Alexander Type Invariants of Tangles, Skew Howe Duality for Crystals and The Cactus Group

by

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Abstract

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This thesis consists of two parts, the first part is in the setting of algebraic knot theory while the second studies ideas in representation theory.

In the first part of this work, we study generalizations of a classical link invariant–the multivariable Alexander polynomial–to tangles. The starting point is Archibald's tMVA invariant for virtual tangles which lives in the setting of circuit algebras. Using the Hodge star map, we define a reduction of the tMVA to an invariant (rMVA) which is valued in matrices with entries equal to certain Laurent polynomials. When restricted to tangles without closed components, we show the rMVA has the structure of a metamonoid morphism and is further equivalent to another tangle invariant defined by Bar-Natan. This invariant also reduces to the Gassner representation on braids and has a partially defined trace operation for closing open strands of a tangle.

In the second part, we look at crystals and the cactus group. The crystals for a finite-dimensional, complex, reductive Lie algebra $\mathfrak{g}$ encode the structure of its representations, yet can also reveal surprising new structure of their own. In this work, we construct a group $J_{\mathfrak{g}}$, the "cactus group", using the Dynkin diagram of $\mathfrak{g}$ and show that it acts combinatorially on any $\mathfrak{g}$-crystal via the Schützenberger involutions. For $\mathfrak{g} = \mathfrak{g}l_n$, the cactus group was studied by Henriques and Kamnitzer, who construct an action of it on n-tensor products of $\mathfrak{g}$-crystals. We study the crystal corresponding to the $\mathfrak{g}l_n \times \mathfrak{g}l_m$-representation $\Lambda^N(\mathbb{C}^n \otimes \mathbb{C}^m)$, derive skew Howe duality on the crystal level and show that the two cactus group actions agree in this setting. An application of this result is discussed in studying a family maximal commutative subalgebras of the universal enveloping algebra, the shift of argument and Gaudin algebras, where an algebraically constructed monodromy action is expected to match that of the cactus group.

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Part I

Alexander Type Invariants of Tangles

Chapter 1

Introduction

The main question knot theorists have studied since the beginnings of the field, which has also been of interest to algebraists and geometers who have worked on applications in topology, is how to tell *knots*, or smooth embeddings of S^1 in $\mathbb{R}^3$, apart. The standard approach is to study "nice" two-dimensional projections of knots, called *knot diagrams*, and look for invariants. Namely, functions on these diagrams which also remain invariant under the Reidemeister moves representing the possible isotopies in $\mathbb{R}^3$ which might produce a different diagram for the same knot. Such functions are then honest knot invariants and are used as the key tool in distinguishing knots. A classical invariant of knots and links is the Alexander polynomial, originally introduced by James Alexander in 1928, [Alex28]. A generalization of it to a multivariable polynomial invariant of links, with a different variable for each strand, was later presented by Torres [Tor53] and is known as the multivariable Alexander polynomial, or MVA. Here, we will work with a version of it, vMVA, which is an invariant of oriented, regular, long, virtual knots and links, i.e. those sitting in a thickened surface rather than $\mathbb{R}^3$. A collection of objects which is more general than knots and links is that of *tangles*, having knotted components which are either an embedding of S^1 or of an interval, with its endpoints fixed at two different planes. In her thesis [Arch10], Archibald defines an extension tMVA–an invariant of oriented, regular, virtual tangles from which can be recovered both the MVA and vMVA when closing the open strands of the tangle. In fact, it satisfies the additional "Overcrossing Commute" relation which makes it an invariant of so called welded tangles. This invariant is computed from the same type of Alexander matrix constructed for the MVA, through an exponential time algorithm and provides a convenient setting for proving most of the local relations satisfied by the classical invariant it generalizes. More recently, Bar-Natan defined several versions of another invariant for tangles, called β-calculus in [BNS13] and [DBN1], and Γ-calculus in [DBN2]. It originates in invariants of ribbon-knotted copies of S^1 and S^2 in $\mathbb{R}^4$ (see [DBN3]), and associates to a pure tangle (i.e. one with no closed components), a (scalar, matrix) pair with entries certain Laurent polynomials. In this thesis, we show that although the two invariants come from, on first sight, very different places, they carry essentially the same information for pure tangles. For the purpose, we start with Archibald's invariant in the setting of circuit algebras, and transform it to a reduced, more compact version rMVA. This reduced invariant fits in the algebraic structure of metamonoids when restricted to pure tangles, is readily computable, and also recovers the MVA. In addition, it is closer in shape to the invariant of Bar-Natan, and we produce in the context of metamonoids an isomorphism between their target spaces. The main result of this work is that, after the listed transformations, the two invariants are equivalent as metamonoid morphisms. The result provides a more efficient way of computing tMVA, through the matrix-valued reduction rMVA. In addition, it provides a more direct proof that the Bar-Natan invariant is also a generalization of the Alexander polynomial. We further show these invariants restrict to the Gassner (and Burau) representations on braids, and have a partially defined trace operation allowing the closure of individual tangle components. The fact that they are readily computable by decomposing a tangle into its building blocks, and map into Laurent polynomials, also makes these invariants prime candidates for categorification. In Chapter 2, we discuss the preliminaries on tangles, circuit algebras, and the tMVA tangle invariant, as well as set the stage for reducing it. In Chapter 3 we define the reduced invariant rMVA and derive its values on positive and negative crossings, as well as the gluing or multiplication operation in the target space. In Chapter 4, we discuss the algebraic structure of metamonoids, show that rMVA is a metamonoid morphism, recall the Bar-Natan invariant and show the two are equivalent as metamonoid morphisms. Recovering the original MVA invariant for links and the Gassner representation for braids from the rMVA is described in Chapter 5. Some further results, including a partial trace operation and future directions are discussed in Chapter 6. Some of the longer proofs are delegated to the end in Chapter 7.

Chapter 2

Preliminaries

2.1 Circuit algebras

The tangle invariant tMVA described by Archibald in [Arch10] is defined in the context of circuit algebras, described in detail by Bar-Natan and Dancso [BND1], so we will start by discussing those. The operations in a circuit algebra are defined via circuit diagrams:

Definition 2.1.1 *An **oriented circuit diagram (OCD)** encodes the oriented pairing among a collection of "input" or "internal" and "output" or "external" points.*

More precisely, let $[n_i]$ denote a set with $n_i \in \mathbb{Z}_{\geq 0}$ elements, say $\{i_1, i_2, \ldots, i_{n_i}\}$. Given the integers $k,\ l \in \mathbb{Z}_{\geq 0}$ and pairs $(n_i, m_i) \in \mathbb{Z}^2_{\geq 0},\ i = 0, 1, \ldots, k$, an OCD prescribes a pairing among the elements of $\sqcup_{i=0}^{k} [n_i] \cup [m_i]$, and l counts the number of closed oriented loops (we will elaborate on that later). The pairing must be oriented in the sense that a point in $\sqcup_{i=0}^{k} [n_i]$ must be paired with a point in $\sqcup_{i=0}^{k} [m_i]$ and conversely (so in particular if $\sum_{i=0}^{k} n_i \neq \sum_{i=0}^{k} m_i$ then no such pairing exists). Moreover, we think of $[n_0] \sqcup [m_0]$ as the "output" points and the rest, $\sqcup_{i=1}^{k} [n_i] \cup [m_i]$, as the "input" in the OCD. Formally, it is an oriented compact 1-manifold with boundary $([n_i], [m_i]), i = 0, 1, \ldots, k$, up to homeomorphism.

One way to represent such an OCD is with one "external" circle with marked points $([n_0], [m_0])$ and k "internal" circles, with marked points $([n_i], [m_i]), i = 1, \ldots, k$, together with l oriented circles. In each such pair $([n_i], [m_i])$, the first number indicates "arrow tails" and the second "arrow heads". We also number and place a dot on each circle from which we go counterclockwise when considering the marked points. Then an OCD can be expressed as a pairing of the marked points by oriented arrows connecting combinatorially (i.e. it is not important how they intersect) to the same circle or to another, which can only start at an arrow tail and end at an arrow head. Informally, it is a pairing of the points $\sqcup_{i=0}^{k} [n_i]$ and $\sqcup_{i=0}^{k} [m_i]$, much like an electric circuit board with "internal" circles being placeholders for chips. In Figure 2.1 can be seen an example of an ocd with $k = 2,\ l = 1,\ n_0 = 1,\ m_0 = 1,\ n_1 = 2,\ m_1 = 1,\ n_2 = 2,\ m_2 = 3$.

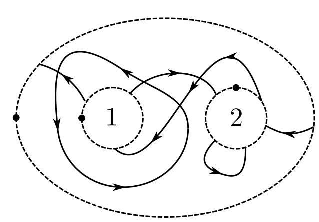


Figure 2.1: An example of an oriented circuit diagram.

Oriented OCDs can also be composed, as long as the marked points match. More precisely, we can compose an OCD D with parameters $(k; ([n_i], [m_i]), i = 0, 1, \ldots, k)$ with OCDs $D_1, D_2, \ldots, D_k$ with *parameters* $(d_i; ([n_{ij}], [m_{ij}]), i = 1, \ldots, k, j = 0, 1, \ldots, d_i)$ *to obtain an OCD* $D(D_1, D_2, \ldots, D_k)$ *with parameters* $(\sum_{i=1}^{k} d_i; ([n_0], [m_0]), ([n_{ij}], [m_{ij}]), i = 1, \ldots, k, j = 1, \ldots, d_i)$ *precisely when* $n_i = n_{i0}, m_i = m_{i0}\ \forall\ i = 1, \ldots, k$. *In Figure 2.2 we give an example of a composition of compatible ocd's.*

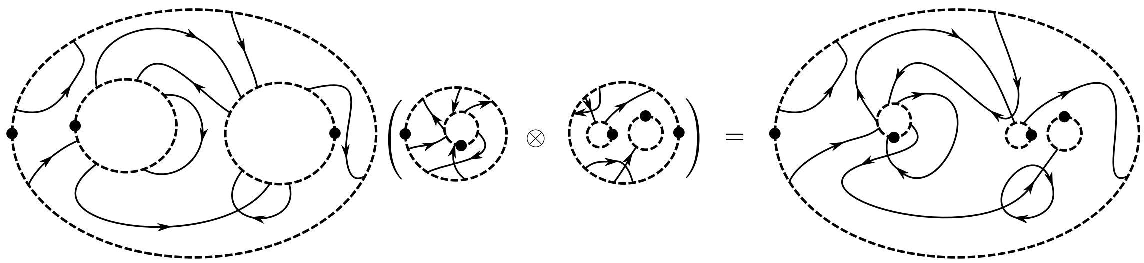


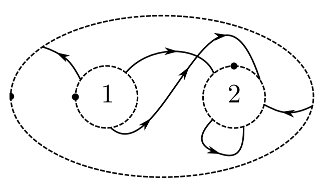
Figure 2.2: An example of a composition of oriented circuit diagrams.

The strands connect combinatorially to the same circle or to another, and the diagram represents an operation $V_{n_1,m_1} \otimes \ldots \otimes V_{n_k,m_k} \longrightarrow V_{n_0,m_0}$ (or $V_{n_1,m_1} \times \ldots \times V_{n_k,m_k} \longrightarrow V_{n_0,m_0}$ if we are dealing with sets).

These combinatorial diagrams correspond to algebraic operations in the structure of an oriented circuit algebra, which we define next.

Definition 2.1.2 *An **oriented circuit algebra** is an algebraic structure, whose operations are indexed by oriented circuit diagrams. Namely, it is a collection $\mathcal{V}$ of objects and a collection $\mathcal{F}$ of operations, where:*

- *For each pair $(n, m) \in \mathbb{Z}^2_{\geq 0}$ there is a set of objects $V_{n,m}$. The objects themselves can be sets, vector spaces, modules, etc.*
- *To every oriented circuit diagram D (see Figure 2.3), there is a corresponding operation, or morphism (of sets, vector spaces, modules, etc.) F_D.*



$: V_{3,0} \otimes V_{1,4} \longrightarrow V_{1,1}$

Figure 2.3: A circuit diagram representing an operation on a pair of objects.

Analogously to planar algebras for usual tangles, introduced by Vaughan Jones [Jon99], circuit algebras provide a natural setting for virtual tangles. Indeed, virtual tangles $v\mathcal{T}$ have a simple presentation as a circuit algebra with just two generators:

$$\mathrm{CA}\left\langle ⤯,\ ⤰ \right\rangle / (\text{Real Reidemeister Moves})$$

The Virtual Reidemeister Moves required to represent virtual tangles as diagrams come for free, from the internal structure of circuit diagrams. In this oriented circuit algebra quotient:

- The collections of objects are indexed by $(n, m) \in \mathbb{Z}^2_{\geq 0}$ such that:

$$V_{n,m} = \begin{cases} \emptyset, \text{ if } n \neq m \\ \text{tangles with } n \text{ open strands and possibly some closed components}, \text{ if } n = m \end{cases}$$

- The "gluing" operation within circuit diagrams is the concatenation of tangle strands.

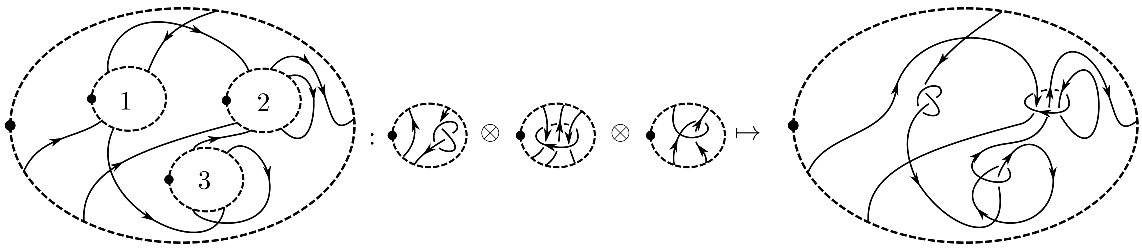


Figure 2.4: A circuit algebra operation on three tangles.

2.2 The construction of tMVA

The target space for Archibald's tMVA invariant [Arch10] is also a circuit algebra, $\mathcal{AHD}$, with the only nonempty sets of objects being $V_n := V_{n,n}$, given by:

$$V_n = \text{AHD}([n^{\text{in}}], [n^{\text{out}}]) := \Lambda^n([n^{\text{out}}]) \otimes \Lambda^{n/2}([n^{\text{in}}] \cup [n^{\text{out}}])$$

where $[n^{\text{in}}] = \{a_1,\ a_2, \ldots,\ a_n\}$ and $[n^{\text{out}}] = \{b_1,\ b_2, \ldots, b_n\}$ are n-element sets. Given $k \in \mathbb{Z}_{\geq 0}$ and S a set of size m,

$$\Lambda^k(S) := \text{the } k\text{-th exterior power of the vector space with (formal) basis } S.$$ [1]

The morphisms are given by "gluing" via the interior multiplication. Namely, for a morphism F_D : $V_{n_1} \otimes \ldots \otimes V_{n_m} \longrightarrow V_{n_0}$ with gluing prescribed by its circuit diagram D, assign the same labels to the elements to be glued in $p_j \otimes q_j \in V_{n_j},\ j = 1, \ldots, m$ [2] , call that set $S = \bigcup_{j=1}^m [n_j^{\text{in}}] \cap \bigcup_{j=1}^m [n_j^{\text{out}}]$. Then the gluing map is:

$$i_S\left(\bigwedge_{j=1}^m p_j\right) \otimes i_S\left(\bigwedge_{j=1}^m q_j\right) \in \text{AHD}\left(\bigcup_{j=1}^m [n_j^{\text{in}}] - S, \bigcup_{j=1}^m [n_j^{\text{out}}] - S\right)$$

where i_S is interior multiplication.

The tMVA invariant then gives a circuit algebra morphism between tangles and Alexander half densities:

$$\text{tMVA} : (v\mathcal{T}, \text{concatenation}) \longrightarrow (\mathcal{AHD}, \text{interior multiplication})$$

To compute the tMVA on a regular, oriented v-tangle T, we first need the **Alexander matrix** $M(D_T)$ for a diagram D_T of T. It is indexed by the *arcs* of the tangle, i.e. segments of the strands

[1] Strictly speaking, for a tangle with K strands in total (open and closed), we are taking the exterior power of the $\mathbb{R}[t_1, \ldots, t_K]$-module with basis S.

[2] More formally, we also need to keep track of the closed components in the tangle.

beginning and ending at an undercrossing or the outer circle. Let X^{in} be the set labelling the incoming arcs, and X^{out} the set of outgoing arcs of the tangle, $|X^{\text{in}}| = |X^{\text{out}}| = n$, then the Alexander matrix is of the form:

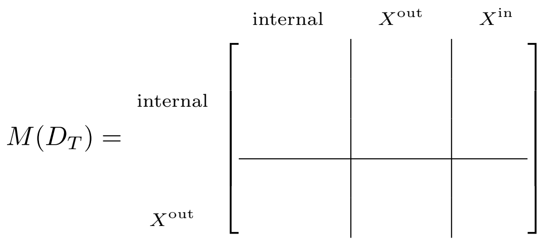


Figure 2.5: The structure of the Alexander matrix $M(D_T)$ of a tangle diagram D_T.

Remark 2.2.1 *To guarantee we have an equal number of distinct incoming and outgoing labels, we might also need to artificially break an arc b into two differently labeled arcs b and a. This results in an additional row and column of the matrix, both indexed a, and does not affect the final expression for the invariant (See [Arch10], Lemma 3.9).*

In addition, we assign a variable to each strand of the tangle, so we have $\{t_i\}_{i=1}^{n+m}$ where n is the number of open and m is the number of closed components. The rows of the matrix are then obtained as below for each local piece of the tangle. If a column index does not appear, it is understood that the corresponding entry in the given row is zero.

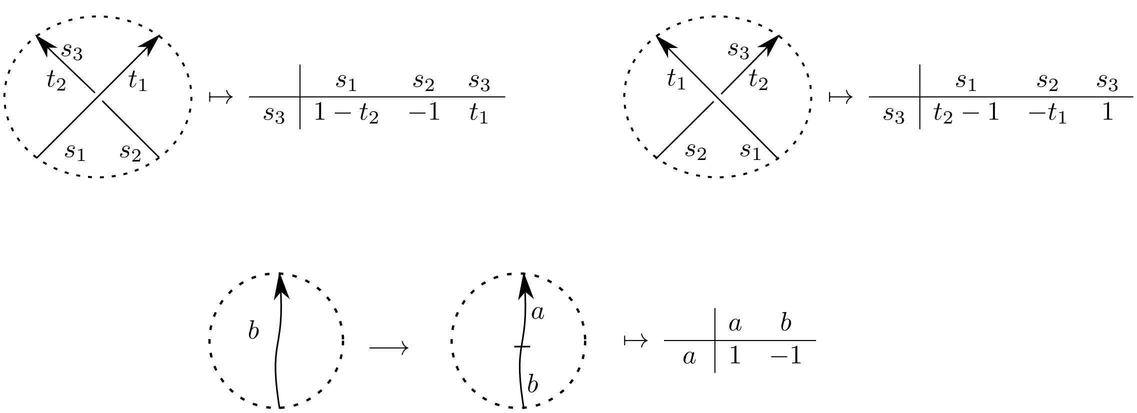


Figure 2.6: The local rules for building the Alexander matrix $M(D_T)$ of a tangle diagram D_T.

Here s_i denotes the label of an arc, and t_j is the variable corresponding to the j^{th} strand of the tangle. From the Alexander matrix, Archibald defines the tangle invariant:

$$\text{tMVA}(T) = \prod_{s=1}^{n+m} t_s^{-\frac{\mu(s)}{2}} w \otimes \sum_{k=0}^{n} \sum_{\substack{i_1<\ldots<i_{n-k} \\ j_1<\ldots<j_k}} \det M(D_T)^{i_1,\ldots,i_{n-k};j_1,\ldots,j_k} b_{i_1} \wedge \ldots \wedge b_{i_{n-k}} \wedge a_{j_1} \wedge \ldots \wedge a_{j_k}$$

The formula consists of the following ingredients:

1) $\mu(s)$ counts the number of times the s^{th} strand is the overstrand in a crossing.

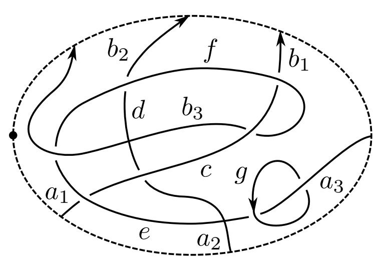
2) $w \in \Lambda^n(X^{\text{out}})$ is a choice of ordering of the elements of X^{out} as they appear in the rows (and columns) of the Alexander matrix.

3) $M(D_T)^{i_1,\ldots,i_{n-k};j_1,\ldots,j_k}$ is the submatrix of $M(D_T)$ with columns indexed by all the internal arcs, as well as $\{b_{i_1},\ldots,b_{i_{n-k}}\} \in X^{\text{out}}$ and $\{a_{j_1},\ldots,a_{j_k}\} \in X^{\text{in}}$.

The normalizing factor $\prod_{s=1}^{n+m} t_s^{-\frac{\mu(s)}{2}}$ belongs to $\mathbb{R}[t_1^{\pm\frac{1}{2}},\ldots,t_{n+m}^{\pm\frac{1}{2}}]$, while the remaining expression lives in the Alexander half density space:

$$\text{AHD}(X^{\text{in}}, X^{\text{out}}) = \Lambda^n(X^{\text{out}}) \otimes \Lambda^n(X^{\text{in}} \cup X^{\text{out}})$$

Example. Below we compute the Alexander matrix for the given tangle diagram and the tMVA invariant of the corresponding tangle. The expression might appear considerable but the computations are straightforward and can be delegated to a computer.

$D_T =$ 

$M(D_T) =$

	c	d	e	f	g	b_1	b_2	b_3	a_1	a_2	a_3
c	1	0	t_1-1	0	0	0	0	0	$-t_3$	0	0
d	$1-t_2$	t_1	0	0	0	0	0	0	0	-1	0
e	0	0	1	0	t_3-1	0	0	0	0	0	$-t_4$
f	0	0	$-t_3$	1	0	0	0	t_3-1	0	0	0
g	0	0	0	0	$1-t_3$	0	0	0	0	0	t_4-1
b_1	-1	0	0	$1-t_1$	0	t_3	0	0	0	0	0
b_2	0	-1	0	$1-t_2$	0	0	t_3	0	0	0	0
b_3	$1-t_3$	0	0	-1	0	0	0	t_1	0	0	0

$$
\begin{aligned}
\mathrm{tMVA}(T) \quad &= t_1^{-1}t_3^{-\frac{5}{2}}t_4^{-\frac{1}{2}}\; b_1\wedge b_2\wedge b_3\otimes (t_3-1)t_3\,[\,(t_1-1)^2(t_3-1)\; b_3\wedge a_2\wedge a_3+\\
&+(t_3t_1-t_1-2t_3+1)\; b_1\wedge a_2\wedge a_3 - t_3(t_3t_1-t_1-t_3)\; a_1\wedge a_2\wedge a_3+\\
&+(t_2-1)(t_3t_1-t_1-1)t_3^2\; b_1\wedge a_1\wedge a_3+t_1(t_3-1)t_3^2\; b_1\wedge b_2\wedge a_1-\\
&t_1(t_3t_1-t_1-t_3)t_3^2\; b_2\wedge a_1\wedge a_3-t_1(t_3t_1-2t_1-t_3+1)t_3^2\; b_2\wedge b_3\wedge a_1+\\
&+(t_3-1)t_3\; b_1\wedge a_1\wedge a_2-t_1(t_3t_1-t_1-2t_3+1)t_3\; b_1\wedge b_2\wedge a_3+\\
&+(t_2-1)(t_1t_3^2-2t_1t_3-t_3+1)t_3\; b_1\wedge b_3\wedge a_1+(t_1-1)^2t_1(t_3-1)t_3\; b_2\wedge b_3\wedge a_3+\\
&+(t_3t_1-2t_1-t_3+1)t_3\; b_3\wedge a_1\wedge a_2-t_1(t_2-1)t_3\; b_3\wedge a_1\wedge a_3-\\
&-(t_1+t_3-1)\; b_1\wedge b_3\wedge a_2-t_1(t_1+t_3-1)t_3\; b_1\wedge b_2\wedge b_3\\
&-(t_2-1)(t_3^2t_1^2-t_3t_1^2-t_3^2t_1+t_3t_1+t_1+t_3-1)\; b_1\wedge b_3\wedge a_3\;]
\end{aligned}
$$

Figure 2.7: The tMVA invariant evaluated on the tangle T.

2.3 The Hodge star operator

We can apply the Hodge star operator $*_w$, for a given choice of w (i.e. choice of ordering of the elements of X^{out}), to the second tensor component of $\mathrm{AHD}(X^{\text{out}}, X^{\text{in}})$:

$$
\begin{array}{ccc}
\bigoplus_{k=0}^n \Lambda^k(X^{\text{in}})\otimes\Lambda^{n-k}(X^{\text{out}}) & \xrightarrow{\quad *_w \quad} & \bigoplus_{k=0}^n \Lambda^k(X^{\text{in}})\otimes\Lambda^{k}(X^{\text{out}})\\
\uparrow\cong & & \downarrow\cong\\
\Lambda^n(X^{\text{in}}\cup X^{\text{out}}) & & \bigoplus_{k=0}^n \mathrm{Hom}(\Lambda^k(X^{\text{in}}),\Lambda^k(X^{\text{out}}))
\end{array}
\tag{2.1}
$$

It acts as the identity on $\Lambda^k(X^{\text{in}})$, on the basis $\{b_{i_1}\wedge\ldots\wedge b_{i_{n-k}}\}_{1\le i_1<\ldots<i_{n-k}\le n}$ of $\Lambda^{n-k}(X^{\text{out}})$ (where $X^{\text{out}}=\{b_1,\ldots,b_n\}$) as described below, and we extend linearly:

$$b_{i_1}\wedge\ldots\wedge b_{i_{n-k}} \mapsto (-1)^{*_w} b_{i_{n-k+1}}\wedge\ldots\wedge b_{i_n} \;\Leftrightarrow\; (-1)^{*_w} b_{i_1}\wedge\ldots\wedge b_{i_{n-k}}\wedge b_{i_{n-k+1}}\wedge\ldots\wedge b_{i_n}=w$$

Let $p_2:\Lambda^n(X^{\text{out}})\otimes\Lambda^n(X^{\text{in}}\cup X^{\text{out}})\longrightarrow\Lambda^n(X^{\text{in}}\cup X^{\text{out}})$ denote the projection onto the second tensor factor in $\mathrm{AHD}(X^{\text{in}},X^{\text{out}})$.

If we consider, for a tangle T, and a given $w\in\Lambda^n(X^{\text{out}})$ the image of $p_2(\mathrm{tMVA}(T))$ in the space $\bigoplus_{k=0}^n \mathrm{Hom}(\Lambda^k(X^{\text{in}}),\Lambda^k(X^{\text{out}}))$ under $*_w$, it turns out that it is determined by the degree 0 and 1 components.

Theorem 2.3.1 *For a tangle T, let $\lambda\in Hom(\Lambda^0(X^{in}),\Lambda^0(X^{out}))$ denote the degree 0 component, and $\phi\in Hom(\Lambda^1(X^{in}),\Lambda^1(X^{out}))$ the degree 1 component of the image of $p_2(tMVA(T))$ in the space $\bigoplus_{k=0}^n Hom(\Lambda^k(X^{in}),\Lambda^k(X^{out}))$ under the Hodge star operator $*_w$. If $\lambda\neq 0$, then the image is determined by λ and ϕ, it is precisely $\lambda\cdot\Lambda(\phi/\lambda)$.*

The proof will be postponed until Chapter 7.

Let us now consider the degree 0 and 1 components of the image of $p_2(\mathrm{tMVA}(T))$ in the space $\bigoplus_{k=0}^n \Lambda^k(X^{\text{in}})\otimes\Lambda^k(X^{\text{out}})$ using $*_w$.

The degree 0 component is $\lambda=\det M(D_T)^{1,\ldots,n;\emptyset}$ with the earlier notation, namely the submatrix $M(D_T)^{1,\ldots,n;\emptyset}$ of $M(D_T)$ has columns indexed by all internal arcs, as well as all n elements of X^{out} but none from X^{in}.

The degree 1 elements can be expressed as the entries of an $n \times n$ matrix $\mathcal{A}$, where $\mathcal{A}_{i,j}$ is the determinant of $M(D_T)^{1,2,\dots,n;\emptyset}$ with column j in X^{out} replaced by column i from X^{in}:

$$\mathcal{A}_{i,j} = (-1)^{n-i} \det M(D_T)^{1,\dots,\hat{i},\dots,n;j}$$

This $n \times n$ matrix, with the additional degree 0 element and the normalizing factor $\prod_{s=1}^{n} t_s^{-\mu(s)/2}$ in the definition of the tMVA invariant carries the information of tMVA(T).

Theorem 2.3.2 *From the pair $(\lambda, \mathcal{A})$ for a tangle T, when $\lambda \neq 0$ we can recover the coefficients of the tangle invariant tMVA(T) with the formula:*

$$\det M(D_T)^{\{1,\dots,n\}\setminus\{i_1,\dots,i_k\};j_1,\dots,j_k} = (-1)^{nk-\sum_{p=1}^{k} i_p-(k-1)k/2} \frac{\det \mathcal{A}_{i_1,\dots,i_k}^{j_1,\dots,j_k}}{\lambda^{k-1}}$$

where $\mathcal{A}_{i_1,\dots,i_k}^{j_1,\dots,j_k}$ denotes the submatrix of $\mathcal{A}$ with rows $i_1,\dots,i_k$ and columns $j_1,\dots,j_k$.

This proof will also be postponed until Chapter 7. In the next section, we will explore this version of the tMVA invariant further.

Chapter 3

The Reduction of tMVA to rMVA

Since the new pair $(\lambda, \mathcal{A})$ preserves the information of the tMVA, as expected it is also a tangle invariant that we'll denote rMVA ("r" for reduced). Its computation can be simplified similarly to that of tMVA by breaking up a tangle into the generating pieces, i.e. positive and negative crossings, computing the invariant on each piece, and then gluing the pieces back together. For the purpose, we need to know the values of the rMVA on positive and negative crossings, and its behaviour under gluing strands and taking the disjoint union of tangles.

To define rMVA, we restrict to **pure**, regular, v-tangles, i.e. ones without closed components. Then in particular, as we'll see in Proposition 3.3.4, $\lambda \neq 0$ and by Theorem 2.3.2 we can recover tMVA from rMVA. Given such an n-tangle T with incoming and outgoing labels for the strands being the sets X^{in} and X^{out} respectively, we can identify $X^{\text{in}} \cong X^{\text{out}} = X$ from the underlying permutation of the tangle, using the strand variables $\{t_i\}_{i=1}^{n}$. The set X then also provides a labelling for set of strands of the tangle.

If $(\lambda_T, A_T) \in \mathbb{R}(t_i) \times M_{X\times X}(\mathbb{R}(t_i))$ are the pair of degree 0 and 1 components for the image of T as defined above for a fixed order w of X^{out}, by Theorem 2.3.2, tMVA reduces to:

$$\text{rMVA}(T) := \prod_{k=1}^{m} t_k^{-\mu(k)/2}(\lambda, A) \in R_X = \mathbb{R}(\sqrt{t_i}) \times M_{X\times X}(\mathbb{R}(\sqrt{t_i})) \tag{3.1}$$

3.1 Tangle invariant

Since the tMVA is a virtual tangle invariant, and rMVA is a function of it, it also has that property. We will nevertheless check for illustration purposes that it satisfies the real Reidemeister II and III moves as well as the "Overcrossings Commute" relation, making it a welded tangle invariant. We include here the last verification and delegate the remaining ones to Chapter 7. We need to verify that the values of rMVA on each side of the move agree. As the virtual crossings don't contribute to the Alexander matrix, rMVA is automatically invariant under the virtual and mixed Reidemeister moves.

Overcrossings Commute moves.

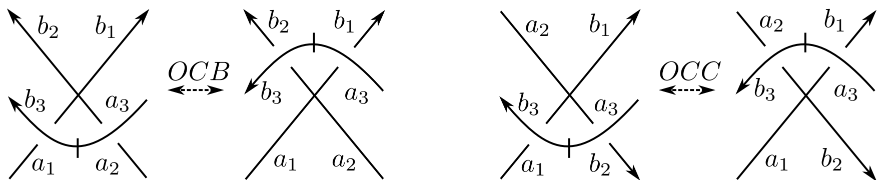


Figure 3.1: The braid-like and cyclic Overcrossings Commute moves.

Starting with the braid-like move, the diagram on the left side D_{LB} gives the following Alexander matrix and corresponding value of the rMVA:

$$M_{LB} = \begin{array}{c|cccccc} & b_1 & b_2 & b_3 & a_1 & a_2 & a_3 \\ \hline b_1 & 1 & 0 & t_1-1 & -t_3 & 0 & 0 \\ b_2 & 0 & 1 & 0 & 0 & -t_3 & t_2-1 \\ b_3 & 0 & 0 & 1 & 0 & 0 & -1 \end{array}$$

$$\Rightarrow \mathrm{rMVA}(D_{LB}) = \prod_{s=1}^{3} t_s^{-\frac{\mu(s)}{2}} \cdot \begin{array}{c|c} \lambda_{LB} & X^{\mathrm{in}} \\ \hline X^{\mathrm{out}} & A_{LB} \end{array} = t_3^{-1} \cdot \begin{array}{c|ccc} 1 & a_1 & a_2 & a_3 \\ \hline b_1 & -t_3 & 0 & t_1-1 \\ b_2 & 0 & -t_3 & t_2-1 \\ b_3 & 0 & 0 & -1 \end{array}$$

The diagram on the right side of the braid-like Overcrossings Commute move, D_{RB}, produces analogously:

$$M_{RB} = \begin{array}{c|cccccc} & b_1 & b_2 & b_3 & a_1 & a_2 & a_3 \\ \hline b_1 & 1 & 0 & 0 & -t_3 & 0 & t_1-1 \\ b_2 & 0 & 1 & t_2-1 & 0 & -t_3 & 0 \\ b_3 & 0 & 0 & 1 & 0 & 0 & -1 \end{array}$$

$$\Rightarrow \mathrm{rMVA}(D_{RB}) = \prod_{s=1}^{3} t_s^{-\frac{\mu(s)}{2}} \cdot \begin{array}{c|c} \lambda_{RB} & X^{\mathrm{in}} \\ \hline X^{\mathrm{out}} & A_{RB} \end{array} = t_3^{-1} \cdot \begin{array}{c|ccc} 1 & a_1 & a_2 & a_3 \\ \hline b_1 & -t_3 & 0 & t_1-1 \\ b_2 & 0 & -t_3 & t_2-1 \\ b_3 & 0 & 0 & -1 \end{array}$$

Analogously, for the left side of the cyclic Overcrossings Commute move, we have:

$$M_{LB} = \begin{array}{c|cccccc} & b_1 & b_2 & b_3 & a_1 & a_2 & a_3 \\ \hline b_1 & 1 & 0 & t_1 - 1 & -t_3 & 0 & 0 \\ b_2 & 0 & t_3 & 0 & 0 & -1 & 1 - t_2 \\ b_3 & 0 & 0 & 1 & 0 & 0 & -1 \end{array}$$

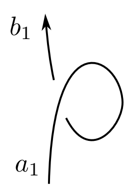
$$\Rightarrow \text{rMVA}(D_{LB}) = \prod_{s=1}^{3} t_s^{-\frac{\mu(s)}{2}} \cdot \begin{array}{c|c} \lambda_{LB} & X^{\text{in}} \\ \hline X^{\text{out}} & A_{LB} \end{array} = t_3^{-1} \cdot \begin{array}{c|ccc} t_3 & a_1 & a_2 & a_3 \\ \hline b_1 & -t_3^2 & 0 & t_3(1-t_1) \\ b_2 & 0 & -1 & 1-t_2 \\ b_3 & 0 & 0 & -T_3 \end{array}$$

The diagram on the right side of the cyclic move produces:

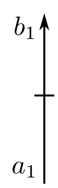
$$M_{RB} = \begin{array}{c|cccccc} & b_1 & b_2 & b_3 & a_1 & a_2 & a_3 \\ \hline b_1 & 1 & 0 & 0 & -t_3 & 0 & t_1 - 1 \\ b_2 & 0 & t_3 & 1 - t_2 & 0 & -1 & 0 \\ b_3 & 0 & 0 & 1 & 0 & 0 & -1 \end{array}$$

$$\Rightarrow \text{rMVA}(D_{RB}) = \prod_{s=1}^{3} t_s^{-\frac{\mu(s)}{2}} \cdot \begin{array}{c|c} \lambda_{RB} & X^{\text{in}} \\ \hline X^{\text{out}} & A_{RB} \end{array} = t_3^{-1} \cdot \begin{array}{c|ccc} t_3 & a_1 & a_2 & a_3 \\ \hline b_1 & -t_3^2 & 0 & t_3(1-t_1) \\ b_2 & 0 & -1 & 1-t_2 \\ b_3 & 0 & 0 & -T_3 \end{array}$$

Remark 3.1.1 *Note that rMVA does not satisfy the real Reidemeister I move. Comparing the values of rMVA on the two diagrams, we see that they differ:*



$$\xrightarrow{M} \begin{array}{c|cc} & b_1 & a_1 \\ \hline b_1 & t_1 & -t_1 \end{array} \qquad \xrightarrow{\text{rMVA}} t_1^{-1/2} \cdot \begin{array}{c|c} t_1 & a_1 \\ \hline b_1 & -t_1 \end{array}$$

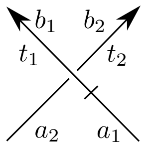


$$\xrightarrow{M} \begin{array}{c|cc} & b_1 & a_1 \\ \hline b_1 & 1 & -1 \end{array} \qquad \xrightarrow{\text{rMVA}} 1 \cdot \begin{array}{c|c} 1 & a_1 \\ \hline b_1 & -1 \end{array}$$

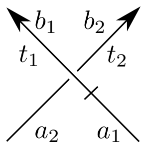
After verifying that rMVA is indeed a tangle invariant, we simplify its description by recovering its values on positive and negative crossings, as well as the strand gluing and disjoint union maps in the target space induced from those for the tMVA invariant in the Alexander half-density spaces.

3.2 Positive and negative crossings

The Alexander matrices for the positive and negative crossings are obtained after splitting the overcrossing arc into two. From them, we find the degree 0 and 1 matrices for rMVA:



$$\xrightarrow{M} \begin{array}{c|cccc} & b_1 & b_2 & a_1 & a_2 \\ \hline b_1 & 1 & 0 & -1 & 0 \\ b_2 & 1-t_2 & t_1 & 0 & -1 \end{array} \xrightarrow{\text{rMVA}} t_1^{-1/2} \cdot \begin{array}{c|cc} t_1 & a_1 & a_2 \\ \hline b_1 & -t_1 & 0 \\ b_2 & 1-t_2 & -1 \end{array}$$



$$\xrightarrow{M} \begin{array}{c|cccc} & b_1 & b_2 & a_1 & a_2 \\ \hline b_1 & 1 & 0 & -1 & 0 \\ b_2 & t_2-1 & 1 & 0 & -t_1 \end{array} \xrightarrow{\text{rMVA}} t_1^{-1/2} \cdot \begin{array}{c|cc} 1 & a_1 & a_2 \\ \hline b_1 & -1 & 0 \\ b_2 & t_2-1 & -t_1 \end{array}$$

Figure 3.2: The Alexander matrices and resulting degree 0 and 1 pairs for the positive and negative crossing tangles.

3.3 Gluing and disjoint union

The operations of gluing and disjoint union on tMVA(T) in the target space AHD(X) of tMVA for a pure tangle T with labels X ($= X^{\text{out}} \cong X^{\text{in}}$), corresponding to gluing two strands of the tangle or taking the disjoint union of two tangles, can also be studied on the level of the Alexander matrix. We will do so to find their reduction to analogous operations on the pairs:

$$\{\text{rMVA}(T) = (\lambda_T, \mathcal{A}_T)\}_{T\text{-pure tangle}} = \left\{ \left(\prod_{s=1}^{|X|} t_s^{-\frac{\mu(s)}{2}} \widetilde{\lambda}_T, \prod_{s=1}^{|X|} t_s^{-\frac{\mu(s)}{2}} \widetilde{\mathcal{A}}_T \right) \right\}_{T\text{-pure tangle}}$$

in the target space $R_X = \mathbb{R}(\sqrt{t_i}) \times M_{X\times X}(\mathbb{R}(\sqrt{t_i}))$ of rMVA under the map taking the Alexander matrix to such a pair, $M(D_T) \longrightarrow (\lambda_T, \mathcal{A}_T)$, where $(\widetilde{\lambda}_T, \widetilde{\mathcal{A}_T})$ are the original degree 0 and 1 components corresponding to T.

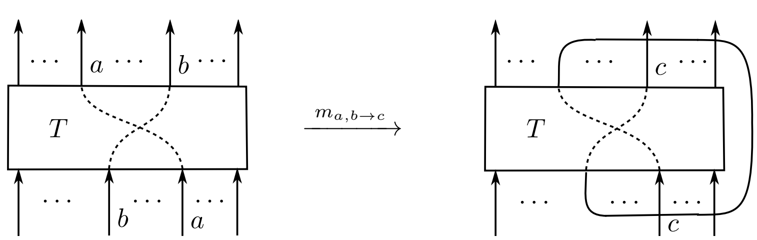
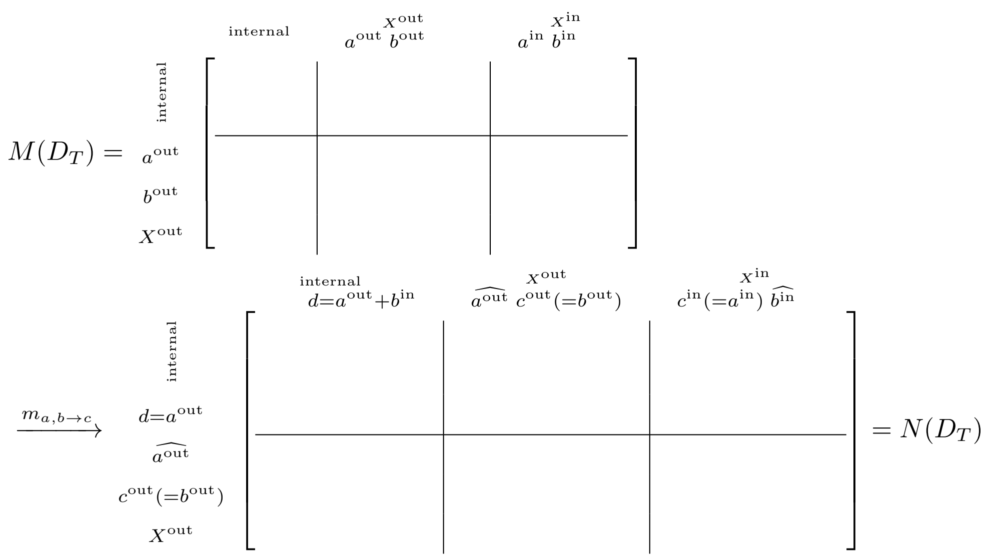


Figure 3.3: Gluing the strands labelled "a" and "b" of a tangle T, and calling the resulting strand "c".

Lemma 3.3.1 ***(Gluing)*** *The result of gluing the outgoing strand labeled "a" $\in X$ in a tangle T, with ends labeled X, to the incoming strand labeled "b" $\in X$ and calling the resulting strand c for instance, as illustrated in Figure 3.3, corresponds to the following map on the level of the target space R_X of rMVA. (For the moment we restrict to gluing different strands so we don't get closed components.)*

$$\begin{array}{c|ccc} \lambda & a & b & X^{\text{in}} \\ \hline a & \alpha & \beta & \theta \\ b & \gamma & \delta & \epsilon \\ X^{\text{out}} & \phi & \psi & \Xi \end{array} \xrightarrow{m_{a,b\to c}} \left(\begin{array}{c|cc} \lambda+\beta & c & X^{\text{in}} \\ \hline c & \gamma+\frac{\beta\gamma-\alpha\delta}{\lambda} & \epsilon+\frac{\beta\epsilon-\delta\theta}{\lambda} \\ X^{\text{out}} & \phi+\frac{\beta\phi-\alpha\psi}{\lambda} & \Xi+\frac{\beta\Xi-\psi\theta}{\lambda}\end{array}\right)_{t_a,t_b\to t_c}$$

Proof: The gluing operation $m_{a,b\to c}$ on a tangle T pictured in Figure 3.3 has the following effect on the Alexander matrix:



Denote by $M(D_T)$ and rMVA$(T) = (\lambda, \mathcal{A})$ the original Alexander matrix and rMVA invariant values, and the result after gluing by $N(D_T) := m_{a,b\to c}(M(D_T))$ and rMVA$(m_{a,b\to c}(T)) = m_{a,b\to c}(\lambda, \mathcal{A}) =: (\omega, \mathcal{B})$. To obtain $N(D_T)$, we delete column a^{out} in X^{out} and column b^{in} in X^{in} and include a new internal arc with column d, which is the sum of the original columns a^{out} and b^{in}. Furthermore, we need to remove row a^{out} in X^{out} and include it, now labeled l, with the rows labeled by the internal arcs. We call the overall new strand obtained after gluing a and b by c, so in the columns we'll have c^{out} be equal the old b^{out}, c^{in} be the old a^{in}, and in the rows c^{out} is equal to the old b^{out}. From the matrix $N(D_T)$, we can compute directly the resulting invariant $(\omega, \mathcal{B})$ from the definition:

$$
\begin{aligned}
\omega &= \det N(T)^{1,\dots,n;\emptyset} \\
&= \det \begin{array}{c|c}
 & \text{(internal) } l{=}a^{out}{+}b^{in} \;\; \vert \;\; (X^{out})\; \widehat{a^{out}}\; c^{out}(=b^{out}) \\
\text{internal} & \left[\begin{array}{c|c} & \\ \hline & \end{array}\right] \\
l{=}a^{out},\; \widehat{a^{out}},\; c^{out}(=b^{out}),\; X^{out} &
\end{array} \\
&= \det \begin{array}{c|c}
 & \text{(internal) } \widehat{l} \;\; \vert \;\; (X^{out})\; a^{out}\; b^{out} \\
\text{internal} & \left[\begin{array}{c|c} & \\ \hline & \end{array}\right] \\
\widehat{l},\; a^{out},\; b^{out},\; X^{out} &
\end{array}
+ \det \begin{array}{c|c}
 & \text{(internal) } \widehat{l} \;\; \vert \;\; (X^{out})\; b^{in}\; b^{out} \\
\text{internal} & \left[\begin{array}{c|c} & \\ \hline & \end{array}\right] \\
\widehat{l},\; a^{out},\; b^{out},\; X^{out} &
\end{array} \\
&= \lambda + \mathcal{A}_{a,b} = \lambda + \beta
\end{aligned}
$$

Similarly, for the entries of $\mathcal{B}$, from the definition we have that $\mathcal{B}_{d,e}$ for $d \in (X^{out} \setminus \{a\})_{b\to c}$ and $e \in (X^{in} \setminus \{b\})_{a\to c}$ is the determinant of the matrix $N(T)^{1,\dots,n;\emptyset}$ with column "d" in X^{out} replaced with column "e" from X^{in}:

$$
\begin{aligned}
\mathcal{B}_{d,e} &= \det \begin{array}{c|c}
 & \text{(internal) } l{=}a^{out}{+}b^{in} \;\; \vert \;\; (X^{out})\; \widehat{a^{out}}\; c^{out}(=b^{out})\; (\widehat{d^{out}})\, e^{in} \\
\text{internal} & \left[\begin{array}{c|c} & \\ \hline & \end{array}\right] \\
l{=}a^{out},\; \widehat{a^{out}},\; c^{out}(=b^{out}),\; X^{out} &
\end{array} \\
&= \det \begin{array}{c|c}
 & \text{(internal) } \widehat{l} \;\; \vert \;\; (X^{out})\; a^{out}\; b^{out}\; (\widehat{d^{out}})\, e^{in} \\
\text{internal} & \left[\begin{array}{c|c} & \\ \hline & \end{array}\right] \\
\widehat{l},\; a^{out},\; b^{out},\; X^{out} &
\end{array}
+ \det \begin{array}{c|c}
 & \text{(internal) } \widehat{l} \;\; \vert \;\; (X^{out})\; b^{in}\; b^{out}\; (\widehat{d^{out}})\, e^{in} \\
\text{internal} & \left[\begin{array}{c|c} & \\ \hline & \end{array}\right] \\
\widehat{l},\; a^{out},\; b^{out},\; X^{out} &
\end{array} \\
&= \mathcal{A}_{d',e'} + \frac{\det \mathcal{A}^{b,e'}_{a,d'}}{\lambda}
\end{aligned}
$$

Where the last equality follows from the definition of the degree 1 matrix $\mathcal{A}$ and Theorem 2.3.2:

$$d' = \begin{cases} d, \text{ if } d \in X^{\text{out}} \setminus \{a,b\} \\ b, \text{ if } d = c \end{cases} \qquad e' = \begin{cases} e, \text{ if } e \in X^{\text{in}} \setminus \{a,b\} \\ a, \text{ if } e = c \end{cases}$$

□

Lemma 3.3.2 ***(Disjoint Union)*** *When taking the disjoint union of two tangles T_1 and T_2, i.e. putting them side by side to create a new tangle, the resulting map in the target space of rMVA is:*

$$\begin{array}{c|c} \lambda_1 & X_1^{in} \\ \hline X_1^{out} & A_1 \end{array} \cup \begin{array}{c|c} \lambda_2 & X_2^{in} \\ \hline X_2^{out} & A_2 \end{array} = \begin{array}{c|cc} \lambda_1 \cdot \lambda_2 & X_1^{in} & X_2^{in} \\ \hline X_1^{out} & \lambda_2 A_1 & 0 \\ X_2^{out} & 0 & \lambda_1 A_2 \end{array}$$

Proof: The Alexander matrix for a diagram of the disjoint union $T_1 \cup T_2$ of the two tangles will be of the form:

$$M(D_{T_1 \cup T_2}) = \begin{array}{c} \\ \text{int}_1 \\ \text{int}_2 \\ X_1^{\text{out}} \\ X_2^{\text{out}} \end{array} \begin{array}{c} \begin{array}{cccccc} \text{int}_1 & \text{int}_2 & X_1^{\text{out}} & X_2^{\text{out}} & X_1^{\text{in}} & X_2^{\text{in}} \end{array} \\ \left[\begin{array}{c|c|c|c|c|c} & O & & O & & O \\ \hline O & & O & & O & \\ \hline & O & & O & & O \\ \hline O & & O & & O & \end{array}\right] \end{array}$$

Suppose $|X_1^{\text{out}}| = |X_1^{\text{in}}| = n_1,\ |X_2^{\text{out}}| = |X_2^{\text{in}}| = n_2$, and $\text{rMVA}(T_1) = (\lambda_1, \mathcal{A}_1),\ \text{rMVA}(T_2) = (\lambda_2, \mathcal{A}_2)$. Then for the resulting invariant $\text{rMVA}(T_1 \cup T_2) = (\omega, \mathcal{B})$, we have:

$$\omega = \det \begin{array}{c} \\ \text{int}_1 \\ \text{int}_2 \\ X_1^{\text{out}} \\ X_2^{\text{out}} \end{array} \begin{array}{c} \begin{array}{cccc} \text{int}_1 & \text{int}_2 & X_1^{\text{out}} & X_2^{\text{out}} \end{array} \\ \left[\begin{array}{c|c|c|c} & O & & O \\ \hline O & & O & \\ \hline & O & & O \\ \hline O & & O & \end{array}\right] \end{array} = \det \begin{array}{c} \\ \text{int}_1 \\ X_1^{\text{out}} \\ \text{int}_2 \\ X_2^{\text{out}} \end{array} \begin{array}{c} \begin{array}{cccc} \text{int}_1 & X_1^{\text{out}} & \text{int}_2 & X_2^{\text{out}} \end{array} \\ \left[\begin{array}{c|c|c|c} & & O & O \\ \hline & & O & O \\ \hline O & O & & \\ \hline O & O & & \end{array}\right] \end{array}$$

Namely, $\omega = \lambda_1 \lambda_2$. The (i,j)-th entry of the $(n_1+n_2) \times (n_1+n_2)$ matrix $\mathcal{B}$ is obtained by taking the determinant of $M(D_{T_1 \cup T_2})^{1,\ldots,n_1+n_2;\emptyset}$ with the the i-th column among the labels $X_1^{\text{out}} \cup X_2^{\text{out}}$ replaced with the j-th column among the labels $X_1^{\text{in}} \cup X_2^{\text{in}}$. If we replace the i-th column from X_2^{out} with the j-th column, a_j^1 from X_1^{in} we get a zero determinant, i.e. $\mathcal{B}_{n_1+i,j} = 0,\ \forall\, 1 \le i \le n_2,\ 1 \le j \le n_1$:

$$
\begin{aligned}
\mathcal{B}_{n_1+i,j} &= \det \begin{array}{c} \\ \text{int}_1 \\ \text{int}_2 \\ X_1^{\text{out}} \\ X_2^{\text{out}} \end{array}
\begin{array}{c|c|c|ccc}
\text{int}_1 & \text{int}_2 & X_1^{\text{out}} & X_2^{\text{out}} & a_j^1 & \\
\hline
 & O & & 0 & * & 0 \\
\hline
O & & O & * & 0 & * \\
\hline
 & O & & 0 & * & 0 \\
\hline
O & & O & * & 0 & *
\end{array} \\
&= (-1)^r \det \begin{array}{c} \\ \text{int}_1 \\ X_1^{\text{out}} \\ \text{int}_2 \\ X_2^{\text{out}} \end{array}
\begin{array}{c|c|c|ccc}
\text{int}_1 & X_1^{\text{out}} & \text{int}_2 & X_2^{\text{out}} & a_j^1 & \\
\hline
 & & O & 0 & * & 0 \\
\hline
 & & O & 0 & * & 0 \\
\hline
O & O & & * & 0 & * \\
\hline
O & O & & * & 0 & *
\end{array} \\
&= (-1)^r \det \begin{array}{c} \\ \text{int}_1 \\ X_1^{\text{out}} \end{array}
\begin{array}{c|c}
\text{int}_1 & X_1^{\text{out}} \\
\hline
 & \\
\hline
 &
\end{array}
\det \begin{array}{c} \\ \text{int}_2 \\ X_2^{\text{out}} \end{array}
\begin{array}{c|ccc}
\text{int}_2 & X_2^{\text{out}} & a_j^1 & \\
\hline
 & * & 0 & * \\
\hline
 & * & 0 & *
\end{array} = 0
\end{aligned}
$$

Where $(-1)^r$ is the sign obtained from rearranging the rows and columns. We can show analogously for a choice of columns from X_1^{out} and X_2^{in}, that $\mathcal{B}_{i,n_1+j} = 0,\ \forall\, 1 \leq i \leq n_1,\ 1 \leq j \leq n_2$.

In the more interesting case, if we replace the i-th column from X_1^{out} with the j-th column, a_j^1 from X_1^{in} we get:

$$
\begin{aligned}
\mathcal{B}_{i,j} &= \det \begin{array}{c} \\ \text{int}_1 \\ \text{int}_2 \\ X_1^{\text{out}} \\ X_2^{\text{out}} \end{array}
\begin{array}{c|c|cc|c}
\text{int}_1 & \text{int}_2 & X_1^{\text{out}} & a_j^1 & X_2^{\text{out}} \\
\hline
 & O & & & O \\
\hline
O & & O & 0 & \\
\hline
 & O & & & O \\
\hline
O & & O & 0 &
\end{array} \\
&= \det \begin{array}{c} \\ \text{int}_1 \\ X_1^{\text{out}} \\ \text{int}_2 \\ X_2^{\text{out}} \end{array}
\begin{array}{c|cc|c|c}
\text{int}_1 & X_1^{\text{out}} & a_j^1 & \text{int}_2 & X_2^{\text{out}} \\
\hline
 & & & O & O \\
\hline
 & & & O & O \\
\hline
O & O & 0 & & \\
\hline
O & O & 0 & &
\end{array} \\
&= \det \begin{array}{c} \\ \text{int}_1 \\ X_1^{\text{out}} \end{array}
\begin{array}{c|cc}
\text{int}_1 & X_1^{\text{out}} & a_j^1 \\
\hline
 & & \\
\hline
 & &
\end{array}
\det \begin{array}{c} \\ \text{int}_2 \\ X_2^{\text{out}} \end{array}
\begin{array}{c|c}
\text{int}_2 & X_2^{\text{out}} \\
\hline
 & \\
\hline
 &
\end{array} \\
&= (\mathcal{A}_1)_{i,j} \cdot \lambda_2
\end{aligned}
$$

Similarly, replacing the i-th column from X_2^{out} with the j-th column from X_2^{in} we get:

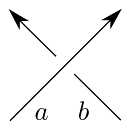
$$\mathcal{B}_{n_1+i,n_1+j} = (\mathcal{A}_2)_{i,j} \cdot \lambda_1,\ \forall\, 1 \leq i, j \leq n_2.$$

□

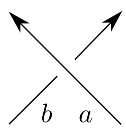
Remark 3.3.3 *For the gluing operation of rMVA to be well-defined and for this invariant to determine* *tMVA we need, when computing the normalized pair $(\lambda_T, \mathcal{A}_T)$ for a tangle T, to have $\lambda_T \neq 0$. Note that in the positive and negative crossings, if we set all the variables $t_i = 1$, we get the pair $(1, -I_{2\times 2})$. Furthermore, the operations of gluing (as long as we're gluing different strands and not creating closed components) and disjoint union preserve that property, as we show in Proposition 3.3.4, so for pure tangles it is always true that $\lambda \neq 0$.*

Proposition 3.3.4 *For any pure tangle T, in the corresponding normalized degree 0 and 1 pair $(\lambda_T, \mathcal{A}_T)$, we always have $\lambda_T \neq 0$.*

Proof: It suffices to check that this property is true for the positive and negative crossing generators, and that it is preserved by the disjoint union and gluing maps.



$$\xrightarrow{\text{rMVA}} t_a^{-1/2} \cdot \begin{array}{c|cc} t_a & a & b \\ \hline a & -t_a & 0 \\ b & 1-t_b & -1 \end{array} \xrightarrow{t_i=1} \begin{array}{c|cc} 1 & a & b \\ \hline a & -1 & 0 \\ b & 0 & -1 \end{array}$$



$$\xrightarrow{\text{rMVA}} t_a^{-1/2} \cdot \begin{array}{c|cc} 1 & a & b \\ \hline a & -1 & 0 \\ b & t_b-1 & -t_a \end{array} \xrightarrow{t_i=1} \begin{array}{c|cc} 1 & a & b \\ \hline a & -1 & 0 \\ b & 0 & -1 \end{array}$$

In particular, since the λ_T values for the positive and negative crossings satisfy $\lambda_+(1) = \lambda_-(1) = 1$, then they are not identically zero. We will show that for any n-component pure tangle T, $(\lambda_T, \mathcal{A}_T)_{t_i=1} = (1, -I_{n\times n})$. That would mean in particular that λ_T is not identically zero. It suffices to check this property is preserved under gluing and disjoint union.

$$\begin{array}{c|ccc} \lambda & a & b & X \\ \hline a & \alpha & \beta & \theta \\ b & \gamma & \delta & \epsilon \\ X & \phi & \psi & \Xi \end{array} \xrightarrow{m_{a,b\to c}} \left(\begin{array}{c|cc} \lambda+\beta & c & X \\ \hline c & \gamma + \frac{\beta\gamma-\alpha\delta}{\lambda} & \epsilon + \frac{\beta\epsilon-\delta\theta}{\lambda} \\ X & \phi + \frac{\beta\phi-\alpha\psi}{\lambda} & \Xi + \frac{\beta\Xi-\psi\theta}{\lambda} \end{array} \right)_{t_a,t_b\to t_c}$$

Assuming that in the original matrix $\lambda(1) = 1,\ \alpha(1) = \delta(1) = -1,\ \Xi(1) = -I_{(n-2)\times(n-2)},\ \beta(1) = \gamma(1) = 0,\ \theta(1) = \epsilon(1) = (\phi(1))^{\text{tr}} = (\psi(1))^{\text{tr}} = (0,\ldots,0)$, then the element and matrix pair resulting from the gluing operation $m_{a,b\to c}$ evaluated at $t_i = 1$ is if the form $(1, -I_{(n-1)\times(n-1)})$:

$$(\lambda+\beta)(1) = 1 + 0 = 1,\ \left(\gamma + \frac{\beta\gamma-\alpha\delta}{\lambda}\right)(1) = 0 + \frac{0-1}{1} = -1$$

$$\left(\epsilon + \frac{\beta\epsilon-\delta\theta}{\lambda}\right)(1) = (0,\ldots,0) + \frac{(0,\ldots,0)}{1} = (0,\ldots,0)$$

$$\left(\left(\phi + \frac{\beta\phi-\alpha\psi}{\lambda}\right)(1)\right)^{\text{tr}} = (0,\ldots,0) + \frac{(0,\ldots,0)}{1} = (0,\ldots,0)$$

$$\left(\Xi + \frac{\beta\Xi-\psi\theta}{\lambda}\right)(1) = -I_{(n-2)\times(n-2)} + \frac{0_{(n-2)\times(n-2)}}{1} = -I_{(n-2)\times(n-2)}$$

Similarly, when taking the disjoint union:

$$\begin{array}{c|c} \lambda_1 & X_1 \\ \hline X_1 & A_1 \end{array} \cup \begin{array}{c|c} \lambda_2 & X_2 \\ \hline X_2 & A_2 \end{array} = \begin{array}{c|cc} \lambda_1 \cdot \lambda_2 & X_1 & X_2 \\ \hline X_1 & \lambda_2 A_1 & 0 \\ X_2 & 0 & \lambda_1 A_2 \end{array}$$

Assuming $\lambda_1(1) = \lambda_2(2) = 1$ and $A_1(1) = -I_{|X_1|\times|X_1|},\ A_2(1) = -I_{|X_2|\times|X_2|}$, then for the resulting pair, we have $\lambda_1\lambda_2(1) = 1,\ (\lambda_2 A_1 \oplus \lambda_1 A_2)(1) = -I_{(|X_1|+|X_2|)\times(|X_1|+|X_2|)}$. □

Chapter 4

Metamonoids and Γ-calculus

In this section, we relate the rMVA invariant to another tangle invariant defined by Bar-Natan ([BNS13, DBN1]) which also generalizes the MVA. We start by discussing its algebraic structure. Since the gluing operation in the target space of rMVA is not defined on a pair $(\lambda, \mathcal{A})$ where $\lambda = 0$, it does not make it a circuit algebra, and in this setting rMVA is not a circuit algebra morphism. We can however consider it in a different algebraic setting which also describes pure virtual tangles, namely that of metamonoids.

4.1 Metamonoid structure

Definition 4.1.1 *([BNS13, ?, DBN1, DBN2]) A **meta-monoid** is a collection of objects $\{G_X\}$ indexed by finite sets X, together with maps between them:*

m^{xy}_z	$: G_{\{x,y\}\cup X} \longrightarrow G_{\{z\}\cup X}$	*"multiplication"*	$\eta_x \quad : G_{\{x\}\cup X} \longrightarrow G_X$	*"deletion"*
$\sqcup$	$: G_X \times G_Y \longrightarrow G_{X\cup Y}$	*"union"*	$\sigma^x_z \quad : G_{\{x\}\cup X} \longrightarrow G_{\{z\}\cup X}$	*"renaming"*
e_x	$: G_X \longrightarrow G_{\{x\}\cup X}$	*"identity"*		

Satisfying the following relations:

Monoid axioms		**Set axioms**		
$m^{dc}_f \circ m^{ab}_c = m^{cb}_f \circ m^{da}_c$	(associativity)	$\sigma^a_b \circ e_a = e_b$	$\sigma^b_c \circ \sigma^a_b = \sigma^a_c$	$\sigma^b_a \circ \sigma^a_b = \mathrm{Id}$
$m^{ab}_c \circ e_a = \sigma^b_c$	(left identity)	$\eta_b \circ \sigma^a_b = \eta_a$	$\eta_a \circ e_a = \mathrm{Id}$	$\eta_c \circ m^{ab}_c = \eta_b \circ \eta_a$
$m^{ab}_c \circ e_b = \sigma^a_c$	(right identity)	$\sigma^c_d \circ m^{ab}_c = m^{ab}_d$	$m^{bc}_d \circ \sigma^a_b = m^{ac}_d$	

As well as the commuting of operations involving labels which don't interact, for instance $\sigma^a_b \circ \sigma^c_d = \sigma^c_d \circ \sigma^a_b$.

Remark 4.1.2 *A metamonoid $\{G_X\}$ can be thought of as elements in the collection G_X labelled by X. For instance, the collection $G^X = \{f : X \to G\}$ where G is a group and X is a finite set forms a metamonoid. Another example is $pv\mathcal{T}_X$, the collection of X-labeled, pure, regular, virtual tangles with the following operations:*

$$\begin{aligned} m_c^{ab} &= \textit{join the outgoing end of the strand labelled "a" to the incoming end of the strand} \\ & \quad \textit{labelled "b" and call the resulting strand "c"} \\ \sqcup &= \textit{take the union of the two tangles by placing them side by side} \\ e_a &= \textit{add a trivially knotted strand labelled "a"} \\ \eta_a &= \textit{delete/remove the strand labelled "a" in the tangle} \\ \sigma_b^a &= \textit{change the label of strand "a" to "b".} \end{aligned}$$

Notice in particular that $\eta_x(T) \sqcup \eta_y(T) \neq T$, *unlike the previous example.*

The target space of our invariant, together with the gluing and disjoint union maps described in Lemmas 3.3.1 and 3.3.2 and the remaining operations defined in a natural way is also a metamonoid, as we discuss below.

Lemma 4.1.3 *The collection* $R_X = \mathbb{R}(\sqrt{t_i}) \times M_{X\times X}(\mathbb{R}(\sqrt{t_i}))$ *is a metamonoid under the above gluing and disjoint union operations, together with:*

$$\eta_a \left(\begin{array}{c|cc} \lambda & X & a \\ \hline X & M & \phi \\ a & \epsilon & \alpha \end{array} \right) = \left(\begin{array}{c|c} \lambda & X \\ \hline X & M \end{array} \right)_{t_a=1} \qquad \sigma_b^a \left(\begin{array}{c|cc} \lambda & X & a \\ \hline X & M & \phi \\ a & \epsilon & \alpha \end{array} \right) = \left(\begin{array}{c|cc} \lambda & X & b \\ \hline X & M & \phi \\ b & \epsilon & \alpha \end{array} \right)_{t_a\to t_b}$$

$$e_a \left(\begin{array}{c|c} \lambda & X \\ \hline X & M \end{array} \right) = \begin{array}{c|cc} \lambda & X & a \\ \hline X & M & 0 \\ a & 0 & -\lambda \end{array}$$

Proof: The set and commuting axioms follow directly from the definition of the operations on R_X, so we'll verify that the monoid axioms are satisfied:

<u>Checking the "left identity" axiom:</u>

$$m_c^{ab} \circ e_a \left(\begin{array}{c|cc} \lambda & b & X \\ \hline b & \delta & \epsilon \\ X & \psi & \Xi \end{array} \right) = m_c^{ab} \left(\begin{array}{c|ccc} \lambda & a & b & X \\ \hline a & -\lambda & 0 & 0_{|X|} \\ b & 0 & \delta & \epsilon \\ X & 0^{\text{tr}}_{|X|} & \psi & \Xi \end{array} \right) =$$

$$= \left(\begin{array}{c|cc} \lambda & c & X \\ \hline c & 0 + \frac{0-\delta(-\lambda)}{\lambda} & \epsilon + \frac{0\cdot\epsilon-\delta\cdot 0_{|X|}}{\lambda} \\ X & 0^{\text{tr}}_{|X|} + \frac{0^{\text{tr}}_{|X|}-(-\lambda)\psi}{\lambda} & \Xi + \frac{0\cdot\Xi-\psi\cdot 0_{|X|}}{\lambda} \end{array} \right)_{t_a,t_b\to t_c} = \left(\begin{array}{c|cc} \lambda & c & X \\ \hline c & \delta & \epsilon \\ X & \psi & \Xi \end{array} \right)_{t_b\to t_c} = \sigma_c^b \left(\begin{array}{c|cc} \lambda & b & X \\ \hline b & \delta & \epsilon \\ X & \psi & \Xi \end{array} \right)$$

<u>Checking the "right identity" axiom:</u>

$$m_c^{ab} \circ e_b \left(\begin{array}{c|cc} \lambda & a & X \\ \hline a & \alpha & \theta \\ X & \phi & \Xi \end{array}\right) = m_c^{ab} \left(\begin{array}{c|ccc} \lambda & a & b & X \\ \hline a & \alpha & 0 & \theta \\ b & 0 & -\lambda & 0_{|X|} \\ X & \phi & 0_{|X|}^{\mathrm{tr}} & \Xi \end{array}\right) =$$

$$= \left(\begin{array}{c|cc} \lambda & c & X \\ \hline c & 0 + \frac{0-\alpha(-\lambda)}{\lambda} & 0_{|X|} + \frac{0_{|X|}-(-\lambda)\theta}{\lambda} \\ X & \phi + \frac{0\cdot\phi - \alpha\cdot 0_{|X|}^{\mathrm{tr}}}{\lambda} & \Xi + \frac{0\cdot\Xi - 0_{|X|}^{\mathrm{tr}}\cdot\theta}{\lambda} \end{array}\right)_{t_a,t_b\to t_c} = \left(\begin{array}{c|cc} \lambda & c & X \\ \hline c & \alpha & \theta \\ X & \phi & \Xi \end{array}\right)_{t_a\to t_c} = \sigma_c^a \left(\begin{array}{c|cc} \lambda & a & X \\ \hline a & \alpha & \theta \\ X & \phi & \Xi \end{array}\right)$$

<u>Checking the "associativity" axiom:</u>

$$m_f^{dc} \circ m_c^{ab} \left(\begin{array}{c|cccc} \lambda & a & b & d & X \\ \hline a & \alpha & \beta & \tau & \theta \\ b & \gamma & \delta & \nu & \epsilon \\ d & \kappa & \rho & \xi & \sigma \\ X & \phi & \psi & \mu & \Xi \end{array}\right) = m_f^{dc} \left(\begin{array}{c|ccc} \lambda+\beta & c & d & X \\ \hline c & \gamma + \frac{\beta\gamma-\alpha\delta}{\lambda} & \nu + \frac{\beta\nu-\delta\tau}{\lambda} & \epsilon + \frac{\beta\epsilon-\delta\theta}{\lambda} \\ d & \kappa + \frac{\beta\kappa-\alpha\rho}{\lambda} & \xi + \frac{\beta\xi-\rho\tau}{\lambda} & \sigma + \frac{\beta\sigma-\rho\theta}{\lambda} \\ X & \phi + \frac{\beta\phi-\alpha\psi}{\lambda} & \mu + \frac{\beta\mu-\tau\psi}{\lambda} & \Xi + \frac{\beta\Xi-\psi\theta}{\lambda} \end{array}\right)$$

$$= \left(\begin{array}{c|cc} \frac{(\beta+\lambda)(\kappa+\lambda)-\alpha\rho}{\lambda} & f & X \\ \hline f & \frac{\alpha(\delta\xi-\rho\nu)-(\beta+\lambda)\gamma\xi+(\kappa+\lambda)((\beta+\lambda)\nu-\tau\delta)+\rho\tau\gamma}{\lambda^2} & \frac{\alpha(\delta\sigma-\rho\epsilon)-(\beta+\lambda)\gamma\sigma+(\kappa+\lambda)((\beta+\lambda)\epsilon-\delta\theta)+\rho\gamma\theta}{\lambda^2} \\ X & \frac{\alpha(\xi\psi-\rho\mu)-(\beta+\lambda)\xi\phi+(\kappa+\lambda)((\beta+\lambda)\mu-\tau\psi)+\rho\tau\phi}{\lambda^2} & \frac{\alpha(\psi\sigma-\rho\Xi)+(\kappa+\lambda)((\beta+\lambda)\Xi-\psi\theta)-(\beta+\lambda)\phi\sigma+\rho\phi\theta}{\lambda^2} \end{array}\right)$$

Where in the first step we set $t_a, t_b \to t_c$ and in the second step we set $t_c, t_d \to t_f$. Similarly, for the right side of the axiom we get:

$$m_f^{cb} \circ m_c^{da} \left(\begin{array}{c|cccc} \lambda & a & b & d & X \\ \hline a & \alpha & \beta & \tau & \theta \\ b & \gamma & \delta & \nu & \epsilon \\ d & \kappa & \rho & \xi & \sigma \\ X & \phi & \psi & \mu & \Xi \end{array}\right) = m_f^{cb} \left(\begin{array}{c|ccc} \lambda+\kappa & c & b & X \\ \hline c & \tau + \frac{\kappa\tau-\alpha\xi}{\lambda} & \beta + \frac{\kappa\beta-\alpha\rho}{\lambda} & \theta + \frac{\kappa\theta-\alpha\sigma}{\lambda} \\ b & \nu + \frac{\kappa\nu-\xi\gamma}{\lambda} & \delta + \frac{\kappa\delta-\gamma\rho}{\lambda} & \epsilon + \frac{\kappa\epsilon-\gamma\sigma}{\lambda} \\ X & \mu + \frac{\kappa\mu-\xi\phi}{\lambda} & \psi + \frac{\kappa\psi-\rho\phi}{\lambda} & \Xi + \frac{\kappa\Xi-\phi\sigma}{\lambda} \end{array}\right)$$

$$= \left(\begin{array}{c|cc} \frac{(\beta+\lambda)(\kappa+\lambda)-\alpha\rho}{\lambda} & f & X \\ \hline f & \frac{\alpha(\delta\xi-\rho\nu)-(\beta+\lambda)\gamma\xi+(\kappa+\lambda)((\beta+\lambda)\nu-\tau\delta)+\rho\tau\gamma}{\lambda^2} & \frac{\alpha(\delta\sigma-\rho\epsilon)-(\beta+\lambda)\gamma\sigma+(\kappa+\lambda)((\beta+\lambda)\epsilon-\delta\theta)+\rho\gamma\theta}{\lambda^2} \\ X & \frac{\alpha(\xi\psi-\rho\mu)-(\beta+\lambda)\xi\phi+(\kappa+\lambda)((\beta+\lambda)\mu-\tau\psi)+\rho\tau\phi}{\lambda^2} & \frac{\alpha(\psi\sigma-\rho\Xi)+(\kappa+\lambda)((\beta+\lambda)\Xi-\psi\theta)-(\beta+\lambda)\phi\sigma+\rho\phi\theta}{\lambda^2} \end{array}\right)$$

Where in the first step we set $t_a, t_d \to t_c$ and in the second step we set $t_c, t_b \to t_f$. □

Thus, both the domain $pv\mathcal{T}_X$ and the target space R_X of rMVA are metamonoids and in this more general algebraic setting, by construction the tangle invariant rMVA : $pv\mathcal{T}_X \longrightarrow R_X$ is a **metamonoid morphism**. We will next relate it to another tangle invariant which is also defined in this algebraic context.

4.2 Γ- or Gassner-calculus

Bar-Natan defines another tangle invariant, "Γ-calculus", which generalizes the multivariable Alexander polynomial in [DBN1],[DBN2]. It is also an invariant for pure, virtual, regular X-labelled tangles, where X is a finite set, so its domain is the metamonoid $pv\mathcal{T}_X$. The target space is the metamonoid $\Gamma_X = \mathbb{Z}(t_i)_{i\in X} \times M_{X\times X}(\mathbb{Z}(t_i))$ with operations given by:

Multiplication:

$$\begin{array}{c|ccc} \lambda & a & b & X \\ \hline a & \alpha & \beta & \theta \\ b & \gamma & \delta & \epsilon \\ X & \phi & \psi & \Xi \end{array} \xrightarrow{m_{a,b\to c}} \left(\begin{array}{c|cc} \lambda(1-\beta) & c & X \\ \hline c & \gamma + \frac{\alpha\delta}{1-\beta} & \epsilon + \frac{\delta\theta}{1-\beta} \\ X & \phi + \frac{\alpha\psi}{1-\beta} & \Xi + \frac{\psi\theta}{1-\beta} \end{array} \right)_{t_a,t_b\to t_c}$$

Remark 4.2.1 *Multiplication is well-defined on the image of pure tangles as, similar to the case of R_X, for a pair $(\lambda_T, \mathcal{A}_T)$ which is the image of a pure tangle T, we have $\lambda_T(1) = 1, \mathcal{A}_T(1) = Id$. Namely, setting all the variables $t_i = 1$, we get the pair $(1, Id)$. So, in the multiplication operation, the division term $1-\beta$ where β is an off-diagonal element is never identically zero.*

Union:

$$\begin{array}{c|c} \lambda_1 & X_1 \\ \hline X_1 & A_1 \end{array} \sqcup \begin{array}{c|c} \lambda_2 & X_2 \\ \hline X_2 & A_2 \end{array} = \begin{array}{c|cc} \lambda_1\cdot\lambda_2 & X_1 & X_2 \\ \hline X_1 & A_1 & 0 \\ X_2 & 0 & A_2 \end{array}$$

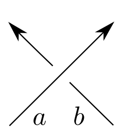
Identity:

$$e_a\left(\begin{array}{c|c} \lambda & X \\ \hline X & A \end{array} \right) = \begin{array}{c|cc} \lambda & X & a \\ \hline X & A & 0 \\ a & 0 & 1 \end{array}$$

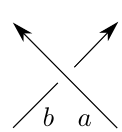
The deletion and renaming operations for Γ_X are the same as those for R_X and we will not repeat them here. The Bar-Natan invariant is then defined as the metamonoid morphism:

$$Z : pv\mathcal{T}_X \longrightarrow \Gamma_X$$

which maps the positive and negative crossing generators as indicated below.



$$\mapsto \begin{array}{c|cc} 1 & a & b \\ \hline a & 1 & 1-t_a \\ b & 0 & t_a \end{array}$$



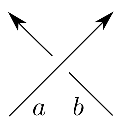
$$\mapsto \begin{array}{c|cc} 1 & a & b \\ \hline a & 1 & 1-t_a^{-1} \\ b & 0 & t_a^{-1} \end{array}$$

4.3 Relating rMVA and Z

Both tangle invariants rMVA and Z are metamonoid morphisms mapping into almost identical spaces, so not surprisingly there is a close relationship between them. We will explore that relationship next by first defining a slight variation of Bar-Natan's invariant. Note that the elements in both R_X and Γ_X are $X \times X$ matrices with an additional element, and if we equip R_X with the operations from Γ_X instead, it is still a metamonoid so we get a second metamonoid structure on it, which we will denote by $\widetilde{R}_X$. Consider the following tangle invariant $\widetilde{Z}$, which is a slight modification of Bar-Natan's Z invariant, namely the metamonoid morphism:

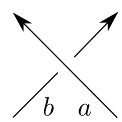
$$\widetilde{Z} : pv\mathcal{T}_X \longrightarrow \widetilde{R}_X$$

which is defined on the generators as:



$\mapsto$

$t_a^{\frac{1}{2}}$	a	b
a	1	0
b	$\frac{t_b-1}{t_a}$	t_a^{-1}



$\mapsto$

$t_a^{-\frac{1}{2}}$	a	b
a	1	0
b	$1-t_b$	t_a

Figure 4.1: The images under $\widetilde{Z}$ of $\overcrossing$ and $\undercrossing$

Note that, analogously to the previous cases, for each of these pairs $(\lambda, \mathcal{A})$, setting all the variables t_i to 1, we get $(\lambda(1), \mathcal{A}(1)) = (1, \mathrm{Id})$, so for any pure tangle T, $\lambda_T \neq 0$. This version of the Bar-Natan invariant turns out to be equivalent to rMVA, as we show next.

Theorem 4.3.1 *There is a metamonoid morphism $\widetilde{R}_X \to R_X$, which is an isomorphism on the level of the images of rMVA and $\widetilde{Z}$, taking the positive and negative crossing generators for $\widetilde{Z}$ in $\widetilde{R}$ to those for rMVA in R_X.*

Proof: The map $F : \widetilde{R}_X \longrightarrow R_X$ defined by $F(\lambda, \mathcal{A}) = (\lambda, -\lambda \cdot \mathcal{A})$ takes the positive and negative crossing generators for rMVA in Figure 3.2 precisely to the positive and negative crossing generators for $\widetilde{Z}$ in Figure 4.1. Furthermore, for any pair $(\lambda, \mathcal{A})$ in the image $\widetilde{Z}(pv\mathcal{T}_X)$ or rMVA$(pv\mathcal{T}_X)$ of pure tangles using either invariant, we've seen that $\lambda \neq 0$ so restricted to these subsets F is a bijection. It remains to show that it is a metamonoid morphism, i.e. we need to check that the following diagram commutes, as well as the analogous ones with m_c^{ab} replaced by $\sqcup$, e_a, η_a, σ_b^a.

$$\begin{array}{ccc} \widetilde{R}_{X\cup\{a,b\}} & \xrightarrow{F} & R_{X\cup\{a,b\}} \\ {\scriptstyle (m_c^{ab})_{\widetilde{R}_X}}\downarrow & & \downarrow{\scriptstyle (m_c^{ab})_{R_X}} \\ \widetilde{R}_{X\cup\{c\}} & \xrightarrow{F} & R_{X\cup\{c\}} \end{array}$$

For a given element in $\widetilde{R}_{X\cup\{a,b\}}$, we first follow the diagram horizontally then vertically:

$$(m_c^{ab})_{R_X} \circ F \left(\begin{array}{c|ccc} \lambda & a & b & X \\ \hline a & \alpha & \beta & \theta \\ b & \gamma & \delta & \epsilon \\ X & \phi & \psi & \Xi \end{array}\right) = (m_c^{ab})_{R_X} \left(\begin{array}{c|ccc} \lambda & a & b & X \\ \hline a & -\lambda\alpha & -\lambda\beta & -\lambda\theta \\ b & -\lambda\gamma & -\lambda\delta & -\lambda\epsilon \\ X & -\lambda\phi & -\lambda\psi & -\lambda\Xi \end{array}\right)$$

$$= \left(\begin{array}{c|cc} \lambda - \lambda\beta & c & X \\ \hline c & -\lambda\gamma + \frac{(-\lambda\beta)(-\lambda\gamma)-(-\lambda\delta)(-\lambda\alpha)}{\lambda} & -\lambda\epsilon + \frac{(-\lambda\beta)(-\lambda\epsilon)-(-\lambda\delta)(-\lambda\theta)}{\lambda} \\ X & -\lambda\phi + \frac{(-\lambda\beta)(-\lambda\phi)-(-\lambda\alpha)(-\lambda\psi)}{\lambda} & -\lambda\Xi + \frac{(-\lambda\beta)(-\lambda\Xi)-(-\lambda\psi)(-\lambda\theta)}{\lambda} \end{array}\right)_{t_a,t_b\to t_c}$$

$$= \left(\begin{array}{c|cc} \lambda(1-\beta) & c & X \\ \hline c & -\lambda\gamma + \lambda\beta\gamma - \lambda\delta\alpha & -\lambda\epsilon + \lambda\beta\epsilon - \lambda\delta\theta \\ X & -\lambda\phi + \lambda\beta\phi - \lambda\alpha\psi & -\lambda\Xi + \lambda\beta\Xi - \lambda\psi\theta \end{array}\right)_{t_a,t_b\to t_c}$$

$$= \left(\begin{array}{c|cc} \lambda(1-\beta) & c & X \\ \hline c & -\lambda(1-\beta)\gamma - \lambda\alpha\delta & -\lambda(1-\beta)\epsilon - \lambda\delta\theta \\ X & -\lambda(1-\beta)\phi - \lambda\alpha\psi & -\lambda(1-\beta)\Xi - \lambda\psi\theta \end{array}\right)_{t_a,t_b\to t_c}$$

Alternatively, following the map going down first, then right, we get:

$$F \circ (m_c^{ab})_{\widetilde{R}_X} \left(\begin{array}{c|ccc} \lambda & a & b & X \\ \hline a & \alpha & \beta & \theta \\ b & \gamma & \delta & \epsilon \\ X & \phi & \psi & \Xi \end{array}\right) = F \left(\begin{array}{c|cc} \lambda(1-\beta) & c & X \\ \hline c & \gamma + \frac{\alpha\delta}{1-\beta} & \epsilon + \frac{\delta\theta}{1-\beta} \\ X & \phi + \frac{\alpha\psi}{1-\beta} & \Xi + \frac{\psi\theta}{1-\beta} \end{array}\right)_{t_a,t_b\to t_c}$$

$$= \left(\begin{array}{c|cc} \lambda(1-\beta) & c & X \\ \hline c & -\lambda(1-\beta)\gamma - \lambda\alpha\delta & -\lambda(1-\beta)\epsilon - \lambda\delta\theta \\ X & -\lambda(1-\beta)\phi - \lambda\alpha\psi & -\lambda(1-\beta)\Xi - \lambda\psi\theta \end{array}\right)_{t_a,t_b\to t_c}$$

The proof that the diagram commutes for η_a and σ_b^a is immediate from the definition, so we'll additionally only check it for $\sqcup$ and e_a, whose verification is also straightforward. The following diagram verifies that F preserves disjoint union:

$$\begin{array}{ccc}
\left(\begin{array}{c|c}\lambda_1 & X_1\\ \hline X_1 & A_1\end{array}\,,\;\begin{array}{c|c}\lambda_2 & X_2\\ \hline X_2 & A_2\end{array}\right) & \xrightarrow{F} & \left(\begin{array}{c|c}\lambda_1 & X_1\\ \hline X_1 & -\lambda_1 A_1\end{array}\,,\;\begin{array}{c|c}\lambda_2 & X_2\\ \hline X_2 & -\lambda_2 A_2\end{array}\right)\\
{\scriptstyle (\sqcup)_{\widetilde{R}_X}}\Big\downarrow & & \Big\downarrow{\scriptstyle (\sqcup)_{R_X}}\\
\begin{array}{c|cc}\lambda_1\cdot\lambda_2 & X_1 & X_2\\ \hline X_1 & A_1 & 0\\ X_2 & 0 & A_2\end{array} & \xrightarrow{F} & \begin{array}{c|cc}\lambda_1\cdot\lambda_2 & X_1 & X_2\\ \hline X_1 & -\lambda_1\lambda_2 A_1 & 0\\ X_2 & 0 & -\lambda_1\lambda_2 A_2\end{array}
\end{array}$$

Similarly, we confirm with the diagram below that F commutes with the "identity" map e_a of the two metamonoids:

$$\begin{array}{ccc}
\begin{array}{c|c}\lambda & X\\ \hline X & A\end{array} & \xrightarrow{F} & \begin{array}{c|c}\lambda & X\\ \hline X & -\lambda A\end{array}\\
{\scriptstyle (e_a)_{\widetilde{R}_X}}\Big\downarrow & & \Big\downarrow{\scriptstyle (e_a)_{R_X}}\\
\begin{array}{c|cc}\lambda & X & a\\ \hline X & A & 0\\ a & 0 & 1\end{array} & \xrightarrow{F} & \begin{array}{c|cc}\lambda & X & a\\ \hline X & -\lambda A & 0\\ a & 0 & -\lambda\end{array}
\end{array}$$

□

Remark 4.3.2 *Let $P_X = rMVA(pv\mathcal{T}_X)$ and $\widetilde{P}_X = \widetilde{Z}(pv\mathcal{T}_X)$ denote the image sets of the two invariants. Then we can summarize the main result from Theorem 4.3.1 in Figure 4.2, namely that the target spaces of rMVA and $\widetilde{Z}$ are isomorphic and the isomorphism takes the images of the tangle metamonoid generators using $\widetilde{Z}$ to those using rMVA. In particular, this tells us that rMVA and $\widetilde{Z}$ are equivalent as pure tangle invariants.*

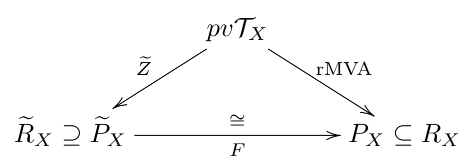


Figure 4.2: A commutative diagram of metamonoid morphisms.

Chapter 5

Links, Braids, and Tangles with Closed Components

5.1 Recovering the MVA

One of the main motivations for studying the rMVA tangle invariant is that, since it comes from Archibald's tMVA invariant, it is a generalization to tangles of the multivariable Alexander polynomial on links. It is therefore of interest to study how to recover the MVA directly from the rMVA.

In order to consider links, we need to look at tangles with closed components, so setting aside the algebraic structure of metamonoids, let us consider the larger space $v\mathcal{T}_X$ of (not necessarily pure) regular, virtual tangles labelled using the set X. Note that we can also define rMVA on $v\mathcal{T}_X$, not in terms of a metamonoid morphism presented in Lemmas 3.3.1, 3.3.2, and 4.1.3, but directly from the degree 0 and 1 components using the Hodge star operator on the image of tMVA (which can also be found through the Alexander matrix) and the normalizing factor, as in Equation 3.1. More precisely, for a tangle $T \in v\mathcal{T}_X$, possibly with some closed components, we can define:

$$\text{rMVA}(T) := \prod_k t_k^{-\frac{\mu(k)}{2}} (\lambda_T, \mathcal{A}_T)$$

This has the disadvantage of no longer providing the "divide-and-conquer" approach of computing the invariant by splitting a tangle in pieces through the generators and then gluing in the context of metamonoid morphisms. However, it allows us to consider the invariant on a bigger space which includes tangles with closed components, and recover from it the multivariable Alexander polynomial for links.

The MVA on links is in fact defined using the same construction for the Alexander matrix as described in Figure 2.6. The version of the link invariant that we will work with and for which it is simplest to describe the relation to rMVA is **vMVA** or the multivariable Alexander polynomial for regular, virtual, long links, i.e. links in which exactly one strand has both ends going to infinity.

Definition 5.1.1 *[Arch10] The multivariable Alexander polynomial vMVA for a regular, virtual, long link L is defined as:*

$$vMVA(L) = \frac{1}{t_l - 1} \prod_s t_s^{-\frac{\mu(s)}{2}} \det M(D_L)^{l^{out};\widehat{l^{in}}}$$

where t_l is the variable associated to the long strand, $\mu(s)$ as before is the number of times the strand labelled "s" overcrosses in a crossing, D_L is a diagram of the link L, and $M(D_L)^{l^{out};\widehat{l^{in}}}$ is the submatrix of the Alexander matrix with all the columns corresponding to the internal arcs included, as well as the column corresponding to the outgoing arc of strand "l", but not the incoming one.

Going back to our consideration of rMVA, another disadvantage of considering it on the bigger space $v\mathcal{T}_X$ as above is that for a tangle $T \in v\mathcal{T}_X$, we are no longer guaranteed that $\lambda_T \neq 0$, and so we cannot apply Theorem 2.3.2 to conclude that rMVA(T) determines tMVA(T). We can however say more if we restrict our attention to the space $v\mathcal{T}_X^1$ of 1-tangles, i.e. tangles with possibly some closed components and exactly one open strand component, and the set X labels both the closed components and the open component. In that case the Alexander matrix has a single outgoing and incoming arc column, so tMVA$(v\mathcal{T}_X^1)$ lives in only the degree 0 and 1 components and is therefore completely determined by rMVA directly from the definition, for $T \in v\mathcal{T}_X^1$:

$$\mathrm{rMVA}\left(\xrightarrow{a^{\mathrm{in}}}\boxed{\mathrm{T}}\xrightarrow{a^{\mathrm{out}}}\right) = \begin{array}{c|c} \lambda_T & a^{\mathrm{in}} \\ \hline a^{\mathrm{out}} & \mathcal{A}_T \end{array}$$

where $\lambda_T = \prod_k t_k^{-\frac{\mu(k)}{2}} \det M(D_T)^{a^{\mathrm{out}};\emptyset}$, $\mathcal{A}_T = \prod_k t_k^{-\frac{\mu(k)}{2}} \det M(D_T)^{\emptyset;a^{\mathrm{in}}}$. Thus,

$$\mathrm{tMVA}(T) = \prod_k t_k^{-\frac{\mu(k)}{2}} a^{\mathrm{out}} \otimes \left(\det M(D_T)^{a^{\mathrm{out}};\emptyset} a^{\mathrm{out}} + \det M(D_T)^{\emptyset;a^{\mathrm{in}}} a^{\mathrm{in}}\right) = a^{\mathrm{out}} \otimes \left(\lambda_T\, a^{\mathrm{out}} + \mathcal{A}_T\, a^{\mathrm{in}}\right).$$

To make the connection to links, we note that long links, i.e. those with exactly one component having ends going to infinity, coincide with 1-tangles, so we can relate the values of rMVA on 1-tangles and vMVA on long links.

Archibald makes that connection between tMVA and vMVA in [Arch10] by noting that the Alexander matrix for the two invariants is constructed by the same rules, and furthermore, as can be deduced from Figure 2.6, the columns $\{C_s\}$ of the matrix satisfy the relation $\sum_s (t_s - 1)C_s = 0$ and since the incoming and outgoing ends of a tangle correspond to the same strand and so the same variable t_s, the relation is:

Theorem 5.1.2 *[Arch10, Theorem 6.8] For a 1-tangle T:*

$$\frac{1}{t_a - 1} tMVA\left(\xrightarrow{a^{in}}\boxed{T}\xrightarrow{a^{out}}\right) = vMVA\left(\xrightarrow{a^{in}}\boxed{T}\xrightarrow{a^{out}}\right) a^{out} \otimes (a^{out} - a^{in})$$

The theorem relies in particular on the conclusion from the relation among the columns of the Alexander matrix that $\det M(D_T)^{a^{\mathrm{out}};\emptyset} = -\det M(D_T)^{\emptyset;a^{\mathrm{in}}}$, which leads to the connection between rMVA and vMVA:

Proposition 5.1.3 *For $\xrightarrow{a^{in}}\boxed{T}\xrightarrow{a^{out}}$ a regular, virtual 1-tangle (i.e. a long knot or link) and $rMVA(T) = (\lambda_T, \mathcal{A}_T)$:*

$$\lambda_T = -\mathcal{A}_T \qquad \frac{\lambda_T}{t_a - 1} = vMVA(T)$$

For long knots, i.e. 1-tangles without closed components, we can still benefit from the metamonoid structure in using this proposition to break down the knot into smaller pieces and consequently glue their images, making computations more efficient.

5.2 Braids and the Gassner representation

Another specialization of the rMVA invariant leads again to familiar terrirory. For this section we will restrict our attention to a subset of the collection of X-labelled tangles, namely that of pure virtual braids, where "pure" in this case is used in the more standard meaning of the underlying permutation being the identity.

Definition 5.2.1 *[Ba04][BND1] The pure virtual braid group pvB_X with strands labelled by X is generated by σ_{ab} for all $a \neq b \in X$, corresponding to the positive crossing where strand "a" crosses over strand "b", and relations:*

$$\sigma_{ab}\sigma_{ac}\sigma_{bc} = \sigma_{bc}\sigma_{ac}\sigma_{ab} \qquad \forall\, a, b, c \text{ pairwise distinct}$$
$$\sigma_{ab}\sigma_{cd} = \sigma_{cd}\sigma_{ab} \qquad \forall\, a, b, c, d \text{ pairwise distinct}$$

Remark 5.2.2 *Note that in this notation, the labels record the "identity" of the strand rather than its location. So, unlike the usual braid group where only adjacent strands can cross and two strands crossing exchange their labels (which indicate their position), in the notation above for this pure virtual setting a strand labelled "a" continues to carry that label even after crossing with another strand.*

In her thesis [G59], Gassner defines a multivariable version of the Burau representation for the braid group, which can be generalized to pvB_X by mapping each generator $\sigma_{ab},\ a, b, \in X$ to the following matrix (See also [DBN4]):

$$\sigma_{ab} \xrightarrow{G} \left[\begin{array}{c|ccc} & a & b & X\setminus\{a,b\} \\ \hline a & t_a & 1-t_b & O \\ b & 0 & 1 & O \\ X\setminus\{a,b\} & O & O & I \end{array}\right]$$

In the next result, we show that when restricted to pvB_X, the rMVA invariant is equivalent to the Gassner representation (and so also the Burau representation when we set all the variables to be equal).

Proposition 5.2.3 *On pure v-Braids pvB_X, rMVA reduces to the Gassner representation, respectively the Burau representation in the single variable case (letting $t_a = t,\ \forall\, a \in X$).*

Proof: We will first verify that, when restricted to pvB_X, the rMVA invariant with a small correction is a group homomorphism, the group structure on the target space being given by $(\lambda_1, \mathcal{A}_1) \cdot (\lambda_2, \mathcal{A}_2) = (\lambda_1\lambda_2, \mathcal{A}_1\mathcal{A}_2)$, with matrix multiplication in the second factor. We will first verify that:

$$\text{rMVA}(B \cdot \sigma_{cd}) = -\text{rMVA}(\sigma_{cd})\text{rMVA}(B) \tag{5.1}$$

for any braid $B \in pvB_{X\cup\{a,b\}}$ and generator σ_{cd}. Note that the operation of multiplying a pair $(\lambda, \mathcal{A})$ by a scalar commutes with both metamonoid multiplication or gluing and disjoint union, so we'll suppress the normalizing factor $1/\sqrt{t_c}$ for σ_{cd} in the following computations for the sake of simplicity. The left side of Equation 5.1 then becomes:

$$
\begin{array}{c|cc}
t_c & c & d \\ \hline
c & -t_c & 0 \\
d & 1-t_d & -1
\end{array}
\sqcup
\begin{array}{c|ccc}
\lambda & a & b & X \\ \hline
a & \alpha & \beta & \theta \\
b & \gamma & \delta & \epsilon \\
X & \phi & \psi & \Xi
\end{array}
=
\begin{array}{c|ccccc}
t_c\lambda & c & d & a & b & X \\ \hline
c & -t_c\lambda & 0 & 0 & 0 & 0_{|X|} \\
d & (1-t_d)\lambda & -\lambda & 0 & 0 & 0_{|X|} \\
a & 0 & 0 & t_c\alpha & t_c\beta & t_c\theta \\
b & 0 & 0 & t_c\gamma & t_c\delta & t_c\epsilon \\
X & 0^{\text{tr}}_{|X|} & 0^{\text{tr}}_{|X|} & t_c\phi & t_c\psi & t_c\Xi
\end{array}
\xrightarrow{m^{ac}_a}
$$

$$
\begin{array}{c|cccc}
t_a\lambda & a & d & b & X \\ \hline
a & t_a\alpha & 0 & t_a\beta & t_a\theta \\
d & (t_d-1)\alpha & -\lambda & (t_d-1)\beta & (t_d-1)\theta \\
b & t_a\gamma & 0 & t_a\delta & t_a\epsilon \\
X & t_a\phi & 0^{\text{tr}}_{|X|} & t_a\psi & t_a\Xi
\end{array}
\xrightarrow{m^{bd}_b}
\begin{array}{c|ccc}
t_a\lambda & a & b & X \\ \hline
a & t_a\alpha & t_a\beta & t_a\theta \\
b & (t_b-1)\alpha+\gamma & (t_b-1)\beta+\delta & (t_b-1)\theta+\epsilon \\
X & t_a\phi & t_a\psi & t_a\Xi
\end{array}
$$

Strictly speaking, rather than computing $m^{bd}_d \circ m^{ac}_a((\lambda_{\sigma_{cd}}, \mathcal{A}_{\sigma_{cd}}) \sqcup (\lambda_B, \mathcal{A}_B))$, if we think of σ_{cd} as a braid on $|X|+2$ strands, we need to compute $m^{x_{|X|}y_{|X|}}_{x_{|X|}} \circ \ldots \circ m^{x_1 y_1}_{x_1} \circ m^{bd}_d \circ m^{ac}_a(e_{y_{|X|}} \circ \ldots \circ e_{y_1}(\lambda_{\sigma_{cd}}, \mathcal{A}_{\sigma_{cd}}) \sqcup (\lambda_B, \mathcal{A}_B))$, i.e. include the image of $|X|$ unknotted strands using rMVA. However, from the metamonoid axioms, since operations on non-overlapping labels commute and $m^{ab}_a \circ e_b = \text{Id}$, the computation reduces to the one above.

On the other hand, to compute the matrix product of the matrices for the two braids on the right side of Equation 5.1, we need them to be of the same size so we need to write σ_{cd} formally as a braid on $|X|+2$ strands. For that purpose we take, on the level of the image of the invariant, the disjoint union of the positive crossing labelled by c, d with $|X|$ unknotted strands. As before, we suppress the normalizing factor $1/\sqrt{t_c}$.

$$
\begin{array}{c|cc}
t_c & c & d \\ \hline
c & -t_c & 0 \\
d & 1-t_d & -1
\end{array}
\sqcup \bigsqcup_{x\in X}
\begin{array}{c|c}
1 & x \\ \hline
x & -1
\end{array}
=
\begin{array}{c|ccc}
t_c & c & d & X \\ \hline
c & -t_c & 0 & 0_{|X|} \\
d & 1-t_d & -1 & 0_{|X|} \\
X & 0^{\text{tr}}_{|X|} & 0^{\text{tr}}_{|X|} & -t_c I_{|X|\times|X|}
\end{array}
$$

Then, the negative of the matrix product (in opposite order) agrees with the matrix from the earlier operation corresponding to stitching together the two braids:

$$
(\lambda_{\sigma_{cd}}, \mathcal{A}_{\sigma_{cd}}) \cdot (\lambda_B, \mathcal{A}_B) = \left(t_c\lambda, \begin{bmatrix} -t_c & 0 & 0_{|X|} \\ 1-t_d & -1 & 0_{|X|} \\ 0^{\text{tr}}_{|X|} & 0^{\text{tr}}_{|X|} & -t_c I_{|X|\times|X|} \end{bmatrix} \cdot \begin{bmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \Xi \end{bmatrix} \right)
$$

$$
\xrightarrow{t_c\to t_a,\, t_d\to t_b}
\begin{array}{c|ccc}
t_a\lambda & a & b & X \\ \hline
a & -t_a\alpha & -t_a\beta & -t_a\theta \\
b & (1-t_b)\alpha-\gamma & (1-t_b)\beta-\delta & (1-t_b)\theta-\epsilon \\
X & -t_a\phi & -t_a\psi & -t_a\Xi
\end{array}
$$

Both this and the matrix from the earlier computation needs to be multiplied by the normalizing factor $1/\sqrt{t_a}$. Thus, taking the negative transpose (or negative inverse) of the image makes rMVA a group homomorphism. Analogous computations can be performed for σ_{cd}^{-1}. Since the $\sigma_{cd},\ c,d \in Y := X \cup \{a,b\}$ generate pvB_Y, we get more generally:

$$\mathrm{rMVA}(B_1 \cdot B_2) = -\mathrm{rMVA}(B_2) \cdot \mathrm{rMVA}(B_1) \quad \forall\, B_1, B_2 \in pvB_Y$$

Furthermore, the image of the generators σ_{cd} using rMVA is equivalent to that for the Gassner representation, after taking the negative transpose:

$$\mathrm{rMVA}(\sigma_{cd}) = t_c^{-\frac{1}{2}} \begin{array}{c|cc} t_c & c & d \\ \hline c & -t_c & 0 \\ d & 1-t_d & -1 \end{array} \xrightarrow{-(\ \cdot\ )^{\mathrm{tr}}} t_c^{-\frac{1}{2}} \begin{array}{c|cc} t_c & c & d \\ \hline c & t_c & t_d - 1 \\ d & 0 & 1 \end{array}$$

□

Remark 5.2.4 *Since the rMVA invariant satisfies the "Overcrossings Commute" relation described in Figure 3.1, it is also a Gassner-type invariant for pure welded braids pwB_X.*

Chapter 6

Extensions

6.1 Partial trace

We have mainly discussed the rMVA invariant in the context of pure tangles where it is well defined, recovers the tMVA, and is a metamonoid morphism. A brief digression in Section 5.1 discussed the special case of 1-tangles with possibly some additional closed components. Since the rMVA invariant is obtained from tMVA, which allows gluing of the ends of the same strand and can be evaluated on non-pure tangles albeit in the circuit algebra setting, there is hope that the rMVA invariant can also be extended to non-pure tangles. A step in that direction is the following "strand closure" or trace operation induced from the corresponding operation on the target space of tMVA.

$$\begin{array}{c|cc} \lambda & a & X \\ \hline a & \alpha & \theta \\ X & \phi & \Xi \end{array} \xrightarrow{\mathrm{tr}_a} \begin{array}{c|c} \lambda+\alpha & X \\ \hline X & \frac{(\lambda+\alpha)\Xi-\phi\theta}{\lambda} \end{array}$$

Figure 6.1: A partial trace operation on the target space of rMVA.

It is only partially defined, as after applying this operation to a pair $(\lambda, \mathcal{A})$, in the resulting pair $(\mathrm{tr}_a(\lambda), \mathrm{tr}_a(\mathcal{A}))$ it is no longer guaranteed that $\mathrm{tr}_a(\lambda) \neq 0$, and so we cannot apply the metamonoid operations, and we can no longer recover the tMVA using Theorem 2.3.2. This operation nonetheless appears promising as it and recovers the MVA for all links with up to 7 crossings in the Knot Atlas [DBN5]. Hence, if the domain of the trace is understood better, it could allow to generalize the rMVA invariant to non-pure tangles.

6.2 Future directions

Several further questions can be pursued in order to understand the rMVA invariant better and explore possible generalizations. Among them are:

i) Since the rMVA invariant fits in the setting of metamonoids, but is originally obtained through Archibald's tMVA invariant which is a circuit algebra morphism, a more general algebraic description of the relation between oriented circuit algebras and metamonoids would aid in understanding the connection better.

ii) To clarify more the connection of rMVA with tMVA and the original multivariable Alexander polynomial on links, it would be beneficial to study further the trace operation in Figure 6.1 and the circumstances under which we get $\lambda = 0$ in a pair $(\lambda, \mathcal{A})$ in the target space of rMVA.

iii) As described in Theorem 4.3.1, the rMVA is equivalent to a version of Bar-Natan's Z invariant, so the partial trace operation can be considered in that context. The invariant Z itself is a reduction of an invariant of ribbon-knotted copies of S^1 and S^2 in $\mathbb{R}^4$ mapping to certain free Lie and cyclic words, defined by Bar-Natan in [DBN3], so it would be of interest to explore how the trace manifests there.

iv) The Alexander polynomial satisfies several skein relations, most of which have been verified by Archibald in [Arch10] through the tMVA invariant in a more straightforward way that the standard setting, and whose proofs might be further simplified using rMVA.

v) Finally, since the image of a tangle under the studied invariants is essentially a collection of Laurent polynomials, it would be interesting to seek a categorification of rMVA and Z.

Chapter 7

Proofs

7.1 Invariance of rMVA

<u>*Reidemeister II moves.*</u>

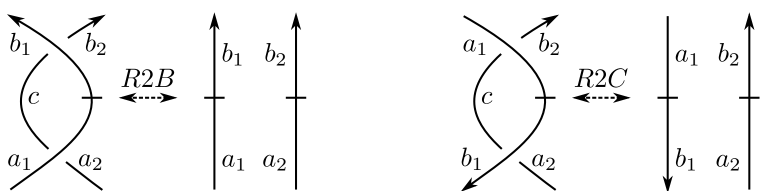


Figure 7.1: The braid-like and cyclic Reidemeister II moves.

For the diagram on the right side of both Reidemeister II moves, we have:

$$M_R = \begin{array}{c|cccc} & b_1 & b_2 & a_1 & a_2 \\ \hline b_1 & 1 & 0 & -1 & 0 \\ b_2 & 0 & 1 & 0 & -1 \end{array} \Rightarrow \lambda_R = 1 \; A_R = \begin{array}{c|cc} & a_1 & a_2 \\ \hline b_1 & -1 & 0 \\ b_2 & 0 & -1 \end{array}$$

$$\Rightarrow \text{rMVA}(D_R) = \prod_{s=1}^{2} t_s^{-\frac{\mu(s)}{2}} \cdot \begin{array}{c|c} \lambda_R & X^{\text{in}} \\ \hline X^{\text{out}} & A_R \end{array} = 1 \cdot \begin{array}{c|cc} 1 & a_1 & a_2 \\ \hline b_1 & -1 & 0 \\ b_2 & 0 & -1 \end{array} = \begin{array}{c|cc} 1 & a_1 & a_2 \\ \hline b_1 & -1 & 0 \\ b_2 & 0 & -1 \end{array}$$

For the braid-like move, the left side diagram D_{LB} gives the following Alexander matrix and rMVA value:

$$M_{LB} = \begin{array}{c|ccccc} & c & b_1 & b_2 & a_1 & a_2 \\ \hline c & t_1 & 0 & 0 & 1-t_2 & -1 \\ b_1 & 0 & 1 & 0 & -1 & 0 \\ b_2 & -t_1 & t_2-1 & 1 & 0 & 0 \end{array} \Rightarrow \lambda_{LB} = t_1,\ A_{LB} = \begin{array}{c|cc} & a_1 & a_2 \\ \hline b_1 & -t_1 & 0 \\ b_2 & 0 & -t_1 \end{array}$$

$$\Rightarrow \text{rMVA}(D_{LB}) = \prod_{s=1}^{2} t_s^{-\frac{\mu(s)}{2}} \cdot \frac{\lambda_{LB}}{X^{\text{out}}} \left| \frac{X^{\text{in}}}{A_{LB}} \right. = t_1^{-1} \cdot \begin{array}{c|cc} t_1 & a_1 & a_2 \\ \hline b_1 & -t_1 & 0 \\ b_2 & 0 & -t_1 \end{array} = \begin{array}{c|cc} 1 & a_1 & a_2 \\ \hline b_1 & -1 & 0 \\ b_2 & 0 & -1 \end{array}$$

Similarly, for the cyclic move, the left side D_{LC} gives the following Alexander matrix and rMVA value:

$$M_{LC} = \begin{array}{c|ccccc} & c & b_1 & b_2 & a_1 & a_2 \\ \hline c & 1 & t_2-1 & 0 & 0 & -t_1 \\ b_1 & 0 & 1 & 0 & -1 & 0 \\ b_2 & -1 & 0 & t_1 & 1-t_2 & 0 \end{array} \Rightarrow \lambda_{LC} = t_1,\ A_{LC} = \begin{array}{c|cc} & a_1 & a_2 \\ \hline b_1 & -t_1 & 0 \\ b_2 & 0 & -t_1 \end{array}$$

$$\Rightarrow \text{rMVA}(D_{LC}) = \prod_{s=1}^{2} t_s^{-\frac{\mu(s)}{2}} \cdot \frac{\lambda_{LC}}{X^{\text{out}}} \left| \frac{X^{\text{in}}}{A_{LB}} \right. = t_1^{-1} \cdot \begin{array}{c|cc} t_1 & a_1 & a_2 \\ \hline b_1 & -t_1 & 0 \\ b_2 & 0 & -t_1 \end{array} = \begin{array}{c|cc} 1 & a_1 & a_2 \\ \hline b_1 & -1 & 0 \\ b_2 & 0 & -1 \end{array}$$

Next, we'll consider the braid-like and cyclic Reidemeister III moves.

Reidemeister III moves.

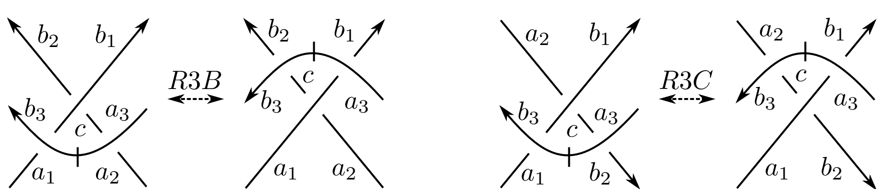


Figure 7.2: The braid-like and cyclic Reidemeister III moves.

Starting with the braid-like move, the diagram on the left side D_{LB} gives the following Alexander matrix and corresponding value of the rMVA:

$$M_{LB} = \begin{array}{c|ccccccc} & c & b_1 & b_2 & b_3 & a_1 & a_2 & a_3 \\ \hline c & 1 & 0 & 0 & 0 & 0 & -t_3 & t_2-1 \\ b_1 & 0 & 1 & 0 & t_1-1 & -t_3 & 0 & 0 \\ b_2 & -1 & 1-t_2 & t_1 & 0 & 0 & 0 & 0 \\ b_3 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{array}$$

$$\Rightarrow \text{rMVA}(D_{LB}) = \prod_{s=1}^{3} t_s^{-\frac{\mu(s)}{2}} \cdot \frac{\lambda_{LB}}{X^{\text{out}}} \left| \frac{X^{\text{in}}}{A_{LB}} \right. = t_1^{-\frac{1}{2}} t_3^{-1} \cdot \begin{array}{c|ccc} t_1 & a_1 & a_2 & a_3 \\ \hline b_1 & -t_1 t_3 & 0 & t_1(t_1-1) \\ b_2 & t_3(1-t_2) & -t_3 & t_1(t_2-1) \\ b_3 & 0 & 0 & -t_1 \end{array}$$

The diagram on the right side of the braid-like Reidemeister III move, D_{RB}, produces analogously:

$$M_{RB} = \begin{array}{c|ccccccc} & c & b_1 & b_2 & b_3 & a_1 & a_2 & a_3 \\ \hline c & t_1 & 0 & 0 & 0 & 1-t_2 & -1 & 0 \\ b_1 & 0 & 1 & 0 & 0 & -t_3 & 0 & t_1-1 \\ b_2 & -t_3 & 0 & 1 & t_2-1 & 0 & 0 & 0 \\ b_3 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{array}$$

$$\Rightarrow \mathrm{rMVA}(D_{RB}) = \prod_{s=1}^{3} t_s^{-\frac{\mu(s)}{2}} \cdot \begin{array}{c|c} \lambda_{RB} & X^{\mathrm{in}} \\ \hline X^{\mathrm{out}} & A_{RB} \end{array} = t_1^{-\frac{1}{2}} t_3^{-1} \cdot \begin{array}{c|ccc} t_1 & a_1 & a_2 & a_3 \\ \hline b_1 & -t_1t_3 & 0 & t_1(t_1-1) \\ b_2 & t_3(1-t_2) & -t_3 & t_1(t_2-1) \\ b_3 & 0 & 0 & -t_1 \end{array}$$

Looking next at the second type of Reidemeister III move, the cyclic one, the diagram on the left side of the move D_{LC} gives rise to the following Alexander matrix and rMVA value:

$$M_{LC} = \begin{array}{c|ccccccc} & c & b_1 & b_2 & b_3 & a_1 & a_2 & a_3 \\ \hline c & 1 & t_2-1 & 0 & 0 & 0 & -t_1 & 0 \\ b_1 & 0 & 1 & 0 & t_1-1 & -t_3 & 0 & 0 \\ b_2 & -1 & 0 & t_3 & 0 & 0 & 0 & 1-t_2 \\ b_3 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{array}$$

$$\Rightarrow \mathrm{rMVA}(D_{LC}) = \prod_{s=1}^{3} t_s^{-\frac{\mu(s)}{2}} \cdot \begin{array}{c|c} \lambda_{LC} & X^{\mathrm{in}} \\ \hline X^{\mathrm{out}} & A_{LC} \end{array} = t_1^{-\frac{1}{2}} t_3^{-1} \cdot \begin{array}{c|ccc} t_3 & a_1 & a_2 & a_3 \\ \hline b_1 & -t_3^2 & 0 & t_3(t_1-1) \\ b_2 & t_3(t_2-1) & -t_1 & t_1(1-t_2) \\ b_3 & 0 & 0 & -t_3 \end{array}$$

For the diagram on the right side of the same move, D_{RC}, we get:

$$M_{RC} = \begin{array}{c|ccccccc} & c & b_1 & b_2 & b_3 & a_1 & a_2 & a_3 \\ \hline c & t_3 & 0 & 0 & 1-t_2 & 0 & -1 & 0 \\ b_1 & 0 & 1 & 0 & 0 & -t_3 & 0 & t_1-1 \\ b_2 & -t_1 & 0 & 1 & 0 & t_2-1 & 0 & 0 \\ b_3 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{array}$$

$$\Rightarrow \mathrm{rMVA}(D_{RC}) = \prod_{s=1}^{3} t_s^{-\frac{\mu(s)}{2}} \cdot \begin{array}{c|c} \lambda_{RC} & X^{\mathrm{in}} \\ \hline X^{\mathrm{out}} & A_{RC} \end{array} = t_1^{-\frac{1}{2}} t_3^{-1} \cdot \begin{array}{c|ccc} t_3 & a_1 & a_2 & a_3 \\ \hline b_1 & -t_3^2 & 0 & t_3(t_1-1) \\ b_2 & t_3(t_2-1) & -t_1 & t_1(1-t_2) \\ b_3 & 0 & 0 & -t_3 \end{array}$$

7.2 Proof of Theorem 2.3.2

Theorem 2.3.2 follows from the proof of the generalized Cramer's rule by Gong, Aldeen and Elsner [GAE], which we reproduce here in our setting. Throughout, we'll use the notation:

$$A^{j_1,\ldots,j_k}_{i_1,\ldots,i_k} \quad = \quad \text{the } k\times k \text{ submatrix of } A \text{ with columns } j_1,\ldots,j_k \text{ and rows } i_1,\ldots,i_k$$

$$A^{p_1,\ldots,p_k}_{B(q_1,\ldots,q_k)} \quad = \quad \text{the matrix } A \text{ with column } p_s \text{ replaced by column } q_s \text{ of } B,\ s=1,\ldots,k$$

Suppose $T \in pv\mathcal{T}_X$ is a pure, regular, v-tangle with m internal arcs and $|X| = n$, with $(m+n)\times(m+2n)$ Alexander matrix $M(D_T)$ for a diagram D_T of T. For convenience, we will distinguish $X^{\text{in}} \cong X^{\text{out}} \cong X$, the labels of the incoming and outgoing arcs of T. As discussed above, the degree 0 and 1 components for T using the tMVA invariant and the Hodge $*$ operator are:

$$\lambda = \det M(D_T)^{1,\ldots,n;\emptyset} \qquad \mathcal{A}_{i,j} = (-1)^{n-i}\det M(D_T)^{1,\ldots,\hat{i},\ldots,n;j},\ 1\le i,j\le n$$

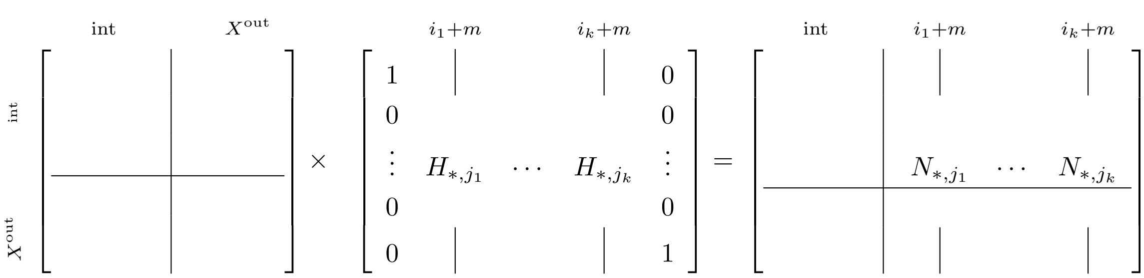
Take the $(m+n)\times(m+n)$ matrix $D = M(D_T)^{1,\ldots,n;\emptyset}$ ($\det D = \lambda$) and the $(m+n)\times n$ matrix N, $N_{i,j} = M(D_T)_{i,(m+n)+j}$, namely the matrix made up of the columns of the Alexander matrix labelled by X^{in}.

Now, consider the $(m+n)\times n$ matrix H defined by $H_{i,j} = \frac{\det D^i_{N(j)}}{\det D}$. Note that $H_{i+m,j} = \frac{\mathcal{A}_{i,j}}{\det D}$ for $1\le i\le n$.

Lemma 7.2.1 *Let I denote the $(m+n)\times(m+n)$ identity matrix. Then:*

$$D\cdot I^{i_1+m,\ldots,i_k+m}_{H(j_1,\ldots,j_k)} = D^{i_1+m,\ldots,i_k+m}_{N(j_1,\ldots,j_k)} \tag{7.1}$$

Proof: Let $A_{*,j}$ denote the jth column vector of the matrix A. Then the equation in the lemma can be visualized as:



Let $B = I^{i_1+m,\ldots,i_k+m}_{X(j_1,\ldots,j_k)}$. Then the equation follows directly from the classical Cramer's rule, from which we can conclude that:

$$D\cdot B_{*,s} = \begin{cases} D\cdot H_{*,j_p} = D\cdot \dfrac{\det D^*_{N(j_p)}}{\det D} = N_{*,j_p} & s = i_p+m,\ (p=1,\ldots,k) \\ D\cdot I_{*,s} = D_{*,s} & \text{otherwise.}\end{cases}$$

□

We take determinants of the equality in Lemma 7.2.1, and use:

Lemma 7.2.2 *For any $1\le i_1,\ldots,i_k,j_1,\ldots,j_k\le n$:*

$$\det I^{i_1+m,\ldots,i_k+m}_{H(j_1,\ldots,j_k)} = \frac{1}{\lambda^k}\det \mathcal{A}^{j_1,\ldots,j_k}_{i_1,\ldots,i_k}$$

Proof: Let $B = I^{i_1+m,\dots,i_k+m}_{X(j_1,\dots,j_k)}$. Then, using $\delta_{x,y}$ to denote the Kronecker delta ($\delta_{x,y} = 1$ if $x = y$ and is zero otherwise):

$$\begin{aligned}
\det B &= \sum_{\sigma\in S_{n+m}} (-1)^{\sigma} b_{\sigma(1),1}\dots b_{\sigma(i_1+m),i_1+m}\dots b_{\sigma(i_k+m),i_k+m}\dots b_{\sigma(n+m),n+m}\\
&= \sum_{\sigma\in S_{n+m}} (-1)^{\sigma} \delta_{\sigma(1),1}\dots b_{\sigma(i_1+m),i_1+m}\dots b_{\sigma(i_k+m),i_k+m}\dots \delta_{\sigma(n+m),n+m}\\
&= \sum_{\sigma\in S_k} (-1)^{\sigma} b_{\sigma(i_1+m),i_1+m} b_{\sigma(i_2+m),i_2+m}\dots b_{\sigma(i_k+m),i_k+m}\\
&= \det H^{j_1,\dots,j_k}_{i_1+m,\dots,i_k+m} = \frac{1}{\lambda^k}\det \mathcal{A}^{j_1,\dots,j_k}_{i_1,\dots,i_k}
\end{aligned}$$

□

The determinant version of Equation 7.1 then becomes:

$$\lambda\frac{1}{\lambda^k}\det \mathcal{A}^{j_1,\dots,j_k}_{i_1,\dots,i_k} = \det D^{i_1+m,\dots,i_k+m}_{N(j_1,\dots,j_k)} = (-1)^{nk-\sum_{p=1}^k i_p - \frac{(k-1)k}{2}} \det M(D_T)^{\{1,\dots,n\}\setminus\{i_1,\dots,i_k\};j_1,\dots,j_k}$$

The coefficients of the tMVA invariant are thus obtained from the pair $(\lambda, \mathcal{A})$ whenever $\lambda \neq 0$ using the formula:

$$\det M(D_T)^{\{1,\dots,n\}\setminus\{i_1,\dots,i_k\};j_1,\dots,j_k} = (-1)^{nk-\sum_{p=1}^k i_p-(k-1)k/2}\frac{\det \mathcal{A}^{j_1,\dots,j_k}_{i_1,\dots,i_k}}{\lambda^{k-1}}$$

7.3 Proof of Theorem 2.3.1

Suppose T is a pure, regular, virtual tangle. Then, using the earlier notation:

$$\begin{aligned}
\mathrm{tMVA}(T) &= \prod_k t_k^{-\frac{\mu(k)}{2}} w \otimes \sum_{k=0}^{n} \sum_{\substack{i_1<\dots<i_{n-k}\\ j_1<\dots<j_k}} \det M(D_T)^{i_1,\dots,i_{n-k};j_1,\dots,j_k} x^{\mathrm{in}}_{j_1}\wedge\dots\wedge x^{\mathrm{in}}_{j_k}\otimes x^{\mathrm{out}}_{i_1}\wedge\dots\wedge x^{\mathrm{out}}_{i_{n-k}}\\
\Rightarrow p_2(\mathrm{tMVA}(T)) &= \sum_{k=0}^{n} \sum_{\substack{i_1<\dots<i_{n-k}\\ j_1<\dots<j_k}} \det M(D_T)^{i_1,\dots,i_{n-k};j_1,\dots,j_k} x^{\mathrm{in}}_{j_1}\wedge\dots\wedge x^{\mathrm{in}}_{j_k}\otimes x^{\mathrm{out}}_{i_1}\wedge\dots\wedge x^{\mathrm{out}}_{i_{n-k}}
\end{aligned}$$

To shorten the notation, we will denote $[n] = \{1,\dots,n\}$, $\bar{i} = \{i_1 < \dots < i_{n-k}\}$, $\underline{j} = \{j_1 \dots, j_k\}$ and $x_{\bar{i}} = x_{i_1}\wedge\dots\wedge x_{i_{n-k}}$, respectively $y_{\underline{j}} = y_{j_1}\wedge\dots\wedge y_{j_k}$, as well as $x_{\underline{s}}(y_{\underline{t}}) = x_{s_1}(y_{t_1})\cdot\dots\cdot x_{s_k}(y_{t_k})$, as X^{in}

and X^{out} are self-dual. Following the diagram in Equation 2.1, for $p_2(\text{tMVA}(T))$, we get:

$$p_2(\text{tMVA}(T)) \xrightarrow{*_w} \sum_{k=0}^{n} \sum_{\bar{i},\underline{j}} (-1)^{*_w} \det M(D_T)^{\bar{i};\underline{j}} x^{\text{in}}_{\underline{j}} \otimes x^{\text{out}}_{[n]\setminus\bar{i}} \quad \in \bigoplus_{k=0}^{n} \Lambda^k(X^{\text{in}}) \otimes \Lambda^k(X^{\text{out}})$$

$$\longrightarrow \left(\phi_k : x^{\text{in}}_{\underline{p}} \mapsto \sum_{\bar{i},\underline{j}} (-1)^{*_w} \det M(D_T)^{\bar{i};\underline{j}} \left[\sum_{\sigma \in S_k} \text{sgn}(\sigma) x^{\text{in}}_{\sigma(\underline{j})}(x^{\text{in}}_{\underline{p}}) \right] x^{\text{out}}_{[n]\setminus\bar{i}} \right)_{k=0}^{n} =$$

$$= \left(\phi_k : x^{\text{in}}_{\underline{p}} \mapsto \sum_{\bar{i}} (-1)^{*_w} \det M(D_T)^{\bar{i};\underline{p}} x^{\text{out}}_{[n]\setminus\bar{i}} \right)_{k=0}^{n} \quad \in \bigoplus_{k=0}^{n} \text{Hom}(\Lambda^k(X^{\text{in}}), \Lambda^k(X^{\text{out}}))$$

In particular, the degree zero and one components are:

$$\phi_0(1) = \det M(D_T)^{[n];\emptyset} = \lambda \qquad \phi(x^{\text{in}}_j) := \phi_1(x^{\text{in}}_j) = \sum_{i=1}^{n} (-1)^{n-i} \det M(D_T)^{[n]\setminus\{i\};j} x^{\text{out}}_i$$

Then, taking $\psi = \phi/\lambda$, we have for any $0 \le k \le n$, $1 \le j_1 < \ldots < j_k \le n$:

$$\begin{aligned}
&\Lambda(\psi)(x^{\text{in}}_{j_1} \wedge \ldots \wedge x^{\text{in}}_{j_k}) = \psi(x^{\text{in}}_{j_1}) \wedge \ldots \wedge \psi(x^{\text{in}}_{j_k}) \\
&= \left(\sum_{i_1=1}^{n} (-1)^{n-i_1} \frac{\det M(D_T)^{[n]\setminus\{i_1\};j_1}}{\lambda} x^{\text{out}}_{i_1} \right) \wedge \ldots \wedge \left(\sum_{i_k=1}^{n} (-1)^{n-i_k} \frac{\det M(D_T)^{[n]\setminus\{i_k\};j_k}}{\lambda} x^{\text{out}}_{i_k} \right) \\
&= \left(\sum_{i_1=1}^{n} \frac{\mathcal{A}_{i_1,j_1}}{\lambda} x^{\text{out}}_{i_1} \right) \wedge \ldots \wedge \left(\sum_{i_k=1}^{n} \frac{\mathcal{A}_{i_k,j_k}}{\lambda} x^{\text{out}}_{i_k} \right) \\
&= \frac{1}{\lambda^k} \sum_{i_1<\ldots<i_k} \sum_{\sigma \in S_k} \text{sgn}(\sigma) \prod_{s=1}^{k} \mathcal{A}_{\sigma(i_s),j_s} x^{\text{out}}_{i_1} \wedge \ldots \wedge x^{\text{out}}_{i_k} = \sum_{i_1<\ldots<i_k} \frac{\det \mathcal{A}^{j_1,\ldots,j_k}_{i_1,\ldots,i_k}}{\lambda^k} x^{\text{out}}_{i_1} \wedge \ldots \wedge x^{\text{out}}_{i_k} \\
&\overset{Thm 2.3.2}{=} \frac{1}{\lambda} \sum_{i_1<\ldots<i_k} (-1)^{nk - \sum_{p=1}^{k} i_p - (k-1)k/2} \det M(D_T)^{\{1,\ldots,n\}\setminus\{i_1,\ldots,i_k\};j_1,\ldots,j_k} x^{\text{out}}_{i_1} \wedge \ldots \wedge x^{\text{out}}_{i_k} \\
&= \frac{1}{\lambda} \phi_k(x^{\text{in}}_{j_1} \wedge \ldots \wedge x^{\text{in}}_{j_k})
\end{aligned}$$

Therefore, $\lambda\Lambda(\phi/\lambda) = (\phi_k)_{k=0}^{n}$, concluding the proof.

Part II

Skew Howe duality for crystals and the cactus group

Chapter 8

Introduction

One approach to studying the representation theory of a finite-dimensional, complex reductive Lie algebra $\mathfrak{g}$ is through crystals–a combinatorial tool which encodes the inner structure of a given $\mathfrak{g}$-representation. A crystal can be represented as a coloured, directed graph and has nice behaviour in terms of irreducible components, direct sum, and tensor product. The category of $\mathfrak{g}$-crystals has been studied by Henriques and Kamnitzer in [HeKa06], where they describe its structure as a coboundary category, with an action of the cactus group J_n on n-tensor products of $\mathfrak{g}$-crystals, analogous to the action of the braid group B_n in a braided category. In this thesis, we define the cactus group $J_{\mathfrak{g}}$ for any $\mathfrak{g}$, with $J_{\mathfrak{g}} = J_n$ for $\mathfrak{g} = \mathfrak{g}l_n (= \mathfrak{g}l_n(\mathbb{C}))$, and show that it acts on any $\mathfrak{g}$-crystal via the so-called Schützenberger involutions. We then study the set $\Lambda^N B_{n,m}$ of $n \times m$-matrices with N ones and the rest zeros, show it has the structure of both a $\mathfrak{g}l_n$- and a $\mathfrak{g}l_m$-crystal and the two structures commute, corresponding to the representation $\Lambda^N(\mathbb{C}^n \otimes \mathbb{C}^m)$. We then derive $(\mathfrak{g}l_n, \mathfrak{g}l_m)$ skew Howe duality on the level of crystals, also studied by van Leeuwen [vL06] and Danilov-Koshevoy [DK04], analogous to skew Howe duality on the level of representations. Using this duality, and the decomposition:

$$\Lambda^N B_{n,m} \cong \bigsqcup_{\underline{l} \in \mathbb{N}^n, |\underline{l}|=N} \mathcal{B}^{\mathfrak{g}l_m}_{\omega_{l_1}} \otimes \ldots \otimes \mathcal{B}^{\mathfrak{g}l_m}_{\omega_{l_n}}$$

where ω_i is the i-th fundamental weight of $\mathfrak{g}l_m$, the "outer" action of the cactus group J_n on $\Lambda^N B_{n,m}$, acting on n-tensor products of $\mathfrak{g}l_m$-crystals as defined by Henriques and Kamnitzer, is then shown to agree with the "inner" cactus group action on $\mathfrak{g}l_n$-crystals defined here. One of the applications of this work is demonstrating the equivalence of two conjectured results, relating the "inner" cactus group action to certain maximal commutative subalgebras in $U(\mathfrak{g}l_n)$ known as "shift of argument algebras" on one hand, and on the other hand relating the "outer" cactus group action to maximal commutative subalgebras in $(U(\mathfrak{g}l_m))^{\otimes n}$ known as "Gaudin algebras". Chapter 9 provides the setup and some background information on crystals, Chapter 10 defines the cactus group $J_{\mathfrak{g}}$ for general $\mathfrak{g}$ and verifies that it acts on any $\mathfrak{g}$-crystal. In Chapter 11 we discuss the crystal structure of $\Lambda^N B_{n,m}$. The main result of establishing skew Howe duality and the equivalence of the outer and inner cactus group actions on $\Lambda^N B_{n,m}$ is in Chapter 12. Finally, an application where the cactus group action is realized geometrically is described in Chapter 13.

Chapter 9

Preliminaries

Crystals are a useful combinatorial tool for encoding the representations of a complex, reductive Lie algebra $\mathfrak{g}$. We present some of the necessary background here, further information can be obtained in [HoKa02], [J95], and [Kas91]. Informally, a crystal gives a "nice" basis for a $U_q(\mathfrak{g})$-module at "$q = 0$", where $U_q(\mathfrak{g})$ denotes the quantum group corresponding to $\mathfrak{g}$. More precisely, let $\mathfrak{h}$ denote a Cartan subalgebra of $\mathfrak{g}$, $\Lambda \subset \mathfrak{h}^*$ the weight lattice, Λ_+ the dominant weights, I the Dynkin diagram, $\{\alpha_i\}_{i\in I}$ and $\{\alpha_i^\vee\}_{i\in I}$ the simple roots and coroots, and $E_i,\ F_i,\ i \in I$, the Chevalley generators of $\mathfrak{g}$. We consider V belonging to the $\mathcal{O}^q_{\text{int}}$ category of $U_q(\mathfrak{g})$-modules, i.e. satisfying:

- The action of E_i and F_i on V is locally nilpotent.
- $V = \bigoplus_{\lambda\in\Lambda} V_\lambda$ where each of the weight spaces V_λ is finite-dimensional over $\mathbb{C}(q)$.
- For the set of weights of V, we have $\text{wt}(V) \subset \bigcup_{i=1}^{k}\{\lambda \in \Lambda \mid \lambda \leq \lambda_i\}$ for some finite set $\{\lambda_1, \ldots, \lambda_k\} \subset \Lambda$.

The operators on crystals $e_i,\ f_i,\ i \in I$, called **Kashiwara operators**, are modified versions of the root operators which preserve the structure of the representation. They map weight spaces into weight spaces $e_i(V_\lambda) \subset V_{\lambda+\alpha_i},\ f_i(V_\lambda) \subset V_{\lambda-\alpha_i}$ and commute with the $U_q(\mathfrak{g})$-module morphisms. These properties follow directly from their definition:

A weight vector $v \in V_\lambda$, can be written as $v = \sum_{k=1}^{p} F_i^{(k)} v_k$ for some $p \in \mathbb{N}$, in terms of the divided powers $F_i^{(k)}$ of F_i, where $E_i(v_k) = 0$ and $v_k \in V_{\lambda+k\alpha_i}$. Then the Kashiwara raising and lowering operators act as:

$$e_i(v) = \sum_{k=1}^{p} F_i^{(k-1)}(v_k) \quad f_i(v) = \sum_{k=1}^{p} F_i^{(k+1)}(v_k)$$

To define the corresponding crystal basis, we consider the localization of $\mathbb{C}[q]$ at the ideal (q): $\mathbb{C}[q]_0 = \{\frac{f}{g} \mid f, g \in \mathbb{C}[q], g(0) \neq 0\}$.

Definition 9.0.1 *A **crystal lattice** L of V is a $\mathbb{C}[q]_0$-submodule such that:*

$$(1)\ V = \mathbb{C}(q) \otimes_{\mathbb{C}[q]_0} L \qquad (2)\ e_i(L),\ f_i(L) \subset L\ \forall\, i \in I \qquad (3)\ L = \bigoplus_{\lambda\in\Lambda}(L \cap V_\lambda).$$

*A **crystal base** of V is a pair (L, B) where L is a crystal lattice of V and B is a set such that:*

(i) B *is a* $\mathbb{C}$*-basis of* L/qL

(ii) $B = \bigsqcup_{\lambda \in \Lambda} B \cap [(L \cap V_\lambda)/q(L \cap V_\lambda)]$

(iii) $e_i(B), f_i(B) \subset B \cup \{0\} \ \forall\, i \in I$

(iv) *For all* $b, c \in B,\ f_i(b) = c \Leftrightarrow b = e_i(b)$

Using Kashiwara's *grand loop argument* [Kas91], it can be shown that every representation in $\mathcal{O}^q_{\text{int}}$, which in turn corresponds to a $\mathfrak{g}$-representation, has a crystal base that is unique up to isomorphism.

Properties: Crystals have several nice properties which make them well-suited for studing the structure of representations of $\mathfrak{g}$.

1. For a $\mathfrak{g}$-representation V with crystal base (L, B), let $B_\lambda = B \cap [(L \cap V_\lambda)/q(L \cap V_\lambda)]$ where λ is a weight of V. Then the character of V can be recovered by:

$$\text{ch}(V) = \sum_{\lambda \in \Lambda} |B_\lambda| e^\lambda$$

2. A useful combinatorial tool that comes from a crystal base (L, B) is the **crystal graph**, whose vertices are the set B and whose edges are I-coloured and correspond to the action of the Kashiwara operators: for $b_1, b_2 \in B$, we have $b_1 \xrightarrow{i} b_2$ if $f_i(b_1) = b_2$ (equivalently $e_i(b_2) = b_1$). For instance, the crystal graph for the adjoint representation of $\mathfrak{sl}_3$ is shown in Figure 9.1. Moreover, irreducible components of the representation are easy to spot in the crystal graph as they correspond to connected components of the graph.

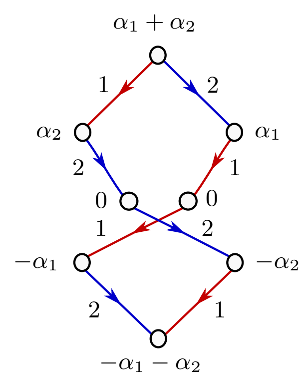


Figure 9.1: The crystal graph for the adjoint representation of $\mathfrak{sl}_3$.

More abstractly, a $\mathfrak{g}$**-crystal** B is a finite set B together with the maps:

$$e_i,\ f_i : B \to B \cup \{0\},\ \epsilon_i, \phi_i : B \to \mathbb{Z} \quad (i \in I)$$
$$\text{wt} : B \to \Lambda$$

Satisfying the properties:

(a) For all $b, c \in B,\ f_i(b) = c$ if and only if $b = e_i(c)$.

(b) For $b \in B$, if $e_i(b) \in B$ (i.e. $e_i(b) \neq 0$), then $\text{wt}(e_i(b)) = \text{wt}(b) + \alpha_i$ and if $f_i(b) \in B$, then $\text{wt}(f_i(b)) = \text{wt}(b) - \alpha_i$.

(c) For all $b \in B,\ i \in I,\ \epsilon_i(b) = \max\{k \,|e_i^k(b) \in B\}$ and $\phi_i(b) = \max\{k \,|f_i^k(b) \in B\}$.

(d) For all $b \in B,\ i \in I$, we have $\phi_i(b) - \epsilon_i(b) = \langle \text{wt}(b), \alpha_i^\vee \rangle$.

1. Taking the **tensor product** of two $\mathfrak{g}$-crystals A and B can be done using the following rules–the underlying set of $A \otimes B$ is $A \times B$ and for any element $a \otimes b \in A \otimes B$ we have:

$$e_i(a \otimes b) = \begin{cases} e_i(a) \otimes b & \text{if } \phi_i(a) \geq \epsilon_i(b) \\ a \otimes e_i(b) & \text{otherwise} \end{cases}$$

$$f_i(a \otimes b) = \begin{cases} f_i(a) \otimes b & \text{if } \phi_i(a) > \epsilon_i(b) \\ a \otimes f_i(b) & \text{otherwise} \end{cases}$$

$$\text{wt}(a \otimes b) = \text{wt}(a) + \text{wt}(b)$$
$$\epsilon_i(a \otimes b) = \max(\epsilon_i(a), \epsilon_i(b) - \langle \text{wt}(a), \alpha_i^\vee \rangle)$$
$$\phi_i(a \otimes b) = \max(\phi_i(b), \phi_i(a) + \langle \text{wt}(b), \alpha_i^\vee \rangle)$$

We will discuss more generally the tensor product of m crystals in Chapter 11. In addition, the **direct sum** of two crystals is their disjoint union.

2. Among the abstract $\mathfrak{g}$-crystals, we wish to only consider those corresponding to a representation of $\mathfrak{g}$. For that purpose, we take a *closed family* of **highest weight crystals** $\{B_\lambda |\ \lambda \in \Lambda_+\}$, namely in each B_λ there is a (highest weight) element b_λ such that $\mathrm{wt}(b_\lambda) = \lambda,\ e_i(b_\lambda) = 0\ \forall\ i \in I$, and B_λ is generated by f_i acting on b_λ, together with the additional property that the connected crystal generated by $b_\lambda \otimes b_\mu \in B_\lambda \otimes B_\mu$ is isomorphic to $B_{\lambda+\mu}$, for all $\lambda, \mu \in \Lambda_+$ (see [J95]). The **category of $\mathfrak{g}$-crystals** we then want to consider has objects each of whose connected components is B_λ for some $\lambda \in \Lambda_+$, while the morphisms are defined below.

Definition 9.0.2 *Suppose $\mathcal{B}_1$ and $\mathcal{B}_2$ are $\mathfrak{g}$-crystals. A* ***crystal morphism*** $F : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ *is a map of sets* $F : \mathcal{B}_1 \cup \{0\} \longrightarrow \mathcal{B}_2 \cup \{0\}$ *such that:*

1. $F(0) = 0$.
2. *If* $b \in \mathcal{B}_1$ *s.t.* $F(b) \in \mathcal{B}_2$*, then* $wt(F(b)) = wt(b),\ \epsilon_i(F(b)) = \epsilon_i(b),\ \phi_i(F(b)) = \phi_i(b)$.
3. *If* $b \in \mathcal{B}_1$ *such that* $F(b) \in \mathcal{B}_2$*, and* $f_i(b) \in \mathcal{B}_1$ *(i.e. it is nonzero), then* $F(f_i(b)) = f_i(F(b))$*, respectively if* $e_i(b) \in \mathcal{B}_1$ *then* $F(e_i(b)) = e_i(F(b))$.

We will recall some further properties of crystals in the context of the following sections. Next, we define the general cactus group and its action on crystals.

Chapter 10

The cactus group

As earlier, $\mathfrak{g}$ denotes a finite-dimensional, complex, reductive Lie algebra, I is the set of vertices in its Dynkin diagram, and $\{\alpha_i\}_{i\in I}$ are the simple roots.

As described in [HeKa06], there is a Dynkin diagram automorphism $\theta : I \longrightarrow I$ defined by $\alpha_{\theta(i)} = -w_0 \cdot \alpha_i$, where w_0 is the longest element in the Weyl group of $\mathfrak{g}$. For instance, in the case of $\mathfrak{g}l_n$, $\theta(i) = n - i$. Let B_λ be the highest weight $\mathfrak{g}$-crystal (corresponding to the irreducible highest weight representation V_λ) with highest weight λ and the corresponding crystal maps be $\mathrm{wt}, e_i, f_i, \epsilon_i, \phi_i$, where $i \in I$. Then, let $\xi_\lambda : B_\lambda \to B_\lambda$ denote the unique map of sets satisfying, for all $b \in B_\lambda$:

$$\begin{aligned} e_i \cdot \xi_\lambda(b) &= \xi_\lambda(f_{\theta(i)} \cdot b) \\ f_i \cdot \xi_\lambda(b) &= \xi_\lambda(e_{\theta(i)} \cdot b) \\ \mathrm{wt}(\xi_\lambda(b)) &= w_0 \cdot \mathrm{wt}(b) \end{aligned}$$

In particular, it interchanges the highest and lowest weight elements of B_λ, $b_\lambda \xrightarrow{\xi_\lambda} b_\lambda^{\mathrm{low}} \xrightarrow{\xi_\lambda} b_\lambda$. More generally we have:

Definition 10.0.1 *The* ***Schützenberger involution*** $\xi : B \to B$ *for any* $\mathfrak{g}$*-crystal* B *acts by applying the corresponding map* ξ_λ *to each connected component* B_λ *of* B*.*

We define the **cactus group** $J_\mathfrak{g}$ corresponding to $\mathfrak{g}$ in terms of generators and relations. Its generators are s_J, where J is a connected subdiagram of I, and the relations are:

1. $s_J^2 = 1 \ \ \forall J \subset I$

2. $s_J s_{J'} = s_{\theta_J(J')} s_J \ \ \forall J' \subset J$

3. $s_J s_{J'} = s_{J'} s_J \ \ \forall J', J \subset I$ disjoint subdiagrams (i.e. not connected by an edge)

In Relation 2, θ_J is the Dynkin diagram automorphism of J, namely $\alpha_{\theta_J(j)} = -w_0^J \cdot \alpha_j, \ \forall\, j \in J$, where w_0^J is the longest element in the Weyl group for $\mathfrak{g}_J$ ($\mathfrak{g}$ restricted to J). In our original notation, $\theta = \theta_I$. This definition is a generalization of the cactus group defined in [HeKa06]–Henriques and Kamnitzer's cactus group corresponds to $J_{\mathfrak{g}l_n}$.

Remark 10.0.2 *The cactus group $J_{\mathfrak{g}}$ is a (distant) cousin of the braid group as it also surjects onto the Weyl group of $\mathfrak{g}$, by mapping $s_J \mapsto w_0^J$ where w_0^J is the longest element corresponding to the Weyl group of $\mathfrak{g}$ restricted to J. The kernel of this surjection is called the* ***pure cactus group*** *$PJ_{\mathfrak{g}}$:*

$$1 \longrightarrow PJ_{\mathfrak{g}} \longrightarrow J_{\mathfrak{g}} \longrightarrow W_{\mathfrak{g}} \longrightarrow 1$$

Proposition 10.0.3 *There is a natural action of the cactus group $J_{\mathfrak{g}}$ on any $\mathfrak{g}$-crystal B given by, for any connected subdiagram $J \subset I$:*

$$s_J(b) = \xi_{B_J}(b)$$

where B_J is the restriction of B to the subdiagram J of I, as a graph it has the same set of vertices but only arrows labelled by J.

Proof: We need to show that the Schützenberger involutions satisfy the cactus relations. Relation 1, claiming the cactus generators indeed act as involutions, is already shown in [HeKa06] as they verify that $\xi_B^2 = 1$ for any $\mathfrak{g}$-crystal B.

Note that as Relation 2 does not depend on the ambient Dynkin diagram of which $J' \subset J$ are subsets, it suffices to check the relation for the pair $J \subset I$, where J is a connected subdiagram:

$$\xi_B \xi_{B_J} = \xi_{B_{\theta(J)}} \xi_B \tag{10.1}$$

Take any $b \in B$. It lies in some connected component A of B_J of highest weight μ say, and in turn A is inside a connected component C of $B = B_I$ of some highest weight λ. Let $b_\mu, b_\mu^{\text{low}}$ and b_λ, b_λ^{low} denote the highest and lowest weight elements in A and C respectively. Then $b = f_{i_1} \cdots f_{i_k} b_\mu$ and $b_\mu^{\text{low}} = e_{j_1} \cdots e_{j_m} b_\lambda^{\text{low}}$ for some $i_1, \ldots, i_k \in J,\ j_1, \ldots, j_m \in I$. Evaluating the left side of Equation (10.1) at b, we get:

$$\begin{aligned}
\xi_B \xi_{B_J}(b) &= \xi_B \xi_{B_J}(f_{i_1} \cdots f_{i_k} b_\mu) \\
&= \xi_B(e_{\theta_J(i_1)} \cdots e_{\theta_J(i_k)} b_\mu^{\text{low}}) \\
&= \xi_B(e_{\theta_J(i_1)} \cdots e_{\theta_J(i_k)} e_{j_1} \cdots e_{j_m} b_\lambda^{\text{low}}) \\
&= f_{\theta\theta_J(i_1)} \cdots f_{\theta\theta_J(i_k)} f_{\theta(j_1)} \cdots f_{\theta(j_m)} b_\lambda
\end{aligned}$$

Note that ξ_B takes the connected component A of B_J to a connected component D of $B_{\theta(J)}$, call its highest weight ν, by exchanging the highest and lowest weight elements, $\xi_B(b_\mu) = b_\nu^{\text{low}}$ and $\xi_B(b_\mu^{\text{low}}) = b_\nu$ since for any $j \in J,\ b \in B$:

$$f_j^{\phi_j(b)}(b) \in B_J \ \Rightarrow\ \xi_B(f_j^{\phi_j(b)}(b)) = e_{\theta(j)}^{\phi_j(b)}(\xi_B(b)) \in B_{\theta(J)} \ \Rightarrow\ \epsilon_{\theta(j)}(\xi_B(b)) \geq \phi_j(b)$$

$$\begin{aligned}
e_{\theta(j)}^{\epsilon_{\theta(j)}(\xi_B(b))}(\xi_B(b)) \in B_{\theta(J)} \ &\Rightarrow\ \xi_B\left[e_{\theta(j)}^{\epsilon_{\theta(j)}(\xi_B(b))}(\xi_B(b))\right] = f_j^{\epsilon_{\theta(j)}(\xi_B(b))}(b) \in B_J \\
&\Rightarrow \phi_j(b) \geq \epsilon_{\theta(j)}(\xi_B(b))
\end{aligned}$$

In particular, $\epsilon_{\theta(j)}(\xi_B(b_\mu^{\text{low}})) = \phi_j(b_\mu^{\text{low}}) = 0$ and similarly $\phi_{\theta(j)}(\xi_B(b_\mu)) = \epsilon_j(b_\mu) = 0$, since θ and ξ_B are involutions.

These highest and lowest weight elements lie in the same connected component of $B_{\theta(J)}$ since there exist $s_1, \dots, s_p \in J$ such that:

$$b_\nu^{\text{low}} = \xi_B(b_\mu) = \xi_B(e_{s_1} \dots e_{s_p} b_\mu^{\text{low}}) = f_{\theta(s_1)} \dots f_{\theta(s_p)} \xi_B(b_\mu^{\text{low}}) = f_{\theta(s_1)} \dots f_{\theta(s_p)} b_\nu$$

Finally, the Dynkin diagram automorphism on $\theta(J)$ defined above can be expressed as $\psi = \theta\theta_J\theta$. Indeed, the Dynkin diagram isomorphism $\theta : J \xrightarrow{\sim} \theta(J)$ induces an isomorphism on the Weyl groups of $\mathfrak{g}_J$ and $\mathfrak{g}_{\theta(J)}$, $W^J \cong W^{\theta(J)}$ given by $w \mapsto w_0 w w_0$ for any $w \in W^J$, so the longest element in $W^{\theta(J)}$ is $w_0^{\theta(J)} = w_0 w_0^J w_0$. Using this, we verify the defining property for ψ is satisfied, since for any $j \in J$:

$$\alpha_{\psi(\theta(j))} = \alpha_{\theta\theta_J\theta\theta(j)} = -w_0 w_0^J w_0 \cdot \alpha_{\theta(j)} = -w_0^{\theta(J)} \cdot \alpha_{\theta(j)}$$

Putting things together, we get for the right side of Equation (10.1):

$$\begin{aligned} \xi_{B_{\theta(J)}} \xi_B(b) &= \xi_{B_{\theta(J)}} \xi_B(f_{i_1} \cdots f_{i_k} b_\mu) = \xi_{B_{\theta(J)}}(e_{\theta(i_1)} \cdots e_{\theta(i_k)} \xi_B(b_\mu)) \\ &= \xi_{B_{\theta(J)}}(e_{\theta(i_1)} \cdots e_{\theta(i_k)} b_\nu^{\text{low}}) = f_{\theta\theta_J(i_1)} \cdots f_{\theta\theta_J(i_k)} b_\nu \\ &= f_{\theta\theta_J(i_1)} \cdots f_{\theta\theta_J(i_k)} \xi_B(b_\mu^{\text{low}}) = f_{\theta\theta_J(i_1)} \cdots f_{\theta\theta_J(i_k)} \xi_B(e_{j_1} \cdots e_{j_m} b_\lambda^{\text{low}}) \\ &= f_{\theta\theta_J(i_1)} \cdots f_{\theta\theta_J(i_k)} f_{\theta(j_1)} \cdots f_{\theta(j_m)} b_\lambda \end{aligned}$$

This matches our computations for the left side above, and so Relation 2 is satisfied.

For the third relation, consider J' and J disjoint, connected Dynkin subdiagrams of I.

Lemma 10.0.4 *The Kashiwara operators e_i, f_i of B commute with e_j, f_j, and preserve ϵ_j, ϕ_j for any $i \in J$, $j \in J'$.*

Proof: Take any two nodes $i \in J$, $j \in J'$ and consider $B_{\{i\} \cup \{j\}}$, the crystal B restricted to the Dynkin diagram $\{i\} \cup \{j\}$ of $\mathfrak{sl}_2 \times \mathfrak{sl}_2$. The connected components of $B_{\{i\} \cup \{j\}}$ correspond to irreducible representations of $\mathfrak{sl}_2 \times \mathfrak{sl}_2$, which are precisely of the form $V(n) \otimes V(m)$, $n, m \in \mathbb{Z}_{\geq 0}$, where $V(k)$ is the $(k+1)$-dimensional irreducible $\mathfrak{sl}_2$-module. The crystal for this representation is an $(n+1) \times (m+1)$ grid of rectangles, where in particular we have the desired commutativity and preservation of ϵ_j, ϕ_j. □

Using Lemma 10.0.4, we can show:

Lemma 10.0.5 *The Schützenberger involution ξ_{B_J} commutes with e_k, f_k for any $k \in J'$ with J' disjoint from J, and similarly for $\xi_{B_{J'}}$.*

Proof: Take any $b \in B$. It belongs to some connected component of B_J of highest weight λ, with highest and lowest weight elements b_λ and b_λ^{low} respectively, so we can find $i_1, \dots i_m \in J$ such that $b = f_{i_1} \dots f_{i_m} b_\lambda$.

Note that by Lemma 10.0.4, for any $k \in J'$, $e_k(b_\lambda)$ and $e_k(b_\lambda^{\text{low}})$ are highest and lowest weight elements in B_J since for any $i \in J$, $\epsilon_i(e_k(b_\lambda)) = \epsilon_i(b_\lambda) = 0$ and $\phi_i(e_k(b_\lambda^{\text{low}})) = \phi_i(b_\lambda^{\text{low}}) = 0$.

Furthermore, they belong to the same connected component: $e_k(b_\lambda^{\text{low}}) = e_k(f_{j_1} \dots f_{j_k} b_\lambda) = f_{j_1} \dots f_{j_k} e_k(b_\lambda)$ for some $j_1, \dots, j_k \in J$. As the Schützenberger involution interchanges the lowest and highest weight elements in a connected component, we have $\xi_{B_J}(e_k(b_\lambda)) = e_k(b_\lambda^{\text{low}})$. Hence:

$$\begin{aligned}\xi_{B_J} e_k(b) &= \xi_{B_J} e_k(f_{i_1}\dots f_{i_m} b_\lambda) = e_{\theta_J(i_1)}\dots e_{\theta_J(i_m)}\xi_{B_J}(e_k(b_\lambda))\\ &= e_{\theta_J(i_1)}\dots e_{\theta_J(i_m)} e_k(b_\lambda^{\text{low}}) = e_k(e_{\theta_J(i_1)}\dots e_{\theta_J(i_m)}(b_\lambda^{\text{low}})) = e_k\xi_{B_J}(b)\end{aligned}$$

□

Using the notation and the results of Lemma 10.0.5, we can show similarly that $\xi_{B_{J'}}(b_\lambda)$ and $\xi_{B_{J'}}(b_\lambda^{\text{low}})$ are the highest and lowest weight elements in the same connected component of B_J: $b_\lambda = e_{n_1}\dots e_{n_l} b_\lambda^{\text{low}}$ for some $n_1,\dots,n_l \in J$, so $\xi_{B_{J'}}(b_\lambda) = \xi_{B_{J'}}(e_{n_1}\dots e_{n_l} b_\lambda^{\text{low}}) = e_{n_1}\dots e_{n_l p}\xi_{B_{J'}}(b_\lambda^{\text{low}})$. Furthermore, b_λ belongs to some connected component of $B_{J'}$ with highest and lowest elements $b_\mu, b_\mu^{\text{low}}$, say, so:

$$f_{s_1}\dots f_{s_p} b_\mu = b_\lambda = e_{t_1}\dots e_{t_q} b_\mu^{\text{low}}$$

for some $s_1,\dots,s_p,t_1,\dots,t_q \in J'$. Using Lemma 10.0.4, this means $0 = \epsilon_j(b_\lambda) = \epsilon_j(b_\mu) = \epsilon_j(b_\mu^{\text{low}})\ \forall\ j \in J$. Furthermore:

$$\begin{aligned}\epsilon_j(\xi_{B_{J'}}(b_\lambda)) &= \epsilon_j(\xi_{B_{J'}}(e_{t_1}\dots e_{t_q} b_\mu^{\text{low}})) = \epsilon_j(f_{\theta_{J'}(t_1)}\dots f_{\theta_{J'}(t_q)}(\xi_{B_{J'}}(b_\mu^{\text{low}})))\\ &= \epsilon_j(f_{\theta_{J'}(t_1)}\dots f_{\theta_{J'}(t_q)}(b_\mu)) = \epsilon_j(b_\mu) = 0\end{aligned}$$

So indeed $\xi_{B_{J'}}(b_\lambda)$ is highest weight in B_J and similarly $\xi_{B_{J'}}(b_\lambda^{\text{low}})$ is lowest weight. Therefore, $\xi_{B_J}(\xi_{B_{J'}}(b_\lambda)) = \xi_{B_{J'}}(b_\lambda^{\text{low}})$ and so:

$$\begin{aligned}\xi_{B_J}\xi_{B_{J'}}(b) &= \xi_{B_J}\xi_{B_{J'}}(f_{i_1}\dots f_{i_m} b_\lambda) = e_{\theta_J(i_1)}\dots e_{\theta_J(i_m)}\xi_{B_J}(\xi_{B_{J'}}(b_\lambda))\\ &= e_{\theta_J(i_1)}\dots e_{\theta_J(i_m)}\xi_{B_{J'}}(b_\lambda^{\text{low}}) = \xi_{B_{J'}}(e_{\theta_J(i_1)}\dots e_{\theta_J(i_m)} b_\lambda^{\text{low}}) = \xi_{B_{J'}}\xi_{B_J}(b)\end{aligned}$$

This verifies the last relation in the cactus group and so concludes the proof that it acts on a $\mathfrak{g}$-crystal. □

An application of this general cactus group action will be the discussion of Chapter 13. In the next section we'll focus our attention on $\mathfrak{gl}_k$-crystals and their tensor products.

Chapter 11

Tensor products of $\mathfrak{g}l_k$-crystals

Let us first generalize the tensor product rule of $\mathfrak{g}$-crystals in Chapter 9 from two to any number of crystals.

11.1 General tensor product rule

[J95]
If $\mathcal{B}_1, \ldots, \mathcal{B}_m$ are $\mathfrak{g}$-crystals, then so is $\mathcal{B}_1 \otimes \ldots \otimes \mathcal{B}_m$ with the following operations:

1. $\mathrm{wt}(b_1 \otimes \ldots \otimes b_m) = \mathrm{wt}(b_1) + \ldots + \mathrm{wt}(b_m)$

2. $\epsilon_i(b_1 \otimes \ldots \otimes b_m) = \max_k\{\epsilon_i^k(b_1 \otimes \ldots \otimes b_m), 0\}$, where
   $\epsilon_i^k(b_1 \otimes \ldots \otimes b_m) = \epsilon_i(b_k) - \langle \alpha_i^\vee, \mathrm{wt}(b_1) + \ldots + \mathrm{wt}(b_{k-1})\rangle$

3. $\phi_i(b_1 \otimes \ldots \otimes b_m) = \max_k\{\phi_i^k(b_1 \otimes \ldots \otimes b_m), 0\}$, where
   $\phi_i^k(b_1 \otimes \ldots \otimes b_m) = \phi_i(b_k) + \langle \alpha_i^\vee, \mathrm{wt}(b_{k+1}) + \ldots + \mathrm{wt}(b_m)\rangle$

4. $e_i(b_1 \otimes \ldots \otimes b_m) = b_1 \otimes \ldots \otimes e_i(b_s) \otimes \ldots \otimes b_m$, where
   $s \in \{1, \ldots, m\}$ is the smallest number such that $\epsilon_i^s = \epsilon_i$

5. $f_i(b_1 \otimes \ldots \otimes b_m) = b_1 \otimes \ldots \otimes f_i(b_t) \otimes \ldots \otimes b_m$, where
   $t \in \{1, \ldots, m\}$ is the largest number such that $\phi_i^t = \phi_i$

11.2 The $\mathfrak{g}l_n \times \mathfrak{g}l_m$-crystal $\Lambda^N B_{n,m}$

Consider in particular $\Lambda^N B_{n,m}$, the set of $n \times m$ matrices with entries in $\{0, 1\}$ and precisely N 1's, which is the underlying set for the crystal of $\Lambda^N(\mathbb{C}^n \otimes \mathbb{C}^m)$ both as a $\mathfrak{g}l_n$- and as a $\mathfrak{g}l_m$-representation.

Indeed, we have both a $\mathfrak{gl}_n$- and a $\mathfrak{gl}_m$-crystal structure on $\Lambda^N B_{n,m}$ coming from the decompositions:

$$\Lambda^N B_{n,m} \cong \bigsqcup_{\underline{k}\in\mathbb{N}^m, |\underline{k}|=N} \mathcal{B}^{\mathfrak{gl}_n}_{\omega_{k_1}} \otimes \ldots \otimes \mathcal{B}^{\mathfrak{gl}_n}_{\omega_{k_m}}$$
$$\begin{bmatrix} | & & | \\ c_1 & \ldots & c_m \\ | & & | \end{bmatrix} \mapsto c_m \otimes \ldots \otimes c_1 \tag{11.1}$$

where in Equation 11.1 above, $\mathcal{B}^{\mathfrak{gl}_n}_{\omega_i}$ is the crystal for the fundamental $\mathfrak{gl}_n$-representation $\Lambda^i(\mathbb{C}^n)$, and in Equation 11.2 below $\mathcal{B}^{\mathfrak{gl}_m}_{\omega_j}$ is the crystal for the $\mathfrak{gl}_m$-representation $\Lambda^j(\mathbb{C}^m)$:

$$\Lambda^N B_{n,m} \cong \bigsqcup_{\underline{l}\in\mathbb{N}^n, |\underline{l}|=N} \mathcal{B}^{\mathfrak{gl}_m}_{\omega_{l_1}} \otimes \ldots \otimes \mathcal{B}^{\mathfrak{gl}_m}_{\omega_{l_n}}$$
$$\begin{bmatrix} — & r_1 & — \\ & \vdots & \\ — & r_n & — \end{bmatrix} \mapsto r_1 \otimes \ldots \otimes r_n \tag{11.2}$$

We will denote its $\mathfrak{gl}_n$-crystal maps (which act on the columns of the matrices) by $C\epsilon_i$, $C\phi_i$, Ce_i, Cf_i and $C\mathrm{wt}$, and the $\mathfrak{gl}_m$-crystal maps by $R\epsilon_i, R\phi_i, Re_i, Rf_i$ and $R\mathrm{wt}$, all coming from the rule in Chapter 11.1.

Proposition 11.2.1 *The $\mathfrak{gl}_n$- and $\mathfrak{gl}_m$-crystal structures of $\Lambda^N B_{n,m}$ commute, i.e. the Kashiwara operators Re_i and Rf_i, $i \in \{1,\ldots,m-1\}$, are $\mathfrak{gl}_n$-crystal morphisms and analogously Ce_j and Cf_j, $j \in \{1,\ldots,n-1\}$ are $\mathfrak{gl}_m$-crystal morphisms. This gives $\Lambda^N B_{n,m}$ the structure of a $\mathfrak{gl}_n \times \mathfrak{gl}_m$-crystal.*

Proof: It suffices to check the following for each $M \in \Lambda^N B_{n,m}$ such that, in the left column of equations $Re_i(M) \neq 0$, in the right column $Rf_i(M) \neq 0$, and additionally $Ce_j(M) \neq 0$ or $Cf_j(M) \neq 0$ if Ce_j, respectively Cf_j appears in the equation:

$$C\mathrm{wt}Re_i = C\mathrm{wt} \qquad C\mathrm{wt}Rf_i = C\mathrm{wt} \tag{11.1}$$
$$C\epsilon_j Re_i = C\epsilon_j \qquad C\epsilon_j Rf_i = C\epsilon_j \tag{11.2}$$
$$C\phi_j Re_i = C\phi_j \qquad C\phi_j Rf_i = C\phi_j \tag{11.3}$$
$$Ce_j Re_i = Re_i Ce_j \qquad Ce_j Rf_i = Rf_i Ce_j \tag{11.4}$$
$$Cf_j Re_i = Re_i Cf_j \qquad Cf_j Rf_i = Rf_i Cf_j \tag{11.5}$$

We also need to include Equations 11.1, 11.2, and 11.3 with the roles of "R" and "C" switched. Those are analogous and we will omit their proof, with the exception of the analogue of 11.2, which we show in Lemma 11.2.4.

Note that in each line, once we know the equation for Re_i, it follows for Rf_i from:

$$Rf_i(M) = M' \neq 0 \;\Rightarrow\; M = Re_i(M')$$

For instance, $C\mathrm{wt}(Rf_i(M)) = C\mathrm{wt}(M') = C\mathrm{wt}(Re_i(M')) = C\mathrm{wt}(M)$. Similarly, using that Ce_j and Cf_j are partial inverses, Equations 11.2 and 11.4 imply 11.5. Indeed, by definition of the Kashiwara operators:

$$Cf_j(M) = M'' \neq 0 \ \Rightarrow M = Ce_j(M'')$$

We also assume that $Re_i(M) \neq 0$. This implies that $Re_i(M'') \neq 0$, as $R\epsilon_i(M) \neq 0$ and Cf_j preserves $R\epsilon_i$ by the analogue of Equation 11.2 with R and C switched, so $Re_i(M'') = Re_i(Cf_j(M)) \neq 0$. Then we can apply Equation 11.4 for M'' to get:

$$Cf_jRe_i(M) = Cf_jRe_i(Ce_j(M'')) \overset{(11.4)}{=} Cf_jCe_j(Re_i(M'')) = Re_i(M'') = Re_iCf_j(M)$$

Furthermore, equations (1) and (2) imply (3). For $Re_i(M) \neq 0$:

$$C\phi_j(Re_i(M)) = C\epsilon_j(Re_i(M)) + \left\langle C\mathrm{wt}(Re_i(M)), C\alpha_j^\vee \right\rangle = C\epsilon_j(M) + \left\langle C\mathrm{wt}(M), C\alpha_j^\vee \right\rangle = C\phi_j(M)$$

We will break the proof of Equations 11.1, 11.2 and 11.3 into several lemmas below. □

Lemma 11.2.2 *The operator Re_i, $i \in \{1, \dots, m-1\}$, preserves the $\mathfrak{gl}_n$-weight of each element:*

$$\forall\ M \in \Lambda^N B_{n,m} \ \text{s.t.}\ Re_i(M) \neq 0 \Rightarrow Cwt(Re_i(M)) = Cwt(M)$$

Proof: Given a matrix M, if Re_i acts in the l-th row, then for $\widetilde{M} = Re_i(M)$, $\widetilde{M}_{pq} = M_{pq}$ for $(p,q) \neq (l,i),(l,i+1)$ and $\widetilde{M}_{l,i} = M_{l,i}+1$, $\widetilde{M}_{l,i+1} = M_{l,i+1}-1$. Note that the $\mathfrak{gl}_n$-weight of M is given by $C\mathrm{wt}(M) = (\sum_{s=1}^m M_{1,s}, \sum_{s=1}^m M_{2,s}, \dots, \sum_{s=1}^m M_{n,s})$, and so:

$$C\mathrm{wt}(\widetilde{M}) = \left(\sum_{s=1}^m M_{1,s}, \sum_{s=1}^m M_{2,s}, \dots, \sum_{s=1}^m M_{l,s} + 1 - 1, \sum_{s=1}^m M_{n,s} \right) = C\mathrm{wt}(M)$$

□

Lemma 11.2.3 *Re_i preserves the value of $C\epsilon_j$ for all $i \in \{1, \dots, m-1\}$, $j \in \{1, \dots, n-1\}$:*

$$\forall\ M \in \Lambda^N B_{n,m} \ \text{s.t.}\ Re_i(M) \neq 0 \Rightarrow C\epsilon_j(Re_i(M)) = C\epsilon_j(M)$$

Proof: Given a matrix M in $\Lambda^N B_{n,m}$, suppose Re_i acts in the l-th row:

$$M = \begin{array}{c} \\ \\ l \\ \\ \end{array} \left[\begin{array}{ccccc} & \scriptstyle i & \scriptstyle i+1 & & \\ & & & & \\ & 0 & 1 & & \\ & & & & \end{array} \right] \xrightarrow{Re_i} \begin{array}{c} \\ \\ l \\ \\ \end{array} \left[\begin{array}{ccccc} & \scriptstyle i & \scriptstyle i+1 & & \\ & & & & \\ & 1 & 0 & & \\ & & & & \end{array} \right]$$

Note that, from the description of $C\epsilon_j$ above, we have, for $\widetilde{k} = m+1-k$ (for simplicity of notation):

$$C\epsilon_j^{\widetilde{k}}(M) = C\epsilon_j^{\widetilde{k}}\left(\begin{bmatrix} | & & | \\ c_1 & \dots & c_m \\ | & & | \end{bmatrix}\right) = C\epsilon_j^{\widetilde{k}}(c_m \otimes c_{m-1} \otimes \dots \otimes c_1) =$$

$$= C\epsilon_j(c_k) - \langle \alpha_j^\vee, \mathrm{wt}(c_m) + \dots + \mathrm{wt}(c_{k+1}) \rangle = \delta_{M_{j,k},0}\delta_{M_{j+1,k},1} + \sum_{s=k+1}^{m} M_{j+1,s} - \sum_{s=k+1}^{m} M_{j,s}$$

Diagramtically, this can be represented on the matrix as:

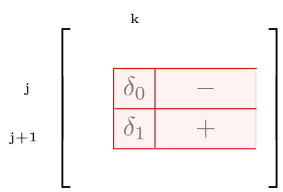


Figure 11.1: An short-hand illustration of how $C\epsilon_j^{\widetilde{k}}$ is computed on a given matrix.

This only involves rows j and $j+1$, and $C\epsilon_j = \max_{\widetilde{k}} C\epsilon_j^{\widetilde{k}}$, so Re_i acting in the l-th row could only affect $C\epsilon_{l-1}$ and $C\epsilon_l$. It doesn't affect $C\epsilon_{l-1}^{\widetilde{k}}, C\epsilon_l^{\widetilde{k}}$ for $k > i+1$ and $k < i$, as illustrated by the pictures below:

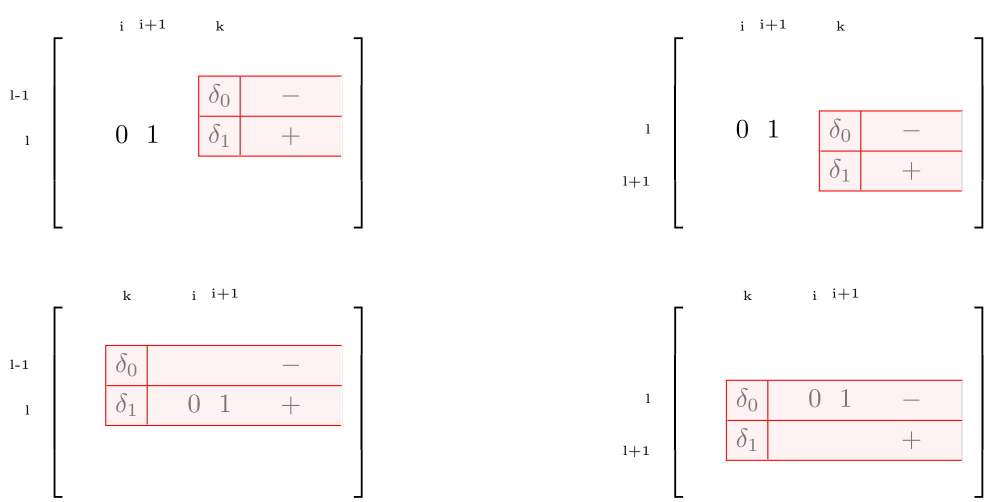


Figure 11.2: A diagrammatic illustration that Re_i acting in the l-th row doesn't affect $C\epsilon_{l-1}^{\widetilde{k}}, C\epsilon_l^{\widetilde{k}}$ for $k > i+1$ and $k < i$.

So, only the values of $C\epsilon_{l-1}^{\widetilde{i}}, C\epsilon_{l-1}^{\widetilde{i+1}}, C\epsilon_l^{\widetilde{i}}, C\epsilon_l^{\widetilde{i+1}}$ could be affected by the action of Re_i, (the first pair in the case when $l > 1$ and the second in the case $l < n$).

First, let's look at the pair $C\epsilon_{l-1}^{\widetilde{i}}$, $C\epsilon_{l-1}^{\widetilde{i+1}}$. For simplicity, denote:

$$A := \sum_{s=i+2}^{m} M_{l,s} \qquad x := M_{l-1,i} \qquad B := \sum_{s=i+2}^{m} M_{l-1,s} \qquad y := M_{l-1,i+1}$$

Then we have:

$$M = \begin{array}{c} \\ \text{l-1} \\ \text{l} \end{array} \left[\begin{array}{ccc} \text{i} & \text{i+1} & \\ x & y & B \\ 0 & 1 & A \end{array} \right] \xrightarrow{Re_i} \begin{array}{c} \\ \text{l-1} \\ \text{l} \end{array} \left[\begin{array}{ccc} \text{i} & \text{i+1} & \\ x & y & B \\ 1 & 0 & A \end{array} \right]$$

$$C\epsilon_{l-1}^{\widetilde{i}}(M) = A - B - y + 1 \qquad C\epsilon_{l-1}^{\widetilde{i}}(Re_i(M)) = A - B - y + \delta_{x,0}$$

$$M = \begin{array}{c} \\ \text{l-1} \\ \text{l} \end{array} \left[\begin{array}{ccc} \text{i} & \text{i+1} & \\ x & y & B \\ 0 & 1 & A \end{array} \right] \xrightarrow{Re_i} \begin{array}{c} \\ \text{l-1} \\ \text{l} \end{array} \left[\begin{array}{ccc} \text{i} & \text{i+1} & \\ x & y & B \\ 1 & 0 & A \end{array} \right]$$

$$C\epsilon_{l-1}^{\widetilde{i+1}}(M) = A - B + \delta_{y,0} \qquad C\epsilon_{l-1}^{\widetilde{i+1}}(Re_i(M)) = A - B$$

We get the following values of the pair in the two cases $y = 0$ and $y = 1$.

Note that if $y = 0$, we must also have $x = 0$: If $(x, y) = (1, 0)$, this would mean $R\epsilon_i^{l-2} = R\epsilon_i^l$ for $l > 2$, or $R\epsilon_i^l = 0$. That would contradict the assumption that Re_i acts in row l, i.e. that l is the smallest number such that:

$$R\epsilon_i^l = \max_k R\epsilon_i^k > 0$$

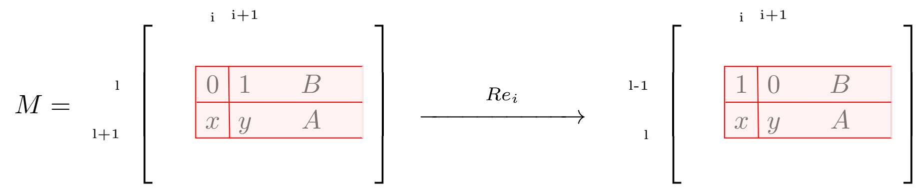
As a result,

	Case 1: $y = 0$		Case 2: $y = 1$	
	M	$Re_i(M)$	M	$Re_i(M)$
$C\epsilon_{l-1}^{\widetilde{i}}$	$A - B + 1$	$A - B + 1$	$A - B$	$A - B - 1 + \delta_{x,0}$
$C\epsilon_{l-1}^{\widetilde{i+1}}$	$A - B + 1$	$A - B$	$A - B$	$A - B$

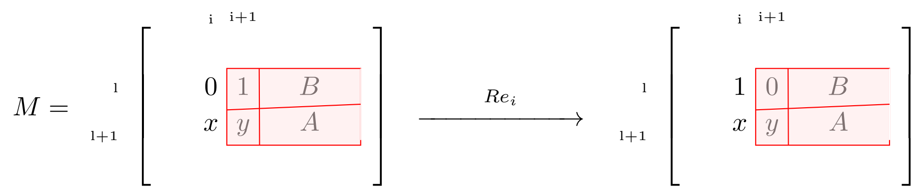
In either case, we see that $C\epsilon_{l-1} = \max_{\widetilde{k}}\{C\epsilon_{l-1}^{\widetilde{k}}, 0\}$ remains unchanged.

Next, we consider the pair $C\epsilon_l^{\widetilde{i}}$, $C\epsilon_l^{\widetilde{i+1}}$. In this case, we denote:

$$A := \sum_{s=i+2}^{m} M_{l+1,s} \qquad x := M_{l+1,i} \qquad B := \sum_{s=i+2}^{m} M_{l,s} \qquad y := M_{l+1,i+1}$$



$C\epsilon_l^{\tilde{i}}(M) = A - B + y - 1 + x$ $\qquad$ $C\epsilon_l^{\tilde{i}}(Re_i(M)) = A - B + y$



$C\epsilon_l^{\widetilde{i+1}}(M) = A - B$ $\qquad$ $C\epsilon_l^{\widetilde{i+1}}(Re_i(M)) = A - B + y$

We get the following values of the pair in the two cases $y = 0$ and $y = 1$.

Note that if $y = 1$, we must also have $x = 1$: If $(x, y) = (0, 1)$, this would mean $R\epsilon_i^{l+1} > R\epsilon_i^l$, which would contradict the assumption that Re_i acts in row l, i.e. that l is the smallest number such that:

$$R\epsilon_i^l = \max_k R\epsilon_i^k > 0$$

As a result,

	Case 1: $y = 0$		Case 2: $y = 1$	
	M	$Re_i(M)$	M	$Re_i(M)$
$C\epsilon_l^{\tilde{i}}$	$A - B - 1 + x$	$A - B$	$A - B + 1$	$A - B + 1$
$C\epsilon_l^{\widetilde{i+1}}$	$A - B$	$A - B$	$A - B$	$A - B + 1$

In either case, we see that $C\epsilon_l = \max_{\tilde{k}}\{C\epsilon_l^{\tilde{k}}, 0\}$ remains unchanged. □

Lemma 11.2.4 *Ce_i preserves the value of $R\epsilon_j$ for all $i \in \{1, \ldots, n-1\}$, $j \in \{1, \ldots, m-1\}$:*

$$\forall\, M \in \Lambda^N B_{n,m},\ Ce_i(M) \neq 0 \Rightarrow R\epsilon_j(Ce_i(M)) = R\epsilon_j(M)$$

Proof: Given a matrix M in $\Lambda^N B_{n,m}$, suppose Ce_i acts in the l-th column:

$$M = \begin{array}{c} \\ \text{i} \\ \text{i+1} \end{array} \begin{bmatrix} & \text{l} & \\ & 0 & \\ & 1 & \end{bmatrix} \xrightarrow{Ce_i} \begin{array}{c} \\ \text{i} \\ \text{i+1} \end{array} \begin{bmatrix} & \text{l} & \\ & 1 & \\ & 0 & \end{bmatrix}$$

From the description of $R\epsilon_j$, we have:

$$R\epsilon_j^k(M) = R\epsilon_j^k\left(\begin{bmatrix} — & r_1 & — \\ & \vdots & \\ — & r_n & — \end{bmatrix}\right) = R\epsilon_j^k(r_1 \otimes r_2 \otimes \ldots \otimes r_n) =$$

$$= R\epsilon_j(r_k) - \langle \alpha_j^\vee, \mathrm{wt}(r_1) + \ldots + \mathrm{wt}(r_{k-1})\rangle = \delta_{M_{k,j},0}\delta_{M_{k,j+1},1} + \sum_{s=1}^{k-1} M_{s,j+1} - \sum_{s=1}^{k-1} M_{s,j}$$

Diagramatically, this can be represented on the matrix as:

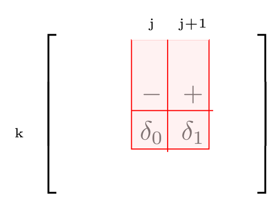


Figure 11.3: A short-hand notation for the value of $R\epsilon_j^k$ on a matrix.

This only involves columns j and $j+1$, and $R\epsilon_j = \max_k R\epsilon_j^k$, so Ce_i acting in the l-th column could only affect $R\epsilon_{l-1}$ and $R\epsilon_l$. It doesn't affect $R\epsilon_{l-1}^k, R\epsilon_l^k$ for $k < i$ and $k > i+1$, as illustrated by the pictures below:

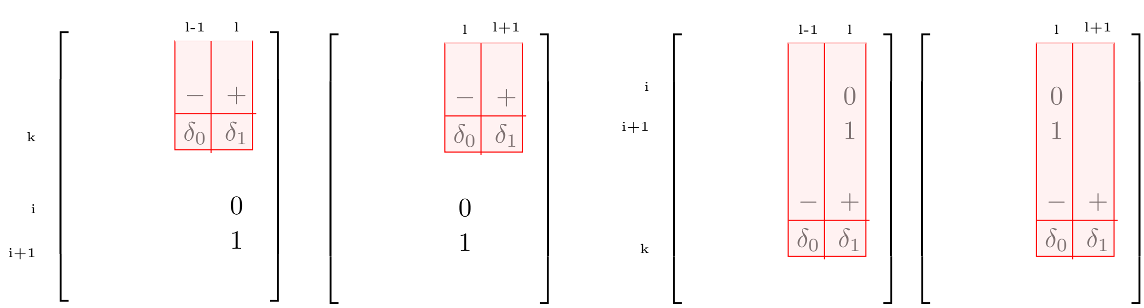
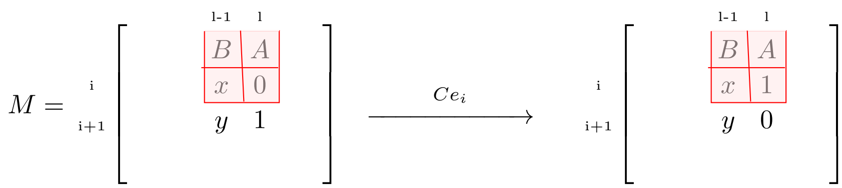


Figure 11.4: A diagrammatic illustration that Ce_i acting in the l-th column doesn't affect $R\epsilon_{l-1}^k, R\epsilon_l^k$ for $k < i$ and $k > i+1$.

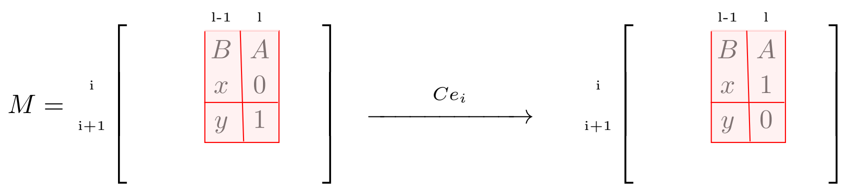
So, only the values of $R\epsilon_{l-1}^i$, $R\epsilon_{l-1}^{i+1}$, $R\epsilon_l^i$, $R\epsilon_l^{i+1}$ could be affected by the action of Ce_i, (the first pair in the case when $l > 1$, the second when $l < m$).

First, let's look at the pair $R\epsilon^i_{l-1}$, $R\epsilon^{i+1}_{l-1}$. For simplicity, denote:

$$A := \sum_{s=1}^{i-1} M_{s,l} \qquad x := M_{i,l-1} \qquad B := \sum_{s=1}^{i-1} M_{s,l-1} \qquad y := M_{i+1,l-1}$$



$$R\epsilon^i_{l-1}(M) = A - B \qquad\qquad R\epsilon^i_{l-1}(Ce_i(M)) = A - B - x + 1$$



$$R\epsilon^{i+1}_{l-1}(M) = A - B - x + 1 - y \qquad\qquad R\epsilon^{i+1}_{l-1}(Ce_i(M)) = A - B - x + 1$$

We get the following values of the pair in the two cases $x = 0$ and $x = 1$.

Note that if $x = 0$, we must also have $y = 0$: If $(x, y) = (0, 1)$, this would mean $C\epsilon_i^{\widetilde{l-1}} = C\epsilon_i^{\widetilde{l}}$ (in this setting $l > 1$). That would contradict the assumption that Ce_i acts in column l, i.e. that l is closest to m such that:

$$C\epsilon_i^{\widetilde{l}} = \max_{\widetilde{k}} C\epsilon_i^{\widetilde{k}} > 0$$

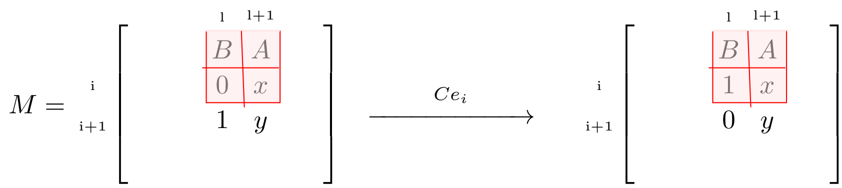
As a result,

	Case 1: $x = 0$		Case 2: $x = 1$	
	M	$Ce_i(M)$	M	$Ce_i(M)$
$R\epsilon^i_{l-1}$	$A - B$	$A - B + 1$	$A - B$	$A - B$
$R\epsilon^{i+1}_{l-1}$	$A - B + 1$	$A - B + 1$	$A - B - y$	$A - B$

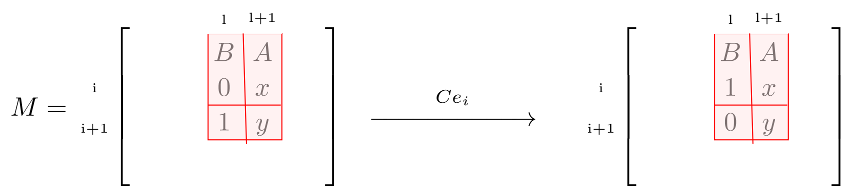
In either case, we see that $R\epsilon_{l-1} = \max_k\{R\epsilon^k_{l-1}, 0\}$ remains unchanged, while the earliest the maximum is achieved might switch from $R\epsilon^{i+1}_{l-1}$ to $R\epsilon^i_{l-1}$.

Next, we consider $R\epsilon^i_l$, $R\epsilon^{i+1}_l$. Analogously to before, denote:

$$A := \sum_{s=1}^{i-1} M_{s,l+1} \qquad x := M_{i,l+1} \qquad B := \sum_{s=1}^{i-1} M_{s,l} \qquad y := M_{i+1,l+1}$$



$Re_l^i(M) = A - B + x$ $\qquad$ $Re_l^i(Ce_i(M)) = A - B$



$Re_l^{i+1}(M) = A - B + x$ $\qquad$ $Re_l^{i+1}(Ce_i(M)) = A - B + x - 1 + y$

We get the following values of the pair in the two cases $x = 0$ and $x = 1$.

Note that if $x = 1$, we must also have $y = 1$: If $(x, y) = (1, 0)$, this would mean $C\epsilon_i^{\tilde{l}} = C\epsilon_i^{\widetilde{l+2}}$ (if $l + 2 \leq m$), or $C\epsilon_i^{\tilde{l}} = 0$. This contradicts the assumption that Ce_i acts in column l, i.e. that l is closest to m such that:

$$C\epsilon_i^{\tilde{l}} = \max_{\tilde{k}} C\epsilon_i^{\tilde{k}} > 0$$

As a result,

	Case 1: $x = 0$		Case 2: $x = 1$	
	M	$Ce_i(M)$	M	$Ce_i(M)$
$R\epsilon_l^i$	$A - B$	$A - B$	$A - B + 1$	$A - B$
$R\epsilon_l^{i+1}$	$A - B$	$A - B - 1 + y$	$A - B + 1$	$A - B + 1$

In either case, we see that $R\epsilon_{l-1} = \max_k \{R\epsilon_{l-1}^k, 0\}$ remains unchanged, while the earliest the maximum is achieved might switch from $R\epsilon_l^i$ to $R\epsilon_l^{i+1}$ or conversely. $\square$

Lemma 11.2.5 *For all $i \in \{1, \ldots, m-1\}$, $j \in \{1, \ldots, n-1\}$:*

$$Re_i Ce_j(M) = Ce_j Re_i(M) \;\; \forall\, M \in \Lambda^N B_{n,m} \;\; s.t. \;\; Ce_j(M),\; Re_i(M) \neq 0$$

Proof: Given an element $M \in \Lambda^N B_{n,m}$, suppose Re_i acts in the l-th row. Namely, l is the first row counting from the top such that $R\epsilon_i = R\epsilon_i^l > 0$ and $M_{l,i} = 0$, $M_{l,i+1} = 1$. Then the image, $Re_i(M)$ has the same entries as M except $Re_i(M)_{l,i} = 0$, $Re_i(M)_{l,i+1} = 1$:

$$Re_i(M)_{p,q} = \begin{cases} M_{p,q} & (p,q) \neq (l,i),\ (l,i+1) \\ M_{l,i} + 1 & (p,q) = (l,i) \\ M_{l,i+1} - 1 & (p,q) = (l,i+1) \end{cases}$$

Therefore, by Lemma 11.2.3, Re_i affects only $C\epsilon_{l-1}^{\widetilde{i}}$, $C\epsilon_{l-1}^{\widetilde{i+1}}$, $C\epsilon_{l}^{\widetilde{i}}$, $C\epsilon_{l}^{\widetilde{i+1}}$ and doesn't change the overall value of $C\epsilon_s$, $\forall\ s \in \{1, \dots, n-1\}$.

Suppose also that Ce_j acts in the k-th column. Namely k is the closest column to the right-most one such that $C\epsilon_j = C\epsilon_j^{\widetilde{k}} > 0$ and $M_{j,k} = 0,\ M_{j+1,k} = 1$. Then the image, $Ce_j(M)$ has the same entries as M except $Ce_j(M)_{j,k} = 1,\ Ce_j(M)_{j+1,k} = 0$:

$$Ce_j(M)_{p,q} = \begin{cases} M_{p,q} & (p,q) \neq (j,k),\ (j+1,k) \\ M_{j,k}+1 & (p,q) = (j,k) \\ M_{j+1,k}-1 & (p,q) = (j+1,k) \end{cases}$$

Ce_j affects only $R\epsilon_{k-1}^{j}$, $R\epsilon_{k-1}^{j+1}$, $R\epsilon_{k}^{j}$, $R\epsilon_{k}^{j+1}$ and doesn't change the overall value of $R\epsilon_t$, $\forall\ t \in \{1, \dots, m-1\}$.

So, the actions of Re_i and Ce_j on M are determined respectively by $R\epsilon_i^l$ and $C\epsilon_j^{\widetilde{k}}$, and interact with each other if:

$$R\epsilon_i^l \overset{(1)}{=} R\epsilon_{k,k-1}^{j,j+1} \qquad\qquad C\epsilon_j^{\widetilde{k}} \overset{(2)}{=} C\epsilon_{l,l-1}^{\widetilde{i,i+1}}$$

Namely, we need to study if the maximum of $R\epsilon_i$ and $C\epsilon_j$ at $R\epsilon_i^l$ and $C\epsilon_j^{\widetilde{k}}$ gets shifted.

Case 1. $j < l-1$ or $j > l$.

In this case, neither (1) nor (2) can be true, so the actions of Re_i and Ce_j are unaffected by one another and hence commute.

Case 2. $j = l-1$.

2.1 If $i < k-1$ or $i > k$, (1) or (2) still can't be true, so Re_i and Ce_j don't interact and hence commute.

2.2 If $i = k-1$, then $R\epsilon_i^l \overset{(1)}{=} R\epsilon_{k-1}^{j+1}$, $C\epsilon_j^{\widetilde{k}} \overset{(2)}{=} C\epsilon_{l-1}^{\widetilde{i+1}}$ and the action of Re_i and Ce_j happens in the 2×2 matrix with rows $j,\ j+1$ and columns $i,\ i+1$:

$$M_{j,j+1}^{i,i+1} = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$$

If $a = 1$, then $R\epsilon_i^{j+1} = R\epsilon_i^{j-1}$ for $j > 1$ or $R\epsilon_i^{j+1} = 0$, which contradicts the assumption that $l = j+1$ is the smallest number for which $R\epsilon_i = R\epsilon_i^{j+1} > 0$. Therefore, we must have $a = 0$. Then the actions commute, in the following way, $M_{j,j+1}^{i,i+1}$ being in the upper left corner:

$$\begin{array}{ccc} \begin{matrix}0&0\\0&1\end{matrix} & \xrightarrow{Re_i} & \begin{matrix}0&0\\1&0\end{matrix} \\ {\scriptstyle Ce_j}\big\downarrow & & \big\downarrow{\scriptstyle Ce_j} \\ \begin{matrix}0&1\\0&0\end{matrix} & \xrightarrow{Re_i} & \begin{matrix}1&0\\0&0\end{matrix} \end{array}$$

2.3 If $i = k$, on one hand we must have $M_{j+1,i} = 0,\ M_{j+1,i+1} = 1$ for Re_i to act in the $l = (j+1)$-st row, on the other we must have $M_{j,i} = 0,\ M_{j+1,i} = 1$ for Ce_j to act in the $i = k$-th column. This leads to a contradiction, so $i = k$ is not possible.

Case 3. $j = l$.

3.1 If $i < k-1$ or $i > k$, (1) or (2) still can't be true, so Re_i and Ce_j don't interact and hence commute.

3.2 If $i = k - 1$, then on one hand we must have $M_{j,i} = 0,\ M_{j,i+1} = 1$ for Re_i to act in the $l = j$-th row, on the other we must have $M_{j,i+1} = 0,\ M_{j+1,i+1} = 1$ for Ce_j to act in the $k = (i+1)$-st column. This leads to a contradiction, so $i = k - 1$ is not possible.

3.3 If $i = k$, $R\epsilon_i^l \overset{(1)}{=} R\epsilon_k^j$, $C\epsilon_j^{\tilde{k}} \overset{(2)}{=} C\epsilon_l^{\tilde{i}}$ and the action of Re_i and Ce_j happens in the 2×2 matrix with rows $j,\ j+1$ and columns $i,\ i+1$:

$$M_{j,j+1}^{i,i+1} = \begin{bmatrix} 0 & 1 \\ 1 & b \end{bmatrix}$$

If $b = 0$, then $C\epsilon_j^{\tilde{i}} = C\epsilon_j^{\widetilde{i+2}}$ for $i+2 \leq m$, or $C\epsilon_j^{\tilde{i}} = 0$, which contradicts that Ce_j acts in the $k = i$-th column. So, we must have $b = 1$. Then the actions commute in the following way, $M_{j,j+1}^{i,i+1}$ being in the upper left corner:

$$\begin{array}{ccc} \begin{matrix} 0 & 1 \\ 1 & 1 \end{matrix} & \xrightarrow{Re_i} & \begin{matrix} 1 & 0 \\ 1 & 1 \end{matrix} \\ {\scriptstyle Ce_j}\downarrow & & \downarrow{\scriptstyle Ce_j} \\ \begin{matrix} 1 & 1 \\ 0 & 1 \end{matrix} & \xrightarrow{Re_i} & \begin{matrix} 1 & 1 \\ 1 & 0 \end{matrix} \end{array}$$

□

Chapter 12

Skew Howe duality

12.1 Skew Howe duality for crystals

As we saw in Chapter 11, the set of matrices:

$$\Lambda^N B_{n,m} = \{M \in \mathrm{Mat}_{n\times m}(\{0,1\}) \mid \#1's = N\}$$

is the $\mathfrak{gl}_n \times \mathfrak{gl}_m$-crystal for the representation $\Lambda^N(\mathbb{C}^n \otimes \mathbb{C}^m)$. There is an isomorphism of $\mathfrak{gl}_n \times \mathfrak{gl}_m$-crystals, whose construction is also discussed by van Leeuwen [vL06] and Danilov-Koshevoy [DK04], also known as **skew-Howe duality** for crystals:

$$\Lambda^N B_{n,m} \cong \bigsqcup_{\substack{\lambda\subset n\times m\\ |\lambda|=N}} B_\lambda^{\mathfrak{gl}_n} \otimes B_{\lambda^{\mathrm{tr}}}^{\mathfrak{gl}_m} \tag{12.1}$$

We can express it by mapping to pairs of $n \times m$ matrices or pairs of tableaux of transpose shape, by considering the first map, or alternatively the composition of the first and second maps as follows, for $M \in \Lambda^N B_{n,m}$:

$$M \xrightarrow{(Re^{\max}, Cf^{\max})} (P, Q) \xrightarrow{(\phi,\psi)} (T_P, T_Q) \tag{12.2}$$

The operation $Re^{\max}$ represents a sequence of raising operations Re_i that maps M to a $\mathfrak{gl}_m$-highest weight matrix P and similarly $Cf^{\max}$ represents a sequence of lowering operations Cf_j that maps M to a $\mathfrak{gl}_n$-lowest weight matrix Q. The final answer (P, Q) does not depend on the particular sequence of operations we apply, as each connected component of a crystal has a single highest and lowest weight element.

Lemma 12.1.1 *An element $M \in \Lambda^N B_{n,m}$ which is both $\mathfrak{gl}_n$- and $\mathfrak{gl}_m$-highest weight must be a matrix with all N 1's in the upper left corner filling the shape of a partition λ_M. Similarly, if $L \in \Lambda^N B_{n,m}$ is $\mathfrak{gl}_n$-lowest weight and $\mathfrak{gl}_m$-highest weight, then all N 1's fill the shape of a partition λ_L that is aligned Southwest.*

Proof: It suffices to show that pairs of the type $M_{i,j} = 0,\ M_{i,j+1} = 1$ and $M_{i,j} = 0,\ M_{i+1,j} = 1$ are not possible.

Suppose first that a pair of the first type exists in M, choose the one which is furthest Northeast, of the form $M_{i,j}=0,\ M_{i,j+1}=1$. If $i=1$, we can apply Re_j, which contradicts that M is $\mathfrak{gl}_m$-highest weight. Consider $i>1$, then by assumption $R\epsilon_j^i = 1+\sum_{s=1}^{i-1} M_{s,j+1} - \sum_{s=1}^{i-1} M_{s,j} \leq 0$, so we must have $\sum_{s=1}^{i-1} M_{s,j+1} < \sum_{s=1}^{i-1} M_{s,j}$. For that to be true, we must have $M_{k,j+1}=0$ for at least one $1\leq k\leq i-1$. Take the lowest (Southernmost) such entry, then $M_{k+1,j+1}=1$ and since our $(0,1)$ pair was the furthest Northeast, we must have $M_{k,s}=0\ \forall\ s=j+2,\dots,m$ (possibly $j+2>m$ and there are no such elements). But this tells us that $C\widetilde{\epsilon_k^{j+1}} \geq 1$, which contradicts the assumption that M is $\mathfrak{gl}_n$-highest weight.

Similarly, suppose a pair of the second type exists and pick the one which is the furthest Northeast, $M_{i,j}=0,\ M_{i+1,j}=1$. If $j=m$, then we can apply Ce_i, which contradicts the assumption that M is $\mathfrak{gl}_n$-highest weight. Consider $j<m$, then by our assumption, $C\widetilde{\epsilon_i^j} = 1+\sum_{s=j+1}^{m} M_{i+1,s} - \sum_{s=j+1}^{m} M_{i,s} \leq 0$, hence $\sum_{s=j+1}^{m} M_{i+1,s} < \sum_{s=j+1}^{m} M_{i,s}$. This can only occur if $M_{i,k}=1$ for at least one $i+1\leq k\leq m$, pick the left-most such. Then $M_{i,k-1}=0$ and since $M_{i,j}=0,\ M_{i+1,j}=1$ was the furthest Northeast such pair, we must have $M_{s,k}=1\ \forall\ 1\leq s\leq i-1$. Hence, $R\epsilon_{k-1}^i\geq 1$, which contradicts the assumption that M is $\mathfrak{gl}_m$-highest weight.

The case when a matrix L is $\mathfrak{gl}_n$-lowest weight and $\mathfrak{gl}_m$-highest weight is analogous, and there we can show that pairs of the type $L_{i,j}=0,\ L_{i,j+1}=1$ and $L_{i,j}=1,\ L_{i+1,j}=0$ are not possible. □

The map $\phi: P\mapsto T_P$ in Equation 12.2 maps the $\mathfrak{gl}_m$-highest weight matrix P to a top-left aligned semi-standard Young tableau (SSYT) T_P with N boxes, in the following way:

T_P is defined by a sequence of partitions $\emptyset=\lambda^{(0)}\subset\lambda^{(1)}\subset\lambda^{(2)}\subset\ldots\subset\lambda^{(n)}=\lambda$. Each of the partitions is determined by $(\lambda^{(i)})^{\text{tr}}=\sum_{j=1}^{i} rP_j\in\mathbb{Z}_{\geq 0}^m$, where rP_j denotes the j-th row of the matrix P. The SSYT T_P has shape $\lambda\subset n\times m$ and filling obtained by placing the number i in the each box of the skew-shape $\lambda^{(i)}/\lambda^{(i-1)}$, for each $i=1,2,\dots,n$.

Indeed, since P is of size $n\times m$, by construction $\lambda^{\text{tr}}\subset m\times n$. Furthermore, we have that $(\lambda^{(i)})^{\text{tr}} = ((\lambda^{(i)})_1^{\text{tr}},(\lambda^{(i)})_2^{\text{tr}},\dots,(\lambda^{(i)})_m^{\text{tr}})$ is a partition for each $i=1,\dots,n$ because P is $\mathfrak{gl}_m$-highest weight, i.e. $R\epsilon_k(P)=0\ \forall\ k=1,2,\dots,m-1$ so in particular:

$$0\geq R\epsilon_k^i(P)=\sum_{j=1}^{i}(P_{j,k+1}-P_{j,k})+\delta_{P_{i,k},1}\delta_{P_{i,k+1},0}\qquad\forall\ i=1,2,\dots,n,\ \ k=1,2,\dots,m-1$$

which in turn implies that $(\lambda^{(i)})_k^{\text{tr}}=\sum_{j=1}^{i}P_{j,k}\geq\sum_{j=1}^{i}P_{j,k+1}=(\lambda^{(i)})_{k+1}^{\text{tr}}$. Because the skew-shape $(\lambda^{(i)})^{\text{tr}}/(\lambda^{(i-1)})^{\text{tr}}$ has at most one box in each row (as the entries of P are 0 and 1), $\lambda^{(i)}/\lambda^{(i-1)}$ has at most one box in each column, so each number i appears at most once in each column. Together with $\lambda^{(i-1)}\subset\lambda^{(i)}$, we get that T_P is strictly increasing in the columns and weakly increasing in the rows, so it is a SSYT.

The map $\psi: Q\mapsto T_Q$ in Equation 12.2 is defined similarly. For a $\mathfrak{gl}_n$-lowest weight matrix $Q=Cf^{\max}(M)$, it gives a SSYT T_Q obtained as follows: The shape $\lambda\subset m\times n$ and its filling are determined analogously to ϕ, with the difference that $(\lambda^{(i)})^{\text{tr}}=\sum_{j=1}^{i}\overline{cQ}_j\in\mathbb{Z}_{\geq 0}^n,\ \forall\ i=1,2,\dots,m$, where $\overline{cQ}_j$ denotes the j-th column of Q traversed from bottom to top ($Q_{i,j}=\overline{Q}_{n+1-i,j}$). Namely, $(\lambda^{(i)})^{\text{tr}}=((\lambda^{(i)})_1^{\text{tr}},(\lambda^{(i)})_2^{\text{tr}},\dots,(\lambda^{(i)})_n^{\text{tr}})$ where $(\lambda^{(i)})_k^{\text{tr}}=\sum_{j=1}^{i}Q_{n+1-k,j}$. The matrix Q is $\mathfrak{gl}_n$-lowest weight, i.e. $C\phi_{n-k}(Q)=0\ \forall\ k=1,2,\dots,n-1$ so in particular:

$$0\geq C\phi_{n-k}^i(Q)=\sum_{j=1}^{i}(Q_{n-k,j}-Q_{n+1-k,j})+\delta_{Q_{n+1-k,i},1}\delta_{Q_{n-k,i},0}\qquad \begin{aligned} i&=1,2,\dots,m,\\ k&=1,2,\dots,n-1\end{aligned}$$

which in turn implies that $(\lambda^{(i)})^{\text{tr}}_k = \sum_{j=1}^{i} Q_{n+1-k,j} \geq \sum_{j=1}^{i} Q_{n-k,j} = (\lambda^{(i)})^{\text{tr}}_{k+1}$. So $(\lambda^{(i)})^{\text{tr}}$ and hence $\lambda^{(i)}$ is a partition for all i. The remaining arguments that T_Q is a SSYT are identical to those for T_P.

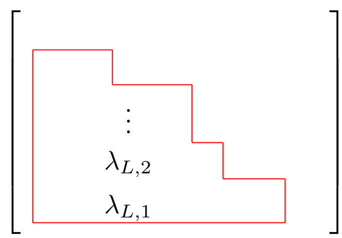
Moreover, using the two maps ϕ and ψ, the tableau we obtain from the pair of matrices $(P, Q) = (Re^{\max}(M), Cf^{\max}(M))$ are of transpose shape.

Lemma 12.1.2 *The partitions (λ_P, λ_Q) associated to the two matrices (P, Q) as above are of transpose shapes: $(\lambda_P)^{tr} = \lambda_Q$.*

Proof: Let $L = Cf^{\max}(P) = Cf^{\max}Re^{\max}(M) = Re^{\max}Cf^{\max}(M) = Re^{\max}(Q)$. Then, by Lemma 12.1.1, the N 1's of L fill the shape of a partition λ_L aligned Southwest and so:

$$\lambda_L = \left(\sum_{k=1}^{m} L_{n,k}, \ldots, \sum_{k=1}^{m} L_{2,k}, \sum_{k=1}^{m} L_{1,k}\right) \qquad (\lambda_L)^{\text{tr}} = \left(\sum_{j=1}^{n} L_{j,1}, \ldots, \sum_{j=1}^{n} L_{j,m-1}, \sum_{j=1}^{n} L_{j,m}\right)$$

$$L = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & \ldots & 0 & 0 & 0 & 0 \\ \vdots & & & \ddots & & & & \vdots \\ 1 & 1 & 1 & \ldots & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & \ldots & 1 & 1 & 1 & 0 \end{bmatrix}$$

$\lambda_L =$ 

By definition, we have:

$$\begin{aligned}(\lambda_P)^{\text{tr}} &= \sum_{j=1}^{n} rP_j = \left(\sum_{j=1}^{n} P_{j,1}, \ldots, \sum_{j=1}^{n} P_{j,m}\right) \\ &= R\text{wt}(P) \overset{(*)}{=} R\text{wt}(L) = \left(\sum_{j=1}^{n} L_{j,1}, \ldots, \sum_{j=1}^{n} L_{j,m}\right) = (\lambda_L)^{\text{tr}}\end{aligned}$$

where $(*)$ follows from the fact that Cf_i is a $\mathfrak{gl}_m$-crystal morphism for all i. Similarly,

$$\begin{aligned}(\lambda_Q)^{\text{tr}} &= \sum_{k=1}^{m} c\overline{Q}_k = \left(\sum_{k=1}^{m} Q_{n,k}, \ldots, \sum_{k=1}^{m} Q_{1,k}\right) \\ &= \overline{C\text{wt}(L)} \overset{(**)}{=} \overline{C\text{wt}(Q)} = \left(\sum_{k=1}^{m} L_{n,k}, \ldots, \sum_{k=1}^{m} L_{1,k}\right) = \lambda_L\end{aligned}$$

where $\overline{C\text{wt}(A)} = \overline{(a_1, \ldots, a_n)} = (a_n, \ldots, a_1)$ for a matrix A, and $(**)$ follows from the fact that Re_j is a $\mathfrak{gl}_n$-crystal morphism for all j. Therefore, indeed $(\lambda_P)^{\text{tr}} = \lambda_Q$. □

Next, we verify that the map in Equation 12.2 is an $\mathfrak{gl}_n \times \mathfrak{gl}_m$-crystal morphism. We already know that $Re^{\max}$ is an $\mathfrak{gl}_n$-crystal morphism and $Cf^{\max}$ is an $\mathfrak{gl}_m$-crystal morphism, so it remains to check that for ϕ and ψ.

Lemma 12.1.3 *The map defined above*

$$\phi : \{P \in \Lambda^N B_{n,m} \mid Re_i(P) = 0 \quad \forall\, i = 1, 2, \ldots, m-1\} \longrightarrow \bigsqcup_{\substack{\lambda \in n\times m \\ |\lambda| = N}} B_\lambda^{\mathfrak{gl}_n}$$

is an $\mathfrak{gl}_n$-crystal morphism.

The crystal $B_\lambda^{\mathfrak{gl}_n}$ with $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{Z}^n_{\geq 0}$, $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$, corresponds to the irreducible $\mathfrak{gl}_n$-representation with highest weight $\lambda = \lambda_1\epsilon_1 + \lambda_2\epsilon_2 + \ldots + \lambda_n\epsilon_n$. The elements of the crystal are the SSYT with shape λ and filling from the alphabet $\{1, 2, \ldots, n\}$. The Kashiwara operators Ce_i, Cf_i act by: For each tableau, create a list by looking at the columns from left to right and adding a "+" if the column contains $i+1$, and a "−" if the column contains i (if it contains both, put "+" in front of "−"). Then consecutively cancel all the "+−" pairs until all the "−" are on the left and all the "+" are on the right. Then $C\epsilon_i$ is equal to the number of unpaired "+" and $C\phi_i$ is the number of unpaired "−". The operator Ce_i then changes the $i+1$ corresponding to the leftmost unpaired "+" to an i, and Cf_i changes the i corresponding to the rightmost "−" to an $i+1$. The weight of a tableau can be recovered from considering the number of times each element appears in the filling, it is $(\#1's, \#2's, \ldots, \#n's)$. [HoKa02]

Proof: (of Lemma 12.1.3) To verify that $C\mathrm{wt}(\phi(P)) = C\mathrm{wt}(P)$, recall that we get the $\mathfrak{gl}_n$-weight of P by adding up the elements in each row, $C\mathrm{wt}(P) = (\sum_{s=1}^m P_{1,s}, \sum_{s=1}^m P_{2,s}, \ldots, \sum_{s=1}^m P_{n,s})$. But by the definition of ϕ, the number of "1"s in row i of P becomes the number of entries "i" in $T_P = \phi(P)$, which gives precisely $C\mathrm{wt}(T_P)$.

We want to also show that for any Ce_k, $k = 1, 2, \ldots, n-1$ and any $P \in \Lambda^N B_{n,m}$ such that $Re_i(P) = 0\ \forall\ i = 1, 2, \ldots, m-1$ and $Ce_k(P) \neq 0$, the following diagram commutes. The proof is analogous for Cf_k and we omit it.

$$\begin{array}{ccc} P & \xrightarrow{Ce_k} & P' \\ \phi\downarrow & & \downarrow\phi \\ T_P & \xrightarrow{Ce_k} & T_{P'} \end{array}$$

On the matrix level, to determine the action of Ce_k, we need to look at rows k and $k+1$. Notice that the "+/−" rule on the tableau T_P can be translated to the matrix P since $P_{i,j} = 1$ precisely when i appears in the j-th column of T_P. So, when looking at rows k and $k+1$ of P, we obtain the same "+/−" list as in the tableau by going left to right along those rows of the matrix and using the rule:

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto + \qquad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto - \qquad \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mapsto +-$$

Suppose Ce_k acts in the l-th column of the matrix P, where $P_{k,l} = 0,\ P_{k+1,l} = 1$. We need to check that the corresponding "+" is the leftmost unpaired one in the list obtained from rows k and $k+1$, i.e. that the Ce_k action on the tableau T_P is in the same location.

First, we verify that the "+" in column l is unpaired. Suppose on the contrary that it is paired with a "−" in column $b > l$. Then all the "+" in columns l to b must be paired with "−" in the same interval. Namely, on the interval of columns $[l, b]$, $\#\{+\} \leq \#\{-\}$. In other words, $\sum_{j=l}^b (P_{k+1,j} - P_{k,j}) \leq 0$, which in turn implies that:

$$C\epsilon_k^{\widetilde{l}}(P) = \sum_{j=l}^{m}(P_{k+1,j} - P_{k,j}) \leq \sum_{j=b+1}^{m}(P_{k+1,j} - P_{k,j}) = C\epsilon_k^{\widetilde{b}}(P)$$

where $\widetilde{s} = m + 1 - s$, contradicting the fact that l is the number closest to m where the maximum $C\epsilon_k$ is achieved. Hence, the "+" in column l must be unpaired.

Next, we check that it is the leftmost unpaired "+". Suppose on the contrary there is an unpaired "+" in column $a < l$. Then in particular, $P_{k+1,a} = 1,\ P_{k,a} = 0$ and any "−" on the interval $[a, l-1]$ must be paired with a "+" in the same interval, so $\#\{-\} < \#\{+\}$, i.e. $\sum_{j=a}^{l-1}(P_{k+1,j} - P_{k,j}) > 0$. This implies that:

$$C\epsilon_k^{\widetilde{a}}(P) = \sum_{j=a}^{m}(P_{k+1,j} - P_{k,j}) > \sum_{j=l}^{m}(P_{k+1,j} - P_{k,j}) = C\epsilon_k^{\widetilde{l}}(P)$$

which contradicts that the maximum $C\epsilon_k$ is achieved at $C\epsilon_k^{\widetilde{l}}$.

So, we showed that the location where Ce_k acts on P, changing it to $P' = Ce_k(P)$ with $(P')_{k,l} = 1,\ (P')_{k+1,l} = 0,\ (P')_{i,j} = P_{i,j}\ \forall\ (i,j) \neq (k,l), (k+1,l)$, corresponds to the location where Ce_k acts on $\phi(P) = T_P$, changing it to a tableau identical in shape and filling to T_P except in the box where Ce_k acts, which now contains k instead of $k+1$. From the above arguments and the construction of ϕ, the tableau for P' is then exactly the tableau $Ce_k(T_P)$, so the diagram commutes.

Finally, we verify that for any $C\epsilon_k,\ k = 1, 2, \ldots, n-1$ and any $P \in \Lambda^N B_{n,m}$ such that $Re_i(P) = 0\ \forall\ i = 1, 2, \ldots, m-1$, we have $C\epsilon_k(\phi(P)) = C\epsilon_k(P)$. The proof is analogous for $C\phi_k$. If $C\epsilon_k(P) = 0$, then for every $1 \leq l \leq m$, we have

$$C\epsilon_k^{\widetilde{l}}(P) = \sum_{j=l}^{m}(P_{k+1,j} - P_{k,j}) + \delta_{P_{k,l},1}\delta_{P_{k+1,l},0} \leq 0$$

so, $\sum_{j=l}^{m}(P_{k+1,j} - P_{k,j}) \leq 0$ which means on every interval $[l, m]$, $\#\{+\} \leq \#\{-\}$, so there are no unpaired "+" and $C\epsilon_k(\phi(P)) = C\epsilon_k(T_P) = 0$.

Suppose that $C\epsilon_k(P) > 0$ and let l denote the number closest to m where the maximum $\max_s C\epsilon_k^{\widetilde{s}}(P)$ is achieved, $C\epsilon_k(P) = C\epsilon_k^{\widetilde{l}}(P)$. Then, following the considerations about for the action of Ce_k concerning rows k and $k+1$ in P, there is an unpaired "+" in column l, all "+" on the left are paired with "−" on that side, and all "−" to the right of column l are paired with "+" on the same side. Therefore, the number of unpaired "+" for rows k and $k+1$, which corresponds to $C\epsilon_k(T_P) = C\epsilon_k(\phi(P))$ is:

$$\#\{+ \text{ in columns } l \text{ to } m\} - \#\{- \text{ in columns } l \text{ to } m\} = \sum_{s=l}^{m}(P_{k+1,s} - P_{k,s}) = C\epsilon_k^{\widetilde{l}}(P) = C\epsilon_k(P).$$

□

Lemma 12.1.4 *The map defined above*

$$\psi : \{Q \in \Lambda^N B_{n,m} \mid Cf_i(Q) = 0 \quad \forall\ i = 1, 2, \ldots, n\} \longrightarrow \bigsqcup_{\substack{\lambda \in n\times m \\ |\lambda| = N}} B_{\lambda^{tr}}^{\mathfrak{gl}_m}$$

is a $\mathfrak{gl}_m$-crystal morphism.

Proof: We want to show that for any $Re_k,\ k = 1, 2, \dots, m-1$ and any $Q \in \Lambda^N B_{n,m}$ such that $Cf_i(Q) = 0\ \forall\ i = 1, 2, \dots, n-1$ and $Re_k(Q) \neq 0$, the following diagram commutes. The proof is analogous for Rf_k.

$$\begin{array}{ccc} Q & \xrightarrow{Re_k} & Q' \\ \psi \downarrow & & \downarrow \psi \\ T_Q & \xrightarrow{Re_k} & T_{Q'} \end{array}$$

When considering the action of Re_k, we are looking at columns k and $k+1$ of Q. The setup here is the same as in Lemma 12.1.3 if we consider Q rotated 90° clockwise, and everything else follows through analogously, including that ψ preserves the values of $R\epsilon_k$, $R\phi_k$ and Rwt. □

Finally, we'll check that the map is indeed an isomorphism, i.e. it remains to show it is a bijection on sets.

Lemma 12.1.5 ([vL06]) *The $\mathfrak{gl}_n \times \mathfrak{gl}_m$ crystal morphism $\Lambda^N B_{n,m} \longrightarrow \bigsqcup_{\substack{\lambda \subset n \times m \\ |\lambda| = N}} B_\lambda^{\mathfrak{gl}_n} \otimes B_{\lambda^{tr}}^{\mathfrak{gl}_m}$ given below is an isomorphism:*

$$M \xrightarrow{(Re^{max}, Cf^{max})} P \otimes Q \xrightarrow{(\phi,\psi)} T_P \otimes T_Q$$

Proof: Let's first consider the map $(Re^{\max}, Cf^{\max})$. As it doesn't map any nonzero element to zero, it suffices to check it is surjective. For a given $\lambda \subset n \times m$, $|\lambda| = N$, take any pair $P \otimes Q \subset B_\lambda^{\mathfrak{gl}_n} \otimes B_{\lambda^{\mathrm{tr}}}^{\mathfrak{gl}_m}$. Namely, $P, Q \in \mathrm{Mat}_{n\times m}(\{0,1\})$ such that $Re_i(P) = 0\ \forall\ i = 1, 2, \dots, m-1$, $Cf_j(Q) = 0\ \forall\ j = 1, 2, \dots, n-1$ and $\lambda^{\mathrm{tr}} = \sum_{s=1}^{n} rP_s$, $\lambda = \sum_{s=1}^{m} \overline{cQ}_s$. Then, as discussed in Lemma 12.1.1, $Cf^{\max}P = Re^{\max}Q = L$ is the matrix with N 1's filling the shape of λ aligned Southwest. Suppose in particular, two sequences that realize L are:

$$Cf_{j_1}Cf_{j_2}\dots Cf_{j_p}P = Re_{i_1}Re_{i_2}\dots Re_{i_q}Q$$

Using the commutativity properties of the row and column Kashiwara operators, we take:

$$M := Rf_{i_q}\dots Rf_{i_2}Rf_{i_1}P = Ce_{j_p}\dots Ce_{j_2}Ce_{j_1}Q \neq 0$$

Then $M \in \Lambda^N B_{n,m}$ has the property that $(Re^{\max}, Cf^{\max})(M) = P \otimes Q$. When considering (ϕ, ψ), given a tableau $T_P \subset B_\lambda^{\mathfrak{gl}_n}$, we recover the $n \times m$ matrix P such that $\phi(P) = T_P$ by, for each $k = 1, \dots, n$, looking at the entries labelled k in T_P, and if such an entry appears in the j-th column of the tableau, then set $P_{k,j} = 1$ while the remaining elements in the k-th row of the matrix are zero. Similarly, given a tableau $T_Q \subset B_{\lambda^{\mathrm{tr}}}^{\mathfrak{gl}_m}$, to recover the $n \times m$ matrix Q such that $\psi(Q) = T_Q$, for each $l = 1, \dots, m$, looking at the entries labelled l in T_Q, and if such an entry appears in the j-th column of the tableau, then set $Q_{m+1-j,l} = 1$ while the remaining elements in the l-th column of the matrix Q are set to zero. □

12.2 Two cactus group actions

Henriques and Kamnitzer [HeKa06] define an action of the cactus group J_n on n-tensor products of $\mathfrak{g}$-crystals (for any finite-dimensional complex reductive Lie algebra $\mathfrak{g}$) given by, for any $s_{p,q} \in J_n = J_{\mathfrak{gl}_n}$, $1 \le p < q \le n$ corresponding to the Dynkin subdiagram $\{p, \dots, q-1\}$ of $\mathfrak{gl}_n$, and any element in a $\mathfrak{g}$-crystal $B_1 \otimes B_2 \otimes \ldots \otimes B_n$:

$$s_{p,q}(b_1 \otimes b_2 \otimes \ldots \otimes b_n) = b_1 \otimes \ldots b_{p-1}\xi(\xi(b_q) \otimes \xi(b_{q-1}) \otimes \ldots \otimes \xi(b_{p+1}) \otimes \xi(b_p)) \otimes b_{q+1} \otimes \ldots \otimes b_n$$

Here ξ is the Schützenberger involution as in Definition 10.0.1. If $\mathrm{flip}^{p,q}$ denotes the permutation that flips the interval $[p,q]$, then $s_{p,q}$ acts as:

$$s^{\mathrm{o}}_{p,q} = \mathrm{Id} \otimes \ldots \otimes \mathrm{Id} \otimes \xi_{p,q} \circ (\xi \otimes \ldots \otimes \xi) \circ \mathrm{flip}^{p,q} \otimes \mathrm{Id} \otimes \ldots \otimes \mathrm{Id}$$

We claim that this "outer" cactus group action agrees with the "inner" cactus group action from Chapter 10 using skew Howe duality for $\mathfrak{gl}_n$ and $\mathfrak{gl}_m$.

Lemma 12.2.1 *For any $b = (b_1, b_2, \ldots, b_m) \in B^{\mathfrak{gl}_m}_{\omega_k}$, ω_k denoting the k-th fundamental weight of $\mathfrak{gl}_m$, and the Schützenberger involution $\xi = \xi_{B^{\mathfrak{gl}_m}_{\omega_k}}$:*

$$\xi(b)_i = b_{m+1-i}, \qquad i.e. \quad \xi(b_1,\ b_2,\ \ldots,\ b_m) = (b_m,\ \ldots,\ b_2,\ b_1)$$

Proof: An element in $B^{\mathfrak{gl}_m}_{\omega_k}$ can be expressed as a vector in $(\{0,1\})^m$ with k of its entries equal to 1. If they occupy positions $1 \le i_1 < i_2 < \ldots < i_k \le m$, that element can be represented as $\{i_1, \ldots, i_k\}$. In particular, the highest and lowest weight elements are $(1,1,\ldots,1,0,\ldots,0)$ and $(0,\ldots,0,1,\ldots,1)$, represented by $\{1,2,\ldots,k\}$ and $\{m+1-k,\ldots,m-1,m\}$. Additionally, let $f_{j_1,\ldots,j_s} := f_{j_s} \ldots f_{j_1}$ and $e_{j_1,\ldots,j_s} := e_{j_s} \ldots e_{j_1}$ represent a composition of Kashiwara operators for the crystal $B^{\mathfrak{gl}_m}_{\omega_k}$. Then, any element $\{i_1, \ldots, i_k\}$ can be expressed in terms of the highest weight element and a sequence of lowering Kashiwara operators, so we get:

$$\begin{aligned}\{i_1,\ldots,i_k\} &= f_{1,2,\ldots,i_1-1} \ldots f_{k-1,k,\ldots,i_{k-1}-1} f_{k,k+1,\ldots,i_k-1}\{1,2,\ldots,k\} \\ &\xrightarrow{\xi} e_{m-1,m-2,\ldots,m-i_1+1} \ldots e_{m-k,m-k-1,\ldots,m-i_k+1}\{m-k+1, m-k+2, \ldots, m\} \\ &= \{m+1-i_k, m+1-i_{k-1}, \ldots, m+1-i_1\}\end{aligned}$$

□

Lemma 12.2.2 *The $\mathfrak{gl}_k$-crystal Schützenberger involution $\xi = \xi^{\mathfrak{gl}_m}_{1,k} : \Lambda^N B_{n,m} \longrightarrow \Lambda^N B_{n,m}$, corresponding to the Dynkin subdiagram $\overline{k} = \{1, \ldots, k-1\}$ of $\mathfrak{gl}_m$, is a $\mathfrak{gl}_n$-crystal morphism. Analogously, the $\mathfrak{gl}_l$-crystal Schützenberger involution $\xi^{\mathfrak{gl}_n}_{1,l}$ is a $\mathfrak{gl}_m$-crystal morphism.*

Proof: We will focus on the commutativity of ξ with Ce_i, the remaining morphism properties are analogous. Take any element $0 \neq b \in \Lambda^N B_{n,m}$ such that $Ce_i(b) \neq 0$, it belongs to some irreducible $\mathfrak{gl}_k$-crystal $B^{\mathfrak{gl}_k}_\lambda \subset \left(\Lambda^N B_{n,m}\right)_{\overline{k}}$, i.e. a connected component of $\Lambda^N B_{n,m}$ restricted to $\overline{k}$, with highest weight λ, say, and highest and lowest weight elements b_λ and b^{low}_λ respectively. We want to check that $\xi Ce_i(b) = Ce_i\xi(b)$. Expanding each side using that there is a sequence of Kashiwara operators such that $b = Re_{j_s}Re_{j_{s-1}} \ldots Re_{j_1}(b^{\mathrm{low}}_\lambda)$, where $j_1, \ldots, j_s \in \overline{k}$, together with $Ce_iRe_j = Re_jCe_i$, $Ce_iRf_j = Rf_jCe_i$ from Proposition 11.2.1 and $\xi Re_j = Rf_{k-j}\xi$, $\xi(b^{\mathrm{low}}_\lambda) = b_\lambda$, $\xi(b_\lambda) = b^{\mathrm{low}}_\lambda$ from Definition 10.0.1, we get:

$$b \xrightarrow{Ce_i} Re_{j_s}Re_{j_{s-1}} \ldots Re_{j_1}Ce_i(b^{\mathrm{low}}_\lambda) \xrightarrow{\xi} Rf_{k-j_s}Rf_{k-j_{s-1}} \ldots Rf_{k-j_1}\xi Ce_i(b^{\mathrm{low}}_\lambda)$$

$$b \xrightarrow{\xi} Rf_{k-j_s}Rf_{k-j_{s-1}} \ldots Rf_{k-j_1}(b_\lambda) \xrightarrow{Ce_i} Rf_{k-j_s}Rf_{k-j_{s-1}} \ldots Rf_{k-j_1}Ce_i(b_\lambda)$$

It remains to show that $\xi Ce_i(b_\lambda^{\mathrm{low}}) = Ce_i(b_\lambda)$. Since $Ce_i(b) \neq 0$, we also have $Ce_i(b_\lambda) \neq 0,\ Ce_i(b_\lambda^{\mathrm{low}}) \neq 0$, as Re_j and Rf_j preserve $C\epsilon_i$, and we can apply Proposition 11.2.1, which tells us that Ce_i commutes with Re_j and preserves $R\epsilon_j,\ R\phi_j,\ R\mathrm{wt}$. In particular, this means $c_\lambda = Ce_i(b_\lambda)$ and $c_\lambda^{\mathrm{low}} = Ce_i(b_\lambda^{\mathrm{low}})$ are the highest and lowest weight elements of an irreducible $\mathfrak{gl}_k$-crystal in $\left(\Lambda^N B_{n,m}\right)_{\overline{k}}$, and so get exchanged by ξ. □

Lemma 12.2.3 *For all $i = 1,\ \ldots,\ k-1,\ M \in \Lambda^N B_{n,m}$ s.t. $Ce_i(M) \neq 0$, resp. $Cf_i(M) \neq 0$:*

$$s^o_{1,k}(Ce_i(M)) = Cf_{k-i}(s^o_{1,k}(M)) \qquad s^o_{1,k}(Cf_i(M)) = Ce_{k-i}(s^o_{1,k}(M))$$

Proof: Let $M_{[k]}$ denote the submatrix made up of the first k rows of M, $r_1,\ \ldots,\ r_k \in (\{0,1\})^m$. Since $s^{\mathrm{o}}_{1,k}$ and $Ce_i,\ Cf_i,\ i = 1,\ \ldots,\ k-1$, are determined by and affect only the first k rows of M, we have $s^{\mathrm{o}}_{1,k}(M)_{[k]} = s^{\mathrm{o}}_{1,k}(M_{[k]})$ and similarly for $Ce_i,\ Cf_i$ and so we'll work with $L := M_{[k]}$.

Let $p_k := \xi^{\otimes k} \circ \mathrm{flip}^{1,k}$. Note that $s^{\mathrm{o}}_{1,k}(Ce_i(L)) = \xi^{\mathfrak{gl}_m}_{1,k}(p_k(Ce_i(L)))$, while on the other hand $Cf_{k-i}(s^{\mathrm{o}}_{1,k}(L)) = Cf_{k-i}(\xi^{\mathfrak{gl}_m}_{1,k}(p_k(L))) = \xi^{\mathfrak{gl}_m}_{1,k}(Cf_{k-i}(p_k(L)))$ since, as shown in Lemma 12.2.2, $\xi^{\mathfrak{gl}_m}_{1,k}$ is a $\mathfrak{gl}_n$-crystal morphism. So, we need to check that $p_k(Ce_i(L)) = Cf_{k-i}(p_k(L))$. To verify that, we'll first check that $C\epsilon_i^{\widetilde{j}}(L) = C\phi_{k-i}^{\widetilde{m+1-j}}(p_k(L))$. From the definition, $p_k(L)_{s,t} = L_{k+1-s,m+1-t}$, so:

$$\begin{aligned}
C\epsilon_i^{\widetilde{j}}(L) &= \sum_{r=j}^{m}(L_{i+1,r} - L_{i,r}) + \delta_{L_{i,j},1}\delta_{L_{i+1,j},0}\\
C\phi_{k-i}^{\widetilde{m+1-j}}(p_k(L)) &= \sum_{r=1}^{m+1-j}(p_k(L)_{k-i,r} - p_k(L)_{k+1-i,r}) + \delta_{p_k(L)_{k+1-i,m+1-j},1}\delta_{p_k(L)_{k-i,m+1-j},0}\\
&= \sum_{r=1}^{m+1-j}(L_{i+1,m+1-r} - L_{i,m+1-r}) + \delta_{L_{i,j},1}\delta_{L_{i+1,j},0}\\
&= C\epsilon_i^{\widetilde{j}}(L)
\end{aligned}$$

So, if l is the closest number to m where the maximum $C\epsilon_i^{\widetilde{l}}(L) = \max_{\widetilde{j}} C\epsilon_i^{\widetilde{j}}(L)$ is achieved, then $m+1-l$ is the closest number to 1 where $C\phi_{k-i}^{\widetilde{m+1-l}}(p_k(L)) = \max_{\widetilde{j}} C\phi_{k-i}^{\widetilde{m+1-j}}(p_k(L))$ is achieved. Namely, if Ce_i acts in column l of L, then Cf_{k-i} acts in column $m+1-l$ of $p_k(L)$, and:

$$p_k(Ce_i(L))_{s,t} = \begin{cases} L_{k+1-s,m+1-t} & (s,t) \neq (k+1-i, m+1-l),\ (k-i, m+1-l)\\ 1 & (s,t) = (k+1-i, m+1-l)\\ 0 & (s,t) = (k-i, m+1-l)\end{cases}$$

$$\begin{aligned}
Cf_{k-i}(p_k(L))_{s,t} &= \begin{cases} p_k(L)_{s,t} & (s,t) \neq (k+1-i, m+1-l),\ (k-i, m+1-l)\\ 1 & (s,t) = (k+1-i, m+1-l)\\ 0 & (s,t) = (k-i, m+1-l)\end{cases}\\
&= \begin{cases} L_{k+1-s,m+1-t} & (s,t) \neq (k+1-i, m+1-l),\ (k-i, m+1-l)\\ 1 & (s,t) = (k+1-i, m+1-l)\\ 0 & (s,t) = (k-i, m+1-l)\end{cases}
\end{aligned}$$

□

Theorem 12.2.4 *The outer action of J_n on the n-tensor product of $\mathfrak{gl}_m$-crystals defined by Henriques-Kamnitzer agrees with the inner cactus group action on a $\mathfrak{gl}_n$-crystal under the crystal isomorphism: $\Lambda^N B_{n,m} \cong \bigsqcup_{\substack{\lambda\subset n\times m\\ |\lambda|=N}} B_\lambda^{\mathfrak{gl}_n} \otimes B_{\lambda^{tr}}^{\mathfrak{gl}_m}$. More precisely, for any $s_{p,q}\in J_n$ and any $M \in \Lambda^N B_{n,m}$:*

$$s^o_{p,q}(M) = s^i_{p,q}(M)$$

Proof: Note that it suffices to check the claim on elements of the form $s_{1,k} \in J_n,\ k = 1, 2, \ldots, n$ since they generate the group. Indeed, from the cactus relations, we have that for any $1 \le p < q \le n$, $s_{p,q} = s_{1,q}s_{1,q+1-p}s_{1,q}$.

We will use the matrix version of the isomorphism:

$$\Lambda^N B_{n,m} \cong \bigsqcup_{\substack{\lambda\subset n\times m\\ |\lambda|=N}} B_\lambda^{\mathfrak{gl}_n} \otimes B_{\lambda^{\rm tr}}^{\mathfrak{gl}_m} \tag{12.3}$$

namely the map $(Re^{\max}, Cf^{\max})$. On the left side of Equation 12.3, we have the tensor product action of $s^{\rm o}_{1,k}$ on n-tensors of $\mathfrak{gl}_m$-crystals:

$$\Lambda^N B_{n,m} \cong \bigsqcup_{\underline{l}\in\mathbb{N}^n, |\underline{l}|=N} \mathcal{B}^{\mathfrak{sl}_m}_{\omega_{l_1}} \otimes \ldots \otimes \mathcal{B}^{\mathfrak{sl}_m}_{\omega_{l_n}}$$

On the right side of Equation 12.3, we have the internal action of $s^{\rm i}_{1,k}$ on any element $b\otimes c \in B_\lambda^{\mathfrak{gl}_n} \otimes B_{\lambda^{\rm tr}}^{\mathfrak{gl}_m}$, given by:

$$s^{\rm i}_{1,k}\cdot (b\otimes c) = (\xi_{1,k}\otimes {\rm id})\ (b\otimes c) = \xi_{1,k}(b)\otimes c$$

Since $s_{1,k} \in J_n$ acting on n-tensor products of $\mathfrak{gl}_m$-crystals is an $\mathfrak{gl}_m$-crystal isomorphism (see [HeKa06], Theorem 2), it suffices to check the statement of the theorem on $\mathfrak{gl}_m$-highest weight elements, on which $Re^{\max}$ acts as the identity. So we need to check that the following diagram commutes for every $P \in \Lambda^N B_{n,m}$ s.t. $Re_i(P) = 0,\ \forall\, i = 1, 2, \ldots, m-1$:

$$\begin{array}{ccc}
\Lambda^N B_{n,m} & \xrightarrow{(Re^{\max}, Cf^{\max})} & \bigsqcup_{\substack{\lambda\subset n\times m\\ |\lambda|=N}} B_\lambda^{\mathfrak{gl}_n} \otimes B_{\lambda^{\rm tr}}^{\mathfrak{gl}_m} \\
{\scriptstyle s^{\rm o}_{1,k}}\downarrow & & \downarrow{\scriptstyle s^{\rm i}_{1,k}} \\
\Lambda^N B_{n,m} & \xrightarrow{(Re^{\max}, Cf^{\max})} & \bigsqcup_{\substack{\lambda\subset n\times m\\ |\lambda|=N}} B_\lambda^{\mathfrak{gl}_n} \otimes B_{\lambda^{\rm tr}}^{\mathfrak{gl}_m}
\end{array}$$

Starting from the top left, going first right and then down we get:

$$s^{\rm i}_{1,k}\circ (Re^{\max}, Cf^{\max})(P) = s^{\rm i}_{1,k}(P\otimes Cf^{\max}(P)) = s^{\rm i}_{1,k}(P)\otimes Cf^{\max}(P)$$

Whereas, applying first $s^{\rm o}_{1,k}$ and then the horizontal map gives us:

$$(Re^{\max}, Cf^{\max})\circ s^{\rm o}_{1,k}(P) = Re^{\max}(s^{\rm o}_{1,k}(P))\otimes Cf^{\max}(s^{\rm o}_{1,k}(P)) =$$
$$= s^{\rm o}_{1,k}(Re^{\max}(P))\otimes Cf^{\max}(s^{\rm o}_{1,k}(P)) = s^{\rm o}_{1,k}(P)\otimes Cf^{\max}(s^{\rm o}_{1,k}(P))$$

So, we need to verify that:

1. $s^{\mathrm{o}}_{1,k}(P) = s^{\mathrm{i}}_{1,k}(P)$
2. $Cf^{\max}(s^{\mathrm{o}}_{1,k}(P)) = Cf^{\max}(P)$

For (1), both $s^{\mathrm{o}}_{1,k}$ and $s^{\mathrm{i}}_{1,k}$ are determined by and change only the top k rows of P, so we'll restrict our attention to them and call that submatrix $P_{[k]}$. Consider $L = Ce^{\max}(P_{[k]}) = Ce_{i_1} \ldots Ce_{i_{s-1}} Ce_{i_s}(P_{[k]})$ where $\{i_1,\ i_2, \ldots,\ i_s\}$ is some path to the $\mathfrak{gl}_k$-highest weight element associated to $P_{[k]}$. Then $P_{[k]} = Cf_{i_s} Cf_{i_{s-1}} \ldots Cf_{i_1}(L)$, and so using Lemma 12.2.3:

$$\begin{aligned} s^{\mathrm{o}}_{1,k}(P_{[k]}) &= s^{\mathrm{o}}_{1,k}(Cf_{i_s} Cf_{i_{s-1}} \ldots Cf_{i_1}(L)) = Ce_{k-i_s} s^{\mathrm{o}}_{1,k}(Cf_{i_{s-1}} \ldots Cf_{i_1}(L)) = \\ &= \ldots = Ce_{k-i_s} Ce_{k-i_{s-1}} \ldots Ce_{k-i_1} s^{\mathrm{o}}_{1,k}(L) \end{aligned}$$

And similarly, $s^{\mathrm{i}}_{1,k}(P_{[k]}) = Ce_{k-i_s} Ce_{k-i_{s-1}} \ldots Ce_{k-i_1} s^{\mathrm{i}}_{1,k}(L)$ by the definition of the inner action in Proposition 10.0.3. So, it suffices to check that $s^{\mathrm{o}}_{1,k}(L) = s^{\mathrm{i}}_{1,k}(L)$. Since L is both $\mathfrak{gl}_m$- and $\mathfrak{gl}_k$-highest weight, as discussed in Lemma 12.1.1 it is a matrix filled with zeros except the upper left corner, where it has 1's filling the shape of a partition λ with $\lambda^{\mathrm{tr}} = (\sum_{i=1}^k P_{i,1}, \sum_{i=1}^k P_{i,2}, \ldots, \sum_{i=1}^k P_{i,m})$.

We have that $L \in \Lambda B_{k,m} \cong \bigsqcup_{\underline{i}} B^{\mathfrak{gl}_m}_{\omega_{i_1}} \otimes \ldots \otimes B^{\mathfrak{gl}_m}_{\omega_{i_k}}$. Under that correspondence, we can express L as $b_1 \otimes b_2 \otimes \ldots b_k$, where $b_j \in (\{0,1\})^m$ is the j-th row of L. Furthermore, L having entries "1" filling the shape of the partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ implies that $b_1 \otimes b_2 \otimes \ldots \otimes b_k \in B^{\mathfrak{gl}_m}_{\omega_{\lambda_1}} \otimes \ldots \otimes B^{\mathfrak{gl}_m}_{\omega_{\lambda_k}}$ and each b_i is a $\mathfrak{gl}_m$-highest weight element in $B^{\mathfrak{gl}_m}_{\omega_{\lambda_i}}$, i.e. $\mathrm{wt}(b_i) = \omega_{\lambda_i}$. Therefore, $\xi(b_i) = b_i^{\mathrm{low}} \in B^{\mathfrak{gl}_m}_{\omega_{\lambda_i}}$, where b_i^{low} with $\mathrm{wt}(b_i^{\mathrm{low}}) = w_0^{\mathfrak{gl}_m} \cdot \omega_{\lambda_i}$ is the lowest weight element in that crystal. Then:

$$s^{\mathrm{o}}_{1,k}(L) = s^{\mathrm{o}}_{1,k}(b_1 \otimes b_2 \otimes \ldots \otimes b_k) = \xi(\xi(b_k) \otimes \ldots \otimes \xi(b_2) \otimes \xi(b_1)) = \xi(b_k^{\mathrm{low}} \otimes \ldots \otimes b_2^{\mathrm{low}} \otimes b_1^{\mathrm{low}})$$

For the final computation, we will consider the weights of the elements in the crystal we're working with. In any crystal of the form $B^{\mathfrak{gl}_m}_{\omega_{i_1}} \otimes \ldots \otimes B^{\mathfrak{gl}_m}_{\omega_{i_k}}$, let b_j^{high} denote the highest weight element in $B^{\mathfrak{gl}_m}_{\omega_{i_j}}$, $\mathrm{wt}(b_j^{\mathrm{high}}) = \omega_{i_j}$, and b_j^{low} denote the lowest weight element, $\mathrm{wt}(b_j^{\mathrm{low}}) = w_0^{\mathfrak{gl}_m} \cdot \omega_{i_j}$. Then, for any other element $b = b_1 \otimes b_2 \otimes \ldots \otimes b_k \in B^{\mathfrak{gl}_m}_{\omega_{i_1}} \otimes \ldots \otimes B^{\mathfrak{gl}_m}_{\omega_{i_k}}$:

$$\sum_{j=1}^{k} w_0^{\mathfrak{gl}_m} \cdot \omega_{i_j} \leq \mathrm{wt}(b) = \sum_{j}^{k} \mathrm{wt}(b_j) \leq \sum_{j=1}^{k} \omega_{i_j}$$

We get equality on the left precisely when $b_j = b_j^{\mathrm{low}}\ \forall\ j = 1, \ldots, k$ and equality on the right precisely when $b_j = b_j^{\mathrm{high}}\ \forall\ j = 1, \ldots, k$. Namely, $b_1^{\mathrm{low}} \otimes b_2^{\mathrm{low}} \otimes \ldots \otimes b_k^{\mathrm{low}}$ and $b_1^{\mathrm{high}} \otimes b_2^{\mathrm{high}} \otimes \ldots \otimes b_k^{\mathrm{high}}$ are the unique elements in $B^{\mathfrak{gl}_m}_{\omega_{i_1}} \otimes \ldots \otimes B^{\mathfrak{gl}_m}_{\omega_{i_k}}$ with weights $\sum_{j=1}^k w_0^{\mathfrak{gl}_m} \cdot \omega_{i_j}$ and $\sum_{j=1}^k \omega_{i_j}$ respectively.

Back to our case, since:

$$\mathrm{wt}(\xi(b_k^{\mathrm{low}} \otimes \ldots \otimes b_2^{\mathrm{low}} \otimes b_1^{\mathrm{low}})) = w_0^{\mathfrak{gl}_m} \cdot \mathrm{wt}(b_k^{\mathrm{low}} \otimes \ldots \otimes b_2^{\mathrm{low}} \otimes b_1^{\mathrm{low}}) = w_0^{\mathfrak{gl}_m} \cdot \sum_{i=1}^{k} w_0^{\mathfrak{gl}_m} \cdot \omega_{\lambda_i} = \sum_{i=1}^{k} \omega_{\lambda_i}$$

we must have $\xi(b_k^{\mathrm{low}} \otimes \ldots \otimes b_2^{\mathrm{low}} \otimes b_1^{\mathrm{low}}) = b_k^{\mathrm{high}} \otimes \ldots \otimes b_2^{\mathrm{high}} \otimes b_1^{\mathrm{high}}$.

So, we get that for a matrix L with filling determined by $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ as above:

$s^{\mathrm{o}}_{1,k}(L) =$ the $k \times m$ matrix with entries equal to zero except in the lower left corner which has 1's filling the shape of the reflected partition $(\lambda_k, \ldots, \lambda_2, \lambda_1)$ (see Figure 12.2)

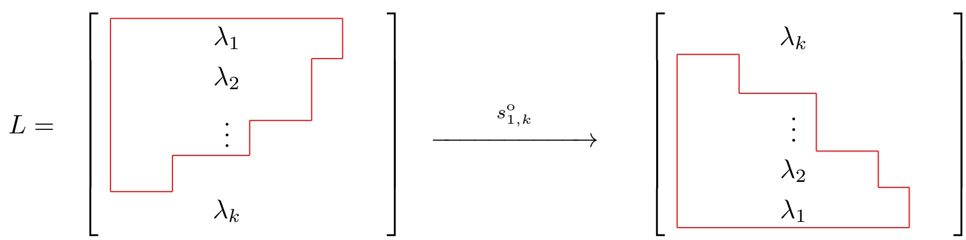


Figure 12.1: The result of applying $s^{\mathrm{o}}_{1,k}$ to an $\mathfrak{gl}_k$- and $\mathfrak{gl}_m$-highest weight matrix.

For the action of $s^{\mathrm{i}}_{1,k}$, we consider $L \in \Lambda B_{k,m} \cong \bigsqcup_{\underline{j}} B^{\mathfrak{gl}_k}_{\omega_{j_1}} \otimes \ldots \otimes B^{\mathfrak{gl}_k}_{\omega_{j_m}}$ as a tensor product of $\mathfrak{gl}_k$ crystals. Let $\lambda^{\mathrm{tr}} =: \tau = (\tau_1, \tau_2, \ldots, \tau_m)$ be the conjugate partition to the partition λ described above. Then, an analogous argument about the filling of L as before tells us that $L = c^{\mathrm{high}}_m \otimes c^{\mathrm{high}}_{m-1} \otimes \ldots \otimes c^{\mathrm{high}}_1 \in B^{\mathfrak{gl}_k}_{\omega_{\tau_m}} \otimes \ldots \otimes B^{\mathfrak{gl}_k}_{\omega_{\tau_1}}$, with $\mathrm{wt}(c^{\mathrm{high}}_i) = \omega_{\tau_i}$. By weight considerations similar to the earlier ones:

$$\mathrm{wt}(s^{\mathrm{i}}_{1,k}(L)) = \mathrm{wt}(s^{\mathrm{i}}_{1,k}(c^{\mathrm{high}}_m \otimes c^{\mathrm{high}}_{m-1} \otimes \ldots \otimes c^{\mathrm{high}}_1)) = w^{\mathfrak{gl}_k}_0 \cdot \mathrm{wt}(c^{\mathrm{high}}_m \otimes c^{\mathrm{high}}_{m-1} \otimes \ldots \otimes c^{\mathrm{high}}_1) =$$
$$= w^{\mathfrak{gl}_k}_0 \cdot \sum_{i=1}^{m} \tau_i = \sum_{i=1}^{m} w^{\mathfrak{gl}_k}_0 \cdot \tau_i = \mathrm{wt}(c^{\mathrm{low}}_m \otimes c^{\mathrm{low}}_{m-1} \otimes \ldots \otimes c^{\mathrm{low}}_1)$$

And as $c^{\mathrm{low}}_m \otimes c^{\mathrm{low}}_{m-1} \otimes \ldots \otimes c^{\mathrm{low}}_1$ is the unique element in $B^{\mathfrak{gl}_k}_{\omega_{\tau_m}} \otimes \ldots \otimes B^{\mathfrak{gl}_k}_{\omega_{\tau_1}}$ with this weight, $s^{\mathrm{i}}_{1,k}(L) = c^{\mathrm{low}}_m \otimes c^{\mathrm{low}}_{m-1} \otimes \ldots \otimes c^{\mathrm{low}}_1$. Since $c^{\mathrm{low}} = (c^{\mathrm{high}}$ reflected through the middle), i.e. $(c^{\mathrm{low}})_i = (c^{\mathrm{high}})_{k+1-i}$, $\forall\ i = 1, \ldots, k$:

$s^{\mathrm{i}}_{1,k}(L) =$ the matrix L reflected through the middle row, i.e. the $k \times m$ matrix with entries equal to zero except in the lower left corner which has 1's filling the shape of the reflected partition $(\lambda_k, \ldots, \lambda_2, \lambda_1)$

This operation gives the same result on L as applying $s^{\mathrm{o}}_{1,k}$ (shown in Figure 12.2), thus $s^{\mathrm{o}}_{1,k}(L) = s^{\mathrm{i}}_{1,k}(L)$ and so $s^{\mathrm{o}}_{1,k}(P) = s^{\mathrm{i}}_{1,k}(P)$.

To show (2), we know that $s^{\mathrm{o}}_{1,k}$ is a $\mathfrak{gl}_m$-crystal isomorphism (as shown in [HeKa06]), so takes $\mathfrak{gl}_m$-highest weight elements to $\mathfrak{gl}_m$-highest weight elements, namely $s^{\mathrm{o}}_{1,k}(P)$ is also an $\mathfrak{gl}_m$-highest weight element, and so $Cf^{\max}(s^{\mathrm{o}}_{1,k}(P))$ and $Cf^{\max}(P)$ are both obtained by "pushing" all the way down all the 1's in $s^{\mathrm{o}}_{1,k}(P)$, respectively P, obtaining matrices with 0's everywhere except the lower left corner where the 1's fill a reflected partition, i.e. one which is aligned Southwest. All we have to confirm then, is that the action of $s^{\mathrm{o}}_{1,k}$ preserves the number of 1's in each column of P, which determine the partition. But since $s^{\mathrm{o}}_{1,k}$ is an $\mathfrak{gl}_m$-crystal isomorphism, it preserves the $\mathfrak{gl}_m$-crystal weights $(\sum_{i=1}^{n} P_{i,1}, \ldots, \sum_{i=1}^{n} P_{i,m})$ and $\sum_{i=1}^{n} P_{i,j}$ is the number of 1's in the j-th column of P, which therefore is preserved by $s^{\mathrm{o}}_{1,k}$.

□

Corollary 12.2.5 *The outer action of J_m on the m-th tensor product of $\mathfrak{gl}_n$-crystals agrees with the inner action on the $\mathfrak{gl}_m$-crystal $\Lambda^N B_{n,m}$ under the crystal isomorphism: $\Lambda^N B_{n,m} \cong \bigsqcup_{\substack{\lambda \subset n \times m \\ |\lambda| = N}} B^{\mathfrak{gl}_n}_{\lambda} \otimes B^{\mathfrak{gl}_m}_{\lambda^{tr}}$. More precisely, for any $s_{p,q} \in J_m$ and any $M \in \Lambda^N B_{n,m}$:*

$$s^o_{m+1-q,m+1-p}(M) = s^i_{p,q}(M)$$

Proof: Consider the map $F : \Lambda^N B_{m,n} \longrightarrow \Lambda^N B_{n,m}$ which rotates a given matrix counterclockwise by 90 degrees. It is a bijection, and we will consider its effect on the "outer" and "inner" cactus group actions. We will first show that for $M \in \Lambda^N B_{m,n}$, $Re_i(F(M)) = F(Ce_i(M))\ \forall\ i = 1, \ldots, m-1$. Let $N = F(M)$ and first consider:

$$\begin{aligned} R\epsilon_i^k(N) &= \sum_{s=1}^{k-1} (N_{s,i+1} - N_{s,i}) + \delta_{N_{k,i},0}\delta_{N_{k,i+1},1} \\ &= \sum_{s=1}^{k-1} (M_{i+1,n+1-s} - M_{i,n+1-s}) + \delta_{M_{i,n+1-k},0}\delta_{M_{i+1,n+1-k},1} \\ &= \sum_{t=n+2-k}^{n} (M_{i+1,t} - M_{i,t}) + \delta_{M_{i,n+1-k},0}\delta_{M_{i+1,n+1-k},1} = C\epsilon_i^k(M) \end{aligned}$$

So, if Ce_i acts in the $(n+1-k)$-th column in M, changing $(M_{i,n+1-k}, M_{i+1,n+1-k}) = (N_{k,i}, N_{k,i+1})$ from $(0,1)$ to $(1,0)$, then Re_i acts in the k-th row of N, changing $(N_{k,i}, N_{k,i+1})$ from $(0,1)$ to $(1,0)$. Therefore, $Re_i(F(M)) = F(Ce_i(M))$ and analogously, we can show that $Rf_i(F(M)) = F(Cf_i(M))$. We'll study the inner cactus group action first. Given $M \in \Lambda^N B_{m,n}$ and $s_{p,q} \in J_m$, there exist two decompositions:

$$Cf_{i_s} \ldots Cf_{i_1}(H) = M = Ce_{j_t} \ldots Ce_{j_1}(L)$$

For some $i_1, \ldots, i_s, j_1, \ldots, j_t \in \{p, \ldots, q-1\}$ and $H \in \Lambda^N B_{m,n}$ such that $Ce_i(H) = 0\ \forall\ i \in \{p, \ldots, q-1\}$, $L \in \Lambda^N B_{m,n}$ such that $Cf_i(L) = 0\ \forall\ i \in \{p, \ldots, q-1\}$. Then let $d = q+1-p$, and using that $s^{\mathrm{i}}_{p,q}(Cf_i(M)) = Ce_{d-i}(s^{\mathrm{i}}_{p,q}(M))$, $s^{\mathrm{i}}_{p,q}(Ce_i(M)) = Cf_{d-i}(s^{\mathrm{i}}_{p,q}(M))$, $s^{\mathrm{i}}_{p,q}(H) = L$ from Definition 10.0.1 of the Schützenberger involution, we get:

$$\begin{aligned} &M \xrightarrow{s^{\mathrm{i}}_{p,q}} Ce_{d-i_s} \ldots Ce_{d-i_1} L \xrightarrow{F} Re_{d-i_s} \ldots Re_{d-i_1} F(L) \\ &M \xrightarrow{F} Rf_{i_s} \ldots Rf_{i_1} F(H) \xrightarrow{s^{\mathrm{i}}_{p,q}} Re_{d-i_s} \ldots Re_{d-i_1} F(L) \end{aligned}$$

The last equality follows since $Re_i F((H)) = F(Ce_i(H)) = 0 = F(Cf_i(L)) = Rf_i(F(L))\ \forall\ i \in \{p, \ldots, q-1\}$, so $F(H)$ and $F(L)$ are the highest and lowest weight elements of an irreducible component of the crystal resricted to the interval $\{p, q\}$, and so get switched by $s^{\mathrm{i}}_{p,q}$. Overall:

$$F(s^{\mathrm{i}}_{p,q}(M)) = s^{\mathrm{i}}_{p,q}(F(M)) \tag{12.4}$$

Next, we'll consider the outer cactus group action. Since the "outer" action of an element $s_{p,q} \in J_m$ on $M \in \Lambda^N_{B_{m,n}}$ is determined by the $\mathfrak{gl}_n$-crystal structure, we'll study that, which is determined by the rows of M, $r_1, \ldots, r_m$. Under the decomposition in Equation 11.2, M corresponds to $r_1 \otimes \ldots \otimes r_m$ as an $\mathfrak{gl}_n$-crystal. For $N = F(M)$, we have $N_{k,j} = M_{j,n+1-k}$, so the j-th column of N corresponds to the reflected j-th row of M, which by Lemma 12.2.1 is $\xi(r_j)$. So, the columns of N are $\xi(r_1), \ldots, \xi(r_m)$. Under the decomposition in Equation 11.1, N then corresponds to $\xi(r_m) \otimes \ldots \otimes \xi(r_1)$ as an $\mathfrak{gl}_n$-crystal. Diagrammatically,

$$\begin{bmatrix} - & r_1 & \longrightarrow \\ & \vdots & \\ - & r_m & \longrightarrow \end{bmatrix} \xrightarrow{F} \begin{bmatrix} \uparrow & & \uparrow \\ r_1 & \ldots & r_m \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ \xi(r_1) & \ldots & \xi(r_m) \\ \downarrow & & \downarrow \end{bmatrix}$$

So the map F on the level of rows can be described as $\xi^{\otimes m} \circ \mathrm{flip}^{1,m}$:

$$F(r_1 \otimes \ldots \otimes r_m) = \xi(r_m) \otimes \ldots \otimes \xi(r_1)$$

Combining this with the outer action of the cactus group, we have:

$$\begin{aligned} r_1 \otimes \ldots \otimes r_m \quad & \xrightarrow{s^{\mathrm{o}}_{p,q}} r_1 \otimes \ldots \otimes r_{p-1} \otimes \xi(\xi(r_q) \otimes \ldots \otimes \xi(r_p)) \otimes r_{q+1} \otimes \ldots \otimes r_m \\ & (\widetilde{r_q} \otimes \ldots \otimes \widetilde{r_p} := \xi(\xi(r_q) \otimes \ldots \otimes \xi(r_p))) \\ & \xrightarrow{F} \xi(r_m) \otimes \ldots \otimes \xi(r_{q+1}) \otimes \xi(\widetilde{r_p}) \otimes \ldots \otimes \xi(\widetilde{r_q}) \otimes \xi(r_{p-1}) \otimes \ldots \otimes \xi(r_1) \\ r_1 \otimes \ldots \otimes r_m \quad & \xrightarrow{F} \xi(r_m) \otimes \ldots \otimes \xi(r_q) \otimes \ldots \otimes \xi(r_p) \otimes \ldots \otimes \xi(r_1) \\ & \xrightarrow{s^{\mathrm{o}}_{m+1-q,m+1-p}} \xi(r_m) \otimes \ldots \otimes \xi(r_{q+1}) \otimes \xi(r_p \otimes \ldots \otimes r_q) \otimes \xi(r_{p-1}) \otimes \ldots \otimes \xi(r_1) \end{aligned}$$

The first $(m-q)$ (i.e. $\xi(r_m) \otimes \ldots \otimes \xi(r_{q+1})$) elements and the last $(p-1)$ (i.e. $\xi(r_{p-1}) \otimes \ldots \otimes \xi(r_1)$) elements of the resulting tensor product agree under both operations, so let's concentrate on what happens to $r_p \otimes \ldots \otimes r_q \in B_p \otimes \ldots \otimes B_q$ (denoting the $\mathfrak{gl}_n$-crystals to which the elements belong). They end up in positions $m+1-q$ to $m+1-p$ in the overall tensor product in both compositions, so disregarding that, the action of the first composition $F \circ s^{\mathrm{o}}_{p,q}$ is:

$$(\xi_{B_p} \otimes \ldots \otimes \xi_{B_q}) \circ \mathrm{flip}^{p,q} \circ \xi_{B_q \otimes \ldots \otimes B_p} \circ (\xi_{B_q} \otimes \ldots \otimes \xi_{B_p}) \circ \mathrm{flip}^{p,q}$$

Similarly, we can describe the second composition $s^{\mathrm{o}}_{m+1-q,m+1-p} \circ F$ acting on $r_p \otimes \ldots \otimes r_q$ as:

$$\begin{aligned} & \xi_{B_p \otimes \ldots \otimes B_q} \circ (\xi_{B_p} \otimes \ldots \otimes \xi_{B_q}) \circ \mathrm{flip}^{p,q} \circ (\xi_{B_q} \otimes \ldots \otimes \xi_{B_p}) \circ \mathrm{flip}^{p,q} \\ = & \xi_{B_p \otimes \ldots \otimes B_q} \circ \mathrm{flip}^{p,q} \circ (\xi_{B_q} \otimes \ldots \otimes \xi_{B_p}) \circ (\xi_{B_q} \otimes \ldots \otimes \xi_{B_p}) \circ \mathrm{flip}^{p,q} \\ = & (\xi_{B_p} \otimes \ldots \otimes \xi_{B_q}) \circ \mathrm{flip}^{p,q} \circ \xi_{B_q \otimes \ldots \otimes B_p} \circ (\xi_{B_q} \otimes \ldots \otimes \xi_{B_p}) \circ \mathrm{flip}^{p,q} \end{aligned}$$

The last equality follows from Proposition 2 in [HeKa06], where they prove that $\xi_{B_p \otimes \ldots \otimes B_q} \circ \mathrm{flip}^{p,q} \circ (\xi_{B_q} \otimes \ldots \otimes \xi_{B_p}) = (\xi_{B_p} \otimes \ldots \otimes \xi_{B_q}) \circ \mathrm{flip}^{p,q} \circ \xi_{B_q \otimes \ldots \otimes B_p}$ as part of the result that the outer cactus group action is an involution. This tells us:

$$F(s^{\mathrm{o}}_{p,q}(M)) = s^{\mathrm{o}}_{m+1-q,m+1-p}(F(M)) \tag{12.5}$$

Finally, given any matrix $N \in \Lambda^N B_{n,m}$, there is a matrix $M \in \Lambda^N B_{m,n}$ such that $F(M) = N$. From Theorem 12.2.4, $s^{\mathrm{o}}_{p,q}(M) = s^{\mathrm{i}}_{p,q}(M)$. Applying F to both sides and using Equations 12.4 and 12.5, we get:

$$s^{\mathrm{o}}_{m+1-q,m+1-p}(N) = s^{\mathrm{i}}_{p,q}(N).$$

□

In the next section we discuss one setting in which crystal structures and the action of the cactus group can be applied.

Chapter 13

Shift of argument and Gaudin algebras

One of the main motivations for this work, which also puts it into more context, has to do with $\mathfrak{g}$-crystal structures appearing in the action of certain commutative subalgebras of the universal enveloping algebra $U(\mathfrak{g})$, and the question of how that action relates to the cactus group action. Below we describe the two families of algebras for general $\mathfrak{g}$ and their connection using skew Howe duality for $\mathfrak{gl}_n$ and $\mathfrak{gl}_m$.

13.1 Shift of argument algebras

The symmetric algebra of $\mathfrak{g}$, $S(\mathfrak{g}) \cong \mathbb{C}[\mathfrak{g}]$ has the structure of a Poisson algebra by extending the Lie bracket on $\mathfrak{g}$ via the Leibniz rule. As set earlier, let $\mathfrak{h}$ denote a Cartan subalgebra of $\mathfrak{g}$. Originally defined by Mishchenko and Fomenko in [MF78], there is a family of maximal Poisson-commutative subalgebras of $S(\mathfrak{g})$ indexed by the elements of $\mathfrak{h}$: $\{A_\mu\}_{\mu\in\mathfrak{h}}$. Let $\{F_p\}_{p=1}^{\mathrm{rk}(\mathfrak{g})}$ be a set of algebraically independent generators of the center $Z(S(\mathfrak{g})) = S(\mathfrak{g})^{\mathfrak{g}}$. Then, the shift of argument algebra A_μ is generated by:

$$A_\mu = \langle F_p, \partial_\mu^n F_p \rangle$$

Work by Rybnikov, Shuvalov, Tarasov, and Vinberg (see [R06], [R05], [S02], [Ta00], [Ta02], [Ta03], [V91]) shows that, at least in the case $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ and conjecturally for all reductive, finite-dimensional $\mathfrak{g}$, there exists a compactification and a lifting of this family to a family $\{\mathcal{A}_\mu\}$ of maximal commutative subalgebras of $U(\mathfrak{g})$, indexed by $\mu \in \overline{\mathbb{P}\left(\mathfrak{h}_{\mathbb{R}}^{\mathrm{reg}}\right)}$, where $\mathfrak{h}_{\mathbb{R}}^{\mathrm{reg}}$ are the regular, real points of $\mathfrak{h}$. The connection to the cactus group comes from a result in [DJS03] relating this space to the pure cactus group, for any Lie algebra $\mathfrak{g}$:

$$PJ_{\mathfrak{g}} \cong \pi_1\left(\overline{\mathbb{P}\left(\mathfrak{h}_{\mathbb{R}}^{\mathrm{reg}}\right)}\right)$$

For any irreducible highest weight $\mathfrak{g}$-representation V_λ with corresponding crystal B_λ, work by Rybnikov leads to the result that $\mathcal{A}_\mu \subset U(\mathfrak{g})$ acts on V_λ with simple spectrum, therefore we obtain a covering $E(\lambda) \xrightarrow{\phi_\lambda} \overline{\mathbb{P}\left(\mathfrak{h}_{\mathbb{R}}^{\mathrm{reg}}\right)}$ and so have a monodromy action of the fundamental group $\pi_1(\mathbb{P}\left(\mathfrak{h}_{\mathbb{R}}^{\mathrm{reg}}\right)) = PJ_{\mathfrak{g}}$ on a fiber $\phi_\lambda^{-1}(\mu)$ of $\mathcal{A}_\mu$-eigenlines. Joint work with J. Kamnitzer, L. Rybnikov and A. Weekes, has lead to the conjecture and so far partial proof of the following result, relating this geometric monodromy action of the pure cactus group with the combinatorial cactus group action on B_λ defined in Chapter 10 :

Conjecture 13.1.1 *For $\mathfrak{g}$ a finite-dimensional, complex, reductive Lie algebra, $\lambda \in \Lambda_+$ and $\mu \in \overline{\mathbb{P}(\mathfrak{h}_{\mathbb{R}}^{reg})}$, $B_\lambda \cong \phi_\lambda^{-1}(\mu)$ as $PJ_{\mathfrak{g}}$-sets.*

Note that in the case of $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$, we have $PJ_n \cong \pi_1\left(\overline{\mathbb{P}(\mathfrak{h}_{\mathbb{R}}^{\text{reg}})}\right) \cong \pi_1\left(\overline{M_0^{n+1}(\mathbb{R})}\right)$, where $\overline{M_0^{n+1}(\mathbb{R})}$ is the Deligne-Knudson-Mumford moduli space of stable real curves of genus 0 with $n+1$ marked points, also appearing in the work of Kapranov [Kap93] and Devadoss [D99].

13.2 Gaudin algebras

Given an n-tuple $\underline{z} = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$ of pairwise distinct complex numbers and an orthonormal basis $\{x_k\}_{k=1}^{\dim(\mathfrak{g})}$ of $\mathfrak{g}$ with respect to the Killing form, the quadratic Gaudin hamiltonians are given by the formula:

$$H_i = \sum_{j \neq i} \frac{\sum_{k=1}^{\dim(\mathfrak{g})} x_k^{(i)} x_k^{(j)}}{z_i - z_j}$$

where $x_k^{(i)} = 1 \otimes \ldots \otimes 1 \otimes x_k \otimes 1 \otimes \ldots \otimes 1 \subset U(\mathfrak{g})^{\otimes n}$, x_k being in the i-th position. Feigin, Frenkel, and Reshetikhin [FFR94] prove the existence of a large commutative subalgebra $G_{\underline{z}} \subset (U(\mathfrak{g}))^{\otimes n}$ known as the Gaudin algebra, which is generated by the central elements of $(U(\mathfrak{g}))^{\otimes n}$ and $\{H_i\}_{i=1}^n$ in the case of $\mathfrak{g} = \mathfrak{sl}_2$, and by additional generators known as higher Gaudin Hamiltonians in other cases. If we take $\underline{z} \in \overline{M_0^{n+1}((R))}$, then work in [R14], [MTV10], [MTV14], and [R06] leads to the conclusion that $G(z)$ acts with simple spectrum on $V_{\lambda_1} \otimes \ldots V_{\lambda_n}$, and moreover on $\mathrm{Hom}(V_\nu, V_{\lambda_1} \otimes \ldots V_{\lambda_n})$ for any $(n+1)$-tuple of highest weights $(\nu, \underline{\lambda}) = (\nu, (\lambda_1, \ldots, \lambda_n))$ and the corresponding irreducible highest weight $\mathfrak{g}$-representations. Let $B_\nu, B_{\lambda_1}, \ldots, B_{\lambda_n}$ denote the crystals for those representations. Work in progress by J. Kamnitzer and L. Rybnikov shows that this gives, analogously to the shift of argument setting a covering $E(\nu, \underline{\lambda}) \xrightarrow{\psi_{\nu,\underline{\lambda}}} \overline{M_0^{n+1}((R))}$ and so a monodromy action of the fundamental group $\pi_1(\overline{M_0^{n+1}((R))}) = PJ_n$ on a fiber $\psi_{\nu,\underline{\lambda}}^{-1}(\underline{z})$ of $\mathcal{G}_{\underline{z}}$-eigenlines. Kamnitzer and Rybnikov further relate (in preparation) this action to the outer cactus group action of J_n on an n-tensor product of crystals defined in [HeKa06]:

Conjecture 13.2.1 *For $\mathfrak{g}$ a finite-dimensional, semisimple Lie algebra, $\nu, \underline{\lambda} \subset \Lambda_+$, and $\underline{z} \in \overline{M_0^{n+1}((R))}$, $Hom_{\mathfrak{g}-crystals}(B_\nu, B_{\lambda_1} \otimes \ldots \otimes B_{\lambda_n}) \cong \psi_{\nu,\underline{\lambda}}^{-1}(\underline{z})$ as PJ_n-sets.*

13.3 An application of skew Howe duality

Classical skew Howe duality on $\Lambda^N(\mathbb{C}^n \otimes C^m)$ allows us to consider this space as both a $\mathfrak{gl}_n$- and $\mathfrak{gl}_m$-representation, and we have the decomposition:

$$\Lambda^N(\mathbb{C}^n \otimes \mathbb{C}^m) \cong \bigoplus_{\underline{k} \in \mathbb{N}^m, |\underline{k}| = N} \Lambda^{k_1}\mathbb{C}^n \otimes \ldots \otimes \Lambda^{k_m}\mathbb{C}^n$$

So, given $\underline{z} \in \overline{M_0^{n+1}((R))}$, we can consider the action of both $\mathcal{A}_{\underline{z}} \subset U(\mathfrak{gl}_m)$ and $G_{\underline{z}} \subset U(\mathfrak{gl}_n)^{\otimes m}$ on $W = \Lambda^{k_1}\mathbb{C}^n \otimes \ldots \otimes \Lambda^{k_m}\mathbb{C}^n$. Their images in $\mathrm{End}(W)$ are expected to agree, and so together with Theorem 12.2.4, make Conjectures 13.1.1 and 13.2.1 equivalent for the $(\mathfrak{gl}_n, \mathfrak{gl}_m)$ setting.

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