DRAFT
Superconducting Interfaces

Equivariant Solutions to a System of Nonlinear Wave Equations with Ginzburg-Landau Type Potential

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Abstract

In this thesis, we look for solutions to a two-component system of nonlinear wave equations with the properties that one component has an interface and the other is exponentially small except near the interface of the first component. The second component can be identified with a superconducting current confined to an interface. A formal analysis suggests that for suitable initial data, the energy of solutions concentrate about a codimension one timelike surface Γ whose dynamics are coupled in a highly nonlinear way to the phase of the superconducting current. We provide a rigorous verification of the predictions these formal arguments make for solutions with an equivariant symmetry in two dimensions subject to a non-degeneracy condition. The contents of this thesis are based on the results presented in [31].

Dedication

To Linda.

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1 Introduction

1.1 Synopsis

For this thesis we consider two-component systems of hyperbolic PDEs qualitatively similar to

$$\begin{cases} \partial_{tt}\phi - \Delta\phi + \frac{\lambda_\phi}{\epsilon^2}(\phi^2 - 1)\phi = -\frac{\beta}{\epsilon^2}|\sigma|^2\phi \\ \partial_{tt}\sigma - \Delta\sigma + \frac{\lambda_\sigma}{\epsilon^2}(|\sigma|^2 - 1)\sigma = -\frac{\beta}{\epsilon^2}\phi^2\sigma \end{cases} \tag{1.1}$$

where $\Phi := (\phi, \sigma) : \mathbb{R}^{1+n} \to \mathbb{R} \times \mathbb{C}$, $0 < \epsilon \ll 1$ is a small parameter of the model, and $(\lambda_\phi, \lambda_\sigma, \beta)$ are real, non-negative constants. We are interested in solutions to (1.1) with the properties that

(1a) ϕ has an interface

(1b) σ is exponentially small except near the interface

For the first equation of (1.1), if $\beta = 0$ or if $\sigma = 0$, then the right hand side of (1.1) vanishes and the system decouples. In this case, ϕ satisfies

$$\partial_{tt}\phi - \Delta\phi + \frac{\lambda_\phi}{\epsilon^2}(\phi^2 - 1)\phi = 0$$

and it is was shown in [16] that there exists ϕ with an interface solving this equation. We, however, would like to consider regimes where ϕ and σ are coupled (i.e. $\beta \neq 0$) and where $(\lambda_\phi, \lambda_\sigma, \beta)$ are chosen so that (ϕ, σ) have the properties described above, which in particular stipulate that $\sigma \neq 0$ near the interface of ϕ. For these regimes, it follows from the physics literature on superconducting strings, reviewed in section 2 below, that the σ-field can naturally be identified with a superconducting current confined to the interface of ϕ. Hence, we call (1.1) the **superconducting interface model**. A goal of this thesis is to understand the coupling between the current and the interface and, in particular, understand how the current affects the dynamics of the interface.

As discussed in appendix A, a formal asymptotic expansion suggests that in suitable local coordinates $(y^\tau, y^\nu) = (y_0, .., y_n)$ near a codimension one timelike surface Γ, with $y^\tau = (y_0, .., y_{n-1})$ parameterizing Γ and with $\{y^\nu = 0\}$ corresponding to Γ, then there *should* exist a solution to (1.1) satisfying

$$\begin{cases} \phi(y^\tau, y^\nu) \approx \phi_0(\frac{y^\nu}{\epsilon}; \zeta(y^\tau)) \\ \sigma(y^\tau, y^\nu) \approx e^{\frac{i}{\epsilon}\theta(y^\tau)}\sigma_0(\frac{y^\nu}{\epsilon}; \zeta(y^\tau)) \end{cases} \tag{1.2}$$

where

(2a) θ is a function of y^τ only

(2b) $\zeta(y^\tau) := \gamma(\nabla_\tau\theta, \nabla_\tau\theta)$, where ∇_τ denotes the tangential gradiant along Γ and γ_{ij} is the induced metric on Γ (the ambient metric for this problem is the Minkowski metric - denoted η).

(2c) For each $\rho \in \mathbb{R}$, $\Phi_0(\cdot; \rho) := (\phi_0, \sigma_0)(\cdot; \rho) : \mathbb{R} \to \mathbb{R}^2$ satisfies the minimization problem

$$\mu(\rho) = \inf_{(f,s)\in\mathcal{A}} \int \left\{ \frac{1}{2}|(f', s')|^2 + \frac{\lambda_\phi}{4}(f^2 - 1)^2 + \frac{\lambda_\sigma}{4}(s^2 - 2)s^2 + \frac{\beta}{2}f^2 s^2 + \frac{1}{2}\rho s^2 \right\}$$

$$\mathcal{A} := \left\{ (f, s) \in H^1_{loc}(\mathbb{R}, \mathbb{R}^2) \ : \ \lim_{y^\nu \to \pm\infty} f(y^\nu) = \pm 1, \ f(0) = 0 \right\}$$

In particular, the profiles ϕ_0 and σ_0 in (1.2) are determined by $\zeta(y^\tau)$.

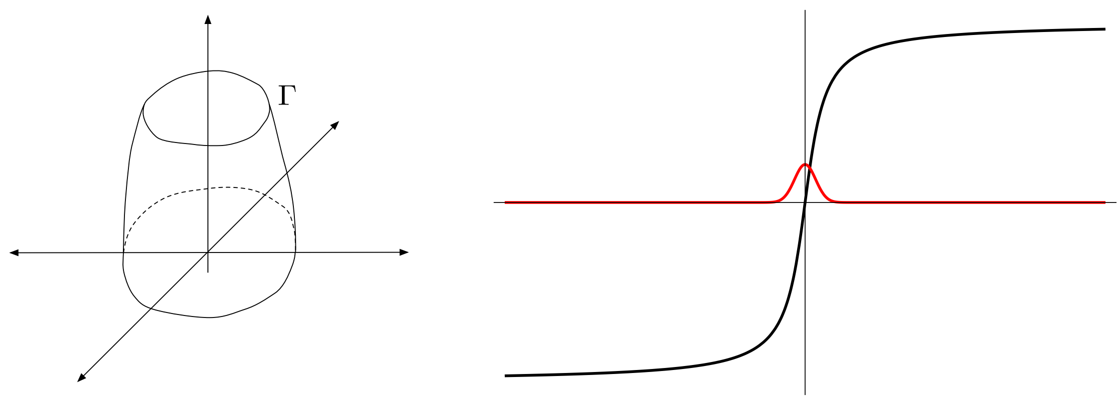
(2d) θ and Γ satisfy the highly nonlinear, coupled system of PDEs

$$\Box_\Gamma \theta = -\gamma \left(\nabla_\tau \log \left[\frac{1}{2} \left\| \sigma_0 \right\|_2^2 \right], \nabla_\tau \theta \right) \tag{1.3}$$

$$\text{Mean Curvature of } \Gamma = \frac{\left\| \sigma_0 \right\|_2^2}{\left\| \Phi_0' \right\|_2^2} \eta \left(\nu, \mathbb{II}(\nabla_\tau \theta, \nabla_\tau \theta) \right) \tag{1.4}$$

where we have supressed Φ_0's dependence on ζ and where ν is normal to Γ of unit length and $\mathbb{II}$ is the second fundamental form of Γ (we choose ν to be the same unit normal used to define the mean curvature of Γ and the second fundamental form).

In this thesis, we verify, subject to a non-degeneracy condition, that there does indeed exist a solution to (1.1) satisfying (1.2) when $n = 2$ and when Φ is an equivariant map.



(a) Nodal Set (b) Profiles

Figure 1: The formal asymptotic expansion suggests that there exists a solution $\Phi = (\phi, \sigma)$ to (1.1) so that for θ and Γ satisfying (1.3 - 1.4), then at each $p \in \Gamma$ we expect that as we move away from Γ in the transverse direction ϕ looks like the black curve in (1b) and σ looks like $e^{\frac{i}{\epsilon}\theta(s_M(p))}\sigma_0$ where σ_0 looks like the red curve in (1b). Looking at figure 1a, this means that σ is exponentially small except near Γ, $\phi \approx -1$ inside Γ, $\phi \approx 1$ outside of Γ, and ϕ transitions from -1 to 1 near Γ.

It can be shown that if the winding number density $\gamma^{ij}\partial_i\theta\partial_j\theta$ is sufficiently large, then the σ_0-field of the approximate solution is 0. It is believed that there are regimes where a solution may initially have a non-zero current (i.e. $\sigma_0(\frac{\cdot}{\epsilon}; \gamma^{ij}\partial_i\theta\partial_j\theta) \neq 0$), but as the system evolves the solution may lose its current. This type of phenomena is referred to as **current quenching** [32] and we show in section 3.4.2 below that given suitable initial conditions that the solutions we find undergo current quenching.

To the best of our knowledge, this is the first work to consider solutions to a two-component hyperbolic system with interfaces.

Mathematical Background

There is an extensive mathematical literature with results that are of the type we obtain in this thesis. The unifying theme of these types of results is

- For certain PDEs, there exist solutions which have interfaces, point vortices, or vortex filaments whose dynamics are approximately described by solutions to some associated geometric problem.

See [16] for a detailed account of these types of results for the scalar elliptic, scalar parabolic, and scalar hyperbolic counterparts of (1.1).

The scalar analogue of (1.1) is

$$\partial_{tt}u - \Delta u + \frac{\lambda}{\epsilon^2}(|u|^2 - 1)u = 0 \tag{1.5}$$

where $u : \mathbb{R}^{1+n} \to \mathbb{R}$. In [23], a formal analysis of solutions to (1.5) with interfaces is carried out in the case when $n = 3$. In [26], the authors further develop the techniques and arguments used in [23] to carry out a formal analysis of solutions with interfaces to other more complicated nonlinear scalar wave equations when $n = 2$. In appendix A, we will use some of the ideas introduced in [23] and [26] to carry out a formal analysis of solutions to (1.1) with superconducting interfaces. As a result of this formal analysis, we obtain an effective action which, at least formally, describes to leading order the evolution of the interface and current associated to solutions with a superconducting interface.

In [9], the author shows that there exists solutions to (1.5) with an interface when $n = 3$ and $\epsilon = 1$. In particular, he looks for solutions of the form

$$u(t,x) = \tanh(\frac{x^3}{\sqrt{2}}) + w(t,x) \tag{1.6}$$

and shows that for suitable initial data, then there exists a global in time solution to (1.5) of the form (1.6) with $w(t,x)$ satisfying the bound

$$\left|w(t,x)\right| \lesssim \frac{1}{\sqrt{1+t}}$$

More recently, it has been shown [16] that for any smooth codimension one timelike minimal surface Γ, then for suitable initial data, there exists a solution to (1.5) with an interface located near Γ, at least up to some time T independent of the initial data and ϵ. A similar result was then obtained in [11] for nonlinear wave equations like (1.5), but whose potentials are qualitatively similar to

$$V'(u) = \frac{\lambda}{\epsilon^2}(|u|^2 - 1)(2u - \kappa)$$

where κ is some positive constant of the model. Roughly speaking, the results obtained in [16] and [11] use weighted energy estimates to show that for suitable initial data, there exists an exact solution to (1.5) satisfying

$$u \approx q(\frac{d_M}{\epsilon})$$

where q is some appropriately chosen profile and d_M is the Minkowski distance to a codimension one timelike minimal surface Γ.

We should also mention the results of [7]. In this paper, the authors study the $\epsilon \to 0^+$ limit of solutions to (1.5). However, to obtain their main result, they need to make a technical assumption that is not easily verified.

One can also consider a version of (1.5) for which $u : \mathbb{R}^{1+n} \to \mathbb{C}$. In this case, the goal is to find and describe solutions to (1.5) that have vortices or vortex filaments. Results describing point vortices and/or vortex filaments in (1.5) and a gauged version of (1.5) have been obtained in [15, 20, 16] and [13, 10], respectively. Of particular interest to us are some the results found in [16]. In this paper, the author shows that for any smooth codimension two timelike minimal surface Γ, then for suitable initial data, there exists a solution to (1.5) that has a vortex filament located near Γ, at least up to some time T independent of the initial data and ϵ. A noteworthy observation is that the dynamical law associated to the Nambu-Goto action [22, 12] describes exactly how the vortex filaments of solutions found in [16] evolve.

Similarly for us, we could consider the case when $\phi : \mathbb{R}^{1+n} \to \mathbb{C}$. We would then like to find solutions to (1.1) so that ϕ has a vortex filament and σ is exponentially small except near the vortex filament of ϕ and we would like to find a geometric description of the evolution of the vortex filament. For now, though, we focus our attention on the case when $\phi : \mathbb{R}^{1+n} \to \mathbb{R}$ and look for solutions with an interface.

Two-component systems have been considered in the physics literature as models for interfaces, point vortices, or vortex filaments in various physical systems [19, 14]. However, rigorous mathematical descriptions of solutions to two-component systems of the type we consider are sparse in the math literature. For example, progress on the existence and classification of solutions with interfaces or vortices has been made for various two-component, elliptic systems [3, 5, 6, 4] and (potentially very complicated) ground states of other two-component models subject to physically relevant forcing has been studied [21, 1, 2].

In contrast to [16, 11], who use weighted energy estimates, we linearize (1.1) about an approximate solution obtained using a formal asymptotic expansion and use spectral properties of the linearized operator to show that there exists an exact solution of (1.1) which is close to this approximate solution. We were inspired to take this approach by works such as [28, 29, 13] who use a similar approach. The reason we use this "linearization" approach is that in order to resolve the new complexities introduced by the coupling of the current to the interface of ϕ, a more detailed description of solutions is required that seems hard to obtain using weighted energy estimates.

1.2 Description of Results

1.2.1 Assumptions

In this thesis, we are not necessarily considering (1.1), but rather systems *qualitatively* similar to (1.1). Furthermore, we will only be considering the $n = 2$ case and we will only be looking for solutions to these systems that are equivariant maps. That is, for $x \in \mathbb{R}^2$, we will look for solutions of the form

$$\begin{aligned} \phi(t,x) &= \tilde{\phi}(t,|x|) \\ \sigma(t,x) &= e^{i\frac{d}{\epsilon}\arg(x)}\tilde{\sigma}(t,|x|) \end{aligned} \tag{1.7}$$

where $(\tilde{\phi}, \tilde{\sigma}) : \mathbb{R}^{1+1} \to \mathbb{R}^2$ and $d \in \mathbb{R}/2\pi\epsilon\mathbb{Z}$ is a fixed constant.

Using assumption (1.7), the systems we are interested in reduce to the following

$$\partial_{tt}\begin{pmatrix} \tilde{\phi} \\ \tilde{\sigma} \end{pmatrix} - \partial_{rr}\begin{pmatrix} \tilde{\phi} \\ \tilde{\sigma} \end{pmatrix} - \frac{1}{r}\partial_r\begin{pmatrix} \tilde{\phi} \\ \tilde{\sigma} \end{pmatrix} + \frac{1}{\epsilon^2}\begin{pmatrix} \partial_\phi V(\tilde{\phi}, \tilde{\sigma}) \\ \partial_\sigma V(\tilde{\phi}, \tilde{\sigma}) \end{pmatrix} + \frac{1}{\epsilon^2}\frac{d^2}{r^2}\begin{pmatrix} 0 \\ \tilde{\sigma} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1.8}$$

where $(t, r) \in \mathbb{R} \times \mathbb{R}_+$ and $V : \mathbb{R}^2 \to \mathbb{R}$ and $\nabla_\Phi V := (\partial_\phi V, \partial_\sigma V) : \mathbb{R}^2 \to \mathbb{R}^2$ satisfy

(1.9)

1. $V \in C^\infty(\mathbb{R}^2, \mathbb{R})$ and $V(\phi, \sigma) = V(|\phi|, \sigma) = V(\phi, |\sigma|) = V(|\phi|, |\sigma|)$.

2. V has local minima at $(1, 0)$ and $(-1, 0)$ and $V(\pm 1, 0) = 0$, V has saddle points at $(0, 1)$ and $(0, -1)$, and V has a local max at $(0, 0)$ with $\nabla_\Phi V \neq (0, 0)$ otherwise. Also, for each fixed $\phi \in [0, 1]$, we want $V(\phi, 1) \leq V(\phi, \sigma)$ for all $\sigma \geq 1$. Similarily, for each fixed $\sigma \in [0, 1]$, we want $V(1, \sigma) \leq V(\phi, \sigma)$ for all $\phi \geq 1$.

3. $|\text{Hess}_\Phi V(\Phi)| \lesssim 1 + |\Phi|^2$ and $\text{Hess}_\Phi V(\pm 1, 0) \geq cI$ where I is the 2×2 identity matrix and $c > 0$.

4. V satisfies a non-degeneracy condition - see (1.20) below.

5. Solutions to an associated minimization problem determined by V are unique - see (1.22).

Note that the angular part of the Laplacian gives rise to the $\frac{d^2}{r^2}$ term. The initial data of (1.8) that we consider in this thesis will be described in section 1.3 below.

For the rest of the thesis, we will be looking for and studying solutions to (1.8) with properties (1a) and (1b). For notational convenience, we drop the $\sim$'s from $\tilde{\phi}$ and $\tilde{\sigma}$. We also define

$$W(\Phi, R) := V(\Phi) + \frac{1}{2}\frac{d^2}{R^2}\sigma^2 \tag{1.10}$$

We call W the **shifted potential** and we will denote the gradient of the shifted potential as

$$w(\Phi, r) := \nabla_\Phi W(\Phi, r) \tag{1.11}$$

1.2.2 Formal Analysis

Before moving on, we introduce some notation. Let Γ be a smooth codimension one timelike surface parameterized by $(\tau, R(\tau))$ for $0 \leq \tau \leq T$.

Lemma 1.2.1. *There exists a neighbourhood $\mathcal{N}$ of Γ on which there exists a differentiable solution to*

$$\begin{cases} -\partial_t d_M^2 + \partial_r d_M^2 = 1 & \text{on } \mathcal{N} \\ d_M = 0 & \text{on } \Gamma \end{cases} \tag{1.12}$$

Furthermore, there exists $s_M : \mathbb{R}^{1+1} \to \mathbb{R}$ satisfying

$$
\begin{aligned}
-\partial_t d_M \partial_t s_M + \partial_r d_M \partial_r s_M = 0 \quad &\text{on } \mathcal{N} \\
(s_M(t,r), R(s_M(t,r))) = (t,r) \quad &\text{on } \Gamma
\end{aligned}
\tag{1.13}
$$

so that

$$
(t,r) = (s_M(t,r), R(s_M(t,r))) + \frac{d_M(t,r)}{\sqrt{1 - R'(s_M(t,r))^2}}(R'(s_M(t,r)), 1) \tag{1.14}
$$

*We call d_M the **Minkowski distance to** Γ and s_M the **Minkowski projection to** Γ.*

A proof for lemma 1.2.1 can be found in [17].

Following our discussion in appendix A, we expect that for suitable surfaces Γ and profiles F_0, that there exists a solution to (1.8) with properties (1a) and (1b) satisfying

$$
\Phi(t,r) \approx F_0(\frac{d_M}{\epsilon}; R(s_M))
$$

However, when $\sigma \neq 0$ the coupling between the ϕ-field and σ-field introduces new subtleties into the nature of the solutions that necessitates a more detailed description. Thus, we include the leading order correction in our analysis and look for solutions of the form

$$
\Phi(t,r) \approx F_0(\frac{d_M}{\epsilon}; R(s_M)) + \epsilon F_1(\frac{d_M}{\epsilon}; R(s_M), R'(s_M)) \tag{1.15}
$$

Notice that F_1 depends on R *and* R'. We will comment further on this later in this section. As an aside, we could use the same notation as we use in appendix A and write $F_0 = F_0(x; \frac{d^2}{R^2})$, but we write $F_0 = F_0(x; R)$ for convenience.

Using formal arguments, see section 3.2 below, we find F_0, F_1, and R for which we believe the right hand side of (1.15) is a good approximate solution. Specifically, these formal arguments suggest that for each $R \in \mathbb{R}_+$, we should choose the profiles $F_0(\cdot; R)$ to be solutions of the minimization problem

$$
\begin{gathered}
\inf_{F \in \mathcal{A}} \int \left\{ \frac{1}{2}(F'(x))^2 + W(F(x), R) \right\} dx \\
\mathcal{A} := \{F = (f,\ s) \in H^1_{loc} \ : \ \lim_{x \to \pm\infty} F(x) = (\pm 1, 0),\ f(0) = 0 \right\}
\end{gathered}
\tag{1.16}
$$

and that we should choose the leading order correction $F_1(\cdot; R, R')$ to be a solution of

$$
\begin{gathered}
L_1(F_0(x;R), R)F_1 = H(R)F_0'(x;R) + 2\frac{x}{\sqrt{1 - R'^2}} \frac{d^2}{R^3} \begin{pmatrix} 0 \\ s_0(x;R) \end{pmatrix} \\
\lim_{x \to \pm\infty} F_1(x; R, R') = 0
\end{gathered}
\tag{1.17}
$$

where we have defined $L_1(F_0, R)$ and $H(R)$ to be

$$
L_1(F_0, R) := -\frac{d^2}{dx^2} I_{2\times 2} + \text{Hess}_\Phi\, W(F_0; R) \tag{1.18}
$$

$$H(R) := \frac{1}{\sqrt{1-(R')^2}}\left(\frac{R''}{1-(R')^2}+\frac{1}{R}\right) \tag{1.19}$$

Note that a necessary condition for (1.17) to have a solution is that

$$H(R)F_0'(x;R) + 2\frac{x}{\sqrt{1-R'^2}}\frac{d^2}{R^3}\begin{pmatrix} 0 \\ s_0(x;R) \end{pmatrix} \in \ker(L_1(F_0,R))^{\perp}$$

where $\perp = \perp_{L^2}$ (see proposition 3.3.4 below for more details). We will see in a moment that this solvability criteria suggests a good choice for R. Also note that since F_0 has an interface, then the ϕ-field of any solution to (1.8) satisfying (1.15) also has an interface and the interface of the ϕ-field is near a codimension one timelike surface parameterized by R.

Suppose F_0 satisfies the minimization problem (1.16). Then, it solves

$$-(F_0)'' + \nabla_\Phi W(F_0, R) = 0$$

Differentiating this with respect to x, we see that

$$L_1(F_0,R)F_0' = 0$$

and hence $F_0' \in \ker(L_1(F_0,R))$. In fact, for this thesis, we only consider potentials V satisfying

$$\text{If } r_0 \le R \le r_1, \text{ then } \ker(L_1(F_0;R)) = \operatorname{span}\{F_0'\}. \tag{1.20}$$

This is the non-degeneracy condition we alluded to in (1.9). In particular, this implies that if $r_0 \le R \le r_1$, then for (1.17) to have a solution, it is necessary that $R(\tau)$ satisfies

$$H(R)\int \left|F_0(x;R)\right|^2 dx - \frac{1}{\sqrt{1-R'^2}}\frac{d^2}{R^3}\int \left|s_0(x;R)\right|^2 dx = 0 \tag{1.21}$$

One might expect that by how we choose F_1 that it might depend on x as well as R, R', and R''. However, (1.21) tells us that R'' is completely determined by R and R'. Hence, F_1 depends only on x, R, and R'.

Note that all these arguments are *formal* and only *suggest* an approximate solution. Furthermore, it should be noted that the approximate solution $F_0(\frac{d_M}{\epsilon}; R(s_M)) + \epsilon F_1(\frac{d_M}{\epsilon}; R(s_M), R'(s_M))$ is only guarenteed to exist as long as $R(s_M)$ exists and is in $[r_0, r_1]$.

In section 3.3, we will show that for every $R \in \mathbb{R}_+$, there exists a minimizer of (1.16), see proposition 3.3.2. We will also show that there exists R solving (1.21), but in order to show this, we need to know that $F_0(\cdot;R)$ is continuous in R. In order to show this, it suffices to know that for each fixed R, there exists a unique minimizer of (1.16), see proposition 3.3.3. The final condition we assume V satisfies in this thesis is

Let r_0 and r_1 be the constants from (1.20). We assume that for each $R \in [r_0, r_1]$, if $F_0(\cdot;R)$ and $\tilde{F}_0(\cdot;R)$ are both solutions of the minimization problem (1.16), then $F_0(\cdot;R) = \tilde{F}_0(\cdot;R)$. (1.22)

Once we have that $F_0(\cdot;R)$ and R both exist, then we can use the non-degeneracy condition to show that F_1 exists, see proposition 3.3.4.

1.2.3 Main Result

The main result of this thesis verifies the predictions of the formal analysis outlined in the previous section. Roughly speaking, our main result says that for F_0 minimizing (1.16) and F_1 and R satisfying (1.17) and (1.21), respectively, then for suitable initial data, there exists $0 < \overline{T}$, independent of ϵ, and a solution to (1.8) satisfying (1.15) for all $0 \le t \le \overline{T}$. Most importantly, if we chose a potential V satisfing (1.9) for which $s_0 \neq 0$, then we have found a class of solutions to (1.1) satisfying properties (1a) and (1b).

More formally,

Theorem 1.2.2. *Let V be a potential satisfying (1.9) and suppose F_0 minimizes (1.16) and F_1 and R solve (1.17) and (1.21), respectively, with $R(0) \in (r_0, r_1)$ and $R'(0) = 0$. Let T be the maximal time of existence of R, chosen so that $R(\tau) \in [r_0, r_1]$ for all $0 \le \tau < T$, and for $\Gamma := \{(\tau, R(\tau)) : 0 \le \tau < T\}$, let $\mathcal{N}$ be the neighbourhood from lemma 1.2.1. Then, for suitable initial data, there exists a solution Φ to (1.8), a function $a : \mathbb{R} \to \mathbb{R}$, and a constant $0 \le \overline{T} \le T$, independent of ϵ, so that for*

$$\mathcal{N}_{\overline{T}} := \mathcal{N} \cap [0, \overline{T}] \times \mathbb{R}_+$$

we have

$$\text{For all } (t,r) \in \mathcal{N}_{\overline{T}},\ \frac{|a(s_M)|}{\epsilon},\ \frac{|a'(s_M)|}{\epsilon} \lesssim 1 \tag{1.23}$$

$$\left\|\Phi - F_0(\frac{d_M - a(s_M)}{\epsilon}; R(s_M)) - \epsilon F_1(\frac{d_M - a(s_M)}{\epsilon}; R(s_M), R'(s_M))\right\|_{L^1_t H^1_r(\mathcal{N}_{\overline{T}})} \lesssim \epsilon^2 \tag{1.24}$$

$$\left\|\partial_t \left[\Phi - F_0(\frac{d_M - a(s_M)}{\epsilon}; R(s_M)) - \epsilon F_1(\frac{d_M - a(s_M)}{\epsilon}; R(s_M), R'(s_M))\right]\right\|_{L^1_t L^2_r(\mathcal{N}_{\overline{T}})} \lesssim \epsilon^2 \tag{1.25}$$

$$\Phi(t,r) = (-1, 0) \text{ for } (t,r) \in [0, \overline{T}] \times [0, R(0) - \overline{T}) \tag{1.26}$$

$$\Phi(t,r) = (1, 0) \text{ for } (t,r) \in [0, \overline{T}] \times (R(0) + \overline{T}, \infty) \tag{1.27}$$

where d_M is the Minkowski distance to Γ and s_M is the Minkowski projection to Γ, both defined in lemma 1.2.1.

1.3 Initial Data and the Existence of Solutions

In appendix B, we prove that for suitable initial data that is sufficiently regular and which decays to $(1, 0)$ at ∞ fast enough, (1.1) is globally well posed. The proof is completely standard and is done for the reader's convenience.

Recall, for each $R \in \mathbb{R}_+$, we have picked $F_0(\cdot; R)$ to be a minimizer of (1.16) and we have chosen R to be a solution of (1.21) with $R(0) \in (r_0, r_1)$ and $R'(0) = 0$. Let T be the maximal time of existence of R, chosen so that $R(\tau) \in (r_0, r_1)$ for all $0 \le \tau < T$. Define $\Gamma := \{(\tau, R(\tau)) : 0 \le \tau < T\}$. Since $R(\tau)$ is smooth for $0 \le \tau < T$, then so is Γ. Let $\mathcal{N}$ be the neighbourhood of Γ from lemma 1.2.1 on which d_M, also defined in lemma 1.2.1, is well defined.

Since d_M is continuous, we can find a maximal $0 < \tilde{T} \le T$ so that

$$\mathcal{B} := [0, \frac{1}{2}\tilde{T}) \times (R(0) - \tilde{T}, R(0) + \tilde{T}) \subset \mathcal{N}$$

We choose the initial data of Φ as

$$\Phi(0,r) = \begin{cases} (-1,0) & \text{for } 0 \leq r < \frac{1}{2}\tilde{T} \\ (1,0) & \text{for } r > \frac{1}{2}\tilde{T} \end{cases} \tag{1.28}$$

$$\partial_t \Phi(0,r) = 0 \quad \text{for } 0 \leq r < \tfrac{1}{2}\tilde{T} \text{ and } r > \tfrac{1}{2}\tilde{T} \tag{1.29}$$

Since (1.8) is a wave equation, there is a finite speed of propogation of data. For the systems we consider in this thesis, the speed of propogation is always 1. With this choice of initial data, we have that

$$\Phi(t,r) = \begin{cases} (-1,0) & \text{for } 0 \leq r < R(0) - \tilde{T} \\ (1,0) & \text{for } r > R(0) + \tilde{T} \end{cases}$$

for all $0 \leq t \leq \frac{1}{2}\tilde{T}$. This reduces the analysis to controlling the error between Φ and the right hand side of (1.15) on the region where Φ transitions from $(-1,0)$ to $(1,0)$ - the region $\mathcal{N}_{\frac{1}{2}\tilde{T}}$.

Roughly speaking, we will choose the initial data of Φ on $(R(0) - \frac{1}{2}\tilde{T}, R(0) + \frac{1}{2}\tilde{T})$ so that Φ is sufficiently close, in some sense, to the right hand side of (1.15) at $t = 0$. A precise specification of initial data on this set is not important at the moment and so we save this for later..

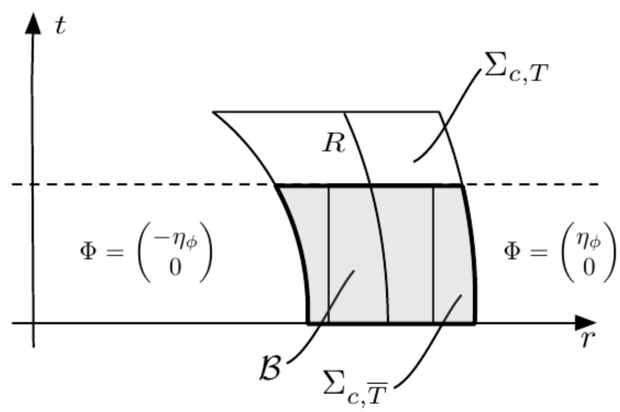


Figure 2: $\Sigma_{c,T}$ is a tubular neighbourhood on which the Minkowski normal coordinates are well defined. Pick a rectangle $\mathcal{B} = [0, \overline{T}] \times (R(0) - \delta, R(0) + \delta)$ that lays within $\Sigma_{c,T}$. We then pick the initial data of (1.8 so that $\Phi = (-1,0)$ to the left of $\mathcal{B}$ and $\Phi = (1,0)$ to the right of $\mathcal{B}$.

2 Physical Motivation: Superconducting Strings

Motivated by [24], Witten introduced a two-component model, closely related to the abelian-Higgs model, to describe finite energy solutions with vortex filaments supporting superconducting currents [33]. We call this model the **superconducting string model**. It was our initial consideration of this model that lead us to study (1.1) - the superconducting interface model. We will describe what led us to consider (1.1), but in order to do so we will first need to describe the superconducting interface model.

In [33], an effective action for the superconducting string model using formal arguments was derived. The effective action found suggests that

(3a) there should be solutions to this model with a vortex filament supporting a superconducting current

(3b) the vortex filament is near a codimension two timelike surface Γ, where Γ satisfies a geometric equation that is coupled in a highly nonlinear way to the phase of the current and an ambient vector potential representing an external electromagnetic field

Initially, we were interested in rigorously verifying that there exists a solution to the superconducting string model satisfying (3a) and (3b). However, we did not find certain aspects of the coupling between the phase of the current, the local string profile, and the geometric description of the vortex filament coming from these effective dynamics to be completely accurate.

By simplifying to the superconducting interface model we are able to obtain an effective action, different in nature to the effective action proposed for the "neutral current" superconducting string in [33]. In fact, theorem 1.2.2 verifies that for $n = 2$, in the equivariant case, and subject to a non-degeneracy condtion, that there exists a solution to (1.1) with an interface supporting a current whose dynamics are approximately described by the effective action we find in appendix A. The coupling as described in (3b) is much more complicated in our description compared to what is proposed in [33]. Furthermore, we expect that when the phase of the current is decoupled from the ambient vector potential, then the behaviour of superconducting strings *should be* qualitatively similar to the behaviour of superconducting interfaces as described in (2a) - (2d).

To illustrate how the superconducting interface model is related to the superconducting string model, we first need to state the superconducting string model. At a high level, the superconducting string model as presented in [33] consists of two weakly coupled complex scalar fields with two independent $U(1)$ gauge symmetries. More precisely, for $\phi, \sigma : \mathbb{R}^{1+3} \to \mathbb{C}$ let $A_\phi, A_\sigma : \mathbb{R}^{1+3} \to \mathbb{R}^4$ denote their associated gauge fields. Also, define the covariant derivatives associated to the ϕ and σ fields as $\nabla_\phi = \nabla - iq_\phi A_\phi$ and $\nabla_\sigma = \nabla - iq_\sigma A_\sigma$, respectively, where q_ϕ, $q_\sigma \in \mathbb{R}$ are the coupling constants between (ϕ, σ) and their associated gauge fields. As is standard notation, we define the field strength tensor of the A_ϕ-field as $F_{\phi,\mu\nu} := \partial_\mu A_{\phi,\nu} - \partial_\nu A_{\phi,\mu}$ and similarily define $F_{\sigma,\mu\nu}$ as the field strength tensor of the A_σ-field. Finally, for $(\lambda_\phi, \lambda_\sigma, \beta) \in \mathbb{R}^3_+$ we define the **superconducting string potential** as

$$V_S(\phi, \sigma) = \frac{\lambda_\phi}{4}(|\phi|^2 - 1)^2 + \frac{\lambda_\sigma}{4}(|\sigma|^2 - 2)\,|\sigma|^2 + \frac{\beta}{2}\,|\phi|^2\,|\sigma|^2 \tag{2.1}$$

The Lagrangian of the superconducting string model is

$$\mathcal{L} = \frac{1}{2}\eta^{\alpha\beta}\overline{\nabla_{\phi,\alpha}\phi}\nabla_{\phi,\beta}\phi + \frac{1}{2}\eta^{\alpha\beta}\overline{\nabla_{\sigma,\alpha}\sigma}\nabla_{\sigma,\beta}\sigma + \frac{1}{\epsilon^2}V_S(\phi, \sigma) + \frac{\epsilon^2}{4}F_{\phi,\mu\nu}F_\phi^{\mu\nu} + \frac{\epsilon^2}{4}F_{\sigma,\mu\nu}F_\sigma^{\mu\nu} \tag{2.2}$$

where $0 < \epsilon \ll 1$ and $\eta = \mathrm{diag}(-1, 1, 1, 1)$ is the Minkowski metric. As stated earlier, an important feature of this model is that the ϕ and σ fields have two, independent $U(1)$ gauge symmetries. For appropriately chosen λ_ϕ, λ_σ, and β we can arrange it so that ϕ has a broken gauge symmetry while the gauge symmetry associated to the field σ field remains unbroken. We can identify this unbroken gauge symmetry with electromagnetism. See [32] for an in depth discussion of the physics behind this model.

To obtain (1.1), two changes to the superconducting string model will be made. The first is to consider $\phi : \mathbb{R}^{1+n} \to \mathbb{R}$. In this case, ϕ loses its $U(1)$ gauge symmetry and gains a discrete symmetry. In particular, this allows for ϕ to have an interface. The second change we make is to simplify the problem by decoupling the current from the ambient vector potential. To do this, set $q_\sigma = 0$. Applying these changes to (2.2), one obtains the Lagrangian for the superconducting interface model

$$\mathcal{L} = \frac{1}{2}\eta^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + \frac{1}{2}\eta^{\alpha\beta}\overline{\partial_\alpha\sigma}\partial_\beta\sigma + \frac{1}{\epsilon^2}V_S(\phi,\sigma) \tag{2.3}$$

In this thesis, we consider the $n = 2$ case.

3 Effective Equations

3.1 Change to Minkowski Normal Coordinates

Suppose Γ is codimension 1 timelike surface parameterized by $\left\{(y^0, R(y^0))\right\}$ in polar coordinates. Define a new coordinate system (y^0, y^1) centred about Γ as follows. Set

$$(t, r) = (y^0, R(y^0)) + y^1\nu(y^0)$$

where

$$\eta\left(\partial_{y^0}(y^0, R(y^0)), \nu(y^0)\right) \quad \text{and} \quad \eta\left(\nu(y^0), \nu(y^0)\right) = 1$$

If $\nu(y^0)$ satisfies these conditions, then $-\nu(y^0)$ does too. Thus, we have a choice of $\nu(y^0)$ to make and choose

$$\nu(y^0) = \frac{1}{\sqrt{1-(R'(y^0))^2}}(R'(y^0), 1)$$

We call these new coordinates (y^0, y^1) **Minkowski normal coordinates**.

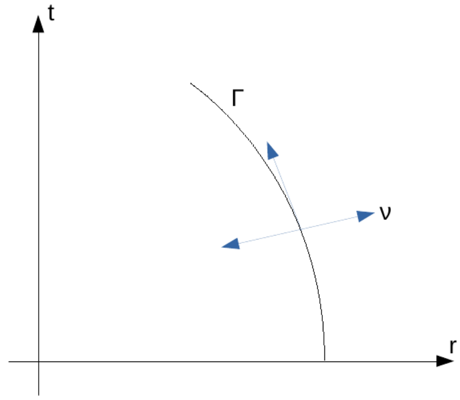


Figure 3: Given $(t, r) \in \mathbb{R} \times \mathbb{R}_+$ sufficiently close to Γ, we can uniquely identify (t, r) with a point $p(t, r)$ on Γ by taking the Minkowski projection of (t, r) onto Γ. Since Γ is parameterized by $(y^0, R(y^0))$, then there exists a unique y^0 so that $p(t, r) = (y^0, R(y^0))$. Furthermore, we can identify y^1 as the Minkowski distance of (t, r) to Γ.

In this new coordinate system, $\Gamma = \left\{ y^1 = 0 \right\}$. By changing to Minkowski normal coordinates we "straightened Γ out". Also, recall the definition of s_M and d_M from lemma 1.2.1. We can identify y^0, which we call the **tangential coordinate**, with s_M and we can identify y^1, which we call the **normal** or **transverse coordinate**, with d_M.

Motivated by the work of Jerrard in [16] and the formal asymptotics from appendix A, we expect that for suitable profiles (f_0, s_0) and Γ, there exsists a solution to (1.8) of the form

$$\Phi(y^0, y^1) \approx \begin{pmatrix} f_0(\frac{y^1}{\epsilon}; R(y^0)) \\ s_0(\frac{y^1}{\epsilon}; R(y^0)) \end{pmatrix}$$

Thus, we would like to consider (1.8) in Minkowski normal coordinates. To rewrite (1.8) in Minkowski normal coordinates, consider the action integral associated of (1.8)

$$S(\Phi) := \int \left\{ -\frac{1}{2}\partial_t \Phi^2 + \frac{1}{2}\partial_r \Phi^2 + \frac{1}{\epsilon^2} W(\Phi, r) \right\} r dt dr \tag{3.1}$$

where $\Phi = (\phi, \sigma)$ and where W is the shifted potential defined in (1.10). Define

$$m := (1 - (R')^2)^{-1/2} \quad \text{and} \quad n := 1 + y^1 m^3 R'' \tag{3.2}$$

A computation shows that

$$\begin{pmatrix} \partial_t \\ \partial_r \end{pmatrix} = \frac{m}{n} \begin{pmatrix} m & -n\,R' \\ -m\,R' & n \end{pmatrix} \begin{pmatrix} \partial_{y^0} \\ \partial_{y^1} \end{pmatrix} \tag{3.3}$$

In Minkowski normal coordinates, S is

$$S(\Phi) = \int \left\{ -\frac{m^2}{2n^2}\partial_{y^0}\Phi^2 + \frac{1}{2}\partial_{y^1}\Phi^2 + \frac{1}{\epsilon^2} W\left(\Phi, R + y^1 m\right) \right\} \left(R + y^1 m\right) \frac{n}{m} dy^0 dy^1 \tag{3.4}$$

The equations of motion of (3.4) are thus

$$\frac{m^2}{n^2}\partial_{y^0y^0}\Phi + B^\alpha\partial_\alpha\Phi - \partial_{y^1y^1}\Phi + \frac{1}{\epsilon^2}w\left(\Phi, R(y^0) + y^1 m(y^0)\right) = 0 \tag{3.5}$$

where w was defined in (1.11) and we have defined

$$B^0 := \frac{m}{n}\partial_{y^0}(\frac{m}{n}) + \frac{1}{(R+y^1m)}\frac{m^2}{n}R' \tag{3.6}$$

$$B^1 := -\frac{m^3}{n}R'' - \frac{1}{(R+y^1m)}m \tag{3.7}$$

It turns out that Minkowski normal coordinates may not be well defined everywhere and thus the domain of (3.5) is not necessarily all of $\mathbb{R}^{1+1}$. Minkowski normal coordinates are, however, well defined on $[0, y^0_*] \times [-y^1_*, y^1_*]$, where y^0_* and y^1_* are determined by R and its domain of existence. We may need to pick y^0_* even smaller so that $R(y^0) \in [r_0, r_1]$ for all $y^0 \in [0, y^0_*]$ in order to satisfy the non-degeneracy condition of (1.9). By picking the initial data of (1.8) as we did in section 1.3, the potential breakdown of Minkowski normal coordinates isn't an obstruction as we can use the finite speed of propogation of (1.8) to solve for Φ outside of $[0, y^0_*]\times[-y^1_*, y^1_*]$ up to some time $t = T$. Thus, we are just left to find solutions to (3.5) on $[0, y^0_*]\times[-y^1_*, y^1_*]$.

3.2 Expansion

As we stated in the previous section, for suitable profiles (f_0, s_0) and surfaces Γ, we would like to find solutions to (3.5) of the form

$$\Phi(y^0, y^1) \approx \begin{pmatrix} f_0(\frac{y^1}{\epsilon}; R) \\ s_0(\frac{y^1}{\epsilon}; R) \end{pmatrix} \tag{3.8}$$

It turns out that in order to prove that there exists a solution to (3.5) satisfying (3.8), we need to consider the leading order correction. Thus, for suitable (f_0, s_0), (f_1, s_1), and surfaces Γ we would actually like to look for solutions to (3.5) satisfying

$$\Phi(y^0, y^1) \approx \begin{pmatrix} f_0(\frac{y^1}{\epsilon}; R) \\ s_0(\frac{y^1}{\epsilon}; R) \end{pmatrix} + \epsilon \begin{pmatrix} f_1(\frac{y^1}{\epsilon}; R, R') \\ s_1(\frac{y^1}{\epsilon}; R, R') \end{pmatrix} \tag{3.9}$$

Set $F_0(y^1; R) = (f_0(y^1; R), s_0(y^1; R))$ and $F_1(y^1; R, R') = (f_1(y^1; R, R'), s_1(y^1; R, R'))$. Using the results of this section, we showed, in section 1.2.2, that F_1 only depends on y^1, R, and R'.

If we plug the right hand side of (3.9) into (3.4) and group terms together by their power in ϵ, we find that the first few terms are

ϵ^{-2} term:

$$\int \left(\int \left\{ \frac{1}{2}\partial_{y^1}F_0^2 + W(F_0, R) \right\} dy^1 \right) \frac{R}{m} dy^0 \tag{3.10}$$

$\underline{\epsilon^{-1}\text{ term}}$:

$$\begin{aligned}&\int\left(\int\left\{\partial_{y^1}F_1\cdot\partial_{y^1}F_0+w(F_0,R)F_1\right\}dy^1\right)\frac{R}{m}dy^0\\&-\int\left(\int\frac{y^1}{\epsilon}m\frac{d^2}{R^3}s_0^2dy^1\right)\frac{R}{m}dy^0+\int\left(\int\left\{\frac{1}{2}\partial_{y^1}F_0^2+W(F_0,R)\right\}\frac{y^1}{\epsilon}dy^1\right)(1+m^2RR'')dy^0\end{aligned}\tag{3.11}$$

$\underline{\epsilon^{0}\text{ term}}$:

$$\begin{aligned}&\int\left(\int\left\{\frac{1}{2}\partial_{y^1}F_1^2+\frac{1}{2}F_1\cdot\mathrm{Hess}_\Phi\,W(F_0,R)F_1-2\frac{y^1}{\epsilon}m\frac{d^2}{R^3}s_0s_1\right\}dy^1\right)\frac{R}{m}dy^0\\&+\int\left(\int\left\{\partial_{y^1}F_1\cdot\partial_{y^1}F_0+w(F_0,R)F_1\right\}\frac{y^1}{\epsilon}\right)(1+m^2RR'')dy^0\\&+\int\left(\int\left\{\frac{3}{2}\frac{(y^1)^2}{\epsilon^2}m^2\frac{d^2}{R^4}s_0^2-\frac{m^2}{2}\partial_RF_0^2(R')^2\right\}dy^1\right)\frac{R}{m}dy^0\\&+\int\left(\int\left\{\frac{1}{2}\partial_{y^1}F_0^2+W(F_0,R)\right\}\frac{y^1}{\epsilon}dy^1\right)m^3R''dy^0\end{aligned}\tag{3.12}$$

where every F_0 is evaluated at $(\frac{y^1}{\epsilon};R)$ and every F_1 is evaluated at $(\frac{y^1}{\epsilon};R,R')$. We omitted the arguments of these functions for notational convenience.

As we discuss in appendix A, it is natural to choose profiles F_0 so that f_0 has an interface and so that F_0 is energy minimizing. Also, we want F_0 to be independent of ϵ and we want to be able to take ϵ arbitrarily small. Thus, upon examining the leadering order term of the above expansion (3.10), for each $R\in\mathbb{R}_+$, we pick $F_0(\cdot;R)$ satisfying the minimization problem

$$\begin{aligned}&\inf_{F\in\mathcal{A}}\int\left\{\frac{1}{2}F'(x)^2+W(F(x),R)\right\}dx\\&\mathcal{A}:=\{F=(f,\ s)\in H^2_{loc}\ :\ \lim_{x\to\pm\infty}F(x)=(\pm1,0),\ f(0)=0\Big\}\end{aligned}\tag{3.13}$$

Without the requirement that $f(0)=0$, then any translation of a minimizer would be another minimizer. This condition kills this degeneracy. Note that if F_0 satisfies this minimization problem, then it also satisfies

$$-\partial_{y^1y^1}F_0+w(F_0,R)=0\tag{3.14}$$

Also, we see that for this choice of F_0, then the first term of (3.11) will always be 0, regardless of what we choose F_1 to be.

Next, we choose F_1. Again, it is natural to pick F_1 that is energy minimizing. That is, we choose F_1 satisfying

$$\begin{aligned}&\inf_{F\in\mathcal{B}}\int\left\{\frac{1}{2}F'(x)^2+\frac{1}{2}F(x)\cdot\mathrm{Hess}_\Phi\ W(F_0,R)F(x)-2\frac{y^1}{\epsilon}m\frac{d^2}{R^3}s_0s(x)-H(R)\partial_{y^1}F_0\cdot F(x)\right\}dx\\&\mathcal{B}:=\left\{F\in H^1_{loc}\ :\ \lim_{x\to\pm\infty}F(x)=(0,0)\right\}\end{aligned}\tag{3.15}$$

It turns out that if F_1 satisfies this minimization problem, then it also satisfies

$$L_1(F_0,R)F_1 = H(R)\partial_{y^1}F_0 - \frac{y^1}{\sqrt{1-(R')^2}}\partial_R w(F_0,R) \tag{3.16}$$

where

$$L_\epsilon(F_0,R) := -\partial_{y^1y^1} + \frac{1}{\epsilon^2}\operatorname{Hess}_\Phi W(F_0,R) \tag{3.17}$$

$$H(R) := \frac{1}{\sqrt{1-(R')^2}}\left(\frac{R''}{1-(R')^2} + \frac{1}{R}\right) \tag{3.18}$$

$$\partial_R w(\Phi,r) := \lim_{\Delta r\to 0}\frac{w(\Phi,r+\Delta r)-w(\Phi,r)}{\Delta r} \tag{3.19}$$

The operator $L_\epsilon(F_0,R)$ is the linearization of (3.5) about F_0 and $H(R)$ is the **mean curvature** of the surface of rotation generated by R in $\mathbb{R}^{1+2}$.

As we discussed in section 1.2.2, a necessary condition for (3.16) to be solvable is that the right hand side of (3.16) must be orthogonal to the kernel of $L_\epsilon(F_0,R)$. The non-degeneracy condition, see (1.20), then tells us that whenever $R\in[r_0,r_1]$, then for (3.16) to have a solution, R must satisfy

$$H(R)\int_{\mathbb{R}}\partial_{y^1}F_0(\cdot;R)^2 - \frac{d^2}{R^3}m\int_{\mathbb{R}}s_0(\cdot;R)^2 = 0 \tag{3.20}$$

Since the approximate solution only makes sense for as long as R exists and $R\in[r_0,r_1]$, then we choose $R(0)\in(r_0,r_1)$ and $R'(0)=0$ and we pick the maximal $y^0_*>0$ so that for all $y^0\in[0,y^0_*)$, (3.18) has a solution and $R(y^0)\in(r_0,r_1)$. In particular, y^0_* is independent of ϵ. Also, (3.20) is the geometric relation we were seeking and set

$$\Gamma := \left\{(y^0,R(y^0)) \;:\; 0\le y^0<y^0_*\right\}$$

In section 3.3 below, we establish the existence of solutions to the minimization problems (3.13) and (3.15) and the existence of solutions R to (3.20). It turns out, though, that the non-degeneracy condition implies that the functional in (3.15) is suitably convex. This implies, in particular, that any solution of (3.16) is unique and will be a global minimizer of (3.15).

Based on these *formal* calculations, we believe that if F_0 and F_1 satisfy the minimization problems (3.13) and (3.15), respectively, and if R satisfies (3.20), then for suitable initial data, there exists a solution to (1.8) satisfying (3.9). In section 4, we verify this for potentials V satisfying (1.9).

Before moving on, though, we establish the following important estimate regarding the linearized operator $L_\epsilon(F_0;R)$ that will be used to verify that there exists a solution to (3.5) satisfying (3.9). Throughout this thesis, we will use the following notation

We say $a\lesssim b$ if and only if there exists a constant $C>0$ independent of ϵ so that $a\le Cb$. (3.21)

Theorem 3.2.1 (Spectral Estimate)**.** *Suppose F_0 and R satisfy (3.14) and (3.20), respectively. By assumption 4 of (1.9),* $\ker(L_\epsilon(F_0, R)) \cap H^1(\mathbb{R}) = \operatorname{span}\left\{\partial_{y^1} F_0\right\}$*. In particular, this implies that for $\perp = \perp_{L^2}$, then for any $\xi \in \ker(L_\epsilon(F_0, R))^\perp$ we have*

$$\frac{1}{\epsilon^2}\left\|\xi\right\|^2_{L^2(\mathbb{R})} \lesssim \int\limits_{\mathbb{R}} \xi \cdot L_\epsilon(F_0; R)\xi \tag{3.22}$$

Proof of theorem 3.2.1: For fixed $R \in [r_0, r_1]$ define,

$$X = \left\{\xi \in H^1(\mathbb{R}; \mathbb{R}^2) \ : \ \left\|\xi\right\|_2 = 1 \text{ and } \left\langle \xi, \partial_{y^1} F_0\right\rangle_2 = 0\right\}$$

$$I(\xi) := \int\limits_{\mathbb{R}} \xi \cdot L_\epsilon(F_0, R)\xi$$

We want to show that

$$m_\epsilon := \inf_{\xi \in X} I(\xi) > 0 \tag{3.23}$$

Clearly, if $m_\epsilon \geq \frac{1}{\epsilon^2}c$, where c is from assumption 3 of (1.9), then there is nothing to be done. Suppose $m_\epsilon < \frac{1}{\epsilon^2}c$. If $m_\epsilon = 0$ and there exists $\xi \in X$ at which this infimum is attained, then by the non-degeneracy condition (1.20) we have that $\xi \propto \partial_{y^1} F_0$. Since $\xi \in X$, then $\xi \perp \partial_{y^1} F_0$ which implies that $\xi = 0$. This contradicts the fact that $\left\|\xi\right\|_2 = 1$. Thus, if we show that there exists $\xi \in X$ at which the infimum of I is attained, then we are done.

Let $\xi_n \in X$ be a minimizing sequence. First, observe

$$\begin{aligned}
\left\|\xi_n'\right\|_2^2 &= I(\xi_n) + \frac{1}{\epsilon^2}\int \xi_n \operatorname{Hess}_\Phi W(F_0, R)\xi_n \\
&\leq I(\xi_n) + \frac{1}{\epsilon^2}\left\|\operatorname{Hess}_\Phi W(F_0, R)\right\|_\infty \left\|\xi_n\right\|_2^2 \\
&\leq C
\end{aligned}$$

where we used the fact that $\left\|\operatorname{Hess}_\Phi W(F_0, R)\right\|_\infty < \infty$ and $\left\|\xi_n\right\|_2^2 = 1$ for all n. Thus, we have that there exists some constant C, possibly depending on ϵ, so that

$$\left\|\xi_n\right\|_2 = 1 \leq C \quad \text{and} \quad \left\|\xi_n'\right\|_2 \leq C \quad \text{and} \quad [\xi_n]_{1/2} \leq \left\|\xi_n'\right\|_2 \leq C \tag{3.24}$$

where we used the Sobolev embedding $\mathring{H}^1 \hookrightarrow C^{0,1/2}$ to obtain the third inequality. Thus, by possibly passing to a subsequence we have that $\xi_{n_k} \to \xi$ locally uniformly and weakly in H^1.

Note, we still have that $\xi \perp \partial_{y^1} F_0$. Suppose $\left\|\xi\right\|_2 = t \in [0, 1]$. We have that

$$I(\xi) = I(\frac{\xi}{t})t^2 \geq m_\epsilon t^2$$

Also, for $0 < \delta \ll 1$, choose ρ so that

$$\begin{aligned}
&\text{(i)} \quad \left\|\xi\right\|_{L^2(\mathbb{R}\setminus B_\rho)} < \delta \\
&\text{(ii)} \quad \text{For all } |y^1| > \rho, \text{ then } \operatorname{Hess}_\Phi W(F_0, R) \geq (c - \delta)I
\end{aligned} \tag{3.25}$$

For n_k sufficiently large, we have that

$$\begin{aligned} 1 &= \left\|\xi_{n_k}\right\|_2^2 \\ &= \left\|\xi_{n_k}\right\|_{L^2(B_\rho)}^2 + \left\|\xi_{n_k}\right\|_{L^2(\mathbb{R}\setminus B_\rho)}^2 \\ &\leq \left(\left\|\xi_{n_k} - \xi\right\|_{L^2(B_\rho)} + \left\|\xi\right\|_{L^2(B_\rho)}\right)^2 + \left\|\xi_{n_k}\right\|_{L^2(\mathbb{R}\setminus B_\rho)}^2 \\ &\leq (\delta + t)^2 + \left\|\xi_{n_k}\right\|_{L^2(\mathbb{R}\setminus B_\rho)}^2 \end{aligned}$$

Furthermore, we have that

$$\begin{aligned} m_\epsilon &= \lim_{n_k\to\infty} I(\xi_{n_k}) \\ &\geq \lim_{n_k\to\infty} \int \frac{1}{2}(\xi'_{n_k})^2 + \lim_{n_k\to\infty} \int \xi_{n_k} \operatorname{Hess}_\Phi W(F_0, R)\xi_{n_k} \\ &\geq \frac{1}{2}\left\|\xi'\right\|_2^2 + \lim_{n_k\to\infty} \int_{B_\rho} \xi_{n_k} \operatorname{Hess}_\Phi W(F_0, R)\xi_{n_k} + \lim_{n_k\to\infty} \int_{\mathbb{R}\setminus B_\rho} \xi_{n_k} \operatorname{Hess}_\Phi W(F_0, R)\xi_{n_k} \\ &\geq I(\xi) + \lim_{n_k\to\infty} \int_{\mathbb{R}\setminus B_\rho} \xi_{n_k} \operatorname{Hess}_\Phi W(F_0, R)\xi_{n_k} - \int_{\mathbb{R}\setminus B_\rho} \xi \operatorname{Hess}_\Phi W(F_0, R)\xi \\ &\geq I(\xi) + \frac{(c-\delta)}{\epsilon^2}\left[\lim_{n_k\to\infty} \int_{\mathbb{R}\setminus B_\rho} \xi_{n_k}^2 - \int_{\mathbb{R}\setminus B_\rho} \xi^2\right] \\ &\geq m_\epsilon t^2 + \frac{(c-\delta)}{\epsilon^2}\left[(1-(\delta+t)^2) - \delta\right] \end{aligned}$$

where we needed to use (3.25) to go from the 4th line to the 5th. This is true for all $0 < \delta \ll 1$ and so we actually have that

$$m_\epsilon = \lim_{n_k\to\infty} I(\xi_{n_k}) \geq m_\epsilon t^2 + \frac{c}{\epsilon^2}(1-t^2) \geq m_\epsilon$$

with equality if and only if $t = 1$. Thus, we necessarily have that $t = 1$.

□

3.3 Existence of F_0, F_1, and R

Recall the minimization problem set out in (3.13)

$$\mu(R) = \inf_{F\in\mathcal{A}} \int_{\mathbb{R}} \mu(F; R) \tag{3.26}$$

$$\mathcal{A} = \left\{F = (f, s) \in H^1_{loc}(\mathbb{R}) \ : \ F(\pm\infty) = (\pm 1, 0),\ f(0) = 0\right\} \tag{3.27}$$

where

$$\mu(F; R) := \frac{1}{2}(F')^2 + W(F; R) \tag{3.28}$$

We will now show that there exists $(f, s) \in \mathcal{A}$ at which $\mu(f, s; R)$ attains its infimum.

Lemma 3.3.1. *Set*

$$\tilde{\mathcal{A}} = \Big\{(f,s) \in \mathcal{A} \ : \ f(y^1) = 0 \textit{ iff } y^1 = 0, |f| \leq 1, 0 \leq s \leq 1, f \textit{ is odd}, s \textit{ is even}\Big\}$$

Then

$$\inf_{(f,s)\in\mathcal{A}} \int_{\mathbb{R}} \mu(f,s;R) \quad = \quad \inf_{(f,s)\in\tilde{\mathcal{A}}} \int_{\mathbb{R}} \mu(f,s;R)$$

Thus, we will look for minimizers to (3.26) in $\tilde{\mathcal{A}}$. The proof of lemma 3.3.1 is given below.

Proposition 3.3.2. *There exists $(f,s) \in \tilde{\mathcal{A}}$ that solves the minimization problem (3.26).*

The proof of proposition 3.3.2 is completely standard, but we include it below for the sake of completeness.

Next, we now establish the existence of solutions R to (3.20).

Proposition 3.3.3. *Suppose $R(0) \in (r_0, r_1)$ and $R'(0) = 0$ and suppose that $F_0(\cdot;R)$ is a solution to the minimization problem (3.13). Then, there exists $R : \mathbb{R} \to \mathbb{R}$ and a y^0_*, independent of ϵ, so that $R(y^0)$ is a soution to (3.20) for all $0 \leq y^0 < y^0_*$ and $R(y^0) \in [r_0, r_1]$ for all $0 \leq y^0 < y^0_*$.*

Proof of Proposition 3.3.3: Using assumption 5 of (1.9), we can show that for $R \in (r_0, r_1)$, then $F_0(\cdot, R)$ is a unique minimizer of (3.13), see step 1 of the proof of proposition 3.6.1. With this, one is then able to use standard ODE techniques to obtain the local existence of a solution to (3.20), as long as R remains in the interval (r_0, r_1) [8].

□

As discuseed in section 3.2, finding a solution to (3.16) is equivalent to finding a minimizer of (3.15). Thus,

Proposition 3.3.4 (Existence of F_1)**.** *Define*

$$g(R,v) := \frac{1}{\sqrt{1-v^2}} \frac{d^2}{R^3} \frac{\left\|s_0(\cdot;R)\right\|_2^2}{\left\|\partial_{y^1} F_0(\cdot;R)\right\|_2^2} \tag{3.29}$$

For every $(R,v) \in [r_0, r_1] \times (-1,1)$, there exists a unique $F_1 = F_1(y^1;R,v) \in H^1(\mathbb{R};\mathbb{R}^2)$ solving

$$L_1(F_0,R)F_1 = g(R,v)\partial_{y^1} F_0(\cdot;R) + 2\frac{y^1}{\sqrt{1-v^2}}\partial_R w(F_0,R)$$
$$\int F_1 \cdot \partial_{y^1} F_0(\cdot;R) = 0$$

where $L_1(F_0,R)$ was defined in (3.17) and $\partial_R w(F_0,R)$ was defined in (3.19).

Proof of Proposition 3.3.4: Since $L_1(F_0, R)$ is self-adjoint, then the spectral theorem [25, 18] implies that there exists a family of spectral projection operators $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ associated to $L_1(F_0, R)$ satisfying

$$\langle \phi, L_1(F_0, R)\psi\rangle_{L^2} = \int\limits_{\sigma(L_1(F_0,R))} \lambda \, d\,\langle \phi, E_\lambda \psi\rangle$$

for all ϕ and ψ in H^1, where $\sigma(L_1(F_0, R))$ is the spectrum of $L_1(F_0, R)$. By theorem 3.2.1, the eigenvalue 0 of $L_1(F_0, R)$ is simple. Recall that m_1, defined in (3.23), is the smallest, non-zero value in the spectrum of $L_1(F_0, R)$. Since

$$g(R, v)\partial_{y^1} F_0(\cdot; R) + \frac{y^1}{\sqrt{1-v^2}}\partial_R w(F_0, R) \perp \ker(L_1(F_0, R))$$

then $F_1 \in H^1$ satisfying

$$\langle \psi, F_1\rangle := \int\limits_{\frac{1}{2}m_1}^{\infty} \frac{1}{\lambda} d\left\langle \psi, E_\lambda \left(g(R, v)\partial_{y^1} F_0(\cdot; R) + \frac{y^1}{\sqrt{1-v^2}}\partial_R w(F_0, R)\right)\right\rangle \tag{3.30}$$

for all $\psi \in H^1$ solves (3.16). Moreover, from (3.30) we can see that that $F_1 \perp_{L^2} \partial_{y^1} F_0$ too.

□

Proof of Lemma 3.3.1: Since $\tilde{\mathcal{A}} \subset \mathcal{A}$, then

$$\inf_{(f,s)\in\tilde{\mathcal{A}}} \int\limits_{\mathbb{R}} \mu(f, s; R) \geq \inf_{(f,s)\in\mathcal{A}} \int\limits_{\mathbb{R}} \mu(f, s; R)$$

We will show that

$$\inf_{(f,s)\in\tilde{\mathcal{A}}} \int\limits_{\mathbb{R}} \mu(f, s; R) \leq \inf_{(f,s)\in\mathcal{A}} \int\limits_{\mathbb{R}} \mu(f, s; R)$$

Define

$$\tilde{\mathcal{A}}_1 = \{(f, s) \in \mathcal{A} \;:\; |f| \leq 1, 0 \leq s \leq 1\}$$

and for $(f, s) \in \mathcal{A}$ set

$$\tilde{f}(y^1) = \max(-1, \min(f(y^1), 1)) \quad \text{and} \quad \tilde{s}(y^1) = \max(0, \min(|s(y^1)|, 1))$$

Based on condition 2 of (1.9), we have that $\mu(\tilde{f}, \tilde{s}; R) \leq \mu(f, s; R)$ and so

$$\inf_{(f,s)\in\tilde{\mathcal{A}}_1} \int\limits_{\mathbb{R}} \mu(f, s; R) \leq \inf_{(f,s)\in\mathcal{A}} \int\limits_{\mathbb{R}} \mu(f, s; R) \tag{3.31}$$

Next, let $(f,s) \in \tilde{A}_1$. Let λ_1 be the smallest zero of f and let λ_2 be the largest zero of f. Note that it is possible that λ_1 may the same as λ_2. Also, since $\lim_{y^1\to\pm\infty} f(y^1) = \pm 1$, then $\lambda_1 > -\infty$ and $\lambda_2 < \infty$. We necessarily have that

$$\int_{-\infty}^{\lambda_1} \mu(f,s;R) \leq \frac{M}{2} \quad \text{or} \quad \int_{\lambda_2}^{\infty} \mu(f,s;R) \leq \frac{M}{2} \tag{3.32}$$

Assume the first inequality of (3.32) holds. By translating, we still have that $(f(\cdot - \lambda_1), s(\cdot - \lambda_1)) \in \tilde{\mathcal{A}}_1$. Thus, we may assume that $\lambda_1 = 0$. Define $(\tilde{f}, \tilde{s})$ so that $\tilde{f}$ is an odd function and $\tilde{s}$ is an even function and so that $(\tilde{f}, \tilde{s}) = (f,s)$ on $(-\infty, 0]$. Then $(\tilde{f}, \tilde{s}) \in \tilde{\mathcal{A}}$ and

$$\int \mu(\tilde{f}, \tilde{s}; R) \leq \int \mu(f,s;R)$$

If the second inequality of (3.32) holds, then a slight modification to the above argument gives the desired result.

□

Proof of proposition 3.3.2: Let $(f_j, s_j) \in \tilde{\mathcal{A}}$ be a minimizing sequence. That is, let (f_j, s_j) be a sequence satisfying

$$\mu(R) = \lim_{j\to\infty} \int_{\mathbb{R}} \mu(f_j, s_j, R) \tag{3.33}$$

Observe that

$$\int_{\mathbb{R}} \mu(f_j, s_j, R) \geq \frac{1}{2}\left\|f_j'\right\|_2^2 + \frac{1}{2}\left\|s_j'\right\|_2^2$$

and so $\left\{f_j'\right\}$ and $\left\{s_j'\right\}$ are unformly bounded sequences in L^2.

Since $\left\|(f_j', s_j')\right\|_2$ is uniformly bounded, then the Sobolev embedding $\mathring{H}^1 \hookrightarrow C^{0,1/2}$ implies that $\{(f_j, s_j)\}_{j=1}^{\infty}$ is an equicontinuous family. Since $(f_j, s_j) \in \tilde{\mathcal{A}}$ for all j, then this family is also uniformly bounded. By a variant of the Arzela-Ascoli theorem, we have that there exists a subsequence (f_{j_l}, s_{j_l}) converging locally uniformly to some uniformly continuous function (f,s). This implies, in particular, that (f_{j_l}, s_{j_l}) converges pointwise to (f,s). By Fatou's lemma, we have that

$$\int_{\mathbb{R}} W(f,s,R) \leq \liminf_{j_l\to\infty} \int_{\mathbb{R}} W(f_{j_l}, s_{j_l}, R) \tag{3.34}$$

Since $\left\|(f_{j_l}', s_{j_l}')\right\|_2$ is uniformly bounded, then by the Banach Alaoglu theorem we have that there exists a further subsequence so that (f_{j_l}', s_{j_l}') converges weakly to some (p,q) in $L^2(\mathbb{R})$. Observe that for $\phi \in C_0^{\infty}(\mathbb{R})$ we have

$$\int_{\mathbb{R}} \phi p = \lim_{j_l\to\infty} \int_{\mathbb{R}} \phi f_{j_l}' = -\lim_{j_l\to\infty} \int_{\mathbb{R}} \phi' f_{j_l} = -\int_{\mathbb{R}} \phi' f$$

where we used the compact support of ϕ and the fact that $f_{j_l} \to f$ locally uniformly to obtain the last equality. Thus, $f' = p$. Similarily, we have that $s' = q$. Using the identity $x^2 \geq a^2 + 2a(x-a)$, we have that

$$\left\|(f'_{j_l}, s'_{j_l})\right\|_2^2 \geq \left\|(f', s')\right\|_2^2 + 2\int_{\mathbb{R}} (f', s') \cdot (f' - f'_{j_l}, s' - s'_{j_l})$$

Since $(f'_{j_l}, s'_{j_l}) \rightharpoonup (f', s')$ in $L^2(\mathbb{R})$, we have that

$$\left\|(f', s')\right\|_2 \leq \liminf_{j_l \to \infty} \left\|(f'_{j_l}, s'_{j_l})\right\|_2 \tag{3.35}$$

Combining (3.34) and (3.35) completes the proof.

□

3.4 Properties of F_0 and R

As we stated in the introduction, we are interested in regimes where $\sigma \neq 0$. In section 3.4.1, we show that there exists a potential satisfying conditions 1-3 of (1.9) so that if $(f_0, s_0)(\cdot; R)$ is a minimizer of (3.13) with this potential, then $s_0(\cdot; R) \neq 0$ for some range of R. In section 3.4.2, we show that our main theorem is consistent with the phenomena of current quenching described in the physics literature when the winding number density if sufficiently large [32]. The results of this section are not used anywhere else in this thesis.

3.4.1 Regimes where s_0 is nonzero

As we discussed in the introduction, we would like to find solutions to (1.1) so that the ϕ-field has an interface and so that the σ-field is exponentially small except near the interface of ϕ. Furthermore, we want these solutions to be of the form

$$\begin{pmatrix} \phi \\ \sigma \end{pmatrix} \approx \begin{pmatrix} f_0(\frac{y^\nu}{\epsilon}; R) \\ e^{\frac{i}{\epsilon} d\psi} s_0(\frac{y^\nu}{\epsilon}; R) \end{pmatrix} \tag{3.36}$$

where $y^\tau = (y^0, \psi)$ and $y^\nu = y^1$ and (f_0, s_0) solve (3.14).

In this section we will verify that there exists potentials V satisfying conditions 1-3 of (1.9) for which $s_0 \neq 0$. In proposition 3.5.1 below, we show that $s_0(y^1; R)$ goes to zero exponentially fast in y^1. Thus, if (ϕ, σ) is a solution to (1.1) satisfying (3.36) and if V is a potential for which $s_0 \neq 0$, then we would have found solutions with the properties that we want.

For this section we choose $V = V_S$, where V_S was defined in (2.1). We will choose

$$0 < \lambda_\sigma < \lambda_\phi \quad \text{and} \quad 0 < \beta < \sqrt{\lambda_\phi \lambda_\sigma} \tag{3.37}$$

In this case, it can easily be checked that this potential satisfies all the conditions set out in (1.9) except for condition 4. We believe that (2.1) satisfies this condition as well, but we did not verify this.

Set

$$E(f, s) = \int_{\mathbb{R}} \left\{\frac{1}{2}(f')^2 + \frac{1}{2}(s')^2 + W(f, s, R)\right\}$$

When $s = 0$, then it is known that $E_0(f) := E(f, 0)$ is uniquely minimized when

$$f_{min} = \tanh(\sqrt{\frac{\lambda_\phi}{2}} x)$$

The goal is to find parameters so that $E(f_{min}, s) < E(f_{min}, 0)$ for some non-zero s with $(f_{min}, s) \in \tilde{\mathcal{A}}$.

Proposition 3.4.1. *Suppose that the constants of V_S satisfy (3.37) and*

$$\lambda_\phi + \beta < 2\lambda_\sigma \tag{3.38}$$

For sufficiently large R there exists a minimizer $(f, s) \in \tilde{\mathcal{A}}$ of

$$\int_{\mathbb{R}} \mu(f, s; R)$$

with $s \neq 0$.

Proof of proposition 3.4.1: Note that

$$E(f_{min}, s) = E(f_{min}, 0) + \int_{\mathbb{R}} \left\{ \frac{1}{2}(s')^2 + \frac{\lambda_\sigma}{4}(s^2 - 2)s^2 + \frac{\beta}{2} f^2 s^2 + \frac{d^2}{2R^2} s^2 \right\} \tag{3.39}$$

If we can show that the second term is negative we would be done. To this end, take

$$s = \frac{1}{\cosh(Bx)}$$

where $B = \sqrt{\frac{\lambda_\phi}{2}}$. Thus, pluggin s into the second term we have that

$$\begin{aligned}
& \int_{\mathbb{R}} \left\{ \frac{1}{2}(s')^2 + \frac{\lambda_\sigma}{4}(s^2 - 2)s^2 + \frac{\beta}{2} f^2 s^2 + \frac{d^2}{2R^2} s^2 \right\} \\
= & \int_{\mathbb{R}} \left[B^2 - \frac{\lambda_\sigma}{4} + \frac{\beta}{2} \right] \frac{\sinh^2(By^1)}{\cosh^4(By^1)} + \int_{\mathbb{R}} \left[\frac{d^2}{2R^2} - \frac{\lambda_\sigma}{4} \right] \frac{1}{\cosh^2(By^1)} \\
= & \frac{1}{3B} \left\{ \lambda_\phi + \beta + 3\frac{d^2}{R^2} - 2\lambda_\sigma \right\}
\end{aligned}$$

The additional constraint (3.38) ensures that the second term is indeed negative when R is sufficiently large which allows us to conclude that $E(f_{min}, s) < E(f_{min}, 0)$.

□

3.4.2 Interface Evolution and Current Quenching

We start this section with two observations about the approximate solution F_0 and the surface R.

The first observation we make is that when $s_0 \neq 0$, that is when the interface has a current, then the interface moves towards the origin. To see why this is true, recall that the surface R satisfies the geometric relation

$$\frac{1}{\sqrt{1-(R')^2}}\left(\frac{R''}{1-(R')^2}+\frac{1}{R}\right)=\frac{1}{\sqrt{1-(R')^2}}\frac{d^2}{R^3}\frac{\left\|s_0\right\|_2^2}{\left\|F_0'\right\|_2^2}$$

Rearranging, we have that

$$R''=\left[\frac{d^2}{R^2}\frac{\left\|s_0\right\|_2^2}{\left\|F_0'\right\|_2^2}-1\right]\frac{1-(R')^2}{R}$$

Since F_0 minimizes $\mu(f,s;R)$, then F_0 satisfies

$$-F_0''+\nabla_\Phi W(F_0,R)=0$$

Multiplying this by F_0' and integrating, we see that F_0 also satisfies

$$\frac{1}{2}(F_0')^2=W(F_0,R)$$

Thus,

$$R''=\left[\frac{1}{2}\frac{d^2}{R^2}\frac{\left\|s_0\right\|_2^2}{\left\|W(F_0,R)\right\|_{L^1}}-1\right]\frac{1-(R')^2}{R} \tag{3.40}$$

Since $V(F_0)>0$, we have that

$$W(F_0,R)=V(F_0)+\frac{d^2}{2R^2}s_0^2>\frac{d^2}{2R^2}s_0^2$$

and hence

$$\frac{1}{2}\frac{d^2}{R^2}\frac{\left\|s_0\right\|_2^2}{\left\|W(F_0,R)\right\|_{L^1}}>1$$

Thus, as long as $\left|R'\right|<1$, we have that $R''<0$. Since $R'(0)=0$, this implies that R is moving towards the origin.

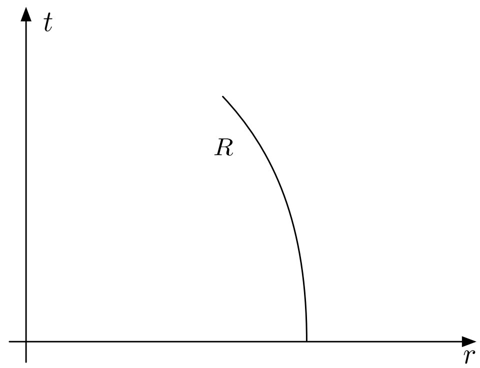


Figure 4: Since $R'(0) = 0$ and $R'' < 0$ whenver $s_0 \neq 0$, at least for a short time, then R is decreasing.

The second observation we make is that for R sufficiently small and for $F = (f, s) \in \tilde{\mathcal{A}}$ satisfying the minimization problem (3.26), then we necessarily have that $s = 0$. To see why this is true, consider the following expansion

$$W(f, s; R) = W(f, 0; R) + \partial_\sigma V(f, 0)s + \left[\int_0^1 \partial_{\sigma\sigma} V(f, \lambda s) d\lambda + \frac{1}{2} \frac{d^2}{R^2} \right] s^2$$

Since $|f| \leq 1$ and $0 \leq s \leq 1$, then for R sufficiently small the second term is non-negative. Further, condition 1 of (1.9) implies that $\partial_\sigma V(f, 0) = 0$ for all f. Thus, for R sufficiently small we have that

$$W(f, s; R) > W(f, 0; R)$$

and this in turn implies that $s = 0$.

Suppose $(f, s) \in \tilde{\mathcal{A}}$ satisfies the minimization problem (3.26) and suppose we have a potential V for which there exists a range of R so that $s(y^1; R) \neq 0$. By the second observation, we see that even though $s(y^1; R) \neq 0$ for some R, there exists R sufficiently small for which $s(y^1; R) = 0$. Suppose R_* is the largest value of R so that $s(y^1; R_*) = 0$. Note that

$$\int \mu(f, s; R) = \int \mu(f, s; R_*) + \int \frac{1}{2} \left(\frac{d^2}{R^2} - \frac{d^2}{R_*^2} \right) s^2$$

For $R < R_*$, the second term on the right hand side is positive and hence

$$\int \mu(f, s; R) \geq \inf_{(f,s) \in \tilde{\mathcal{A}}} \int \mu(f, s; R_*) = \inf_{(f,0) \in \tilde{\mathcal{A}}} \int \mu(f, 0; R_*)$$

where a necessary condition for equality is that $s = 0$. Thus, $s(y^1; R) = 0$ for all $R \leq R_*$ and $s(y^1; R) \neq 0$ for $R > R_*$. Since we are assuming that there exist some R for which $s \neq 0$, then $R_* < \infty$.

We will examine the case where $R_* < R(0) < R_* + \delta$ for some $0 < \delta \ll 1$. By the first observation, R becomes smaller as the system evolves. Since our solution only makes sense when $|R'| < 1$, an interesting question to ask is: does R become smaller than R_* before $|R'| = 1$? In other words, can the natural evolution of this system kill or quench a current? In the arguments to follow, we will need that (f_0, s_0) is continuous in R for R close to R_*. We will see in section 3.6 below that this is indeed the case if $R_* \in (r_0, r_1)$, where $r_0 < r_1$ come from the non-degeneracy condition (1.20).

Pick r_1 possibly smaller so that $R_* \in (r_0, r_1)$ and

$$-1 \leq \frac{1}{2}\frac{d^2}{R^2}\frac{\left\|s_0\right\|_2^2}{\left\|W(F_0, R)\right\|_1} - 1 \leq -\frac{1}{2}$$

for all $r_0 \leq R \leq r_1$. Define $I := \left\{y^0 \mid r_0 \leq R(y^0) \leq r_1\right\}$. For $y^0 \in I$, we have that

$$-\frac{1-(R')^2}{r_0} \leq R'' \leq -\frac{1}{2}\frac{1-(R')^2}{r_1}$$

where we used the previous observation to conclude that R is decreasing which then allows us to obtain the upper bound. It then follows that for $y^0 \in I$ and for $R'(0) = 0$, then

$$-\tanh(\frac{y^0}{r_0}) \leq R' \leq -\tanh(\frac{1}{2}\frac{y^0}{r_1}) \tag{3.41}$$

Note that $\left|R'(y^0)\right| < 1$ for all $y^0 \in I$. Integrating once more in y^0, we find that

$$R(0) - r_0 \log\cosh(\frac{y^0}{r_0}) \leq R \leq R(0) - 2r_1 \log\cosh(\frac{1}{2}\frac{y^0}{r_1}) \tag{3.42}$$

Choosing $r_0 < R_* < R(0) = R_* + \delta < r_1$ and $0 < \delta$ sufficiently small, then (3.42) and (3.41) imply that there exists $\tilde{y}^0 \in I$ so that $R(\tilde{y}^0) = R_*$.

3.5 Asymptotics of F_0 and F_1

Proposition 3.5.1. *Suppose* $(f, s) \in \tilde{\mathcal{A}}$ *minimizes*

$$\int_{\mathbb{R}} \mu(f, s; R) \tag{3.43}$$

Then there exists $\alpha > 0$ *so that*

$$\begin{cases} 1 - |f| & \lesssim e^{-\alpha|y^1|} \\ |f'| & \lesssim e^{-\alpha|y^1|} \end{cases} \quad \text{and} \quad \begin{cases} |s| & \lesssim e^{-\alpha|y^1|} \\ |s'| & \lesssim e^{-\alpha|y^1|} \end{cases} \tag{3.44}$$

Proposition 3.5.2. *Suppose* F_1 *solves (3.16). Then there exists* $\alpha > 0$ *so that for* $\beta = 0, 1$ *we have*

$$\left|\partial_{y^1}^{\beta} F_1\right| \lesssim e^{-\alpha|y^1|} \tag{3.45}$$

Obtaining these asymptotics is a standard exercise, but we include the proofs since they are used quite a bit in the proof of theorem 1.2.2. Before moving on to the proofs of these propositions, we make a few key observations.

First, observe that (3.44) implies that $(1 - f, s) \in L^2([0, \infty))$ and $(-1 - f, s) \in L^2((-\infty, 0])$.

Second, since (f, s) minimizes (3.43), then (f, s) satisfies

$$-\begin{pmatrix} f \\ s \end{pmatrix}'' + w(f, s, R) = 0 \tag{3.46}$$

Thus, once we know that (f, s) is smooth in y^1 (see section 3.6), then this relation tells us that for $\beta \in \mathbb{Z}_+$, there exists $\alpha > 0$ so that $\left|\partial^\beta_{y^1}(f, s)\right| \lesssim e^{-\alpha|y^1|}$. Once we have that (f, s) is differentiable in R and that $\partial_R(f, s)$ is smooth in y^1, then by differentiating (3.46) with respect to R we have that

$$-\begin{pmatrix} \partial_R f \\ \partial_R s \end{pmatrix}'' + \mathrm{Hess}_\Phi\, W((f, s), R)\begin{pmatrix} \partial_R f \\ \partial_R s \end{pmatrix} = 2\frac{d^2}{R^3}\begin{pmatrix} 0 \\ s \end{pmatrix}$$

Note that in order to show that (f, s) is differentiable with respect to R, we need that (3.44) holds. Using arguments similar to those used to prove proposition 3.5.1, one can show that for $\beta \in \mathbb{Z}_+$, there exists $\alpha > 0$ so that $\left|\partial^\beta_{y^1}\partial_R(f, s)\right| \lesssim e^{-\alpha|y^1|}$. Proceeding inductively, we have that for $\beta, \gamma \in \mathbb{Z}_+$, there exists $\alpha > 0$ so that

$$\left|\partial^\beta_{y^1}\partial^\gamma_R(f, s)\right| \lesssim e^{-\alpha|y^1|} \tag{3.47}$$

Third, recall that $F_1 = F_1(y^1; R, R')$ satisfies

$$L_1(F_0, R)F_1 = g(R, v)\partial_{y^1}F_0(\cdot; R) + 2\frac{y^1}{\sqrt{1 - v^2}}\partial_R w(F_0, R)$$

where $g(R, v)$ was defined in (3.29). Once we have that F_1 is differentiable with respect to R and v, a fact that requires (3.45) in order to establish, then we have that for β, γ, and $\lambda \in \mathbb{Z}_+$, there exists $\alpha > 0$ so that

$$\left|\partial^\beta_{y^1}\partial^\gamma_R\partial^\lambda_v F_1\right| \lesssim e^{-\alpha|y^1|} \tag{3.48}$$

This can be shown using the same arguments used to show that (3.47) holds.

Proof of proposition 3.5.1: Using the parity of (f, s), it suffices to show that (3.44) holds for $y^1 \to \infty$. Set

$$F = (f, s) \quad \text{and} \quad v = \tfrac{1}{2}\left|F(y^1) - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right|^2$$

Differentiating twice, we have

$$v'(y^1) = \left(F(y^1) - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \cdot F'(y^1) \quad \text{and} \quad v''(y^1) = \left|F'(y^1)\right|^2 + \left(F(y^1) - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \cdot F''(y^1)$$

Examining v'', we have that for sufficiently large y^1

$$\begin{aligned} v'' &\geq \left(F - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \cdot \nabla_\Phi W(F,R) \\ &= \left(F - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \cdot \left(\operatorname{Hess}_\Phi W(\begin{pmatrix} 1 \\ 0 \end{pmatrix}; R)\left(F - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) + O(v)\right) \\ &\geq 2cv + O(v^{3/2}) \end{aligned}$$

where we used condition 3 of (1.9) to obtain the last inquality. Since $v \to 0$ as $y^1 \to \infty$, then by possibly taking $2c$ smaller we have that

$$v'' \geq 2cv$$

We have that since $v > 0$ for all y^1, then $v' \leq 0$ for all y^1 sufficiently large. For if there exists some large $\tilde{y}^1$ so that $v'(\tilde{y}^1) > 0$, then v would have a local max somewhere because $v'(y^1) \to 0$ as $y^1 \to \infty$. Note that $v' \to 0$ as $y^1 \to \infty$ because $F \in \tilde{\mathcal{A}}$ and F minimizes (3.43). At local maxes we have that $v'' \leq 0$. However, this contradicts the fact that $v''(\tilde{y}^1) \geq 2cv > 0$. Thus, $v' \leq 0$ for sufficiently large y^1.

Examining the estimate $v'' \geq 2cv$ more closely, we have that

$$(v' - \sqrt{2c}v)' + \sqrt{2c}(v' - \sqrt{2c}v) \geq 0$$

Set $w = v' - \sqrt{2c}v$. By the previous observation, we have that $w < 0$ for sufficiently large y^1. Then

$$\begin{aligned} w' + \sqrt{2c}w &\geq 0 \\ \left(e^{\sqrt{2c}y^1} w\right)' &\geq 0 \\ e^{-\sqrt{2c}y^1} &\gtrsim -v' + \sqrt{2c}v \\ e^{-2\sqrt{2c}y^1} &\gtrsim -\left(e^{-\sqrt{2c}y^1} v\right)' \\ e^{-\sqrt{2c}y^1} &\gtrsim v \end{aligned}$$

giving us the desired asymptotics of f and s.

Since $F \in \tilde{\mathcal{A}}$ is a solution of the minimization problem (3.26), then F satisfies $F'' = \nabla_\Phi W(F,R)$. Multiplying both sides by F' and integrating, we have that $\frac{1}{2}|F'|^2 = W(F,R)$. Using the identity $g(t) = g(0) + \int_0^1 g'(t)dt$ and the fact that $W((1,0),R) = 0$, we have

$$|F'|^2 \lesssim \left\|\nabla_\Phi W(F,R)\right\|_{L^\infty(y^1,\infty)} \left|F - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right|$$

Since $\nabla_\Phi W(F,R)$ is bounded for $|(x,y)| < 1$, then

$$|F'|^2 \lesssim e^{-\sqrt{2c}y^1}$$

for y^1 sufficiently large. This gives us the desired asymptotics for f' and s'.

□

Proof of proposition 3.5.2: Set

$$v = \frac{1}{2}|F_1|^2$$

Then

$$v' = F_1 \cdot F_1' \quad \text{and} \quad v'' = \left|F_1'\right|^2 + F_1 \cdot F_1''$$

Examining v'', we have

$$\begin{aligned} v'' &\geq F_1 \cdot \operatorname{Hess}_\Phi W(F_0, R) F_1 + \left[-H(R)\partial_{y^1} F_0 + y^1 m(y^0)\partial_R w(F_0, R)\right] \cdot F_1 \\ &\geq F_1 \cdot \operatorname{Hess}_\Phi W(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, R) F_1 + F_1 \cdot P(y^1) F_1 + Q(y^1) \cdot F_1 \end{aligned}$$

where we have defined

$$P(y^1) = \operatorname{Hess}_\Phi W(F_0, R) - \operatorname{Hess}_\Phi W(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, R) \quad \text{and} \quad Q(y^1) = -H(R)\partial_{y^1} F_0 + y^1 m(y^0)\partial_R w(F_0, R)$$

Observe that proposition 3.5.1 implies that

$$\left|P(y^1)\right| \to 0 \quad \text{and} \quad \left|Q(y^1)\right| \to 0$$

exponentially fast as $\left|y^1\right| \to \infty$. By assumption 3 of (1.9)

$$v'' \geq 2cv - 2\left|P(y^1)\right| v - \sqrt{2}\left|Q(y^1)\right| v^{1/2}$$

Cauchy-Schwarz can then be used to show that

$$v'' \geq \left[2c - 2\left|P(y^1)\right| - \frac{\delta^2}{\sqrt{2}}\right] v - \frac{1}{\sqrt{2}\delta^2}\left|Q(y^1)\right|^2$$

for all $0 < \delta \ll 1$. Since $\left|P(y^1)\right|, \left|Q(y^1)\right| \to 0$ as $\left|y^1\right| \to \infty$, then by choosing δ sufficiently small, we can find a constant $C > 0$ so that

$$v'' \geq 2Cv - \frac{1}{\sqrt{2}\delta^2}\left|Q(y^1)\right|^2$$

for all $\left|y^1\right|$ sufficiently large. The rest of the proof then proceeds as the proof of proposition 3.5.1.

□

3.6 Regularity of F_0, F_1, and R

For each fixed R, standard ODE theory tells us that solutions to (3.14) are smooth in y^1 [8]. However, it does not tell us anything about the differentiability of F_0 with respect to R. It does not even tell us if F_0 is continuous in R. In this section, we will show that F_0 is actually smooth in R.

Once we have that F_0 is smooth in R, standard ODE theory tells us that $R(y^0)$ satisfying (3.20) is also smooth as long as $\left|R'\right| < 1$. Once we have that F_0 is smooth in (y^1, R), we would like to then use standard ODE theory to conclude that F_1 is smooth in (y^1, R, v). Unfortunately, things are not so simple. Using these types of arguments, one can only show that for each fixed R, F_1 is smooth in (y^1, v). It turns out, though, that once the smoothness of F_0 in (y^1, R) has been established, then the arguments from step 3 of the proof of proposition 3.6.1 can be used to show that F_1 is smooth in (y^1, R, v).

Proposition 3.6.1. *Suppose $F_0 \in \tilde{\mathscr{A}}$ is a solution to the minimization problem (3.13), then F_0 is smooth in y^1 and R for $(y^1, R) \in \mathbb{R} \times (r_0, r_1)$.*

Proof of proposition 3.6.1: As we stated before, F_0 is smooth in y^1. The first step in this proof is to show that F_0 is differentiable with respect to R and that $\partial_R F_0(R)$ is smooth in y^1. Once we have established this, we will show that F_0 is actually smooth in y^1 and R.

Step 1: In this step, we will show that F_0 is continuous with respect to R. We need to know this in order to show that F_0 is differentiable with repect ot R.

Define

$$I_R(F) := \int \mu(F; R) \tag{3.49}$$

where $\mu(F; R)$ was defined in (3.28). Let $F_R \in \tilde{\mathscr{A}}$ denote a minimizer of I_R. We have that

$$\begin{aligned} I_R(F_R) = &\ I_{\tilde{R}}(F_R) + \frac{1}{2}\left(\frac{d^2}{R^2} - \frac{d^2}{\tilde{R}^2}\right)\int s_R^2 \\ \geq &\ I_{\tilde{R}}(F_{\tilde{R}}) + \frac{1}{2}\left(\frac{d^2}{R^2} - \frac{d^2}{\tilde{R}^2}\right)\int s_R^2 \end{aligned} \tag{3.50}$$

and similarily

$$I_{\tilde{R}}(F_{\tilde{R}}) \ \geq \ I_R(F_R) - \left(\frac{d^2}{R^2} - \frac{d^2}{\tilde{R}^2}\right)\int s_{\tilde{R}}^2 \tag{3.51}$$

Putting these together, we have that

$$\left(\frac{d^2}{R^2} - \frac{d^2}{\tilde{R}^2}\right)\int s_R^2 \leq I_R(F_R) - I_{\tilde{R}}(F_{\tilde{R}}) \leq \left(\frac{d^2}{R^2} - \frac{d^2}{\tilde{R}^2}\right)\int s_{\tilde{R}}^2 \tag{3.52}$$

By (3.50), we have that $I_R(F_R) \leq C$ for all $R \in [r_0, r_1]$. Thus, there exists some constant C so that for all $R \in [r_0, r_1]$, then $\left\|s_R\right\|_2^2 \leq C$. This implies that

$$\lim_{\tilde{R} \to R} I_{\tilde{R}}(F_{\tilde{R}}) = I_R(F_R) \tag{3.53}$$

Similar to (3.50) and (3.51), we have the bound

$$I_R(F_R) \le I_R(F_{\tilde R}) = I_{\tilde R}(F_{\tilde R}) + \frac{1}{2}\left(\frac{d^2}{R^2} - \frac{d^2}{\tilde R^2}\right)\int s_{\tilde R}^2 \tag{3.54}$$

Using (3.53), the fact that $\left\|s_R\right\|_2^2 \le C$ for all $R \in [r_0, r_1]$, and (3.54), we see that if R_k is any sequence in $[r_0, r_1]$ convering to $R \in [r_0, r_1]$, then F_{R_k} is a minimizing sequence of I_R. Using the same arguments found in the proof of proposition 3.3.2, we have that there exists a subsequence of F_{R_k} converging to a minimizer of I_R weakly in $\mathring{H}^1$ and locally uniformly. Since we are assuming that minimizers of I_R are unique, this is the 5^{th} property we are assuming V satisifes (1.22), then in fact we have that if $\tilde R \to R$, then $F_{\tilde R} \to F_R$ weakly in $\mathring{H}^1$ and locally uniformly.

Step 2: In this step, we will show that F_R is differentiable in R.

First, we will show that for all $\xi \in \ker(L_1(F_R;R))^\perp$, there exists a constant $C > 0$ so that

$$I_R(F_R) + \frac{m_1}{2}\left\|\xi\right\|_2^2 - C\left\|\xi'\right\|_2^{1/2}\left\|\xi\right\|_2^{5/2} \le I_R(F_R + \xi) \tag{3.55}$$

where m_1 was defined in (3.23).

This identity is obtained as follows. We have that

$$\begin{aligned}
&I_R(F_R + \xi)\\
&= I_R(F_R) + \int \xi \cdot (-F_R'' + \nabla_\Phi W(F_R, R)) + \frac{1}{2}\int \xi L_1(F_R;R)\xi\\
&\quad + \int \left[W(F_R + \xi, R) - W(F_R, R) - \nabla_\Phi W(F_R, R)\xi - \frac{1}{2}\xi \operatorname{Hess}_\Phi W(F_R, R)\xi\right]\\
&\ge I_R(F_R) + \frac{m_1}{2}\left\|\xi\right\|_2^2 - C\left\|\xi\right\|_3^3
\end{aligned}$$

where we used the spectral estimate (3.22), m_1 was defined in (3.23), and the fact that $-F_R'' + \nabla_\Phi W(F_R, R) = 0$ to obtain the last inequality. Using the Gagliardo-Nirenberg inequality, we have that

$$I_R(F_R) + \frac{1}{2}\left\|\xi\right\|_2^2 - c\left\|\xi'\right\|_2^{1/2}\left\|\xi\right\|_2^{5/2} \le I_R(F_R + \xi)$$

establishing the identity (3.55).

Suppose F_R and $F_{\tilde R}$ minimize I_R and $I_{\tilde R}$, respectively. Then, we have that

$$\begin{aligned}
I_R(F_{\tilde R}) &= I_{\tilde R}(F_{\tilde R}) + \left(\frac{d^2}{R^2} - \frac{d^2}{\tilde R^2}\right)\int s_{\tilde R}^2\\
&\le I_{\tilde R}(F_R) + \left(\frac{d^2}{R^2} - \frac{d^2}{\tilde R^2}\right)\int s_{\tilde R}^2\\
&= I_R(F_R) + \left(\frac{d^2}{R^2} - \frac{d^2}{\tilde R^2}\right)\int \left(s_{\tilde R}^2 - s_R^2\right)
\end{aligned} \tag{3.56}$$

Setting $\xi = F_{\tilde{R}} - F_R$ and using 3.55, we have that

$$\begin{aligned} I_R(F_{\tilde{R}}) &= I_R(F_R + \xi) \\ &\geq I_R(F_R) + \frac{1}{2}\|\xi\|_2^2 - c\|\xi'\|_2^{1/2}\|\xi\|_2^{5/2} \end{aligned}$$

Since $I_R(F_R) < \infty$ for all $R \in [r_0, r_1]$, then $\|F_R'\|_2 \leq C$ for all $R \in [r_0, r_1]$. Also, since F_R is continuous in R, then for all $\tilde{R}$ sufficiently close to R, we have that

$$I_R(F_{\tilde{R}}) \geq I_R(F_R) + C\|\xi\|_2^2 \tag{3.57}$$

where $C > 0$ is some constant. Combining (3.56) and (3.57) and using Cauchy-Schwarz, we have that

$$\begin{aligned} C\|\xi\|_2^2 &\leq \left(\frac{d^2}{R^2} - \frac{d^2}{\tilde{R}^2}\right)\int \left(s_{\tilde{R}}^2 - s_R^2\right) \\ &\leq \frac{d^2(R+\tilde{R})}{R^2\tilde{R}^2}|R-\tilde{R}| \int |s_{\tilde{R}} + s_R||s_{\tilde{R}} - s_R| \\ &\leq \frac{d^2(R+\tilde{R})}{R^2\tilde{R}^2}|R-\tilde{R}| \left[\|s_{\tilde{R}}\|_2 + \|s_R\|_2\right] \|s_{\tilde{R}} - s_R\|_2 \\ &\leq \frac{d^2(R+\tilde{R})}{R^2\tilde{R}^2}|R-\tilde{R}| \left[\|s_{\tilde{R}}\|_2 + \|s_R\|_2\right] \|\xi\|_2 \end{aligned}$$

for all $\tilde{R}$ sufficiently close to R. Again, since $I_R(F_R) < \infty$ for all $R \in [r_0, r_1]$, then $\|s_R\|_2 \leq C$ for all $R \in [r_0, r_1]$. Thus,

$$\|F_{\tilde{R}} - F_R\|_2 \leq C|\tilde{R} - R| \tag{3.58}$$

for some constant C.

Set

$$\Delta_R(\tilde{R}) := \frac{F_{\tilde{R}} - F_R}{\tilde{R} - R}$$

Using parity arguments, we see that $F_{\tilde{R}} - F_R \in \ker(L_1(F_R;R))^\perp$. Thus, $\Delta_R(\tilde{R}) \in \ker(L_1(F_R;R))^\perp$ too. From (3.58), we see that $\|\Delta_R(\tilde{R})\| \leq C$ for all $\tilde{R}$ sufficiently close to R. Furthermore, $\Delta_R(\tilde{R})$ satisfies the following relation

$$L_1(F_R;R)\Delta_R(\tilde{R}) = \frac{d^2(\tilde{R}+R)}{\tilde{R}^2R^2}\begin{pmatrix} 0 \\ s_{\tilde{R}} \end{pmatrix} - \int_0^1 \left[\mathrm{Hess}_\Phi V(F_R + t(F_{\tilde{R}} - F_R)) - \mathrm{Hess}_\Phi V(F_R)\right]\Delta_R(\tilde{R})dt \tag{3.59}$$

With this identity, we obtain the estimate

$$\begin{aligned} &\|\Delta_R(\tilde{R})'\|_2^2 \\ &\leq \left|\left\langle \Delta_R(\tilde{R}), L_1(F_R;R)\Delta_R(\tilde{R})\right\rangle\right| + c\|\Delta_R(\tilde{R})\|_2^2 \\ &\leq \sup_{t\in[0,1]} \left\|\mathrm{Hess}_\Phi V(F_R + t(F_{\tilde{R}} - F_R)) - \mathrm{Hess}_\Phi V(F_R)\right\|_\infty \|\Delta_R(\tilde{R})\|_2^2 + \frac{d^2(\tilde{R}+R)}{\tilde{R}^2R^2}\|s_{\tilde{R}}\|_2\|\Delta_R(\tilde{R})\|_2 + c\|\Delta_R(\tilde{R})\|_2^2 \\ &\leq C \end{aligned}$$

where the last inequality only holds for $\tilde{R}$ sufficiently close to R.

To summarize, we have shown that $\left\|\Delta_R(\tilde{R})\right\|_{H^1} \leq C$ for all $\tilde{R}$ sufficiently close to R. This bound also implies that $\left[\Delta_R(\tilde{R})\right]_{1/2} \leq C$. Thus, for every sequence $\tilde{R}_k \to R$, there exists some $G \in H^1$ and a subsequence of $\tilde{R}_k$ so that $\Delta_R(\tilde{R}_{k_l}) \to G$ weakly in H^1 and locally uniformly. Since each $\Delta_R(\tilde{R}) \perp \ker(L_1(F_R, R))$, then $G \perp \ker(L_1(F_R, R))$ as well. Furthermore, we have that for each fix $\phi \in C_0^\infty$, then

$$\begin{aligned}
0 &= \left\langle \phi, L_1(F_R, R)\Delta_R(\tilde{R}_{k_l}) + \int_0^1 \left[\mathrm{Hess}_\Phi V(F_R + t(F_{\tilde{R}_{k_l}} - F_R)) - \mathrm{Hess}_\Phi V(F_R)\right]\Delta_R(\tilde{R}_{k_l})dt - \frac{d^2(\tilde{R}_{k_l} + R)}{\tilde{R}_{k_l}^2 R^2}\begin{pmatrix} 0 \\ s_{\tilde{R}_{k_l}} \end{pmatrix} \right\rangle \\
&= \left\langle L_1(F_R, R)\phi + \int_0^1 \left[\mathrm{Hess}_\Phi V(F_R + t(F_{\tilde{R}_{k_l}} - F_R)) - \mathrm{Hess}_\Phi V(F_R)\right]\phi dt, \Delta_R(\tilde{R}_{k_l})\right\rangle - \left\langle \phi, \frac{d^2(\tilde{R}_{k_l} + R)}{\tilde{R}_{k_l}^2 R^2}\begin{pmatrix} 0 \\ s_{\tilde{R}_{k_l}} \end{pmatrix} \right\rangle \\
&\to \langle L_1(F_R, R)\phi, G\rangle - \left\langle \phi, 2\frac{d^2}{R^3}\begin{pmatrix} 0 \\ s_R \end{pmatrix} \right\rangle
\end{aligned}$$

In particular, we have that

$$L_1(F_R, R)G = 2\frac{d^2}{R^3}\begin{pmatrix} 0 \\ s_R \end{pmatrix} \tag{3.60}$$

Suppose G_1 and G_2 both solve (3.60), then

$$L_1(F_R, R)(G_1 - G_2) = 0$$

and hence $G_1 - G_2 \in \ker(L_1(F_R, R))$. By assumption 4 of (1.9), this implies that $G_1 - G_2 = \alpha F_0'$ for some constant α. Recall, however, that $G_1, G_2 \perp \ker(L_1(F_R, R))$ and so $\alpha = 0$. Thus, $G_1 = G_2$. This means every sequence $\tilde{R}_k$ has a subseqence for which $\Delta_R(\tilde{R}_{k_l}) \to G$ weakly in H^1 and locally uniformly. In particular, this implies that $\Delta_R(\tilde{R}) \to G$ weakly in H^1 and locally uniformly. In particular, the locally uniform convergence implies that F_0 is differentiable in R for $R \in (r_0, r_1)$.

Since G satisfies (3.60), then standard ODE theory tells us that G is smooth in y^1. This completes step 2.

Step 3: In this step, we will show that F_0 is smooth in y^1 and R. Actually, we will only show that $\partial_R F_0(R)$ is differentiable in R and that $\partial_{RR} F_0(R)$ is smooth in y^1. This argument can then be applied inductively to show that F_0 is smooth in y^1 and R.

Set $G_R = \partial_R F_R$, where $F_R \in \tilde{\mathscr{A}}$ minimizes I_R. In the previous step we showed that $G_R \in L^2$ for all $R \in (r_0, r_1)$. For

$$h(R) := 2\frac{d^2}{R^3}\begin{pmatrix} 0 \\ s_R \end{pmatrix}$$

we have that G_R satisfies

$$L_1(F_R, R)G_R = h(R)$$

For $0 < |\lambda| \ll 1$, we have that

$$L_1(F_{R+\lambda}, R+\lambda)\frac{G_{R+\lambda} - G_R}{\lambda} + \frac{L_1(F_{R+\lambda}, R+\lambda) - L_1(F_R, R)}{\lambda} G_R = \frac{h(R+\lambda) - h(R)}{\lambda} \tag{3.61}$$

This implies that

$$\begin{aligned}
&\Big|\Big\langle \frac{G_{R+\lambda} - G_R}{\lambda}, L_1(F_{R+\lambda}, R+\lambda)\frac{G_{R+\lambda} - G_R}{\lambda}\Big\rangle\Big| \\
&= \Big|\Big\langle \frac{G_{R+\lambda} - G_R}{\lambda}, \frac{h(R+\lambda) - h(R)}{\lambda} - \frac{L_1(F_{R+\lambda}, R+\lambda) - L_1(F_R, R)}{\lambda} G_R\Big\rangle\Big| \\
&\leq \Big\|\frac{G_{R+\lambda} - G_R}{\lambda}\Big\|_2 \Big\|\frac{h(R+\lambda) - h(R)}{\lambda} - \frac{L_1(F_{R+\lambda}, R+\lambda) - L_1(F_R, R)}{\lambda} G_R\Big\|_2
\end{aligned} \tag{3.62}$$

We also have, using parity arguments, that

$$\frac{G_{R+\lambda} - G_R}{\lambda} \perp \ker(L_1(F_{R+\lambda}, R+\lambda))$$

and hence we apply the spectral estimate (3.22) to (3.62) to obtain the following estimate

$$\Big\|\frac{G_{R+\lambda} - G_R}{\lambda}\Big\|_2 \lesssim \Big\|\frac{h(R+\lambda) - h(R)}{\lambda} - \frac{L_1(F_{R+\lambda}, R+\lambda) - L_1(F_R, R)}{\lambda} G_R\Big\|_2$$

Since $h(R)$ and F_R are differentiable with respect to R, we have that for all $0 < |\lambda| \ll 1$, there exists $C > 0$ so that

$$\Big\|\frac{G_{R+\lambda} - G_R}{\lambda}\Big\|_2 \leq C$$

With this L^2 bound, we can also bound the $\mathring{H}^1$ norm of $\frac{G_{R+\lambda} - G_R}{\lambda}$ for sufficiently small λ as

$$\begin{aligned}
&\Big\|\frac{G_{R+\lambda} - G_R}{\lambda}\Big\|_{\mathring{H}^1}^2 \\
&\leq \Big|\Big\langle \frac{G_{R+\lambda} - G_R}{\lambda}, L_1(F_{R+\lambda}, R+\lambda)\frac{G_{R+\lambda} - G_R}{\lambda}\Big\rangle\Big| + c\Big\|\frac{G_{R+\lambda} - G_R}{\lambda}\Big\|_2^2 \\
&\leq \Big\|\frac{G_{R+\lambda} - G_R}{\lambda}\Big\|_2 \Big\|\frac{h(R+\lambda) - h(R)}{\lambda} - \frac{L_1(F_{R+\lambda}, R+\lambda) - L_1(F_R, R)}{\lambda} G_R\Big\|_2 + c\Big\|\frac{G_{R+\lambda} - G_R}{\lambda}\Big\|_2^2 \\
&\leq C
\end{aligned}$$

where we used (3.61) and the L^2 bound of $\frac{G_{R+\lambda} - G_R}{\lambda}$ to obtain this estimate.

We then proceed as we did at the end of step 1 to show that $\frac{G_{R+\lambda} - G_R}{\lambda}$ has a limit as $\lambda \to 0$. This limit is the derivative of G_R with respect to R, which is the second derivative of F_R with respect to R. Furthermore, $\partial_R G_R$ satisfies

$$L_1(F_R, R)\partial_R G_R = \partial_R h(R) - \partial_R L_1(F_R, R) G_R$$

and since $\partial_R h(R)$ and $\partial_R L_1(F_R, R) G_R$ are smooth in y^1, then standard ODE theory implies that $\partial_R G_R$ is smooth in y^1 too. Thus, $\partial_{RR} F_0$ exists and is smooth in y^1.

□

4 Effective Dynamics

4.1 Approximation Using Profiles Coming From the Formal Asymptotics

The question we would now like to answer is: Suppose Φ is a solution to (3.5) with the following properties

- $\Phi = (-1, 0)$ for $y_1 < -y^1_*$
- $\Phi = (1, 0)$ for $y^1 > y^1_*$
- $\Phi(0, y^1)$ is "close" to $F_0(\frac{y^1}{\epsilon}; R(0)) + \epsilon F_1(\frac{y^1}{\epsilon}; R(0), R'(0))$
- $\partial_{y^0}\Phi(0, y^1)$ is "close" to $\partial_{y^0}\big|_{y^0=0}(F_0(\frac{y^1}{\epsilon}; R) + \epsilon F_1(\frac{y^1}{\epsilon}; R, R'))$

Then, does there exists y^0_* independent of ϵ so that if $0 \le y^0 \le y^0_*$, then $\Phi(y^0)$ and $\partial_{y^0}\Phi(y^0)$ are "close" to $F_0(\frac{y^1}{\epsilon}; R) + \epsilon F_1(\frac{y^1}{\epsilon}; R, R')$ and $\partial_{y^0}(F_0(\frac{y^1}{\epsilon}; R) + \epsilon F_1(\frac{y^1}{\epsilon}; R, R'))$, respectively?

Suppose Φ is a solution to (3.4). We will use the translation symmetry of the profiles F_0 to find a function $a : [0, y^0_*] \to \mathbb{R}$ so that for each $y^0 \in [0, y^0_*]$, the $L^2_{y^1}$ distance between $\Phi(y^0)$ and $F_0(\frac{-a(y^0)}{\epsilon}; R(y^0))$ is minimized.

For each $R \in \mathbb{R}$ and $\Psi \in L^2(\mathbb{R}, \mathbb{R}^2)$, define

$$h_\epsilon(\Psi, R, a) := \left\|\Psi - F_0(\frac{y^1 - a}{\epsilon}; R)\right\|^2_{L^2_{y^1}(\mathbb{R})} \tag{4.1}$$

$$G_\epsilon(\Psi, R, a) := \partial_a h_\epsilon(\Psi, R, a) = -\frac{2}{\epsilon}\left\langle \Psi - F_0(\frac{y^1 - a}{\epsilon}; R), \partial_{y^1} F_0(\frac{y^1 - a}{\epsilon}; R)\right\rangle_{L^2(\mathbb{R})} \tag{4.2}$$

For each $y^0 \in [0, y^0_*]$, we want to find a sufficiently regular $a(y^0)$ so that

$$G_\epsilon(\Phi(y^0), R(y^0), a(y^0)) = 0$$

Define

$$U_{\delta,\epsilon} := \left\{(\Psi, R) \in L^2(\mathbb{R}, \mathbb{R}^2) \times \mathbb{R} \mid \inf_{a\in\mathbb{R}} h_\epsilon(\Psi, R, a) < \delta\right\} \tag{4.3}$$

$$V_{\delta,\epsilon}(a_0) := \left\{(\Psi, R) \in L^2(\mathbb{R}, \mathbb{R}^2) \times \mathbb{R} \mid h_\epsilon(\Psi, R, a_0) < \delta\right\} \tag{4.4}$$

Lemma 4.1.1. *There exists $\delta > 0$ and a unique C^∞ map $\tilde{a} : U_{\delta,\epsilon} \to \mathbb{R}$, C^∞ with respect to the $L^2 \times \mathbb{R}$ topology, so that $G_\epsilon(\Psi, R, \tilde{a}(\Psi, R)) = 0$ where both δ and $\tilde{a}$ possibly depend on ϵ.*

Proof: Since

1. $G_{R,\epsilon}$ is C^∞ because it is linear in Ψ and because F_0 is smooth in both y^1 and R
2. $G_\epsilon(F_0(\frac{-a_0}{\epsilon}; R), R, a_0) = 0$

3.
$$\partial_a\Big|_{a=a_0} G_\epsilon(F_0(\frac{\cdot - a_0}{\epsilon};R),R,a) = \left\langle \partial_{y^1}F_0(\frac{\cdot - a_0}{\epsilon};R), \partial_{y^1}F_0(\frac{\cdot - a_0}{\epsilon};R)\right\rangle_{L^2(\mathbb{R})} > 0$$

then we can apply the implicit function. That is, there exists $\delta > 0$ and a unique C^∞ map $a : V_{\delta,\epsilon}(a_0) \to \mathbb{R}$, both δ and a possibly depending on ϵ, so that $G_\epsilon(\Psi, R, a(\Psi, R)) = 0$ for all $(\Psi, R) \in V_{\delta,\epsilon}(a_0)$.

Observe that
$$U_{\delta,\epsilon} = \bigcup_{b\in\mathbb{R}} V_{\delta,\epsilon}(a_0 + b)$$
For each $(\Psi, R) \in U_{\delta,\epsilon}$ there exists $b \in \mathbb{R}$ so that for $\tau_b\Psi := \Psi(\cdot - b)$, then $(\tau_b\Psi, R) \in V_{\delta,\epsilon}(a_0)$. Define $\tilde{a}_b(\Psi, R) := a(\tau_b\Psi, R) - b$, then $G_\epsilon(\Psi, R, \tilde{a}_b(\Psi, R)) = 0$. If $(\tau_b\Psi, R), (\tau_c\Psi, R) \in V_{\delta,\epsilon}(a_0)$, then by the uniqueness of a one has that $\tilde{a}_b(\Psi, R) = \tilde{a}_c(\Psi, R)$. Thus, one can find a unique $\tilde{a} : U_{\delta,\epsilon} \to \mathbb{R}$ so that $G_\epsilon(\Psi, R, \tilde{a}(\Psi, R)) = 0$.

□

Suppose Φ is a solution to (1.8) and that $(\Phi(0), R(0)) \in U_{\delta,\epsilon}$ with $\delta > 0$ coming from lemma 4.1.1. Then, there exists some maximal $0 < y^0_*(\epsilon)$ so that $(\Phi(y^0), R(y^0)) \in U_{\delta,\epsilon}$ for all $0 \leq y^0 \leq y^0_*(\epsilon)$. We emphasize here that because $U_{\delta,\epsilon}$ depends on ϵ, then $y^0_*(\epsilon)$ may or may not depend on ϵ. Thus,
$$G_\epsilon(\Phi(y^0), R(y^0), \tilde{a}(\Phi(y^0), R(y^0))) = 0$$
for $0 \leq y^0 \leq y^0_*(\epsilon)$. While proving theorem 1.2.2 we will actually show that $y^0_*(\epsilon)$ does not depend on ϵ. We will see later that if $y^0_*(\epsilon)$ does depend on ϵ, then $(\Phi(y^0_*(\epsilon), R(y^0_*(\epsilon)) \in U_{\delta,\epsilon}$ which contradicts the maximality of $y^0_*(\epsilon)$.

Corollary 4.1.2. *Suppose Φ solves (1.8) and suppose that $(\Phi(0), R(0)) \in U_{\delta,\epsilon}$ with $\delta > 0$ from lemma 4.1.1. Then, there exists $y^0_*(\epsilon) > 0$ and a unique C^∞ function $a(y^0)$ so that $G_\epsilon(\Phi(y^0), R(y^0), a(y^0)) = 0$ for all $0 \leq y^0 \leq y^0_*(\epsilon)$.*

Given a solution Φ to (1.8), we define
$$\tilde{F}_0(y^0, y^1) = F_0(\frac{y^1 - a(y^0)}{\epsilon}; R(y^0)) \tag{4.5}$$
where $a(y^0)$ is from corollary 4.1.2.

Examining (3.16), we see that F_1 does not have a translation symmetry in y^1 as the inhomogeneity of (3.16) depends explicity on y^1. Instead, we have that for $v = R'$, then $F_1(\frac{y^1-a}{\epsilon}; R, v)$ from proposition 3.3.4 solves
$$\epsilon L_\epsilon(\tilde{F}_0; R)\left[F_1(\frac{y^1 - a}{\epsilon}; R, R')\right] = H(R)\partial_{y^1}\tilde{F}_0 - \frac{1}{\epsilon}m(y^0)(\frac{y^1 - a}{\epsilon})\partial_R w(\tilde{F}_0, R) \tag{4.6}$$
where m was defined (3.2) and $H(R)$ was defined in (3.18). Remember that F_1 was defined independent of ϵ. We define
$$\tilde{F}_1(y^0, y^1) = F_1(\frac{y^1 - a(y^0)}{\epsilon}; R(y^0), R'(y^0)) \tag{4.7}$$

For our result, we will need to control two quantities. The first is the error and the second is the **shift** $a(y^0)$. Define the **error** between Φ and our approximation $\tilde{F}_0 + \epsilon\tilde{F}_1$ as

$$\xi := \Phi - \tilde{F}_0 - \epsilon\tilde{F}_1 \tag{4.8}$$

Also, define the quantity

$$\underline{A}(y^0) := \left(1 + \frac{\left|a(y^0)\right|}{\epsilon} + \frac{\left|a'(y^0)\right|}{\epsilon}\right)^3 \tag{4.9}$$

An observation that we will make use of later is the following. Since $0 = \partial_a h(a(y^0))$, we can use (4.2) to get that

$$\begin{aligned} 0 &= \int_{\mathbb{R}} (\xi + \epsilon\tilde{F}_1) \cdot \partial_{y^1}\tilde{F}_0 \\ &= \int_{\mathbb{R}} \xi\partial_{y^1}\tilde{F}_0 + \epsilon\int_{\mathbb{R}} \tilde{F}_1 \cdot \partial_{y^1}\tilde{F}_0 \\ &= \int_{\mathbb{R}} \xi \cdot \partial_{y^1}\tilde{F}_0 \end{aligned}$$

where we needed to use the fact that $\tilde{F}_1 \perp \partial_{y^1}\tilde{F}_0$, from proposition 3.3.4, to go from the second line to the third. That is, we have that $\xi \perp \partial_{y^1}\tilde{F}_0$.

When we will plug $\Phi = \tilde{F}_0 + \epsilon\tilde{F}_1 + \xi$ into (3.5), we find that ξ solves

$$\frac{m^2}{n^2}\partial_{y^0y^0}\xi + B^\alpha\partial_\alpha\xi + L_\epsilon(\tilde{F}_0, R)\xi + S_{-1} + S_0 + N = 0 \tag{4.10}$$

where we used the fact that

$$-\partial_{y^1y^1}\tilde{F}_0 + \frac{1}{\epsilon^2}w(\tilde{F}_0, R) = 0$$

to simplify and for $\tilde{F}_\xi = \epsilon\tilde{F}_1 + \xi$ we defined

$$S_{-1} := \epsilon L_\epsilon(\tilde{F}_0, R)\tilde{F}_1 + B^1\partial_{y^1}\tilde{F}_0 + \frac{y^1}{\epsilon^2}m(y^0)\partial_R w(\tilde{F}_0, R) \tag{4.11}$$

$$S_0 := \frac{m^2}{n^2}\partial_{y^0y^0}(\tilde{F}_0 + \epsilon\tilde{F}_1) + B^0\partial_{y^0}(\tilde{F}_0 + \epsilon\tilde{F}_1) + \epsilon B^1\partial_{y^1}\tilde{F}_1 \tag{4.12}$$

$$N := \frac{1}{\epsilon^2}\left[w(\tilde{F}_0 + \tilde{F}_\xi, R + y^1 m) - w(\tilde{F}_0, R) - \mathrm{Hess}_\Phi\, W(\tilde{F}_0, R)\tilde{F}_\xi - y^1 m(y^0)\partial_R w(\tilde{F}_0, R)\right] \tag{4.13}$$

We would have liked to use (4.10) to show that if ξ starts off small, then it remains small under the evolution of (4.10). However, this equation only makes sense on $(0, y^0_*) \times (-y^1_*, y^1_*)$. By picking the initial data of Φ as we did in section 1.3, we are able to use the finite speed of propogation property of the wave equation to control the size of ξ on $[0, y^0_*) \times \left\{ y^1 \;\middle|\; |y^1| > y^1_* \right\}$. More precisely, since Φ and $\tilde{F}_0 + \epsilon \tilde{F}_\xi$ are both defined on $(0, y^0_*) \times \mathbb{R}$, then we actually have that ξ is defined on this set too even though ξ only satisfies (4.10) for $(0, y^0_*) \times (-y^1_*, y^1_*)$. Furthermore, since for $0 \leq y^0 \leq y^0_*$ we have that $\Phi(y^0) = (1, 0)$ for $y^1 > y^1_*$ and $\Phi(y^0) = (-1, 0)$ for $y^1 < -y^1_*$, then outside of $\left\{ y^1 \;\middle|\; |y^1| \leq y^1_* \right\}$ we can use the asymptotics derived in proposition 3.5.1 to show that

$$\begin{aligned} \left\| \xi \right\|_{L^2(|y^1| > y^1_*)} &\leq \left\| \begin{pmatrix} \frac{y^1}{|y^1|} \\ 0 \end{pmatrix} - \tilde{F}_0 \right\|_{L^2(|y^1| > y^1_*)} + \epsilon \left\| \tilde{F}_1 \right\|_{L^2(|y^1| > y^1_*)} \\ &\lesssim e^{-\alpha \frac{y^1_* - a}{\epsilon}} \end{aligned} \tag{4.14}$$

for some $\alpha > 0$. Thus, if we can control the size of a and if ϵ is taken sufficiently small, then we have that $\xi(y^0)$ is small outside of $\left\{ y^1 \;\middle|\; \left| y^1 \right| \leq y^1_* \right\}$ for $y^0 \in [0, y^0_*)$.

We are then left to show that if ξ starts off small, then ξ remains small on $[0, y^0_*) \times (-y^1_*, y^1_*)$. We will use the following quatities to control the size of ξ on this set.

Definition 4.1.3. *For $Q = (Q_1, Q_2) : (0, y^0_*) \times (-y^1_*, y^1_*) \to \mathbb{R}^2$ define the energy density*

$$e(Q) = \frac{1}{2} \frac{m^2}{n^2} \partial_{y^0} Q^2 + \frac{1}{2} \partial_{y^1} Q^2 + \frac{1}{2\epsilon^2} Q \cdot \mathrm{Hess}_\Phi W(\tilde{F}_0, R) Q \tag{4.15}$$

Using the energy density, we define the energy of Q as

$$E(Q) = \int_{|y^1| \leq y^1_*} e(Q) dy^1 \tag{4.16}$$

For convenience we set $E(y^0) := E(\xi)(y^0)$.

Using this new definition, we obtain a very useful corollary to theorem 3.2.1 that we will use to control the error term ξ

Corollary 4.1.4. *Suppose Φ is a solution to (3.5) with initial data as described in section 1.3. For ξ as defined in (4.8), we have that*

$$\frac{1}{\epsilon^2} \int_{-y^1_*}^{y^1_*} |\xi|^2 \lesssim E + \frac{1}{\epsilon^2} e^{-\alpha \frac{y^1_* - a}{\epsilon}} \tag{4.17}$$

for some $\alpha > 0$

Proof of corollary 4.1.4: Use theorem 3.2.1 along with (4.14) to obtain the estimate.

□

4.2 Main Theorem

The main theorem we want to prove is the following.

Theorem 4.2.1. *Suppose that* Φ *solves (3.5) and suppose that* $\Phi(0)$ *satifies the conditions outlined in section 1.3. Further assume that* $\Phi(0)$, $\partial_{y^0}\Phi(0)$, $a(0)$, *and* $a'(0)$ *satisfy*

$$\underline{A}(0) \lesssim 1 \quad \textit{and} \quad E(0) \lesssim \epsilon^2$$

Then there exists $0 < \overline{y}^0 < y^0_*$, $\overline{y}^0$ *independent of* ϵ, *and* $a : (0, \overline{y}^0) \to \mathbb{R}$ *so that*

$$\underline{A}(y^0) \lesssim 1 \quad \textit{and} \quad E(y^0) \lesssim \epsilon^2$$

for all $0 \leq y^0 \leq \overline{y}^0$.

Theorem 1.2.2 is then obtained from this by applying the spectral estimate (3.22) to the estimate $E(y^0) \lesssim \epsilon^2$.

The proof of theorem 4.2.1 relies on the following two estimates

Theorem 4.2.2. *Suppose that* Φ *solves (3.5) and that* $\Phi(0)$ *satisfies the conditions outlined in section 1.3. Then, for as long as* $a(y^0)$ *is well defined we have*

$$\begin{aligned} E \lesssim E(0) &+ \left[\frac{y^1_*}{(R - y^1_* m)^3} + \frac{(y^1_*)^2}{(R - y^1_* m)^4}\right] E\Big|_0^{y^0} + \epsilon^{1/2} e^{-\alpha\frac{y^1_*-a}{\epsilon}} \underline{A}\Big|_0^{y^0} \\ &+ (\epsilon + (\epsilon E^{1/2} + e^{-\alpha\frac{y^1_*-a}{\epsilon}})^{1/2} E^{1/4})(\epsilon^{3/2} + \epsilon E^{1/2} + e^{-\alpha\frac{y^1_*-a}{\epsilon}})\underline{A}\frac{E^{1/2}}{\epsilon}\Big|_0^{y^0} \\ &+ \frac{1}{\epsilon^2}\int_0^{y^0} (1 + \epsilon + (\epsilon E^{1/2} + e^{-\alpha\frac{y^1_*-a}{\epsilon}})(\epsilon^2 + \epsilon E^{1/2} + e^{-\alpha\frac{y^1_*-a}{\epsilon}})(\epsilon E^{1/2} + e^{-\alpha\frac{y^1_*-a}{\epsilon}})\underline{A} \end{aligned} \tag{4.18}$$

Theorem 4.2.3. *Suppose that* Φ *solves (3.5) and that* $\Phi(0)$ *satisfies the conditions outlined in section 1.3. Then, for as long as* $a(y^0)$ *is well defined we have*

$$\begin{aligned} \frac{1}{\epsilon}|a''| \lesssim \frac{1}{\epsilon}|a''|\left[\epsilon\underline{A} + \sqrt{\epsilon}\underline{A}E^{1/2} + e^{-\alpha\frac{y^1_*-a}{\epsilon}}\right] \\ + \left[\frac{1}{\epsilon^{5/2}}(\epsilon\underline{A} + (\epsilon E^{1/2} + e^{-\alpha\frac{y^1_*-a}{\epsilon}})^{1/2}E^{1/4})(\epsilon^{3/2} + \epsilon E^{1/2} + e^{-\alpha\frac{y^1_*-a}{\epsilon}})\right] \end{aligned} \tag{4.19}$$

Proof of Theorem 4.2.1: In order to close the argument to be outlined, we may need to choose y^0_* and y^1_* smaller so that

$$\frac{y^1_*}{(R - y^1_* m)^3} \leq \frac{1}{4} \quad \text{and} \quad \frac{(y^1_*)^2}{(R - y^1_* m)^4} \leq \frac{1}{4} \tag{4.20}$$

We will also make use of the following two estimates

$$\begin{aligned} \left|a(y^0)\right| &\leq |a(0)| + \int_0^{y^0} |a'| \\ \left|a'(y^0)\right| &\leq |a'(0)| + \int_0^{y^0} |a''| \end{aligned} \tag{4.21}$$

Next, suppose $a(y^0)$ is well defined on the interval $I = (0, b)$. Define

$$E_M(I) := \max_{y^0 \in I} E(y^0) \quad \text{and} \quad \underline{A}_M(I) := \max_{y^0 \in I} \underline{A}(y^0)$$

Using theorem 4.2.2 we have that

$$\begin{aligned} E_M(I) \;\lesssim\; & E(0) + \tfrac{1}{2}E_M(I) + \epsilon^{3/2}\underline{A}_M(I)E_M(I)^{1/2} + \epsilon^{1/2}E_M(I)^{3/2} + \epsilon^{1/2}e^{-\frac{\alpha}{\epsilon}y_*^1}e^{\alpha\underline{A}_M(I)} \\ & + \tfrac{1}{\epsilon}E_M(I)^{1/2}e^{-\frac{\alpha}{\epsilon}y_*^1}e^{\alpha\underline{A}_M(I)} + |I|\left[E_M(I) + \epsilon E_M(I)^{1/2} + \tfrac{1}{\epsilon^2}e^{-\frac{\alpha}{\epsilon}y_*^1}e^{\alpha\underline{A}_M(I)}\right]A_M(I) \end{aligned} \tag{4.22}$$

Using theorem 4.2.3 and (4.21), we also have that

$$\underline{A}_M(I)^{1/3} \lesssim (1 + |I|)\underline{A}(0)^{1/3} + \frac{|I| + |I|^2}{1 - \epsilon\underline{A}_M(I) - \sqrt{\epsilon}E_M(I)^{1/2}\underline{A}_M(I) + e^{-\frac{\alpha}{\epsilon}y_*^1}e^{-\alpha\underline{A}_M(I)}}\underline{B}_M(I) \tag{4.23}$$

where $\underline{B}(I)$ is defined as

$$\underline{B}_M(I) \;:=\; \tfrac{1}{\epsilon^{5/2}}\left[\epsilon\underline{A}_M(I) + (\epsilon E_M(I)^{1/2} + e^{-\frac{\alpha}{\epsilon}y_*^1}e^{\alpha\underline{A}_M(I)})^{1/2}E_M(I)^{1/4}\right]\left[\epsilon^{3/2} + \epsilon E_M(I)^{1/2} + e^{-\frac{\alpha}{\epsilon}y_*^1}e^{\alpha\underline{A}_M(I)}\right] \tag{4.24}$$

By corollary 4.1.2, $a(y^0)$ is well defined up to some time $\widehat{y}^0$, where $\widehat{y}^0$ may or may not depend on ϵ. Using (4.22) and (4.23), we see that there exists $\overline{y}^0$, independent of ϵ, so that for $I = \left(0, \min\left\{\widehat{y}^0, \overline{y}^0\right\}\right)$

$$E_M(I) \lesssim \epsilon^2 \quad \text{and} \quad \underline{A}_M(I) \lesssim 1$$

If $\min\left\{\widehat{y}^0, \overline{y}^0\right\} = \overline{y}^0$, then we are done. If not, then because

$$\underline{E}_M(I) \lesssim \epsilon^2$$

we can use corollary 4.1.2 to show that $a(y^0)$ actually exists beyond $\widehat{y}^0$. Boot strapping then allows us to conclude that $a(y^0)$ exists and is well defined on $I = \left(0, \overline{y}^0\right)$ and that

$$E_M(I) \lesssim \epsilon^2 \quad \text{and} \quad \underline{A}_M(I) \lesssim 1$$

on I.

□

4.3 Proof of Energy Estimate (Theorem 4.2.2)

We will need to estimate $(y^1)^\gamma \partial_{y^1}^\alpha \partial_{y^0}^\beta \tilde{F}_i$, for α, β, $\gamma \in \mathbb{N} \cup \{0\}$, in ordre to prove this theorem and theorem 4.2.3. A point on notation before continuing. We have that $F_1 = F_1(y^1; R, v)$ where the third slot of F_1 is called v.

Lemma 4.3.1. *For α, β, κ, and $\gamma \in \mathbb{N} \cup \{0\}$, then as long as $a(y^0)$ exists, we have*

$$\left\|(y^1)^\gamma (\frac{\partial}{\partial y^1})^\alpha (\frac{\partial}{\partial R})^\beta \left[F_0(\frac{\cdot - a}{\epsilon}; R)\right]\right\|_2 \lesssim \frac{1}{\epsilon^{\alpha-\gamma-\frac{1}{2}}}\left(1 + \frac{|a(y^0)|}{\epsilon}\right)^\gamma \tag{4.25}$$

$$\left\|(y^1)^\gamma (\frac{\partial}{\partial y^1})^\alpha (\frac{\partial}{\partial R})^\beta (\frac{\partial}{\partial v})^\kappa \left[F_1(\frac{\cdot - a}{\epsilon}; R, v)\right]\right\|_2 \lesssim \frac{1}{\epsilon^{\alpha-\gamma-\frac{1}{2}}}\left(1 + \frac{|a(y^0)|}{\epsilon}\right)^\gamma \tag{4.26}$$

where the constant in the estimate depends on y^0_*, but not ϵ.

Proof of lemma 4.3.1: For (4.25) we have

$$\begin{aligned}
\left\|(y^1)^\gamma (\frac{\partial}{\partial y^1})^\alpha (\frac{\partial}{\partial R})^\beta \left[F_0(\frac{\cdot - a}{\epsilon}; R)\right]\right\|_2 &= \frac{1}{\epsilon^\alpha}\left\|(y^1)^\gamma \frac{\partial^{\alpha+\beta} F_0}{\partial (y^1)^\alpha \partial R^\beta}(\frac{\cdot - a}{\epsilon}; R)\right\|_2 \\
&= \frac{1}{\epsilon^{\alpha-\gamma}}\left\|\left(\frac{y^1 - a + a}{\epsilon}\right)^\gamma \frac{\partial^{\alpha+\beta} F_0}{\partial (y^1)^\alpha \partial R^\beta}(\frac{\cdot - a}{\epsilon}; R)\right\|_2 \\
&= \frac{1}{\epsilon^{\alpha-\gamma}}\left\|\left[\sum_{j=0}^{\gamma}\binom{\gamma}{j}\left(\frac{y^1 - a}{\epsilon}\right)^{\gamma-j}\left(\frac{a}{\epsilon}\right)^j\right]\frac{\partial^{\alpha+\beta} F_0}{\partial (y^1)^\alpha \partial R^\beta}(\frac{\cdot - a}{\epsilon}; R)\right\|_2 \\
&\lesssim \frac{1}{\epsilon^{\alpha-\gamma}}\sum_{j=0}^{\gamma}(\frac{|a|}{\epsilon})^j\left\|\left(\frac{y^1 - a}{\epsilon}\right)^{\gamma-j}\frac{\partial^{\alpha+\beta} F_0}{\partial (y^1)^\alpha \partial R^\beta}(\frac{\cdot - a}{\epsilon}; R)\right\|_2 \\
&\lesssim \frac{1}{\epsilon^{\alpha-\gamma-\frac{1}{2}}}\left(1 + \frac{|a(y^0)|}{\epsilon}\right)^\gamma
\end{aligned}$$

where we did a change of variables and used the exponential decay of F_0 and it's derivatives to obtain the last inequality.

We estimate (4.26) in the same way.

□

We will use lemma 4.3.1 to obtain the more useful estimates

Corollary 4.3.2. *For α, $\gamma \in \mathbb{N} \cup \{0\}$ and $\beta = 0, 1, 2$, then as long as $a(y^0)$ exists we have for $i = 1, 2$*

$$\left\|(y^1)^\gamma (\frac{\partial}{\partial y^1})^\alpha (\frac{\partial}{\partial y^0})^\beta \tilde{F}_i\right\|_2 \lesssim \frac{1}{\epsilon^{\alpha-\gamma-1/2}}(1 + \delta^{\beta 2}\left|\frac{a''}{\epsilon}\right|)(1 + \frac{|a|}{\epsilon} + \left|\frac{a'}{\epsilon}\right|)^{\gamma+\beta} \tag{4.27}$$

where δ^{ij} is the Kronecker-delta and the constant in the estimate depends on y^0_*, but not ϵ.

Proof of corollary 4.3.2:

$\beta = 0$: This immediately follows from lemma 4.3.1.

$\beta = 1$: We have that

$$\frac{\partial}{\partial y^0}\left[\tilde{F}_0\right] = -\frac{a'}{\epsilon}\frac{\partial F_0}{\partial y^1} + R'\,\frac{\partial F_0}{\partial R} \tag{4.28}$$

$$\frac{\partial}{\partial y^0}\left[\tilde{F}_1\right] = -\frac{a'}{\epsilon}\frac{\partial F_1}{\partial y^1} + R'\frac{\partial F_1}{\partial R} + R''\frac{\partial F_1}{\partial v} \tag{4.29}$$

where F_0 and all of its partial derivatives are evaluated at $(\frac{y^1-a}{\epsilon}; R)$ and F_1 and all of its partial derivatives evaluated at $(\frac{y^1-a}{\epsilon}; R, R')$ (we supress the arguments of these quantities for notational convenience). Estimating $(y^1)^\gamma \partial_{y^0}\partial^\alpha_{y^1}\tilde{F}_0$ first, we have

$$\begin{aligned}\left\|(y^1)^\gamma \frac{\partial}{\partial y^0}(\frac{\partial}{\partial y^1})^\alpha \tilde{F}_0\right\|_2 \;&\lesssim\; \left\|(y^1)^\gamma a'(\frac{\partial}{\partial y^1})^{\alpha+1}\left[F_0(\frac{\cdot - a}{\epsilon}; R)\right]\right\|_2 + \left\|(y^1)^\gamma R'(\frac{\partial}{\partial y^1})^\alpha \frac{\partial}{\partial R}\left[F_0(\frac{\cdot - a}{\epsilon}; R)\right]\right\|_2 \\ &\lesssim\; \frac{1}{\epsilon^{\alpha-\gamma-1/2}}\left(1 + \frac{|a|}{\epsilon} + \frac{|a'|}{\epsilon}\right)^{\gamma+1}\end{aligned}$$

where we used (4.25) to obtain the last inequality. Estimating $(y^1)^\gamma \partial_{y^0}\partial^\alpha_{y^1}\tilde{F}_1$ in the same way, we find that

$$\left\|(y^1)^\gamma \partial^\alpha_{y^1}\partial^\beta_{y^0}\tilde{F}_1\right\|_2 \;\lesssim\; \frac{1}{\epsilon^{\alpha-\gamma-1/2}}\left(1 + \frac{|a|}{\epsilon} + \frac{|a'|}{\epsilon}\right)^{\gamma+1}$$

$\beta = 2$: We have that

$$\frac{\partial^2}{\partial (y^0)^2}\left[\tilde{F}_0\right] = -\frac{a''}{\epsilon}\frac{\partial F_0}{\partial y^1} + (\frac{a'}{\epsilon})^2\frac{\partial^2 F_0}{\partial (y^1)^2} + 2R'\frac{a'}{\epsilon}\frac{\partial^2 F_0}{\partial y^1 \partial R} + (R')^2\frac{\partial^2 F_0}{\partial R^2} + R''\frac{\partial F_0}{\partial R} \tag{4.30}$$

$$\begin{aligned}\frac{\partial}{\partial (y^0)^2}\left[\tilde{F}_1\right] = &-\frac{a''}{\epsilon}\frac{\partial F_1}{\partial y^1} + \left(\frac{a'}{\epsilon}\right)^2\frac{\partial^2 F_1}{\partial (y^1)^2} - 2R'\frac{a'}{\epsilon}\frac{\partial^2 F_1}{\partial y^1 \partial R} \\ &- 2R''\frac{a'}{\epsilon}\frac{\partial^2 F_1}{\partial y^1 \partial v} + R''\frac{\partial F_1}{\partial R} + (R')^2\frac{\partial^2 F_1}{\partial R^2} + 2R'R''\frac{\partial^2 F_1}{\partial R \partial v} \\ &+ 2R'R'''\frac{\partial^2 F_1}{\partial R \partial v} + R'''\frac{\partial F_1}{\partial v} + (R'')^2\frac{\partial^2 F_1}{\partial v^2}\end{aligned} \tag{4.31}$$

where again F_0 and all of its partial derivatives are evaluated at $(\frac{y^1-a}{\epsilon}; R)$ and F_1 and all of its partial derivatives evaluated at $(\frac{y^1-a}{\epsilon}; R, R')$. The arguments used in the $\beta = 1$ case can again be used to obtain the estimates

$$\left\|(y^1)^\gamma \partial^2_{y^0}\partial^\alpha_{y^1}\tilde{F}_0\right\|_2 \;\lesssim\; \frac{1}{\epsilon^{\alpha-\gamma-1/2}}(1 + \left|\frac{a''}{\epsilon}\right|)(1 + \frac{|a|}{\epsilon} + \left|\frac{a'}{\epsilon}\right|)^{\gamma+2}$$

$$\left\|(y^1)^\gamma \partial^2_{y^0}\partial^\alpha_{y^1}\tilde{F}_1\right\|_2 \;\lesssim\; \frac{1}{\epsilon^{\alpha-\gamma-1/2}}(1 + \left|\frac{a''}{\epsilon}\right|)(1 + \frac{|a|}{\epsilon} + \left|\frac{a'}{\epsilon}\right|)^{\gamma+2}$$

□

To begin the energy estimate, we will use the following divergence identity.

Lemma 4.3.3.

$$\partial_{y^0}\xi \cdot \left[\frac{m^2}{n^2}\partial_{y^0y^0}\xi + B^\alpha\partial_\alpha\xi + L_\epsilon(\tilde{F}_0, R)\xi\right] = div_{y^0,y^1}\overrightarrow{X} + Y \tag{4.32}$$

where

$$\overrightarrow{X} = (e(\xi), -\partial_{y^0}\xi \cdot \partial_{y^1}\xi)$$

$$Y = -\frac{1}{\epsilon^2}\xi \cdot \left[\partial_{y^0}\,\mathrm{Hess}_\Phi\, W(\tilde{F}_0, R)\right]\xi + B^\alpha\partial_{y^0}\xi \cdot \partial_\alpha\xi - \frac{1}{2}\partial_{y^0}(\frac{m^2}{n^2})\partial_{y^0}\xi^2$$

We omit the proof of lemma 4.3.3 as the proof is a straightforward computation. Using the divergence identity (4.32) and (4.10), we have

$$\begin{aligned}
\partial_{y^0}E &= \int\limits_{|y^1|\le y^1_*} \partial_{y^0}e \\
&= \int\limits_{|y^1|\le y^1_*} div_{y^0,y^1}\overrightarrow{X} + \partial_{y^0}\xi \cdot \partial_{y^1}\xi\Big|_{-y^1_*}^{y^1_*} \\
&= -\int\limits_{|y^1|\le y^1_*} \partial_{y^0}\xi \cdot [S_{-1} + S_0 + N] - \int\limits_{|y^1|\le y^1_*} Y + \partial_{y^0}\xi \cdot \partial_{y^1}\xi\Big|_{-y^1_*}^{y^1_*}
\end{aligned}$$

We then integrate $\partial_{y^0}E$ with respect to y^0 to get

$$E(y^0) - E(0) = -\int_0^{y^0}\int\limits_{|y^1|\le y^1_*} \partial_{y^0}\xi \cdot [S_{-1} + S_0 + N] - \int_0^{y^0}\int\limits_{|y^1|\le y^1_*} Y + \int_0^{y^0} \partial_{y^0}\xi \cdot \partial_{y^1}\xi\Big|_{-y^1_*}^{y^1_*} \tag{4.33}$$

We will use this energy identity in order to establish the estimate (4.18).

We will break the analysis up to simplify things. We will estimate each term on the right hand side of (4.33) individually and then add all of the individual estimates back up to obtain the energy estimate.

Lemma 4.3.4.

$$-\int_0^{y^0}\int\limits_{|y^1|\le y^1_*} \partial_{y^0}\xi \cdot S_{-1} \lesssim (\epsilon E^{1/2} + e^{-\alpha\frac{y^1_*-a}{\epsilon}})\sqrt{\epsilon}\underline{A}\Big|_0^{y^0} + \int_0^{y^0}(\epsilon E^{1/2} + e^{-\alpha\frac{y^1_*-a}{\epsilon}})\sqrt{\epsilon}\underline{A} \tag{4.34}$$

Proof of lemma 4.3.4: Recall that $\tilde{F}_1$ solves

$$\epsilon L_\epsilon(\tilde{F}_0, R)\tilde{F}_1 = H(R)\partial_{y^1}\tilde{F}_0 - (\frac{y^1 - a(y^0)}{\epsilon})m(y^0)\partial_R w(\tilde{F}_0, R)$$

Using (3.7) and (3.18), we see that

$$B^1 = -H(R) + O(y^1)$$

Upon examining the definition of S_{-1}, see (4.11), we see that

$$S_{-1} = \left[H(R) + B^1\right]\partial_{y^1}\tilde{F}_0 + \frac{a(y^0)}{\epsilon}m(y^0)\partial_R w(\tilde{F}_0, R) \tag{4.35}$$

Integrating by parts in y^0 we have

$$-\int_0^{y^0}\int_{|y^1|\le y^1_*}\partial_{y^0}\xi\cdot S_{-1} = -\int_{|y^1|\le y^1_*}\xi\cdot S_{-1}\Big|_0^{y^0} + \int_0^{y^0}\int_{|y^1|\le y^1_*}\xi\cdot\partial_{y^0}S_{-1} \tag{4.36}$$

For $j = 0, 1$ we have

$$\begin{aligned}\int_{-y^1_*}^{y^1_*}\xi\cdot\partial^j_{y^0}S_{-1} &\lesssim \left(\int_{-y^1_*}^{y^1_*}|\xi|^2\right)^{1/2}\left\|\partial^j_{y^0}S_{-1}\right\|_2 \\ &\lesssim (\epsilon E^{1/2} + e^{-\alpha\frac{y^1_*-a}{\epsilon}})\left\|\partial^j_{y^0}S_{-1}\right\|_2\end{aligned} \tag{4.37}$$

We estimate $\left\|S_{-1}\right\|_2$ first as

$$\begin{aligned}\left\|S_{-1}\right\|_2 &\lesssim \left\|(y^1)\partial_{y^1}\tilde{F}_0\right\|_2 + \left\|\frac{a(y^0)}{\epsilon}m(y^0)\partial_R w(\tilde{F}_0, R)\right\|_2 \\ &\lesssim \sqrt{\epsilon}\underline{A}\end{aligned} \tag{4.38}$$

where we used corollary 4.3.2 to obtain the last inequality.

We estimate the second term of (4.36) as

$$\begin{aligned}\left\|\partial_{y^0}S_{-1}\right\|_2 &\lesssim \left\|\partial_{y^0}\left(H(R) + B^1\right)\partial_{y^1}\tilde{F}_0\right\|_2 + \left\|\left(H(R) + B^1\right)\partial_{y^0}\partial_{y^1}\tilde{F}_0\right\|_2 + \left\|\frac{a'}{\epsilon}\partial_{y^0}m(y^0)\ \partial_R w(\tilde{F}_0, R)\right\|_2 \\ &\quad + \left\|\frac{a(y^0)}{\epsilon}m(y^0)\nabla_\Phi\partial_R w(\tilde{F}_0, R)\cdot\partial_{y^0}\tilde{F}_0\right\|_2 + \left\|\frac{a(y^0)}{\epsilon}m(y^0)R'\partial_{RR}w(\tilde{F}_0, R)\right\|_2 \\ &\lesssim \sqrt{\epsilon}\underline{A}\end{aligned} \tag{4.39}$$

where we again used corollary 4.3.2 to obtain the last inequality. Combining (4.37), (4.38), and (4.39) finishes the proof.

$\square$

Lemma 4.3.5.

$$-\int_0^{y^0}\int_{|y^1|\leq y^1_*} \partial_{y^0}\xi\cdot S_0 \lesssim (\epsilon E^{1/2}+e^{-\alpha\frac{y^1-a}{\epsilon}})\sqrt{\epsilon}\underline{A}\Big|_0^{y^0} + \int_0^{y^0}(1+\frac{|a''|}{\epsilon})(\epsilon E^{1/2}+e^{-\alpha\frac{y^1-a}{\epsilon}})\sqrt{\epsilon}\underline{A} \tag{4.40}$$

Proof of lemma 4.3.5: Using the definition of S_0, see (4.12), we see that

$$\int_0^{y^0}\int_{|y^1|\leq y^1_*} \partial_{y^0}\xi\cdot S_0 = \int_0^{y^0}\int_{|y^1|\leq y^1_*} \partial_{y^0}\xi\cdot\left[\frac{m^2}{n^2}\partial_{y^0y^0}(\tilde{F}_0+\epsilon\tilde{F}_1)+B^0\partial_{y^0}(\tilde{F}_0+\epsilon\tilde{F}_1)+\epsilon B^1\partial_{y^1}\tilde{F}_1\right]$$

We would like to integrate by parts in y^0 to move the derivative from ξ to S_0 and use corollaries 4.1.4 and 4.3.2 to estimate. However, then we would have to estimate $\partial_{y^0y^0y^0}(\tilde{F}_0+\epsilon\tilde{F}_1)$ which will give rise to $\partial_{y^0y^0y^0}a$ terms, which we would rather avoid. So, we need to take special care when estimating these two problematic terms and proceed as we would like to for the other terms.

1. $\partial_{y^0y^0}\tilde{F}_0$ term: Recall that $\tilde{F}_0 = F_0(\frac{y^1-a(y^0)}{\epsilon};R(y^0))$ and so

$$\begin{aligned}&\int_0^{y^0}\int_{|y^1|\leq y^1_*} \frac{m^2}{n^2}\partial_{y^0}\xi\cdot\partial_{y^0y^0}\tilde{F}_0\\ =\ &\int_0^{y^0}\int_{|y^1|\leq y^1_*} \frac{m^2}{n^2}\partial_{y^0}\xi\cdot\Big[\frac{a''}{\epsilon}\partial_{y^1}F_0-(\frac{a'}{\epsilon})^2\partial_{y^1y^1}F_0\\ &\qquad +2R'\frac{a'}{\epsilon}\partial_R\partial_{y^1}F_0-R''\partial_RF_0-(R')^2\partial_{RR}F_0\Big]\end{aligned} \tag{4.41}$$

where we use the notation $\partial^\alpha_{y^1}\partial^\beta_RF_0=\partial^\alpha_{y^1}\partial^\beta_RF_0(\frac{y^1-a}{\epsilon};R)$ for convenience.

First, we will bound the a'' term. Recall that $m=(1-(R')^2)^{-1/2}$ and $n=1+y^1m^3R''$. Consider the $\partial_{y^0}\xi\cdot\partial_{y^1}F_0$ term. We have that

$$\begin{aligned}\int_0^{y^0}\int_{|y^1|\leq y^1_*} \frac{m^2}{n^2}\frac{a''}{\epsilon}\partial_{y^0}\xi\cdot\partial_{y^1}F_0 &= \int_0^{y^0}\int_{|y^1|\leq y^1_*} m^2\frac{a''}{\epsilon}\partial_{y^0}\xi\cdot\partial_{y^1}F_0\\ &+\int_0^{y^0}\int_{|y^1|\leq y^1_*}\left[\frac{m^2}{n^2}-m^2\right]\frac{a''}{\epsilon}\partial_{y^0}\xi\cdot\partial_{y^1}F_0\end{aligned}$$

Also, by differentiating

$$0 = \int \xi \cdot \partial_{y^1} \tilde{F}_0$$

with respect to y^0, we see that

$$\int\limits_{|y^1| \le y^1_*} \partial_{y^0}\xi \cdot \partial_{y^1} F_0 = \frac{a'}{\epsilon} \int\limits_{|y^1| \le y^1_*} \xi \cdot \partial_{y^1 y^1} F_0 - R' \int\limits_{|y^1| \le y^1_*} \xi \cdot \partial_R \partial_{y^1} F_0 \tag{4.42}$$

Using (4.42), we control the m^2 term as follows

$$\begin{aligned} \Big|\int\limits_0^{y^0} \int\limits_{|y^1| \le y^1_*} m^2 \frac{a''}{\epsilon} \partial_{y^0}\xi \partial_{y^1} F_0\Big| &\lesssim \int\limits_0^{y^0} |m|^2 \int\limits_{|y^1| \le y^1_*} \left| \frac{a''}{\epsilon} \partial_{y^0}\xi \cdot \partial_{y^1} F_0 \right| \\ &\lesssim \int\limits_0^{y^0} \left| \frac{a''}{\epsilon} \right| \left[\frac{|a'|}{\epsilon} \int\limits_{|y^1| \le y^1_*} \left| \xi \cdot \partial_{y^1 y^1} F_0 \right| + \int\limits_{|y^1| \le y^1_*} \left| \xi \cdot \partial_R \partial_{y^1} F_0 \right| \right] \end{aligned}$$

where we used the fact that $|m|^2 \lesssim 1$ for all $0 \le y^0 \le y^0_*$ to obtain the second estimate. We then use the Cauchy-Schwarz inequality, (4.3.2), and (4.17) to conclude that

$$\Big|\int\limits_0^{y^0} \int\limits_{|y^1| \le y^1_*} m^2 \frac{a''}{\epsilon} \partial_{y^0}\xi \cdot \partial_{y^1} F_0\Big| \lesssim \int\limits_0^{y^0} \frac{|a''|}{\epsilon} (\epsilon E^{1/2} + e^{-\alpha \frac{y^1 - a}{\epsilon}}) \sqrt{\epsilon} \underline{A}$$

To control the $\frac{m^2}{n^2} - m^2$, observe that $\frac{m^2}{n^2} - m^2 = O(y^1)$. We use the Cauchy-Schwarz inequality and the definition of the energy (4.16) to see that

$$\begin{aligned} \Big|\int\limits_0^{y^0} \int\limits_{|y^1| \le y^1_*} \left[\frac{m^2}{n^2} - m^2 \right] \frac{a''}{\epsilon} \partial_{y^0}\xi \cdot \partial_{y^1} F_0\Big| &\lesssim \int\limits_0^{y^0} \frac{|a''|}{\epsilon} \left\| \partial_{y^0}\xi \right\|_{L^2(-y^1_*, y^1_*)} \left\| y^1 \partial_{y^1} F_0 \right\|_2 \\ &\lesssim \int\limits_0^{y^0} \frac{|a''|}{\epsilon} \epsilon^{3/2} E^{1/2} \underline{A} \end{aligned}$$

To summarize, we have the following estimate

$$\int\limits_0^{y^0} \int\limits_{|y^1| \le y^1_*} \frac{m^2}{n^2} \frac{a''}{\epsilon} \partial_{y^0}\xi \cdot \partial_{y^1} F_0 \lesssim \int\limits_0^{y^0} \frac{|a''|}{\epsilon} (\epsilon E^{1/2} + e^{-\alpha \frac{y^1 - a}{\epsilon}}) \sqrt{\epsilon} \underline{A}$$

We deal with the rest of the terms of (4.41) by shifting the ∂_{y^0} off of ξ and use (4.17) along with corollary 4.3.2. We only estimate the $\partial_{y^1y^1}F_0$ term of (4.41). The estimation of the other three terms of (4.41) are done in the same way yielding the same bounds.

Estimating the $\partial_{y^1y^1}F_0$ term, we see that

$$
\begin{aligned}
&\int_0^{y^0}\int_{|y^1|\leq y^1_*} \frac{m^2}{n^2}\partial_{y^0}\xi\cdot\left[-(\frac{a'}{\epsilon})^2\partial_{y^1y^1}F_0\right] \\
&= -\int_{|y^1|\leq y^1_*} \frac{m^2}{n^2}(\frac{a'}{\epsilon})^2\xi\cdot\partial_{y^1y^1}F_0\Big|_0^{y^0} + \int_0^{y^0}\int_{|y^1|\leq y^1_*}\xi\cdot\partial_{y^0}\left[\frac{m^2}{n^2}(\frac{a'}{\epsilon})^2\partial_{y^1y^1}F_0\right] \\
&\lesssim (\frac{a'}{\epsilon})^2\|\xi\|_{L^2(-y^1_*,y^1_*)}\|\partial_{y^1y^1}F_0\|_2\Big|_0^{y^0} + \int_0^{y^0}\|\xi\|_{L^2(-y^1_*,y^1_*)}\Big[(\frac{a'}{\epsilon})^2\|\partial_{y^1y^1}F_0\|_2 + 2\frac{|a'|}{\epsilon}\frac{|a''|}{\epsilon}\|\partial_{y^1y^1}F_0\|_2 \\
&\qquad + \frac{|a'|^3}{\epsilon^3}\|\partial_{y^1y^1y^1}F_0\|_2 + \frac{|a'|^2}{\epsilon^2}|R'|\,\|\partial_{y^1y^1}\partial_R F_0\|_2\Big] \\
&\lesssim (\epsilon E^{1/2} + e^{-\alpha\frac{y^1-a}{\epsilon}})\sqrt{\epsilon}\underline{A}\Big|_0^{y^0} + \int_0^{y^0}(1+\frac{|a''|}{\epsilon})(\epsilon E^{1/2} + e^{-\alpha\frac{y^1-a}{\epsilon}})\sqrt{\epsilon}\underline{A}
\end{aligned}
$$

Putting together the above estimates yields

$$
-\int_0^{y^0}\int_{|y^1|\leq y^1_*}\frac{m^2}{n^2}\partial_{y^0}\xi\cdot\partial_{y^0y^0}\tilde{F}_0 \lesssim (\epsilon E^{1/2} + e^{-\alpha\frac{y^1-a}{\epsilon}})\sqrt{\epsilon}\underline{A}\Big|_0^{y^0} + \int_0^{y^0}(1+\frac{|a''|}{\epsilon})(\epsilon E^{1/2} + e^{-\alpha\frac{y^1-a}{\epsilon}})\sqrt{\epsilon}\underline{A} \tag{4.43}
$$

2. $\underline{\partial_{y^0y^0}\tilde{F}_1 \text{ term}}$: Since $\frac{m^2}{n^2}\lesssim 1$ on $(0,y^0_*)\times(-y^1_*,y^1_*)$, we have

$$
\begin{aligned}
-\epsilon\int_0^{y^0}\int_{|y^1|\leq y^1_*}\frac{m^2}{n^2}\partial_{y^0}\xi\cdot\partial_{y^0y^0}\tilde{F}_1 &\lesssim \epsilon\int_0^{y^0}E^{1/2}\|\partial_{y^0y^0}\tilde{F}_1\|_2 \\
&\lesssim \int_0^{y^0}(1+\frac{|a''|}{\epsilon})\epsilon^{3/2}E^{1/2}\underline{A}
\end{aligned} \tag{4.44}
$$

where we used Cauchy-Schwarz and the definition of the energy (4.16) to obtain the first inequality and corollary 4.3.2 to obtain the last inequality.

3. $\underline{B^0 \partial_{y^0} \tilde{F}_0}$: Using the boundedness of B^0 and $\partial_{y^0} B^0$ on $(0, y^0_*) \times (-y^1_*, y^1_*)$, we see that

$$\begin{aligned}
&-\int_0^{y^0} \int_{|y^1| \leq y^1_*} B^0 \partial_{y^0} \xi \cdot \partial_{y^0} \tilde{F}_0 \\
&= -\int_{|y^1| \leq y^1_*} B^0 \xi \cdot \partial_{y^0} \tilde{F}_0 \Big|_0^{y^0} + \int_0^{y^0} \int_{|y^1| \leq y^1_*} \xi \cdot \partial_{y^0} \left[B^0 \partial_{y^0} \tilde{F}_0 \right] \\
&\lesssim (\epsilon E^{1/2} + e^{-\alpha \frac{y^1 - a}{\epsilon}}) \sqrt{\epsilon} \underline{A} \Big|_0^{y^0} + \int_0^{y^0} (1 + \frac{|\partial_{y^0 y^0} a|}{\epsilon})(\epsilon E^{1/2} + e^{-\alpha \frac{y^1 - a}{\epsilon}}) \sqrt{\epsilon} \underline{A}
\end{aligned} \tag{4.45}$$

where we used Cauchy-Schwarz, corollary 4.1.4, and corollary 4.3.2 to obtain the last inequality.

4. $\underline{B^0 \partial_{y^0} \tilde{F}_1 \text{ term}}$: We estimate this term in the same way as the $B^0 \partial_{y^0} \tilde{F}_0$ term.

$$-\epsilon \int_0^{y^0} \int_{-y^1_*}^{y^1_*} B^0 \partial_{y^0} \xi \cdot \partial_{y^0} \tilde{F}_1 \lesssim \int_0^{y^0} (\epsilon E^{1/2} + e^{-\alpha \frac{y^1 - a}{\epsilon}}) \sqrt{\epsilon} \underline{A} \tag{4.46}$$

5. $\underline{B^1 \partial_{y^1} \tilde{F}_1 \text{ term}}$: Using the boundedness of B^1 and $\partial_{y^0} B^1$ on $(0, y^0_*) \times (-y^1_*, y^1_*)$ we get

$$\begin{aligned}
&-\int_0^{y^0} \int_{|y^1| \leq y^1_*} \epsilon B^1 \partial_{y^0} \xi \cdot \partial_{y^1} \tilde{F}_1 \\
&= -\int_{|y^1| \leq y^1_*} \xi \cdot \left[\epsilon B^1 \partial_{y^1} \tilde{F}_1 \right] \Big|_0^{y^0} + \int_0^{y^0} \int_{|y^1| \leq y^1_*} \xi \cdot \partial_{y^0} \left[\epsilon B^1 \partial_{y^1} \tilde{F}_1 \right] \\
&\lesssim (\epsilon E^{1/2} + e^{-\alpha \frac{y^1 - a}{\epsilon}}) \sqrt{\epsilon} \underline{A} \Big|_0^{y^0} + \int_0^{y^0} (\epsilon E^{1/2} + e^{-\alpha \frac{y^1 - a}{\epsilon}}) \sqrt{\epsilon} \underline{A}
\end{aligned} \tag{4.47}$$

where we used Cauchy-Schwarz, corollary 4.1.4, and corollary 4.3.2 to obtain the last inequality.

Putting together the estimates obtained from steps 1-5 we obtain (4.40).

□

Lemma 4.3.6.

$$-\int_0^{y^0}\int_{-y^1_*}^{y^1_*}\partial_{y^0}\xi\cdot N \lesssim \int_0^{y^0}(1+\epsilon+(\epsilon E^{1/2}+e^{-\alpha\frac{y^1_*-a}{\epsilon}})(\epsilon^{5/2}+\epsilon E^{1/2}+e^{-\alpha\frac{y^1_*-a}{\epsilon}})(\epsilon E^{1/2}+e^{-\alpha\frac{y^1_*-a}{\epsilon}})\underline{A}$$
$$+\frac{E^{1/2}}{\epsilon}\left(\epsilon+(\epsilon E^{1/2}+e^{-\alpha\frac{y^1_*-a}{\epsilon}})^{1/2}E^{1/4}\right)\left(\epsilon^{3/2}+(\epsilon E^{1/2}+e^{-\alpha\frac{y^1_*-a}{\epsilon}})\right)\underline{A}\Big|_0^{y^0} \tag{4.48}$$
$$+\left[\frac{y^1_*}{(R-y^1_*m)^3}+\frac{(y^1_*)^2}{(R-y^1_*m)^4}\right]E\Big|_0^{y^0}$$

In order to close the bootstrap argument appearing in the proof of theorem (4.2.1), we will need to choose y^1_* sufficiently small so that the factor multiplying E in (4.18) is less than one. This is why we need to include the $\frac{y^1_*}{(R-y^1_*m)^3}$ and $\frac{(y^1_*)^2}{(R-y^1_*m)^4}$ factors in (4.48).

<u>Proof of lemma 4.3.6</u>: Recall that we defined N as

$$N=-\frac{1}{\epsilon^2}\left[w(\tilde{F}_0+\tilde{F}_\xi,R+y^1m)-w(\tilde{F}_0,R)-\text{Hess}_\Phi(\tilde{F}_0,R)\tilde{F}_\xi-y^1m(y^0)\partial_R w(\tilde{F}_0,R))\right]$$

and $\tilde{F}_\xi$ as $\tilde{F}_\xi=\epsilon\tilde{F}_1+\xi$. Using the identity

$$g(t)=g(0)+g'(0)t+\int_0^1(1-t)g''(t)dt$$

we can rewrite N as

$$N=-\frac{1}{\epsilon^2}\int_0^1(1-t)\frac{d^2}{dt^2}w(\tilde{F}_0+t\tilde{F}_\xi,R+ty^1m)dt \tag{4.49}$$

Examine now the left hand side of (4.48). First, we integrate by parts to move ∂_{y^0} from ξ onto N. That is,

$$-\int_0^{y^0}\int_{-y^1_*}^{y^1_*}\partial_{y^0}\xi\cdot N=-\int_{-y^1_*}^{y^1_*}\xi\cdot N\Big|_0^{y^0}+\int_0^{y^0}\int_{-y^1_*}^{y^1_*}\xi\cdot\partial_{y^0}N$$

Using Cauchy-Schwarz, we see that

$$-\int_0^{y^0}\int_{-y^1_*}^{y^1_*}\partial_{y^0}\xi\cdot N\lesssim\epsilon E^{1/2}\|N\|_{L^2(-y^1_*,y^1_*)}\Big|_0^{y^0}+\Big|\int_0^{y^0}\int_{-y^1_*}^{y^1_*}\xi\cdot\partial_{y^0}N\Big| \tag{4.50}$$

To simplify things, we will estimate the two terms appearing on the right hand side of (4.50) seperately.

Before continuing, though, recall that $\tilde{F}_\xi$ has two-components: one corresponding to the ϕ-field and the other corresponding to the σ-field. We write $\tilde{F}_\xi$ as

$$\tilde{F}_\xi=((\tilde{F}_\xi)_\phi,(\tilde{F}_\xi)_\sigma)$$

$\underline{\|N\|_{L^2(-y^1_*,y^1_*)}\text{ Estimate}}$: First, examine the $\frac{d^2}{dt^2}w(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1m)$ term in N. Expanding things out, we have that

$$\begin{aligned}
& \frac{d^2}{dt^2}w(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1m) \\
= & \frac{d}{dt}\left[\operatorname{Hess}_\Phi W(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1m)\tilde{F}_\xi + \partial_R w(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1m)y^1m\right] \\
= & (\tilde{F}_\xi)_\phi\partial_\phi \operatorname{Hess}_\Phi W(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1m)\tilde{F}_\xi + (\tilde{F}_\xi)_\sigma\partial_\sigma \operatorname{Hess}_\Phi W(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1m)\tilde{F}_\xi \\
& +2y^1m\partial_R \operatorname{Hess}_\Phi W(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1m)\tilde{F}_\xi + (y^1m)^2\partial_{RR}w(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1m)
\end{aligned} \tag{4.51}$$

Using 4.49) and (4.51), we have that

$$\begin{aligned}
\|N\|_{L^2(-y^1_*,y^1_*)} \lesssim & \Big\|\frac{1}{\epsilon^2}\int_0^1 (1-t)(\tilde{F}_\xi)_\phi\partial_\phi \operatorname{Hess}_\Phi W(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1m)\tilde{F}_\xi dt\Big\|_{L^2(-y^1_*,y^1_*)} \\
& +\Big\|\frac{1}{\epsilon^2}\int_0^1 (1-t)(\tilde{F}_\xi)_\sigma\partial_\sigma \operatorname{Hess}_\Phi W(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1m)\tilde{F}_\xi dt\Big\|_{L^2(-y^1_*,y^1_*)} \\
& +\Big\|\frac{1}{\epsilon^2}\int_0^1 (1-t)2y^1m\,\partial_R \operatorname{Hess}_\Phi W(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1m)\tilde{F}_\xi\Big\|_{L^2(-y^1_*,y^1_*)} \\
& +\Big\|\frac{1}{\epsilon^2}\int_0^1 (1-t)(y^1m)^2\partial_{RR}w(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1m)\Big\|_{L^2(-y^1_*,y^1_*)}
\end{aligned}$$

We will estimate each of the four terms appearing on the right hand side of the above seperately.

1. $\underline{(\tilde{F}_\xi)_\phi\text{ term}}$:

$$\Big\|\int_0^1 (1-t)(\tilde{F}_\xi)_\phi\partial_\phi \operatorname{Hess}_\Phi W(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1m)\tilde{F}_\xi dt\Big\|_{L^2(-y^1_*,y^1_*)} \lesssim \big\|(\tilde{F}_\xi)_\phi\big\|_{L^\infty(-y^1_*,y^1_*)}\big\|\tilde{F}_\xi\big\|_{L^2(-y^1_*,y^1_*)}$$

   where we used the boundedness of $\tilde{F}_0 + t\tilde{F}_\xi$ and $R + ty^1m$ on $(0, y^0_*) \times (-y^1_*, y^1_*)$ for $0 \le t \le 1$ to control the operator norm $\big\|\partial_\phi \operatorname{Hess}_\Phi W(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1m)\big\|$. Next, we use the Gagliardo-Nirenberg inequality and corollary 4.1.4 to show that

$$\begin{aligned}
\big\|\tilde{F}_\xi\big\|_{L^\infty(-y^1_*,y^1_*)} & \lesssim \epsilon\big\|\tilde{F}_1\big\|_\infty + \|\xi\|_{L^\infty(-y^1_*,y^1_*)} \\
& \lesssim \epsilon + \|\xi\|^{1/2}_{L^2(-y^1_*,y^1_*)}\|\partial_{y^1}\xi\|^{1/2}_{L^2(-y^1_*,y^1_*)} \\
& \lesssim \epsilon + (\epsilon E^{1/2} + e^{-\alpha\frac{y^1_*-a}{\epsilon}})^{1/2}E^{1/4}
\end{aligned} \tag{4.52}$$

Furthermore, using corollary 4.1.4 and corollary 4.3.2 we see that

$$\begin{aligned}\|\tilde F_\xi\|_{L^2(-y^1_*,y^1_*)} &\lesssim \epsilon\|\tilde F_1\|_2 + \|\xi\|_{L^2(-y^1_*,y^1_*)}\\ &\lesssim \epsilon^{3/2} + (\epsilon E^{1/2} + e^{-\alpha\frac{y^1_*-a}{\epsilon}})\end{aligned}$$

Thus, we obtain the estimate

$$\begin{aligned}&\Big\|\frac{1}{\epsilon^2}\int_0^1 (1-t)(\tilde F_\xi)_\phi \partial_\phi \operatorname{Hess}_\Phi W(\tilde F_0 + t\tilde F_\xi, R + ty^1 m)\tilde F_\xi dt\Big\|_{L^2(-y^1_*,y^1_*)}\\ \lesssim\ &\frac{1}{\epsilon^2}\Big(\epsilon + (\epsilon E^{1/2} + e^{-\alpha\frac{y^1_*-a}{\epsilon}})^{1/2}E^{1/4}\Big)\Big(\epsilon^{3/2} + (\epsilon E^{1/2} + e^{-\alpha\frac{y^1_*-a}{\epsilon}})\Big)\end{aligned}$$

2. $\underline{(\tilde F_\xi)_\sigma \text{ term}}$: Similarily, we have that

$$\begin{aligned}&\Big\|\frac{1}{\epsilon^2}\int_0^1 (1-t)(\tilde F_\xi)_\sigma \partial_\sigma \operatorname{Hess}_\Phi W(\tilde F_0 + t\tilde F_\xi, R + ty^1 m)\tilde F_\xi dt\Big\|_{L^2(-y^1_*,y^1_*)}\\ \lesssim\ &\frac{1}{\epsilon^2}\Big(\epsilon + (\epsilon E^{1/2} + e^{-\alpha\frac{y^1_*-a}{\epsilon}})^{1/2}E^{1/4}\Big)\Big(\epsilon^{3/2} + (\epsilon E^{1/2} + e^{-\alpha\frac{y^1_*-a}{\epsilon}})\Big)\end{aligned}$$

3. $\underline{\partial_R \operatorname{Hess}_\Phi W \text{ term}}$: Using the boundedness of $\partial_R \operatorname{Hess}_\Phi W(\tilde F_0 + t\tilde F_\xi, R + ty^1 m)$ and $\frac{1}{R-y^1_* m}$ on $(0, y^0_*) \times (-y^1_*, y^1_*)$, corollary 4.1.4, and corollary 4.3.2, we see that

$$\begin{aligned}&\Big\|\frac{1}{\epsilon^2}\int_0^1 (1-t)2y^1 m\ \partial_R \operatorname{Hess}_\Phi W(\tilde F_0 + t\tilde F_\xi, R + ty^1 m)\tilde F_\xi\Big\|_{L^2(-y^1_*,y^1_*)}\\ \lesssim\ &\frac{1}{\epsilon^2}\Big\|\frac{y^1}{(R-y^1_* m)^3}\tilde F_\xi\Big\|_{L^2(-y^1_*,y^1_*)}\\ \lesssim\ &\frac{1}{\epsilon^2}\Big(\epsilon\|y^1\tilde F_1\|_2 + \Big\|\frac{y^1}{(R-y^1_* m)^3}\xi\Big\|_{L^2(-y^1_*,y^1_*)}\Big)\\ \lesssim\ &\frac{1}{\epsilon^2}\Big(\epsilon^{5/2}\underline{A} + \frac{y^1_*}{(R-y^1_* m)^3}\|\xi\|_{L^2(-y^1_*,y^1_*)}\Big)\\ \lesssim\ &\frac{1}{\epsilon^2}\Big(\epsilon^{5/2}\underline{A} + \epsilon\frac{y^1_*}{(R-y^1_* m)^3}E^{1/2} + e^{-\alpha\frac{y^1_*-a}{\epsilon}}\Big)\end{aligned}$$

4. $\underline{(y^1 m)^2\partial_{RR}w \text{ term}}$: Since

$$\partial_{RR} w(\tilde F_0 + t\tilde F_\xi, R + ty^1 m) = \begin{pmatrix} 0 \\ \frac{3d^2}{(R+ty^1 m)^4}(\tilde s_0 + t(\tilde F_\xi)_\sigma)\end{pmatrix}$$

then we have

$$
\begin{aligned}
&\Big\|\frac{1}{\epsilon^2}\int_0^1 (1-t)(y^1m)^2\partial_{RR}w(\tilde F_0+t\tilde F_\xi,R+ty^1m)\Big\|_{L^2(-y^1_*,y^1_*)}\\
\lesssim\;& \frac{1}{\epsilon^2}\Big(\big\|(y^1)^2\tilde s_0\big\|_2+\epsilon\big\|(y^1)^2\tilde s_1\big\|_2+\Big\|\frac{(y^1)^2}{(R-y^1_*m)^4}\xi\Big\|_{L^2(-y^1_*,y^1_*)}\Big)\\
\lesssim\;& \frac{1}{\epsilon^2}\Big(\epsilon^{5/2}\underline{A}+\epsilon^{7/2}\underline{A}+\epsilon\frac{(y^1_*)^2}{(R-y^1_*m)^4}E^{1/2}+e^{-\alpha\frac{y^1_*-a}{\epsilon}}\Big)
\end{aligned}
$$

where again we used corollary 4.3.2 to obtain the last inequality.

To summarize, we have that

$$
\begin{aligned}
\big\|N\big\|_{L^2(-y^1_*,y^1_*)} \lesssim\;& \left[\frac{y^1_*}{(R-y^1_*m)^3}+\frac{(y^1_*)^2}{(R-y^1_*m)^4}\right]\frac{E^{1/2}}{\epsilon}\\
&+\frac{1}{\epsilon^2}\left(\epsilon+(\epsilon E^{1/2}+e^{-\alpha\frac{y^1_*-a}{\epsilon}})^{1/2}E^{1/4}\right)\left(\epsilon^{3/2}+(\epsilon E^{1/2}+e^{-\alpha\frac{y^1_*-a}{\epsilon}})\right)\underline{A}
\end{aligned}
$$

$\underline{\left|\left\langle\xi,\partial_{y^0}N\right\rangle_{L^2(-y^1_*,y^1_*)}\right| \text{ Estimate:}}$ Notice that

$$
\begin{aligned}
\partial_{y^0}N =\;& -\frac{1}{\epsilon^2}\int_0^1(1-t)\frac{d^2}{dt^2}\left[\operatorname{Hess}_\Phi W(\tilde F_0+t\tilde F_\xi,R+ty^1m)\cdot\partial_{y^0}(\tilde F_0+t\tilde F_\xi)\right]\\
&-\frac{1}{\epsilon^2}\int_0^1(1-t)\frac{d^2}{dt^2}\left[\partial_R w(\tilde F_0+t\tilde F_\xi,R+ty^1m)\partial_{y^0}(R+ty^1m(y^0))\right]
\end{aligned}
\tag{4.53}
$$

Consider the first term on the right hand side first. Since

$$
\begin{aligned}
&\frac{d^2}{dt^2}\left[\operatorname{Hess}_\Phi W(\tilde F_0+t\tilde F_\xi,R+ty^1m)\cdot\partial_{y^0}(\tilde F_0+t\tilde F_\xi)\right]\\
=\;& (\tilde F_\xi)^2_\phi\partial_{\phi\phi}\operatorname{Hess}_\Phi W(\tilde F_0+t\tilde F_\xi,R+ty^1m)\cdot\partial_{y^0}(\tilde F_0+t\tilde F_\xi)\\
&+2(\tilde F_\xi)_\phi(\tilde F_\xi)_\sigma\partial_\phi\partial_\sigma\operatorname{Hess}_\Phi W(\tilde F_0+t\tilde F_\xi,R+ty^1m)\cdot\partial_{y^0}(\tilde F_0+t\tilde F_\xi)\\
&+(\tilde F_\xi)^2_\sigma\partial_{\sigma\sigma}\operatorname{Hess}_\Phi W(\tilde F_0+t\tilde F_\xi,R+ty^1m)\cdot\partial_{y^0}(\tilde F_0+t\tilde F_\xi)\\
&+(y^1m)^2\partial_{RR}\operatorname{Hess}_\Phi W(\tilde F_0+t\tilde F_\xi,R+ty^1m)\cdot\partial_{y^0}(\tilde F_0+t\tilde F_\xi)\\
&+(\tilde F_\xi)_\phi\partial_\phi\operatorname{Hess}_\Phi W(\tilde F_0+t\tilde F_\xi,R+ty^1m)\cdot\partial_{y^0}\tilde F_\xi\\
&+(\tilde F_\xi)_\sigma\partial_\sigma\operatorname{Hess}_\Phi W(\tilde F_0+t\tilde F_\xi,R+ty^1m)\cdot\partial_{y^0}\tilde F_\xi\\
&+(y^1m)\partial_R\operatorname{Hess}_\Phi W(\tilde F_0+t\tilde F_\xi,R+ty^1m)\cdot\partial_{y^0}\tilde F_\xi
\end{aligned}
$$

where we needed to use the fact that $\partial_\phi\partial_R\operatorname{Hess}W(F,R)=\partial_\phi\partial_R\operatorname{Hess}W(F,R)=0$ to obtain the above. Using Cauchy-Schwarz, corollary 4.1.4, corollary 4.3.2, the boundedness of $\operatorname{Hess}_\Phi W(\tilde F_0+t\tilde F_\xi,R+ty^1m)$ and all of its derivatives on $(0,y^0_*)\times(-y^1_*,y^1_*)$ for all $0\le t\le 1$, and (4.52), we have that

$$
\begin{aligned}
& \Big|\int_0^{y^0}\Big\langle \xi, \frac{d^2}{dt^2}\Big[\mathrm{Hess}_\Phi\, W(\tilde F_0+t\tilde F_\xi, R+ty^1m)\cdot\partial_{y^0}(\tilde F_0+t\tilde F_\xi)\Big]\Big\rangle_{L^2(-y^1_*,y^1_*)}\Big| \\
\lesssim\ & \int_0^{y^0}\Big[\big\|\tilde F_\xi\big\|_{L^\infty(-y^1_*,y^1_*)}\big\|\partial_{y^0}(\tilde F_0+t\tilde F_\xi)\big\|_{L^2(-y^1_*,y^1_*)}+\big\|\partial_{y^0}\tilde F_\xi\big\|_{L^2(-y^1_*,y^1_*)}\Big]\big\|\tilde F_\xi\big\|_{L^\infty(-y^1_*,y^1_*)}\big\|\xi\big\|_{L^2(-y^1_*,y^1_*)} \\
& +\Big|\int_0^{y^0}\int_{-y^1_*}^{y^1_*}\frac{6d^2(y^1m)^2}{(R+ty^1m)^4}\xi\cdot\partial_{y^0}(\tilde F_0+t\tilde F_\xi)\Big|+\Big|\int_0^{y^0}\int_{-y^1_*}^{y^1_*}\frac{2d^2(y^1m)}{(R+ty^1m)^3}\xi\cdot\partial_{y^0}\tilde F_\xi\Big| \\
\lesssim\ & \int_0^{y^0}(\epsilon+(\epsilon E^{1/2}+e^{-\alpha\frac{y^1_*-a}{\epsilon}})^{1/2}E^{1/4})^2(\epsilon^{1/2}\underline{A}+E^{1/2})(\epsilon E^{1/2}+e^{-\alpha\frac{y^1_*-a}{\epsilon}}) \\
& +\int_0^{y^0}(\epsilon+(\epsilon E^{1/2}+e^{-\alpha\frac{y^1_*-a}{\epsilon}})^{1/2}E^{1/4})(\epsilon^{3/2}\underline{A}+E^{1/2})(\epsilon E^{1/2}+e^{-\alpha\frac{y^1_*-a}{\epsilon}}) \\
& +\int_0^{y^0}\Big[(\epsilon E^{1/2}+e^{-\alpha\frac{y^1_*-a}{\epsilon}})\epsilon^{5/2}\underline{A}+\epsilon^2E+e^{-\alpha\frac{y^1_*-a}{\epsilon}}\Big]+\Big[\frac{y^1_*}{(R-y^1_*m)^3}+\frac{(y^1_*)^2}{(R-y^1_*m)^4}\Big]\epsilon^2E\Big|_0^{y^0}
\end{aligned}
$$

Now consider the second term of (4.53). Since $\partial_R w(F,R)=(0,-2\frac{d^2}{R^3}s)$, we have that

$$
\begin{aligned}
& \frac{d^2}{dt^2}\Big[\partial_R w(\tilde F_0+t\tilde F_\xi,R+ty^1m)(R'+ty^1m')\Big] \\
=\ & 2(\tilde F_\xi)^2_\sigma(y^1m)\partial_\sigma\partial_{RR}w(\tilde F_0+t\tilde F_\xi,R+ty^1m)(R'+ty^1m') \\
& +(y^1m)^2\partial_{RRR}w(\tilde F_0+t\tilde F_\xi,R+ty^1m)(R'+ty^1m') \\
& +(\tilde F_\xi)_\sigma\partial_\sigma\partial_R w(\tilde F_0+t\tilde F_\xi,R+ty^1m)(y^1m') \\
& +(y^1m)\partial_{RR}w(\tilde F_0+t\tilde F_\xi,R+ty^1m)(y^1m')
\end{aligned}
$$

Using Cauchy-Schwarz, corollary 4.1.4, corollary 4.3.2, the boundedness of $\partial_R w(\tilde F_0+t\tilde F_\xi,R+ty^1m)$ and all of its derivatives on $(0, y^0_*) \times (-y^1_*, y^1_*)$ for all $0 \le t \le 1$, and (4.52), we have that

$$\begin{aligned}
&|\int_0^{y^0} \left\langle \xi, \frac{d^2}{dt^2}\left[\partial_R w(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1 m)(R' + ty^1 m')\right]\right\rangle_{L^2(-y^1_*, y^1_*)}| \\
\lesssim\ & \int_0^{y^0} \left[1 + \|\tilde{F}_\xi\|_{L^\infty(-y^1_*, y^1_*)}\right]\left[\epsilon\|(y^1)\tilde{F}_1\|_{L^2(-y^1_*, y^1_*)} + \|\xi\|_{L^2(-y^1_*, y^1_*)}\right]\|\xi\|_{L^2(-y^1_*, y^1_*)} \\
& \int_0^{y^0} \left[\|(y^1)^2 \tilde{s}_0\|_{L^2(-y^1_*, y^1_*)} + \epsilon\|(y^1)^2\tilde{F}_1\|_{L^2(-y^1_*, y^1_*)} + \|\xi\|_{L^2(-y^1_*, y^1_*)}\right]\|\xi\|_{L^2(-y^1_*, y^1_*)} \\
\lesssim\ & \int_0^{y^0} (1 + \epsilon + (\epsilon E^{1/2} + e^{-\alpha\frac{y^1_* - a}{\epsilon}})^{1/2} E^{1/4})(\epsilon^{5/2} + \epsilon E^{1/2} + e^{-\alpha\frac{y^1_* - a}{\epsilon}})(\epsilon E^{1/2} + e^{-\alpha\frac{y^1_* - a}{\epsilon}})
\end{aligned}$$

To summarize, we have that

$$\begin{aligned}
|\int_0^{y^0} \left\langle \xi, \partial_{y^0} N\right\rangle_{L^2(-y^1_*, y^1_*)}| \ \lesssim\ & \left[\frac{y^1_*}{(R - y^1_* m)^3} + \frac{(y^1_*)^2}{(R - y^1_* m)^4}\right] E\Big|_0^{y^0} \\
& + \frac{1}{\epsilon^2}\int_0^{y^0} (1 + \epsilon + (\epsilon E^{1/2} + e^{-\alpha\frac{y^1_* - a}{\epsilon}})(\epsilon^{5/2} + \epsilon E^{1/2} + e^{-\alpha\frac{y^1_* - a}{\epsilon}})(\epsilon E^{1/2} + e^{-\alpha\frac{y^1_* - a}{\epsilon}})\underline{A}
\end{aligned}$$

Putting together the estimates for the two terms on the right hand side of (4.50), we see that

$$\begin{aligned}
-\int_0^{y^0}\int_{-y^1_*}^{y^1_*} \partial_{y^0}\xi \cdot N \ \lesssim\ & \frac{1}{\epsilon^2}\int_0^{y^0} (1 + \epsilon + (\epsilon E^{1/2} + e^{-\alpha\frac{y^1_* - a}{\epsilon}})(\epsilon^{5/2} + \epsilon E^{1/2} + e^{-\alpha\frac{y^1_* - a}{\epsilon}})(\epsilon E^{1/2} + e^{-\alpha\frac{y^1_* - a}{\epsilon}})\underline{A} \\
& + \frac{E^{1/2}}{\epsilon}\left(\epsilon + (\epsilon E^{1/2} + e^{-\alpha\frac{y^1_* - a}{\epsilon}})^{1/2} E^{1/4}\right)\left(\epsilon^{3/2} + (\epsilon E^{1/2} + e^{-\alpha\frac{y^1_* - a}{\epsilon}})\right)\underline{A}\Big|_0^{y^0} \\
& + \left[\frac{y^1_*}{(R - y^1_* m)^3} + \frac{(y^1_*)^2}{(R - y^1_* m)^4}\right] E\Big|_0^{y^0}
\end{aligned}$$

□

Lemma 4.3.7.

$$-\int_0^{y^0}\int_{-y^1_*}^{y^1_*} Y \lesssim \int_0^{y^0}\left(E + \frac{1}{\epsilon^2} e^{-\alpha\frac{y^1 - a}{\epsilon}}\right) \tag{4.54}$$

Proof of lemma 4.3.7: (See lemma 4.32 for the definition of Y)

$$-\int_0^{y^0}\int_{-y^1_*}^{y^1_*} Y = \frac{1}{\epsilon^2}\int_0^{y^0}\int_{-y^1_*}^{y^1_*} \xi\cdot\left[\partial_{y^0}\operatorname{Hess}_\Phi W(\tilde F_0,R)\right]\xi - \int_0^{y^0}\int_{|y^1|\le y^1_*} B^\alpha\partial_{y^0}\xi\cdot\partial_\alpha\xi - \int_0^{y^0}\int_{-y^1_*}^{y^1_*}\frac{1}{2}\partial_{y^0}\left(\frac{m^2}{n^2}\right)\partial_{y^0}\xi^2$$

$$\lesssim \int_0^{y^0}\left(E+\frac{1}{\epsilon^2}e^{-\alpha\frac{y^1-a}{\epsilon}}\right)$$

where we used corollary 4.1.4 and the boundedness of the operator $\partial_{y^0}\operatorname{Hess}_\Phi W(\tilde F_0,R)$ and the B^α's on $(0,y^0_*)\times(-y^1_*,y^1_*)$ to obtain the estimate.

□

Lemma 4.3.8.

$$\int_0^{y^0}\partial_{y^0}\xi(y^0,\pm y^1_*)\cdot\partial_{y^1}\xi(y^0,\pm y^1_*)\lesssim\int_0^{y^0}\frac{1}{\epsilon}e^{-\alpha\frac{y^1-a}{\epsilon}}\underline{A} \tag{4.55}$$

Proof of lemma 4.3.8: It follows from our choice of initial conditions, see section 1.3, that

$$\xi(y^0,\pm y^1_*)=\begin{pmatrix}\pm 1\\ 0\end{pmatrix}-\tilde F_0(y^0,\pm y^1_*)-\epsilon\tilde F_1(y^0,\pm y^1_*)$$

for all $0\le y^0\le y^0_*$. Using (4.28), (4.29), and (3.5.1), we have that

$$\left|\partial_{y^0}\xi(y^0,\pm y^1_*)\right|\lesssim e^{-\alpha\frac{y^1_*-a(y^0)}{\epsilon}}\underline{A}$$

Similarily, using (3.5.1) we see that

$$\left|\partial_{y^1}\xi(y^0,\pm y^1_*)\right| = \left|-\frac{1}{\epsilon}\partial_{y^1}F_0\left(\frac{y^1_*-a(y^0)}{\epsilon};R(y^0)\right)-\partial_{y^1}F_1\left(\frac{y^1_*-a(y^0)}{\epsilon};R(y^0),R'(y^0)\right)\right|$$

$$\lesssim \frac{1}{\epsilon}e^{-\alpha\frac{y^1_*-a(y^0)}{\epsilon}}$$

□

Combining the estimates obtained in lemma 4.3.4 to lemma 4.3.8 allows us to conclude the proof of theorem 4.2.2.

□

4.4 Proof of Bounded Shift Theorem (Theorem 4.2.3)

To prove this theorem we will use the fact that $\xi \perp \partial_{y^1} \tilde{F}_0$. Differentiating this quantity with respect to y^0 twice we find

$$\begin{aligned}
0 &= \partial_{y^0 y^0} \int\limits_{|y^1| \le y^1_*} \xi \cdot \partial_{y^1} \tilde{F}_0 \\
&= \partial_{y^0} \left(\int\limits_{|y^1| \le y^1_*} \partial_{y^0} \xi \cdot \partial_{y^1} \tilde{F}_0 + \int\limits_{|y^1| \le y^1_*} \xi \cdot \partial_{y^0 y^1} \tilde{F}_0 \right) \\
&= \int\limits_{|y^1| \le y^1_*} \partial_{y^0 y^0} \xi \cdot \partial_{y^1} \tilde{F}_0 + 2 \int\limits_{|y^1| \le y^1_*} \partial_{y^0} \xi \cdot \partial_{y^0 y^1} \tilde{F}_0 + \int\limits_{|y^1| \le y^1_*} \xi \cdot \partial_{y^0 y^0} \partial_{y^1} \tilde{F}_0 \\
&= \xi \cdot \partial_{y^0 y^0} \tilde{F}_0 \Big|_{-y^1_*}^{y^1_*} + \int\limits_{|y^1| \le y^1_*} \left(\partial_{y^0 y^0} \xi \cdot \partial_{y^1} \tilde{F}_0 + 2 \partial_{y^0} \xi \cdot \partial_{y^0} \partial_{y^1} \tilde{F}_0 - \partial_{y^1} \xi \cdot \partial_{y^0 y^0} \tilde{F}_0 \right)
\end{aligned} \tag{4.56}$$

On the other hand, we can use equation (4.10) for ξ to show that

$$\int\limits_{|y^1| \le y^1_*} \partial_{y^0 y^0} \xi \cdot \partial_{y^1} \tilde{F}_0 = - \int_{-y^1_*}^{y^1_*} \frac{n^2}{m^2} \left[B^\alpha \partial_\alpha \xi + L_\epsilon(\tilde{F}_0, R) \xi + S_{-1} + S_0 + N \right] \cdot \partial_{y^1} \tilde{F}_0 \tag{4.57}$$

Examining the S_0 term on the right hand side of (4.57) more closely, we see that

$$\int_{-y^1_*}^{y^1_*} \frac{n^2}{m^2} S_0 \cdot \partial_{y^1} \tilde{F}_0 = \int_{-y^1_*}^{y^1_*} \left(\partial_{y^0 y^0} \tilde{F}_0 + \epsilon \partial_{y^0 y^0} \tilde{F}_1 + \frac{n^2}{m^2} \partial_{y^0} (\tilde{F}_0 + \epsilon \tilde{F}_1) + \epsilon \frac{n^2}{m^2} B^1 \partial_{y^1} \tilde{F}_1 \right) \cdot \partial_{y^1} \tilde{F}_0 \tag{4.58}$$

Next, examine the term containing $\partial_{y^0 y^0} \tilde{F}_0$. Since $\tilde{F}_0(y^0, y^1) = F_0(\frac{y^1 - a(y^0)}{\epsilon}; R(y^0))$, we see that

$$\begin{aligned}
\int_{-y^1_*}^{y^1_*} \partial_{y^0 y^0} \tilde{F}_0 \cdot \partial_{y^1} \tilde{F}_0 &= -\frac{a''}{\epsilon^2} \int_{-y^1_*}^{y^1_*} \partial_{y^1} F_0^2 + \frac{1}{\epsilon} \int_{-y^1_*}^{y^1_*} \left[(\frac{a'}{\epsilon})^2 \partial_{y^1 y^1} F_0 - 2R' \frac{a'}{\epsilon} \partial_{y^1} \partial_R F_0 \right] \\
&\quad + \frac{1}{\epsilon} \int_{-y^1_*}^{y^1_*} \left[R'' \partial_R F_0 + (R')^2 \partial_{RR} F_0 \right] \cdot \partial_{y^1} F_0
\end{aligned} \tag{4.59}$$

where again we have suppressed the arguments of the F_0's and used the notation

$$\partial^\alpha_{y^1} \partial^\beta_R F_0 = \partial^\alpha_{y^1} \partial^\beta_R F_0 (\frac{y^1 - a}{\epsilon}; R)$$

Remember, the goal is to obtain a bound for $\frac{|a''|}{\epsilon}$. To do this, we will use (4.56 - 4.59) to isolate the $\partial_{y^1} F_0^2$ term appearing in (4.59). We will then estimate each of the remaining terms. We will break the analysis up into three parts and then combine them to obtain (4.19).

1. $\underline{\partial_{y^0 y^0}\xi \cdot \partial_{y^1}\tilde{F}_0 \text{ upper bound estimate}}$: Using (4.56), we can isolate the $\partial_{y^0 y^0}\xi \cdot \partial_{y^1}\tilde{F}_0$ term. Estimating the resulting expression, we have that

$$\begin{aligned} \Big|\int_{-y_*^1}^{y_*^1} \partial_{y^0 y^0}\xi \cdot \partial_{y^1}\tilde{F}_0\Big| &= \Big|\xi \cdot \partial_{y^0 y^0}\tilde{F}_0\Big|_{-y_*^1}^{y_*^1} + \int_{-y_*^1}^{y_*^1} 2\partial_{y^0}\xi \cdot \partial_{y^0}\partial_{y^1}\tilde{F}_0 - \partial_{y^1}\xi \cdot \partial_{y^0 y^0}\tilde{F}_0)\Big| \\ &\lesssim \Big|\xi \cdot \partial_{y^0 y^0}\tilde{F}_0\Big|_{-y_*^1}^{y_*^1}\Big| + \frac{1}{\sqrt{\epsilon}}E^{1/2}\underline{A} + \sqrt{\epsilon}(1 + \frac{|a''|}{\epsilon})E^{1/2}\underline{A} \end{aligned}$$

where we used Cauchy-Schwarz and corollary 4.3.2 to obtain the last inequality. Using (3.47), we can show that

$$\Big|\int_{-y_*^1}^{y_*^1} \partial_{y^0 y^0}\xi \cdot \partial_{y^1}\tilde{F}_0\Big| \lesssim (1 + \frac{|a''|}{\epsilon})e^{-\alpha\frac{y_*^1 - a(y^0)}{\epsilon}} + \frac{1}{\sqrt{\epsilon}}E^{1/2}\underline{A} + \sqrt{\epsilon}(1 + \frac{|a''|}{\epsilon})E^{1/2}\underline{A}$$

2. $\underline{\partial_{y^0 y^0}\xi \cdot \partial_{y^1}\tilde{F}_0 \text{ lower bound estimate}}$: Estimating $\partial_{y^0 y^0}\xi \cdot \partial_{y^1}\tilde{F}_0$ using (4.57), we get

$$\begin{aligned} \Big|\int_{-y_*^1}^{y_*^1} \partial_{y^0 y^0}\xi \cdot \partial_{y^1}\tilde{F}_0\Big| \gtrsim{}& \Big|\int_{-y_*^1}^{y_*^1} \frac{n^2}{m^2}S_0 \cdot \partial_{y^1}\tilde{F}_0\Big| - \Big|\int_{-y_*^1}^{y_*^1} \frac{n^2}{m^2}L_\epsilon(\tilde{F}_0, R)\xi \cdot \partial_{y^1}\tilde{F}_0\Big| \\ &- \Big|\int_{-y_*^1}^{y_*^1} \frac{n^2}{m^2}B^\alpha\partial_\alpha\xi \cdot \partial_{y^1}\tilde{F}_0\Big| - \Big|\int_{-y_*^1}^{y_*^1} \frac{n^2}{m^2}S_{-1} \cdot \partial_{y^1}\tilde{F}_0\Big| - \Big|\int_{-y_*^1}^{y_*^1} \frac{n^2}{m^2}N \cdot \partial_{y^1}\tilde{F}_0\Big| \end{aligned}$$

We will estimate the $B^\alpha\partial_\alpha\xi$, $L_\epsilon(\tilde{F}_0, R)\xi$, S_{-1}, and N terms seperately.

   (a) $\underline{B^\alpha\partial_\alpha\xi \cdot \partial_{y^1}\tilde{F}_0 \text{ term}}$: Recall the definitions for B^0 (3.6), B^1 (3.7), and E (4.16). Using Cauchy-Schwarz and the boundedness of $\frac{n^2}{m^2}B^\alpha$ on $(0, y_*^0) \times (-y_*^1, y_*^1)$, we have

$$\Big|\int_{|y^1| \le y_*^1} \frac{n^2}{m^2}B^\alpha\partial_\alpha\xi \cdot \partial_{y^1}\tilde{F}_0\Big| \lesssim E^{1/2}\left\|\partial_{y^1}\tilde{F}_0\right\|_2 \lesssim \frac{1}{\sqrt{\epsilon}}E^{1/2}\underline{A}$$

   where we used Cauchy-Schwarz and corollary 4.3.2 to obtain the last inequality.

(b) $\underline{L_\epsilon(\tilde F_0, R)\xi\cdot\partial_{y^1}\tilde F_0 \text{ term}}$: Recall that $\text{Hess}_\Phi\, W(\tilde F_0, R)$ is symmetric. By integrating by parts with respect to y^1 twice, we can move the operator $L_\epsilon(\tilde F_0, R)$ onto $\frac{n^2}{m^2}\partial_{y^1}\tilde F_0$. Doing so, we see that

$$\int\limits_{|y^1|\le y^1_*} \frac{n^2}{m^2} L_\epsilon(\tilde F_0, R)\xi\cdot\partial_{y^1}\tilde F_0 = \int\limits_{|y^1|\le y^1_*} \xi\cdot L_\epsilon(\tilde F_0, R)\left(\frac{n^2}{m^2}\partial_{y^1}\tilde F_0\right) + \left[\frac{n^2}{m^2}\partial_{y^1}\xi\cdot\partial_{y^1}\tilde F_0 - \xi\cdot\partial_{y^1}\left(\frac{n^2}{m^2}\partial_{y^1}\tilde F_0\right)\right]_{-y^1_*}^{y^1_*}$$

Using corollary 3.47 we can easily bound the second term as

$$\left|\left[\frac{n^2}{m^2}\partial_{y^1}\xi\cdot\partial_{y^1}\tilde F_0 - \xi\cdot\partial_{y^1}\left(\frac{n^2}{m^2}\partial_{y^1}\tilde F_0\right)\right]_{-y^1_*}^{y^1_*}\right| \lesssim \frac{1}{\epsilon^2} e^{-\alpha\frac{y^1_*}{\epsilon}} e^{\alpha\underline{A}}$$

Using the fact that $\frac{n^2}{m^2}$ is bounded on $(0, y^0_*)\times(-y^1_*, y^1_*)$, Cauchy-Schwarz, and corollary 4.3.2 we estimate the first term as follows

$$\begin{aligned}\left|\int\limits_{|y^1|\le y^1_*} \xi\cdot L_\epsilon(\tilde F_0, R)\left(\frac{n^2}{m^2}\partial_{y^1}\tilde F_0\right)\right| &\lesssim \left|\int\limits_{|y^1|\le y^1_*} \frac{n^2}{m^2}\xi\cdot L_\epsilon(\tilde F_0, R)\partial_{y^1}\tilde F_0\right| \\ &\quad + \int\limits_{|y^1|\le y^1_*}\left|\left[\partial_{y^1}\left(\frac{n^2}{m^2}\right)\xi\cdot\partial_{y^1y^1}\tilde F_0 + \partial_{y^1y^1}\left(\frac{n^2}{m^2}\right)\xi\cdot\partial_{y^1}\tilde F_0\right]\right| \\ &\lesssim \frac{1}{\sqrt{\epsilon}}E^{1/2}\end{aligned}$$

where we needed to use the fact that $\partial_{y^1}\tilde F_0 \in \ker(L_\epsilon(\tilde F_0, R))$ to kill the $L_\epsilon(\tilde F_0, R)\partial_{y^1}\tilde F_0$ term. Thus, we have that

$$\left|\int\limits_{|y^1|\le y^1_*} \frac{n^2}{m^2} L_\epsilon(\tilde F_0, R)\xi\cdot\partial_{y^1}\tilde F_0\right| \lesssim \frac{1}{\sqrt{\epsilon}}E^{1/2} + \frac{1}{\epsilon^2}e^{-\alpha\frac{y^1_*}{\epsilon}}e^{\alpha\underline{A}}$$

(c) $\underline{S_{-1}\cdot\partial_{y^1}\tilde F_0 \text{ term}}$: Recall the definition of S_{-1} (4.11). Using the boundedness of $\frac{n^2}{m^2}$ on $(0, y^0_*)\times(-y^1_*, y^1_*)$, Cauchy-Schwarz, and corollary 4.3.2, we have that

$$\begin{aligned}\left|\int\limits_{|y^1|\le y^1_*} S_{-1}\cdot\partial_{y^1}\tilde F_0\right| &= \left|\int\limits_{|y^1|\le y^1_*} \frac{n^2}{m^2}\partial_{y^1}\tilde F_0\cdot\left[B^1 + H(R)\right]\partial_{y^1}\tilde F_0 + \frac{n^2}{m^2}\frac{a}{\epsilon}\partial_R w(\tilde F_0, R)\cdot\partial_{y^1}\tilde F_0\right| \\ &\lesssim \int_{-y^1_*}^{y^1_*} |y^1|\left|\partial_{y^1}\tilde F_0\right|^2 + \frac{|a|}{\epsilon}\left\|\partial_R w(\tilde F_0, R)\right\|_2\left\|\partial_{y^1}\tilde F_0\right\|_2 \\ &\lesssim \underline{A}\end{aligned}$$

where we used the fact that $B^1 = -H(R) + O(y^1)$ to obtain first inequality.

(d) $\underline{N \cdot \partial_{y^1}\tilde{F}_0 \text{ term}}$: For this term, we proceed as we did in the proof of lemma 4.3.6 when we estimated the $N \cdot \partial_{y^0}\xi$ term. Again, we will use the identity

$$g(t) = g(0) + g'(0)t + \int_0^1 (1-t)g''(t)$$

to rewrite N. Thus, we have that

$$\begin{aligned}
&\int_{-y^1_*}^{y^1_*} \frac{n^2}{m^2} N \cdot \partial_{y^1}\tilde{F}_0 \\
= &-\frac{1}{\epsilon^2}\int_0^1 (1-t)\int_{-y^1_*}^{y^1_*} \frac{n^2}{m^2}(\tilde{F}_\xi)_\phi \partial_\phi \operatorname{Hess}_\Phi W(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1 m)\tilde{F}_\xi \cdot \partial_{y^1}\tilde{F}_0 \\
&-\frac{1}{\epsilon^2}\int_0^1 (1-t)\int_{-y^1_*}^{y^1_*} \frac{n^2}{m^2}(\tilde{F}_\xi)_\sigma \partial_\sigma \operatorname{Hess}_\Phi W(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1 m)\tilde{F}_\xi \cdot \partial_{y^1}\tilde{F}_0 \\
&-\frac{1}{\epsilon^2}\int_0^1 (1-t)\int_{-y^1_*}^{y^1_*} \frac{n^2}{m^2} 2y^1 m\, \partial_R \operatorname{Hess}_\Phi W(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1 m)\tilde{F}_\xi \cdot \partial_{y^1}\tilde{F}_0 \\
&-\frac{1}{\epsilon^2}\int_0^1 (1-t)\int_{-y^1_*}^{y^1_*} \frac{n^2}{m^2}(y^1 m)^2 \partial_{RR} w(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1 m) \cdot \partial_{y^1}\tilde{F}_0
\end{aligned}$$

We will estimate each of the four terms appearing on the right hand side of the above individually. That is,

i. Since $\partial_\phi \operatorname{Hess}_\Phi W(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1 m)$ is bounded on $(0, y^0_*) \times (-y^1_*, y^1_*)$ for $0 \le t \le 1$, then

$$\begin{aligned}
&\Big|\frac{1}{\epsilon^2}\int_0^1 (1-t)\int_{-y^1_*}^{y^1_*} \frac{n^2}{m^2}(\tilde{F}_\xi)_\phi \partial_\phi \operatorname{Hess}_\Phi W(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1 m)\tilde{F}_\xi \cdot \partial_{y^1}\tilde{F}_0\Big| \\
\lesssim\ & \frac{1}{\epsilon^2}\int_{-y^1_*}^{y^1_*} \big|\tilde{F}_\xi\big|^2 \big|\partial_{y^1}\tilde{F}_0\big| \\
\lesssim\ & \frac{1}{\epsilon^2}\big\|\tilde{F}_\xi\big\|_{L^\infty(-y^1_*, y^1_*)}\big\|\tilde{F}_\xi\big\|_{L^2(-y^1_*, y^1_*)}\big\|\partial_{y^1}\tilde{F}_0\big\|_2 \\
\lesssim\ & \frac{1}{\epsilon^{5/2}}(\epsilon\big\|\tilde{F}_1\big\|_\infty + \big\|\xi\big\|_{L^\infty(-y^1_*, y^1_*)})(\epsilon\big\|\tilde{F}_1\big\|_2 + \big\|\xi\big\|_{L^2(-y^1_*, y^1_*)}) \\
\lesssim\ & \frac{1}{\epsilon^{5/2}}(\epsilon + \big\|\xi\big\|_{L^\infty(-y^1_*, y^1_*)})(\epsilon^{3/2} + \epsilon E^{1/2} + e^{-\alpha\frac{y^1_* - a}{\epsilon}})
\end{aligned}$$

where we needed to use corollaries 4.1.4 and 4.3.2 to obtain the last and second to last inequalities. Next, we estimate the $\|\xi\|_\infty$ term. Using Gagliardo-Nirenberg, we see that

$$\begin{aligned}\|\xi\|_{L^\infty(-y^1_*,y^1_*)} &\lesssim \|\xi\|^{1/2}_{L^2(-y^1_*,y^1_*)}\|\partial_{y^1}\xi\|^{1/2}_{L^2(-y^1_*,y^1_*)} \\ &\lesssim (\epsilon E^{1/2} + e^{-\alpha\frac{y^1_*-a}{\epsilon}})^{1/2}E^{1/4}\end{aligned}$$

Thus, we have that

$$\begin{aligned}&\Big|\frac{1}{\epsilon^2}\int_0^1(1-t)\int_{-y^1_*}^{y^1_*}\frac{n^2}{m^2}(\tilde{F}_\xi)_\phi\partial_\phi \operatorname{Hess}_\Phi W(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1m)\tilde{F}_\xi\cdot\partial_{y^1}\tilde{F}_0\Big| \\ \lesssim\ & \frac{1}{\epsilon^{5/2}}(\epsilon + (\epsilon E^{1/2} + e^{-\alpha\frac{y^1_*-a}{\epsilon}})^{1/2}E^{1/4})(\epsilon^{3/2} + \epsilon E^{1/2} + e^{-\alpha\frac{y^1_*-a}{\epsilon}})\end{aligned}$$

ii. Similarily, we have that

$$\begin{aligned}&\Big|\frac{1}{\epsilon^2}\int_0^1(1-t)\int_{-y^1_*}^{y^1_*}\frac{n^2}{m^2}(\tilde{F}_\xi)_\sigma\partial_\sigma \operatorname{Hess}_\Phi W(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1m)\tilde{F}_\xi\cdot\partial_{y^1}\tilde{F}_0\Big| \\ \lesssim\ & \frac{1}{\epsilon^{5/2}}(\epsilon + (\epsilon E^{1/2} + e^{-\alpha\frac{y^1_*-a}{\epsilon}})^{1/2}E^{1/4})(\epsilon^{3/2} + \epsilon E^{1/2} + e^{-\alpha\frac{y^1_*-a}{\epsilon}})\end{aligned}$$

iii. Using Cauchy-Schwarz, the boundedness of $\frac{n^2}{m^2}m(y^0)\partial_R \operatorname{Hess}_\Phi W(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1m)$ on $(0, y^0_*)\times(-y^1_*, y^1_*)$ for all $0 \le t \le 1$, corollary 4.1.4, and corollary 4.3.2 we find that

$$\begin{aligned}&\Big\|\frac{1}{\epsilon^2}\int_0^1(1-t)\int_{-y^1_*}^{y^1_*}\frac{n^2}{m^2}2y^1m\ \partial_R \operatorname{Hess}_\Phi W(\tilde{F}_0 + t\tilde{F}_\xi, R + ty^1m)\tilde{F}_\xi\cdot\partial_{y^1}\tilde{F}_0\Big\| \\ \lesssim\ & \frac{1}{\epsilon^2}(\|\epsilon\tilde{F}_1\|_2 + \|\xi\|_{L^2(-y^1_*,y^1_*)})\|y^1\partial_{y^1}\tilde{F}_0\|_2 \\ \lesssim\ & \frac{1}{\epsilon^{3/2}}(\epsilon^{3/2} + \epsilon E^{1/2} + e^{-\alpha\frac{y^1_*-a}{\epsilon}})\end{aligned}$$

iv. In our notation

$$\partial_{RR}w(\tilde{F}_0, R) = 3\frac{d^2}{R^4}\tilde{s}_0$$

Thus, we have that

$$
\begin{aligned}
&\Big|\frac{1}{\epsilon^2}\int_0^1(1-t)\int_{-y_*^1}^{y_*^1}\frac{n^2}{m^2}(y^1m)^2\partial_{RR}w(\tilde{F}_0+t\tilde{F}_\xi,R+ty^1m)\cdot\partial_{y^1}\tilde{F}_0\Big|\\
=&\ \Big|\frac{1}{\epsilon^2}\int_0^1(1-t)\int_{-y_*^1}^{y_*^1}(y^1)^2\ 6\frac{d^2n^2}{(R+ty^1m)^4}(\tilde{s}_0+t\tilde{s}_\xi)\partial_{y^1}\tilde{s}_0\Big|\\
\lesssim&\ \frac{1}{\epsilon^2}\big\|(y^1)^2\tilde{s}_0\partial_{y^1}\tilde{s}_0\big\|_2+\frac{1}{\epsilon^2}\big\|\epsilon\tilde{s}_1\big\|_2\big\|(y^1)^2\partial_{y^1}\tilde{s}_0\big\|_2+\frac{1}{\epsilon^2}\big\|\xi\big\|_{L^2(-y_*^1,y_*^1)}\big\|(y^1)^2\partial_{y^1}\tilde{s}_0\big\|_2\\
\lesssim&\ 1+\epsilon+\sqrt{\epsilon}E^{1/2}+\frac{1}{\sqrt{\epsilon}}e^{-\alpha\frac{y_*^1-a}{\epsilon}}
\end{aligned}
$$

where we used Cauchy-Schwarz and the boundedness of $\frac{n^2}{(R+ty^1m)^4}$ on $(0,y_*^0)\times(-y_*^1,y_*^1)$ for $0\le t\le 1$ to obtain the second last inequality and corollaries 4.1.4 and 4.3.2 to obtain the last inequality.

Thus, we have that

$$
\Big|\int_{-y_*^1}^{y_*^1}\frac{n^2}{m^2}N\cdot\partial_{y^1}\tilde{F}_0\Big|\ \lesssim\ 1+\frac{1}{\epsilon^{5/2}}(\epsilon+(\epsilon E^{1/2}+e^{-\alpha\frac{y_*^1-a}{\epsilon}})^{1/2}E^{1/4})(\epsilon^{3/2}+\epsilon E^{1/2}+e^{-\alpha\frac{y_*^1-a}{\epsilon}})
$$

Putting the estimates from steps (a)-(d) together, we have that

$$
\begin{aligned}
\Big|\int_{|y^1|\le y_*^1}\partial_{y^0y^0}\xi\cdot\partial_{y^1}\tilde{F}_0\Big|\ \gtrsim&\ \Big|\int_{|y^1|\le y_*^1}\frac{n^2}{m^2}S_0\cdot\partial_{y^1}\tilde{F}_0\Big|\\
&-\left[\frac{1}{\epsilon^{5/2}}(\epsilon\underline{A}+(\epsilon E^{1/2}+e^{-\alpha\frac{y_*^1-a}{\epsilon}})^{1/2}E^{1/4})(\epsilon^{3/2}+\epsilon E^{1/2}+e^{-\alpha\frac{y_*^1-a}{\epsilon}})\right]
\end{aligned}
$$

3. <u>$S_0\cdot\partial_{y^1}\tilde{F}_0$ lower bound estimate:</u> Using (4.58) and Cauchy-Schwarz, we see that

$$
\begin{aligned}
\Big|\int_{-y_*^1}^{y_*^1}\frac{n^2}{m^2}S_0\cdot\partial_{y^1}\tilde{F}_0\Big|\ \gtrsim&\ \Big|\int_{-y_*^1}^{y_*^1}\partial_{y^0y^0}\tilde{F}_1\cdot\partial_{y^1}\tilde{F}_0\Big|\\
&-\Big[\epsilon\big\|\partial_{y^0y^0}\tilde{F}_1\big\|_2+\big\|\partial_{y^0}\tilde{F}_0\big\|_2+\epsilon\big\|\partial_{y^0}\tilde{F}_1\big\|_2+\epsilon\big\|\partial_{y^1}\tilde{F}_0\big\|_2\Big]\big\|\partial_{y^1}\tilde{F}_0\big\|_2\\
\gtrsim&\ \Big|\int_{-y_*^1}^{y_*^1}\partial_{y^0y^0}\tilde{F}_1\cdot\partial_{y^1}\tilde{F}_0\Big|-[\underline{A}+|a''|\,\underline{A}]
\end{aligned}
$$

where we needed to use the fact that $\frac{n^2}{m^2}B^\alpha$ is bounded on $(0, y^0_*) \times (-y^1_*, y^1_*)$ to obtain the first inequality and where we used corollary 4.3.2 and the facts that R' and R'' are bounded for $y^0 \in (0, y^0_*)$ to obtain the last inequality.

4. $\underline{\partial_{y^0y^0}\tilde{F}_0 \cdot \partial_{y^1}\tilde{F}_0 \text{ lower bound estimate}}$: This is the last term we need to estimate. Using (4.59), we see that

$$\begin{aligned}
\epsilon\Big|\int_{-y^1_*}^{y^1_*} \partial_{y^0y^0}\tilde{F}_0 \cdot \partial_{y^1}\tilde{F}_0\Big| \quad &\gtrsim \quad \left|\frac{a''}{\epsilon}\right|\Big|\int_{-y^1_*}^{y^1_*} \partial_{y^1}F_0^2 - (\frac{a'}{\epsilon})^2\Big| \int_{|y^1|\le y^1_*} \partial_{y^1y^1}F_0 \cdot \partial_{y^1}F_0\Big| \\
&\quad -2R'\left|\frac{a'}{\epsilon}\right|\Big|\int_{-y^1_*}^{y^1_*} \partial_R\partial_{y^1}F_0 \cdot \partial_{y^1}F_0\Big| - |R''|\Big|\int_{-y^1_*}^{y^1_*} \partial_R F_0 \cdot \partial_{y^1}F_0\Big| \\
&\quad -(R')^2\Big|\int_{-y^1_*}^{y^1_*} \partial_{RR}F_0 \cdot \partial_{y^1}F_0\Big| \\
&\gtrsim \quad \epsilon\left|\frac{a''}{\epsilon}\right| - \epsilon\underline{A}
\end{aligned}$$

where we used the boundedness of R' and R'' on $(0, y^0_*)$, Cauchy-Schwarz, and corollary 4.3.2 to obtain the last inequality.

Combining the estimates obtained in steps 1-4, we obtain the estimate

$$\begin{aligned}
\left|\frac{a''}{\epsilon}\right| \quad &\lesssim \quad \frac{|a''|}{\epsilon}\left[\epsilon\underline{A} + \sqrt{\epsilon}\underline{A}E^{1/2} + e^{-\alpha\frac{y^1_*-a}{\epsilon}}\right] \\
&\quad +\frac{1}{\epsilon^{5/2}}(\epsilon + (\epsilon E^{1/2} + e^{-\alpha\frac{y^1_*-a}{\epsilon}})^{1/2}E^{1/4})(\epsilon^{3/2} + \epsilon E^{1/2} + e^{-\alpha\frac{y^1_*-a}{\epsilon}})
\end{aligned}$$

Rearranging this inequality, we obtain (4.19)

□

A Formal Asymptotics

Let η be the Minkowski metric on $\mathbb{R}^{1+n}$ and let $\Gamma \subset (\mathbb{R}^{1+n}, \eta)$ be an n-dimensional time-like surface in space-time. Suppose that Γ is parameterized by some map $H : \Omega \subset \mathbb{R}^n \to \mathbb{R}^{1+n}$. Define a new coordinate system $(y^\tau, y^\nu) \in \mathbb{R}^n \times \mathbb{R}$, called Minkowski normal coordinates, as

$$(t, x) = \psi(y^\tau, y^\nu) = H(y^\tau) + y^\nu\nu(y^\tau)$$

where $\nu(y^\tau) \perp_\eta \partial_{y^\tau} H(y^\tau)$ and $|\nu(y^\tau)|_\eta = 1$. We call $y^\tau \in \mathbb{R}^n$ "tangential coordinates" and $y^\nu \in \mathbb{R}$ the "normal coordinate". Note that this coordinate system may only be well defined on a neighbourhood $\mathcal{N}$ of Γ.

Recall that we want to find solutions of (1.1) so that ϕ has an interface and so that σ is exponentially small except near the interface of ϕ. Based on [16], we expect that for suitable Γ, $\theta : \Omega \to \mathbb{R}$, and $\Phi_0 := (\phi_0, \sigma_0) : \mathbb{R} \to \mathbb{R}^2$ there exists a solution with these characteristics of the form

$$\Phi(y^\tau, y^\nu) \approx \begin{pmatrix} \phi_0(\frac{y^\nu}{\epsilon}) \\ e^{\frac{i}{\epsilon}\theta(y^\tau)}\sigma_0(\frac{y^\nu}{\epsilon}) \end{pmatrix} \tag{A.1}$$

We will now carry out a formal asymptotic analysis to find Φ_0 so that ϕ_0 has an interface and to find Γ and θ for which we expect (A.1) to hold. To do this, we will expand the action integral associated to (1.1) about the right hand side of (A.1). From this expansion, we obtain an **effective action**. We will then make a choice for the profile Φ_0 and for this choice of Φ_0, we expect, heuristically, that the correction terms coming from expanding the action about the right hand side of (A.1) will be of lower order when Γ and θ are critical points of the effective action.

The Lagrangian associated to (1.1) in Minkowski normal coordinates is

$$\mathcal{L} := \frac{1}{2} g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + \frac{1}{2} g^{\alpha\beta}\overline{\partial_\alpha\sigma}\partial_\beta\sigma + \frac{1}{\epsilon^2}V(\phi,\sigma) \tag{A.2}$$

where $g_{\alpha\beta} := \eta_{\lambda\omega}\partial_\alpha\psi^\lambda\partial_\beta\psi^\omega$ is the Minkowski metric in normal coordinates and, only for this appendix,

$$V = \frac{\lambda_\phi}{4}(\phi^2-1)^2 + \frac{\lambda_\sigma}{4}(|\sigma|^2-2)|\sigma|^2 + \frac{\beta^2}{\phi}|\sigma|^2$$

Note that

$$\begin{cases} g_{ij} = \gamma_{ij} + (y^\nu)\eta_{\lambda\omega}\left(\partial_i H^\lambda \partial_j \nu^\omega + \partial_j H^\lambda \partial_i \nu^\omega\right) + (y^\nu)^2\eta_{\lambda\omega}\partial_i\nu^\lambda\partial_j\nu^\omega \\ g_{i\nu} = 0 \\ g_{\nu\nu} = 1 \end{cases}$$

where $\gamma_{ij} := \eta_{\alpha\beta}\partial_i H^\alpha \partial_j H^\beta$ denotes the induced metric on the surface Γ (latin indices range over the tangential coordinates and Greek indices will range over both tangential and normal coordinates). For $\xi = (\xi_\phi, \xi_\sigma) : \mathbb{R}^{1+n} \to \mathbb{R}\times\mathbb{R}$, we plug

$$\begin{aligned} \phi &= \phi_0(\frac{y^\nu}{\epsilon}) + \xi_\phi \\ \sigma &= e^{\frac{i}{\epsilon}\theta(y^\tau)}\left[\sigma_0(\frac{y^\nu}{\epsilon}) + \xi_\sigma\right] \end{aligned}$$

into the action integral to get

$$S(\Phi) = \frac{1}{\epsilon^2}\int \left\{\frac{1}{2}\Phi_0'(\frac{y^\nu}{\epsilon})^2 + V(\Phi_0)(\frac{y^\nu}{\epsilon}) + \frac{1}{2}\gamma^{ij}\partial_i\theta\partial_j\theta\ \sigma_0(\frac{y^\nu}{\epsilon})^2\right\}\sqrt{-\gamma(y^\tau)}dy^\tau dy^\nu + \text{“}\xi \text{ terms''} \tag{A.3}$$

Note: A goal of this thesis is to rigorously verify that for suitable intial conditions the "ξ terms" are small, in a sense. The effective action we obtain from this expansion is

$$\tilde{S} := \int \left\{ \frac{1}{2}(\Phi_0'(\frac{y^\nu}{\epsilon}))^2 + V(\Phi_0)(\frac{y^\nu}{\epsilon}) + \frac{1}{2}\gamma^{ij}\partial_i\theta\partial_j\theta\sigma_0(\frac{y^\nu}{\epsilon})^2 \right\} \sqrt{-\gamma(y^\tau)}dy^\tau dy^\nu \tag{A.4}$$

Consider the $\frac{1}{\epsilon^2}$ term. It is natural to choose Φ_0 so that in transverse directions to Γ, Φ_0 is energy minimizing and so that ϕ_0 has an interface. To this end, suppose for $\rho \in \mathbb{R}$, $F = (f, s)(\cdot; \rho)$ satisfies the minimization problem

$$\mu(\rho) := \inf_{(f,s)\in\mathcal{A}} \int \left\{ \frac{1}{2}\left|(f', s')\right|^2 + V(f, s) + \frac{1}{2}\rho s^2 \right\} dy^\nu \tag{A.5}$$

$$\mathcal{A} := \left\{ (f, s) \in H^1_{loc} \; : \; \lim_{y^\nu\to\pm\infty} f(y^\nu) = \pm 1, \; f(0) = 0 \right\} \tag{A.6}$$

In this case, the boundary conditions imposed results in f having an interface and the condition that $f(0) = 0$ kills the translation symmetry that would otherwise be present in any solutions to this minimization problem. Furthermore, for suitable potentials V, s is exponentially small except near the interface of f. We pick $\Phi_0 = (f, s)(\cdot; \zeta)$, where $\zeta(y^\tau) := \gamma^{ij}\partial_i\theta\partial_j\theta$. *Important*: The profile Φ_0 that we chose actually depends on ζ. That is, in contrast to our initial hypothesis (A.1), we expect that there should exist a solution to (1.1) satisfying

$$\Phi \approx \begin{pmatrix} \phi_0(\frac{y^\nu}{\epsilon}; \zeta(y^\tau)) \\ e^{\frac{i}{\epsilon}\theta(y^\tau)}\sigma_0(\frac{y^\nu}{\epsilon}; \zeta(y^\tau)) \end{pmatrix} \tag{A.7}$$

for $\Phi_0 = \Phi_0(\cdot; \zeta)$ minimizing (A.5) and for suitable Γ and θ. Note: The differentiability of Φ_0 in both y^ν and ζ is important and needs to be established (we did this in section 3.6 for potentials satisfying (1.9)). For this section we assume that Φ_0 is sufficiently regular so that we may carry out the calculations that are to follow.

For this choice of Φ_0, the effective action becomes

$$\tilde{S}(H, \theta) = \int \mu(\zeta) \sqrt{-\gamma} dy^\tau \tag{A.8}$$

Heuristically, we expect that when θ and H are critical points of $\tilde{S}$, then ξ will be of lower order than the right hand side of (A.1). That is, for θ and H satisfying the nonlinear, coupled system

$$0 = \frac{\delta\tilde{S}}{\delta\theta} = -2\partial_j\left(\mu'(\zeta)\sqrt{-\gamma}\gamma^{ij}\partial_i\theta\right) \tag{A.9}$$

$$0 = \frac{\delta S}{\delta H} = -\eta_{\alpha\beta}\partial_j\left(\mu(\zeta)\sqrt{-\gamma}\gamma^{ij}\partial_i H^\alpha\right) + 2\eta_{\alpha\beta}\partial_j\left(\mu'(\zeta)\sqrt{-\gamma}\gamma^{ik}\gamma^{lj}\partial_k\theta\partial_l\theta\partial_i H^\beta\right) \tag{A.10}$$

then $\Phi_0(\frac{y^\nu}{\epsilon}; \zeta)$ should be a good approximate solution. We expect that the coupled system for θ and H is a hyperbolic system, but this isn't completely clear by just looking at it. By expanding (A.10) and taking its inner product with ν^β, we can rewrite this system as

$$\Box_\Gamma \theta = -\gamma(\nabla_\tau \log\left[\mu'(\zeta)\right], \nabla_\tau \theta) \tag{A.11}$$

$$\text{mean curvature of } \Gamma = 2\frac{\mu'(\zeta)}{\mu(\zeta)}\eta\left(\nu, \mathrm{I\!I}(\nabla_\tau\theta, \nabla_\tau\theta)\right) \tag{A.12}$$

where $\mathrm{I\!I}$ is the second fundamental form of Γ. We can simplify this further as follows. For Φ_0 minimizing (A.5 - A.6), then Φ_0 satisfies

$$-\Phi_0'' + v(\Phi_0) + \zeta\begin{pmatrix} 0 \\ \sigma_0 \end{pmatrix} = 0 \tag{A.13}$$

where

$$v(\Phi_0) := \begin{pmatrix} \lambda_\phi(\phi_0^2 - 1)\phi_0 + \beta\sigma_0^2\phi_0 \\ \lambda_\sigma(\sigma_0^2 - 1)\sigma_0 + \beta\phi_0^2\sigma_0 \end{pmatrix}$$

Multiplying (A.13) by Φ_0' and integrating we find the equipartition of energy identity

$$\frac{1}{2}\left|\Phi_0'\right|^2 = V(\Phi_0) + \frac{1}{2}\zeta\sigma_0^2 \tag{A.14}$$

Using this equipartition identity we find that

$$\mu(\zeta) = \left\|\Phi_0'\right\|^2$$

Further, if we differentiate $\mu(\zeta)$ with respect to ζ we find

$$\begin{aligned}
\mu'(\zeta) &= \int\left\{\Phi_0'\partial_\zeta\Phi_0 + v(\Phi_0)\partial_\zeta\Phi_0 + \zeta\sigma_0\partial_\zeta\sigma_0 + \frac{1}{2}\sigma_0^2\right\} \\
&= \int\left\{\left(-\Phi_0'' + v(\Phi_0) + \zeta\begin{pmatrix} 0 \\ \sigma_0 \end{pmatrix}\right)\partial_\zeta\Phi_0 + \frac{1}{2}\sigma_0^2\right\} \\
&= \int \frac{1}{2}\sigma_0^2
\end{aligned}$$

where we used (A.13) to obtain the last reduction. Thus, we obtain a nice geometric relation relating the surface about which our approximate surface is concentrated and the phase of σ-field

$$\Box_\Gamma \theta = -\gamma(\nabla_\tau \log\left[\frac{1}{2}\left\|\sigma_0\right\|_2^2\right], \nabla_\tau \theta) \tag{A.15}$$

$$\text{mean curvature of } \Gamma = \frac{\left\|\sigma_0\right\|_2^2}{\left\|\Phi_0'\right\|_2^2}\eta\left(\nu, \mathrm{I\!I}(\nabla_\tau\theta, \nabla_\tau\theta)\right) \tag{A.16}$$

B Well Posedness of the Interface with a Current Model

B.1 Local Well Posedness

To show that the superconducting interface model is LWP we will implement a fixed point argument. This argument is classical, but we include it for the sake of completeness. See [27] for a review of wave equations and [30] for a more complete review of fixed point arguments in PDEs.

We will require the following estimate in order to carry out this argument. That is, for $(w_0, w_1) \in H_x^1 \times L_x^2$ and w satisfying

$$\begin{cases} \Box w &= F(w) \\ w(0) &= w_0 \\ \partial_t w(0) &= w_1 \end{cases}$$

then

$$\|w\|_{C_t^0 H_x^1(I\times\mathbb{R}^2)} + \|\partial_t w\|_{C_t^0 L_x^2(I\times\mathbb{R}^2)} \lesssim \langle|I|\rangle \left(\|w_0\|_{H_x^1} + \|w_1\|_{L_x^2}\right) + \|F(w)\|_{L_t^1 L_x^2} \tag{B.1}$$

For our application u is a 2-component vector (i.e. $u : \mathbb{R}^{1+2} \to \mathbb{R}^2$). We are interested in a problem whose initial data satisfy

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} - u_0 \in H_x^1 \quad \text{and} \quad u_1 \in L_x^2$$

and obeys

$$\begin{cases} \Box u &= \nabla_\Phi V(u) \\ u(0) &= u_0 \\ \partial_t u(0) &= u_1 \end{cases} \tag{B.2}$$

We want to use (B.1) and our problem doesn't quite satisfy the hypothesis to directly apply it. So we will need to be careful.

Set $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. We can rewrite (B.2) as

$$\begin{cases} \Box(u-v) &= \nabla_\Phi V(v + (u-v)) \\ u(0) - v &= u_0 - v \in H_x^1 \\ \partial_t u(0) &= u_1 \in L_x^2 \end{cases}$$

Define

$$\Psi(u-v, \partial_t u) = S(t)\begin{pmatrix} u_0 - v \\ u_1 \end{pmatrix} + \int_0^t S(t-s)\begin{pmatrix} 0 \\ \nabla_\Phi V(u)(s) \end{pmatrix} ds \tag{B.3}$$

where $S(t)$ is the semi-group associated to the linear operator $\Box w = 0$ (Note: $\Psi(u-v, \partial_t u)(t,x) \in \mathbb{R}^2 \times \mathbb{R}^2$). We will show that for sufficiently small T, Ψ has a unique fixed point in $X = X((0,T)\times\mathbb{R}^2)$ using the Banach Fixed Point Theorem, where

$$X = \left\{G \;:\; \|G\|_X < \infty\right\}$$

$$\|G\|_X = \|G_1\|_{C_t^0 H_x^1} + \|G_2\|_{C_t^0 L_x^2}$$

with $G = (G_1, G_2) : \mathbb{R}^{1+2} \to \mathbb{R}^2 \times \mathbb{R}^2$.

1. For $\overline{B_R(0)} \subset X$ and R sufficiently large, we will show that $\Psi : \overline{B_R(0)} \hookrightarrow \overline{B_R(0)}$. Using (B.3), we have that

$$\|\Psi(u-v, \partial_t u)\|_X \lesssim \left\|S(t)\begin{pmatrix} u_0 - v \\ u_1 \end{pmatrix}\right\|_X + \left\|\int_0^t S(t-s)\begin{pmatrix} 0 \\ \nabla_\Phi V(u)(s) \end{pmatrix}\right\|_X$$

(a) Using (B.1), we have that

$$\left\|S(t)\begin{pmatrix} u_0 - v \\ u_1 \end{pmatrix}\right\|_X \lesssim \langle T\rangle \left(\|u_0 - v\|_{H^1_x} + \|u_1\|_{L^2_x}\right)$$

(b) Again using (B.1), we have that

$$\begin{aligned}
\left\|\int_0^t S(t-s)\begin{pmatrix} 0 \\ \nabla_\Phi V(u)(s) \end{pmatrix} ds\right\|_X &\lesssim \int_0^t \left\|S(t-s)\begin{pmatrix} 0 \\ \nabla_\Phi V(u)(s) \end{pmatrix}\right\|_X ds \\
&\lesssim \int_0^t \langle t-s\rangle \left\|\nabla_\Phi V(u)(s)\right\|_{L^2_x} \\
&\lesssim \langle T\rangle \left\|\nabla_\Phi V(u)\right\|_{L^1_t L^2_x}
\end{aligned}$$

We are left to now estimate $\|\nabla_\Phi V(u)\|_{L^1_t L^2_x}$. By assumption 4 of (1.9), we have

$$\begin{aligned}
\left\|\nabla_\Phi V(u)\right\|_{L^1_t L^2_x} &= \left\|\nabla_\Phi V(v) + \int_0^1 \frac{d}{ds}\nabla_\Phi V(v + s(u-v))ds\right\|_{L^1_t L^2_x} \\
&= \left\|\nabla_\Phi V(v) + \int_0^1 \mathrm{Hess}_\Phi\, V(s(u-v)+v)(u-v)ds\right\|_{L^1_t L^2_x}
\end{aligned}$$

Using now assumptions 2 and 4 of (1.9), we have

$$\begin{aligned}
\left\|\nabla_\Phi V(u)\right\|_{L^1_t L^2_x} &\lesssim \int_0^1 \left\| \left[1 + |v + s(u-v)|^2\right] |u-v| \right\|_{L^1_t L^2_x} ds \\
&\lesssim \|u-v\|_{L^1_t L^2_x} + \left\| |u-v|^3 \right\|_{L^1_t L^2_x} \\
&\lesssim \|u-v\|_{L^1_t L^2_x} + \|u-v\|^3_{L^3_t L^6_x} \\
&\lesssim T\left[\|u-v\|_{C^0_t L^2_x} + \|u-v\|^3_{C^0_t L^6_x}\right]
\end{aligned}$$

Using the Sobolev embedding theorem, we have that

$$\left\|\nabla_\Phi V(u)\right\|_{L^1_t L^2_x} \lesssim T\left[\|(u-v,\partial_t u)\|_X + \|(u-v,\partial_t u)\|^3_X\right]$$

Thus, we obtain the estimate

$$\|\Psi(u-v,\partial_t u)\|_X \le c\langle T\rangle\left(\|u_0-v\|_{H^1_x} + \|u_1\|_{L^2_x}\right) + c\langle T\rangle T\left[\|(u-v,\partial_t u)\|_X + \|(u-v,\partial_t u)\|^3_X\right]$$

where c is the constant independent of T coming from the above estimates. Choosing R so that

$$c\langle T\rangle(\|u_0-v\|_{H^1_x}+\|u_1\|_{L^2_x})\leq\frac{R}{2}$$

and taking T sufficiently small, depending on R, we have that

$$\|\Psi(u-v,\partial_t u)\|_X\leq R$$

2. By possibly taking T smaller, we will next show that the map Ψ is a contraction mapping. Fix u and $\tilde{u}$ with $(u-v,\partial_t u),(\tilde{u}-v,\partial_t\tilde{u})\in\overline{B_R(0)}$ and examine

$$\|\Psi(u-v,\partial_t u)-\Psi(\tilde{u}-v,\partial_t\tilde{u})\|_X$$

Using (B.1) again and estimating as we did in step 1, then

$$\begin{aligned}\|\Psi(u-v,\partial_t u)-\Psi(\tilde{u}-v,\partial_t\tilde{u})\|_X &\lesssim \langle T\rangle\|\nabla_\Phi V(u)-\nabla_\Phi V(\tilde{u})\|_{L^1_tL^2_x}\\ &\lesssim \langle T\rangle\int_0^1\Big\|\frac{d}{ds}\nabla_\Phi V(s(u-v)+(1-s)(\tilde{u}-v))\,ds\Big\|_{L^1_tL^2_x}\\ &\lesssim \langle T\rangle\int_0^1\|\operatorname{Hess}_\Phi V(s(u-v)+(1-s)(\tilde{u}-v))(u-\tilde{u})\|_{L^1_tL^2_x}ds\\ &\lesssim \langle T\rangle\int_0^1\Big\|\left[1+|s(u-v)+(1-s)(\tilde{u}-v)|^2\right]|u-\tilde{u}|\Big\|_{L^1_tL^2_x}\end{aligned}$$

where we used assumption 4 of (1.9) to obtain the last inequality. Using the Holder inequality we thus have that

$$\begin{aligned}&\|\Psi(u-v,\partial_t u)-\Psi(\tilde{u}-v,\partial_t\tilde{u})\|_X\\ \lesssim\ &\langle T\rangle\left[\|u-\tilde{u}\|_{L^1_tL^2_x}+\big\||u-v|^2|u-\tilde{u}|\big\|_{L^1_tL^2_x}+\big\||\tilde{u}-v|^2|u-\tilde{u}|\big\|_{L^1_tL^2_x}\right]\\ \lesssim\ &\langle T\rangle T\left[\|u-\tilde{u}\|_{C^0_tL^2_x}+\big\||u-v|^2|u-\tilde{u}|\big\|_{C^0_tL^2_x}+\big\||\tilde{u}-v|^2|u-\tilde{u}|\big\|_{C^0_tL^2_x}\right]\\ \lesssim\ &\langle T\rangle T\left[\|u-\tilde{u}\|_{C^0_tL^2_x}+\|u-v\|^2_{C^0_tL^8_x}\|u-\tilde{u}\|_{C^0_tL^4_x}+\|\tilde{u}-v\|^2_{C^0_tL^8_x}\|u-\tilde{u}\|_{C^0_tL^4_x}\right]\\ \lesssim\ &\langle T\rangle T\left[1+\|(u-v,\partial_t u)\|^2_X+\|(\tilde{u}-v,\partial_t\tilde{u})\|^2_X\right]\|(u-\tilde{u},\partial_t(u-\tilde{u}))\|_X\end{aligned}$$

where we used the Sobolev embedding theorem to obtain the last estimate. Thus,

$$\|\Psi(u-v,\partial_t u)-\Psi(\tilde{u}-v,\partial_t\tilde{u})\|_X\lesssim\langle T\rangle T\left[1+\|(u-v,\partial_t u)\|^2_X+\|(\tilde{u}-v,\partial_t\tilde{u})\|^2_X\right]\|(u-\tilde{u},\partial_t(u-\tilde{u})\|_X$$

and so by taking T sufficiently small we then get that Ψ is also a contraction mapping. Thus, Ψ is a contraction mapping.

Thus (B.2) is locally wellposed.

B.2 Global Well Posedness

We would like to show that a solution to (B.2) exists for all time. To show this, we first iterate the process outlined above. That is, we find a solution from $(0, T_1)$ using the fixed point argument given above. We then repeat this process, but now we take the initial data to be $u(T_1) = u_0$ and $\partial_t u(T_1) = u_1$. This will then give us another time interval (T_1, T_2) for which (B.2) has a solution. We have thus found a slution on $(0, T_2)$. We can keep repeating this argument as long as

$$\left\|u(T_i)\right\|_{H^1_x} + \left\|\partial_t u(T_i)\right\|_{L^2_x} < \infty$$

Let $(0, T_*)$ be the maximal time of existence and suppose $T_* < \infty$. This means that

$$\lim_{t \nearrow T_*} \left[\left\|u(t)\right\|_{H^1_x} + \left\|\partial_t u(t)\right\|_{L^2_x}\right] = \infty$$

but for all times $0 < t < T_*$, then

$$\left\|u(t)\right\|_{H^1_x} + \left\|\partial_t u(t)\right\|_{L^2_x} < \infty$$

We will use a conservation law to show that T_* can't be finite. The energy of (B.2) is given by

$$E(u)(t) = \int_{\mathbb{R}^2} \left\{\frac{1}{2}\partial_t u^2 + \frac{1}{2}\left|\nabla u\right|^2 + V(u)\right\}$$

and this is a conserved quantity (i.e. $E(u)(t) = E(u)(0)$). First, we have

$$\begin{aligned}
\left\|u\right\|_{L^2_x} &\lesssim \left\|u(0)\right\|_{L^2_x} + \int_0^t \left\|\partial_t u\right\|_{L^2_x} \\
&\lesssim \left\|u(0)\right\|_{L^2_x} + \int_0^t E(u)^{1/2} \\
&\lesssim \left\|u(0)\right\|_{L^2_x} + tE(u)(0)^{1/2}
\end{aligned}$$

Secondly, we have that

$$\left\|\partial_t u\right\|_{L^2_x} + \left\|\nabla u\right\|_{L^2_x} \lesssim E(u)(0)^{1/2}$$

Combining these two estimates yields

$$\left\|u\right\|_{H^1_x} + \left\|\partial_t u\right\|_{L^2_x} \lesssim \left\|u(0)\right\|_{L^2_x} + (1 + t)E(u)(0)^{1/2}$$

Thus, we have that

$$\lim_{t \nearrow T_*} \left[\left\|u\right\|_{H^1_x} + \left\|\partial_t u\right\|_{L^2_x}\right] \lesssim \left\|u(0)\right\|_{L^2_x} + (1 + T_*)E(u)(0)^{1/2} < \infty$$

Thus, we can extend the solution to (B.2) contradicting the maximality of the time of existence. Thus, $T_* = \infty$ and so (B.2) is globally well posed.

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