (η, h, ξ) is an $\mathfrak{sl}_2(\mathbb{C})$-triple and $\mathfrak{a}$ is an isomorphic copy of $\mathfrak{sl}_2(\mathbb{C})$.

As an $\mathfrak{a} \cong \mathfrak{sl}_2(\mathbb{C})$-module, $\mathfrak{g}$ decomposes into irreducible highest-weight representations. Kostant showed this decomposition to be

$$\mathfrak{g} \cong \bigoplus_{j=1}^{r} V(n_j) \tag{2.53}$$

for some even positive integers $n_1, n_2, \dots, n_r$, where $r = \text{rank}(\mathfrak{g})$ and $V(n_j)$ is the irreducible $\mathfrak{a}$-module of highest weight n_j (see [52], Theorem 5.2). Perhaps surprisingly, Kostant found the following relationship between the representation-theoretic statement (2.53) and the Poincaré polynomial of G, $P_G(t)$:

$$P_G(t) = \prod_{j=1}^{r}(1 + t^{n_j+1}).$$

- **The Minimal Nilpotent Orbit:** On the opposite extreme, one finds $\{0\}$ to be the unique minimal element of $\mathcal{N}/G$. While this is neither surprising nor interesting, it turns out that the set of non-zero nilpotent orbits has a unique minimal element $\mathcal{O}_{\text{min}}$, called the *minimal nilpotent orbit*.

  Like the regular nilpotent orbit, $\mathcal{O}_{\text{min}}$ is known to have standard representatives. Indeed, recalling that λ denotes the highest root, one has

$$\mathfrak{g}_\lambda \setminus \{0\} \subseteq \mathcal{O}_{\text{min}}. \tag{2.54}$$

  (see [18]).

  While $\mathcal{O}_{\text{min}}$ is independently interesting, it is often studied in conjunction with its projectivization $\mathbb{P}(\mathcal{O}_{\text{min}})$. This latter variety has several rich geometric structures. Indeed, $\mathbb{P}(\mathcal{O}_{\text{min}})$ is the G-orbit in $\mathbb{P}(\mathfrak{g})$ having minimal dimension. It follows that $\mathbb{P}(\mathcal{O})$ is a closed (hence projective) orbit, and therefore also a partial flag variety of G (see Proposition 9 for more details). At the same time, the symplectic structure on $\mathcal{O}_{\text{min}}$ descends to give a complex contact structure on $\mathbb{P}(\mathcal{O}_{\text{min}})$. It is then a famous conjecture of LeBrun and Salamon that minimal nilpotent orbit projectivizations constitute the only examples of Fano contact varieties having second Betti number equal to 1. This conjecture has significant implications in quaternionic Kähler geometry, where these Fano contact varieties arise as twistor spaces. For more details, we direct the reader to Chapter 6.

2.8.2 Nilpotent Orbits in Type A

We now specialize aspects of 2.8.1 to $G = SL_n(\mathbb{C})$ and its nilpotent orbits. Indeed, as discussed at the beginning of 2.8.1, $X \in \mathfrak{sl}_n(\mathbb{C})$ is nilpotent in our Lie algebraic sense if and only if it is nilpotent as a matrix. Hence, by (2.1), the nilpotent orbits of $SL_n(\mathbb{C})$ are precisely the ($SL_n(\mathbb{C})$-) conjugacy classes of nilpotent $n \times n$ matrices. The latter are indexed by the partitions of n by means of Jordan canonical forms. In particular, one sees directly that there are only finitely many nilpotent orbits of $SL_n(\mathbb{C})$.

Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of n, let $\mathcal{O}_\lambda \subseteq \mathfrak{sl}_n(\mathbb{C})$ denote the conjugacy class of a nilpotent Jordan matrix with k blocks of respective sizes $\lambda_1, \lambda_2, \ldots, \lambda_k$. It is a classical result of Gerstenhaber [34] that

$$O_\lambda \leq \mathcal{O}_\mu \iff \lambda \leq \mu, \tag{2.55}$$

where λ and μ are compared in the dominance order (see [70]). Studying the dominance order, one finds that

$$\mathcal{O}_{\text{reg}} = \mathcal{O}_{(n)} \tag{2.56}$$

and

$$\mathcal{O}_{\text{min}} = \mathcal{O}_{(2,1,1,\ldots,1)}. \tag{2.57}$$

Chapter 3

The T-Equivariant Cohomology of Nilpotent Orbits

In what follows, we obtain concrete, self-contained presentations of the T-equivariant cohomology rings of $\mathcal{O}_{\text{reg}}$ and $\mathcal{O}_{\text{min}}$. Our exposition is largely based on the author's work [20].

3.1 Introduction and Structure

As discussed in 2.8.1, the study of nilpotent orbits invokes algebraic geometry, representation theory, and symplectic geometry. However, nilpotent orbits have also been studied in the context of algebraic topology. The work of Springer, Steinberg, and others has led to a computation of the fundamental group of every nilpotent orbit in the classical Lie algebras (see [18]). Also, Juteau's paper [46] gives the integral cohomology groups of the minimal nilpotent orbit in each of the finite-dimensional complex simple Lie algebras. Additionally, Biswas and Chatterjee compute $\mathsf{H}^2(\mathcal{O};\mathbb{R})$ for $\mathcal{O}$ any nilpotent orbit in a finite-dimensional complex simple Lie algebra (see their paper [10]).

We examine aspects of the equivariant algebraic topology and geometry of nilpotent orbits, with the following descriptions of $\mathsf{H}^*_\mathsf{T}(\mathcal{O}_{\text{reg}})$ and $\mathsf{H}^*_\mathsf{T}(\mathcal{O}_{\text{min}})$ constituting the the main results of this section.

Theorem 7. *(i)* $\mathsf{H}^*_\mathsf{T}(\mathcal{O}_{\text{reg}}) \cong \mathsf{H}^*(\mathsf{G}/\mathsf{B})$

(ii) Let Λ denote the collection of simple roots that are orthogonal to the highest root λ. The

algebra $H_T^*(\mathcal{O}_{min})$ *is isomorphic to the quotient of*

$$\left\{ (f_{[w]}) \in \bigoplus_{[w]\in W/W_\Lambda} H_T^*(pt) : \begin{array}{c} (w\gamma)|(f_{[w]} - f_{[ws_\gamma]}) \\ \forall w \in W \\ \forall \gamma \in \Delta_+ \text{ s.t. } \langle \gamma, \lambda \rangle \neq 0 \end{array} \right\}$$

by the ideal generated by $(w\lambda)_{[w]\in W/W_\Lambda}$.

Our arguments begin in 3.2, which provides a direct computation of $H_T^*(\mathcal{O}_{reg})$. Section 3.3 then treats the case of the minimal nilpotent orbit, but the approach differs considerably from that adopted when studying $\mathcal{O}_{reg}$. In particular, we make extensive use of the intermediate variety $\mathbb{P}(\mathcal{O}_{min})$ from 2.8.1. This variety has interesting properties beyond those materially relevant to computing $H_T^*(\mathcal{O}_{min})$. Notably, $\mathbb{P}(\mathcal{O}_{min})$ is naturally a symplectic manifold, and the T-action on $\mathcal{O}_{min}$ descends to a Hamiltonian action (of the real part of T) on $\mathbb{P}(\mathcal{O}_{min})$. We give an explicit description of $\mathbb{P}(\mathcal{O}_{min})^T$ and use it to find the moment polytope of $\mathbb{P}(\mathcal{O}_{min})$.

We next give a GKM-theoretic description of $H_T^*(\mathbb{P}(\mathcal{O}_{min}))$. This algebra is then related to $H_T^*(\mathcal{O}_{min})$ via a Thom-Gysin sequence in T-equivariant cohomology, completing the proof of Theorem 7.

3.2 The Regular Nilpotent Orbit

Recall that an element $\xi \in \mathfrak{g}$ is called *regular* if the dimension of the Lie algebra centralizer $C_{\mathfrak{g}}(\xi) = \{X \in \mathfrak{g} : [X, \xi] = 0\}$ coincides with the rank of $\mathfrak{g}$. The regular nilpotent elements of $\mathfrak{g}$ constitute $\mathcal{O}_{reg}$.

Fix a point $\eta \in \mathcal{O}_{reg}$, and let $C_G(\eta) = \{g \in G : \mathrm{Ad}_g(\eta) = \eta\}$ be the G-stabilizer of η. This gives a G-equivariant isomorphism $\mathcal{O}_{reg} \cong G/C_G(\eta)$. Having realized $\mathcal{O}_{reg}$ in this way, we turn our attention to $C_G(\eta)$. Indeed, note that $\mathcal{O}_{reg}$ is a *distinguished* nilpotent orbit (see [18]), meaning that the unipotent radical of $C_G(\eta)$ coincides with the identity component $C_G(\eta)^0$. It follows that

$$C_G(\eta) = C_G(\eta)^0 \times Z(G)$$

is the Levi decomposition of $C_G(\eta)$, where $Z(G)$ is the (finite) centre of G. Noting that $C_G(\eta)^0$ is a copy of affine space, one has the affine bundle

$$EG/Z(G) \to EG/C_G(\eta)$$

and therefore also an isomorphism between the cohomology rings of the base and total spaces. The ordinary cohomology of $EG/C_G(\eta)$ is precisely $H_G^*(G/C_G(\eta)) = H_G^*(\mathcal{O}_{\text{reg}})$, while that of $EG/Z(G)$ is isomorphic to $H_{Z(G)}^*(\text{pt})$. In other words,

$$H_G^*(\mathcal{O}_{\text{reg}}) \cong H_{Z(G)}^*(\text{pt}).$$

However, since $Z(G)$ is a finite group, $H_{Z(G)}^*(\text{pt})$ is isomorphic to $\mathbb{Q}$ (by which we mean the graded $\mathbb{Q}$-algebra equal to $\mathbb{Q}$ in degree zero and vanishing in all other grading degrees). We thus have

$$H_G^*(\mathcal{O}_{\text{reg}}) \cong \mathbb{Q}. \tag{3.1}$$

Now, let K be a compact real form of G such that $T_{\mathbb{R}} := K \cap T$ is a real form of T. Recall that the $T_{\mathbb{R}}$-equivariant cohomology of a smooth K-manifold is obtained by tensoring its K-equivariant cohomology with $H_{T_{\mathbb{R}}}^*(\text{pt})$ over $H_{T_{\mathbb{R}}}^*(\text{pt})^W$ (see Proposition 1(iii) of [14]). In our case, this gives

$$H_T^*(\mathcal{O}_{\text{reg}}) = H_{T_{\mathbb{R}}}^*(\mathcal{O}_{\text{reg}}) \cong H_{T_{\mathbb{R}}}^*(\text{pt}) \otimes_{H_{T_{\mathbb{R}}}^*(\text{pt})^W} H_K^*(\mathcal{O}_{\text{reg}}). \tag{3.2}$$

Using (3.1) and recalling the standard equivariant cohomology isomorphisms (2.27) and (2.30) we obtain

$$H_K^*(\mathcal{O}_{\text{reg}}) \cong H_G^*(\mathcal{O}_{\text{reg}}) \cong \mathbb{Q} \cong H_K^*(K).$$

Accordingly, we may replace $H_K^*(\mathcal{O}_{\text{reg}})$ with $H_K^*(K)$ in (3.2). Applying Proposition 1(iii) of [14] again, the right-hand side of (3.2) is seen to be isomorphic to $H_{T_{\mathbb{R}}}^*(K) \cong H^*(K/T_{\mathbb{R}}) \cong H^*(G/B)$. In other words,

$$H_T^*(\mathcal{O}_{\text{reg}}) \cong H^*(G/B),$$

as claimed in the statement of Theorem 7.

Remark. Without modification, our arguments establish the slightly more general fact that the T-equivariant cohomology of a distinguished nilpotent orbit coincides with the ordinary cohomology of the full flag variety.

3.3 The Minimal Nilpotent Orbit

We now address the matter of computing $H_T^*(\mathcal{O}_{\text{min}})$. Note that we could try to proceed in analogy with Section 3.2 by fixing $\eta \in \mathcal{O}_{\text{min}}$, taking L to be the reductive part in a Levi decomposition of $C_G(\eta)$, and so forth. While this approach is certainly legitimate, we will compute the T-equivariant cohomology of $\mathcal{O}_{\text{min}}$ by first determining that of the variety $\mathbb{P}(\mathcal{O}_{\text{min}})$ mentioned in 2.8.1. With this in mind, it will be informative to give nilpotent orbit projectivizations some context in equivariant geometry.

3.3.1 The Equivariant Geometry of $\mathbb{P}(\mathcal{O})$

Fix a non-zero nilpotent orbit $\mathcal{O} \subseteq \mathcal{N}$. Also, as before, let K be a maximal compact subgroup of G with the property that $T_{\mathbb{R}} := K \cap T$ is a real form of T. Choose a K-invariant Hermitian inner product $\langle , \rangle : \mathfrak{g} \otimes_{\mathbb{R}} \mathfrak{g} \to \mathbb{C}$. This yields a K-invariant Kähler structure on $\mathbb{P}(\mathfrak{g})$. Since the usual action of $U(n+1)$ on $\mathbb{P}^n$ is Hamiltonian, so too is the action of K on $\mathbb{P}(\mathfrak{g})$. Furthermore, one has the moment map $\Phi : \mathbb{P}(\mathfrak{g}) \to \mathfrak{k}^*$ defined by

$$\Phi([\xi])(X) = -i\frac{\langle [X, \xi], \xi \rangle}{\langle \xi, \xi \rangle},$$

where $X \in \mathfrak{g} \setminus \{0\}$ and $\eta \in \mathfrak{k}$ (see [22] for a derivation of Φ). Note that the Kähler structure on $\mathbb{P}(\mathfrak{g})$ restricts to a K-invariant Kähler structure on the smooth subvariety $\mathbb{P}(\mathcal{O})$, and that the action of K on $\mathbb{P}(\mathcal{O})$ is Hamiltonian.

Let us take a moment to examine the Hamiltonian action of $T_{\mathbb{R}}$ on $\mathbb{P}(\mathcal{O})$. We have

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\beta \in \Delta} \mathfrak{g}_\beta,$$

the weight space decomposition of the adjoint representation. Note that a point in $\mathbb{P}(\mathfrak{g})$ is fixed by T if and only if it is a class of vectors in $\mathfrak{g} \setminus \{0\}$ with the property that T acts by scaling each vector. In other words,

$$\mathbb{P}(\mathfrak{g})^T = \mathbb{P}(\mathfrak{t}) \cup \{\mathfrak{g}_\beta : \beta \in \Delta\}.$$

With this description, we may determine $\mathbb{P}(\mathcal{O})^T$. Indeed, since $\mathfrak{t}$ consists of semisimple elements of $\mathfrak{g}$ while $\mathcal{O}$ consists of non-zero nilpotent elements, we find that $\mathfrak{t} \cap \mathcal{O} = \emptyset$. Hence,

$$\mathbb{P}(\mathcal{O})^T = \{\mathfrak{g}_\beta : \beta \in \Delta, \mathfrak{g}_\beta \cap \mathcal{O} \neq \emptyset\},$$

a finite set. In particular, $\mathbb{P}(\mathcal{O})^T$ is non-empty if and only if $\mathcal{O}$ is the orbit of a root vector.

Let us take a moment to provide a more refined description of $\mathbb{P}(\mathcal{O})^T$. To this end, we will require the following lemma.

Lemma 8. *Let $\beta, \gamma \in \Delta$ be roots. The root spaces $\mathfrak{g}_\beta$ and $\mathfrak{g}_\gamma$ are* G*-conjugate if and only if* β *and* γ *are conjugate under* W.

Proof. Suppose that $w \in W$ and that $\beta = w \cdot \gamma$. Choosing a representative $g \in N_G(T)$ of w, this means precisely that $\beta = \gamma \circ \varphi_{g^{-1}}|_T$, where $\varphi_{g^{-1}} : G \to G$ is conjugation by g^{-1}. Given $t \in T$ and $\xi \in \mathfrak{g}_\beta$, note that

$$\begin{aligned} \mathrm{Ad}(t)(\mathrm{Ad}(g)(\xi)) &= \mathrm{Ad}(g)(\mathrm{Ad}(g^{-1}tg)(\xi)) \\ &= \mathrm{Ad}(g)(\beta(g^{-1}tg)\xi) \\ &= \mathrm{Ad}(g)(\gamma(t)\xi) \\ &= \gamma(t)(\mathrm{Ad}(g)(\xi)). \end{aligned}$$

It follows that $\mathfrak{g}_\gamma = \mathrm{Ad}(g)(\mathfrak{g}_\beta)$.

Conversely, suppose that $g \in G$ and that $\mathfrak{g}_\gamma = \mathrm{Ad}(g)(\mathfrak{g}_\beta)$. Consider the Zariski-closed subgroup

$$L := \{x \in G : \mathrm{Ad}(x)(\mathfrak{g}_\gamma) = \mathfrak{g}_\gamma\},$$

noting that $T, gTg^{-1} \subseteq L$. Since T and gTg^{-1} are maximal tori of L, there exists $x \in L$ for which $xTx^{-1} = gTg^{-1}$. Hence, $x^{-1}g \in N_G(T)$ and $\mathrm{Ad}(x^{-1}g)(\mathfrak{g}_\beta) = \mathfrak{g}_\gamma$. We may therefore assume that $g \in N_G(T)$. Now, let $w \in W$ denote the class of g. Given $t \in T$ and $\xi \in \mathfrak{g}_\beta$, we find that

$$\begin{aligned} (w \cdot \beta)(t)\xi &= \beta(g^{-1}tg)\xi \\ &= \mathrm{Ad}(g^{-1}tg)(\xi) \\ &= \mathrm{Ad}(g^{-1})(\gamma(t)\,\mathrm{Ad}(g)(\xi)) \\ &= \gamma(t)\xi. \end{aligned}$$

It follows that $\gamma = w \cdot \beta$. □

Since $\mathfrak{g}$ is a simple Lie algebra, the root system associated with the pair $(\mathfrak{g}, \mathfrak{t})$ is irreducible. Hence, there are at most two distinct root lengths (namely, those of the long and short roots), and the roots of a given length constitute an orbit of W in Δ. By Lemma 8, there are at most two nilpotent G-orbits $\mathcal{O}$ for which $\mathbb{P}(\mathcal{O})^T$ is non-empty, the orbits of root vectors for the short and long roots. Furthermore, if $\mathcal{O}$ is the orbit of a root vector $e_\beta \in \mathfrak{g}_\beta \setminus \{0\}$, $\beta \in \Delta$, then $\mathbb{P}(\mathcal{O})^T$ is the union of the points $\mathfrak{g}_\gamma$ for all $\gamma \in \Delta$ with length equal to that of β. Since $\mathcal{O}_{\min}$ is the orbit of a highest (hence long) root vector, $\mathbb{P}(\mathcal{O}_{\min})^T = \{\mathfrak{g}_\gamma : \gamma \in \Delta_{\text{long}}\}$, where $\Delta_{\text{long}} \subseteq \Delta$ is the set of long roots.

3.3.2 The Moment Polytope of $\mathbb{P}(\mathcal{O}_{\mathbf{min}})$

Note that the moment map $\Phi : \mathbb{P}(\mathfrak{g}) \to \mathfrak{k}^*$ considered earlier can be modified to obtain a moment map for the Hamiltonian action of $T_{\mathbb{R}}$ on $\mathbb{P}(\mathcal{O}_{\text{min}})$. Indeed, we denote by $\mu : \mathbb{P}(\mathcal{O}_{\text{min}}) \to (\mathfrak{t}_{\mathbb{R}})^*$ the moment map given by the composition

$$\mathbb{P}(\mathcal{O}_{\text{min}}) \hookrightarrow \mathbb{P}(\mathfrak{g}) \xrightarrow{\Phi} \mathfrak{k}^* \to (\mathfrak{t}_{\mathbb{R}})^*$$

(where $\mathfrak{t}_{\mathbb{R}}$ denotes the Lie algebra of $T_{\mathbb{R}}$).

Recall that

$$\mathbb{P}(\mathcal{O}_{\text{min}})^T = \{\mathfrak{g}_\beta : \beta \in \Delta_{\text{long}}\}.$$

Given $\beta \in \Delta_{\text{long}}$, choose a point $e_\beta \in \mathfrak{g}_\beta \setminus \{0\}$. Note that for $X \in \mathfrak{t}_{\mathbb{R}}$,

$$\mu(\mathfrak{g}_\beta)(X) = -i\frac{\langle [X, e_\beta], e_\beta\rangle}{\langle e_\beta, e_\beta\rangle} = -i\frac{d_e\beta(X)\langle e_\beta, e_\beta\rangle}{\langle e_\beta, e_\beta\rangle} = -id_e\beta(X), \tag{3.3}$$

where $d_e\beta : \mathfrak{t}_{\mathbb{R}} \to i\mathbb{R}$ is the morphism of real Lie algebras induced by $\beta : T_{\mathbb{R}} \to U(1)$. If one regards the weight lattice $X^*(T)$ as included into $(\mathfrak{t}_{\mathbb{R}})^*$ in the usual way, then (3.3) becomes

$$\mu(\mathfrak{g}_\beta) = \beta.$$

The moment polytope $\mu(\mathbb{P}(\mathcal{O}_{\text{min}}))$ is then the convex hull of Δ_{long} in $(\mathfrak{t}_{\mathbb{R}})^*$.

3.3.3 A Description of $\mathcal{O}_{\mathbf{min}}$ and $\mathbb{P}(\mathcal{O}_{\mathbf{min}})$

We devote this section to giving explicit descriptions of $\mathcal{O}_{\text{min}}$ and $\mathbb{P}(\mathcal{O}_{\text{min}})$ as homogeneous G-varieties. To this end, fix a highest root vector $e_\lambda \in \mathfrak{g}_\lambda \setminus \{0\} \subseteq \mathcal{O}_{\text{min}}$, and let $[e_\lambda] \in \mathbb{P}(\mathcal{O}_{\text{min}})$ denote its class in $\mathbb{P}(\mathcal{O}_{\text{min}})$. Also, let

$$\Lambda := \{\alpha \in \Pi : \langle \alpha, \lambda\rangle = 0\} \tag{3.4}$$

be the collection of simple roots orthogonal to the highest root λ.

Proposition 9. *The* G*-stabilizer* $C_G([e_\lambda])$ *coincides with* P_Λ*, inducing a* G*-equivariant variety isomorphism* $G/P_\Lambda \cong \mathbb{P}(\mathcal{O}_{min})$.

Proof. Since λ is the highest root, $\text{Ad}(b)(e_\lambda)$ is a non-zero multiple of e_λ for each $b \in B$. It follows that $B \subseteq C_G([e_\lambda])$, so that $C_G([e_\lambda])$ is a standard parabolic subgroup of G. It will therefore suffice to prove that for $\alpha \in \Pi$, $\mathfrak{g}_{-\alpha}$ belongs to $C_{\mathfrak{g}}([e_\lambda])$ if and only if $\langle \alpha, \lambda\rangle = 0$, where $C_{\mathfrak{g}}([e_\lambda])$ denotes the Lie algebra of $C_G([e_\lambda])$. Noting that $C_{\mathfrak{g}}([e_\lambda]) =$

$\{\xi \in \mathfrak{g} : [\xi, \mathfrak{g}_\lambda] \subseteq \mathfrak{g}_\lambda\}$, and that $[\mathfrak{g}_{-\alpha}, \mathfrak{g}_\lambda] \cap \mathfrak{g}_\lambda = \{0\}$, we see that $\mathfrak{g}_{-\alpha} \subseteq C_\mathfrak{g}([e_\lambda])$ if and only if $[\mathfrak{g}_{-\alpha}, \mathfrak{g}_\lambda] = \{0\}$. So, our task is actually to prove that $[\mathfrak{g}_{-\alpha}, \mathfrak{g}_\lambda] = \{0\}$ if and only if $\langle \alpha, \lambda \rangle = 0$.

Now, assume that $[\mathfrak{g}_{-\alpha}, \mathfrak{g}_\lambda] = \{0\}$. Since λ is the highest root, we also have $[\mathfrak{g}_{\alpha}, \mathfrak{g}_\lambda] = \{0\}$. It follows that

$$[[\mathfrak{g}_{-\alpha}, \mathfrak{g}_\alpha], \mathfrak{g}_\lambda] = \{0\}. \tag{3.5}$$

Letting $h_\alpha \in [\mathfrak{g}_{-\alpha}, \mathfrak{g}_\alpha]$ be the coroot associated with α and regarding λ as an element of $\mathfrak{t}^*$, (3.5) gives

$$0 = [h_\alpha, e_\lambda] = \lambda(h_\alpha)e_\lambda = 2\frac{\langle \alpha, \lambda \rangle}{\langle \alpha, \alpha \rangle} e_\lambda.$$

It follows that $\langle \alpha, \lambda \rangle = 0$.

Conversely, assume that $\langle \alpha, \lambda \rangle = 0$. If $[\mathfrak{g}_{-\alpha}, \mathfrak{g}_\lambda] \neq \{0\}$, then $\lambda - \alpha$ is a root. Hence,

$$s_\alpha(\lambda - \alpha) = \lambda - 2\frac{\langle \alpha, \lambda \rangle}{\langle \alpha, \alpha \rangle}\alpha + \alpha = \lambda + \alpha$$

is also a root, contradicting the maximality of λ. □

Let us now address the G-variety structure of $\mathcal{O}_{\min}$. To this end, we denote by $\mathcal{L} \xrightarrow{\pi} \mathbb{P}(\mathfrak{g})$ the tautological line bundle over $\mathbb{P}(\mathfrak{g})$. Recall that for $\xi \in \mathfrak{g} \setminus \{0\}$, we have $\pi^{-1}([\xi]) = \text{span}_\mathbb{C}\{\xi\}$. Furthermore, the tautological bundle is G-equivariant, with the G-action on the total space $\mathcal{L}$ given by

$$g : ([\xi], \nu) \mapsto ([\text{Ad}_g(\xi)], \text{Ad}_g(\nu)),$$

$g \in G$, $\xi \in \mathfrak{g} \setminus \{0\}$, $\nu \in \text{span}_\mathbb{C}\{\xi\}$.

Let $\mathcal{E} \xrightarrow{\varphi} \mathbb{P}(\mathcal{O}_{\min})$ denote the pullback of $\mathcal{L}$ along the inclusion $\mathbb{P}(\mathcal{O}_{\min}) \hookrightarrow \mathbb{P}(\mathfrak{g})$. Note that $\mathcal{O}_{\min}$ G-equivariantly (and also $\mathbb{C}^*$-equivariantly) includes into $\mathcal{E}$ as a smooth open subvariety, namely the complement $\mathcal{E}^\times$ of the zero-section. Accordingly, our analysis of $\mathcal{O}_{\min}$ will benefit from a closer examination of the bundle $\mathcal{E}$. In particular, note that $\mathcal{E}$ inherits from $\mathcal{L}$ the structure of a G-equivariant line bundle over $\mathbb{P}(\mathcal{O}_{\min})$. Having identified $\mathcal{O}_{\min}$ with G/P_Λ via Proposition 9, we may exhibit $\mathcal{E}$ as an associated bundle for the one-dimensional P_Λ-representation $\varphi^{-1}([e_\lambda]) = \mathfrak{g}_\lambda$. Using the notation (2.17), our conclusion becomes $\mathcal{E} \cong \mathcal{L}(\lambda)$. With this in mind, the following is a summary of our discussion.

Theorem 10. *There is an isomorphism of* G*-equivariant* $\mathbb{C}^*$*-bundles over* G/P_Λ *between* $\mathcal{O}_{\min}$ *and* $\mathcal{L}(\lambda)^\times$.

3.3.4 The T-Equivariant Cohomology of $\mathcal{O}_{\min}$

Let us use the description of $\mathcal{O}_{\min}$ provided in Theorem 10 to compute $H^*_T(\mathcal{O}_{\min})$. To this end, we have the equivariant Thom-Gysin sequence

$$\cdots \to H^{i-2}_T(G/P_\Lambda) \to H^i_T(\mathcal{L}(\lambda)) \to H^i_T(\mathcal{L}(\lambda)^\times) \to \cdots \tag{3.6}$$

associated with the zero-section G/P_Λ in $\mathcal{L}(\lambda)$ and its complement $\mathcal{L}(\lambda)^\times$. We can say considerably more about this sequence in our context, but it will require a brief computation of the T-equivariant first Chern class $c_1^T(N) \in H^2_T(G/P_\Lambda)$ of the normal bundle $N \cong \mathcal{L}(\lambda)$ of the zero-section in $\mathcal{L}(\lambda)$. Recalling the notation (2.4), consider the inclusion $i_{[w]} : \{x([w])\} \hookrightarrow G/P_\Lambda$ of the T-fixed point $x([w]) \in (G/P_\Lambda)^T$, $[w] \in W/W_\Lambda$. Let

$$i^*_{[w]} : H^*_T(G/P_\Lambda) \to H^*_T(\mathrm{pt})$$

denote the associated restriction map on T-equivariant cohomology. The following lemma computes the restriction of $c_1^T(\mathcal{L}(\lambda))$ to each T-fixed point in G/P_Λ.

Lemma 11. *If $w \in W$, then $i^*_{[w]}(c_1^T(\mathcal{L}(\lambda)) = w\lambda$.*

Proof. Note that

$$i^*_{[w]}(c_1^T(\mathcal{L}(\lambda))) = c_1^T(\mathcal{L}(\lambda)_{x([w])}),$$

where $\mathcal{L}(\lambda)_{x([w])}$ is the fibre over $x([w])$, viewed as a T-equivariant vector bundle over a point. Recalling (2.32), the T-equivariant first Chern class of this bundle is precisely its weight as a representation of T. To compute this weight, choose a representative $k \in N_G(T)$ of w. Note that by (2.12) and the discussion preceding it, any element of $\mathcal{L}(\lambda)_{x([w])}$ is of the form $[(k, \xi)]$, $\xi \in \mathfrak{g}_\lambda$. For $t \in T$, we have

$$\begin{aligned} t \cdot [(k, \xi)] &= [(tk, \xi)] \\ &= [(k(k^{-1}tk), \xi)] \\ &= [(k, (k^{-1}tk) \cdot \xi)] \\ &= [(k, \lambda(k^{-1}tk)\xi)] \\ &= [(k, (w\lambda)(t)\xi)] \\ &= (w\lambda)(t)[(k, \xi)]. \end{aligned}$$

Hence, $w\lambda = c_1^T(\mathcal{L}(\lambda)_{x([w])}) = i^*_{[w]}(c_1^T(\mathcal{L}(\lambda)))$. □

Now, recall that

$$H_T^*((G/P_\Lambda)^T) = \bigoplus_{[w]\in W/W_\Lambda} H_T^*(\mathrm{pt}) \tag{3.7}$$

as $\mathbb{Q}$-algebras. Lemma 11 is then seen to imply that the image of $c_1^T(\mathcal{L}(\lambda))$ under the restriction map

$$H_T^*(G/P_\Lambda) \to H_T^*((G/P_\Lambda)^T)$$

has a non-zero projection to each direct summand appearing in (3.7). Since restriction gives an inclusion of $H_T^*(G/P_\Lambda)$ into $H_T^*((G/P_\Lambda)^T)$ as a subalgebra, and since $H_T^*(\mathrm{pt})$ has no zero-divisors, we conclude that $c_1^T(\mathcal{L}(\lambda))$ is not a zero-divisor in $H_T^*(G/P_\Lambda)$. It follows that our Thom-Gysin sequence splits into the short-exact sequences

$$0 \to H_T^{i-2}(G/P_\Lambda) \to H_T^i(\mathcal{L}(\lambda)) \to H_T^i(\mathcal{L}(\lambda)^\times) \to 0. \tag{3.8}$$

(For a proof, see [4].)

For a second useful refinement of our Thom-Gysin sequence, we note that restriction to the zero-section gives a T-equivariant homotopy equivalence between $\mathcal{L}(\lambda)$ and G/P_Λ. It follows that the associated restriction map $H_T^*(\mathcal{L}(\lambda)) \to H_T^*(G/P_\Lambda)$ is an isomorphism. Using this isomorphism, we shall replace $H_T^*(\mathcal{L}(\lambda))$ in (3.8) to obtain

$$0 \to H_T^{i-2}(G/P_\Lambda) \to H_T^i(G/P_\Lambda) \to H_T^i(\mathcal{L}(\lambda)^\times) \to 0.$$

The map $H_T^{i-2}(G/P_\Lambda) \to H_T^i(G/P_\Lambda)$ is multiplication by $c_1^T(\mathcal{L}(\lambda))$ (see [9], for instance). Furthermore, the map $H_T^i(G/P_\Lambda) \to H_T^i(\mathcal{L}(\lambda)^\times)$ is the map ψ^* on equivariant cohomology induced by the projection $\psi : \mathcal{L}(\lambda)^\times \to G/P_\Lambda$. (This follows from the fact that the bundle projection $\mathcal{L}(\lambda) \to G/P_\Lambda$ and zero-section $G/P_\Lambda \to \mathcal{L}(\lambda)$ give inverse maps on equivariant cohomology.)

We conclude that $\psi^* : H_T^*(G/P_\Lambda) \to H_T^*(\mathcal{O}_{\mathrm{min}})$ is a surjective graded algebra morphism. Its kernel is $\langle c_1^T(\mathcal{L}(\lambda))\rangle$, the ideal of $H_T^*(G/P_\Lambda)$ generated by the equivariant first Chern class $c_1^T(\mathcal{L}(\lambda)) \in H_T^2(G/P_\Lambda)$. In particular, there is a graded algebra isomorphism

$$H_T^*(\mathcal{O}_{\mathrm{min}}) \cong H_T^*(G/P_\Lambda)/\langle c_1^T(\mathcal{L}(\lambda))\rangle.$$

Using Lemma 11 and Theorem 10, we obtain the second conclusion of Theorem 7.

Remark. In [46], D. Juteau used a non-equivariant version of the Thom-Gysin sequence (3.6) to help compute the ordinary integral cohomology groups of $\mathcal{O}_{\mathrm{min}}$. However, there is an interesting difference between the equivariant and non-equivariant cases. Indeed, Juteau found that multiplication by the ordinary first Chern class $c_1(\mathcal{L}(\lambda)) \in$

$H^2(G/P_\Lambda;\mathbb{Z})$ gave rise to a non-injective map $H^{i-2}(G/P_\Lambda;\mathbb{Z}) \to H^i(G/P_\Lambda;\mathbb{Z})$ for some values of i. This is in contrast to the equivariant setup, as $c_1^T(\mathcal{L}(\lambda))$ is not a zero-divisor in $H_T^*(G/P_\Lambda)$.

3.3.5 An Example

Let us compute the equivariant cohomology of the minimal nilpotent orbit of $G = SL_2(\mathbb{C})$. Note that $\Delta = \{-2, 2\} \subseteq \mathbb{Z} \cong X^*(T)$ is the resulting collection of roots, and that 2 is the highest one. This highest root is not orthogonal to any of the simple roots, so that $\Lambda = \emptyset$ and $P_\Lambda = B$. The Weyl group W is $\mathbb{Z}/2\mathbb{Z}$, and the generator acts by negation on the weight lattice. The subgroup W_Λ is trivial. In particular, G/P_Λ has two T-fixed points.

Since λ is identified with $2x \in \mathbb{Q}[x] \cong H_T^*(pt)$, Theorem 2.46 implies that $H_T^*(G/P_\Lambda)$ includes into $H_T^*(pt)^{\oplus 2} \cong \mathbb{Q}[x]^{\oplus 2}$ as the subalgebra

$$H_T^*(G/P_\Lambda) \cong \{(f_1(x), f_2(x)) \in \mathbb{Q}[x]^{\oplus 2} : 2x | (f_1(x) - f_2(x))\}$$

$$= \{(f_1(x), f_2(x)) \in \mathbb{Q}[x]^{\oplus 2} : f_1(0) = f_2(0)\}.$$

Furthermore, Lemma 11 tells us that $c_1^T(\mathcal{L}(\lambda)) = (2x, -2x)$ when included into $\mathbb{Q}[x]^{\oplus 2}$. Hence,

$$H_T^*(\mathcal{O}_{min}) \cong \frac{\{(f_1(x), f_2(x)) \in \mathbb{Q}[x]^{\oplus 2} : f_1(0) = f_2(0)\}}{\langle (2x, -2x) \rangle}.$$

Note that this is generated as a $\mathbb{Q}$-algebra by $y := [(x, 0)]$. The relation is $y^2 = 0$, so that

$$H_T^*(\mathcal{O}_{min}) \cong \mathbb{Q}[y]/\langle y^2 \rangle,$$

with y an element of grading degree two.

We remark that this is consistent with our findings in Section 3.2. Indeed, if $G = SL_2(\mathbb{C})$, then $\mathcal{O}_{min} = \mathcal{O}_{reg}$. Hence, $H_T^*(\mathcal{O}_{min}) = H_T^*(\mathcal{O}_{reg})$, and the latter is isomorphic to the ordinary cohomology of $G/B \cong \mathbb{P}^1$.

Chapter 4

Hessenberg Varieties for the Minimal Nilpotent Orbit

The following chapter studies the equivariant geometry and topology of certain (Hessenberg) subvarieties G/B associated with $\mathcal{O}_{\text{min}}$. This study is based on joint work with Hiraku Abe, and the exposition largely adheres to our preprint [2]. While Hiraku and I were both fully participant in all aspects of this project, my principal contributions were to the results holding in general Lie type (as opposed to those holding only in Lie type A).

4.1 Introduction and Structure

Hessenberg varieties form a rich and diverse family of subvarieties of the flag variety, including the flag variety itself, the Peterson variety, and Springer fibres. They are of interest to researchers in algebraic geometry [15, 23, 45, 60, 72], combinatorics [29, 36, 41, 65], geometric representation theory [33, 69], and equivariant topology. With respect to the last of these areas, there has been a pronounced emphasis on equivariant cohomology computations for torus actions on Hessenberg varieties (see [3, 30, 40]).

This chapter studies a class of Hessenberg varieties arising from the minimal nilpotent orbit. To this end, recall that each highest root vector $e_\lambda \in \mathfrak{g}_\lambda \setminus \{0\}$ belongs to the minimal nilpotent orbit of G. Accordingly, for a Hessenberg subspace $H \subseteq \mathfrak{g}$, we may consider the Hessenberg variety $X_H(e_\lambda)$. This variety has received some attention in the literature as an example of a *highest weight Hessenberg variety* (see [73]).

As is the case with nilpotent Hessenberg varieties in general, $X_H(e_\lambda)$ is sometimes singular and reducible, and its geometry depends heavily on the choice of H. However, one distinguishing feature is that $X_H(e_\lambda)$ is a union of Schubert varieties. In particular, it is invariant under the full maximal torus T.

While we present a wide array of results on the geometry and topology of $X_H(e_\lambda)$, the following are our main results.

- There are explicit combinatorial procedures for determining the Poincaré polynomial and irreducible components of $X_H(e_\lambda)$ in Lie type A.

- The T-action renders $X_H(e_\lambda)$ a GKM variety. Its GKM graph is the full subgraph of the GKM graph of G/B with vertex set $\{w \in W : \mathfrak{g}_{w^{-1}\lambda} \subseteq H\}$.

- The restriction map $i_T^* : H_T^*(G/B) \to H_T^*(X_H(e_\lambda))$ is surjective. Its kernel is the $H_T^*(\text{pt})$-submodule of $H_T^*(G/B)$ freely generated by the equivariant opposite Schubert classes $\sigma_T^-(w)$ (see (2.35)) for all $w \in W$ with $\mathfrak{g}_{w^{-1}\lambda} \cap H = \{0\}$.

We also prove a similar statement for the ordinary cohomology ring $H^*(X_H(e_\lambda))$.

Let us address the structure of this chapter. We begin with Section 4.2.1, which introduces and motivates $X_H(e_\lambda)$ as an object of study. In 4.2.2, we use a common description of Hessenberg varieties in type A to provide an explicit example of $X_H(e_\lambda)$.

Section 4.3 seeks to introduce $X_H(e_\lambda)$ through the lens of equivariant geometry. Specifically, 4.3.1 shows $X_H(e_\lambda)$ to be T-invariant, and it gives a description of the T-fixed point set $X_H(e_\lambda)^T$. Using this description, 4.3.2 computes $|X_H(e_\lambda)^T|$, the Euler number of $X_H(e_\lambda)$. Also, 4.3.3 uses properties of $X_H(e_\lambda)^T$ to give an upper bound on the codimension of $X_H(e_\lambda)$ in G/B.

Section 4.4 exploits combinatorial descriptions of Hessenberg varieties in type A to investigate the geometry of $X_H(e_\lambda)$. Specifically, 4.4.1 computes the Poincaré polynomial of $X_H(e_\lambda)$ by means of the Hessenberg stair shape diagram (see Figure 4.1). Next, beginning with some partial results in type ADE, 4.4.2 and 4.4.3 introduce the modified Hessenberg stair shape to completely describe the irreducible components of $X_H(e_\lambda)$ in type A.

Section 4.5 studies $X_H(e_\lambda)$ via GKM theory. Specifically, 4.5.1 exhibits the GKM graph of $X_H(e_\lambda)$ as a full subgraph of the GKM graph of G/B. In 4.5.2, we explain how one would implement this result to draw the GKM graph of $X_H(e_\lambda)$ in type A. We then provide the GKM graphs of all five such Hessenberg varieties in type A_2.

Section 4.6 is devoted to the calculation of $H^*(X_H(e_\lambda))$ and $H_T^*(X_H(e_\lambda))$. Specifically, the restriction maps $H^*(G/B) \to H^*(X_H(e_\lambda))$ and $H_T^*(G/B) \to H_T^*(X_H(e_\lambda))$ are shown to be surjective with kernels generated by certain opposite Schubert classes.

4.2 Background

4.2.1 The Hessenberg Varieties of Interest

Suppose that $H \subseteq \mathfrak{g}$ is a *Hessenberg subspace*, namely a $\mathfrak{b}$-invariant subspace of $\mathfrak{g}$ containing $\mathfrak{b}$[1]. Note that

$$H = \mathfrak{t} \oplus \bigoplus_{\gamma \in \Delta_H} \mathfrak{g}_\gamma = \mathfrak{b} \oplus \bigoplus_{\gamma \in \Delta_H^-} \mathfrak{g}_\gamma. \tag{4.1}$$

We shall call the roots in Δ_H *Hessenberg roots*, while calling those in Δ_H^- *negative Hessenberg roots*.

Now, given $\xi \in \mathfrak{g}$, the subset

$$G_H(\xi) := \{g \in G : \mathrm{Ad}_{g^{-1}}(\xi) \in H\}$$

is invariant under the right-multiplicative action of B on G. We may therefore define

$$X_H(\xi) := G_H(\xi)/B.$$

This is a closed (hence projective) subvariety of G/B, called a *Hessenberg variety* (see [23]). If $\xi \in \mathfrak{g}$ is nilpotent, one calls $X_H(\xi)$ a *nilpotent Hessenberg variety*.

The following relationship between nilpotent orbits and nilpotent Hessenberg varieties will help to give context for the Hessenberg varieties studied in this chapter.

Lemma 12. *If ξ and η belong to the same nilpotent G-orbit, then $X_H(\xi)$ and $X_H(\eta)$ are isomorphic as varieties.*

Proof. By assumption $\eta = \mathrm{Ad}_g(\xi)$ for some $g \in G$. Note that left-multiplication by g defines an isomorphism from $G_H(\xi)$ to $G_H(\eta)$. This isomorphism is B-equivariant for the right-multiplicative action of B. Hence, the quotients $G_H(\xi)/B = X_H(\xi)$ and $G_H(\eta)/B = X_H(\eta)$ are isomorphic. □

Fix a non-zero vector in the highest root space, $e_\lambda \in \mathfrak{g}_\lambda \setminus \{0\}$, and consider the nilpotent Hessenberg variety $X_H(e_\lambda)$. Noting that e_λ belongs to the minimal nilpotent orbit $\mathcal{O}_{\min}$ of G, Lemma 12 implies that $X_H(\xi) \cong X_H(e_\lambda)$ for all $\xi \in \mathcal{O}_{\min}$. In this sense, $X_H(e_\lambda)$ is precisely the Hessenberg variety arising from the minimal nilpotent orbit.

Letting the Hessenberg subspace H vary, the $X_H(e_\lambda)$ constitute an interesting family of subvarieties of G/B. With respect to inclusion, the largest and smallest are $X_{\mathfrak{g}}(e_\lambda)$

[1]We emphasize that H need not be a parabolic subalgebra of $\mathfrak{g}$.

and $X_{\mathfrak{b}}(e_\lambda)$, respectively. The former is easily seen to be G/B itself, while the latter is the Springer fibre above e_λ. In particular, $X_H(e_\lambda)$ is sometimes singular and reducible.

To obtain additional examples, we will need to recall a concrete description of Hessenberg varieties in type A.

4.2.2 Examples in Type A

Suppose that $G = SL_n(\mathbb{C})$, and that T are B are the subgroups of diagonal and upper-triangular matrices in $SL_n(\mathbb{C})$, respectively. Recall that for distinct $i, j \in \{1, 2, \dots, n\}$, $t_i - t_j$ denote the root

$$\lambda : T \to \mathbb{C}^*, \quad \begin{bmatrix} t_1 & 0 & 0 & \dots & 0 \\ 0 & t_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & t_n \end{bmatrix} \mapsto t_i t_j^{-1}. \tag{4.2}$$

The highest root is then given by

$$\lambda := t_1 - t_n, \tag{4.3}$$

and

$$e_\lambda := \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

is a choice of highest root vector.

Now, suppose that $H \subseteq \mathfrak{sl}_n(\mathbb{C})$ is a Hessenberg subspace. There exists a unique weakly increasing function $h : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$ with $j \leq h(j)$ for all j, such that

$$H = \{[a_{ij}] \in \mathfrak{sl}_n(\mathbb{C}) : a_{ij} = 0 \text{ for } i > h(j)\}. \tag{4.4}$$

Noting that (4.4) defines a bijective correspondence between the Hessenberg subspaces H and all such functions h, one calls these functions *Hessenberg functions*. We will represent a Hessenberg function h by listing its values, so that $h = (h(1), h(2), \dots, h(n))$.

Now, recall the isomorphism (2.11) between $SL_2(\mathbb{C})/B$ and $\mathcal{F}(1, 2, \dots, n-1; \mathbb{C}^n)$. If

$h = (h(1), h(2), \ldots, h(n))$ is a Hessenberg function corresponding to $H \subseteq \mathfrak{sl}_n(\mathbb{C})$, then

$$X_H(e_\lambda) \cong \{V_\bullet \in \mathcal{F}(1, 2, \ldots, n-1; \mathbb{C}^n) : e_\lambda(V_j) \subseteq V_{h(j)} \text{ for all } j\} \tag{4.5}$$

via the isomorphism (2.11).

Let us use (4.5) to describe $X_H(e_\lambda)$ in the case where $n = 3$ and $h = (2, 3, 3)$. Indeed, we have

$$X_H(e_\lambda) \cong \{V_\bullet \in \mathcal{F}(1, 2; \mathbb{C}^3) : e_\lambda(V_1) \subseteq V_2\}.$$

Letting $\{e_1, e_2, e_3\}$ denote the standard basis of $\mathbb{C}^3$, it is straightforward to see that each $V_\bullet \in X_H(e_\lambda)$ must satisfy $V_1 \subseteq \text{span}\{e_1, e_2\}$ or $e_1 \in V_2$. Let X_1 and X_2 denote the subvarieties of $X_H(e_\lambda)$ defined by these respective conditions, so that $X_H(e_\lambda) = X_1 \cup X_2$. Note that completing $V_1 \subseteq \text{span}\{e_1, e_2\}$ to an element $V_\bullet \in X_H(e_\lambda)$ is equivalent to specifying a 2-dimensional subspace V_2 containing V_1. Also, completing a V_2 containing e_1 to $V_\bullet \in X_H(e_\lambda)$ amounts to specifying a 1-dimensional subspace V_1 contained in V_2. From these observations, we see that each of X_1 and X_2 is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. The intersection of these subvarieties is seen to be two copies of $\mathbb{P}^1$ which themselves intersect in a single point.

In Section 4.5.2, we will study the above-mentioned example as a GKM variety (see Figure 4.9).

4.3 The Equivariant Geometry of $X_H(e_\lambda)$

4.3.1 Algebraic Group Actions on $X_H(e_\lambda)$

In contrast to a general nilpotent Hessenberg variety, $X_H(e_\lambda)$ is a union of Schubert varieties. Equivalently, we have the following proposition.

Proposition 13. *The variety* $X_H(e_\lambda)$ *is invariant under the action of* B *on* G/B.

Proof. It suffices to prove that $G_H(e_\lambda)$ is invariant under left-multiplication by elements of B. To this end, suppose that $b \in B$ and $g \in G_H(e_\lambda)$. We have

$$\text{Ad}_{(bg)^{-1}}(e_\lambda) = \text{Ad}_{g^{-1}}(\text{Ad}_{b^{-1}}(e_\lambda)).$$

Since e_λ belongs to the highest root space, $\text{Ad}_{b^{-1}}(e_\lambda)$ is a scalar multiple of e_λ. Hence, $\text{Ad}_{g^{-1}}(\text{Ad}_{b^{-1}}(e_\lambda))$ is a scalar multiple of $\text{Ad}_{g^{-1}}(e_\lambda)$, and therefore in H. □

As a consequence of Proposition 13, $X_H(e_\lambda)$ carries an action of the maximal torus T. Properties of this T-action will play an essential role in proving the main results in this chapter. The first such property is a description of the T-fixed point set $X_H(e_\lambda)^T$. To this end, recall that the T-fixed points of G/B are enumerated by the bijection

$$W \xrightarrow{\cong} (G/B)^T \tag{4.6}$$

$$w \mapsto x(w) := [g],$$

where $g \in N_G(T)$ represents $w \in W$. Recalling the definition of the Hessenberg roots Δ_H (see (4.1)), we have the following description of $X_H(e_\lambda)^T$.

Proposition 14. *The T-fixed points of $X_H(e_\lambda)$ are given by*

$$X_H(e_\lambda)^T = \{x(w) : w \in W \text{ and } w^{-1}\lambda \in \Delta_H\}.$$

Proof. Suppose that $w \in W$ is represented by $g \in N_G(T)$, so that $x(w) = [g] \in G/B$. Then $x(w) \in X_H(e_\lambda)$ if and only if $g = hb$ for some $h \in G_H(e_\lambda)$ and $b \in B$. Since $G_H(e_\lambda)$ is invariant under right-multiplication by elements of B, this is equivalent to the condition that $g \in G_H(e_\lambda)$. Equivalently, $\mathrm{Ad}_{g^{-1}}(e_\lambda) \in H$, which is precisely the statement that $w^{-1}\lambda \in \Delta_H$. □

For example, suppose that $G = SL_n(\mathbb{C})$ and that $T \subseteq SL_n(\mathbb{C})$ and $B \subseteq SL_n(\mathbb{C})$ are the maximal torus and Borel subgroup considered in 4.2.2, respectively. Recall that $W = S_n$ and let $h : \{1, 2, \dots, n\} \to \{1, 2 \dots, n\}$ be a Hessenberg function corresponding to $H \subseteq \mathfrak{sl}_n(\mathbb{C})$. Since the highest root is as given in (4.3), Proposition 14 implies that $x(w) \in X_H(e_\lambda)^T$ if and only if the root space of $w^{-1}\lambda = t_{w^{-1}(1)} - t_{w^{-1}(n)}$ belongs to H. Noting that this root space is spanned by the matrix with entry 1 in position $(w^{-1}(1), w^{-1}(n))$ and all other entries 0, our characterization becomes

$$x(w) \in X_H(e_\lambda)^T \iff w^{-1}(1) \leq h(w^{-1}(n)). \tag{4.7}$$

4.3.2 The Euler Number of $X_H(e_\lambda)$

This section addresses the computation of $|X_H(e_\lambda)^T|$, the Euler number of $X_H(e_\lambda)$. More precisely, we give a general formula for $|X_H(e_\lambda)^T|$ and then specialize it to cases in which one can be more explicit.

To begin, Proposition 14 implies that

$$|X_H(e_\lambda)^T| = |\{w \in W : w^{-1}\lambda \in \Delta_H\}| = |\{w \in W : w\lambda \in \Delta_H\}|. \tag{4.8}$$

Since W preserves the set of long roots $\Delta_{\text{long}} \subseteq \Delta$, (4.8) becomes

$$|X_H(e_\lambda)^T| = |\{w \in W : w\lambda \in \Delta_{\text{long},H}\}|, \tag{4.9}$$

where $\Delta_{\text{long},H} := \{\alpha \in \Delta_{\text{long}} : \mathfrak{g}_\alpha \subseteq H\}$. Noting that W acts transitively on Δ_{long}, we have

$$|\{w \in W : w\lambda = \alpha\}| = |\{w \in W : w\lambda = \lambda\}| = \frac{|W|}{|\Delta_{\text{long}}|}$$

for all $\alpha \in \Delta_{\text{long},H}$. Hence, (4.9) becomes

Proposition 15.

$$|X_H(e_\lambda)^T| = |W|\frac{|\Delta_{\text{long},H}|}{|\Delta_{\text{long}}|}. \tag{4.10}$$

We now specialize (4.10) to some particularly tractable cases. Firstly, (4.10) is seen to imply that the Springer fibre $X_{\mathfrak{b}}(e_\lambda)$ contains exactly one-half of the T-fixed points in G/B.

Corollary 16. *The Euler number of our Springer fibre* $X_{\mathfrak{b}}(e_\lambda)$ *is given by*

$$|X_{\mathfrak{b}}(e_\lambda)^T| = \frac{|W|}{2}. \tag{4.11}$$

Proof. Since the number of positive long roots coincides with the number of negative long roots, we see that $|\Delta_{\text{long},\mathfrak{b}}| = \frac{1}{2}|\Delta_{\text{long}}|$. The formula (4.11) then follows from (4.10). □

Our second specialization of (4.10) is to the simply-laced case, in which $\Delta_{\text{long}} = \Delta$ and $\Delta_{\text{long},H} = \Delta_H$. Hence, $|\Delta_{\text{long}}| = \dim(\mathfrak{g}) - \text{rank}(\mathfrak{g})$ and $|\Delta_{\text{long},H}| = \dim(H) - \text{rank}(\mathfrak{g})$, so that (4.10) reads as

Corollary 17. *In the simply-laced case, we have*

$$|X_H(e_\lambda)^T| = |W|\left(\frac{\dim(H) - \text{rank}(\mathfrak{g})}{\dim(\mathfrak{g}) - \text{rank}(\mathfrak{g})}\right). \tag{4.12}$$

For example, if $G = SL_n(\mathbb{C})$, then

$$|X_H(e_\lambda)^T| = (n-2)!(\dim(H) - n + 1).$$

4.3.3 The Size of $X_H(e_\lambda)$

Let us write $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, and let $\Pi_i := \Pi - \{\alpha_i\}$ for $i \in \{1, 2, \ldots, n\}$. Denote by $P_i \subseteq G$ be the *maximal* parabolic subgroup corresponding to $\Pi_i \subseteq \Pi$ and recall that W_{Π_i} is the subgroup of W generated by the simple reflections s_{α_k} for $k \neq i$.

Lemma 18. *If $w \in W_{\Pi_i}$, then $w^{-1}\lambda \in \Delta_+$.*

Proof. Note that $\lambda = \sum_{k=1}^{n} m_k \alpha_k$ with $m_k > 0$ for $k \in \{1, 2, \ldots, n\}$. Also, recall that we have

$$s_{\alpha_\ell}(\alpha_k) = \alpha_k - \frac{2\langle \alpha_k, \alpha_\ell \rangle}{\langle \alpha_\ell, \alpha_\ell \rangle}\alpha_\ell. \tag{4.13}$$

If $w \in W_{\Pi_i}$ and we write $w^{-1}\lambda = \sum_{k=1}^{n} d_k \alpha_k$ for some $d_k \in \mathbb{Z}$, then (4.13) implies $d_i = m_i > 0$. Since $w^{-1}\lambda$ is a root, this shows it to be a positive root. □

Proposition 19. *For any maximal parabolic subgroup P_i, we have $P_i/B \subseteq X_H(e_\lambda) \subseteq G/B$.*

Proof. Lemma 18 shows that $\mathfrak{g}_{w^{-1}\lambda} \subseteq \mathfrak{b} \subseteq H$ for any $w \in W_{\Pi_i}$. This means that $x(w) \in X_H(e_\lambda)^T$ for all $w \in W_{\Pi_i}$. Since we have $P_i/B = \coprod_{w \in W_{\Pi_i}} Bx(w)$ and $X_H(e_\lambda) = \coprod_{w \in X_H(e_\lambda)^T} Bx(w)$, we obtain $P_i/B \subseteq X_H(e_\lambda) \subseteq G/B$. □

This proposition has interesting implications for the "size" of $X_H(e_\lambda)$ in G/B. Indeed, $X_H(e_\lambda)^T$ "large" by virtue of containing a copy of W_{Π_i} for all $i \in \{1, 2, \ldots, n\}$. Secondly, when $G = SL_n(\mathbb{C})$, a suitable choice of maximal parabolic P_i gives (by Proposition 19)

$$\mathcal{F}(1, 2, \ldots, n-2; \mathbb{C}^{n-1}) \subseteq X_H(e_\lambda) \subseteq \mathcal{F}(1, 2, \ldots, n-1; \mathbb{C}^n).$$

Hence, the complex codimension of $X_H(e_\lambda)$ in $\mathcal{F}(1, 2, \ldots, n-1; \mathbb{C}^n)$ is at most $n-1$ when $G = SL_n(\mathbb{C})$. Finally, in all Lie types, it is known that the codimension of $X_\mathfrak{b}(e_\lambda)$ in G/B is equal to half the dimension of the minimal nilpotent orbit $\mathcal{O}_{\min}$ ([67] cf. [17, Corollary 3.3.24]). Since $\dim_\mathbb{C}(\mathcal{O}_{\min}) = 2h^\vee - 2$ (see [75])[2], we have

$$\operatorname{codim}_\mathbb{C}(X_\mathfrak{b}(e_\lambda)) = h^\vee - 1.$$

For a general Hessenberg subspace H, the inclusion $X_\mathfrak{b}(e_\lambda) \subseteq X_H(e_\lambda)$ gives

$$\operatorname{codim}_\mathbb{C}(X_H(e_\lambda)) \leq h^\vee - 1.$$

[2]Here, $h^\vee$ is the *dual Coxeter number*.

4.4 Poincaré Polynomials and Irreducible Components

4.4.1 Poincaré Polynomials in Type A

We now compute the Poincaré polynomial $P_H(t)$ of $X_H(e_\lambda)$ when $G = SL_n(\mathbb{C})$. Accordingly, we shall assume all notation to be as in 4.2.2.

Consider an $n \times n$ grid of boxes, and let (i, j) denote the box in row i and column j. If $H \subseteq \mathfrak{sl}_n(\mathbb{C})$ is a Hessenberg subspace with Hessenberg function h, we shall call the stair-shaped sub-grid

$$\{(i,j) \in \{1,2,\dots,n\} \times \{1,2,\dots,n\}\} : i \leq h(j)\}$$

the *Hessenberg stair shape*. One identifies it by drawing a line in the $n \times n$ grid such that the sub-grid consists precisely of the boxes lying above the line.

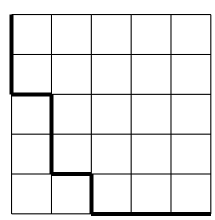

Figure 4.1: The Hessenberg stair shape determined by $h = (2, 4, 5, 5, 5)$ when $n = 5$

Definition 20. For $0 \leq i \leq 2n - 3$, we define $q_H(i)$ to be the number of boxes in the Hessenberg stair shape meeting the diagonal line segment joining $(2, n - i)$ and $(2 + i, n)$. Namely,

$$q_H(i) = |\{(k,j) \in \{1,2,\dots,n\} \times \{1,2,\dots,n\} : 2 \leq k \leq h(n+1-j),\ j + k - 3 = i\}|.$$

The number $q_H(i)$ is easily computed in practice. One starts with the rightmost box in row #2, moves i boxes to the left, and then draws the longest possible diagonal line segment passing through the current box and not passing through a box in row #1. The number of boxes meeting this line segment is precisely $q_H(i)$.

The following figure illustrates the computation of $q_H(2)$ in the case $n = 5$ and $h = (2, 4, 5, 5, 5)$.

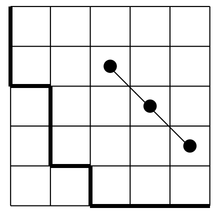

Figure 4.2: The computation of $q_H(2)$ when $h = (2, 4, 5, 5, 5)$ and $n = 5$. One begins by drawing the diagonal line segment connecting $(2, 3)$ and $(4, 5)$. Since this segment meets exactly 3 boxes, we have $q_H(2) = 3$.

It will be convenient to consider the polynomial

$$q_H(t) := \sum_{i=0}^{2n-3} q_H(i) t^{2i}.$$

As is the case with its coefficients, this polynomial can be computed diagrammatically via the Hessenberg stair shape. One simply fills each box involved in the computation of $q_H(i)$ with t^{2i}, and then sums the resulting terms. The following figure illustrates this procedure.

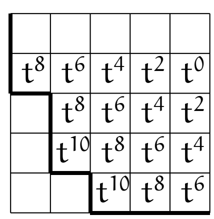

Figure 4.3: The computation of $q_H(t)$ for $n = 5$ and $h = (2, 4, 5, 5, 5)$. One has $q_H(t) = 1 + 2t^2 + 3t^4 + 4t^6 + 4t^8 + 2t^{10}$.

While $q_H(t)$ is not itself the Poincaré polynomial of $X_H(e_\lambda)$, we have the following proposition.

Proposition 21. *The Poincaré polynomial of* $X_H(e_\lambda)$ *is given by*

$$P_H(t) = q_H(t) \prod_{\ell=1}^{n-3} (1 + t^2 + \ldots + t^{2\ell}).$$

Proof. Since $X_H(e_\lambda)$ is a union of Schubert cells, the type A fixed point criterion (4.7) implies that

$$P_H(t) = \sum_{\substack{w \in S_n \\ w^{-1}(1) \le h(w^{-1}(n))}} t^{2\ell(w)}.$$

Writing $j = w^{-1}(n)$ and $k = w^{-1}(1)$, we have

$$P_H(t) = \sum_{j=1}^{n} \Big(\sum_{k=1}^{j-1} \sum_{v \in S_{n-2}} t^{2\ell(v)+2(k-1+n-j)} + \sum_{k=j+1}^{h(j)} \sum_{v \in S_{n-2}} t^{2\ell(v)+2(k-1+n-j-1)} \Big),$$

which can be explained as follows. In the one-line notation for w, if the position of 1 is to the left of the position of n (so $1 \le k \le j-1$), then 1 has $k-1$ inversion pairs and n has $n-j$ inversion pairs. If the position of 1 is to the right of the position of n (so $j+1 \le k \le h(j)$), then 1 has $k-1$ inversion pairs (including the pair $(n,1)$) and n has $n-j-1$ inversion pairs (except for the pair $(n,1)$, which is already counted).

Now, note that

$$\sum_{v \in S_{n-2}} t^{2\ell(v)} = \prod_{\ell=1}^{n-3} (1 + t^2 + \ldots + t^{2\ell}),$$

as each polynomial is the Poincaré polynomial of $\mathcal{F}(1, 2, \ldots, n-3\mathbb{C}^{n-2})$. Hence, a direct computation gives

$$\begin{aligned} P_H(t) &= \sum_{j=1}^{n} \sum_{k=2}^{h(j)} t^{2(k-2+n-j)} \cdot \prod_{\ell=1}^{n-3} (1 + t^2 + \ldots + t^{2\ell}) \\ &= \sum_{j=1}^{n} \sum_{k=2}^{h(n+1-j)} t^{2(j+k-3)} \cdot \prod_{\ell=1}^{n-3} (1 + t^2 + \ldots + t^{2\ell}), \end{aligned}$$

and the claim follows. □

Recall that for $n = 5$ and $h = (2, 4, 5, 5, 5)$ we have $q_H(t) = 1 + 2t^2 + 3t^4 + 4t^6 + 4t^8 + 2t^{10}$. In this case, Proposition 21 yields the Poincaré polynomial

$$\begin{aligned} P_H(t) &= (1 + 2t^2 + 3t^4 + 4t^6 + 4t^8 + 2t^{10}) \cdot (1 + t^2)(1 + t^2 + t^4) \\ &= 1 + 4t^2 + 9t^4 + 15t^6 + 20t^8 + 21t^{10} + 16t^{12} + 8t^{14} + 2t^{16}. \end{aligned}$$

Noting that $h = (1, 2, \ldots, n)$ corresponds to the Hessenberg subspace $\mathfrak{b}$, we obtain the following specialization of Proposition 21.

Corollary 22. *The Poincaré polynomial of our Springer fiber* $X_{\mathfrak{b}}(e_\lambda)$ *is given by*

$$P_{\mathfrak{b}}(t) = (1 + 2t^2 + 3t^4 + \ldots + (n-1)t^{2(n-2)}) \cdot \prod_{\ell=1}^{n-3}(1 + t^2 + \ldots + t^{2\ell}).$$

Additionally, Proposition 21 allows one to deduce the following combinatorial formula for $\dim_{\mathbb{C}}(X_H(e_\lambda))$.

Corollary 23. *The dimension of* $X_H(e_\lambda)$ *is given by*

$$\dim_{\mathbb{C}}(X_H(e_\lambda)) = \frac{1}{2}(n-1)(n-2) + \max\{h(j) - j \mid j = 1, \ldots, n\}.$$

4.4.2 Irreducible Components in Type ADE

We now examine the irreducible components of $X_H(e_\lambda)$. As one might expect, these are precisely the maximal Schubert varieties $X(w) = \overline{Bx(w)}$ contained in $X_H(e_\lambda)$.

Lemma 24. *The irreducible components of* $X_H(e_\lambda)$ *are the Schubert varieties* $X(w)$ *for the maximal* $w \in W$ *satisfying* $w^{-1}\lambda \in \Delta_H$.

Proof. By Proposition 1.5 of [42], our task is to prove the following two statements.

(i) The variety $X_H(e_\lambda)$ is a union of the $X(w)$ for the maximal $w \in W$ satisfying $w^{-1}\lambda \in \Delta_H$.

(ii) If $w_1 \neq w_2$ are two such maximal elements, then neither $X(w_1) \subseteq X(w_2)$ nor $X(w_2) \subseteq X(w_1)$ holds.

Since $X_H(e_\lambda)$ is a union of Schubert varieties (see Proposition 13), one for each T-fixed point in $X_H(e_\lambda)$, Proposition 14 allows us to write

$$X_H(e_\lambda) = \bigcup_{w^{-1}\lambda \in \Delta_H} X(w) \tag{4.14}$$

Furthermore, as $u \leq v$ if and only if $X(u) \subseteq X(v)$, (4.14) holds if the union is taken only over the maximal $w \in W$ satisfying $w^{-1}\lambda \in \Delta_H$. Hence, (i) is true. Of course, the fact $u \leq v \iff X(u) \subseteq X(v)$ also implies that (ii) is true. □

Let us assume G to be of type ADE. If $\beta \in \Delta$, then Lemma 4.4 of [73] allows one to consider the unique maximal $w_\beta \in W$ for which $w_\beta^{-1}\lambda = \beta$. In other words,

$$w_\beta := \max\{w \in W \mid w^{-1}\lambda = \beta\}. \tag{4.15}$$

Note that if $w^{-1}\lambda \in \Delta_H$, then $w \leq w_\beta$ for some $\beta \in \Delta_H$ (e.g. take $\beta = w^{-1}\lambda$). It follows that the maximal elements of $\{w \in W \mid w^{-1}\lambda \in \Delta_H\}$ (the set discussed in Lemma 24) are precisely the maximal elements of

$$\Omega_H := \{w_\beta : \beta \in \Delta_H\}. \tag{4.16}$$

Using Lemma 24, it follows that the maximal elements of Ω_H label the irreducible components of $X_H(e_\lambda)$. However, Ω_H still contains non-maximal elements, and the determination of its maximal elements will involve a few properties of the w_β. To this end, the following proposition is an immediate consequence of the discussion in Section 4 of [73].

Proposition 25. *Suppose that* $\beta, \gamma \in \Delta$.

(i) $\beta = \gamma \iff w_\beta = w_\gamma$

(ii) If β *and* γ *have the same sign, then* $\beta \leq \gamma \iff w_\gamma \leq w_\beta$.

Corollary 26. *(i) If* $\alpha, \beta \in \Pi$ *are distinct simple roots, then* w_α *and* w_β *are incomparable in the Bruhat order.*

(ii) If γ *and* δ *are distinct minimal elements of* Δ_H^-*, then* w_γ *and* w_δ *are incomparable in the Bruhat order.*

(iii) Suppose that $\gamma \in \Delta_H$*. If* w_γ *is a maximal element of* Ω_H*, then* $\gamma \in \Pi$ *or* γ *is a minimal element of* Δ_H^-*.*

Proof. Recognizing (i) and (ii) as immediate consequences of Proposition 25, we prove only (iii). To this end, if γ is positive, then there exists $\alpha \in \Pi$ such that $\alpha \leq \gamma$. Proposition 25 implies that $w_\gamma \leq w_\alpha$, and the maximality of w_γ then yields $w_\gamma = w_\alpha$. It follows that $\gamma = \alpha$ is simple.

Now, assume that γ is negative and let $\delta \in \Delta_H^-$ satisfy $\delta \leq \gamma$. Proposition 25 implies $w_\gamma \leq w_\delta$. Since w_γ is maximal, $w_\gamma = w_\delta$ and we conclude that $\gamma = \delta$. It follows that γ is a minimal element of Δ_H^-. □

In light of Corollary 26, the maximal elements of Ω_H are of the following two types:

1. w_α, where $\alpha \in \Pi$ and $w_\alpha \not< w_\gamma$ for all $\gamma \in \Delta_H^-$, [3]

[3] Strictly speaking, Corollary 26 gives the following different-looking description of the maximal w_α's: w_α, where $\alpha \in \Pi$ and $w_\alpha \not< w_\gamma$ for all minimal $\gamma \in \Delta_H^-$. However, by appealing to Proposition 25, one sees that this is equivalent to the description we have given.

2. w_γ, where γ is a minimal element of Δ_H^- and $w_\gamma \not< w_\alpha$ for all $\alpha \in \Pi$.

In order to refine (2), we will need the following two results.

Lemma 27. *If* $\alpha \in \Pi$, *then* $w_\alpha < w_{-\alpha}$.

Proof. Since $(w_\alpha s_\alpha)^{-1}\lambda = -\alpha$, the maximality of $w_{-\alpha}$ implies that

$$w_\alpha s_\alpha \leq w_{-\alpha}. \tag{4.17}$$

Furthermore, as $(w_\alpha s_\alpha)\alpha = -\lambda \in \Delta_-$, we have

$$w_\alpha < w_\alpha s_\alpha. \tag{4.18}$$

By combining (4.17) and (4.18), we obtain the desired result.

□

Corollary 28. *If* $\alpha \in \Pi$ *and* $\gamma \in \Delta_-$, *then* $w_\gamma \not< w_\alpha$.

Proof. If $w_\gamma < w_\alpha$, then Lemma 27 implies that $w_\gamma < w_{-\alpha}$. Proposition 25 then yields $-\alpha < \gamma$, which is impossible. □

In light of the above, we have the following improved description of the maximal elements of Ω_H:

1. w_α, where $\alpha \in \Pi$ and $w_\alpha \not< w_\gamma$ for all $\gamma \in \Delta_H^-$

2. w_γ, where γ is a minimal element of Δ_H^-.

Remembering that the maximal elements of Ω_H label the irreducible components of $X_H(e_\lambda)$, we have the following immediate corollary.

Corollary 29. *If* γ *is a minimal element of* Δ_H^-, *then* $X(w_\gamma)$ *is an irreducible component of* $X_H(e_\lambda)$.

The next section gives a combinatorial enumeration of the maximal elements of Ω_H (and therefore also the irreducible components of $X_H(e_\lambda)$) in Lie type A_{n-1}.

4.4.3 Complete Description of the Irreducible Components in Type A

Let $\beta = t_i - t_j \in \Delta$. Recall from (4.3), (4.7), and (4.15) that $w_\beta \in S_n$ is the longest permutation satisfying $w_\beta(i) = 1$ and $w_\beta(j) = n$. For a simple root $\alpha = t_{j-1} - t_j$, we have $w_\alpha(j-1) = 1$ and $w_\alpha(j) = n$, i.e. the one-line notation for w_α is

$$w_\alpha = \cdots\ 1\ n\ \cdots$$

where 1 is in the $(j-1)$-st position, n is in the j-th position, and the rest of the ordered sequence $w_\alpha(1), \ldots, w_\alpha(j-2), w_\alpha(j+1), \ldots, w_\alpha(n)$ is given by $n-1, n-2, \ldots, 3, 2$. For $\gamma = t_k - t_\ell$ $(k > \ell)$ a negative root, the one-line notation for w_γ is

$$w_\gamma = \cdots\ n\ \cdots\ 1\ \cdots,$$

where n is in the ℓ-th position and 1 is in the k-th position.

Continuing with our specialization to type A_{n-1}, we will need to introduce the *modified Hessenberg function* and the *modified Hessenberg stair shape*. To this end, let h be a Hessenberg function. We define a function $\overline{h} : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ by

$$\overline{h}(j) := \begin{cases} h(j) - 1 (= j-1) & \text{if } h(j-1) = j-1 \text{ and } h(j) = j, \\ h(j) & \text{otherwise,} \end{cases} \tag{4.19}$$

and we call $\overline{h}$ a *modified Hessenberg function*. Note that while $\overline{h}$ is a weakly increasing function, it might not be an honest Hessenberg function.

As with a Hessenberg function, one can consider the stair-shaped sub-grid

$$\{(i, j) \in \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}\} : i \le \overline{h}(j)\},$$

called the *modified Hessenberg stair shape* (see Figure 4.4).

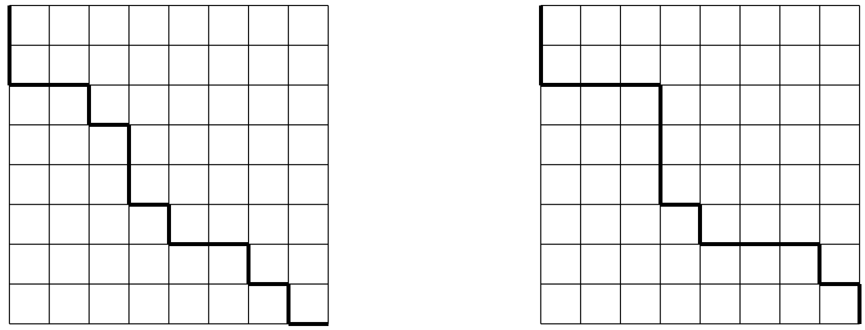

Figure 4.4: A Hessenberg stair shape and its modified Hessenberg stair shape

Lemma 30. *If* $t_k - t_\ell \in \Delta_-$*, then*

$$t_k - t_\ell \in \Delta_H^- \quad \textit{if and only if} \quad k \leq \overline{h}(\ell).$$

Proof. The condition $t_k - t_\ell \in \Delta_H^-$ is equivalent to $\mathfrak{g}_\alpha \subset H$, where $\alpha = t_k - t_\ell$. Also, the latter condition is equivalent to $k \leq h(\ell)$ via the correspondence (4.4) between Hessenberg functions and Hessenberg subspaces. Thus it suffices to show that for $k > \ell$, the condition $k \leq h(\ell)$ is equivalent to $k \leq \overline{h}(\ell)$.

If $k \leq h(\ell)$, then the assumption $k > \ell$ implies $\ell < k \leq h(\ell)$, and hence $h(\ell) \geq \ell + 1$. This implies $h(\ell) = \overline{h}(\ell)$, and hence we obtain $k \leq \overline{h}(\ell)$. If $k \leq \overline{h}(\ell)$, then the definition of $\overline{h}$ gives $\overline{h}(\ell) \leq h(\ell)$ and we obtain $k \leq h(\ell)$. □

Definition 31. For $i, j \in \{1, 2, \ldots, n\}$, we shall call (i, j) a *corner* of the modified Hessenberg stair shape if $i = \overline{h}(j)$ and $\overline{h}(j-1) < \overline{h}(j)$, with the convention $\overline{h}(0) := 0$.

Lemma 32. *If* (i, j) *is a corner of the modified Hessenberg stair shape, then* $\overline{h}(j) \neq j$*.*

Proof. If $\overline{h}(j) = j$, then in particular $\overline{h}(j) \neq j - 1$. The definition (4.19) then implies $h(j) = \overline{h}(j) = j$. Now, (4.19) and $\overline{h}(j) \neq j - 1$ also imply that the two conditions $h(j-1) = j-1$ and $h(j) = j$ cannot hold simultaneously. We therefore have $h(j-1) \neq j - 1$, which together with $h(j-1) \leq h(j) = j$ implies $h(j-1) = j$. So we have $\overline{h}(j-1) = j$ and $\overline{h}(j) = j$, and it follows that (i, j) cannot be a corner. □

Lemma 33. *For a simple root* $\alpha = t_{j-1} - t_j$ $(2 \leq j \leq n)$, $(j-1, j)$ *is a corner of the modified Hessenberg stair shape if and only if* $w_\alpha \not< w_\gamma$ *for all* $\gamma \in \Delta_H^-$*.*

Proof. To begin, assume that $(j-1, j)$ is a corner. By definition, we have $\overline{h}(j) = j - 1$. Suppose in addition that there exists $\gamma = t_k - t_\ell \in \Delta_H^-$ satisfying $w_\alpha < w_\gamma$. In principle, we have the following three cases.

(i) $j < \ell$:

$$w_\alpha = \cdots \ 1\ n\ \cdots\cdots\cdots\cdots,$$
$$w_\gamma = \cdots\cdots\cdots\ n\ \cdots\ 1\ \cdots.$$

(ii) $\ell \leq j$ and $j-1 \leq k$:

$$w_\alpha = \cdots\cdots\cdots\ 1\ n\ \cdots\cdots,$$
$$w_\gamma = \cdots\cdots\ n\ \cdots\cdots\ 1\ \cdots.$$

(iii) $k < j-1$:

$$w_\alpha = \cdots\cdots\cdots\cdots\ 1\ n\ \cdots,$$
$$w_\gamma = \cdots\ n\ \cdots\ 1\ \cdots\cdots\cdots.$$

Case (i) cannot occur, since transposing n to its right is a length-decreasing process. Similarly, Case (iii) cannot occur, since transposing 1 to its left is length-decreasing. Hence, we must have $\ell \leq j$ and $j-1 \leq k$. However, as $t_k - t_\ell$ is not positive, one of the first two inequalities is strict. Hence, by Lemma 30

$$j-1 \leq k \leq \overline{h}(\ell) \leq \overline{h}(j) = j-1.$$

However, since $(\overline{h}(j), j)$ is a corner, one of these inequalities must be strict. This is a contradiction, completing the first half of our proof.

We now prove the converse. Suppose that there is no $\gamma \in \Delta_H^-$ satisfying $w_\alpha < w_\gamma$. We claim that

$$h(j-2) = j-2, \quad h(j-1) = j-1, \quad \text{and} \quad h(j) = j,$$

with the convention $h(0) := 0$. The first of these can be proved as follows. Since the case of $j = 2$ is clear, we can assume $j \geq 3$. If $h(j-2) \geq j-1$, then $\gamma := t_{h(j-2)} - t_{j-2} = t_{\overline{h}(j-2)} - t_{j-2}$ is a negative Hessenberg root by Lemma 30, and $w_\alpha < w_\gamma$ since we are in Case (ii). So $h(j-2) = j-2$ follows. The same argument proves $h(j-1) = j-1$ and $h(j) = j$. Now from the definition of $\overline{h}$, we obtain

$$\overline{h}(j-1) = j-2 \quad \text{and} \quad \overline{h}(j) = j-1.$$

Hence $(j-1, j)$ is a corner of the modified Hessenberg stair shape. $\square$

Now, recall the definition of Ω_H from (4.16), as well as the description of the maximal elements of Ω_H given at the end of 4.4.2. With these considerations in mind, Lemma 33 may be restated in the following way: If $\alpha = t_{j-1} - t_j$ is a simple root, then w_α is a maximal element of Ω_H if and only if $(j-1, j)$ is a corner of the modified Hessenberg stair shape. This is consistent with the following more complete description of the maximal elements of Ω_H in type A. We remind the reader that a corner of the modified Hessenberg stair shape cannot lie on the diagonal (see Lemma 32).

Proposition 34. *If* $i, j \in \{1, 2, \ldots, n\}$ *with* $i \neq j$ *and* $\beta = t_i - t_j$, *then* w_β *is a maximal element of* Ω_H *if and only if* (i, j) *is a corner of the modified Hessenberg stair shape.*

Proof. To prove the backward implication, assume that (i, j) is a corner. We shall distinguish between the cases $\overline{h}(j) = j - 1$ and $\overline{h}(j) \neq j - 1$. In the former, (i, j) being a corner implies that $i = \overline{h}(j) = j - 1$ (so $j - 1 \geq 1$). In particular, $\beta = t_i - t_j = t_{j-1} - t_j$ is a simple root. Lemma 33 then implies that $w_\beta \not< w_\gamma$ for all $\gamma \in \Delta_H^-$. By the discussion at the end of 4.4.2, w_β is a maximal element of Ω_H.

For our second case, suppose that $\overline{h}(j) \neq j - 1$. Since (i, j) is a corner, Lemma 32 implies that $\overline{h}(j) > j$. Again, since (i, j) is a corner, $i = \overline{h}(j)$. In particular, $i > j$ and $t_i - t_j$ is a negative Hessenberg root. As (i, j) is a corner with $i > j$, an application of Lemma 30 establishes that $t_i - t_j$ is a minimal element of Δ_H^-. The discussion at the end of 4.4.2 then shows that w_β is a maximal element of Ω_H.

We now prove the forward implication. Firstly, assume that $\beta = t_i - t_j$ is simple (so $i = j - 1$). By Lemma 33, $(i, j) = (j - 1, j)$ is a corner of the modified Hessenberg stair shape.

Secondly, assume that $\beta = t_i - t_j$ is a minimal element of Δ_H^- (so $i > j \geq 1$). We have $\overline{h}(j) = i$, since $t_{\overline{h}(j)} - t_j$ would otherwise be a strictly less than $t_i - t_j$. A similar argument establishes that $\overline{h}(j - 1) \leq i - 1$ must also hold, so that (i, j) is a corner. □

As noted earlier, the irreducible components of $X_H(e_\lambda)$ correspond to the maximal elements of Ω_H. Noting that these maximal elements are described in Proposition 34, the following theorem gives the irreducible components of $X_H(e_\lambda)$ in Lie type A_{n-1}

Theorem 35. *In type* A_{n-1}, *there is a bijective correspondence between the set of corners of the modified Hessenberg stair shape and the set of irreducible components of* $X_H(e_\lambda)$ *given by*

$$(\overline{h}(j), j) \mapsto X(w_j) = \overline{Bx(w_j)},$$

where w_j *is the longest permutation satisfying* $w_j(\overline{h}(j)) = 1$ *and* $w_j(j) = n$.

Let us implement Theorem 35 in the context of a specific example. Indeed, recall that Figure 4.4 includes the modified Hessenberg stair shape determined by $h = (2, 2, 3, 5, 6, 6, 7, 8)$ when $n = 8$. The corners are $(2, 1)$, $(5, 4)$, $(6, 5)$, $(7, 8)$, as is indicated in the following diagram.

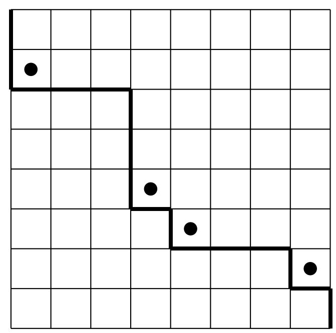

Figure 4.5: The modified Hessenberg stair shape for $h = (2, 2, 3, 5, 6, 6, 7, 8)$ with dots labeling corners

By Theorem 35, the irreducible components of $X_H(e_\lambda)$ are the Schubert varieties $X(w)$ for the following elements $w \in S_8$:

$$8\,1\,7\,6\,5\,4\,3\,2, \quad 7\,6\,5\,8\,1\,4\,3\,2, \quad 7\,6\,5\,4\,8\,1\,3\,2, \quad 7\,6\,5\,4\,3\,2\,1\,8$$

.

4.5 GKM Theory on $X_H(e_\lambda)$

We devote this section to the construction and examination of a GKM variety structure on $X_H(e_\lambda)$.

4.5.1 The GKM Graph of $X_H(e_\lambda)$

Since Proposition 13 shows $X_H(e_\lambda)$ to be a union of Schubert cells, this variety has trivial cohomology in odd grading degrees. It follows that $X_H(e_\lambda)$ is T-equivariantly formal (see [35]), and hence that the GKM structure on G/B restricts to such a structure on $X_H(e_\lambda)$. Accordingly, we will describe $H_T^*(X_H(e_\lambda))$ by exhibiting the GKM graph of $X_H(e_\lambda)$ as a subgraph of the GKM graph of G/B. Noting that the vertices of our subgraph have been determined by Proposition 14, we need only determine the edges. For this latter part, we will need to briefly discuss root strings.

If $\alpha, \beta \in \Delta$ are roots, one has the root string

$$S(\beta, \alpha) := (\Delta \cup \{0\}) \cap \{\beta + n\alpha : n \in \mathbb{Z}\}.$$

If p and q are maximal for the properties $\beta + p\alpha \in S(\beta, \alpha)$ and $\beta - q\alpha \in S(\beta, \alpha)$, respectively, then

$$S(\beta, \alpha) = \{\beta + n\alpha : -q \leq n \leq p\}$$

and

$$q - p = \frac{2(\beta, \alpha)}{(\alpha, \alpha)} \tag{4.20}$$

(see Proposition 2.29 of [49]). The relevance of root strings to our present work is captured by the following lemma.

Lemma 36. *If $w \in W$ and $\alpha \in \Delta_+$ are such that $x(ws_\alpha) \in X_H(e_\lambda)^T$, then*

$$\bigoplus_{\beta \in S(w^{-1}\lambda, \alpha)} \mathfrak{g}_\beta \subseteq H.^4$$

Proof. First note that either $\alpha \in w^{-1}\Delta_+$ or $-\alpha \in w^{-1}\Delta_+$. Since $S(w^{-1}\lambda, \alpha) = S(w^{-1}\lambda, -\alpha)$, we may assume that $\alpha \in w^{-1}\Delta_+$ (ie. that $w\alpha \in \Delta_+$). Noting that λ is the highest root, (4.20) implies that

$$S(\lambda, w\alpha) = \{\lambda - n(w\alpha), \lambda - (n-1)(w\alpha), \ldots, \lambda - w\alpha, \lambda\}$$

for $n = \frac{2(\lambda, w\alpha)}{(w\alpha, w\alpha)} = \frac{2(w^{-1}\lambda, \alpha)}{(\alpha, \alpha)}$. Hence, $S(w^{-1}\lambda, \alpha)$ is given by

$$S(w^{-1}\lambda, \alpha) = \{w^{-1}\lambda - n\alpha, w^{-1}\lambda - (n-1)\alpha, \ldots, w^{-1}\lambda - \alpha, w^{-1}\lambda\}.$$

The lowest root in this string is $w^{-1}\lambda - n\alpha = s_\alpha(w^{-1}\lambda)$. Also, applying Proposition 14 to the condition $x(ws_\alpha) \in X_H(e_\lambda)^T$ gives

$$\mathfrak{g}_{w^{-1}\lambda - n\alpha} = \mathfrak{g}_{s_\alpha(w^{-1}\lambda)} \subseteq H.$$

Since H is $\mathfrak{b}$-invariant, repeated bracketing with $\mathfrak{g}_\alpha \subseteq \mathfrak{b}$ establishes that the root space of each root in $S(w^{-1}\lambda, \alpha)$ lies in H. This completes the proof. □

Theorem 37. *The GKM graph of $X_H(e_\lambda)$ is a full subgraph of the GKM graph of G/B.*

Proof. Equivalently, we claim that if $w \in W$ and $\alpha \in \Delta_+$ are such that $x_w, x(ws_\alpha) \in$

[4] Here, it is understood that $\mathfrak{g}_0 = [\mathfrak{g}_{-\alpha}, \mathfrak{g}_\alpha]$.

$X_H(e_\lambda)^T$, then $Y_{w,\alpha} \subseteq X_H(e_\lambda)$ (where $Y_{w,\alpha}$ is the T-invariant copy of $\mathbb{P}^1$ defined in (2.40)). To this end, fix a representative $g \in N_G(T)$ of w, and let $N_{-\alpha}$ denote the connected closed subgroup of $SL_2(\mathbb{C})_\alpha$ with Lie algebra $\mathfrak{g}_{-\alpha}$. Note that

$$Y_{e,\alpha} = \overline{N_{-\alpha}x_e},$$

the closure of the $N_{-\alpha}$-orbit through x_e. Furthermore, as $Y_{w,\alpha} = gY_{e,\alpha}$ (see (2.40)), we have

$$Y_{w,\alpha} = gY_{e,\alpha} = \overline{(gN_{-\alpha})x_e}.$$

Since $X_H(e_\lambda)$ is a closed subvariety of G/B, proving that $(gN_{-\alpha})x_e \subseteq X_H(e_\lambda)$ will establish that $Y_{w,\alpha} \subseteq X_H(e_\lambda)$. To prove the former, it will suffice to establish that $gh \in G_H(e_\lambda)$ for all $h \in N_{-\alpha}$, namely $\mathrm{Ad}_{(gh)^{-1}}(e_\lambda) \in H$.

Suppose that $h \in N_{-\alpha}$ and note that

$$\mathrm{Ad}_{(gh)^{-1}}(e_\lambda) = \mathrm{Ad}_{h^{-1}}(\mathrm{Ad}_{g^{-1}}(e_\lambda)).$$

Writing $\mathrm{Ad}_{g^{-1}}(e_\lambda) = e_{w^{-1}\lambda} \in \mathfrak{g}_{w^{-1}\lambda}$ and $h = \exp(\xi)$ for $\xi \in \mathfrak{g}_{-\alpha}$, we obtain

$$\mathrm{Ad}_{(gh)^{-1}}(e_\lambda) = \mathrm{Ad}_{\exp(-\xi)}(e_{w^{-1}\lambda}) = e^{\mathrm{ad}_{-\xi}}(e_{w^{-1}\lambda}) = \sum_{k=0}^{\infty} \frac{1}{k!}(\mathrm{ad}_{-\xi})^k(e_{w^{-1}\lambda}) \tag{4.21}$$

Furthermore, if $(\mathrm{ad}_{-\xi})^k(e_{w^{-1}\lambda}) \neq 0$, then it belongs to a root space for a root in $S(w^{-1}\lambda, \alpha)$. Hence,

$$\mathrm{Ad}_{(gh)^{-1}}(e_\lambda) = \sum_{k=0}^{\infty} \frac{1}{k!}(\mathrm{ad}_{-\xi})^k(e_{w^{-1}\lambda}) \in \bigoplus_{\beta \in S(w^{-1}\lambda,\alpha)} \mathfrak{g}_\beta.$$

By Lemma 36, it follows that $\mathrm{Ad}_{(gh)^{-1}}(e_\lambda) \in H$. □

4.5.2 GKM Graphs of $X_H(e_\lambda)$ in Type A

By Theorem 37, finding the GKM graph of $X_H(e_\lambda)$ amounts to determining its T-fixed points. With this in mind, suppose that $G = SL_3(\mathbb{C})$ and that $T \subseteq SL_3(\mathbb{C})$ and $B \subseteq SL_3(\mathbb{C})$ are the maximal torus and Borel considered in 4.2.2, respectively. Recall that $W = S_3$ and let $h : \{1, 2, 3\} \to \{1, 2, 3\}$ be a Hessenberg function corresponding to $H \subseteq \mathfrak{sl}_3(\mathbb{C})$.

The possible Hessenberg functions are $(1, 2, 3)$, $(1, 3, 3)$, $(2, 2, 3)$, $(2, 3, 3)$, and $(3, 3, 3)$. By applying (4.7), one determines the T-fixed points for each corresponding variety $X_H(e_\lambda)$. Noting that Theorem 37 then determines the GKM graph of each variety as a subgraph of Figure 2.1, the following are the GKM graphs of all the $X_H(e_\lambda)$ in type A_2.

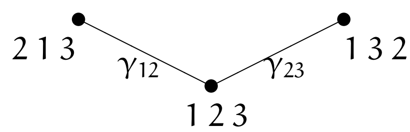

Figure 4.6: The GKM graph of $X_H(e_\lambda)$ for $h = (1, 2, 3)$

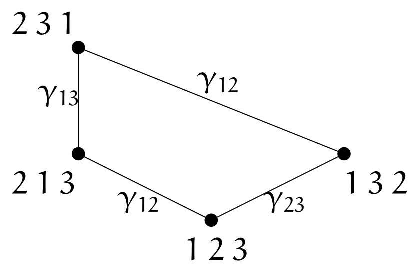

Figure 4.7: The GKM graph of $X_H(e_\lambda)$ for $h = (1, 3, 3)$

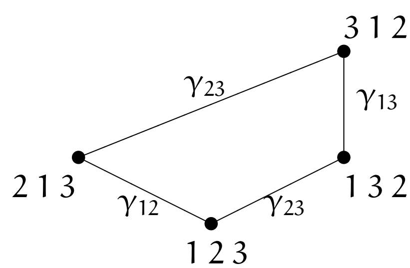

Figure 4.8: The GKM graph of $X_H(e_\lambda)$ for $h = (2, 2, 3)$

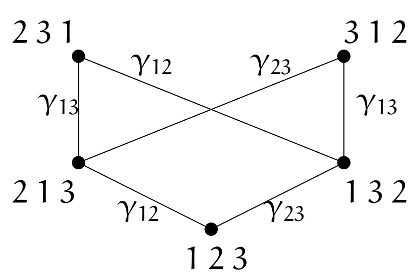

Figure 4.9: The GKM graph of $X_H(e_\lambda)$ for $h = (2, 3, 3)$

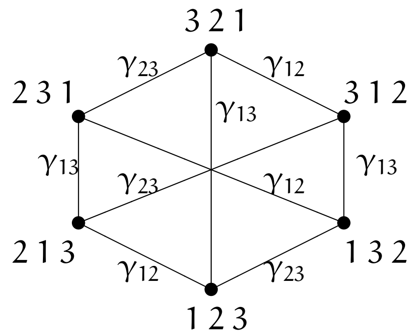

Figure 4.10: The GKM graph of $X_H(e_\lambda)$ for $h = (3, 3, 3)$

4.6 Cohomology Ring Presentations

In this section, we use the restriction maps

$$i^* : H^*(G/B) \to H^*(X_H(e_\lambda)) \text{ and } i_T^* : H_T^*(G/B) \to H_T^*(X_H(e_\lambda))$$

to explicitly present $H^*(X_H(e_\lambda))$ and $H_T^*(X_H(e_\lambda))$ as quotients of $H^*(G/B)$ and $H_T^*(G/B)$, respectively.

4.6.1 Ordinary Cohomology

We begin with the following proposition.

Proposition 38. *The restriction map* $i^* : H^*(G/B) \to H^*(X_H(e_\lambda))$ *is surjective.*

Proof. Since rational singular cohomology is the dual of rational singular homology, it suffices to show that the map $H_*(X_H(e_\lambda)) \to H_*(G/B)$ is injective. Now, consider the commutative diagram

$$\begin{array}{ccc} H_*(X_H(e_\lambda)) & \xrightarrow{\cong} & \overline{H}_*(X_H(e_\lambda)) \\ \downarrow & & \downarrow \\ H_*(G/B) & \xrightarrow{\cong} & \overline{H}_*(G/B) \end{array} , \tag{4.22}$$

where $\overline{H}_*$ denotes Borel-Moore homology (see [31]). The vertical maps in (4.22) are the maps induced by the inclusion $X_H(e_\lambda) \hookrightarrow G/B$, and the horizontal isomorphisms are the ones described in 6.10.14 of [68]. So what we need check is that the induced map $\overline{H}_*(X_H(e_\lambda)) \to \overline{H}_*(G/B)$ is injective. To this end, consider the subsets

$$(G/B)_p := \coprod_{w \in W,\ \ell(w) \le p} Bx(w)$$

and

$$(X_H(e_\lambda))_p := \coprod_{\substack{w \in W,\ \ell(w) \leq p \\ x(w) \in X_H(e_\lambda)^T}} Bx(w).$$

Since $X_H(e_\lambda)$ is a union of Schubert cells, we have the affine pavings

$$G/B = (G/B)_{\dim_{\mathbb{C}}(G/B)} \supseteq \ldots \supseteq (G/B)_1 \supseteq (G/B)_0 = \emptyset$$

and

$$X_H(e_\lambda) = (X_H(e_\lambda))_{\dim_{\mathbb{C}}(X_H(e_\lambda))} \supseteq \ldots \supseteq (X_H(e_\lambda))_1 \supseteq (X_H(e_\lambda))_0 = \emptyset.$$

Also, for each $p = 0, 1, \ldots, \dim_{\mathbb{C}}(G/B)$, we have a commutative diagram (c.f. [31])

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \overline{H}_*((X_H(e_\lambda))_{p-1}) & \longrightarrow & \overline{H}_*((X_H(e_\lambda))_p) & \longrightarrow & \bigoplus_{\substack{\ell(w)=p \\ x(w) \in X_H(e_\lambda)^T}} \overline{H}_*(Bx(w)) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \overline{H}_*((G/B)_{p-1}) & \longrightarrow & \overline{H}_*((G/B)_p) & \longrightarrow & \bigoplus_{\ell(w)=p} \overline{H}_*(Bx(w)) & \longrightarrow & 0. \end{array}$$

The left and the middle vertical maps are those induced by the inclusions, and each component of the right vertical map is the composition $\overline{H}_*(Bx(w)) \to \overline{H}_*(\coprod_{\ell(w)=p} Bx(w)) \to \overline{H}_*(Bx(w))$. It is straightforward to see that each component map is an isomorphism Hence, the right vertical map is an injection, and we see that the induced map $\overline{H}_*(X_H(e_\lambda)) \to \overline{H}_*(G/B)$ is injective by induction on p. This completes the proof. □

By Proposition 38, the map $i^* : H^*(G/B) \to H^*(X_H(e_\lambda))$ is surjective. We shall therefore address ourselves to computing its kernel. To this end, we will need the following proposition.

Proposition 39. *Suppose that $v, w \in W$ satisfy $v \geq w$. If $x(w) \notin X_H(e_\lambda)^T$, then $x(v) \notin X_H(e_\lambda)^T$.*

Proof. Suppose that $x(v) \in X_H(e_\lambda)^T$. Since $X_H(e_\lambda)$ is a closed B-invariant subvariety of G/B, it follows that $X(v) = \overline{Bx(v)} \subseteq X_H(e_\lambda)$. Noting that $v \geq w$, we must have $x(w) \in X(v)$. This contradicts our assumption that $x(w) \notin X_H(e_\lambda)^T$. □

Now, recall the definition of the opposite Schubert variety $X_-(w) := \overline{B_x(w)}$, $w \in W$, as well as the definition of the corresponding opposite Schubert class $\sigma_-(w) \in H^{2\ell(w)}(G/B)$.

Corollary 40. *The subset*

$$J_H := \bigoplus_{x(w)\notin X_H(e_\lambda)^T} \mathbb{Q}\sigma_-(w) \tag{4.23}$$

is an ideal of $H^*(G/B)$.

Proof. If $u \in W$ and $x(w) \notin X_H(e)^T$, then ordinary Schubert calculus gives

$$\sigma_-(u)\sigma_-(w) = \sum_{v\geq u,w} c^v_{uw}\sigma_-(v) \tag{4.24}$$

for some $c^v_{uw} \in \mathbb{Z}$. By Proposition 39, $x(v) \notin X_H(e_\lambda)^T$ for all v appearing in the sum (4.24). Hence, $\sigma_-(u)\sigma_-(w) \in J_H$, proving that J_H is an ideal. □

With these considerations in mind, we offer the following presentation of $H^*(X_H(e_\lambda))$.

Theorem 41. *The map* i^* *induces a graded* $\mathbb{Q}$*-algebra isomorphism*

$$H^*(G/B)/J_H \to H^*(X_H(e_\lambda)).$$

Proof. To begin, we claim that $i^*(\sigma_-(w)) = 0$ for $w \in W$ satisfying $x(w) \notin X_H(e_\lambda)^T$. This will follow from our establishing that

$$X_-(w) \cap X_H(e_\lambda) = \emptyset. \tag{4.25}$$

To this end, we have

$$X_-(w) = \coprod_{w\leq v} B_-x(v) \quad \text{and} \quad X_H(e_\lambda) = \coprod_{x(u)\in X_H(e_\lambda)^T} Bx(u), \tag{4.26}$$

with the latter decomposition being a consequence of Proposition 13. Now, recall that for $u, v \in W$, $B_-x(v) \cap Bx(u) \neq \emptyset$ if and only if $v \leq u$. So, if $X_-(w) \cap X_H(e_\lambda) \neq \emptyset$, then 4.26 implies $w \leq u$ for some $u \in W$ with $x(u) \in X_H(e_\lambda)^T$. Proposition 39 then gives $x(w) \in X_H(e_\lambda)^T$, which is a contradiction. We conclude that 4.25 holds, so that $i^*(\sigma_-(w)) = 0$ whenever $x(w) \notin X_H(e_\lambda)^T$.

In light of our findings, i^* induces a surjective graded $\mathbb{Q}$-algebra homomorphism

$$H^*(G/B)/J_H \to H^*(X_H(e_\lambda)). \tag{4.27}$$

To conclude that (4.27) is an isomorphism, it will suffice to prove that

$$\dim_{\mathbb{Q}}(H^*(G/B)/J_H) = \dim_{\mathbb{Q}}(H^*(X_H(e_\lambda))). \tag{4.28}$$

Noting that

$$H^*(G/B) = \bigoplus_{w \in W} \mathbb{Q}\sigma_-(w),$$

we have $\dim_{\mathbb{Q}}(H^*(G/B)/J_H) = |X_H(e_\lambda)^T|$. Also, the Schubert cell decomposition of $X_H(e_\lambda)$ gives $\dim_{\mathbb{Q}}(H^*(X_H(e_\lambda))) = |X_H(e_\lambda)^T|$. Hence, (4.28) is satisfied and the map $H^*(G/B)/J_H \to H^*(X_H(e_\lambda))$ is an isomorphism. □

For example, suppose that $G = SL_3(\mathbb{C})$ and that all notation is as presented in 4.2.2. We will use Theorem 41 to obtain a presentation of $H^*(X_{\mathfrak{b}}(e_\lambda))$. To this end, let $\mathcal{F}(1,2;\mathbb{C}^3) \times \mathbb{C}^3$ be the trivial vector bundle over $\mathcal{F}(1,2;\mathbb{C}^3)$, and set

$$E_i := \{(V_\bullet, v) \in \mathcal{F}(1,2;\mathbb{C}^3) \times \mathbb{C}^3 \mid v \in V_i\}$$

for $i \in \{1,2,3\}$. Note that E_i is a complex vector bundle over $\mathcal{F}(1,2;\mathbb{C}^3)$. Each quotient $L_i := E_i/E_{i-1}$ is a complex line bundle, allowing us to consider its first Chern class

$$c_1(L_i) \in H^*(\mathcal{F}(1,2;\mathbb{C}^3)).$$

Now, recall that the algebra morphism

$$\mathbb{Q}[x_1,x_2,x_3] \to H^*(\mathcal{F}(1,2;\mathbb{C}^3)), \quad x_i \mapsto c_1(L_i),\ i \in \{1,2,3\}$$

is surjective. Recall also that its kernel is the ideal generated by $e_1(x)$, $e_2(x)$, and $e_3(x)$, where $e_i(x)$ denotes the i-th elementary symmetric polynomial in the variables x_1, x_2, x_3. In particular, we have an algebra isomorphism

$$H^*(\mathcal{F}(1,2;\mathbb{C}^3)) \xrightarrow{\cong} \mathbb{Q}[x_1,x_2,x_3]/(e_1(x), e_2(x), e_3(x)). \tag{4.29}$$

The ideal $J_{\mathfrak{b}} \subseteq H^*(\mathcal{F}(1,2;\mathbb{C}^3))$ is seen to be generated by the opposite Schubert classes $\sigma_-(2\ 3\ 1), \sigma_-(3\ 1\ 2), \sigma_-(3\ 2\ 1) \in H^*(\mathcal{F}(1,2;\mathbb{C}^3))$. Their images under the isomorphism (4.29) are

$$\sigma_-(2\ 3\ 1) = x_1x_2, \quad \sigma_-(3\ 1\ 2) = x_1x_1, \quad \sigma_-(3\ 2\ 1) = x_1x_1x_2,$$

where (by an abuse of notation) x_i is also used to denote its image in the quotient algebra $\mathbb{Q}[x_1, x_2, x_3]/(e_1(x), e_2(x), e_3(x))$. Applying Theorem 41, we obtain

$$H^*(X_{\mathfrak{b}}(e_\lambda)) \cong H^*(G/B)/J_{\mathfrak{b}} \cong \frac{\mathbb{Q}[x_1, x_2, x_3]/(e_1(x), e_2(x), e_3(x))}{\mathbb{Q}x_1x_2 \oplus \mathbb{Q}x_1x_1 \oplus \mathbb{Q}x_1x_1x_2}.$$

A straightforward manipulation of the rightmost ring then yields

$$H^*(X_{\mathfrak{b}}(e_\lambda)) \cong \mathbb{Q}[x_1, x_2, x_3]/(e_1(x), e_2(x), e_3(x), x_1x_2, x_1x_3, x_2x_3),$$

which is exactly Tanisaki's presentation of $H^*(X_{\mathfrak{b}}(e_\lambda))$ (see [71]).

4.6.2 Equivariant Cohomology

As one might expect, we have the following equivariant counterpart of Proposition 38.

Proposition 42. *The restriction map* $i_T^* : H_T^*(G/B) \to H_T^*(X_H(e_\lambda))$ *is surjective.*

Proof. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_T^{>0}(pt) & \longrightarrow & H_T^*(G/B) & \longrightarrow & H^*(G/B) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle i_T^*} & & \downarrow{\scriptstyle i^*} & & \\ 0 & \longrightarrow & H_T^{>0}(pt) & \longrightarrow & H_T^*(X_H(e_\lambda)) & \longrightarrow & H^*(X_H(e_\lambda)) & \longrightarrow & 0, \end{array}$$

where $H_T^{>0}(pt)$ is the ideal of $H_T^*(pt)$ generated by the elements of positive degree (see 2.5.2). The surjectivity of i_T^* then follows from that of i^*. □

Proceeding in analogy with 4.6.1, we now compute the kernel of i_T^*. To this end, recall definition (2.35) of the equivariant opposite Schubert class $\sigma_T^-(w)$. We will need the following well-known description of the image of $\sigma_T^-(w)$ under the restriction map

$$i_w^* : H_T^*(G/B) \to H_T^*(\{x(w)\}) = H_T^*(pt). \tag{4.30}$$

Lemma 43. *If* $w \in W$, *then*

$$i_w^*(\sigma_T^-(w)) = \prod_{\alpha \in \Delta_+ \cap w\Delta_-} \alpha. \tag{4.31}$$

Proof. Since $x(w)$ is a smooth point of $X_-(w)$, $i_w^*(\sigma_T^-(w))$ is precisely the T-equivariant Euler class of the T-representation

$$T_{x(w)}(G/B)/T_{x(w)}(X_-(w)). \tag{4.32}$$

It is therefore equal to the product of the weights occurring in (4.32), which we now determine. To this end, as wBw^{-1} is the G-stabilizer of $x(w)$ and has Lie algebra $w\mathfrak{b}$, we have isomorphisms

$$T_{x(w)}(G/B) \cong \mathfrak{g}/w\mathfrak{b} \cong \bigoplus_{\alpha \in w\Delta_-} \mathfrak{g}_\alpha \tag{4.33}$$

of T-representations. Also, the B_--stabilizer of $x(w)$ is $B_- \cap wBw^{-1}$ and has Lie algebra $\mathfrak{b}_- \cap w\mathfrak{b}$. We therefore have

$$T_{x(w)}(X_-(w)) = T_{x(w)}(B_-wB/B) \cong \mathfrak{b}_-/(\mathfrak{b}_- \cap w\mathfrak{b}) \cong \bigoplus_{\alpha \in \Delta_- \cap w\Delta_-} \mathfrak{g}_\alpha. \tag{4.34}$$

Combining (4.33) and (4.34), one finds that

$$T_{x(w)}(G/B)/T_{x(w)}(X_-(w)) \cong \bigoplus_{\alpha \in \Delta_+ \cap w\Delta_-} \mathfrak{g}_\alpha$$

as T-representations. This completes the proof. □

These equivariant opposite Schubert classes are seen to form an $H_T^*(\text{pt})$-module basis of $H_T^*(G/B)$. With this in mind, the following corollary introduces an important $H_T^*(\text{pt})$-submodule of $H_T^*(G/B)$.

Corollary 44. *The submodule*

$$J_H^T := \bigoplus_{x(w) \notin X_H(e_\lambda)^T} H_T^*(\text{pt})\sigma_T^-(w) \tag{4.35}$$

is an ideal of $H_T^*(G/B)$.

Proof. The argument is similar to that used in the proof of Proposition 40, provided one uses the well-known fact that

$$\sigma_T^-(u)\sigma_T^-(w) = \sum_{v \geq u,w} c_{uw}^v \sigma_T^-(v) \tag{4.36}$$

for $c_{uw}^v \in H_T^*(\text{pt})$. For the reader's convenience, we briefly recount the proof of this fact. To this end, let $v \in W$ be a minimal element with the property that

$$i_v^*(\sigma_T^-(u)\sigma_T^-(w)) \neq 0. \tag{4.37}$$

Note that $vs_\alpha < v$ for all $\alpha \in \Delta_+ \cap v^{-1}\Delta_-$, so that

$$i^*_{vs_\alpha}(\sigma^-_T(u)\sigma^-_T(w)) = 0, \quad \alpha \in \Delta_+ \cap v^{-1}\Delta_-.$$

The GKM conditions defining the image of $H^*_T(G/B) \to H^*_T((G/B)^T)$ (see (2.46)) then give

$$(v\alpha) \mid i^*_v(\sigma^-_T(u)\sigma^-_T(w)), \quad \alpha \in \Delta_+ \cap v^{-1}\Delta_-.$$

Hence, the product

$$\prod_{\alpha \in \Delta_+ \cap v^{-1}\Delta_-} v\alpha \tag{4.38}$$

also divides $i^*_v(\sigma^-_T(u)\sigma^-_T(w))$. Using Lemma 43, one finds that (4.38) coincides with $(-1)^{l(v)} i^*_v(\sigma^-_T(v))$. In particular, $i^*_v(\sigma^-_T(v))$ divides $i^*_v(\sigma^-_T(u)\sigma^-_T(w))$, meaning that

$$i^*_v(\sigma^-_T(u)\sigma^-_T(w) - c^v_{uw}\sigma^-_T(v)) = 0 \tag{4.39}$$

for some $c^v_{uw} \in H^*_T(pt)$.

Continuing the support-reducing process by induction, one eventually obtains a class with no support in the GKM graph. In other words, there exist coefficients $c^v_{uw} \in H^*_T(pt)$ for all $v \geq u, w$ such that

$$\sigma^-_T(u)\sigma^-_T(w) - \sum_{v \geq u,w} c^v_{uw}\sigma^-_T(v)$$

has zero image under the localization map $H^*_T(G/B) \to H^*_T((G/B)^T)$. Since the localization map is injective, we conclude that

$$\sigma^-_T(u)\sigma^-_T(w) = \sum_{v \geq u,w} c^v_{uw}\sigma^-_T(v). \tag{4.40}$$

□

Theorem 45. *The map* $i^*_T : H^*_T(G/B) \to H^*_T(X_H(e_\lambda))$ *induces a graded* $H^*_T(pt)$*-algebra isomorphism*

$$H^*_T(G/B)/J^T_H \to H^*_T(X_H(e_\lambda)).$$

Proof. In the proof of Theorem 41, we showed that $i^*_T(\sigma^-_T(w)) = 0$ for $x(w) \notin X_H(e_\lambda)^T$. Therefore, i^*_T induces a surjective map

$$H^*_T(G/B)/J^T_H \to H^*_T(X_H(e_\lambda)). \tag{4.41}$$

Now, from the definition of J_H^T, it is clear that

$$H_T^*(G/B)/J_H^T \cong \bigoplus_{x(w)\in X_H(e_\lambda)^T} H_T^*(pt)\sigma_T^-(w)$$

as $H_T^*(pt)$-modules. In particular, $H_T^*(G/B)/J_H^T$ is free of rank $|X_H(e_\lambda)^T|$. However, as $X_H(e_\lambda)$ is T-equivariantly formal, $H_T^*(X_H(e_\lambda))$ is also free of rank $|X_H(e_\lambda)^T|$. It follows that (4.41) is actually an isomorphism. $\square$

Chapter 5

Nilpotent Orbit Complexification

The following chapter introduces and studies *nilpotent orbit complexification*, a mechanism for relating real and complex nilpotent orbits. The content is largely based on the preprint [19].

Please note that we will deviate significantly from the conventions and notation in Chapter 2.

5.1 Introduction and Structure

Real nilpotent orbits have been studied in a variety of contexts, including differential geometry, symplectic geometry, and Hodge theory (see [63]). Attention has also been given to the interplay between real and complex nilpotent orbits, with the Kostant-Sekiguchi Correspondence (see [64]) being perhaps the most famous instance. Accordingly, we provide additional points of comparison between real and complex nilpotent orbits. Specifically, let $\mathfrak{g}$ be a finite-dimensional semisimple real Lie algebra with complexification $\mathfrak{g}_{\mathbb{C}}$. Each real nilpotent orbit $\mathcal{O} \subseteq \mathfrak{g}$ lies in a unique complex nilpotent orbit $\mathcal{O}_{\mathbb{C}} \subseteq \mathfrak{g}_{\mathbb{C}}$, the complexification of $\mathcal{O}$. The following is our main result.

Theorem 46. *The process of nilpotent orbit complexification has the following properties.*

(i) Every complex nilpotent orbit is realizable as the complexification of a real nilpotent orbit if and only if $\mathfrak{g}$ is quasi-split and has no simple summand of the form $\mathfrak{so}(2n+1, 2n-1)$.

(ii) If $\mathfrak{g}$ is quasi-split, then a complex nilpotent orbit $\Theta \subseteq \mathfrak{g}_{\mathbb{C}}$ is realizable as the complexification of a real nilpotent orbit if and only if Θ is invariant under conjugation with respect to the real form $\mathfrak{g} \subseteq \mathfrak{g}_{\mathbb{C}}$.

(iii) If $\mathcal{O}_1, \mathcal{O}_2 \subseteq \mathfrak{g}$ are real nilpotent orbits satisfying $(\mathcal{O}_1)_{\mathbb{C}} = (\mathcal{O}_2)_{\mathbb{C}}$, then either $\mathcal{O}_1 = \mathcal{O}_2$ or these two orbits are incomparable in the closure order.

We begin with an overview of nilpotent orbits in semisimple real and complex Lie algebras. In recognition of Theorem 46 (iii), and of the role played by the unique maximal complex nilpotent orbit $\Theta_{\text{reg}}(\mathfrak{g}_{\mathbb{C}})$ throughout the article, Section 5.2.2 reviews the closure orders on the sets of real and complex nilpotent orbits. In Section 5.2.3, we recall some of the details underlying the use of decorated partitions to index nilpotent orbits.

Section 5.3 is devoted to the proof of Theorem 46. In Section 5.3.1, we represent nilpotent orbit complexification as a poset map $\varphi_{\mathfrak{g}}$ between the collections of real and complex nilpotent orbits. Next, we show this map to have a convenient description in terms of decorated partitions. Section 5.3.2 then directly addresses the proof of Theorem 46 (i), formulated as a characterization of when $\varphi_{\mathfrak{g}}$ is surjective. Using Proposition 50, we reduce this exercise to one of characterizing surjectivity for $\mathfrak{g}$ simple. Together with the observation that surjectivity implies $\mathfrak{g}$ is quasi-split and is implied by $\mathfrak{g}$ being split, Proposition 50 allows us to complete the proof of Theorem 46 (i).

We proceed to Section 5.3.3, which provides the proof of Theorem 46 (ii). The essential ingredient is Kottwitz's work [54]. We also include Proposition 55, which gives an interesting sufficient condition for a complex nilpotent orbit to be in the image of $\varphi_{\mathfrak{g}}$.

In Section 5.3.4, we give a proof of Theorem 46 (iii). Our proof makes extensive use of the Kostant-Sekiguchi Correspondence, the relevant parts of which are mentioned.

5.2 Background

5.2.1 Real and Complex Nilpotent Orbits

We begin by fixing some of the objects that will persist throughout this chapter. Let $\mathfrak{g}$ be a finite-dimensional semisimple real Lie algebra with adjoint group G. Also, let $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of $\mathfrak{g}$, whose adjoint group is the complexification $G_{\mathbb{C}}$. One has the adjoint representations

$$\text{Ad} : G \to \text{GL}(\mathfrak{g}) \text{ and } \text{Ad}_{\mathbb{C}} : G_{\mathbb{C}} \to \text{GL}(\mathfrak{g}_{\mathbb{C}})$$

of G and $G_{\mathbb{C}}$, respectively. Differentiation then gives the adjoint representations of $\mathfrak{g}$ and $\mathfrak{g}_{\mathbb{C}}$, namely

$$\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) \text{ and } \mathrm{ad}_{\mathbb{C}} : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{gl}(\mathfrak{g}_{\mathbb{C}}).$$

Recall that an element $\xi \in \mathfrak{g}$ (resp. $\xi \in \mathfrak{g}_{\mathbb{C}}$) is called nilpotent if $\mathrm{ad}(\xi) : \mathfrak{g} \to \mathfrak{g}$ (resp. $\mathrm{ad}_{\mathbb{C}}(\xi) : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$) is a nilpotent vector space endomorphism. The nilpotent cone $\mathcal{N}(\mathfrak{g})$ (resp. $\mathcal{N}(\mathfrak{g}_{\mathbb{C}})$) is then the subvariety of nilpotent elements of $\mathfrak{g}$ (resp. $\mathfrak{g}_{\mathbb{C}}$). A *real* (resp. *complex*) *nilpotent orbit* is an orbit of a nilpotent element in $\mathfrak{g}$ (resp. $\mathfrak{g}_{\mathbb{C}}$) under the adjoint representation of G (resp. $G_{\mathbb{C}}$). Since the adjoint representation occurs by means of Lie algebra automorphisms, a real (resp. complex) nilpotent orbit is equivalently defined to be a G-orbit (resp. $G_{\mathbb{C}}$-orbit) in $\mathcal{N}(\mathfrak{g})$ (resp. $\mathcal{N}(\mathfrak{g}_{\mathbb{C}})$). By virtue of being an orbit of a smooth G-action, each real nilpotent orbit is an immersed submanifold of $\mathfrak{g}$. However, as $G_{\mathbb{C}}$ is a complex linear algebraic group, a complex nilpotent orbit is a smooth locally closed complex subvariety of $\mathfrak{g}_{\mathbb{C}}$.

5.2.2 The Closure Orders

The sets $\mathcal{N}(\mathfrak{g})/G$ and $\mathcal{N}(\mathfrak{g}_{\mathbb{C}})/G_{\mathbb{C}}$ of real and complex nilpotent orbits are finite and carry the so-called closure order. In both cases, this is a partial order defined by

$$\mathcal{O}_1 \leq \mathcal{O}_2 \text{ if and only if } \mathcal{O}_1 \subseteq \overline{\mathcal{O}_2}. \tag{5.1}$$

In the real case, one takes closures in the classical topology on $\mathfrak{g}$. For the complex case, note that a complex nilpotent orbit Θ is a constructible subset of $\mathfrak{g}_{\mathbb{C}}$, so that its Zariski and classical closures agree. Accordingly, $\overline{\Theta}$ shall denote this common closure.

Example 1. Suppose that $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}_n(\mathbb{C})$, whose adjoint group is $G_{\mathbb{C}} = \mathrm{PSL}_n(\mathbb{C})$. The nilpotent elements of $\mathfrak{sl}_n(\mathbb{C})$ are precisely the nilpotent $n \times n$ matrices, so that the nilpotent $\mathrm{PSL}_n(\mathbb{C})$-orbits are exactly the ($\mathrm{GL}_n(\mathbb{C})$-) conjugacy classes of nilpotent matrices. The latter are indexed by the partitions of n via Jordan canonical forms. Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of n, let Θ_λ be the $\mathrm{PSL}_n(\mathbb{C})$-orbit of the nilpotent matrix with Jordan blocks of sizes $\lambda_1, \lambda_2, \ldots, \lambda_k$, read from top-to-bottom. It is a classical result of Gerstenhaber [34] that $\Theta_\lambda \leq \Theta_\mu$ if and only if $\lambda \leq \mu$ in the dominance order. (See [70] for a precise definition of this order.) □

The poset $\mathcal{N}(\mathfrak{g}_{\mathbb{C}})/G_{\mathbb{C}}$ has a unique maximal element $\Theta_{\mathrm{reg}}(\mathfrak{g}_{\mathbb{C}})$, called the regular nilpotent orbit. It is the collection of all elements of $\mathfrak{g}_{\mathbb{C}}$ which are simultaneously regular and nilpotent. In the framework of Example 1, $\Theta_{\mathrm{reg}}(\mathfrak{sl}_n(\mathbb{C}))$ corresponds to the partition (n).

5.2.3 Partitions of Nilpotent Orbits

Generalizing Example 1, it is often natural to associate a partition to each real and complex nilpotent orbit. One sometimes endows these partitions with certain decorations and then uses decorated partitions to enumerate nilpotent orbits. It will be advantageous for us to recall the construction of the underlying (undecorated) partitions. Our exposition will be largely based on Chapters 5 and 9 of [18].

Suppose that $\mathfrak{g}$ comes equipped with a faithful representation $\mathfrak{g} \subseteq \mathfrak{gl}(V) = \mathrm{End}_{\mathbb{F}}(V)$, where V is a finite-dimensional vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.[1] The choice of V determines an assignment of partitions to nilpotent orbits in both $\mathfrak{g}$ and $\mathfrak{g}_{\mathbb{C}}$. To this end, fix a real nilpotent orbit $\mathcal{O} \subseteq \mathcal{N}(\mathfrak{g})$ and choose a point $\xi \in \mathcal{O}$. We may include ξ as the nilpositive element of an $\mathfrak{sl}_2(\mathbb{R})$-triple (ξ, h, η), so that

$$[\xi, \eta] = h, \ [h, \xi] = 2\xi, \ [h, \eta] = -2\eta.$$

Regarding V as an $\mathfrak{sl}_2(\mathbb{R})$-module, one has a decomposition into irreducibles,

$$V = \bigoplus_{j=1}^{k} V_{\lambda_j},$$

where V_{λ_j} denotes the irreducible λ_j-dimensional representation of $\mathfrak{sl}_2(\mathbb{R})$ over $\mathbb{F}$. Let us require that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$, so that $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a partition of $\dim_{\mathbb{F}}(V)$. Accordingly, we define the partition of $\mathcal{O}$ to be

$$\lambda(\mathcal{O}) := (\lambda_1, \lambda_2, \ldots, \lambda_k).$$

It can be established that $\lambda(\mathcal{O})$ depends only on $\mathcal{O}$.

The faithful representation V of $\mathfrak{g}$ canonically gives a faithful representation $\tilde{V}$ of $\mathfrak{g}_{\mathbb{C}}$. Indeed, if V is over $\mathbb{C}$, then one has an inclusion $\mathfrak{g}_{\mathbb{C}} \subseteq \mathfrak{gl}(V)$ (so $\tilde{V} = V$). If V is over $\mathbb{R}$, then the inclusion $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ complexifies to give a faithful representation $\mathfrak{g}_{\mathbb{C}} \subseteq \mathfrak{gl}(V_{\mathbb{C}})$ (ie. $\tilde{V} = V_{\mathbb{C}}$). In either case, one proceeds in analogy with the real nilpotent case, using the faithful representation to yield a partition $\lambda(\Theta)$ of a complex nilpotent orbit $\Theta \subseteq \mathcal{N}(\mathfrak{g}_{\mathbb{C}})$. The only notable difference with the real case is that $\mathfrak{sl}_2(\mathbb{R})$ is replaced with $\mathfrak{sl}_2(\mathbb{C})$.

Example 2. One can use the framework developed above to index the nilpotent orbits in $\mathfrak{sl}_n(\mathbb{C})$ using the partitions of n. This coincides with the indexing given in Example

[1]Since $\mathfrak{g}$ is semisimple, the adjoint representation is a canonical choice of faithful V. Nevertheless, it will be advantageous to allow for different choices.

1. □

Example 3. The nilpotent orbits in $\mathfrak{sl}_n(\mathbb{R})$ are indexed by the partitions of n, after one replaces certain partitions with decorated counterparts. Indeed, if λ is a partition of n having only even parts, we replace λ with the decorated partitions λ_+ and λ_-. Otherwise, we leave λ undecorated. □

Example 4. Suppose that $n \geq 3$ and consider $\mathfrak{g} = \mathfrak{su}(p, q)$ with $1 \leq q \leq p$ and $p+q = n$. This Lie algebra is a real form of $\mathfrak{sl}_n(\mathbb{C})$. Now, let us regard a partition of n as a Young diagram with n boxes. Furthermore, recall that a signed Young diagram is a Young diagram whose boxes are marked with $+$ or $-$, such that the signs alternate across each row (for more details, see Chapter 9 of [18]). We restrict our attention to the signed Young diagrams of signature (p, q), namely those for which $+$ and $-$ appear with respective multiplicities p and q. It turns out that the nilpotent orbits in $\mathfrak{su}(p, q)$ are indexed by the signed Young diagrams of signature (p, q). □

Example 5. Suppose that $\mathfrak{g}_{\mathbb{C}} = \mathfrak{so}_{2n}(\mathbb{C})$ with $n \geq 4$. Taking our faithful representation to be $\mathbb{C}^{2n}$, nilpotent orbits in $\mathfrak{so}_{2n}(\mathbb{C})$ are assigned partitions of $2n$. The partitions realized in this way are those in which each even part appears with even multiplicity. One extends these partitions to an indexing set by replacing each λ having only even parts with the decorated partitions λ_+ and λ_-. □

Example 6. Suppose that $n \geq 3$ and consider $\mathfrak{g} = \mathfrak{so}(p, q)$ with $1 \leq q \leq p$ and $p + q = n$. Note that $\mathfrak{so}(p, q)$ is a real form of $\mathfrak{g}_{\mathbb{C}} = \mathfrak{so}_n(\mathbb{C})$. As with Example 4, we will identify partitions of n with Young diagrams having n boxes. We begin with the signed Young diagrams of signature (p, q) such that each even-length row appears with even multiplicity and has its leftmost box marked with $+$. To obtain an indexing set for the nilpotent orbits in $\mathfrak{so}(p, q)$, we decorate two classes of these signed Young diagrams Y. Accordingly, if Y has only even-length rows, then remove Y and add the four decorated diagrams $Y_{+,+}$, $Y_{+,-}$, $Y_{-,+}$, and $Y_{-,-}$. Secondly, suppose that Y has at least one odd-length row, and that each such row has an even number of boxes marked $+$, or that each such row has an even number of boxes marked $-$. In this case, we remove Y and add the decorated diagrams Y_+ and Y_-. □

5.3 Nilpotent Orbit Complexification

5.3.1 The Complexification Map

There is a natural way in which a real nilpotent orbit determines a complex one. Indeed, the inclusion $\mathcal{N}(\mathfrak{g}) \subseteq \mathcal{N}(\mathfrak{g}_{\mathbb{C}})$ gives rise to a map

$$\varphi_{\mathfrak{g}} : \mathcal{N}(\mathfrak{g})/G \to \mathcal{N}(\mathfrak{g}_{\mathbb{C}})/G_{\mathbb{C}}$$

$$\mathcal{O} \mapsto \mathcal{O}_{\mathbb{C}}.$$

Concretely, $\mathcal{O}_{\mathbb{C}}$ is just the unique complex nilpotent orbit containing $\mathcal{O}$, and we shall call it the *complexification* of $\mathcal{O}$. Let us then call $\varphi_{\mathfrak{g}}$ the *complexification map* for $\mathfrak{g}$.

It will be prudent to note that the process of nilpotent orbit complexification is well-behaved with respect to taking partitions. More explicitly, we have the following proposition.

Proposition 47. *Suppose that $\mathfrak{g}$ is endowed with a faithful representation $\mathfrak{g} \subseteq \mathfrak{gl}(V)$. If $\mathcal{O}$ is a real nilpotent orbit, then $\lambda(\mathcal{O}_{\mathbb{C}}) = \lambda(\mathcal{O})$.*

Proof. Choose a point $\xi \in \mathcal{O}$ and include it in an $\mathfrak{sl}_2(\mathbb{R})$-triple (ξ, h, η) as in Section 5.2.3. Note that (ξ, h, η) is then additionally an $\mathfrak{sl}_2(\mathbb{C})$-triple in $\mathfrak{g}_{\mathbb{C}}$. Hence, we will prove that the faithful representation $\tilde{V}$ of $\mathfrak{g}_{\mathbb{C}}$ decomposes into irreducible $\mathfrak{sl}_2(\mathbb{C})$-representations according to the partition $\lambda(\mathcal{O})$.

Let us write $\lambda(\mathcal{O}) = (\lambda_1, \dots, \lambda_k)$, so that

$$V = \bigoplus_{j=1}^{k} V_{\lambda_j} \tag{5.2}$$

is the decomposition of V into irreducible $\mathfrak{sl}_2(\mathbb{R})$-representations. If V is over $\mathbb{C}$, then $\tilde{V} = V$ and (5.2) is a decomposition of $\tilde{V}$ into irreducible $\mathfrak{sl}_2(\mathbb{C})$-representations. If V is over $\mathbb{R}$, then $\tilde{V} = V_{\mathbb{C}}$ and

$$V_{\mathbb{C}} = \bigoplus_{j=1}^{k} (V_{\lambda_j})_{\mathbb{C}}$$

is the decomposition of $\tilde{V}$ into irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$. In each of these two cases, we have $\lambda(\mathcal{O}_{\mathbb{C}}) = \lambda(\mathcal{O})$. □

Proposition 47 allows us to describe $\varphi_{\mathfrak{g}}$ in more combinatorial terms. To this end, fix a faithful representation $\mathfrak{g} \subseteq \mathfrak{gl}(V)$. As in Examples 2–6, we obtain index sets $I(\mathfrak{g})$

and $I(\mathfrak{g}_{\mathbb{C}})$ of decorated partitions for the real and complex nilpotent orbits, respectively. We may therefore regard $\varphi_{\mathfrak{g}}$ as a map

$$\varphi_{\mathfrak{g}} : I(\mathfrak{g}) \to I(\mathfrak{g}_{\mathbb{C}}).$$

Now, let $P(\mathfrak{g}_{\mathbb{C}})$ be the set of all partitions of the form $\lambda(\Theta)$, with $\Theta \subseteq \mathfrak{g}_{\mathbb{C}}$ a complex nilpotent orbit. One has the map

$$I(\mathfrak{g}_{\mathbb{C}}) \to P(\mathfrak{g}_{\mathbb{C}}),$$

sending a decorated partition to its underlying partition. Proposition 47 is then the statement that the composite map

$$I(\mathfrak{g}) \xrightarrow{\varphi_{\mathfrak{g}}} I(\mathfrak{g}_{\mathbb{C}}) \to P(\mathfrak{g}_{\mathbb{C}})$$

sends an index in $I(\mathfrak{g})$ to its underlying partition. Let us denote this composite map by $\psi_{\mathfrak{g}} : I(\mathfrak{g}) \to P(\mathfrak{g}_{\mathbb{C}})$.

We will later give a characterization of those semisimple real Lie algebras $\mathfrak{g}$ for which $\varphi_{\mathfrak{g}}$ is surjective. To help motivate this, we investigate the matter of surjectivity in some concrete examples.

Example 7. Recall the parametrizations of nilpotent orbits in $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{R})$ and $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}_n(\mathbb{C})$ outlined in Examples 3 and 2, respectively. We see that $I(\mathfrak{g}_{\mathbb{C}}) = P(\mathfrak{g}_{\mathbb{C}})$ and $\varphi_{\mathfrak{g}} = \psi_{\mathfrak{g}}$. The surjectivity of $\varphi_{\mathfrak{g}}$ then follows immediately from that of $\psi_{\mathfrak{g}}$. □

Example 8. Let the nilpotent orbits in $\mathfrak{g} = \mathfrak{su}(n, n)$ be parametrized as in Example 4. We then have $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}_{2n}(\mathbb{C})$, whose nilpotent orbits are indexed by the partitions of $2n$. Given such a partition λ, let Y denote the corresponding Young diagram. Since Y has an even number of boxes, it has an even number, $2k$, of odd-length rows. Label the leftmost box in k of these rows with $+$, and label the leftmost box in each of the remaining k rows with $-$. Now, complete this labelling to obtain a signed Young diagram $\tilde{Y}$, noting that $\tilde{Y}$ then has signature (n, n). Hence, $\tilde{Y}$ corresponds to a nilpotent orbit in $\mathfrak{su}(n, n)$ and $\psi_{\mathfrak{g}}(\tilde{Y}) = \lambda$. It follows that $\psi_{\mathfrak{g}}$ is surjective. Since $I(\mathfrak{g}_{\mathbb{C}}) = P(\mathfrak{g}_{\mathbb{C}})$ and $\varphi_{\mathfrak{g}} = \psi_{\mathfrak{g}}$, we have shown $\varphi_{\mathfrak{g}}$ to be surjective. A similar argument establishes surjectivity when $\mathfrak{g} = \mathfrak{su}(n + 1, n)$. □

Example 9. Let us consider $\mathfrak{g} = \mathfrak{so}(2n + 2, 2n)$, with nilpotent orbits indexed as in Example 6. Noting Example 5, a partition λ of $4n + 2$ represents a nilpotent orbit in $\mathfrak{g}_{\mathbb{C}} = \mathfrak{so}_{4n+2}(\mathbb{C})$ if and only if each even part of λ occurs with even multiplicity. Since $4n+2$ is even and not divisible by 4, it follows that any such λ has exactly $2k$ odd parts for some $k \geq 1$. Let Y be the Young diagram corresponding to λ, and label the leftmost box in $k-1$ of the odd-length rows with +. Next, label the leftmost box in each of $k-1$ different odd-length rows with $-$. Finally, use $+$ to label the leftmost box in each of the two remaining odd-length rows. Let $\tilde{Y}$ be any completion of our labelling to a signed Young diagram, such that the leftmost box in each even-length row is marked with $+$. Note that $\tilde{Y}$ has signature $(2n+2, 2n)$. It follows that $\tilde{Y}$ represents a nilpotent orbit in $\mathfrak{so}(2n+2, 2n)$ and $\psi_{\mathfrak{g}}(\tilde{Y}) = \lambda$. Furthermore, $I(\mathfrak{g}_{\mathbb{C}}) = P(\mathfrak{g}_{\mathbb{C}})$ and $\varphi_{\mathfrak{g}} = \psi_{\mathfrak{g}}$, so that $\varphi_{\mathfrak{g}}$ is surjective. □

Example 10. Suppose that $\mathfrak{g} = \mathfrak{so}(2n+1, 2n-1)$, whose nilpotent orbits are parametrized in Example 6. Let the nilpotent orbits in $\mathfrak{g}_{\mathbb{C}} = \mathfrak{so}_{4n}(\mathbb{C})$ be indexed as in Example 5. There exist partitions of $4n$ having only even parts, with each part appearing an even number of times. Let λ be one such partition, which by Example 6 represents a nilpotent orbit in $\mathfrak{so}_{4n}(\mathbb{C})$. Note that every signed Young diagram with underlying partition λ must have signature $(2n, 2n)$. In particular, λ cannot be realized as the image under $\psi_{\mathfrak{g}}$ of a signed Young diagram indexing a nilpotent orbit in $\mathfrak{so}(2n+1, 2n-1)$. It follows that $\psi_{\mathfrak{g}}$ and $\varphi_{\mathfrak{g}}$ are not surjective. □

5.3.2 Surjectivity

We now address the matter of classifying those semisimple real Lie algebras $\mathfrak{g}$ for which $\varphi_{\mathfrak{g}}$ is surjective. To proceed, we will require some additional machinery. Let $\mathfrak{p} \subseteq \mathfrak{g}$ be the (-1)-eigenspace of a Cartan involution, and let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Also, let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$, and choose a fundamental Weyl chamber $C \subseteq \mathfrak{h}$. Given a complex nilpotent orbit $\Theta \subseteq \mathfrak{g}_{\mathbb{C}}$, there exists an $\mathfrak{sl}_2(\mathbb{C})$-triple (ξ, h, η) in $\mathfrak{g}_{\mathbb{C}}$ with the property that $\xi \in \Theta$ and $h \in C$. The element $h \in C$ is uniquely determined by this property, and is called the characteristic of Θ. Theorem 1 of [26] then states that $\Theta \cap \mathfrak{g} \neq \emptyset$ if and only if $h \in \mathfrak{a}$. If $\mathfrak{g}$ is split, then $\mathfrak{a} = \mathfrak{h}$, and the following lemma is immediate.

Lemma 48. *If $\mathfrak{g}$ is split, then $\varphi_{\mathfrak{g}}$ is surjective.*

Let us now consider necessary conditions for surjectivity. To this end, recall that $\mathfrak{g}$ is called *quasi-split* if there exists a subalgebra $\mathfrak{b} \subseteq \mathfrak{g}$ such that $\mathfrak{b}_{\mathbb{C}}$ is a Borel subalgebra of $\mathfrak{g}_{\mathbb{C}}$. However, the following characterization of being quasi-split will be more suitable for our purposes.

Lemma 49. *The Lie algebra $\mathfrak{g}$ is quasi-split if and only if $\Theta_{\text{reg}}(\mathfrak{g}_{\mathbb{C}})$ is in the image of $\varphi_{\mathfrak{g}}$. In particular, $\mathfrak{g}$ being quasi-split is a necessary condition for $\varphi_{\mathfrak{g}}$ to be surjective.*

Proof. Proposition 5.1 of [61] states that $\mathfrak{g}$ is quasi-split if and only if $\mathfrak{g}$ contains a regular nilpotent element of $\mathfrak{g}_{\mathbb{C}}$. Since $\Theta_{\text{reg}}(\mathfrak{g}_{\mathbb{C}})$ consists of all such elements, this is equivalent to having $\Theta_{\text{reg}}(\mathfrak{g}_{\mathbb{C}})\cap\mathfrak{g} \neq \emptyset$ hold. This latter condition holds precisely when $\Theta_{\text{reg}}(\mathfrak{g}_{\mathbb{C}})$ is in the image of $\varphi_{\mathfrak{g}}$. □

Lemmas 48 and 49 establish that $\varphi_{\mathfrak{g}}$ being surjective is a weaker condition than having $\mathfrak{g}$ be split, but stronger than having $\mathfrak{g}$ be quasi-split. Furthermore, since $\mathfrak{su}(n,n)$ is not a split real form of $\mathfrak{sl}_{2n}(\mathbb{C})$, Example 8 establishes that surjectivity is strictly weaker than $\mathfrak{g}$ being split. Yet, as $\mathfrak{so}(2n+1,2n-1)$ is a quasi-split real form of $\mathfrak{so}_{4n}(\mathbb{C})$, Example 10 demonstrates that surjectivity is strictly stronger than having $\mathfrak{g}$ be quasi-split. To obtain a more precise measure of the strength of the surjectivity condition, we will require the following proposition.

Proposition 50. *Suppose that $\mathfrak{g}$ decomposes as a Lie algebra into*

$$\mathfrak{g} = \bigoplus_{j=1}^{k} \mathfrak{g}_j,$$

where $\mathfrak{g}_1,\dots,\mathfrak{g}_k$ are simple real Lie algebras. Let $G_1,\dots,G_k$ denote the respective adjoint groups.

(i) The map $\varphi_{\mathfrak{g}} : \mathcal{N}(\mathfrak{g})/G \to \mathcal{N}(\mathfrak{g}_{\mathbb{C}})/G_{\mathbb{C}}$ is surjective if and only each orbit complexification map $\varphi_{\mathfrak{g}_j} : \mathcal{N}(\mathfrak{g}_j)/G_j \to \mathcal{N}((\mathfrak{g}_j)_{\mathbb{C}})/(G_j)_{\mathbb{C}}$ is surjective.

(ii) The Lie algebra $\mathfrak{g}$ is quasi-split if and only if each summand $\mathfrak{g}_j$ is quasi-split.

Proof. For each $j \in \{1,\dots,k\}$, let $\pi_j : \mathfrak{g} \to \mathfrak{g}_j$ be the projection map. Note that $\xi \in \mathfrak{g}$ is nilpotent if and only if $\pi_j(\xi)$ is nilpotent in $\mathfrak{g}_j$ for each j. It follows that

$$\pi : \mathcal{N}(\mathfrak{g}) \to \prod_{j=1}^{k} \mathcal{N}(\mathfrak{g}_j)$$

$$\xi \mapsto (\pi_j(\xi))_{j=1}^{k}$$

defines an isomorphism of real varieties. Note that $G = \prod_{j=1}^{k} G_j$, with the former group acting on $\mathcal{N}(\mathfrak{g})$ and the latter group acting on the product of nilpotent cones.

One then sees that π is G-equivariant, so that it descends to a bijection

$$\overline{\pi} : \mathcal{N}(\mathfrak{g})/G \to \prod_{j=1}^{k} \mathcal{N}(\mathfrak{g}_j)/G_j.$$

Analogous considerations give a second bijection

$$\overline{\pi} : \mathcal{N}(\mathfrak{g}_{\mathbb{C}})/G_{\mathbb{C}} \to \prod_{j=1}^{k} \mathcal{N}((\mathfrak{g}_j)_{\mathbb{C}})/(G_j)_{\mathbb{C}}.$$

Furthermore, we have the commutative diagram

$$\begin{array}{ccc} \mathcal{N}(\mathfrak{g})/G & \xrightarrow{\overline{\pi}} & \prod_{j=1}^{k} \mathcal{N}(\mathfrak{g}_j)/G_j \\ \downarrow{\scriptstyle \varphi_{\mathfrak{g}}} & & \downarrow{\scriptstyle \prod_{j=1}^{k} \varphi_{\mathfrak{g}_j}} \\ \mathcal{N}(\mathfrak{g}_{\mathbb{C}})/G_{\mathbb{C}} & \xrightarrow{\overline{\pi}_{\mathbb{C}}} & \prod_{j=1}^{k} \mathcal{N}((\mathfrak{g}_j)_{\mathbb{C}})/(G_j)_{\mathbb{C}} \end{array} \quad . \tag{5.3}$$

Hence, $\varphi_{\mathfrak{g}}$ is surjective if and only if $\prod_{j=1}^{k} \varphi_{\mathfrak{g}_j}$ is so, proving (i).

By Lemma 49, proving (ii) will be equivalent to proving that $\Theta_{\text{reg}}(\mathfrak{g}_{\mathbb{C}})$ is in the image of $\varphi_{\mathfrak{g}}$ if and only if $\Theta_{\text{reg}}(\mathfrak{g}_j)$ is in the image of $\varphi_{\mathfrak{g}_j}$ for all j. Using the diagram (5.3), this will follow from our proving that the image of $\Theta_{\text{reg}}(\mathfrak{g}_{\mathbb{C}})$ under $\overline{\pi}$ is the k-tuple of the regular nilpotent orbits in the $(\mathfrak{g}_j)_{\mathbb{C}}$, namely that

$$\overline{\pi}(\Theta_{\text{reg}}(\mathfrak{g}_{\mathbb{C}})) = (\Theta_{\text{reg}}(((\mathfrak{g}_j)_{\mathbb{C}})))_{j=1}^{k}. \tag{5.4}$$

To see this, note that $\prod_{j=1}^{k} \Theta_{\text{reg}}(((\mathfrak{g}_j)_{\mathbb{C}}))$ is the $G_{\mathbb{C}} = \prod_{j=1}^{k}(G_j)_{\mathbb{C}}$-orbit of maximal dimension in $\prod_{j=1}^{k} \mathcal{N}(((\mathfrak{g}_{\mathbb{C}})_j))$. This orbit is therefore the image of $\Theta_{\text{reg}}(\mathfrak{g}_{\mathbb{C}})$ under the $G_{\mathbb{C}}$-equivariant variety isomorphism $\mathcal{N}(\mathfrak{g}_{\mathbb{C}}) \cong \prod_{j=1}^{k} \mathcal{N}(((\mathfrak{g}_j)_{\mathbb{C}}))$, implying that (5.4) holds. □

In light of Proposition 50, we address ourselves to classifying the simple real Lie algebras $\mathfrak{g}$ with surjective orbit complexification maps $\varphi_{\mathfrak{g}}$. Noting Lemma 49, we may assume $\mathfrak{g}$ to be quasi-split. Since $\mathfrak{g}$ being split is a sufficient condition for surjectivity, we are further reduced to finding those quasi-split simple $\mathfrak{g}$ which are non-split but have surjective $\varphi_{\mathfrak{g}}$. It follows that $\mathfrak{g}$ belongs to one of the four families $\mathfrak{su}(n,n)$, $\mathfrak{su}(n+1,n)$, $\mathfrak{so}(2n+2,2n)$, and $\mathfrak{so}(2n+1,2n-1)$, or that $\mathfrak{g} = EII$, the non-split, quasi-split real form of E_6 (see Appendix C3 of [49]). Our examples establish that $\varphi_{\mathfrak{g}}$ is surjective for $\mathfrak{g} = \mathfrak{su}(n,n)$, $\mathfrak{g} = \mathfrak{su}(n+1,n)$, and $\mathfrak{g} = \mathfrak{so}(2n+2,2n)$, while Example 10 demonstrates that surjectivity does not hold for $\mathfrak{g} = \mathfrak{so}(2n+1,2n-1)$. Also, a brief examination of the computations in [27] reveals that $\varphi_{\mathfrak{g}}$ is surjective for $\mathfrak{g} = EII$. We then have the following characterization of the surjectivity condition.

Theorem 51. *If $\mathfrak{g}$ is a semisimple real Lie algebra, then $\varphi_{\mathfrak{g}}$ is surjective if and only if $\mathfrak{g}$ is quasi-split and has no simple summand of the form $\mathfrak{so}(2n+1,2n-1)$.*

Proof. If $\varphi_{\mathfrak{g}}$ is surjective, then Lemma 49 implies that $\mathfrak{g}$ is quasi-split. Also, Proposition 50 implies that each simple summand of $\mathfrak{g}$ has a surjective orbit complexification map, and the above discussion then establishes that $\mathfrak{g}$ has no simple summand of the form $\mathfrak{so}(2n+1, 2n-1)$. Conversely, assume that $\mathfrak{g}$ is quasi-split and has no simple summand of the form $\mathfrak{so}(2n+1, 2n-1)$. By Proposition 50 (ii), each simple summand of $\mathfrak{g}$ is quasi-split. Furthermore, the above discussion implies that the only quasi-split simple real Lie algebras with non-surjective orbit complexification maps are those of the form $\mathfrak{so}(2n+1, 2n-1)$. Hence, each simple summand of $\mathfrak{g}$ has a surjective orbit complexification map, and Proposition 50 (i) implies that $\varphi_{\mathfrak{g}}$ is surjective. □

5.3.3 The Image of $\varphi_{\mathfrak{g}}$

Having investigated the surjectivity of $\varphi_{\mathfrak{g}}$, let us consider the more subtle matter of characterizing its image. Accordingly, let $\sigma_{\mathfrak{g}} : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ denote complex conjugation with respect to the real form $\mathfrak{g} \subseteq \mathfrak{g}_{\mathbb{C}}$. The following lemma will be useful.

Lemma 52. *If $\Theta \subseteq \mathfrak{g}_{\mathbb{C}}$ is a complex nilpotent orbit, then so is $\sigma_{\mathfrak{g}}(\Theta)$.*

Proof. Note that $\sigma_{\mathfrak{g}}$ integrates to a real Lie group automorphism

$$\tau : G_{sc} \to G_{sc},$$

where $(G)_{sc}$ is the connected, simply-connected Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$. If $g \in G_{sc}$ and $\xi \in \mathfrak{g}_{\mathbb{C}}$, then

$$\sigma_{\mathfrak{g}}(\mathrm{Ad}_{\mathbb{C}}(g)(\xi)) = \mathrm{Ad}_{\mathbb{C}}(\tau(g))(\sigma_{\mathfrak{g}}(\xi)).$$

Hence, $\sigma_{\mathfrak{g}}$ sends the G_{sc}-orbit of ξ to the G_{sc}-orbit of $\sigma_{\mathfrak{g}}(\xi)$. To complete the proof, we need only observe that G_{sc}-orbits coincide with $G_{\mathbb{C}}$-orbits in $\mathfrak{g}_{\mathbb{C}}$, and that $\sigma_{\mathfrak{g}}(\xi)$ is nilpotent whenever ξ is nilpotent. □

We may now use $\sigma_{\mathfrak{g}}$ to explicitly describe the image of $\varphi_{\mathfrak{g}}$ when $\mathfrak{g}$ is quasi-split.

Theorem 53. *If Θ is a complex nilpotent orbit, the condition $\sigma_{\mathfrak{g}}(\Theta) = \Theta$ is necessary for Θ to be in the image of $\varphi_{\mathfrak{g}}$. If $\mathfrak{g}$ is quasi-split, then this condition is also sufficient.*

Proof. Assume that Θ belongs to the image of $\varphi_{\mathfrak{g}}$, so that there exists $\xi \in \Theta \cap \mathfrak{g}$. Note that $\sigma_{\mathfrak{g}}(\Theta)$ is then the complex nilpotent orbit containing $\sigma_{\mathfrak{g}}(\xi) = \xi$, meaning that $\sigma_{\mathfrak{g}}(\Theta) = \Theta$. Conversely, assume that $\mathfrak{g}$ is quasi-split and that $\sigma_{\mathfrak{g}}(\Theta) = \Theta$. The latter means precisely that Θ is defined over $\mathbb{R}$ with respect to the real structure on $\mathfrak{g}_{\mathbb{C}}$ induced by the inclusion $\mathfrak{g} \subseteq \mathfrak{g}_{\mathbb{C}}$. Theorem 4.2 of [54] then implies that $\Theta \cap \mathfrak{g} \neq \emptyset$. □

Using Theorem 53, we will give an interesting sufficient condition for a complex nilpotent orbit to be in the image of $\varphi_{\mathfrak{g}}$ when $\mathfrak{g}$ is quasi-split. In order to proceed, however, we will need a better understanding of the way in which $\sigma_{\mathfrak{g}}$ permutes complex nilpotent orbits. To this end, we have the following lemma.

Lemma 54. *Suppose that $\mathfrak{g}$ comes with the faithful representation $\mathfrak{g} \subseteq \mathfrak{gl}(V)$, where V is over $\mathbb{R}$. If Θ is a complex nilpotent orbit, then $\lambda(\sigma_{\mathfrak{g}}(\Theta)) = \lambda(\Theta)$.*

Proof. Choose an $\mathfrak{sl}_2(\mathbb{C})$-triple (ξ, h, η) in $\mathfrak{g}_{\mathbb{C}}$ with $\xi \in \Theta$. Since $\sigma_{\mathfrak{g}}$ preserves Lie brackets, it follows that $(\sigma_{\mathfrak{g}}(\xi), \sigma_{\mathfrak{g}}(h), \sigma_{\mathfrak{g}}(\eta))$ is also an $\mathfrak{sl}_2(\mathbb{C})$-triple. The exercise is then to show that our two $\mathfrak{sl}_2(\mathbb{C})$-triples give isomorphic representations of $\mathfrak{sl}_2(\mathbb{C})$ on $\tilde{V} = V_{\mathbb{C}}$. For this, it will suffice to prove that h and $\sigma_{\mathfrak{g}}(h)$ act on $V_{\mathbb{C}}$ with the same eigenvalues, and that their respective eigenspaces for a given eigenvalue are equi-dimensional. To this end, let $\sigma_V : V_{\mathbb{C}} \to V_{\mathbb{C}}$ be complex conjugation with respect to $V \subseteq V_{\mathbb{C}}$. Note that

$$\sigma_{\mathfrak{g}}(h) \cdot (\sigma_V(x)) = \sigma_V(h \cdot x)$$

for all $x \in V_{\mathbb{C}}$, where $\cdot$ is used to denote the action of $\mathfrak{g}_{\mathbb{C}}$ on $V_{\mathbb{C}}$. Hence, if x is an eigenvector of h with eigenvalue $\lambda \in \mathbb{R}$, then $\sigma_V(x)$ is an eigenvector of $\sigma_{\mathfrak{g}}(h)$ with eigenvalue λ. We conclude that h and $\sigma_{\mathfrak{g}}(h)$ have the same eigenvalues. Furthermore, their respective eigenspaces for a fixed eigenvalue are related by σ_V, and so are equi-dimensional. □

We now have the following

Proposition 55. *Let $\mathfrak{g}$ be a quasi-split semisimple real Lie algebra endowed with a faithful representation $\mathfrak{g} \subseteq \mathfrak{gl}(V)$, where V is over $\mathbb{R}$. If Θ is the unique complex nilpotent orbit with partition $\lambda(\Theta)$, then Θ is in the image of $\varphi_{\mathfrak{g}}$.*

Proof. By Lemma 54, $\sigma_{\mathfrak{g}}(\Theta)$ is a complex nilpotent orbit with partition $\lambda(\Theta)$, and our hypothesis on Θ gives $\sigma_{\mathfrak{g}}(\Theta) = \Theta$. Theorem 53 then implies that Θ is in the image of $\varphi_{\mathfrak{g}}$. □

A few remarks are in order.

Remark. One can use Proposition 55 to investigate whether $\varphi_{\mathfrak{g}}$ is surjective without appealing to the partition-type description of $\varphi_{\mathfrak{g}}$ discussed in Section 5.3.1. For instance, suppose that $\mathfrak{g} = \mathfrak{so}(2n+2, 2n)$, a quasi-split real form of $\mathfrak{g}_{\mathbb{C}} = \mathfrak{so}_{4n+2}(\mathbb{C})$. We refer the reader to Example 5 for the precise assignment of partitions to nilpotent orbits in $\mathfrak{so}_{4n+2}(\mathbb{C})$. In particular, note that a complex nilpotent orbit is the unique one with its partition if and only if the partition does not have all even parts. Furthermore, as discussed in Example 9, there do not exist partitions of $4n+2$ having only even parts such that each part appears with even multiplicity. Hence, each complex nilpotent orbit is specified by its partition, so Proposition 55 implies that $\varphi_{\mathfrak{g}}$ is surjective.

Remark. The converse of Proposition 55 does not hold. Indeed, suppose that $\mathfrak{g} = \mathfrak{so}(2n, 2n)$, the split real form of $\mathfrak{g}_{\mathbb{C}} = \mathfrak{so}_{4n}(\mathbb{C})$. Recalling Example 5, every partition of $4n$ with only even parts, each appearing with even multiplicity, is the partition of two distinct complex nilpotent orbits. Yet, Lemma 48 implies that $\varphi_{\mathfrak{g}}$ is surjective, so that these orbits are in the image of $\varphi_{\mathfrak{g}}$.

5.3.4 Fibres

In this section, we investigate the fibres of the orbit complexification map $\varphi_{\mathfrak{g}} : \mathcal{N}(\mathfrak{g})/G \to \mathcal{N}(\mathfrak{g}_{\mathbb{C}})/G_{\mathbb{C}}$. In order to proceed, it will be necessary to recall some aspects of the Kostant-Sekiguchi Correspondence. To this end, fix a Cartan involution $\theta : \mathfrak{g} \to \mathfrak{g}$. Letting $\mathfrak{k}$ and $\mathfrak{p}$ denote the 1 and (-1)-eigenspaces of θ, respectively, we obtain the internal direct sum decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.$$

This gives a second decomposition

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_{\mathbb{C}}.$$

Let $K \subseteq G$ and $K_{\mathbb{C}} \subseteq G_{\mathbb{C}}$ be the connected closed subgroups with respective Lie algebras $\mathfrak{k}$ and $\mathfrak{k}_{\mathbb{C}}$. The Kostant-Sekiguchi Correspondence is one between the nilpotent orbits in $\mathfrak{g}$ and the $K_{\mathbb{C}}$-orbits in the ($K_{\mathbb{C}}$-invariant) subvariety $\mathfrak{p}_{\mathbb{C}} \cap \mathcal{N}(\mathfrak{g}_{\mathbb{C}})$ of $\mathfrak{g}_{\mathbb{C}}$.

Theorem 56 (The Kostant-Sekiguchi Correspondence)**.** *There is a bijective correspondence*

$$\mathcal{N}(\mathfrak{g})/G \to (\mathfrak{p}_{\mathbb{C}} \cap \mathcal{N}(\mathfrak{g}_{\mathbb{C}}))/K_{\mathbb{C}}$$

$$\mathcal{O} \mapsto \mathcal{O}^{\vee}$$

with the following properties.

(i) It is an isomorphism of posets, where $(\mathfrak{p}_{\mathbb{C}} \cap \mathcal{N}(\mathfrak{g}_{\mathbb{C}}))/K_{\mathbb{C}}$ is endowed with the closure order (5.1).

(ii) If $\mathcal{O}$ is a real nilpotent orbit, then $\mathcal{O}$ and $\mathcal{O}^{\vee}$ are K-equivariantly diffeomorphic.

The first property was established by Barbasch and Sepanski in [5], while the second was proved by Vergne in [74]. Each paper makes extensive use of Kronheimer's desciption of nilpotent orbits from [55].

We now prove two preliminary results, the first of which is a direct consequence of the Kostant-Sekiguchi Correspondence.

Lemma 57. *If $\mathcal{O}$ is a real nilpotent orbit, then $\mathcal{O}$ is the unique G-orbit of maximal dimension in $\overline{\mathcal{O}}$.*

Proof. Suppose that $\mathcal{O}' \neq \mathcal{O}$ is another G-orbit lying in $\overline{\mathcal{O}}$. By Property (i) in Theorem 56, it follows that $(\mathcal{O}')^\vee$ is an orbit in $\overline{(\mathcal{O}^\vee)}$ different from $\mathcal{O}^\vee$. However, $\mathcal{O}^\vee$ is an orbit of the complex algebraic group $K_\mathbb{C}$ under an algebraic action, and therefore is the unique orbit of maximal dimension in its closure. Hence, $\dim_\mathbb{R}((\mathcal{O}')^\vee) < \dim_\mathbb{R}(\mathcal{O}^\vee)$. Property (ii) of Theorem 56 implies that the Kostant-Sekiguchi Correspondence preserves real dimensions, so that $\dim_\mathbb{R}(\mathcal{O}') < \dim_\mathbb{R}(\mathcal{O})$. □

We will also require some understanding of the relationship between the G-stabilizer of $\xi \in \mathfrak{g}$ and the $G_\mathbb{C}$-stabilizer of ξ, viewed as an element of $\mathfrak{g}_\mathbb{C}$. Denoting these stabilizers by G_ξ and $(G_\mathbb{C})_\xi$, respectively, we have the following lemma.

Lemma 58. *If $\xi \in \mathfrak{g}$, then G_ξ is a real form of $(G_\mathbb{C})_\xi$.*

Proof. We are claiming that the Lie algebra of $(G)_\xi$ is the complexification of the Lie algebra of G_ξ. The former is $(\mathfrak{g}_\mathbb{C})_\xi = \{\eta \in \mathfrak{g}_\mathbb{C} : [\eta, \xi] = 0\}$, while the Lie algebra of G_ξ is $\mathfrak{g}_\xi = \{\eta \in \mathfrak{g} : [\eta, \xi] = 0\}$. If $\eta = \eta_1 + i\eta_2 \in \mathfrak{g}_\mathbb{C}$ with $\eta_1, \eta_2 \in \mathfrak{g}$, then $[\eta, \xi] = [\eta_1, \xi] + i[\eta_2, \xi]$. So, $\eta \in (\mathfrak{g}_\mathbb{C})_\xi$ if and only if $\eta_1, \eta_2 \in \mathfrak{g}_\xi$. This is equivalent to the condition that $\eta \in ((\mathfrak{g})_\xi)_\mathbb{C} \subseteq \mathfrak{g}_\mathbb{C}$, so that $(\mathfrak{g}_\mathbb{C})_\xi = (\mathfrak{g}_\xi)_\mathbb{C}$. □

We may now prove the main result of this section.

Theorem 59. *If $\mathcal{O}_1$ and $\mathcal{O}_2$ are real nilpotent orbits with the property that $(\mathcal{O}_1)_\mathbb{C} = (\mathcal{O}_2)_\mathbb{C}$, then either $\mathcal{O}_1 = \mathcal{O}_2$ or $\mathcal{O}_1$ and $\mathcal{O}_2$ are incomparable in the closure order. In other words, each fibre of $\varphi_\mathfrak{g}$ consists of pairwise incomparable nilpotent orbits.*

Proof. Assume that $\mathcal{O}_1$ and $\mathcal{O}_2$ are comparable. Without the loss of generality, $\mathcal{O}_1 \subseteq \overline{\mathcal{O}_2}$. We will prove that $\mathcal{O}_1 = \mathcal{O}_2$, which by Lemma 57 will amount to showing that the dimensions of $\mathcal{O}_1$ and $\mathcal{O}_2$ agree. To this end, choose points $\xi_1 \in \mathcal{O}_1$ and $\xi_2 \in \mathcal{O}_2$. Since $(\mathcal{O}_1)_\mathbb{C} = (\mathcal{O}_2)_\mathbb{C}$, we have $\dim_\mathbb{C}((G_\mathbb{C})_{\xi_1}) = \dim_\mathbb{C}((G_\mathbb{C})_{\xi_2})$. Using Lemma 58, this becomes $\dim_\mathbb{R}(G_{\xi_1}) = \dim_\mathbb{R}(G_{\xi_2})$. Hence, the (real) dimensions of $\mathcal{O}_1$ and $\mathcal{O}_2$ coincide. □

Chapter 6

Equivariant Contact Geometry and the LeBrun-Salamon Conjecture

In the following chapter, we study G-equivariant contact geometry in the context of LeBrun and Salamon's conjectural classification of Fano contact varieties. This is based on joint work with Steven Rayan, and the exposition follows the manuscript [21]. While Steven and I were both fully participant in all aspects of this project, I generally contributed the Lie-theoretic techniques while Steven provided the geometric ones.

6.1 Introduction and Structure

Fano varieties with complex contact structures have been studied enthusiastically over the last half century, in large part due to their distinguished position at the intersection of complex algebraic geometry and real differential geometry. A compact quaternionic Kähler manifold with positive curvature always supports an S^2-bundle — its twistor space — the total space of which is a Fano contact manifold. As presented in [48], the LeBrun-Salamon conjecture [56] posits that every Fano contact manifold with $b_2 = 1$ arises as the projectivization of the minimal nilpotent orbit of some simple algebraic group. If the conjecture were true, then every compact quaternionic Kähler variety with positive curvature would necessarily be homogeneous, and so progress on the LeBrun-Salamon conjecture is crucial to resolving an outstanding geometric classification problem within Riemannian geometry. The work of Beauville [6] is the strongest evidence thus far for the validity of the conjecture. For multiple points of view on Fano contact varieties, including the minimal rational curves and Mori theory approaches, we refer the reader to [16,44,47,58,59].

On another front, complex contact manifolds have been studied in the context of equivariant geometry. Most notably, Boothby [11,12] gives a complete classification of those compact simply-connected complex contact manifolds which are acted upon transitively by their respective groups of contact automorphisms (the so-called homogeneous complex contact manifolds). He identifies each with the projectivization of a suitable minimal nilpotent orbit.

This article presents some results on equivariant contact geometry for partial flag varieties. Firstly, we give a self-contained proof of the following special case of the LeBrun-Salamon conjecture.

Theorem 60. *Assume that* G *is of type* ADE*, and let* X *be a partial flag variety of* G *with* $b_2(X) = 1$*. If* X *is endowed with a* G*-invariant complex contact structure, then there exists a* G*-equivariant isomorphism* $X \cong \mathbb{P}(\mathcal{O}_{min})$ *of contact varieties.*

While Theorem 60 is deducible from Boothby's work, our argument differs significantly from that offered in [11,12]. Our approach is instead based on a sequence of results concerning the geometry (both equivariant and non-equivariant) of partial flag varieties G/P, where P is a parabolic subgroup. Specifically, we prove that a G-invariant corank-1 subbundle $\mathcal{E}$ of $\mathcal{T}_{G/P}$ is completely determined as such by the isomorphism class of the quotient line bundle $\mathcal{T}_{G/P}/\mathcal{E}$ (see Proposition 62). This leads us to prove Proposition 64, which describes the contact line bundle of a G-invariant contact structure on G/P in terms of the isomorphism between $\text{Pic}(G/P)$ and the group of 1-dimensional P-representations. Proposition 66 and Theorem 68 then combine to give us the desired G-equivariant contact variety isomorphism between G/P and $\mathbb{P}(\mathcal{O}_{\text{min}})$.

Secondly, we offer a detailed description of the contact manifold in Theorem 60 when G is of type D_n. This manifold is precisely the Grassmannian $\text{Gr}_B(2, \mathbb{C}^{2n})$ of those 2-planes in $\mathbb{C}^{2n}$ which are isotropic with respect to the complex-bilinear dot product. While there are descriptions of the $\text{SO}_{2n}(\mathbb{C})$-invariant contact distribution $\mathcal{E}$ on $\text{Gr}_B(2, \mathbb{C}^{2n})$ appearing in the literature (e.g. [44]), ours is global and canonical. Indeed, we use the classical identification of the tangent bundle of the full Grassmannian $\text{Gr}(2, \mathbb{C}^{2n})$ with $\text{Hom}(\mathcal{F}, \mathcal{O}^{\oplus 2n}/\mathcal{F})$, where $\mathcal{F}$ is the tautological bundle on $\text{Gr}(2, \mathbb{C}^{2n})$. We then present $\mathcal{E}$ explicitly as a subbundle of the pullback to $\text{Gr}_B(2, \mathbb{C}^{2n})$ of $\text{Hom}(\mathcal{F}, \mathcal{O}^{\oplus 2n}/\mathcal{F})$.

6.2 Review of Properties of Fano Contact Varieties

Here, we review the salient features of complex contact varieties in general and Fano contact varieties in particular. Let X be a smooth complex variety of complex dimension $2n+1$ for some $n \geq 0$, and let $\iota : \mathcal{E} \hookrightarrow \mathcal{T}_X$ be a rank-$2n$ holomorphic subbundle of the tangent bundle $\mathcal{T}_X$. We say that the pair $(X, \mathcal{E})$ is *contact* if, in the short exact sequence

$$0 \longrightarrow \mathcal{E} \xrightarrow{\iota} \mathcal{T}_X \xrightarrow{\theta} \mathcal{L} \longrightarrow 0 \tag{6.1}$$

induced by ι, the composition of the Lie bracket on sections of $\mathcal{T}_X$ with the quotient map θ is an $\mathcal{L}$-twisted bilinear form that is non-degenerate along $\mathcal{E}$. In keeping with the literature, we call the subbundle $\mathcal{E}$ the *contact distribution* and the quotient $\mathcal{L}$ the *contact line bundle* of $(X, \mathcal{E})$. If there exists an $\mathcal{E} \to X$ for which the pair $(X, \mathcal{E})$ is contact, we say that X *admits a contact structure*.

From now on, we also assume that X is projective, that $b_2(X) = 1$, and that X admits a contact structure, the distribution of which is $\mathcal{E}$. Let $\mathcal{L}$ be the associated contact line bundle. We use $\mathcal{K}_X$ and $\mathcal{K}_X^\vee$, respectively, for the canonical and anticanonical line bundles of X. In this case, X is Fano with $\text{Pic}(X) \cong \mathbb{Z}$ and $\mathcal{K}_X^\vee \cong \mathcal{L}^{\otimes n+1}$. This characterization is a consequence of a theorem of Demailly (Cor. 3 in [24]), applied to an earlier result of Kebekus, Peternell, Sommese, Wiśniewski (Thm. 1.1 in [48]).

There are two possibilities: either the contact line bundle $\mathcal{L}$ is a generator of $\text{Pic}(X)$ or it is not. If it is not, then $\mathcal{L}$ is a holomorphic $(n+1)$-th root of $\mathcal{K}_X^\vee$ and $\mathcal{L}$ itself has nontrivial roots (namely, a generator of $\text{Pic}(X)$). In this case, X must be $\mathbb{P}^N$ for some N, by the well-known Kobayashi-Ochiai characterization of complex projective space [50]. Hence, whenever X is a projective Fano contact variety with $b_2 = 1$ that is *not* a projective space, then it must be that $\text{Pic}(X) = \mathbb{Z} \cdot [\mathcal{L}]$.

Taking these observations together, we have that:

- $\mathcal{L}$ is ample (in particular, it is the ample generator of $\text{Pic}(X)$ whenever $X \not\cong \mathbb{P}^N$), and

- if $\mathcal{L}'$ is any other contact line bundle on X, then there must exist a vector bundle isomorphism $\mathcal{L} \cong \mathcal{L}'$.

The second fact is true because $\text{Pic}(X) \cong \mathbb{Z}$ and $\mathcal{L}$ and $\mathcal{L}'$ are holomorphic roots of the same line bundle (and hence $\deg \mathcal{L} = \deg \mathcal{L}'$).

6.3 Partial Flag Varieties and Contact Structures

6.3.1 Basic Setup

Now we specialize to the case where X is a partial flag variety, namely $X = G/P_S$ for some $S \subseteq \Pi$. In particular, X is necessarily Fano (see, for instance, Thm. V.1.4 in [51]).

With a view to eventually using the isomorphism (2.18), we will need a particular $\mathbb{Z}$-basis of the group $X^*(T)^{W_S}$. For $\beta \in \Pi$, let $h_\beta \in [\mathfrak{g}_\beta, \mathfrak{g}_{-\beta}]$ be the corresponding simple coroot. Note that the h_β form a basis of $\mathfrak{t}$ dual to the basis of fundamental weights $\omega_\beta \in X^*(T)$, $\beta \in \Pi$.

Lemma 61. *The group $X^*(T)^{W_S}$ has a $\mathbb{Z}$-basis of $\{\omega_\beta : \beta \in \Pi \setminus S\}$.*

Proof. Note that $\delta \in X^*(T)$ belongs to $X^*(T)^{W_S}$ if and only if δ is fixed by each simple reflection s_β, $\beta \in S$. This holds if and only if δ is orthogonal to each simple root in S. The desired conclusion then follows from the fact that

$$\delta = \sum_{\beta \in \Pi} \delta(h_\beta)\omega_\beta = \sum_{\beta \in \Pi} 2\frac{(\delta, \beta)}{(\beta, \beta)}\omega_\beta$$

is the expression of δ as a linear combination of the fundamental weights. □

We conclude this section with a proposition that will be of use later.

Proposition 62. *If $\mathcal{F}_1$ and $\mathcal{F}_2$ are G-invariant corank-1 subbundles of $\mathcal{T}_{G/P_S}$ and the quotients $\mathcal{T}_{G/P_S}/\mathcal{F}_1$ and $\mathcal{T}_{G/P_S}/\mathcal{F}_2$ are isomorphic as line bundles, then $\mathcal{F}_1 = \mathcal{F}_2$.*

Proof. To begin, note that $\left(\mathcal{T}_{G/P_S}\right)_{[e]}$ is canonically isomorphic to $\mathfrak{g}/\mathfrak{p}_S$ as a P_S-representation, so that

$$\mathcal{T}_{G/P_S} \cong G \times_{P_S} (\mathfrak{g}/\mathfrak{p}_S). \tag{6.2}$$

The isomorphism (6.2) restricts to isomorphisms

$$\mathcal{F}_1 \cong G \times_{P_S} V_1$$

and

$$\mathcal{F}_2 \cong G \times_{P_S} V_2,$$

where V_1 and V_2 are codimension-1 P_S-subrepresentations of $\mathfrak{g}/\mathfrak{p}_S$. Since each T-weight space of $\mathfrak{g}/\mathfrak{p}_S$ is 1-dimensional, each of V_1 and V_2 is obtained by removing a single weight space from $\mathfrak{g}/\mathfrak{p}_S$. Let $\gamma_1, \gamma_2 \in \Delta_-$ be the weights discarded to obtain V_1 and V_2, respectively. We then have bundle isomorphisms

$$\mathcal{T}_{G/P_S}/\mathcal{F}_1 \cong G \times_{P_S} ((\mathfrak{g}/\mathfrak{p}_S)/V_1) \cong \mathcal{L}(\gamma_1)$$

and

$$\mathcal{T}_{G/P_S}/\mathcal{F}_2 \cong G \times_{P_S} ((\mathfrak{g}/\mathfrak{p}_S)/V_2) \cong \mathcal{L}(\gamma_2).$$

In particular, $\mathcal{L}(\gamma_1) \cong \mathcal{L}(\gamma_2)$ as line bundles over G/P_S, so that $\gamma_1 = \gamma_2$. Hence, $V_1 = V_2$, implying that $\mathcal{F}_1$ and $\mathcal{F}_2$ identify with the same subbundle of $G \times_{P_S} (\mathfrak{g}/\mathfrak{p}_S)$ under (6.2). This completes the proof. □

In the case that $\mathcal{F}_1$ and $\mathcal{F}_2$ define contact structures, we have the following immediate

Corollary 63. *If each of $\mathcal{F}_1$ and $\mathcal{F}_2$ is the distribution of a G-invariant contact structure on G/P_S and Pic$(G/P_S) \cong \mathbb{Z}$, then $\mathcal{F}_1 = \mathcal{F}_2$.*

This is simply the result of combining Proposition 62 with the fact G/P_S is Fano (and then applying the second observation listed at the end of Section 6.2).

6.3.2 The Projectivization of the Minimal Nilpotent Orbit

The material in 6.3.1 facilitates a worthwhile discussion of $\mathbb{P}(\mathcal{O}_{\text{min}})$ and its G-invariant contact structure. To this end, suppose that $\xi \in \mathfrak{g}_\lambda \setminus \{0\}$, which determines a class $[\xi] \in \mathbb{P}(\mathcal{O}_{\text{min}})$. By Proposition 9, the G-stabilizer of $[\xi]$ is the standard parabolic subgroup P_Λ, where Λ is the collection of those simple roots which are orthogonal to λ. We therefore have the G-variety isomorphism

$$\varphi : G/P_\Lambda \xrightarrow{\cong} \mathbb{P}(\mathcal{O}_{\text{min}}) \tag{6.3}$$

$$[g] \mapsto [\text{Ad}(g)(\xi)].$$

It turns out that $\mathbb{P}(\mathcal{O}_{\text{min}})$ carries a distinguished G-invariant contact structure, $\mathcal{E}_{\text{min}} \subseteq \mathcal{T}_{\mathbb{P}(\mathcal{O}_{\text{min}})}$. To obtain it, note that the Killing form on $\mathfrak{g}$ restricts to a G-equivariant variety isomorphism between $\mathcal{O}_{\text{min}}$ and a coadjoint orbit in $\mathfrak{g}^*$. The latter has the Kirillov-Kostant-Souriau symplectic structure, so that $\mathcal{O}_{\text{min}}$ is symplectic. The symplectic form on $\mathcal{O}_{\text{min}}$ has weight 1 with respect to the scaling action of $\mathbb{C}^*$, and Lemma 1.4 of [6] then gives the desired contact structure on $\mathbb{P}(\mathcal{O}_{\text{min}})$.

Using Remark 2.3 from [6], one can more explicitly describe the bundle $\mathcal{E}_{\min}$. Let $[\xi] \in \mathbb{P}(\mathcal{O}_{\min})$ be the class of a highest root vector, as above. Via the isomorphism (6.3), the fibre $(\mathcal{E}_{\min})_{[\xi]}$ identifies with a codimension-1 subspace of $\mathfrak{g}/\mathfrak{p}_{\Lambda}$, the tangent space of G/P_{Λ} at the identity coset. Now, note that $\mathfrak{p}_{\Lambda} \subseteq (\mathfrak{g}_{\lambda})^{\perp}$, the orthogonal complement of $\mathfrak{g}_{\lambda}$ with respect to the Killing form. Our fibre is then given by

$$(\mathcal{E}_{\min})_{[\xi]} = (\mathfrak{g}_{\lambda})^{\perp}/\mathfrak{p}_{\Lambda}. \tag{6.4}$$

Since $\mathcal{E}_{\min}$ is a G-invariant subbundle of $\mathcal{T}_{\mathbb{P}(\mathcal{O}_{\min})}$, (6.4) can be used to determine the fibre of $\mathcal{E}_{\min}$ over any point.

6.3.3 Reduction to the Case of a Maximal Parabolic

Let us begin to directly address the classification of partial flag varieties admitting G-invariant contact structures. To this end, assume $S \subseteq \Pi$ is such that G/P_S admits a G-invariant contact structure $\mathcal{E} \subseteq \mathcal{T}_{G/P_S}$. In light of earlier remarks, we shall also assume that $b_2(G/P_S) = 1$. This second assumption imposes a significant constraint on the subsets S under consideration. Indeed, one has the opposite Schubert cell decomposition (2.6), so that $H^2(G/P_S;\mathbb{Z})$ is free of rank equal to the number of (complex) codimension-1 opposite Schubert cells. Since the codimension of B_-wP_S/P_S in G/P_S is the length of a minimal-length coset representative in $[w] \in W/W_S$, the codimension-1 opposite Schubert cells are those of the form $B_-s_{\beta}P_S/P_S$, $\beta \in \Pi \setminus S$. Hence, the condition $b_2(G/P_S) = 1$ implies that $\Pi \setminus S$ has cardinality 1, so that $S = \Pi \setminus \{\alpha\}$ for some unique $\alpha \in \Pi$. In other words, P_S is a maximal parabolic subgroup of G.

6.3.4 The Contact Line Bundle on G/P_S

We now give a more explicit description of the G-invariant contact structure $\mathcal{E}$. Using the bundle isomorphism (6.2), we will regard the fibre $\mathcal{E}_{[e]}$ as a codimension-1 P_S-subrepresentation of $\mathfrak{g}/\mathfrak{p}_S$. Of course, since $\mathcal{E}$ is a G-invariant subbundle of $\mathcal{T}_{G/P_S}$, we also have

$$\mathcal{E} \cong G \times_{P_S} \mathcal{E}_{[e]}. \tag{6.5}$$

Hence, the contact line bundle $\mathcal{L} = \mathcal{T}_{G/P_S}/\mathcal{E}$ is given by

$$\mathcal{L} \cong G \times_{P_S} \left((\mathfrak{g}/\mathfrak{p}_S)/\mathcal{E}_{[e]}\right). \tag{6.6}$$

Using (6.6), we can present $\mathcal{L}$ in terms of the description of equivariant line bundles from 2.4.3. To The reader will want to recall the isomorphism (2.16) between $\mathrm{Pic}(G/P_S)$ and $X^*(T)^{W_S}$

Proposition 64. *If $\mathfrak{g}$ is simply-laced, then the highest root λ belongs to $X^*(T)^{W_S}$ and $\mathcal{L}$ is isomorphic to $\mathcal{L}(-\lambda)$.*

Proof. We begin with two observations. Firstly, we have $\mathcal{L} \cong \mathcal{L}(\gamma)$ for some $\gamma \in X^*(T)^{W_S}$. Secondly, since $\mathfrak{g}$ is simply-laced, $-\lambda$ is the unique anti-dominant root. Proving the proposition will therefore amount to showing that $\gamma \in \Delta$, and that γ is anti-dominant. For the former, note that each T-weight of $\mathfrak{g}/\mathfrak{p}_S$ is a root. It follows that the weight of the quotient representation $(\mathfrak{g}/\mathfrak{p}_S)/\mathcal{E}_{[e]}$ is also a root. Also, the isomorphisms (6.6) and $\mathcal{L} \cong \mathcal{L}(\gamma)$ together imply that γ is a weight of $\mathfrak{g}/\mathfrak{p}_S$, so that γ must be a root.

To prove that γ is anti-dominant, we note that

$$\mathcal{L}((n+1)\gamma) \cong \mathcal{L}^{\otimes(n+1)} \cong \mathcal{K}^{\vee}_{G/P_S}, \tag{6.7}$$

where $2n+1$ is the (complex) dimension of G/P_S. Also, the isomorphism (6.2) yields

$$\mathcal{K}^{\vee}_{G/P_S} = \wedge^{2n+1}\mathcal{T}_{G/P_S} \cong G \times_{P_S} (\wedge^{2n+1}(\mathfrak{g}/\mathfrak{p}_S)).$$

Letting μ_S be the weight of $\wedge^{2n+1}(\mathfrak{g}/\mathfrak{p}_S)$, we have

$$\mathcal{K}^{\vee}_{G/P_S} \cong \mathcal{L}(\mu_S). \tag{6.8}$$

Combining (6.7) and (6.8), we conclude that $(n+1)\gamma = \mu_S$. Since μ_S is anti-dominant, this implies that γ is anti-dominant. □

Before proceeding to the next section, we note that our arguments allow us to quickly recover the following well-known fact.

Corollary 65. *The subbundle $\mathcal{E}_{min} \subseteq \mathcal{T}_{\mathbb{P}(\mathcal{O}_{min})}$ is the unique G-invariant contact structure on $\mathbb{P}(\mathcal{O}_{min})$.*

Proof. Suppose that $\mathcal{F} \subseteq \mathcal{T}_{\mathbb{P}(\mathcal{O}_{\min})}$is a G-invariant contact structure. Let us first assume G to be of type ADE (so that $\mathfrak{g}$ is simply-laced). Note that both $\mathcal{E}_{\min}$ and $\mathcal{F}$ pull-back to G-invariant contact structures on $G/P_\wedge$ under the isomorphism (6.3). By Proposition 64, both contact line bundles $\mathcal{T}_{\mathbb{P}(\mathcal{O}_{\min})}/\mathcal{E}_{\min}$ and $\mathcal{T}_{\mathbb{P}(\mathcal{O}_{\min})}/\mathcal{F}$ pull-back to $\mathcal{L}(-\lambda)$ under (6.3). In particular, these bundles are isomorphic, and Proposition 62 then implies $\mathcal{F} = \mathcal{E}_{\min}$.

If G is not of type ADE, then $\mathrm{Pic}(\mathbb{P}(\mathcal{O}_{\min})) \cong \mathbb{Z}$. Now, it follows from Corollary 63 that $\mathcal{F} = \mathcal{E}_{\min}$. □

6.4 A Classification of G-Invariant Contact Structures on G/P

6.4.1 The Main Theorem

We now consolidate the results presented in Sections 6.3.3 and 6.3.4. In light of Proposition 64, we will assume G to be of type ADE for the duration of this article. We then have the following relationship between the simple root α from 6.3.3 and the highest root λ.

Proposition 66. *The root α is the unique simple root not orthogonal to λ.*

Proof. By Lemma 61, $X^*(T)^{W_S}$ is freely generated by ω_α. Since $\lambda \in X^*(T)^{W_S}$ by Proposition 64, it follows that $\lambda = k\omega_\alpha$ for some non-zero $k \in \mathbb{Z}$. Also, we may write

$$k\omega_\alpha = \lambda = \sum_{\beta \in \Pi} \lambda(h_\beta)\omega_\beta = \sum_{\beta \in \Pi} 2\frac{\langle \lambda, \beta \rangle}{\langle \beta, \beta \rangle}\omega_\beta.$$

Hence, for $\beta \in \Pi$, we have $\langle \lambda, \beta \rangle = 0$ if and only if $\beta \neq \alpha$. □

Before continuing, we note the following implication of Proposition 66 for partial flag varieties in type A.

Corollary 67. *Suppose that $G = SL_n(\mathbb{C})$ with $n \geq 3$. There does not exist a partial flag variety X of $SL_n(\mathbb{C})$ with $b_2(X) = 1$ admitting an $SL_n(\mathbb{C})$-invariant contact structure. Equivalently, none of the Grassmannians $Gr(k, \mathbb{C}^n)$, $1 \leq k \leq n-1$, supports an $SL_n(\mathbb{C})$-invariant contact structure.*

Proof. By Proposition 66, the existence of such an X would imply that there was a unique simple root not orthogonal to the highest root λ. However, for $G = SL_n(\mathbb{C})$, $n \geq 3$, there are exactly two simple roots not orthogonal to λ. The formulation in terms of Grassmannians follows from their being the partial flag varieties of $SL_n(\mathbb{C})$ having $b_2 = 1$. □

Remark. Corollary 67 has an interesting consequence when n is an even positive integer. Indeed, the odd-dimensional projective space $\mathbb{P}^{n-1}$ is then isomorphic to the projectivization of the minimal nilpotent orbit of $Sp_n(\mathbb{C})$. In particular, $\mathbb{P}^{n-1}$ admits an $Sp_n(\mathbb{C})$-invariant contact structure. Yet, Corollary 67 implies that this contact structure is not $SL_n(\mathbb{C})$-invariant for $n \geq 4$.

Let us return to the matter at hand. Proposition 66 establishes that $S = \Pi \setminus \{\alpha\}$ is the collection of those simple roots which are orthogonal to λ, namely $S = \Lambda$. Hence, $G/P_S = G/P_\Lambda$, which is G-equivariantly isomorphic to $\mathbb{P}(\mathcal{O}_{\text{min}})$ via (6.3). It therefore remains to prove that (6.3) is additionally an isomorphism of contact varieties, recalling that $G/P_S = G/P_\Lambda$ has the G-invariant contact structure $\mathcal{E} \subseteq \mathcal{T}_{G/P_\Lambda}$ fixed in 6.3.3.

Theorem 68. *The map $\varphi : G/P_\Lambda \to \mathbb{P}(\mathcal{O}_{min})$ in* (6.3) *is an isomorphism of contact varieties.*

Proof. We are claiming that $\varphi^*(\mathcal{E}_{\text{min}})$ coincides with $\mathcal{E}$ when the former is regarded as a subbundle of $\mathcal{T}_{G/P_\Lambda}$. Observing that each of $\varphi^*(\mathcal{E}_{\text{min}})$ and $\mathcal{E}$ is a G-invariant corank-1 subbundle of $\mathcal{T}_{G/P_\Lambda}$, Proposition 62 allows us to reduce this to showing $\mathcal{T}_{G/P_\Lambda}/\varphi^*(\mathcal{E}_{\text{min}})$ and $\mathcal{T}_{G/P_\Lambda}/\mathcal{E}$ to be isomorphic as line bundles. The second line bundle is isomorphic to $\mathcal{L}(-\lambda)$ by Proposition 64, so we are further reduced to proving that the fibre $(\mathcal{T}_{G/P_\Lambda}/\varphi^*(\mathcal{E}_{\text{min}}))_{[e]}$ has weight $-\lambda$ as a T-representation.

Let $d_{[e]}\varphi : (T_{G/P_\Lambda})_{[e]} \to (T_{\mathbb{P}(\mathcal{O}_{\text{min}})})_{[\xi]}$ (where $[\xi] = \varphi([e])$) be the differential of φ at $[e]$. Since φ is T-equivariant, $d_{[e]}\varphi$ is an isomorphism of T-representations. Furthermore, $d_{[e]}\varphi(\varphi^*(\mathcal{E}_{\text{min}})_{[e]}) = (\mathcal{E}_{\text{min}})_{[\xi]}$, so that $(\mathcal{T}_{G/P_\Lambda}/\varphi^*(\mathcal{E}_{\text{min}}))_{[e]}$ and $(T_{\mathbb{P}(\mathcal{O}_{\text{min}})})_{[\xi]}/(\mathcal{E}_{\text{min}})_{[\xi]}$ are isomorphic T-representations. We also have an isomorphism $(T_{\mathbb{P}(\mathcal{O}_{\text{min}})})_{[\xi]} \cong \mathfrak{g}/\mathfrak{p}_\Lambda$ from Section 6.3.2, under which $(\mathcal{E}_{\text{min}})_{[\xi]}$ identifies with $(\mathfrak{g}_\lambda)^\perp/\mathfrak{p}_\Lambda$. Putting everything together, we have

$$(\mathcal{T}_{G/P_\Lambda}/\varphi^*(\mathcal{E}_{\text{min}}))_{[e]} \cong (T_{\mathbb{P}(\mathcal{O}_{\text{min}})})_{[\xi]}/(\mathcal{E}_{\text{min}})_{[\xi]} \cong \mathfrak{g}/(\mathfrak{g}_\lambda)^\perp. \tag{6.9}$$

Since we have

$$(\mathfrak{g}_\lambda)^\perp = \mathfrak{b} \oplus \bigoplus_{\beta \in \Delta_- \setminus \{-\lambda\}} \mathfrak{g}_\beta,$$

(6.9) implies that $(\mathcal{T}_{G/P_\Lambda}/\varphi^*(\mathcal{E}_{\text{min}}))_{[e]}$ is indeed the 1-dimensional T-representation of weight $-\lambda$. □

6.4.2 Example: The Grassmannian of Isotropic 2-Planes in $\mathbb{C}^{2n}$

We now describe a class of explicit examples that satisfy the hypotheses of Theorem 60. To this end, let us set $G = SO_{2n}(\mathbb{C})$ with $n \geq 4$. [1] Given $\theta \in \mathbb{R}/(2\pi\mathbb{Z})$, consider the 2×2 matrix

$$R(\theta) := \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

[1]Having developed this chapter for G simply-connected, one really should replace $SO_{2n}(\mathbb{C})$ with its simply-connected double-cover $\text{Spin}_{2n}(\mathbb{C})$. However, the results of this section are readily seen to hold when $G = SO_{2n}(\mathbb{C})$.

For $\theta_1, \theta_2, \dots, \theta_n \in \mathbb{R}/(2\pi\mathbb{Z})$, we define $R(\theta_1, \theta_2, \dots, \theta_n)$ to be the $2n \times 2n$ block-diagonal matrix $R(\theta_1) \oplus R(\theta_2) \oplus \cdots \oplus R(\theta_n)$. Note that the $R(\theta_1, \theta_2, \dots, \theta_n)$ constitute a maximal torus of the compact real form $SO(2n) \subseteq SO_{2n}(\mathbb{C})$. Let $T \subseteq SO_{2n}(\mathbb{C})$ be the complexification of this maximal torus. We then choose our collection of simple roots to be $\Pi := \{\alpha_1, \alpha_2, \dots, \alpha_n\}$, where $\alpha_j : T \to \mathbb{C}^*$ is defined by the property

$$\alpha_j(R(\theta_1, \theta_2, \dots, \theta_n)) = e^{i(\theta_j - \theta_{j+1})}$$

for $j \in \{1, \dots, n-1\}$, while $\alpha_n : T \to \mathbb{C}^*$ satisfies

$$\alpha_n(R(\theta_1, \theta_2, \dots, \theta_n)) = e^{i(\theta_{n-1} + \theta_n)}.$$

The highest root λ is then given by

$$\lambda(R(\theta_1, \theta_2, \dots, \theta_n)) = e^{i(\theta_1 + \theta_2)}.$$

Furthermore, the subset of simple roots orthogonal to λ is $\Lambda = \Pi \setminus \{\alpha_2\}$.

Now, let $B : \mathbb{C}^{2n} \otimes \mathbb{C}^{2n} \to \mathbb{C}$ be the complexification of the dot product on $\mathbb{R}^{2n}$. One then has the Grassmannian of isotropic 2-planes in $\mathbb{C}^{2n}$, $Gr_B(2, \mathbb{C}^{2n})$. More explicitly,

$$Gr_B(2, \mathbb{C}^{2n}) := \{V \in Gr(2, \mathbb{C}^{2n}) : V \subseteq V^\perp\},$$

where $V^\perp$ denotes the complement of $V \in Gr(2, \mathbb{C}^{2n})$ with respect to B. One can verify that $Gr_B(2, \mathbb{C}^{2n})$ has a point whose $SO_{2n}(\mathbb{C})$-stabilizer is P_Λ, so that

$$SO_{2n}(\mathbb{C})/P_\Lambda \cong Gr_B(2, \mathbb{C}^{2n}).$$

By (6.3), we have another $SO_{2n}(\mathbb{C})$-equivariant isomorphism

$$\mathbb{P}(\mathcal{O}_{\text{min}}) \cong Gr_B(2, \mathbb{C}^{2n}), \tag{6.10}$$

where $\mathcal{O}_{\text{min}}$ is the minimal nilpotent orbit of $SO_{2n}(\mathbb{C})$.

It remains to give the contact structure on $Gr_B(2, \mathbb{C}^{2n})$ for which (6.10) is an isomorphism of contact varieties. In other words, it remains to find the unique $SO_{2n}(\mathbb{C})$-invariant contact structure on $Gr_B(2, \mathbb{C}^{2n})$. To this end, let $\mathcal{F}$ denote the tautological bundle on $Gr_B(2, \mathbb{C}^{2n})$, whose fibre over $V \in Gr_B(2, \mathbb{C}^{2n})$ is V itself. Note that $\mathcal{F}$ is a subbundle of the trivial bundle $Gr_B(2, \mathbb{C}^{2n}) \times \mathbb{C}^{2n}$, so that we may consider the subbundle $\mathcal{F}^\perp$ of complements with respect to B. By definition, $\mathcal{F} \subseteq \mathcal{F}^\perp$, and we may define

$$\mathcal{E} := \mathrm{Hom}(\mathcal{F}, \mathcal{F}^{\perp}/\mathcal{F}).$$

Note that $\mathcal{E}$ is canonically a subbundle of $\mathrm{Hom}(\mathcal{F}, \mathcal{O}^{\oplus 2n}/\mathcal{F})$, the pullback to $\mathrm{Gr}_B(2, \mathbb{C}^{2n})$ of $\mathcal{T}_{\mathrm{Gr}(2,\mathbb{C}^{2n})}$. In fact, we have the inclusion

$$\mathcal{E} \subseteq \mathcal{T}_{\mathrm{Gr}_B(2,\mathbb{C}^{2n})}$$

of subbundles of $\mathrm{Hom}(\mathcal{F}, \mathcal{O}^{\oplus 2n}/\mathcal{F})$, giving rise to a short exact sequence

$$0 \to \mathcal{E} \to \mathcal{T}_{\mathrm{Gr}_B(2,\mathbb{C}^{2n})} \to \wedge^2(\mathcal{F}^{\vee}) \to 0 \tag{6.11}$$

(see [38], Chapter 14). Since $\wedge^2(\mathcal{F}^{\vee}) = \det(\mathcal{F}^{\vee})$ is a line bundle, $\mathcal{E}$ is a corank-1 subbundle of $\mathcal{T}_{\mathrm{Gr}_B(2,\mathbb{C}^{2n})}$. Indeed, we have the following proposition.

Proposition 69. *The subbundle $\mathcal{E} \subseteq \mathcal{T}_{\mathrm{Gr}_B(2,\mathbb{C}^{2n})}$ is the unique $\mathrm{SO}_{2n}(\mathbb{C})$-invariant contact structure on $\mathrm{Gr}_B(2, \mathbb{C}^{2n})$.*

Proof. By Proposition 62 and the discussion at the end of Section 6.2, the $\mathrm{SO}_{2n}(\mathbb{C})$-invariant contact structure on $\mathrm{Gr}_B(2, \mathbb{C}^{2n})$ is the unique subbundle of $\mathcal{H} \subseteq \mathcal{T}_{\mathrm{Gr}_B(2,\mathbb{C}^{2n})}$ such $\mathcal{H}$ is $\mathrm{SO}_{2n}(\mathbb{C})$-invariant and $\mathcal{T}_{\mathrm{Gr}_B(2,\mathbb{C}^{2n})}/\mathcal{H}$ is the ample generator of $\mathrm{Pic}(\mathrm{Gr}_B(2, \mathbb{C}^{2n}))$. Accordingly, it will suffice to prove that $\mathcal{E}$ possesses these two properties. For the former, note that $\mathcal{F}^{\perp}/\mathcal{F}$ is an $\mathrm{SO}_{2n}(\mathbb{C})$-invariant subbundle of $\mathcal{O}^{\oplus 2n}/\mathcal{F}$. Hence, $\mathcal{E} = \mathrm{Hom}(\mathcal{F}, \mathcal{F}^{\perp}/\mathcal{F})$ is an $\mathrm{SO}_{2n}(\mathbb{C})$-invariant subbundle of $\mathrm{Hom}(\mathcal{F}, \mathcal{O}^{\oplus 2n}/\mathcal{F})$, and therefore also of $\mathcal{T}_{\mathrm{Gr}_B(2,\mathbb{C}^{2n})}$. For our second property, (6.11) gives a bundle isomorphism

$$\mathcal{T}_{\mathrm{Gr}_B(2,\mathbb{C}^{2n})}/\mathcal{E} \cong \det(\mathcal{F}^{\vee}).$$

The bundle $\det(\mathcal{F}^{\vee})$ is indeed the ample generator of $\mathrm{Pic}(\mathrm{Gr}_B(2, \mathbb{C}^{2n}))$, so our proof is complete. □

We wish to conclude with a comparison of our presentation of the $\mathrm{SO}_{2n}(\mathbb{C})$-invariant contact structure on the isotropic Grassmannian to the one presented in [44] (pp.353–354), whose distribution we will denote by $\mathcal{P}$. There, $\mathrm{Gr}_B(2, \mathbb{C}^{2n})$ is given an alternative presentation, as a parameter space for lines in a hyperquadric of dimension $2n - 2$. If ℓ is line in the hyperquadric representing a point in the parameter space, and if we choose an isomorphism $\ell \cong \mathbb{P}^1$, the fibre $\mathcal{P}_{\ell}$ is the space of global sections of the $(2n-4)$-fold direct sum of the hyperplane bundle on the $\mathbb{P}^1$. One must choose an isomorphism for each point in order to describe $\mathcal{P}$ and so this description — while explicit — is local. Our presentation of the unique $SO_{2n}(\mathbb{C})$-invariant contact structure, with distribution $\mathcal{E}$ given above, does not depend on a family of isomorphisms and uses the tautological bundle on the isotropic Grassmannian directly.

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