Highest weights for truncated shifted Yangians

by

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Abstract

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Truncated shifted Yangians are a family of algebras which are conjectured to quantize slices to Schubert varieties in the affine Grassmannian. In this thesis we study the highest weight theory of these algebras, and explore connections with Nakajima quiver varieties and their cohomology. We give a conjectural parametrization of the set of highest weights in terms of product monomial crystals, which are related to Nakajima's monomial crystal. In type A we prove this conjecture.

Our main tool in describing the set of highest weights is the B–algebra, which is a non-commutative generalization of the notion of torus fixed-point subscheme. We give a conjectural presentation for this algebra based on calculations using Yangians, and show how this presentation admits a natural geometric interpretation in terms of the equivariant cohomology of quiver varieties. We conjecture that this gives an explicit presentation for the equivariant cohomology ring of the Nakajima quiver variety of a finite ADE quiver, and show that this conjecture could be deduced from a special case. We give a proof of this conjecture in type A.

This work can be thought of in the context of symplectic duality. In our case, slices to Schubert varieties in the affine Grassmannian are expected to be symplectic dual to Nakajima quiver varieties. The relationship between B–algebras and equivariant cohomology is part of a general conjecture of Nakajima for symplectic dual varieties. These ideas represent a first approximation to expected connections between the category $\mathcal{O}$'s for a symplectic dual pair of varieties.

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Contents

Chapter 1

Introduction

The highest weight theory of semisimple Lie algebras is important in many areas of mathematics, and for good reason: it allows for the classification of finite dimensional irreducible representations, and thereby a complete description of the ubiquitous category of finite dimensional representations. In this thesis we will consider the highest weight theory of a more exotic family of algebras, which ultimately will have multiple ties back to this foundational Lie algebra case.

Let us further discuss the highest weight theory of a semisimple Lie algebra, as this will provide a framework for our more general setting. We can enlarge our scope from the finite dimensional modules by considering the set of Verma modules, or the even larger category $\mathcal{O}$ of Berstein-Gelfand-Gelfand. The combinatorics of the set of simple objects in category $\mathcal{O}$ is simple: they are in bijection with highest weights, which correspond to elements of the dual space $\mathfrak{h}^*$ of a Cartan subalgebra.

This combinatorial question becomes more interesting upon restricting our attention to a block of category $\mathcal{O}$, where one demands that the centre of $U\mathfrak{g}$ acts by a fixed generalized central character. Here, there are only finitely many simple objects, parametrized by an orbit of the Weyl group acting on $\mathfrak{h}^*$ via the "dot action". In particular, the number of simple objects crucially depends on the choice of central character.

Although this combinatorial picture is simple, category $\mathcal{O}$ and its blocks have a very rich structure. One of the most powerful tools in this study is the localization theorem of Beilinson and Berstein, which moves us into the setting of $\mathcal{D}$–modules on the flag variety G/B. Sheaf-theoretic techniques then permit the description of various categorical properties of interest in representation theory.

We can add another piece to this story by thinking of the enveloping algebra $U\mathfrak{g}$ – or rather, its quotient $\mathcal{A}_\xi = U\mathfrak{g}/Z_\xi$ by a central ideal – as a deformation quantization of the coordinate algebra of the nilpotent cone $\mathcal{N} \subset \mathfrak{g}^*$, with its natural Kostant-Kirillov-Souriau Poisson structure. This is compatible with the above $\mathcal{D}$–module picture: the sheaf $\mathcal{D}_\xi$ of twisted differential operators on G/B has $\mathcal{A}_\xi$ as its algebra of global sections. This connection between $\mathcal{D}_\xi$, which can be thought of as a deformation of T^*G/B, and $\mathcal{A}_\xi$ comes from the moment map $\mu : T^*G/B \longrightarrow \mathcal{N}$. The localization theorem above describes the relationship between $\mathcal{D}_\xi$–modules and $\mathcal{A}_\xi$–modules.

1.1 Generalizing the setting

A modern idea in representation theory is to generalize the above picture by replacing the nilpotent cone with a conical symplectic singularity X. Instead of studying $U\mathfrak{g}$, we consider a deformation quantization of its coordinate ring $\mathbb{C}[X]$. There is a family $\mathcal{A}_\xi$ of such algebras, roughly parametrized by $\xi \in H_2(\widetilde{X}, \mathbb{C})$, where $\widetilde{X} \to X$ is a resolution of singularities.

Given the additional data of a Hamiltonian $\mathbb{C}^\times$–action on $\widetilde{X}$, one can define a highest weight theory and category $\mathcal{O}$ for $\mathcal{A}_\xi$, which consists of modules satisfying certain finiteness conditions. Many well-studied structures and properties of the BGG category $\mathcal{O}$ carry over to this setting, and these categories have been the subject of much recent research [BLPW14], [BPW12], [BDGH16], [Los15].

Let us return to the question of the combinatorics of highest weights. In the case of a conical symplectic *resolution*, it is shown in [BLPW14, Proposition 5.3] that for generic ξ there is a bijection

$$\begin{pmatrix}\text{Highest weights} \\ \text{for } \mathcal{A}_\xi\end{pmatrix} \xrightarrow{\sim} \widetilde{X}^{\mathbb{C}^\times} \tag{1.1}$$

1.2 Symplectic duality

A very interesting aspect of the theory outlined above is symplectic duality, a conjecture stating that conical symplectic singularities should arise in pairs X and $X^!$. Moreover, this duality should relate the categories $\mathcal{O}$ and $\mathcal{O}^!$ in a very specific way [BLPW14], [BDGH16].

Symplectic duality has motivation coming from physics, where the pairs X and $X^!$ should arise as the Coulomb and Higgs branches of the moduli space of vacua in a $3d$ $\mathcal{N} =$ gauge theory [Nak15b], [BDG15]. Mathematically, this is a developing story. While the description of the Higgs branch is known, it is only recently that a mathematical construction of the Coulomb branch was proposed, by Braverman, Finkelberg and Nakajima [Nak15b], [BFN16b].

Returning once again to the topic of highest weights, one expected aspect of symplectic duality is the following: for each (integral) choice of ξ there should be a corresponding $\mathbb{C}^\times$–action on a resolution $\widetilde{X}^! \longrightarrow X^!$, and a bijection

$$\begin{pmatrix}\text{Highest weights} \\ \text{for } \mathcal{A}_\xi\end{pmatrix} \xrightarrow{\sim} \pi_0\big((\widetilde{X}^!)^{\mathbb{C}^\times}\big) \tag{1.2}$$

This idea can be upgraded to an isomorphism $B(A_\xi) \cong H^*_{\mathbb{C}^\times}(\widetilde{X}^!)$ conjectured by Nakajima, where $B(\mathcal{A}_\xi)$ is the B–algebra [KTW$^+$15, §1.6]. This conjecture generalizes one of Hikita [Hik15].

1.3 Our setting in this thesis

In this thesis we consider $X = \mathsf{Gr}^{\overline{\lambda}}_\mu$ a transverse slice to a Schubert variety in the affine Grassmannian of a simple algebraic group G of simply-laced type. These varieties are examples of conical symplectic singularities [KWWY14, Theorem 2.7].

The spaces $\mathsf{Gr}^{\overline{\lambda}}_\mu$, and more generally the Schubert varieties $\overline{\mathsf{Gr}^\lambda}$, are important in geometric representation theory. In particular, they are key players in the celebrated geometric Satake equivalence of Ginzburg [Gin00] and Mirkovic-Vilonen [MV07] (see also [Zhu16]). The geometry of $\mathsf{Gr}^{\overline{\lambda}}_\mu$ is strongly related to the weight space $V(\lambda)_\mu$ of the representation of highest weight λ for the Langlands dual $\mathfrak{g}^\vee$.

The affine Grassmannian can also be considered as a partial flag variety for the affine Kac-Moody group $\widehat{G}$, and the $\overline{\mathsf{Gr}^\lambda}$ are its Schubert varieties. From this point of view, there are natural connections between the spaces $\overline{\mathsf{Gr}}^{\overline{\lambda}}_\mu$ and affine Demazure modules and Weyl modules for the current algebra.

The slice $\overline{\mathsf{Gr}}^{\overline{\lambda}}_\mu$ is the Coulomb branch of a "quiver gauge theory' [BFN16a], while the corresponding Higgs branch is the Nakajima quiver variety $\mathcal{M}(\lambda, \mu)$ (or rather, its affinization $\mathcal{M}_0(\lambda, \mu)$). In other words, the symplectic dual of $X = \overline{\mathsf{Gr}}^{\overline{\lambda}}_\mu$ is expected to be $X^! = \mathcal{M}_0(\lambda, \mu)$. Nakajima quiver varieties are also of great importance in geometric representation theory. For example, they can be used to geometrically construct representations of Lie algebras [Nak98], quantum affine algebras [Nak01a] and Yangians [Var00].

1.4 Overview of results

Inspired by the above considerations, the main aim of this work is to describe the set of highest weights for the algebras $Y^\lambda_\mu(\mathbf{R})$, and connect it with the Nakajima quiver variety $\mathcal{M}(\lambda, \mu)$. More generally, we aim to relate the B–algebra of $Y^\lambda_\mu(\mathbf{R})$ and the (equivariant) cohomology of $\mathcal{M}(\lambda, \mu)$.

Many of the results of this thesis come from joint work with J. Kamnitzer, D. Muthiah, P. Tingley, B. Webster, and O. Yacobi [KWWY14],[KTW$^+$15],[KMW16], [KMWY], [WWY]. In particular, a large portion of the content in this thesis follows [KTW$^+$15]. Any mistakes are mine.

In Chapter 2 we follow [KTW$^+$15, §2], and discuss the product monomial crystal $\mathcal{B}(\lambda, \mathbf{R})$. This set plays a key combinatorial role later: we conjecture that it parametrizes highest weights for $Y^\lambda_\mu(\mathbf{R})$.

In Chapter 3, we give an overview of the theory of deformation quantizations of algebras. We also define the notion of B–algebra, give some its properties, and describe the conjectures of Hikita and Nakajima mentioned above.

Chapter 4 provides an overview of Yangians and the affine Grassmannian, covering our results from [KWWY14]. We show in §4.1.4 how to construct a coproduct for shifted Yangians, and prove that this descends to a coproduct for shifted Yangians in type A. In §4.3, we describe certain elements of the Yangian which are important to our study of highest weights, following[KTW$^+$15, §5.3].

In Chapter 5 our goal is to compute the B–algebra for $Y^\lambda_\mu(\mathbf{R})$. We give partial results in this direction as well as a conjectural description of the B–algebra, and prove that it is correct in type A, building on results in [KTW$^+$15, §5]. In §5.4.1, we show that our algebras have natural "coproduct" maps. We also give a presentation, in Theorem 5.2.4, for the fixed-point subscheme of a scheme $\mathcal{G}^\lambda_\mu$ defined in [KWWY14].

Following this, in Chapter 6 we show that our conjectural B–algebra has points in bijection with the elements of weight μ in the product monomial crystal $\mathcal{B}(\lambda, \mathbf{R})$, following [KTW$^+$15, §6]. We show that the "product" structure on the product monomial crystal corresponds to the coproduct on B–algebras.

In Chapter 7, we change gears, and overview the theory of Nakajima quiver varieties. Using results of Maffei [Maf02], we construct vector bundles on quiver varieties (see Definition 7.2.11) and describe their classes in equivariant K-theory. We end the chapter by discussing graded quiver varieties and their connection to the product monomial crystal, following [KTW$^+$15, §7]. In particular, in type A this proves that (1.2) holds.

In Chapter 8 we combine the results from the previous chapters, and compare our conjectural B–algebras with the equivariant cohomology of Nakajima quiver varieties. Our two main theorems show that our conjectural B–algebra surjects onto the equivariant cohomology ring (Theorem 8.3.1), and give a criteria for proving that this surjection is an isomorphism (Theorem 8.3.6). This conjecture holds in type A, and thus gives an explicit presentation for the equivariant cohomology ring of type A Nakajima quiver varieties. Along the way, we also prove that our coproducts on B–algebras agree with Nakajima's geometric coproducts for quiver varieties.

Remark 1.4.1. *The case when G is type A plays a special role throughout this work. In this case, the RTT presentation for the Yangian in type A gives us sufficient algebraic power to completely compute the B-algebra (see Corollary 5.3.15). A suitable replacement is currently lacking in other types. In addition, work in progress [KMWY] shows that the truncated shifted Yangian $Y_\mu^\lambda(\mathbf{R})$ is an honest quantization of $\overline{\mathsf{Gr}^\lambda_\mu}$ in this case, giving us additional leverage.*

In fact, in this setting it is known that the slice $\overline{\mathsf{Gr}^\lambda_\mu}$ is isomorphic to a slice $\mathcal{S}\cap\overline{\mathbb{O}}$ to a nilpotent orbit closure in $\mathfrak{gl}_N$ [MV03] (and also to a Nakajima quiver variety of type A). The varieties $\mathcal{S}\cap\overline{\mathbb{O}}$ are quantized by parabolic W–algebras, as studied by Losev [Los12] and Webster [Web11]. In forthcoming work with Webster and Yacobi, we compare our construction below with the aforementioned works, using the work of Brundan-Kleshchev relating W–algebras and Yangians [BK06].

1.5 Future directions

1.5.1 Categorification

We outline some goals and expected results regarding the category $\mathcal{O}$ for $Y_\mu^\lambda(\mathbf{R})$. Let us denote this category by $\mathcal{O}_\mu^\lambda(\mathbf{R})$ for brevity. In increasing complexity:

(a) **At the level of crystals:** The union

$$\bigsqcup_\mu \Big(\text{Highest weights for } Y_\mu^\lambda(\mathbf{R})\Big),$$

(or equivalently, the union over μ of the set of equivalence classes of simple objects in $\mathcal{O}_\mu^\lambda(\mathbf{R})$) should have the structure of a $\mathfrak{g}^\vee$–crystal, with μ labelling the weight. Moreover, the highest weight elements of this crystal should correspond to finite-dimensional simple modules. We explored this idea in [KTW+15], and in the current thesis.

(b) **At the level of representations:** The sum of Grothendieck groups

$$\bigoplus_\mu K_0\big(\mathcal{O}_\mu^\lambda(\mathbf{R})\big)$$

should have the structure of a $\mathfrak{g}^\vee$–representation $V(\lambda,\mathbf{R})$, with μ labelling the weight spaces.

(c) **At a categorical level:** The product of (derived) categories $\prod_\mu \mathcal{O}_\mu^\lambda(\mathbf{R})$ should carry a categorical $\mathfrak{g}^\vee$–action, as overviewed e.g. in [Kam14].

These three expectations should be compatible, in the sense that (b) decatorifies (c) via taking Grothendieck rings, while (a) is a combinatorial skeleton for (b). In particular, $\mathcal{B}(\lambda,\mathbf{R})$ should label a basis for $V(\lambda,\mathbf{R})$.

Following the symplectic duality narrative, it is our goal to relate all of the above structures with the Nakajima quiver variety $\mathcal{M}(\lambda,\mu)$. Our crystal $\mathcal{B}(\lambda,\mu)$ works towards this goal, by Theorem 7.3.4. Based on this crystal's definition, it is also reasonable to conjecture that the desired representation $V(\lambda, \mathbf{R})$ should be

$$\bigoplus_{\mu} H^0(\lambda, \mu, \mathbf{R})$$

where $\mathcal{M}(\lambda, \mu, \mathbf{R})$ is the graded quiver variety defined in §7.3.1, and $\mathfrak{g}^\vee$ acts by convolution as in [Nak98]. However, it is not clear yet how to relate this representation to the categories $\mathcal{O}_\mu^\lambda(\mathbf{R})$.

1.5.2 Going outside finite ADE

Some of the discussion in this thesis makes sense in greater generality than finite ADE type. In particular, Yangians and Nakajima quiver varieties can be associated to any finite graph without edge loops (a.k.a. symmetric generalized Cartan data). Likewise, one can attempt to define analogues of the varieties $\overline{\mathsf{Gr}^\lambda_\mu}$ in this generality, although this is quite complicated [BF10].

It would be interesting to better understand any analogous connections between these spaces and algebras. For example, one could hope to present cohomology rings for these Nakajima quiver varieties in greater generality. One obstacle in mimicking our present construction in this generality is that Kirwan surjectivity is not known to hold outside of specific cases.

1.5.3 Quantum cohomology

Hikita and Nakajima's conjectures (see §3.4) tell us a certain explicit way to realize cohomology rings. It is natural to ask: is there an analogous construction which yields the quantum cohomology?

1.6 Notation and Conventions

All vector spaces and varieties/schemes will be over the field $\mathbb{C}$, and $\otimes$ denotes $\otimes_{\mathbb{C}}$. All algebras will be associative and unital, and we will denote the coordinate algebra of an affine scheme M by $\mathcal{O}(M)$.

Let G be a connected simple algebraic group of **simply-laced type**, and let $\mathfrak{g}$ be its Lie algebra. Denote its Cartan matrix by $A = (a_{ij})_{i,j\in I}$ where i, j run over the nodes I of its Dynkin diagram. Since G is simply-laced, A is a symmetric matrix. Write $i \sim j$ if i and j are connected in the Dynkin diagram.

Fix a maximal torus and Borel $T \subset B \subset G$, with corresponding Lie algebras $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$. Denote the weight and coweight lattices by

$$X = \mathrm{Hom}(T, \mathbb{C}^\times), \qquad X^\vee = \mathrm{Hom}(\mathbb{C}^\times, T),$$

where we work in the category of algebraic groups. Denote their canonical pairing $\langle\cdot,\cdot\rangle : X^\vee \times X \longrightarrow \mathbb{Z}$. Denote the set of roots by $\Delta = \Delta_+ \sqcup \Delta_-$ and the root lattice by Q. Similarly denote the coroots and coroot lattice by $\Delta^\vee \subset Q^\vee$. We can identify $Q \subset \mathfrak{h}^*$ and $Q^\vee \subset \mathfrak{h}$. Fixing a choice of simple roots $\Pi = \{\alpha_i\}_{i\in I}$, we have $Q = \bigoplus_{i\in I} \mathbb{Z}\alpha_i$. There is a partial order on weights defined by

$$\lambda \geq \mu \quad \Longleftrightarrow \quad \lambda - \mu \in \bigoplus_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i$$

and similarly for coweights.

Let $W = N_G(T)/T$ be the Weyl group, generated by the simple reflections s_i for $i \in I$. For a (co)weight λ, we write $\lambda^* = -w_0(\lambda)$, where $w_0 \in W$ is the longest element.

Choose Chevalley generators for $\mathfrak{g}$, denoted f_i, h_i, e_i for $i \in I$. There is a unique nondegenerate $\mathfrak{g}$–invariant symmetric bilinear form $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$, such that

$$\langle h_i, h_j\rangle = a_{ij}, \qquad \langle e_i, f_j\rangle = \delta_{ij}$$

The restriction of $\langle\cdot,\cdot\rangle$ to $\mathfrak{h}$ is nondegenerate, so it identifies $\mathfrak{h} \cong \mathfrak{h}^*$. Because G is simply-laced, this isomorphism identifies the set of roots and coroots (as well as Q and $Q^\vee$). Moreover, if we define the absolute weight lattice

$$P = \{\lambda \in \mathfrak{h}^* : (\lambda, \alpha) \in \mathbb{Z},\ \forall \alpha \in Q\},$$

then we **can and will identify** $X, X^\vee \subset P$. Under this identification the forms $\langle\cdot,\cdot\rangle$ coincide, hence our abuse of notation. The distinction between the lattices $X, X^\vee$ and P will only be relevant when discussing the affine Grassmannian in Chapter 4, and the above identification greatly simplifies notation elsewhere.

The lattice P has $\mathbb{Z}$–basis given by the fundamental weights $\{\varpi_i\}_{i\in I}$, which are defined by $\langle\varpi_i, \alpha_j\rangle = \delta_{ij}$. Recall that a weight $\lambda \in P$ is called dominant if $\langle\lambda, \alpha_i\rangle \geq 0$ for all $i \in I$, and regular if $\langle\lambda, \alpha\rangle \neq 0$ for all $\alpha \in \Delta$. Denote the set of dominant weights by P_+, and the set of regular weights by P^{reg}.

Remark 1.6.1. *Throughout this thesis, we will frequently consider a pair $\lambda, \mu \in P$, with $\lambda \in P_+$ and $\lambda \geq \mu$. As a convention, we will denote*

$$\lambda = \sum_{i\in I} \lambda_i \varpi_i, \quad \mu = \sum_{i\in I} \mu_i \varpi_i, \quad \lambda - \mu = \sum_{i\in I} m_i \alpha_i,$$

so $\lambda \geq \mu$ *means that all* $m_i \geq 0$.

This data will be relevant in several contexts, and should generally be thought of as specifying the weight space $V(\lambda)_\mu$ *in the irreducible* $\mathfrak{g}$*–representation of highest weight* λ.

Remark 1.6.2. *The assumption that* G *is simply-laced is not essential in some parts of what follows. However, it simplifies certain definitions and is critical for our discussion of quiver varieties.*

Chapter 2

Monomial crystals

2.1 $\mathfrak{g}$–crystals

A $\mathfrak{g}$–**crystal** is a combinatorial object, consisting of a set B along with maps

$$wt : B \longrightarrow P,$$
$$\varepsilon_i, \varphi_i : B \longrightarrow \mathbb{Z}_{\geq 0},$$
$$\tilde{e}_i, \tilde{f}_i : B \longrightarrow B \cup \{0\}$$

satisfying certain axioms. Intuitively, the set B should be thought of as labelling a basis V for a representation of the Lie algebra $\mathfrak{g}$, and the **Kashiwara operators** $\tilde{e}_i, \tilde{f}_i$ should be thought of as corresponding to the action of the Chevalley generators e_i, f_i on V.

This intuition is not completely accurate: there are many crystals that do not correspond to representations of $\mathfrak{g}$. However, for each dominant weight $\lambda \in P$ there is a corresponding crystal $B(\lambda)$, coming from a crystal basis of the representation $V_q(\lambda)$ of $U_q(\lambda)$ (although there are other characterizations). A crystal B is called **normal** if it is isomorphic to a disjoint union of crystals $B(\lambda)$ (for varying λ).

For more details on the theory of crystals and its connections to representation theory, we refer the reader to Joseph [Jos95, Section 5.2].

2.2 The monomial crystal

We let $\mathbb{C}^\lambda = \prod_i \mathbb{C}^{\lambda_i}/S_i$, the set of all collections of multisets of sizes $(\lambda_i)_{i \in I}$. A point in $\mathbb{C}^\lambda$ will be denoted by $\mathbf{R} = (R_i)_{i \in I}$ where each R_i is a multiset of size λ_i, and it will be called a **set of parameters** of weight λ.

Fix a bipartition $I = I_{\overline{0}} \cup I_{\overline{1}}$ of the Dynkin diagram of $\mathfrak{g}$ (where $\overline{0}, \overline{1} \in \mathbb{Z}/2$). We call vertices in $I_{\overline{0}}$ **even** and those in $I_{\overline{1}}$ **odd**. Thus, we say that $i \in I$ and $k \in \mathbb{Z}$ have the **same parity** if $i \in I_{\overline{k}}$.

We say that $\mathbf{R}$ is **integral**, if for all i, all elements of R_i are integers and have the same parity as i. So an integral set of parameters consists of a multiset of λ_i even integers for every $i \in I_0$ and λ_i odd integers for every $i \in I_1$.

2.2.1 Definition of the monomial crystal

Let $\mathcal{B}$ denote the set of all monomials in the variables $y_{i,k}$, for $i \in I$ and $k \in \mathbb{Z}$ of the same parity. Let

$$z_{i,k} = \frac{y_{i,k} y_{i,k+2}}{\prod_{j \sim i} y_{j,k+1}} \tag{2.1}$$

Given a monomial $p = \prod_{i,k} y_{i,k}^{a_{i,k}}$, let

$$wt(p) = \sum_{i,k} a_{i,k} \varpi_i \quad \varepsilon_i^k(p) = -\sum_{l \le k} a_{i,l} \quad \varphi_i^k(p) = \sum_{l \ge k} a_{i,l}$$

and

$$\varepsilon_i(p) = \max_k \varepsilon_i^k(p) \quad \varphi_i(p) = \max_k \varphi_i^k(p)$$

In each case, the max is taken over those integers k of the same parity as i.

We can define the Kashiwara operators on this set of monomials by the rules:

$$\tilde{e}_i(p) = \begin{cases} 0, & \text{if } \varepsilon_i(p) = 0, \\ z_{i,k} p, & \begin{array}{l}\text{otherwise, where } k \text{ is the smallest integer of the same parity as } i \\ \text{such that } \varepsilon_i^k(p) = \varepsilon_i(p)\end{array} \end{cases}$$

$$\tilde{f}_i(p) = \begin{cases} 0, & \text{if } \varphi_i(p) = 0 \\ z_{i,k-2}^{-1} p, & \begin{array}{l}\text{otherwise, where } k \text{ is the largest integer (of the same parity as } i) \\ \text{such that } \varphi_i^k(p) = \varphi_i(p)\end{array} \end{cases}$$

The set $\mathcal{B}$ with these operations is called the **monomial crystal**. The following result is due to Kashiwara [Kas03, Proposition 3.1].

Theorem 2.2.1. *$\mathcal{B}$ is a normal crystal.*

2.3 Product monomial crystals

For any $c \in \mathbb{Z}$ and $i \in I$ of the same parity, the monomial $y_{i,c} \in \mathcal{B}$ is clearly highest weight and we can consider the monomial subcrystal $\mathcal{B}(\varpi_i, c)$ generated by $y_{i,c}$. Since $y_{i,c}$ has weight ϖ_i, we see that $\mathcal{B}(\varpi_i, c) \cong \mathcal{B}(\varpi_i)$.

The fundamental monomial crystals for different c all look the same: they differ simply by translating the variables. In fact, for any complex number $c \in \mathbb{C}$, we can consider $\mathcal{B}(\varpi_i, c)$, the crystal obtained by translating $y_{i,k} \mapsto y_{i,k+c-c_0}$ all variables appearing in all monomials in $\mathcal{B}(\varpi_i, c_0)$ (for c_0 an integer of the same parity as i). This crystal $\mathcal{B}(\varpi_i, c)$ does not sit inside $\mathcal{B}$, but we will need to use it on occasion in this paper.

Given a dominant coweight λ and an integral set of parameters $\mathbf{R}$ of weight λ as above, we define the **product monomial crystal** $\mathcal{B}(\lambda, \mathbf{R})$ by

$$\mathcal{B}(\lambda, \mathbf{R}) = \prod_{i \in I, c \in R_i} \mathcal{B}(\varpi_i, c)$$

In other words, for each parameter $c \in R_i$, we form its monomial crystal $\mathcal{B}(\varpi_i, c)$ and then take the product of all monomials appearing in all these crystals. The order of the product is irrelevant because taking products of monomials is commutative.

Theorem 2.3.1 ([KTW+15, Theorem 2.1]). *$\mathcal{B}(\lambda, \mathbf{R})$ is a subcrystal of $\mathcal{B}$. In particular it is a normal crystal. Moreover, there exists embeddings $\mathcal{B}(\lambda) \subseteq \mathcal{B}(\lambda, \mathbf{R}) \subseteq \otimes_i \mathcal{B}(\varpi_i)^{\otimes \lambda_i}$.*

2.3.1 Multisets and monomials

Given a collection of multisets $\mathbf{S} = (S_i)_{i \in I}$, which we assume satisfy the same parity conditions as above, we can define

$$y_{\mathbf{S}} = \prod_{i \in I, k \in S_i} y_{i,k}, \quad z_{\mathbf{S}} = \prod_{i \in I, k \in S_i} z_{i,k}.$$

From the definition of the monomial crystal, it is easy to see that every monomial p in $\mathcal{B}(\lambda, \mathbf{R})$ is of the form

$$p = y_{\mathbf{R}} z_{\mathbf{S}}^{-1} = \prod_{i,k \in R_i} y_{i,k} \prod_{i,k \in S_i} \frac{\prod_{j \sim i} y_{j,k+1}}{y_{i,k} y_{i,k+2}} \tag{2.2}$$

for some $\mathbf{S}$. Thus an alternative combinatorics for labelling elements of the monomial crystal are these collections of multisets $\mathbf{S}$.

Remark 2.3.2. *For any $p \in \mathcal{B}(\lambda, \mathbf{R})$, $\mathbf{S}$ is uniquely determined. In fact for any collections of multisets $\mathbf{S}$ and $\mathbf{S}'$, $z_{\mathbf{S}} = z_{\mathbf{S}'}$ implies $\mathbf{S} = \mathbf{S}'$.*

2.3.2 Weyl orbit of the highest weight element

Consider a set of parameters $\mathbf{R}$ of weight λ. Since there is a unique highest weight element

$$p_\lambda := y_{\mathbf{R}} \in \mathcal{B}(\lambda, \mathbf{R})_\lambda,$$

it follows from Theorem 2.3.1 that for any extremal weight $w\lambda \in W\lambda$, there is a unique element $p_{w\lambda} \in \mathcal{B}(\lambda, \mathbf{R})$ of weight $w\lambda$. There is a simple inductive expression for $p_{w\lambda}$, which we will prove in §7.3.3 using graded quiver varieties.

Proposition 2.3.3. *Let $i \in I$ be such that $s_i w\lambda < w\lambda$. If $p_{w\lambda} = \prod_{j,k} y_{j,k}^{a_{j,k}}$, then*

$$p_{s_i w\lambda} = p_{w\lambda} \cdot \prod_k z_{i,k-2}^{-a_{i,k}}$$

2.4 The case of $\mathfrak{sl}_2$

For $\mathfrak{g} = \mathfrak{sl}_2$, the fundamental monomial crystal consists of two elements

$$\mathcal{B}(\varpi, c) : \quad y_c \xrightarrow{\tilde{f}} y_{c-2}^{-1}$$

A weight is simply an integer $\lambda \geq 0$, a set of parameters $\mathbf{R}$ in this case is a single multiset of size λ. Let us number its elements $c_1, \ldots, c_\lambda$. The crystal $\mathcal{B}(\lambda, \mathbf{R})$ admits a surjection from the set $\{\pm\}^\lambda$ of tuples: a $+$ in spot j corresponds to choosing $y_{c_j} \in \mathcal{B}(\varpi, c_j)$, and a $-$ corresponds to choosing $y_{c_j-2}^{-1} \in \mathcal{B}(\varpi, c_j)$.

We then multiply these elements together to produce an element of $\mathcal{B}(\lambda, \mathbf{R})$. Note that $\mathcal{B}(\lambda, \mathbf{R})$ is a set: two tuples may produce the same monomial, and these are not distinguished in $\mathcal{B}(\lambda, \mathbf{R})$.

As described in the previous sections, elements $p \in \mathcal{B}(\lambda, \mathbf{R})$ can be written as $p = y_{\mathbf{R}} z_{\mathbf{S}}^{-1}$ for some (single) multiset $\mathbf{S}$. We will see later in §6.3 that this crystal can be described as follows:

$$\mathcal{B}(\lambda, \mathbf{R}) = \left\{ p = y_{\mathbf{R}} z_{\mathbf{S}}^{-1} \in \mathcal{B} : \mathbf{S} + 1 \subset \mathbf{R} \right\}$$

or in other words, "flags" of multisets (see also [KTW$^+$15, Example 2.6] for a more general case).

Chapter 3

Deformation quantization

3.1 Overview

Let A be a $\mathbb{Z}_{\geq 0}$–graded algebra. By a deformation of A, in this thesis we will mean one of two things:

(a) A **filtered deformation:** a filtered algebra $\mathcal{A} = \bigcup_{n \in \mathbb{Z}_{\geq 0}} \mathcal{A}_{\leq n}$ such that the associated graded $\operatorname{gr} \mathcal{A} \cong A$,

(b) A $\mathbb{C}[\hbar]$–**deformation:** a $\mathbb{Z}_{\geq 0}$–graded $\mathbb{C}[\hbar]$–algebra $\mathcal{A}_\hbar$ such that $\mathcal{A}_\hbar / \hbar \mathcal{A}_\hbar \cong A$,

These two definitions are essentially equivalent: from $\mathcal{A}$ we can construct the Rees algebra,

$$\mathcal{A}_\hbar = \operatorname{Rees}(\mathcal{A}) = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \hbar^n \mathcal{A}_{\leq n} \ \subset \mathcal{A}[\hbar],$$

and from $A_\hbar$ we can take $\mathcal{A} = \mathcal{A}_\hbar / (\hbar - 1) \mathcal{A}_\hbar$. In general, deformations of $\mathcal{A}$ can be classified using Hochschild cohomology.

Let A be commutative, and let $\mathcal{A}_\hbar$ be a $\mathbb{C}[\hbar]$–deformation of A. Then A naturally acquires a Poisson bracket, by letting

$$\{a, b\} := \frac{1}{\hbar} [\overline{a}, \overline{b}] \mod \hbar$$

where $\overline{a}, \overline{b} \in \mathcal{A}_\hbar$ are any lifts of a, b. In this case, we call $\mathcal{A}_\hbar$ **almost-commutative**, in the sense that it is commutative mod $\hbar$.

3.2 Conical symplectic singularities

In this section we briefly overview the technical framework for deformation quantization of a class of varieties relevant to our discussion, following [BPW12], [BLPW14].

Let $\widetilde{X}$ be an algebraic variety, with an algebraic symplectic form ω defined on its smooth locus $\widetilde{X}_{reg}$. Suppose that $\widetilde{X}$ is equipped with an action of $\mathbb{S} = \mathbb{C}^\times$, which rescales the symplectic form: $s^* \omega = s^n \omega$ for all $s \in \mathbb{S}$ and some $n \in \mathbb{Z}_{>0}$. Assume also that $\mathbb{S}$ acts on the coordinate algebra $\mathcal{O}(\widetilde{X}) = \Gamma(\widetilde{X}, \mathcal{O})$ with non-negative weights, and that $\mathcal{O}(\widetilde{X})^{\mathbb{S}} = \mathbb{C}$.

Consider the affinization $X := \operatorname{Spec} \mathcal{O}(\widetilde{X})$. Then the above assumption says that there is a grading $\mathcal{O}(X) = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathcal{O}(X)_n$, with $\mathcal{O}(X)_0 = \mathbb{C}$. In other words, X is a *conical* variety: the $\mathbb{S}$–action contracts it to a unique fixed point. $\mathcal{O}(X)$ also acquires the structure of a graded Poisson algebra, from ω.

To classify deformation quantizations of $\mathcal{O}(X)$, it is best to make additional technical assumptions: that $\widetilde{X}$ is smooth (a **conical symplectic resolution**), or that it is a terminalization. The former case has been extensively studied (e.g. [BLPW14]), while the latter is expected to be a good setting for generalization (see [BPW12, Remark 3.3]). In both of these cases,

In the above cases, one can classify $\mathbb{S}$–equivariant deformation quantizations of $\widetilde{X}$ and X. It is shown in [Los12, Theorem 2.3.3] that these are parametrized by $H^2(\widetilde{X}, \mathbb{C})$ in the case of a conical symplectic resolution (and $H^2(\widetilde{X}_{reg}, \mathbb{C})$ in general [BPW12, Proposition 3.4]).

3.2.1 Category $\mathcal{O}$

In the setting of the previous section, there are two competing notions of category $\mathcal{O}$: a geometric one and an algebraic one. These two categories are equivalent in many cases [BLPW14, §3]. In this paper we work with quantizations of coordinate rings, where the algebraic version of $\mathcal{O}$ is more appropriate.

Let $\mathcal{A}$ be a $\mathbb{Z}$–graded algebra. Denote the grading by $\mathcal{A} = \bigoplus_{k\in\mathbb{Z}} \mathcal{A}_k$, so we have $\mathcal{A}_i \cdot \mathcal{A}_j \subset \mathcal{A}_{i+j}$. Consider $\mathcal{A}_{\geq 0} := \bigoplus_{k\geq 0} \mathcal{A}_k$.

Definition 3.2.1. *The* **algebraic category** $\mathcal{O}$ *for* $\mathcal{A}$ *is defined to be the full subcategory of finitely generated* $\mathcal{A}$*–modules for which* $\mathcal{A}_{\geq 0}$ *acts locally finitely.*

In the definition, $\mathcal{A}_{\geq 0}$ acts locally finitely on an $\mathcal{A}$–module M means that for any $v \in M$, the submodule $\mathcal{A}_{\geq 0} \cdot v$ is finite dimensional.

Remark 3.2.2. *In the case when* $\mathcal{A} = U\mathfrak{g}$ *for* $\mathfrak{g}$ *a semisimple Lie algebra, the above definition does not quite agree with the BGG category* $\mathcal{O}$*. Recall that in BGG category* $\mathcal{O}$ *we demand that* $U\mathfrak{b}$ *acts locally finitely and* $U\mathfrak{h}$ *act semisimply. By using the principal gradation of* $U\mathfrak{g}$*, where* $\deg f_i = -1, \deg h_i = 0, \deg e_i = 1$*, we come close to this definition but lose the semisimplicity assumption.*

3.3 B-algebras

Let $\mathcal{A} = \bigoplus_{k\in\mathbb{Z}} \mathcal{A}_k$ be a $\mathbb{Z}$–graded algebra. The following definition appeared in [BLPW14, §5.1] and can be thought of as a generalization of the notion of $\mathbb{C}^\times$–fixed points, see Section 3.3.2 below.

Definition 3.3.1. *The* **B-algebra** *of* $\mathcal{A}$ *is defined to be the algebra* $B(\mathcal{A}) := \mathcal{A}_0 / \sum_{k>0} \mathcal{A}_{-k}\mathcal{A}_k$

Note that $\sum_{k>0} \mathcal{A}_{-k}\mathcal{A}_k$ is always an ideal in $\mathcal{A}_0$, so this is indeed an algebra.

Remark 3.3.2. *More generally let* $\mathcal{A}$ *be a* $\mathbb{Z}$*–graded* S*–algebra, where* S *is a commutative ring which we give degree zero. Then we can define an* S*–algebra* $B(\mathcal{A})$ *as above, and most of the discussion below goes through. We will stick to* $S = \mathbb{C}$ *for simplicity.*

Given $\mathcal{A}$ and $\mathcal{A}'$ two such algebras, recall that the tensor product $\mathcal{A} \otimes \mathcal{A}'$ has a natural algebra structure defined by

$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$$

Moreover, $\mathcal{A} \otimes \mathcal{A}'$ has a $\mathbb{Z}$-grading with

$$(\mathcal{A} \otimes \mathcal{A}')_k = \bigoplus_{i+j=k} \mathcal{A}_i \otimes \mathcal{A}'_j \tag{3.1}$$

Lemma 3.3.3. *Let $\mathcal{A}, \mathcal{A}'$ be as above.*

(a) If $\varphi : \mathcal{A} \to \mathcal{A}'$ is a homomorphism of graded algebras, then the restriction $\varphi : \mathcal{A}_0 \to \mathcal{A}'_0$ induces a homomorphism

$$B(\varphi) : B(\mathcal{A}) \to B(\mathcal{A}')$$

If φ is surjective, then so is $B(\varphi)$.

(b) The inclusion $\mathcal{A}_0 \otimes \mathcal{A}'_0 \hookrightarrow (\mathcal{A} \otimes \mathcal{A}')_0$ induces a natural isomorphism

$$B(\mathcal{A}) \otimes B(\mathcal{A}') \cong B(\mathcal{A} \otimes \mathcal{A}')$$

3.3.1 Relationship to highest weights

Let $\mathcal{A}$ be a $\mathbb{Z}$–graded algebra, and let M be a graded $\mathcal{A}$–module. Let us call $\mathcal{M}$ a **highest weight module** if there is a homogeneous element $v \in M_0$ such that

(1) v is a cyclic vector (i.e. $M = \mathcal{A} \cdot v$),

(2) $M_0 = \mathbb{C}v$,

(3) $\mathcal{A}_k v = 0$ for all $k > 0$.

For a highest weight representation M, the algebra $B(\mathcal{A})$ naturally acts on the line $\mathbb{C}v$. This induces a homomorphism of algebras $\chi : B(\mathcal{A}) \longrightarrow \mathbb{C}$.

Conversely, given a homomorphism $\chi : B(\mathcal{A}) \longrightarrow$ we can construct a corresponding highest weight module as follows. We can consider $\mathbb{C}$ as a module over $\mathcal{A}_{\geq 0} = \bigoplus_{k \geq 0} \mathcal{A}_k$ via the composition

$$\mathcal{A}_{\geq 0} \twoheadrightarrow \mathcal{A}_0 \twoheadrightarrow B(\mathcal{A}) \xrightarrow{\chi} \mathbb{C}$$

and form the induced $\mathcal{A}$–module

$$M(\chi) := \mathcal{A} \otimes_{\mathcal{A}_{\geq 0}} \mathbb{C}$$

This module is in category $\mathcal{O}$ for $\mathcal{A}$, and is called a **Verma module** (or **standard module**). It is the universal highest weight module corresponding to χ, in the sense that all other such modules are quotients of $M(\chi)$.

3.3.2 The commutative case

Assume that A is a $\mathbb{Z}$–graded commutative algebra. In this case, there is an alternative description of the $B(A)$ which is of a geometric flavour. Indeed, a $\mathbb{Z}$–grading of A corresponds to a $\mathbb{C}^\times$–action on $X = \operatorname{Spec}(A)$. In this setting we can consider the fixed point subscheme, which is the spectrum of the coinvariant ring,

$$X^{\mathbb{C}^\times} = \operatorname{Spec}\left(A/\langle za - a : z \in \mathbb{C}^\times, a \in A\rangle\right),$$

where $z \in \mathbb{C}^\times$ acts on A_k as multiplication by z^k. Equivalently, we can say that the coordinate ring

$$\mathcal{O}(X^{\mathbb{C}^\times}) = A/\langle za - a : z \in \mathbb{C}^\times, a \in A\rangle = A/\langle A_{\neq 0}\rangle$$

where $\langle A_{\neq 0}\rangle$ is the ideal generated by all homogeneous elements of non-zero weight. This is a special case of the general notion of fixed-point subscheme, see Fogarty [Fog73].

Lemma 3.3.4. *If A is commutative, then $B(A) \cong \mathcal{O}(X^{\mathbb{C}^\times})$.*

3.3.3 The almost commutative case

We consider now the case when $\mathcal{A}$ is an almost commutative algebra. More precisely, we assume that:

(1) $\mathcal{A} = \bigoplus_{k\in\mathbb{Z}} \mathcal{A}_k$ is $\mathbb{Z}$–graded,

(2) $\mathcal{A} = \bigcup_{n\in\mathbb{Z}_{\geq 0}} \mathcal{F}_n\mathcal{A}$ has a increasing filtration by graded subspaces, with $\mathcal{F}_0\mathcal{A} = \mathbb{C}$,

(3) the associated graded $A := \operatorname{gr}_{\mathcal{F}} \mathcal{A}$ is commutative.

Then A is naturally bi-graded. In geometric terms, we can think that $X = \operatorname{Spec}(A)$ has a pair of commuting $\mathbb{C}^\times$–actions, and is conical with respect to one of them. Following [BLPW14], let's denote the $\mathbb{C}^\times$–action corresponding to (1) by $\mathbb{T}$, and the conical $\mathbb{C}^\times$–action corresponding to the filtration (2) by $\mathbb{S}$.

We will want to consider B–algebras with respect to the $\mathbb{T}$–grading (1). $B(\mathcal{A})$ naturally inherits the quotient filtration from $\mathcal{A}_0 = \bigcup_{n\geq 0} (\mathcal{F}_n\mathcal{A})_0$.

The next result is a variation on [BLPW14, Proposition 5.1]:

Lemma 3.3.5. *(a) There is a surjection $B(A) \twoheadrightarrow \operatorname{gr} B(\mathcal{A})$.*

(b) If A is finitely generated and $X^{\mathbb{T}}$ consists of finitely many closed points, then $B(\mathcal{A})$ is finite dimensional.

Proof. There is a natural surjection from $\operatorname{gr}\mathcal{A}_0 = A_0$ onto $\operatorname{gr} B(\mathcal{A})$, and $\sum_{k>0} A_{-k}A_k$ lies in its kernel. This proves (a). For (b), we claim that $X^{\mathbb{T}}$ has a single closed point: the unique fixed point of $\mathbb{S}$. By the previous section, $B(A)$ defines the closed subscheme $X^{\mathbb{T}}$ of X, and is therefore finite-dimensional since X is finite-type. Now apply (a). ☐

3.4 Conjectures of Hikita and Nakajima

In this section, we will sketch a general framework in which to describe the conjectures of Hikita and Nakajima descibed in the introduction.

The recent work of Braverman, Finkelberg and Nakajima [BFN16b] provides a recipe for constructing a pair X and $X^!$ of varieties, given the data of a reductive group G and a representation $\mathbf{N}$ of G (we use this redefinition of G locally to this section). It is expected that these varieties will be symplectic dual to each other. Since this is not proven, we will content ourselves to list several properties of these varieties.

The space X is called the **Coulomb branch**, and is defined as the spectrum of a certain algebra A; we refer the reader to [BFN16b, §3(iv)] for the definition. Here are some of its properties:

(a) It is a conical Poisson variety, and there is a natural deformation quantization $\mathcal{A}_\hbar$ of its coordinate ring,

(b) There are injections $H^*_G(pt) \hookrightarrow A$ and $H^*_{G\times\mathbb{C}^\times}(pt) \hookrightarrow \mathcal{A}_\hbar$ with (Poisson) commutative images,

(c) There is a natural action of the torus $T = \pi_1(G)^\wedge$ (Pontryagin dual) on A and $\mathcal{A}_\hbar$, and the image of $H^*_G(pt)$ has degree zero,

(d) Suppose that there is an exact sequence

$$1 \longrightarrow G \longrightarrow \widetilde{G} \longrightarrow G_F \longrightarrow 1$$

such that $\mathbf{N}$ extends to a $\widetilde{G}$–module. Then there is a natural family of deformation quantizations of A, with base $\mathfrak{t}_F/W_F$:

$$H^*_{G_F}(pt) \longrightarrow \mathcal{A}_{\hbar,G_F}$$

whose image is *central*.

The space $X^!$, called the **Higgs branch**, is defined to be the (complex) Hamiltonian reduction $T^*\mathbf{N}/\!\!/\!\!/_0 G$. Assume that we have a resolution $\widetilde{X}^!$ given by $T^*\mathbf{N}/\!\!/\!\!/_\theta G$, for some GIT stability parameter $\theta \in \mathfrak{g}^*$. Since $\widetilde{X}^!$ is a Hamiltonian reduction, there is a Kirwan map

$$H^*_G(pt) \longrightarrow H^*\big(\widetilde{X}^*\big)$$

via the inclusion of the θ–semistable locus into $T^*\mathbf{N}$. *We will assume that this map is surjection.*

The following is a generalization of a conjecture of Hikita [Hik15]:

Conjecture 3.4.1. *There is an isomorphism of $H^*_G(pt)$–algebras*

$$\mathcal{O}\big(X^T\big) \cong H^*\big(\widetilde{X}^!\big)$$

By choosing a generic coweight $\rho : \mathbb{C}^\times \to T$, we get an equality $X^T = X^{\mathbb{C}^\times}$. In particular $\mathcal{O}\big(X^T\big) = B\big(\mathcal{O}(X)\big)$, using the B–algebra notation as in §3.3.2.

Using ρ and B–algebras, we will try to generalize this statement to the deformation quantization $\mathcal{A}_{\hbar,G_F}$. Assume there exists $\widetilde{G}$ as in (d) above; let us make the simplification that $\widetilde{G} = G \times G_F$. Then G_F acts on $X^!$ and $\widetilde{X}^!$, commuting with the action of $\mathbb{C}^\times$ which dilates the cotangent direction of $T^*\mathbf{N}$. We will assume that the equivariant Kirwan map

$$H^*_{G\times G_F\times\mathbb{C}^\times}(pt) \longrightarrow H^*_{G_F\times\mathbb{C}^\times}\big(\widetilde{X}^!\big)$$

is surjective. Based on a conjecture of Nakajima [KTW+15, §1.6], we propose the following:

Conjecture 3.4.2. *There is an isomorphism of $H^*_{G\times G_F\times\mathbb{C}^\times}(pt)$–algebras*

$$B\big(\mathcal{A}_{\hbar,G_F}\big) \cong H^*_{G_F\times\mathbb{C}^\times}\big(\widetilde{X}^!\big)$$

Finally, suppose that we take a coweight $\xi : \mathbb{C}^\times \longrightarrow G_F$, and extend this to $\tilde{\xi} : \mathbb{C}^\times \longrightarrow G_F \times \mathbb{C}^\times$ by $z \mapsto (\xi(z), z)$. There is a corresponding action on $\widetilde{X}^!$. There is also an induced map $H^*_{G_F\times\mathbb{C}^\times}(pt) \to H^*_{\mathbb{C}^\times}(pt)$, and specialization

$$\mathcal{A}_{\hbar,\xi} := \mathcal{A}_{\hbar,G_F} \otimes_{H^*_{G_F\times\mathbb{C}^*}(pt)} H^*_{\mathbb{C}^\times}(pt)$$

Let us further specialize $\hbar = 1$, getting an algebra $\mathcal{A}_\xi$.

We can also make analogous specializations in equivariant cohomology. Assuming favourable cohomological properties (e.g. that $\widetilde{X}^!$ is equivariantly formal for $G_F \times \mathbb{C}^\times$), we have a restriction homomorphism

$$H^*_{G_F\times\mathbb{C}^\times}\big(\widetilde{X}^!\big) \otimes_{H^*_{G_F\times\mathbb{C}^\times}(pt)} H^*_{\mathbb{C}^\times}(pt) \cong H^*_{\mathbb{C}^\times}\big(\widetilde{X}^!\big)$$

where on the right $\mathbb{C}^\times$ acts via $\tilde{\xi}$. Furthermore, by restriction to the fixed-point locus we get

$$H^*_{\mathbb{C}^\times}\big(\widetilde{X}^!\big) \hookrightarrow H^*_{\mathbb{C}^\times}\big((\widetilde{X}^!)^{\mathbb{C}^\times}\big)$$

Let us again specialize $\hbar$ to 1. Denote the connected components of the fixed-point locus by

$$(\widetilde{X}^!)^{\mathbb{C}^\times} = \bigsqcup_a C_a$$

By choosing a point $x \in C_a$, we get inclusions $pt = \{x\} \hookrightarrow C_a \hookrightarrow (\widetilde{X}^!)^{\mathbb{C}^\times}$, and induced maps

$$H^*\big((\widetilde{X}^!)^{\mathbb{C}^\times}\big) \longrightarrow H^*(C_a) \longrightarrow \mathbb{C}$$

These maps correspond precisely to the maximal ideals of $H^*\big((\widetilde{X}^!)^{\mathbb{C}^\times}\big)$ (see also §8.3.2 below).

Conjecture 3.4.3. *Upon specializing as above, the map from Conjecture 3.4.2 is induces a bijection on maximal ideals for $B(\mathcal{A}_\xi)$ and $H^*\big((\widetilde{X}^!)\big)$:*

$$\Big(\textit{Highest weights for } \mathcal{A}_\xi\Big) \xrightarrow{\sim} \pi_0\left((\widetilde{X}^!)^{\mathbb{C}^\times}\right)$$

Remark 3.4.4. *(a) In the case of a "quiver gauge theory", it has recently been shown that $X = \overline{\mathsf{Gr}^\lambda_\mu}$ and $X^! = \mathcal{M}_0(\lambda,\mu)$ [BFN16a]. In this sense, the above conjectures should generalize the setup considered in this thesis.*

(b) Nakajima has proposed other (co)homological dualities between the Coulomb and Higgs branches and their quantizations [Nak15a, §4].

Chapter 4

Yangians and the affine Grassmannian

4.1 Yangians

Yangians first arose in the theory of integrable systems, but quickly spread their influence into other areas of mathematics. They are Hopf algebras, arising as deformations of the enveloping algebras $U(\mathfrak{g}[t])$ of current algebras. We refer the reader to Chari and Pressley [CP95] for a more thorough overview of this theory.

There are several presentations for the Yangian, each providing trade-offs by simplifying some aspects of study while complicating others. Drinfeld's original definition [Dri85] in terms of generators $x, J(x)$ permits explicit calculation of the Hopf algebra structure, but is not particularly well-adapted to studying representation theory.

Drinfeld's second presentation [Dri87] in terms of "loop generators" permits the study of representations and algebra structure, but has the drawback that the coproduct and antipode are very difficult to write down explicitly. Since we will almost exclusively work with the algebra structure, it is this presentation which we will make the most use of here (See §4.1.1 below).

Finally there is a third presentation, the so-called RTT presentation, which is historically the oldest is has been extensively studied in type A [Mol07], as well as in other classical types [AMR06]. Here there are few trade-offs: the full Hopf algebra structure is explicit. We exploit this presentation in §4.3.3 and §5.3.3.

4.1.1 The Yangian and the shifted Yangian

The following algebra is a slight variation on Drinfeld's second presentation of the Yangian (see Remark 4.1.2). It was obtained in [KWWY14] by applying Drinfeld-Gavarini duality (a.k.a the quantum duality principle) to the ordinary Drinfeld Yangian.

Definition 4.1.1. *The* **Yangian** Y *of* $\mathfrak{g}$ *is the* $\mathbb{C}$*-algebra with generators* $E_i^{(r)}, F_i^{(r)}, H_i^{(r)}$ *for* $i \in I$, $r \in \mathbb{Z}_{>0}$, *and relations*

$$\begin{aligned}
[H_i^{(r)}, H_j^{(s)}] &= 0, \\
[E_i^{(r)}, F_j^{(s)}] &= \delta_{ij} H_i^{(r+s-1)}, \\
[H_i^{(1)}, E_j^{(s)}] &= a_{ij} E_j^{(s)}, \\
[H_i^{(r+1)}, E_j^{(s)}] - [H_i^{(r)}, E_j^{(s+1)}] &= \frac{a_{ij}}{2}(H_i^{(r)} E_j^{(s)} + E_j^{(s)} H_i^{(r)}), \\
[H_i^{(1)}, F_j^{(s)}] &= -a_{ij} F_j^{(s)}, \\
[H_i^{(r+1)}, F_j^{(s)}] - [H_i^{(r)}, F_j^{(s+1)}] &= -\frac{a_{ij}}{2}(H_i^{(r)} F_j^{(s)} + F_j^{(s)} H_i^{(r)}), \\
[E_i^{(r+1)}, E_j^{(s)}] - [E_i^{(r)}, E_j^{(s+1)}] &= \frac{a_{ij}}{2}(E_i^{(r)} E_j^{(s)} + E_j^{(s)} E_i^{(r)}), \\
[F_i^{(r+1)}, F_j^{(s)}] - [F_i^{(r)}, F_j^{(s+1)}] &= -\frac{a_{ij}}{2}(F_i^{(r)} F_j^{(s)} + F_j^{(s)} F_i^{(r)}), \\
i \neq j, N = 1 - a_{ij} &\Rightarrow \mathrm{sym}[E_i^{(r_1)}, [E_i^{(r_2)}, \cdots [E_i^{(r_N)}, E_j^{(s)}] \cdots]] = 0 \\
i \neq j, N = 1 - a_{ij} &\Rightarrow \mathrm{sym}[F_i^{(r_1)}, [F_i^{(r_2)}, \cdots [F_i^{(r_N)}, F_j^{(s)}] \cdots]] = 0
\end{aligned}$$

where sym *denotes symmetrization over the indices* $r_1, \ldots, r_N$.

Remark 4.1.2.

(a) This is a filtered version of the algebra appearing in [KWWY14, Theorem 3.5]: we have set $\hbar = 1$. *The appropriate filtration on* Y *is defined by setting* $\deg X^{(r)} = r$, *for* $X = E_i, F_i, H_i$. *The algebra from [KWWY14] is the* $\mathbb{C}[\hbar]$*–algebra* $Y_\hbar = \mathrm{Rees}(Y)$.

(b) As noted above, this is a slight modification of the Yangian as usually defined in the literature: the generators are typically taken to be the elements $x^{(r)} := X^{(r-1)}$ *for* $r \in \mathbb{Z}_{\geq 0}$. *The filtration is also typically taken to be defined by* $\deg_{NC} x^{(r)} = r$. *Drinfeld proved that* $\mathrm{gr}_{NC} Y$ *is isomorphic to the co-Poisson-Hopf algebra* $U(\mathfrak{g}[t])$, *and is essentially its unique filtered deformation (see [CP95, §12.1]).*

There is an embedding $U\mathfrak{g} \hookrightarrow Y$, as the subalgebra generated by the modes $X^{(1)}$. The **principal gradation** of Y and Y_μ is defined by the adjoint action of the element $\rho = \sum_{i \in I} \varpi_i \in \mathfrak{h} \subset Y$, or explicitly by

$$\deg F_i^{(r)} = -1, \quad \deg H_i^{(r)} = 0, \quad \deg E_i^{(r)} = 1 \tag{4.1}$$

The filtration from part (a) of the above remark is by graded subspaces.

We will use the notation $Y^>, Y^0, Y^<$ to denote the (unital) subalgebras of Y generated by the $E_i^{(r)}$, the $H_i^{(r)}$, and the $F_i^{(r)}$, respectively. Denote $Y^\geq = Y^> Y^0$ and $Y^\leq = Y^< Y^0$. We will also sometimes denote $Y^0 = \mathbb{C}[H_\bullet^{(\bullet)}]$.

Definition 4.1.3. *Let* $\mu \in P_+$ *be a dominant weight. The* **shifted Yangian** Y_μ *is defined to be the subalgebra of* Y *generated by all* $E_i^{(r)}$, *all* $H_i^{(r)}$, *and the elements* $F_i^{(s)}$ *for* $s > \mu_i$.

Following [KWWY14, §3B], we define a PBW basis for Y as follows. Fix any order on the Dynkin diagram. Then, for each positive root α we define $\check{\alpha}$ to be the smallest simple root such that $\hat{\alpha} = \alpha - \check{\alpha}$

is a postive root. Inductively, we define

$$E_\alpha^{(r)} = [E_{\hat{\alpha}}^{(r)}, E_{\check{\alpha}}^{(1)}], \quad F_\alpha^{(r)} = [F_{\hat{\alpha}}^{(r)}, F_{\check{\alpha}}^{(1)}]$$

This can be made compatible with $Y_\mu \subset Y$: for $s \leq \langle \mu, \alpha \rangle$ we define $F_\alpha^{(s)}$ as above, while for $s > \langle \mu, \alpha \rangle$ we define

$$F_\alpha^{(s)} = [F_{\hat{\alpha}}^{(s - \langle \mu, \check{\alpha} \rangle)}, F_{\check{\alpha}}^{(\langle \mu, \check{\alpha} \rangle + 1)}]$$

Part (1) of the following theorem is due to Levendorski'i [Lev93], while part (2) is [KWWY14, Proposition 3.11].

Proposition 4.1.4.

(1) Ordered monomials in $E_\alpha^{(r)}, H_i^{(r)}, F_\alpha^{(r)}$, for $r > 0$, form a basis for Y,

(2) Ordered monomials in $E_\alpha^{(r)}, H_i^{(r)}, F_\alpha^{(s)}$, for $r > 0, s > \langle \mu, \alpha \rangle$, form a basis for Y_μ.

Y is a Hopf algebra, see for example Chapter 12 in [CP95]. The coproduct on Y is defined by

$$\Delta(X^{(1)}) = X^{(1)} \otimes 1 + 1 \otimes X^{(1)},$$
$$\Delta(H_i^{(2)}) = H_i^{(2)} \otimes 1 + H_i^{(1)} \otimes H_i^{(1)} + 1 \otimes H_i^{(2)} + \sum_{\beta > 0} c_\beta F_\beta^{(1)} \otimes E_\beta^{(1)}$$

for some constants c_β.

4.1.2 Generating series

In the theory of Yangians, as in the theories of loop algebras and affine Lie algebras, it is frequently convenient (and efficient) to encode relations through the use of formal series.

For Y, we define the following series

$$F_i(u) := \sum_{r>0} F_i^{(r)} u^{-r}, \quad H_i(u) := 1 + \sum_{r>0} H_i^{(r)} u^{-r}, \quad E_i(u) := \sum_{r>0} E_i^{(r)} u^{-r}$$

These are elements of the space of formal Laurent series $Y((u^{-1}))$:

4.1.3 The truncated shifted Yangians

Let $\lambda, \mu \in P_+$ be such that $\lambda \geq \mu$. Thus we can write $\lambda - \mu = \sum m_i \alpha_i$ for some $m_i \in \mathbb{Z}_{\geq 0}$.

Given an integral set of parameters $\mathbf{R} = (R_i)_{i \in I}$ of weight λ as in §2.2, consider the associated monic polynomials

$$R_i(u) = \prod_{c \in R_i} (u - \tfrac{1}{2}c)$$

For each $i \in I$, define a series $r_i(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$ by the formula

$$r_i(u) = \frac{R_i(u)}{u^{\lambda_i}} \prod_{c \in R_i} \left(u - \tfrac{1}{2}c\right) \frac{\prod_{j \sim i} (1 - \frac{1}{2}u^{-1})^{m_j}}{(1 - u^{-1})^{m_i}} \tag{4.2}$$

Define elements $A_i^{(s)} \in Y_\mu$ for $i \in I, s \in \mathbb{Z}_{>0}$ by the formula

$$H_i(u) = r_i(u)\frac{\prod_{j\sim i} A_j(u-\frac{1}{2})}{A_i(u)A_i(u-1)} \tag{4.3}$$

where $A_i(u) = 1 + \sum_{s>0} A_i^{(s)} u^{-s}$. By [GKLO05, Lemma 2.1], this uniquely determines the elements $A_i^{(s)} \in \mathbb{C}[H_\bullet^{(\bullet)}]$.

Definition 4.1.5. *The* **truncated shifted Yangian** $Y_\mu^\lambda(\mathbf{R})$ *is defined to be the quotient a the two-sided ideal,*

$$Y_\mu^\lambda(\mathbf{R}) = \widetilde{Y}_\mu / \langle A_i^{(r)} : i \in I, r > m_i\rangle$$

Remark 4.1.6. *Expressing $A_i(u)$ explicitly in terms of $H_i(u)$ is quite complicated. For example, consider $\mathfrak{g} = \mathfrak{sl}_2$. If we ignore the factor $r(u)$, then we have*

$$H(u) = \frac{1}{A(u)A(u-\hbar)}$$

and the first few elements $A^{(r)}$ are

$$\begin{aligned}
A^{(1)} &= -\tfrac{1}{2}H^{(1)},\\
A^{(2)} &= -\tfrac{1}{2}H^{(2)} + \tfrac{3}{8}(H^{(1)})^2 + \tfrac{1}{4}\hbar H^{(1)},\\
A^{(3)} &= -\tfrac{1}{2}H^{(3)} + \tfrac{3}{4}H^{(1)}H^{(2)} - \tfrac{5}{16}(H^{(1)})^3 + \tfrac{1}{2}\hbar H^{(2)} - \tfrac{3}{8}\hbar(H^{(1)})^2
\end{aligned}$$

In fact, in this case $A(u)$ satisfies a regular abelian difference equation:

$$A(u-2) = \frac{H(u)}{H(u-1)}A(u)$$

As such, the dependence of $A(u)$ on $H(u)$ is related to the Γ–function [GL15, Section 4].

It is also interesting to consider a parametric version of the above construction. Let us denote

$$R_i(u) = u^{\lambda_i} + R_i^{(1)} u^{\lambda_i - 1} + \ldots + R_i^{(\lambda_i)}$$

and consider the coefficients $R_i^{(s)}$ are formal parameters. Consider the polynomial ring

$$\widetilde{Y}_\mu := Y_{\hbar,\mu}[R_i^{(s)} : i \in I, 1 \le s \le \lambda_i] = Y_{\hbar,\mu} \otimes \mathbb{C}[R_i^{(s)}]$$

Note the $\hbar$: we've denoted $Y_{\hbar,\mu} = \mathrm{Rees}(Y_\mu)$ as in Remark 4.1.2(a). Define elements $A_i^{(r)} \in \widetilde{Y}_\mu$ by the analogous equation

$$H_i(u) = \frac{R_i(u)}{u^{\lambda_i}} \frac{\prod_{j\sim i}(1-\frac{1}{2}\hbar u^{-1})^{m_j}}{(1-\hbar u^{-1})^{m_i}} \frac{\prod_{j\sim i} A_j(u-\frac{1}{2}\hbar)}{A_i(u)A_i(u-\hbar)} \tag{4.4}$$

and define the algebra

$$Y_\mu^\lambda = \widetilde{Y}_\mu / \langle A_i^{(r)} : i \in I, r > m_i\rangle \tag{4.5}$$

It can be thought of as a family of algebras over the base $\mathbb{C}^\lambda \times \mathbb{C}^\times = \mathrm{Spec}\,\mathbb{C}[R_i^{(s)}, \hbar]$, having $Y_\mu^\lambda(\mathbf{R})$ as its specialization at the point $\mathbf{R} \times \{1\}$.

Y_μ^λ is naturally $\mathbb{Z}_{\geq 0}$–graded, with

$$\deg R_i^{(s)} = \deg X_i^{(s)} = s \text{ for } X = F, H, E, \quad \deg \hbar = 1$$

Remark 4.1.7. *The definition of $Y_\mu^\lambda(\mathbf{R})$ given here differs from that given in [KWWY14, §4C], where it was instead defined as the image of Y_μ in a certain ring of difference operators. In type A, these two definitions are now known to agree (this follows from Theorem 4.2.12). See also [KTW+15, Remark 4.2].*

It was shown recently that the definition given [KWWY14] is indeed a quantization of $\overline{\mathsf{Gr}^\lambda_\mu}$ [BFN16a, Corollary B.26]. Furthermore this algebra can be realized as a convolution algebra [BFN16b], [BFN16a], which provides a wealth of nice algebraic properties e.g. flatness as a module over $\mathbb{C}[H_\bullet^{(\bullet)}]$. In spite of these recent results, in this thesis we have chosen to use Definition 4.1.5 because of its explicit presentation.

4.1.4 Coproducts

We will discuss here a sort of coproduct for the algebras Y_μ^λ. Strictly speaking it is not one, because it does not map into the tensor square. Rather, there is a family of homomorphisms

$$\Delta^{\lambda',\lambda''}_{\mu',\mu''} : Y_\mu^\lambda \longrightarrow Y_{\mu'}^{\lambda'} \otimes_{\mathbb{C}[\hbar]} Y_{\mu''}^{\lambda''}, \tag{4.6}$$

where $\lambda' \geq \mu'$, $\lambda'' \geq \mu''$ are any elements of P_+ such that $\lambda = \lambda' + \lambda''$ and $\mu = \mu' + \mu''$. This map is inspired by a similar one considered by Brundan and Kleshchev for finite W–algebras [BK06, §11]. It seems likely that our map coincides, in the classical limit, with one defined geometrically recently by Braverman, Finkelberg and Nakajima [BFN16a, §2(vi)].

This map can also be defined for the specialized algebras $Y_\mu^\lambda(\mathbf{R})$. In this case, we must partition $\mathbf{R} = \mathbf{R}' \sqcup \mathbf{R}''$ where $\mathbf{R}'$ and $\mathbf{R}''$ are sets of parameters of the appropriate sizes, and take the usual tensor product over $\mathbb{C}$:

$$\Delta^{\lambda',\lambda''}_{\mu',\mu''} : Y_\mu^\lambda(\mathbf{R}) \longrightarrow Y_{\mu'}^{\lambda'}(\mathbf{R}') \otimes Y_{\mu''}^{\lambda''}(\mathbf{R}'')$$

Remark 4.1.8.

(a) Conjecture 6.1.1 below says that the set of highest weights for $Y_\mu^\lambda(\mathbf{R})$ is a weight space $\mathcal{B}(\lambda, \mathbf{R})_\mu$ of the product monomial crystal (as defined in §2). If this conjecture holds, then every simple highest weight module for $Y_\mu^\lambda(\mathbf{R})$ comes by pull-back of the tensor product of simple modules under some Δ as above. In fact, this is true even if we fix a decomposition $\lambda = \lambda' + \lambda''$, since multiplication is surjective by the definition of the product monomial crystal:

$$\mathcal{B}(\lambda', \mathbf{R}') \times \mathcal{B}(\lambda'', \mathbf{R}'') \twoheadrightarrow \mathcal{B}(\lambda, \mathbf{R})$$

From the point of view of the Yangian, the coproducts explain why the set of highest weights has this "product" structure. See also §6.2.4.

(b) Nakajima gave a similar map for quiver varieties [Nak01b]; we review his construction in §7.1.7. There is a strong connection to our map: in §5.4 we will describe an analogous coproduct for B–algebras, and in §8.3.1 we will show that this map on B–algebras agrees with Nakajima's coproduct in cohomology.

We begin by considering the corresponding map for shifted Yangians. Let us introduce the following notation: for $\mu', \mu'' \in P_+$ let $Y_{\mu',\mu''}$ be the subalgebra of Y generated by

$$\begin{aligned} &F_i^{(r)}, \quad \text{for } i \in I, r > \mu_i'', \\ &H_i^{(r)}, \quad \text{for } i \in I, r > 0, \\ &E_i^{(r)}, \quad \text{for } i \in I, r > \mu_i' \end{aligned}$$

(Here we should consider the Rees algebra $Y_\hbar$, but are writing Y for simplicity).

The following lemma is straightforward:

Lemma 4.1.9. *If $\mu = \mu' + \mu''$, then there is an isomorphism $\Psi_{\mu',\mu''} : Y_\mu \longrightarrow Y_{\mu',\mu''}$ defined by*

$$F_i^{(\mu_i + r)} \longmapsto F_i^{(\mu_i'' + r)}, \quad H_i^{(r)} \longmapsto H_i^{(r)}, \quad E_i^{(r)} \longmapsto E_i^{(\mu_i' + r)}$$

for $i \in I$ and $r > 0$.

Recall that the Yangian Y has the structure of a Hopf algebra, as explained at the end of §4.1.

Lemma 4.1.10. *The comultiplication $\Delta : Y \longrightarrow Y \otimes_{\mathbb{C}[\hbar]} Y$ restricts to a map*

$$\Delta : Y_{\mu',\mu''} \longrightarrow Y_{\mu',0} \otimes_{\mathbb{C}[\hbar]} Y_{0,\mu''}$$

Proof. It suffices to show that the generators of $Y_{\mu',\mu''}$ have coproducts in $Y_{\mu'} \otimes_{\mathbb{C}[\hbar]} Y_{\mu''}$.

As in [CP91], the element $S_i = H_i^{(2)} - \frac{1}{2}(H_i^{(1)})^2$ satisfies $[S_i, E_i^{(r)}] = 2E_i^{(r+1)}$ and

$$\Delta(S_i) = S_i \otimes 1 + 1 \otimes S_i + \sum_{\beta > 0} c_\beta F_\beta^{(1)} \otimes E_\beta^{(1)}$$

for some constants c_β. Also, recall that $\Delta(E_i^{(1)}) = E_i^{(1)} \otimes 1 + 1 \otimes E_i^{(1)}$. Using these two facts as well as the fact that Δ is an algebra homomorphism, it follows by induction that

$$\Delta(E_i^{(r)}) \in E_i^{(r)} \otimes 1 + Y^{\leq} \otimes_{\mathbb{C}[\hbar]} Y^{>}$$

Similarly $\Delta(F_i^{(r)}) \in 1 \otimes F_i^{(r)} + Y^{<} \otimes_{\mathbb{C}[\hbar]} Y^{\geq}$. Note that $[F_i^{(s)}, Y^{>}] \subset Y^{\geq}$ for any s, and so we also have

$$\Delta(H_i^{(r)}) = [\Delta(E_i^{(r)}), \Delta(F_i^{(1)})] \in Y^{\leq} \otimes_{\mathbb{C}[\hbar]} Y^{\geq}$$

∎

Using the maps from the above lemmas, consider

$$Y_\mu \xrightarrow{\Psi_{\mu',\mu''}} Y_{\mu',\mu''} \xrightarrow{\Delta} Y_{\mu',0} \otimes_{\mathbb{C}[\hbar]} Y_{0,\mu''} \xrightarrow{\Psi_{\mu',0}^{-1} \otimes \Psi_{0,\mu''}^{-1}} Y_{\mu'} \otimes_{\mathbb{C}[\hbar]} Y_{\mu''} \tag{4.7}$$

Let us denote this composition by $\Delta_{\mu',\mu''}$.

Fix $\lambda = \lambda' + \lambda'', \mu = \mu' + \mu''$ as above. Recall that the algebra Y_μ^λ is a quotient of the polynomial ring $\widetilde{Y}_\mu = Y_\mu \otimes \mathbb{C}[R_i^{(s)}]$. Extend the map $\Delta_{\mu',\mu''}$ to $\widetilde{Y}_\mu$ by

$$R_i(u) \longmapsto R_i'(u) \otimes R_i''(u)$$

These are polynomials of degree λ_i, λ_i' and λ_i'', respectively, and we have added primes to distinguish them (these are not derivatives!).

Conjecture 4.1.11. *The map* $\Delta_{\mu',\mu''} : \widetilde{Y}_\mu \longrightarrow \widetilde{Y}_{\mu'} \otimes_{\mathbb{C}[\hbar]} \widetilde{Y}_{\mu''}$ *descends to*

$$\Delta_{\mu',\mu''}^{\lambda',\lambda''} : Y_\mu^\lambda \longrightarrow Y_{\mu'}^{\lambda'} \otimes_{\mathbb{C}[\hbar]} Y_{\mu''}^{\lambda''}$$

Corollary 4.1.12. *Assume that the above conjecture holds. Upon specializing to parameters* $\mathbf{R} = \mathbf{R}' \sqcup \mathbf{R}''$ *and setting* $\hbar = 1$*, there is a corresponding homomorphism*

$$\Delta_{\mu',\mu''}^{\lambda',\lambda''} : Y_\mu^\lambda(\mathbf{R}) \longrightarrow Y_{\mu'}^{\lambda'}(\mathbf{R}') \otimes Y_{\mu''}^{\lambda''}(\mathbf{R}'')$$

To prove this conjecture, it suffices to show that the images of the generators $A_i^{(s)}$ of the ideal for Y_μ^λ map to zero in the composite

$$\widetilde{Y}_\mu \longrightarrow \widetilde{Y}_{\mu'} \otimes_{\mathbb{C}[\hbar]} \widetilde{Y}_{\mu''} \twoheadrightarrow Y_{\mu'}^{\lambda'} \otimes_{\mathbb{C}[\hbar]} Y_{\mu''}^{\lambda''}$$

It is not currently clear to the author how to do so.

Proposition 4.1.13. *For* $\mathfrak{g} = \mathfrak{sl}_n$*, the conjecture holds.*

We will postpone the proof until §4.3.3, after we have discussed connections to quantum determinants (whose coproducts are easily computable).

We propose one more conjecture, which is based on the idea that the algebras $Y_\mu^\lambda(\mathbf{R})$ should always have at least one *faithful* simple highest weight module. Combined with the conjecture that all highest weights come by pull-back under one of the maps Δ, as described in Remark 4.1.8(a), we propose:

Conjecture 4.1.14. *For any fixed decomposition* $\lambda = \lambda' + \lambda''$*, the sum of the maps* $\Delta_{\mu',\mu''}^{\lambda',\lambda''}$*,*

$$Y_\mu^\lambda \longrightarrow \bigoplus_{\mu=\mu'+\mu''} Y_{\mu'}^{\lambda'} \otimes_{\mathbb{C}[\hbar]} Y_{\mu''}^{\lambda''}$$

is injective.

4.2 Loop groups and the affine Grassmannian

In this section we will overview the definition of the affine Grassmannian, its Schubert varieties, and certain transverse slices to Schubert varieties. A more thorough discussion of the affine Grassmannian can be found in [Fal03], [Zhu16] or [Kum02, Chapter XIII]. We will also describe a conjectural presentation for the coordinate rings of slices to Schubert varieties, following [KWWY14].

Associated to our algebraic group G, we can consider the **loop groups** $G\big(\mathbb{C}((t^{-1}))\big)$, $G\big(\mathbb{C}[t,t^{-1}]\big)$ and $G\big(\mathbb{C}[t]\big)$; the sets of Laurent series, Laurent polynomial, and polynomial points of the scheme G. Concretely, if $G = SL_2$ then we have

$$SL_2\big(\mathbb{C}((t^{-1}))\big) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a,b,c,d \in \mathbb{C}((t^{-1})),\ \det A = 1 \right\}$$

To simplify notation, we will write $G((t^{-1}))$ instead of $G\big(\mathbb{C}((t^{-1}))\big)$, etc.

There is an associated space Gr_{thin}, called the (thin) affine Grassmannian, which is defined as the homogeneous space $\mathsf{Gr}_{thin} := G[t,t^{-1}]/G[t]$. This space has a natural structure of ind-scheme, coming from the functor

$$\mathrm{Alg}_{\mathbb{C}} \longrightarrow \mathrm{Set}, \qquad R \longmapsto G\big(R[t,t^{-1}]\big)/G\big(R[t]\big)$$

or more precisely, the sheafification of this functor with respect to the fpqc topology.

We will work with the larger space called the **thick affine Grassmannian**, defined by $\mathsf{Gr} := G((t^{-1}))/G[t]$. This is naturally a scheme of infinite type, which is the principal reason for working with Gr as opposed to Gr_{thin}: the coordinate rings of certain subvarieties to be more easily described. Note that there is an embedding of the thin affine Grassmannian into the thick affine Grassmannian

$$G[t,t^{-1}]/G[t] \hookrightarrow G((t^{-1}))/G[t].$$

There is an action of the group $\mathbb{C}^*$ on both spaces by loop roation. Any coweight $\lambda \in X^\vee$ for G can be thought of as a $\mathbb{C}[t,t^{-1}]$–point of G, and hence as a $\mathbb{C}((t^{-1}))$–point of G. We denote the image of this point in Gr by t^λ.

4.2.1 Schubert varieties and transverse slices

The group $G[t]$ acts on Gr by left-multiplication. For each dominant coweight $\lambda \in X_+^\vee$, there is a $G[t]$–orbit called a **Schubert cell**:

$$\mathsf{Gr}^\lambda := G[t]t^\lambda$$

The thin affine Grassmannian has a well-known decomposition $\mathsf{Gr}_{thin} = \bigsqcup_{\lambda \in X_+^\vee} \mathsf{Gr}^\lambda$, although this does not hold for Gr. The Schubert cells satisfy a closure relation

$$\overline{\mathsf{Gr}^\lambda} = \bigcup_{\mu \in X_+^\vee, \lambda \geq \mu} \mathsf{Gr}^\mu$$

The closure $\overline{\mathsf{Gr}^\lambda}$ is called a **Schubert variety**, and is a projective variety of dimension $\langle \lambda, 2\rho \rangle$. It is typically singular.

Consider also the group $G_1[[t^{-1}]]$, defined by the exact sequence

$$1 \longrightarrow G_1[[t^{-1}]] \hookrightarrow G[[t^{-1}]] \overset{t \to \infty}{\twoheadrightarrow} G \longrightarrow 1$$

For $\mu \in X_+^\vee$, consider the orbit

$$\mathsf{Gr}_\mu := G_1[[t^{-1}]]t^{w_0\mu}$$

The group $G_1[[t^{-1}]]$ is prounipotent, and its orbits Gr_μ are infinite-dimensional affine spaces.

We can now define an algebraic variety of central focus in this thesis.

Definition 4.2.1. *Let $\lambda, \mu \in X_+^\vee$ be dominant coweights, and assume that $\lambda \geq \mu$. Define*

$$\overline{\mathsf{Gr}^\lambda_\mu} := \overline{\mathsf{Gr}^\lambda} \cap \mathsf{Gr}_\mu$$

This is a **transverse slice** *to Gr^μ inside $\overline{\mathsf{Gr}^\lambda}$, at the point $t^{w_0\mu}$.*

Proposition 4.2.2 ([KWWY14, Proposition 2.1])**.**

(a) $\overline{\mathsf{Gr}^{\lambda}_{\mu}}$ *is an affine variety of (complex) dimension* $\langle \lambda - \mu, 2\rho\rangle$.

(b) *The action of* $\mathbb{C}^\times$ *on* Gr *by loop rotation preserves* $\overline{\mathsf{Gr}^{\lambda}_{\mu}}$, *and contracts it to the unique fixed point* $t^{w_0\mu}$.

4.2.2 Functions and Poisson structure

Let V be a representation of G, and $\beta \in V^*, \gamma \in V$. Then we have a matrix coefficient $\Delta_{\beta,\gamma} \in \mathbb{C}[G]$, defined by

$$g \longmapsto \langle \beta, g\gamma\rangle,$$

where $\langle \cdot,\cdot\rangle$ denotes the pairing of V^* and V. The group $G_1[[t^{-1}]]$ acts on $V[[t^{-1}]]$, and for $s \in \mathbb{N}$ we define $\Delta^{(s)}_{\beta,\gamma} \in \mathbb{C}[G_1[[t^{-1}]]]$ by

$$\Delta_{\beta,\gamma}(g) = \sum_{s\geq 0} \Delta^{(s)}_{\beta,\gamma}(g) t^{-s}$$

where $g \in G_1[[t^{-1}]]$. We refer to these functions as **generalized minors**. Since $G_1[[t^{-1}]$ is a pro-unipotent group, they are generators of the algebra $\mathcal{O}[G_1[[t^{-1}]]]$. Note that the loop rotation action of $\mathbb{C}^\times$ from in the previous section corresponds to the grading defined by $\deg \Delta^{(s)}_{\beta,\gamma} = s$.

Remark 4.2.3. *It is useful to encode functions in the form of formal series, and we will denote*

$$\Delta_{\beta,\gamma}(u) = \sum_{s\geq 0} \Delta^{(s)}_{\beta,\gamma} u^{-s} \in \mathbb{C}\big[G_1[[t^{-1}]]\big][[u^{-1}]]$$

where u is a formal variable. Many computations below will be done in $\mathbb{C}\big[G_1[[t^{-1}]]\big]((u^{-1}))$. *See also Section 4.1.2.*

As described in [KWWY14], the group $G((t^{-1}))$ has the structure of a Poisson-Lie group. This is inherited from the Manin triple $(\mathfrak{g}((t^{-1})), t^{-1}\mathfrak{g}[[t^{-1}]], \mathfrak{g}[t])$. More explicitly this Poisson structure is (quasi)triangular, with the r–matrix

$$r = \frac{\Omega}{v-u} = -\sum_{r=0}^{\infty}\sum_a J_a v^{-r-1} \otimes J^a u^r \quad \in \quad v^{-1}\mathfrak{g}[[v^{-1}]]\widehat{\otimes}\mathfrak{g}[u]$$

where $\Omega = \sum_a J_a \otimes J^a$ is the Casimir 2-tensor and $\{J_a\}, \{J^a\}$ are dual bases for $\mathfrak{g}$ with respect to the form $\langle\cdot,\cdot\rangle$. The Poisson structure on $G((t^{-1}))$ is defined by the difference $\pi = r^L - r^R$ betweeen left and right translation of the element r considered as a bivector at the identity. The inclusion $G_1[[t^{-1}]] \subset G((t^{-1}))$ is Poisson, while $G[t] \subset G((t^{-1}))$ is anti-Poisson (see [CP95, Proposition 1.4.2]). In particular, Gr inherits the structure of a Poisson homogeneous space.

The Poisson bracket of functions on $G_1[[t^{-1}]]$ can be explicitly described:

Proposition 4.2.4 ([KWWY14, Proposition 2.13])**.**
The Poisson bracket on $\mathcal{O}(G_1[[t^{-1}]])$ is determined by the equality of formal Laurent series

$$(u-v)\left\{\Delta_{\beta_1,\gamma_1}(u), \Delta_{\beta_2,\gamma_2}(v)\right\} = \sum_a \Big(\Delta_{J_a\beta_1,\gamma_1}(u)\Delta_{J^a\beta_2,\gamma_2}(v) - \Delta_{\beta_1,J_a\gamma_1}(u)\Delta_{\beta_2,J^a\gamma_2}(v)\Big)$$

The spaces Gr_μ and $\overline{\mathsf{Gr}^{\lambda}_{\mu}}$ are Poisson subvarieties of Gr, as described in [KWWY14, §2C]. In addition, a terminalization $\widetilde{\overline{\mathsf{Gr}^{\lambda}_{\mu}}} \longrightarrow \overline{\mathsf{Gr}^{\lambda}_{\mu}}$ can be defined using convolution varieties. In particular, $\overline{\mathsf{Gr}^{\lambda}_{\mu}}$ fits into the framework outlined in §3.2. A base for the universal deformation quantization of $\mathsf{Gr}^{\overline{\lambda}}_\mu$ is described in [KWWY14, §4D].

Theorem 4.2.5 ([KWWY14, Theorem 2.7])**.** $\mathsf{Gr}^{\overline{\lambda}}_\mu$ *is a*

4.2.3 The algebras $\mathcal{O}(G_1[[t^{-1}]])$ and $\mathcal{O}(\mathsf{Gr}_\mu)$

As noted in [KWWY14, Section 2.G], even if G is not simply-connected, the minors $\Delta^{(r)}_{\beta,\gamma}$ corresponding to the fundamental representations $V(\varpi_i)$ are still well-defined (in fact, $G_1[[t^{-1}]] \cong G_1^{sc}[[t^{-1}]]$). We will use this observation to describe $\mathcal{O}(G_1[[t^{-1}]])$ and $\mathcal{O}(\mathsf{Gr}_\mu)$.

Following [KTW+15, §5.2], we fix a highest weight vector $v_{\varpi_i} \in V(\varpi_i)$, and the associated Shapovalov form on $V(\varpi_i)$. Recall that this is the unique bilinear form $(\cdot,\cdot) : V(\varpi_i) \times V(\varpi_i) \longrightarrow \mathbb{C}$ such that

$$(x\gamma_1, \gamma_2) = (\gamma_1, \sigma(x)\gamma_2), \qquad \forall x \in \mathfrak{g},\ \gamma_1, \gamma_2 \in V(\varpi_i)$$

Here $\sigma = \tau\omega$, where τ is the anti-involution $x \mapsto -x$ of $\mathfrak{g}$, and ω is the Chevalley involution defined by $e_i \mapsto -f_i, f_i \mapsto -e_i$, and $h \mapsto -h$ for $i \in I$ and $h \in \mathfrak{h}$ (see [Kum02, §2.3]).

The Shapovalov form is non-degenerate, and so identifies $V(\varpi_i) \cong V(\varpi_i)^*$. For $\beta, \gamma \in V(\varpi_i)$, we define a function $\Delta^{(r)}_{\beta,\gamma} \in \mathcal{O}(G_1[[t^{-1}]])$ as above, but as the coefficient of t^{-r} in the map $g \mapsto (\beta, g\gamma)$. We note that the formula from Proposition 4.2.4 holds for these functions as well.

Consider the elements of $\mathcal{O}(G_1[[t^{-1}]])$ defined by the series

$$\begin{aligned} f_i(u) &= \sum_{r>0} f_i^{(r)} u^{-r} := \frac{\Delta_{v_{\varpi_i}, f_i v_{\varpi_i}}(u)}{\Delta_{v_{\varpi_i}, v_{\varpi_i}}(u)}, \\ h_i(u) = 1 &+ \sum_{r>0} h_i^{(r)} u^{-r} := \prod_{j\in I} \Delta_{v_{\varpi_j}, v_{\varpi_j}}(u)^{-a_{ji}}, \\ e_i(u) &= \sum_{r>0} e_i^{(r)} u^{-r} := \frac{\Delta_{f_i v_{\varpi_i}, v_{\varpi_i}}(u)}{\Delta_{v_{\varpi_i}, v_{\varpi_i}}(u)} \end{aligned}$$

Theorem 4.2.6 ([KWWY14, Theorem 3.9])**.**

(a) $\mathcal{O}(G_1[[t^{-1}]])$ *is Poisson generated by the elements* $f_i^{(r)}, h_i^{(r)}, e_i^{(r)}$ *for* $i \in I, r \in \mathbb{Z}_{>0}$*,*

(b) $\mathcal{O}(\mathsf{Gr}_\mu)$ *is the subalgebra of* $\mathcal{O}(G_1[[t^{-1}]])$ *Poisson generated by the elements*

$$\begin{aligned} f_i^{(s)}, &\ \textit{for } i \in I, s > \langle \mu^*, \alpha_i \rangle, \\ h_i^{(r)}, e_i^{(r)}, &\ \textit{for } i \in I, r \in \mathbb{Z}_{>0} \end{aligned}$$

4.2.4 A conjectural description of $\mathcal{O}(\mathsf{Gr}^{\overline{\lambda}}_\mu)$

Let $\lambda \geq \mu \in X^\vee_+$. Recall that $\mathsf{Gr}^{\overline{\lambda}}_\mu = \overline{\mathsf{Gr}^\lambda} \cap \mathsf{Gr}_\mu$, and in particular is a closed subvariety of Gr_μ.

Write $\lambda - \mu = \sum_{i\in I} m_i \alpha_i$. Consider the Poisson ideal J^λ_μ in $\mathcal{O}(\mathsf{Gr}_\mu)$ which is generated by the elements $\Delta^{(r)}_{v_{\varpi_i}, v_{\varpi_i}}$ for all $i \in I$ and $r > m_{i^*}$.

Proposition 4.2.7 ([KWWY14, Proposition 2.21])**.** *The vanishing locus of* J^λ_μ *is* $\mathsf{Gr}^{\overline{\lambda}}_\mu$*.*

Conjecture 4.2.8 ([KWWY14, Conjecture 2.20])**.** *The ideal* J^λ_μ *is radical. In particular, it is the ideal of* $\mathsf{Gr}^{\overline{\lambda}}_\mu$*.*

Building upon joint work with Kamnitzer and Muthiah [KMW16], in collaboration with Kamnitzer, Muthiah and Yacobi we are now able to prove the following result:

Theorem 4.2.9 ([KMWY])**.** *If $\mathfrak{g} = \mathfrak{sl}_n$, then Conjecture 4.2.8 holds.*

4.2.5 Quantization of $\overline{\mathsf{Gr}^\lambda_\mu}$

From §4.2, we have following diagrams of varieties and of coordinate rings:

$$\begin{array}{ccc} G_1[[t^{-1}]] & \twoheadrightarrow & \mathsf{Gr}_\mu \\ & & \uparrow \\ & & \overline{\mathsf{Gr}^\lambda_\mu} \end{array} \qquad \begin{array}{ccc} \mathcal{O}\big(G_1[[t^{-1}]]\big) & \hookleftarrow & \mathcal{O}(\mathsf{Gr}_\mu) \\ & & \downarrow \\ & & \mathcal{O}\big(\overline{\mathsf{Gr}^\lambda_\mu}\big) \end{array}$$

The map $G_1[[t^{-1}]] \twoheadrightarrow \mathsf{Gr}_\mu = G_1[[t^{-1}]]t^{w_0\mu}$ is the surjection of this group onto its orbit, and in particular, the corresponding map on functions is the inclusion of the invariants under the stabilizer of $t^{w_0\mu}$. The inclusion $\overline{\mathsf{Gr}^\lambda_\mu} \hookrightarrow \mathsf{Gr}_\mu$ is as a closed subscheme, and the so the corresponding map on functions is a surjection.

Analogously, from §4.1 we have the diagram

$$\begin{array}{ccc} Y & \hookleftarrow & Y_{\mu^*} \\ & & \downarrow \\ & & Y^{\lambda^*}_{\mu^*}(\mathbf{R}) \end{array}$$

This analogy is of course completely intentional.

Remark 4.2.10. *Recall that $\mu^* = -w_0\mu$. The presence of (perhaps) unexpected $*$'s in this section is due to our decision to avoid them in the definition of $Y^\lambda_\mu(\mathbf{R})$, following [KTW+15]. This is in spite of their appearances in $\mathcal{O}(\mathsf{Gr}_\mu)$ in Theorem 4.2.6(b), as well in the ideal for $\mathcal{O}(\overline{\mathsf{Gr}^\lambda_\mu})$ in §4.2.4.*

The following theorem is an amalgam of Theorems 3.9, 3.12 and 4.8 in [KWWY14]:

Theorem 4.2.11 ([KWWY14])**.**

(a) *There is an isomorphisms of graded Poisson algebras $\operatorname{gr} Y \xrightarrow{\sim} \mathcal{O}(G_1[[t^{-1}]])$, defined by*

$$F_i^{(r)} \longmapsto f_i^{(r)}, \quad H_i^{(r)} \longmapsto h_i^{(r)}, \quad E_i^{(r)} \longmapsto e_i^{(r)}$$

for all $i \in I, r \in \mathbb{Z}_{>0}$.

(b) *The map from (a) restricts to an isomorphism of graded Poisson algebras $\operatorname{gr} Y_{\mu^*} \xrightarrow{\sim} \mathcal{O}(\mathsf{Gr}_\mu)$.*

(c) *The map from (b) induces a surjective map of graded Poisson algebras $\operatorname{gr} Y^{\lambda^*}_{\mu^*}(\mathbf{R}) \twoheadrightarrow \mathcal{O}(\overline{\mathsf{Gr}^\lambda_\mu})$, which is an isomorphism modulo the nilradical of the left-hand side. If Conjecture 4.2.8 holds, then it is an isomorphism.*

It is also conjectured that the algebra $Y^{\lambda^*}_{\mu^*}$ from (4.5) is (a base-change of) the universal deformation quantization of $\overline{\mathsf{Gr}^\lambda_\mu}$, see [KWWY14, Conjecture 4.11].

As a consequence of Theorem 4.2.9, we obtain:

Corollary 4.2.12. *For $\mathfrak{g} = \mathfrak{sl}_n$, the above gives an isomorphism of graded Poisson algebras*

$$\operatorname{gr} Y_{\mu^*}^{\lambda^*}(\mathbf{R}) = \mathcal{O}(\overline{\mathsf{Gr}}_{\mu}^{\lambda})$$

4.3 Lifted minors

In this section, we will show how to canonically define lifts $T_{\beta,\gamma}^{(r)} \in Y$ of the minors $\Delta_{\beta,\gamma}^{(r)}$. These elements will play a key role in Chapter 5, where we describe the B-algebra of $Y_\mu^\lambda(\mathbf{R})$.

Of course, there are many choices of lift of a given element $x \in \mathcal{O}(G_1[[t^{-1}]])$ to an element $X \in Y$. By "canonical", we mean that these elements are uniquely determined by imposing:

(1) $T_{v_{\varpi_i}, v_{\varpi_i}}^{(r)} = A_i^{(r)}$ for all $i \in I$, where $A_i^{(r)}$ is defined as in (4.3).

(2) Compatibility with the adjoint action of the Lie algebra $\mathfrak{g}$.

See Corollary 4.3.6 for a precise statement. In Section 4.3.3, we will show that in type A our lifts $T_{\beta,\gamma}^{(r)}$ are related to quantum determinants (Corollary 4.3.10).

4.3.1 Working on the commutative level

Because of the inclusion $U\mathfrak{g} \hookrightarrow Y$, it follows from Theorem 4.2.11 that there Hamiltonian action of the Lie algebra $\mathfrak{g}$ on $\mathcal{O}(G_1[[t^{-1}]])$. Explicitly, this comes from the (comoment) map extending

$$e_i \longmapsto e_i^{(1)} = \Delta_{f_i v_{\varpi_i}, v_{\varpi_i}}^{(1)}, \qquad f_i \longmapsto f_i^{(1)} = \Delta_{v_{\varpi_i}, f_i v_{\varpi_i}}^{(1)}$$

Using Proposition 4.2.4, this corresponds to the action

$$\Big\{ e_i^{(1)}, \Delta_{\beta,\gamma}(u) \Big\} = \Delta_{f_i\beta,\gamma}(u) - \Delta_{\beta,e_i\gamma}(u)$$
$$\Big\{ f_i^{(1)}, \Delta_{\beta,\gamma}(u) \Big\} = \Delta_{e_i\beta,\gamma}(u) - \Delta_{\beta,f_i\gamma}(u)$$

In particular, the Cartan weight of $\Delta_{\beta,\gamma}^{(r)} \in \mathcal{O}(G_1[[t^{-1}]])$ with respect to this action is $\operatorname{wt}(\gamma - \beta)$.

Recall the Chevalley involution ϖ of $\mathfrak{g}$, as defined in §4.2.3. Define another involution ι of $\mathfrak{g}$ by $e_i \mapsto -e_i$, $f_i \mapsto -f_i$, and $h \mapsto h$.

Corollary 4.3.1. *Consider the cyclic representation V_i of $U\mathfrak{g}$ generated by the vector $\Delta_{\varpi_i,\varpi_i}(u) \in \mathcal{O}(G_1[[t^{-1}]])[[u^{-1}]]$. Then there is an isomorphism*

$$\begin{aligned} V_i &\xrightarrow{\sim} (\iota \circ \gamma)^* \Big(V(\varpi_i) \Big) \otimes \iota^* \Big(V(\varpi_i) \Big) \\ \Delta_{\beta,\gamma}(u) &\longmapsto \beta \otimes \gamma \end{aligned}$$

In particular, $V_i \cong V(\varpi_i^*) \otimes V(\varpi_i)$.

4.3.2 Lifting generalized minors

Fix $\lambda \geq \mu \in P_+$, and $\mathbf{R}$ a set of parameters of weight λ. As in equation (4.3), we define elements $A_i^{(s)}$ in $\widetilde{Y}_\mu$. We will think of these elements as being in Y, under the inclusion $Y_\mu \subset Y$.

Following [GKLO05], consider elements

$$B_i(u) = A_i(u)E_i(u) \quad C_i(u) = F_i(u)A_i(u)$$

$$D_i(u) = H_i(u)A_i(u) + F_i(u)A_i(u)E_i(u).$$

Proposition 4.3.2 (Proposition 2.1, [GKLO05])**.** *We have the relations*

$$\begin{aligned}
[A_i(u), A_j(v)] &= 0, \quad \textit{for all } i,j \in I \\
[A_i(u), B_j(v)] &= [A_i(u), C_j(v)] = 0, \\
[B_i(u), B_i(v)] &= [C_i(u), C_i(v)] = 0, \\
[B_i(u), C_j(v)] &= 0, \\
(u-v)[A_i(u), B_i(v)] &= B_i(u)A_i(v) - B_i(v)A_i(u), \\
(u-v)[A_i(u), C_i(v)] &= A_i(u)C_i(v) - A_i(v)C_i(v), \\
(u-v)[B_i(u), C_i(v)] &= A_i(u)D_i(v) - A_i(v)D_i(u), \\
(u-v)[B_i(u), D_i(v)] &= B_i(u)D_i(v) - B_i(v)D_i(u), \\
(u-v)[C_i(u), D_i(v)] &= B_i(u)C_i(v) - B_i(v)C_i(u), \\
(u-v)[A_i(u), D_i(v)] &= B_i(u)C_i(v) - B_i(v)C_i(u)
\end{aligned}$$

Proof. We note that the series considered here differ from those in [GKLO05] by multiplication by constant series. Indeed, there exist unique $s_i(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$ such that

$$r_i(u) = \frac{s_i(u)s_i(u-1)}{\prod_{j\sim i} s_j(u-\frac{1}{2})}$$

with $r_i(u)$ as defined in (4.2). Then the series $s_i(u)^{-1}X_i(u)$ for $X = A, B, C, D$ are exactly those defined in [GKLO05]. With this observation, the claim is immediate from Proposition 2.1 in [GKLO05]. □

Corollary 4.3.3. *For any $i \in I$ we have*

$$\begin{aligned}
(u-v)[A_i(u), E_i(v)] &= A_i(u)\left(E_i(u) - E_i(v)\right), \\
(u-v)[A_i(u), F_i(v)] &= \left(F_i(v) - F_i(u)\right)A_i(u)
\end{aligned}$$

We can explicitly relate the elements $A_i(u), B_i(u)$, etc., to certain functions on $G_1[[t^{-1}]]$.

Proposition 4.3.4 ([KWWY14, Proposition 4.3])**.** *Under the isomorphism of Poisson algebras* $\operatorname{gr} Y \to \mathcal{O}(G_1[[t^{-1}]])$ *from Theorem 4.2.11, we have*

$$\begin{aligned}
A_i(u) &\mapsto \Delta_{v_{\varpi_i}, v_{\varpi_i}}(u), \\
B_i(u) &\mapsto \Delta_{f_i v_{\varpi_i}, v_{\varpi_i}}(u), \\
C_i(u) &\mapsto \Delta_{v_{\varpi_i}, f_i v_{\varpi_i}}(u), \\
D_i(u) &\mapsto \Delta_{f_i v_{\varpi_i}, f_i v_{\varpi_i}}(u)
\end{aligned}$$

Our goal now is to extend the lifts provided by Proposition 4.3.4 to all generalized minors $\Delta_{\beta,\gamma}(u)$, via an analog of Corollary 4.3.1. We recall that the embedding $U\mathfrak{g} \hookrightarrow Y$ is defined by

$$e_i \longmapsto E_i^{(1)}, \quad f_i \longmapsto F_i^{(1)}$$

Proposition 4.3.5. *The cyclic $U\mathfrak{g}$-module generated by the vector $A_i(u) \in Y[[u^{-1}]]$ under the adjoint action is isomorphic to $V(\varpi_i^*) \otimes V(\varpi_i)$, via the map extending $A_i(u) \mapsto v_{-\varpi_i} \otimes v_{\varpi_i}$.*

Corollary 4.3.6. *There are unique lifts $T_{\beta,\gamma}(u) \in Y[[u^{-1}]]$ of the minors $\Delta_{\beta,\gamma}(u)$ for $\beta, \gamma \in V(\varpi_i)$, satisfying*

(1) $T_{v_{\varpi_i}, v_{\varpi_i}}(u) = A_i(u)$,

(2) Compatibility with the adjoint action of $\mathfrak{g}$:

$$[E_j^{(1)}, T_{\beta,\gamma}(u)] = T_{f_j\beta,\gamma}(u) - T_{\beta, e_j\gamma}(u)$$
$$[F_j^{(1)}, T_{\beta,\gamma}(u)] = T_{e_j\beta,\gamma}(u) - T_{\beta, f_j\gamma}(u)$$

We will denote $T_{\beta,\gamma}(u) = \sum_{r\geq 0} T_{\beta,\gamma}^{(r)} u^{-r}$, and call these elements **lifted minors**. Note that $T_{f_i v_{\varpi_i}, v_{\varpi_i}}(u) = B_i(u)$, $T_{v_{\varpi_i}, f_i v_{\varpi_i}}(u) = C_i(u)$, and $T_{f_i v_{\varpi_i}, f_i v_{\varpi_i}}(u) = D_i(u)$, lifting Proposition 4.3.4.

If β and γ are both weight vectors, then the weight of $T_{\beta,\gamma}^{(r)}$ with the respect to the action of $\mathfrak{h} \subset \mathfrak{g}$ is $\mathrm{wt}(\gamma - \beta)$.

To prove the Proposition, we will use an explicit presentation of $V(\varpi_i^*) \otimes V(\varpi_i)$. Consider the $U(\mathfrak{g})$ module

$$N = \mathrm{Ind}_{U(\mathfrak{h})}^{U(\mathfrak{g})} \mathbb{C}_{\mathrm{triv}}$$

where $\mathbb{C}_{\mathrm{triv}} = \mathbb{C}v_0$ is the trivial $\mathfrak{h}$-module. The following result seems to be known to experts.

Lemma 4.3.7. *There is an isomorphism of $\mathfrak{g}$-modules*

$$V(\varpi_i^*) \otimes V(\varpi_i) \cong N / \left\langle e_i^2 v_0, e_j v_0, f_i^2 v_0, f_j v_0 \middle| j \neq i \right\rangle$$

extending $v_{-\varpi_i} \otimes v_{\varpi_i} \mapsto v_0$.

Proof. Using the BGG resolution, we have two short exact sequences:

$$0 \to \mathrm{Im}\Big(\bigoplus_{j\in I} M(s_j \cdot \varpi_i) \Big) \to M(\varpi_i) \to V(\varpi_i) \to 0$$

$$0 \to \mathrm{Im}\Big(\bigoplus_{j\in I} M_{\mathrm{low}}(-s_j \cdot \varpi_i) \Big) \to M_{\mathrm{low}}(-\varpi_i) \to V(\varpi_i^*) \to 0$$

where $M_{\mathrm{low}}(\lambda)$ denotes the lowest weight Verma for $U(\mathfrak{g})$ of weight $\lambda \in \mathfrak{h}^*$. Hence

$$0 \to \begin{array}{c} \mathrm{Im}\left(\bigoplus M_{\mathrm{low}}(-s_j \cdot \varpi_i)\right) \otimes M(\varpi_i) \\ + \\ M_{\mathrm{low}}(-\varpi_i) \otimes \mathrm{Im}\left(\bigoplus M(s_j \cdot \varpi_i)\right) \end{array} \to M_{\mathrm{low}}(-\varpi_i) \otimes M(\varpi_i) \to V(\varpi_i^*) \otimes V(\varpi_i) \to 0$$

The claim follows by observing that, for any $\lambda \in \mathfrak{h}^*$,

$$M_{\mathrm{low}}(-\lambda) \otimes M(\lambda) \cong N, \quad v_{-\lambda} \otimes v_\lambda \mapsto v_0$$

□

Proof of Proposition 4.3.5. By Proposition 4.3.2, $[\mathfrak{h}, A_i(u)] = 0$, $[E_j^{(1)}, A_i(u)] = [F_j^{(1)}, A_i(u)] = 0$ for $j \neq i$, and

$$[E_i^{(1)}, [E_i^{(1)}, A_i(u)]] = [B_i^{(1)}, B_i(u)] = 0, \quad [F_i^{(1)}, [F_i^{(1)}, A_i(u)]] = -[C_i^{(1)}, C_i(u)] = 0$$

Therefore by the previous Lemma, the cyclic module $U(\mathfrak{g})\cdot A_i(u)$ admits a surjection from $V(\varpi_i^*)\otimes V(\varpi_i)$. But the dimension of $U(\mathfrak{g})\cdot A_i(u)$ is at least that of $U(\mathfrak{g})\cdot \Delta_{v_{\varpi_i}, v_{\varpi_i}}(u)$. Indeed, for any fixed s, the module $U(\mathfrak{g})\cdot A_i^{(s)}$ is contained in the filtered piece $Y_{\leq s}$ of Y. The filtered pieces are $U\mathfrak{g}$-invariant, so there is a surjective morphism $U(\mathfrak{g})\cdot A_i^{(s)} \to U(\mathfrak{g})\cdot \Delta^{(s)}_{v_{\varpi_i}, v_{\varpi_i}}$ of $U(\mathfrak{g})$-modules defined by $x\cdot A_i^{(s)} \mapsto x\cdot \Delta^{(s)}_{v_{\varpi_i}, v_{\varpi_i}}$. □

4.3.3 Connection to quantum determinants in type A

In the case $\mathfrak{g} = \mathfrak{sl}_n$ we can give another description of the lifted minors of the previous section: they correspond with the quantum determinants of the RTT presentation.

To begin, we recall the definition of the Yangian $Y(\mathfrak{gl}_n)$ (see [Mol07], [BK05]): it is the associative algebra with generators $t_{i,j}^{(r)}$ for $1 \leq i, j \leq n$ and $r \geq 1$, and relations

$$[t_{i,j}^{(r+1)}, t_{k,l}^{(s)}] - [t_{i,j}^{(r)}, t_{k,l}^{(s+1)}] = t_{k,j}^{(r)} t_{i,l}^{(s)} - t_{k,j}^{(s)} t_{i,l}^{(r)}$$

where we define $t_{ij}^{(0)} = \delta_{i,j}$. These relations can also be encoded in series form:

$$(u-v)[t_{i,j}(u), t_{k,l}(v)] = t_{k,j}(u)t_{i,l}(v) - t_{k,j}(v)t_{i,l}(u) \tag{4.8}$$

where $t_{i,j}(u) = \delta_{ij} + \sum_{r\geq 1} t_{i,j}^{(r)} u^{-r}$.

Given two i-tuples $\mathbf{a} = (a_1, \ldots, a_i)$ and $\mathbf{b} = (b_1, \ldots, b_i)$ of elements of $\{1, \ldots, n\}$, define the quantum determinant (see [Mol07], [BK05, §8])

$$Q_{\mathbf{a},\mathbf{b}}(u) = \sum_{\sigma\in S_i} (-1)^{\sigma} t_{a_{\sigma(1)}, b_1}(u-i+1) t_{a_{\sigma(2)}, b_2}(u-i+2)\cdots t_{a_{\sigma(i)}, b_i}(u)$$

Considering the left action of the symmetric group S_i on i-tuples, for $\pi \in S_i$ we have

$$Q_{\pi\mathbf{a},\mathbf{b}}(u) = (-1)^{\pi} Q_{\mathbf{a},\mathbf{b}}(u) = Q_{\mathbf{a},\pi\mathbf{b}}(u)$$

In particular if the elements from $\mathbf{a}$ are not pairwise distinct then $Q_{\mathbf{a},\mathbf{b}}(u) = 0$, and similarly for $\mathbf{b}$.

Therefore the definition of the quantum determinant factors through the map taking i-tuples to vectors in $\bigwedge^i \mathbb{C}^n$,

$$\mathbf{a} = (a_1, \ldots, a_i) \longmapsto v_{a_1} \wedge \cdots \wedge v_{a_i}$$

where $v_1, \ldots, v_n$ is the usual basis for $\mathbb{C}^n$. Extending the definition by bilinearity, we define $Q_{\beta,\gamma}(u)$ for $\beta, \gamma \in \bigwedge^i \mathbb{C}^n$. We equip $\bigwedge^i \mathbb{C}^n$ with the usual action of gl_n.

Lemma 4.3.8. *Under the action of* $U(\mathfrak{gl}_n) = \langle t_{i,j}^{(1)} | 1 \leq i, j \leq n\rangle \subset Y(\mathfrak{gl}_n)$*, we have*

$$[t_{i,j}^{(1)}, Q_{\beta,\gamma}(u)] = Q_{e_{ij}\beta,\gamma}(u) - Q_{\beta, e_{ji}\gamma}(u)$$

Proof. It is sufficient to consider the case where $\beta = v_{a_1} \wedge \cdots \wedge v_{a_\ell}$ and $\gamma = v_{b_1} \wedge \cdots \wedge v_{b_\ell}$ for some **a**, **b**. Since $e_{ij} v_a = \delta_{aj} v_i$, formula (4.8) yields

$$[t_{i,j}^{(1)}, t_{a,b}(u)] = \delta_{aj} t_{i,b}(u) - \delta_{ib} t_{a,j}(u)$$

proving the case where $\mathbf{a} = (a)$ and $\mathbf{b} = (b)$ have length 1.

For general **a** and **b** of length ℓ, we apply the $\ell = 1$ case and the definition of $Q_{\mathbf{a},\mathbf{b}}(u)$ to compute

$$\begin{aligned}[t_{i,j}^{(1)}, Q_{\mathbf{a},\mathbf{b}}(u)] = &\sum_{r=1}^{\ell} \sum_{\sigma \in S_\ell} (-1)^\sigma \delta_{j,a_{\sigma(r)}} t_{a_{\sigma(1)},b_1}(u - \ell + 1) \cdots t_{i,b_r}(u - \ell + r) \cdots t_{a_{\sigma(\ell)},b_\ell}(u) \\ &- \sum_{r=1}^{\ell} \sum_{\sigma \in S_\ell} (-1)^\sigma \delta_{i,a_r} t_{a_{\sigma(1)},b_1}(u - \ell + 1) \cdots t_{a_{\sigma(r)},j}(u - \ell + r) \cdots t_{a_{\sigma(\ell)},b_\ell}(u)\end{aligned}$$

If there exists p (necessarily unique) such that $j = a_p$, then the first sum can be written as

$$\sum_{r=1}^{\ell} \sum_{\substack{\sigma \in S_\ell, \\ \sigma(r) = p}} (-1)^\sigma t_{a_{\sigma(1)},b_1}(u - \ell + 1) \cdots t_{i,b_r}(u - \ell + r) \cdots t_{a_{\sigma(\ell)},b_\ell}(u)$$

which is equal to $Q_{e_{ij}\beta,\gamma}(u)$. If no such p exists, then the sum is zero as is $e_{ij}\beta$.

Simiarly, if there exists p such that $i = b_p$ then the second sum can be written as

$$\sum_{\sigma \in S_\ell} (-1)^\sigma t_{a_{\sigma(1)},b_1}(u - \ell + 1) \cdots t_{a_{\sigma(r)},j}(u - \ell + r) \cdots t_{a_{\sigma(\ell)},b_\ell}(u)$$

which equals $Q_{\beta,e_{ji}\gamma}(u)$. If no such p exists then the sum is zero and so is $e_{ji}\gamma$. □

To bring our notation in line with the previous sections we also make the identification $V(\varpi_i) \cong \bigwedge^i \mathbb{C}^n$ of $\mathfrak{sl}_n$ representations, so that the weight space corresponding to a weight in $W\varpi_i$ is identified with the span of a basis element $v_{a_1} \wedge \cdots \wedge v_{a_i}$. For example v_{ϖ_i} corresponds to $v_1 \wedge \cdots \wedge v_i$, while $f_i v_{\varpi_i}$ corresponds to $v_1 \wedge \cdots \wedge v_{i-1} \wedge v_{i+1}$.

The following theorem is well-known, see for example [Mol07, Rk. 3.1.8], or [BK08, §8].

Theorem 4.3.9. *There is an embedding* $\widetilde{\phi} : Y(\mathfrak{sl}_n) \to Y(\mathfrak{gl}_n)$*, defined by*

$$\begin{aligned}\widetilde{\phi}(H_i(u)) &= \frac{Q_{v_{\varpi_{i-1}},v_{\varpi_{i-1}}}(u + \frac{i-1}{2}) Q_{v_{\varpi_{i+1}},v_{\varpi_{i+1}}}(u + \frac{i+1}{2})}{Q_{v_{\varpi_i},v_{\varpi_i}}(u + \frac{i-1}{2}) Q_{v_{\varpi_i},v_{\varpi_i}}(u + \frac{i+1}{2})}, \\ \widetilde{\phi}(E_i(u)) &= Q_{f_i v_{\varpi_i},v_{\varpi_i}}(u + \tfrac{i-1}{2}) Q_{v_{\varpi_i},v_{\varpi_i}}(u + \tfrac{i-1}{2})^{-1}, \\ \widetilde{\phi}(F_i(u)) &= Q_{v_{\varpi_i},v_{\varpi_i}}(u + \tfrac{i-1}{2})^{-1} Q_{v_{\varpi_i},f_i v_{\varpi_i}}(u + \tfrac{i-1}{2})\end{aligned}$$

Moreover, the center of $Y(\mathfrak{gl}_n)$ *is freely generated by the coefficients of* $Q_{v_{\varpi_n},v_{\varpi_n}}(u)$*, and there is an isomorphism of algebras*

$$Y(\mathfrak{gl}_n) \cong Y(\mathfrak{sl}_n) \otimes Z(Y(\mathfrak{gl}_n))$$

Consider the trivial central character χ_0 for $Y(\mathfrak{gl}_n)$, and the central quotient $Y(\mathfrak{gl}_n)/\chi_0$. Define ϕ to be the composition

$$Y(\mathfrak{sl}_n) \xrightarrow{\widetilde{\phi}} Y(\mathfrak{gl}_n) \twoheadrightarrow Y(\mathfrak{gl}_n)/\chi_0$$

Define series $s_1(u), \dots, s_{n-1}(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$ by the property

$$r_i(u) = \frac{s_i(u)s_i(u-1)}{s_{i-1}(u-\frac{1}{2})s_{i+1}(u-\frac{1}{2})}, \quad i = 1, \dots, n-1,$$

viewing $s_0(u) = s_n(u) = 1$. Here $r_i(u)$ is the power series defined using R_i in equation (4.2). It follows from Lemma 2.1 in [GKLO05] that these series are uniquely defined.

The result of this chapter provides the promised relationship between lifted minors and quantum determinants:

Corollary 4.3.10. *$\phi : Y(\mathfrak{sl}_n) \to Y(\mathfrak{gl}_n)/\chi_0$ is an isomorphism. Moreover,*

$$\phi(T_{\beta,\gamma}(u)) = s_i(u) Q_{\beta,\gamma}(u + \tfrac{i+1}{2})$$

for all $\beta, \gamma \in V(\varpi_i)$.

Proof. ϕ is an isomorphism by the previous theorem. For the second claim, we first note that

$$\phi(H_i(u)) = r_i(u) \frac{\phi(A_{i-1}(u-\frac{1}{2}))\phi(A_{i+1}(u-\frac{1}{2}))}{\phi(A_i(u))\phi(A_i(u-1))}$$

By the uniqueness of factorization from Lemma 2.1 in [GKLO05], it follows that $\phi(A_i(u)) = s_i(u)Q_{v_{\varpi_i},v_{\varpi_i}}(u+\frac{i+1}{2})$. Since $\phi(E_i^{(1)}) = t_{i+1,i}^{(1)}$ and $\phi(F_i^{(1)}) = t_{i,i+1}^{(1)}$, the claim follows by applying Corollary 4.3.6 and Lemma 4.3.8. □

We have not made use of the Hopf algebra structure of $Y(\mathfrak{gl}_n)$ so far. A useful property of quantum minors is that their coproducts are easily computable [Mol07, Proposition 1.6.9], and in particular,

$$\Delta\big(Q_{v_{\varpi_i},v_{\varpi_i}}(u)\big) = \sum_\gamma Q_{v_{\varpi_i},\gamma}(u) \otimes Q_{\gamma,v_{\varpi_i}}(u)$$

where γ ranges over a basis for $V(\varpi_i)$. Moreover, the map ϕ is an isomorphism of Hopf algebras [Mol07, Proposition 1.8.4].

Using this, we can now give a proof of Proposition 4.1.13:

Proof of Proposition 4.1.13. We prove the version involving specialization $\mathbf{R} = \mathbf{R}' \sqcup \mathbf{R}''$; the parametric version is similar. In this case there are corresponding series $s_i(u), s_i'(u)$ and $s_i''(u)$, as above, and it is not hard to check that $s_i(u) = s_i'(u)s_i''(u)$. Hence, in $Y(\mathfrak{gl}_n)$ we have

$$\Delta\big(s_i(u)Q_{v_{\varpi_i},\varpi_i}(u)\big) = \sum_\gamma s_i'(u)Q_{v_{\varpi_i},\gamma}(u) \otimes s_i''(u)Q_{\gamma,v_{\varpi_i}}(u)$$

This continues to hold in $Y(\mathfrak{gl}_n)/\chi_0$. Applying ϕ^{-1}, by Corollary 4.3.10 we have

$$\Delta\big(A_i(u)\big) = \sum_\gamma T'_{v_{\varpi_i},\gamma}(u) \otimes T''_{\gamma,v_{\varpi_i}}(u) \tag{4.9}$$

where Δ now denotes the coproduct of Y. The above elements are defined using data $\mathbf{R}, \mathbf{R}'$ and $\mathbf{R}''$, respectively; we have added primes to distinguish them.

We must take into account the map (4.7). Note that $\Psi_{\mu',\mu''}(A_i(u)) = A_i(u)$. Also,

$$T'_{v_{\varpi_i},\gamma}(u) \in U(\mathfrak{b}^-) \cdot A'_i(u), \qquad T''_{\gamma,v_{\varpi_i}}(u) \in U(\mathfrak{b}) \cdot A''_i(u),$$

where by $U(\mathfrak{b}^-)$ we mean the subalgebra of Y generated by the elements $H_i^{(1)}, F_i^{(1)}$. The map $\Psi_{\mu',0}^{-1}$ is the identity on $U(\mathfrak{b}^-)$, and in particular, the coefficient of u^{-r} in $\Psi_{\mu',0}^{-1}\big(T'_{v_{\varpi_i},\gamma}(u)\big)$ is an element of $I_{\mu'}^{\lambda'}$ for $r > m'_i$. Similarly for $T''_{\gamma,v_{\varpi_i}}(u)$.

Therefore, the image of the right-hand side of (4.9) in $Y_{\mu'}^{\lambda'}(\mathbf{R}') \otimes Y_{\mu''}^{\lambda''}(\mathbf{R}'')$ is a polynomial in u^{-1} of degree $m'_i + m''_i = m_i$. Hence the elements $A_i^{(r)}$ map to zero for $r > m_i$, and therefore I_μ^λ is in the kernel of this map. This proves the claim. $\square$

Chapter 5

The B–algebra of $Y_\mu^\lambda(\mathbf{R})$

Our goal in this chapter will be to describe generators for the ideal of the B–algebra for $Y_\mu^\lambda(\mathbf{R})$, with respect to the principal gradation (4.1). We give a conjectural presentation and have partial results in all types, but are only able to give a complete picture in type A.

In Chapter 6, we will use this conjectural presentation to connect the highest weights of $Y_\mu^\lambda(\mathbf{R})$ with the product monomial crystals from §2.3.

5.1 Highest weights for truncated shifed Yangians

By the PBW Theorem 4.1.4, there is a decomposition analogous to that in the Harish-Chandra isomorphism for $U\mathfrak{g}$:

$$Y = \mathbb{C}[H_\bullet^{(\bullet)}] \oplus \sum_{i,r} \left(F_i^{(r)} Y + Y E_i^{(r)} \right)$$

and similarly for Y_μ. We denote the projection $\Pi : Y_\mu \twoheadrightarrow \mathbb{C}[H_\bullet^{(\bullet)}]$.

The following lemma is straightforward using the PBW theorem.

Lemma 5.1.1. *The B–algebras of Y and Y_μ are both canonically isomorphic to $\mathbb{C}[H_\bullet^{(\bullet)}]$, via its inclusion into the zero graded component:*

$$\mathbb{C}[H_\bullet^{(\bullet)}] \subset (Y_\mu)_0 \twoheadrightarrow B(Y_\mu)$$

Intuitively, this result simply says that for any choice of series $J_i(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$ we can find a module with highest weight vector $\mathbf{1}$ such that

$$H_i(u)\mathbf{1} = J_i(u)\mathbf{1}$$

for all $i \in I$. We can think of (the coefficients of) the tuple $(J_i(u))_{i\in I} \in \operatorname{MaxSpec} \mathbb{C}[H_\bullet^{(\bullet)}]$. At least one such module always exists: the Verma module $M(J)$, formed by induction from the subalgebra $Y^{\geq 0}$.

The situation for $Y_\mu^\lambda(\mathbf{R})$ is much more delicate, however. Given J as above, we can form the module

$$M(J, \mathbf{R}) := Y_\mu^\lambda(\mathbf{R}) \otimes_{Y_\mu} M(J)$$

This module represents the functor on the category of $Y_\mu^\lambda(\mathbf{R})$–modules, which takes a module M to the subspace of all vectors $m \in M$ such that

$$H_i(u)m = J_i(u)m, \qquad E_i^{(r)} m = 0$$

However, $M(J,\mathbf{R})$ will typically be zero. More precisely, recall that $Y_\mu^\lambda(\mathbf{R})$ is the quotient of the shifted Yangian Y_μ by the ideal

$$I_\mu^\lambda := \left\langle A_i^{(r)} : i \in I, r > m_i \right\rangle$$

which depends on $\mathbf{R}$ in a subtle way. Then, the module $M(J,\mathbf{R}) \neq 0$ if and only if $I_\mu^\lambda M(J) \neq M(J)$.

The B–algebra provides us with a more convenient way of encoding this condition: $M(J,\mathbf{R}) \neq 0$ iff $J \in \operatorname{MaxSpec} B\big(Y_\mu^\lambda(\mathbf{R})\big)$ (see §3.3.1). Here is a first pass at expressing this algebra:

Lemma 5.1.2. *There is a canonical isomorphism*

$$B\big(Y_\mu^\lambda(\mathbf{R})\big) \cong \mathbb{C}[H_\bullet^{(\bullet)}]/\Pi(I_\mu^\lambda)$$

Proof. By Lemma 3.3.3, there is a surjection $B(Y_\mu) \twoheadrightarrow B(Y_\mu^\lambda(\mathbf{R}))$. Using the previous lemma, consider the composition

$$\mathbb{C}[H_\bullet^{(\bullet)}] \xrightarrow{\sim} B(Y_\mu) \twoheadrightarrow B\big(Y_\mu^\lambda(\mathbf{R})\big)$$

Its kernel is precisely $\Pi(I_\mu^\lambda)$. □

We can allow ourselves slightly more flexibility by working with the larger left ideal $L_\mu^\lambda = Y \cdot I_\mu^\lambda$ in Y, instead of with I_μ^λ. This is justified by the following result:

Lemma 5.1.3. *The B–algebra for $Y_\mu^\lambda(\mathbf{R})$ can be computed using L_μ^λ:*

$$B\big(Y_\mu^\lambda(\mathbf{R})\big) \cong \mathbb{C}[H_\bullet^{(\bullet)}]/\Pi(L_\mu^\lambda)$$

Proof. We need to show that $\Pi(I_\mu^\lambda) \supset \Pi(L_\mu^\lambda)$. By Proposition 4.1.4, the PBW bases for Y and Y_μ are compatible: any $x \in Y$ can be written uniquely as a sum $x = \sum_a F_a x_a$, where each $x_a \in Y_\mu$ and F_a is a monomial in the generators $F_\alpha^{(r)}$ with $r \leq \langle \mu, \alpha \rangle$.

For $x \in L_\mu^\lambda$, we can assume that all $x_a \in I_\mu^\lambda$. But under Π, any summand where F_a is non-trivial is sent to zero. Hence $\Pi(x) \in \Pi(I_\mu^\lambda)$. □

5.2 The commutative case

In this section we will give a conjectural description of the fixed point subscheme $(\overline{\mathsf{Gr}_\mu^\lambda})^{\mathbb{C}^\times}$, corresponding to the conjectural description of $\mathcal{O}(\overline{\mathsf{Gr}_\mu^\lambda})$ from §4.2.4. The $\mathbb{C}^\times$–action we consider corresponds to the principal gradation on Y; geometrically, this is the left action of $\mathbb{C}^\times$ on Gr the under coweight $\rho^\vee : \mathbb{C}^\times \hookrightarrow T$.

Our description will be completely parallel to the non-commutative case discussed in the previously section, and for that reason we will omit proofs. Nonetheless we hope that this geometric picture better motivates our calculations for the Yangian throughout this chapter. Because of §3.3.2, we will continue to use the B–algebra terminology.

Consider the surjection

$$p : G_1[[t^{-1}]] \twoheadrightarrow \mathsf{Gr}_\mu, \quad g \longmapsto g t^{w_0 \mu}$$

From Theorem 4.2.11 (a) and the PBW Theorem 4.1.4 for the Yangian, we identify the corresponding map on functions as the inclusion of polynomial rings

$$\mathbb{C}[f_\alpha^{(s)}, h_i^{(r)}, e_\alpha^{(r)} : r > 0, s > \langle \mu, \alpha\rangle] \hookrightarrow \mathbb{C}[f_\alpha^{(r)}, h_i^{(r)}, e_\alpha^{(r)} : r > 0]$$

In particular, p is locally-trivial fibration.

In §4.2.4, we gave a conjectural description of $\mathcal{O}(\overline{\mathsf{Gr}_\mu^\lambda})$: the algebra $\mathcal{O}(\mathsf{Gr}_\mu)/J_\mu^\lambda$, for a certain Poisson ideal J_μ^λ. Let us denote $\mathcal{G}_\mu^\lambda = \operatorname{Spec}\big(\mathcal{O}(\mathsf{Gr}_\mu)/J_\mu^\lambda\big)$ to distinguish it from $\overline{\mathsf{Gr}_\mu^\lambda}$. Consider now the diagram

$$\begin{array}{ccc} G_1[[t^{-1}]] & \xrightarrow{\;p\;} & \mathsf{Gr}_\mu \\ \uparrow & & \uparrow \\ p^{-1}(\mathcal{G}_\mu^\lambda) & \longrightarrow & \mathcal{G}_\mu^\lambda \end{array}$$

Passing from $\mathcal{G}_\mu^\lambda$ to $p^{-1}(\mathcal{G}_\mu^\lambda)$ makes it simpler to a give a presentation for its defining ideal; in the notation of the previous section, the ideal of $p^{-1}(\mathcal{G}_\mu^\lambda)$ corresponds to L_μ^λ while J_μ^λ corresponds to I_μ^λ. In particular, we proved the following result in [KMW16]:

Proposition 5.2.1 ([KMW16, Proposition 9.4])**.** *The ideal of $p^{-1}(\mathcal{G}_\mu^\lambda)$ in $\mathcal{O}(G_1[[t^{-1}]])$ is generated (as an ordinary ideal) by the elements*

$$\Delta_{\beta,\gamma}^{(s)} \qquad \text{for } s > m_{i^*} + \langle \mu^*, \varpi_i - \operatorname{wt}\gamma\rangle$$

over all $i \in I$ and weight vectors $\beta, \gamma \in V(\varpi_i)$.

Our goal now is to compute the torus-fixed point subscheme of $\mathcal{G}_\mu^\lambda$. The following two lemmas are analogous to Lemmas 5.1.1 and 5.1.3:

Lemma 5.2.2. *The B–algebras of $\mathcal{O}(G_1[[t^{-1}]])$ and $\mathcal{O}(\mathsf{Gr}_\mu)$ are both canonically isomorphic to $\mathbb{C}[h_\bullet^{(\bullet)}]$, via its inclusion into the zero graded component*

$$\mathbb{C}[h_\bullet^{(\bullet)}] \hookrightarrow \mathcal{O}(\mathsf{Gr}_\mu)_0 \twoheadrightarrow B\big(\mathcal{O}(\mathsf{Gr}_\mu)\big)$$

Correspondingly, the map

$$T_1[[t^{-1}]] \hookrightarrow (\mathsf{Gr}_\mu)^{\mathbb{C}^\times}, \quad g \longmapsto g t^{w_0\mu}$$

is an isomorphism.

Lemma 5.2.3. *As quotients of $\mathbb{C}[h_\bullet^{(\bullet)}]$, the B–algebras of $\mathcal{O}(\mathcal{G}_\mu^\lambda)$ and $\mathcal{O}\big(p^{-1}(\mathcal{G}_\mu^\lambda)\big)$ are equal.*

Let us introduce some notation. By the previous two lemmas, we get generators for the B–algebra of $\mathcal{O}\big(\mathcal{G}_\mu^\lambda\big)$ by restricting the generators $\Delta_{\beta,\gamma}^{(s)}$ from Proposition 5.2.1 to $T_1[[t^{-1}]] \subset G_1[[t^{-1}]]$. This restriction is zero unless $\beta, \gamma \in V(\varpi_i)$ have the same weight ν, and in this case the result depends only on ν. Let us denote this function by $h_\nu^{(s)} \in \mathbb{C}[h_\bullet^{(\bullet)}]$; explicitly, it is defined by

$$T_1[[t^{-1}]] \ni g \longmapsto \nu(g) = \sum_{s\geq 0} h_\nu^{(r)}(g) t^{-r} \in 1 + t^{-1}\mathbb{C}[[t^{-1}]]$$

We can now prove the main result of this section:

Theorem 5.2.4. *The B–algebra of $\mathcal{O}(\mathcal{G}^\lambda_\mu)$ is isomorphic to*

$$\mathbb{C}[h_\bullet^{(\bullet)}]/\left\langle h_\nu^{(r)} : i \in I,\ \nu \in W\varpi_i,\quad r > m_{i^*} + \langle \mu^*, \varpi_i - \nu\rangle \right\rangle \tag{5.1}$$

Proof. Suppose that ν is a weight of $V(\varpi_j)$, but $\nu \notin W\varpi_j$. For $\tilde{\nu}$ the dominant W-translate of ν,

$$\langle \lambda - \mu, \tilde{\nu}^*\rangle + \langle \mu^*, \tilde{\nu} - \nu\rangle = \langle \lambda, \tilde{\nu}^*\rangle - \langle \mu, \nu^*\rangle \leq \langle \lambda, \varpi_j^*\rangle - \langle \mu, \nu^*\rangle = \langle \lambda - \mu, \varpi_j^*\rangle + \langle \mu^*, \varpi_j - \nu\rangle$$

since $\tilde{\nu} \leq \varpi_j$. From Proposition 5.2.1, for $h_\nu^{(s)}$ there is a the bound on s corresponding to $V(\varpi_j)$ as well as to $V(\tilde{\nu})$ (approriately extending the statement of the proposition if $\tilde{\nu}$ is not fundamental). The above inequality says that $\tilde{\nu}$'s bound is stronger than ϖ_j's.

If $\tilde{\nu}$ is not a fundamental weight, we write it as a sum thereof. Then ν is a sum of element in the respective Weyl orbits of these fundamentals, and $h_\nu^{(r)}$ decomposes as a product. The bounds on the various factors in this product combine (sum) to imply that for $h_\nu^{(r)}$.

□

By Theorem 4.2.9, in type A we have $\mathcal{G}^\lambda_\mu = \overline{\mathsf{Gr}^\lambda_\mu}$, and so the above gives an explicit description of the B–algebra of $\mathcal{O}(\overline{\mathsf{Gr}^\lambda_\mu})$. In general, the following conjecture is implied by Conjecture 4.2.8, but is possibly weaker:

Conjecture 5.2.5. *For all $\mathfrak{g}$, there is an isomorphism between the B–algebras of $\mathcal{O}(\mathcal{G}^\lambda_\mu)$ and $\mathcal{O}(\overline{\mathsf{Gr}^\lambda_\mu})$.*

5.3 The non-commutative case

Here is an outline of this section.

1. We give a presentation for L^λ_μ as a left Y-ideal

$$L^\lambda_\mu = Y S^\lambda_\mu$$

   where S^λ_μ is an explicit set of elements related to the lifted minors $T_{\beta,\gamma}(u)$ from §4.3. In this step, we have partial results for general type, but complete results only in type A.

2. In type A, we use the generators of L^λ_μ to give generators of the ideal defining the B-algebra.

3. We compute the images of the images of certain lifted minors $T_{\gamma,\gamma}(u)$ under the map Π.

The goal of Step 1 should be thought of as a Yangian version of Proposition 5.2.1, which we achieve in Step 2. Meanwhile, Step 3 is close in spirit to Theorem 5.2.4.

5.3.1 Relation with higher generators

Since the elements $T_{\beta,\gamma}^{(s)}$ are defined using the action of $U(\mathfrak{g})$, and it is not clear what relations they will have with the higher level generators $E_i^{(s)}, F_i^{(s)}, H_i^{(s)} \in Y$. Most important to us will be the interaction with the elements $F_i^{(s)}$ for $s > \mu_i$, since in studying Y_μ we are limited to these modes.

First some notation. For a pair $(\mathbf{i}, \mathbf{s})$ consisting of tuples $\mathbf{i} = (i_1, \dots, i_d) \in I^d$ and $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{Z}^d_{>0}$ of any length $d \geq 0$, we will denote

$$F_{\mathbf{i}}^{(\mathbf{s})} = F_{i_1}^{(s_1)} \cdots F_{i_d}^{(s_d)} \in Y$$

For two such pairs, we will write $(\mathbf{i}', \mathbf{s}') < (\mathbf{i}, \mathbf{s})$ if there exists an increasing sequence $1 \leq a_1 < \ldots < a_{d'} \leq d$, where $d' < d$, such that $\mathbf{i}' = (i_{a_1}, \ldots, i_{a_{d'}})$ and $\mathbf{s}' = (s_{a_1}, \ldots, s_{a_{d'}})$. In this case, $F_{\mathbf{i}'}^{(\mathbf{s}')}$ is a subword of $F_{\mathbf{i}}^{(\mathbf{s})}$.

Let $(\mathbf{i}, \mathbf{s})$ be given. We will denote

$$f_{\mathbf{i}} = f_{i_d} \cdots f_{i_1} \in U(\mathfrak{g})$$

Note that the order is reversed. We will also denote the analogous composition of Kashiwara operators $\tilde{f}_{\mathbf{i}} = \tilde{f}_{i_d} \cdots \tilde{f}_{i_1}$. For any $i \in I$ consider the monomial crystal $\mathcal{B}(\varpi_i, 0)$ from §2.3. If $\tilde{f}_{\mathbf{i}}(y_{i,0}) \neq 0$, define $k_1, \ldots, k_d$ by

$$y_{i,0} \xrightarrow{\tilde{f}_{i_1}} z_{i_1,k_1-2}^{-1} y_{i,0} \xrightarrow{\tilde{f}_{i_2}} z_{i_2,k_2-2}^{-1} z_{i_1,k_1-2}^{-1} y_{i,0} \xrightarrow{\tilde{f}_{i_3}} \cdots \xrightarrow{\tilde{f}_{i_d}} z_{i_d,k_d-2}^{-1} \cdots y_{i_1,k_1-2}^{-1} y_{i,0}$$

Definition 5.3.1. *Let $G_{i,\mathbf{i}}^{\mathbf{s}}(u) \in \mathbb{C}[u]$ be the polynomial*

$$G_{i,\mathbf{i}}^{\mathbf{s}}(u) = \begin{cases} u^{m_i}(u + \frac{1}{2}k_1)^{s_1-1} \cdots (u + \frac{1}{2}k_d)^{s_d-1}, & \text{if } \tilde{f}_{\mathbf{i}}(y_{i,0}) \neq 0, \\ 0, & \text{if } \tilde{f}_{\mathbf{i}}(y_{i,0}) = 0. \end{cases}$$

for $k_1, \ldots, k_d$ as above.

For any formal series $X(u) = \sum_{s \in \mathbb{Z}} X^{(s)} u^{-s} \in Y[[u, u^{-1}]]$, let us denote the principal part by $\underline{X(u)} = \sum_{s>0} X^{(s)} u^{-s}$. The following basic fact will be useful later.

Lemma 5.3.2. *Suppose $X(u) \in Y((u^{-1}))$. For any polynomial $f(u) \in \mathbb{C}[u]$, the coefficients of $\underline{f(u)X(u)}$ are linear combinations of the coefficients of $\underline{X(u)}$.*

We will also denote

$$Y \cdot X(u) = \operatorname{span}_{\mathbb{C}} \left\{ y X^{(s)} : y \in Y, s \in \mathbb{Z} \right\} [[u^{-1}]]$$

Recall from Corollary 4.3.6 that we have

$$[\cdots[[A_i(u), F_{i_1}^{(1)}], F_{i_2}^{(1)}], \cdots, F_{i_d}^{(1)}] = T_{v_{\varpi_i}, f_{i_d} \cdots f_{i_1} v_{\varpi_i}}(u) = T_{v_{\varpi_i}, f_{\mathbf{i}} v_{\varpi_i}}(u)$$

This leads to the formula

$$A_i(u) F_{\mathbf{i}}^{(1,\ldots,1)} = T_{v_{\varpi_i}, f_{\mathbf{i}} v_{\varpi_i}}(u) + \sum_{(\mathbf{i}',\mathbf{s}')<(\mathbf{i},\mathbf{s})} Y \cdot T_{v_{\varpi_i}, f_{\mathbf{i}'} v_{\varpi_i}}(u)$$

We will need to generalize this formula where $(1, \ldots, 1)$ is replaced by an arbitrary sequence $\mathbf{s}$.

Proposition 5.3.3. *Suppose that ϖ_i is a minuscule coweight. Then for any $(\mathbf{i}, \mathbf{s})$, we have*

$$\underline{u^{m_i} A_i(u)} F_{\mathbf{i}}^{(\mathbf{s})} = \underline{G_{i,\mathbf{i}}^{\mathbf{s}}(u) T_{v_{\varpi_i}, f_{\mathbf{i}} v_{\varpi_i}}(u)} + \sum_{(\mathbf{i}',\mathbf{s}')<(\mathbf{i},\mathbf{s})} Y \cdot \underline{G_{i,\mathbf{i}'}^{\mathbf{s}'}(u) T_{v_{\varpi_i}, f_{\mathbf{i}'} v_{\varpi_i}}(u)} \tag{5.2}$$

(In this proposition, we include the case where $f_{\mathbf{i}} v_{\varpi_i} = 0$. In this case, $f_{\mathbf{i}} v_{\varpi_i} = 0$, so even though $G_{i,\mathbf{i}}^{\mathbf{s}}(u)$ is not defined, by convention we have $G_{i,\mathbf{i}}^{\mathbf{s}}(u) T_{v_{\varpi_i}, f_{\mathbf{i}} v_{\varpi_i}}(u) = 0$.)

Note that Proposition 5.3.3 is equivalent to a similar statement where equation (5.2) is replaced by

$$\underline{G^{\mathbf{s}}_{i,\mathbf{i}}(u)T_{v_{\varpi_i},f_{\mathbf{i}}v_{\varpi_i}}(u)} = \underline{u^{m_i}A_i(u)}F_{\mathbf{i}}^{(\mathbf{s})} + \sum_{(\mathbf{i}',\mathbf{s}')<(\mathbf{i},\mathbf{s})} Y\cdot \underline{u^{m_i}A_i(u)}F_{\mathbf{i}'}^{(\mathbf{s}')} \tag{5.3}$$

Proof of the proposition. The proof will proceed by induction on d, which is the length of $\mathbf{i}$ and $\mathbf{s}$.

(i) The case $d=1$: Using Proposition 4.3.3, for any $s>0$ we have the relation

$$[A_i(u),F_i^{(s+1)}] = \Big(\sum_{r>0} F_i^{(s+r)}u^{-r}\Big)A_i(u)$$

Now, by Corollary 4.3.6, $T_{v_{\varpi_i},v_{\varpi_i}}(u) = A_i(u)$ and $T_{v_{\varpi_i},f_iv_{\varpi_i}}(u) = [A_i(u),F_i^{(1)}] = F_i(u)A_i(u)$. Using this, rewrite the above relation:

$$[A_i(u),F_i^{(s+1)}] = u^sT_{v_{\varpi_i},f_iv_{\varpi_i}}(u) - (F_i^{(1)}u^{s-1}+F_i^{(2)}u^{s-2}+\ldots+F_i^{(s)})T_{v_{\varpi_i},v_{\varpi_i}}(u)$$

Rearranging terms, multiplying by u^{m_i}, and taking principal parts, it follows that

$$\begin{aligned}\underline{u^{m_i}A_i(u)}F_i^{(s+1)} &= \underline{u^{s+m_i}T_{v_{\varpi_i},f_iv_{\varpi_i}}(u)} - \underline{(F_i^{(1)}u^{s-1}+\ldots+F_i^{(s)}+F_i^{(s+1)})u^{m_i}T_{v_{\varpi_i},v_{\varpi_i}}(u)} = \\ &= \underline{u^{s+m_i}T_{v_{\varpi_i},f_iv_{\varpi_i}}(u)} + Y\cdot\underline{u^{m_i}T_{v_{\varpi_i},v_{\varpi_i}}(u)}\end{aligned}$$

Since $\tilde{f}_i(y_{i,0}) = z_{i,-2}^{-1}y_{i,0}$ in $\mathcal{B}(\varpi_i,0)$, so for $\mathbf{i}=(i)$ and $\mathbf{s}=(s+1)$ we have $G^{\mathbf{s}}_{i,\mathbf{i}}(u) = u^{m_i+s}$. Hence the claim holds in the case $\mathbf{i}=(i)$. If $\mathbf{i}=(j)$ where $j\neq i$, then $[A_i(u),F_j^{(s+1)}]=0$ and $f_jv_{\varpi_i}=0$, so the claim also holds.

(ii) The inductive step in d: Assume the claim holds for all sequences $(\mathbf{i}',\mathbf{s}')$ of length $\leq d$. We will prove the claim for $(\mathbf{j},\mathbf{r})$ where $\mathbf{j}=(i_1,\ldots,i_d,j)$, $\mathbf{r}=(s_1,\ldots,s_d,r)$, by induction on r. We continue to denote $\mathbf{i}=(i_1,\ldots,i_d)$ and $\mathbf{s}=(s_1,\ldots,s_d)$.

When $r=1$, there is no contribution coming from f_j to any of the polynomials $G^{\mathbf{s}'}_{i,\mathbf{i}'}(u)$, as the exponent of any corresponding linear factor is 0. We multiply equation 5.2 on the right by $F_j^{(1)}$ to give

$$\underline{u^{m_i}A_i(u)}F_{\mathbf{j}}^{(\mathbf{r})} = \underline{G^{\mathbf{s}}_{i,\mathbf{i}}(u)T_{v_{\varpi_i},f_{\mathbf{i}}v_{\varpi_i}}(u)F_j^{(1)}} + \sum_{(\mathbf{i}',\mathbf{s}')<(\mathbf{i},\mathbf{s})} Y\cdot\underline{G^{\mathbf{s}'}_{i,\mathbf{i}'}(u)T_{v_{\varpi_i},f_{\mathbf{i}'}v_{\varpi_i}}(u)F_j^{(1)}}$$

Now we apply Corollary 4.3.6 to prove the statement.

For the inductive step in r, we consider two cases.

(a) The case $f_{\mathbf{j}}v_{\varpi_i}=0$: In this case, we prove a stronger version of equation (5.2):

$$\underline{u^{m_i}A_i(u)}F_{\mathbf{i}}^{(\mathbf{s})}F_j^{(r)} = \sum_{(\mathbf{i}',\mathbf{s}')<(\mathbf{i},\mathbf{s})} Y\cdot\underline{G^{\mathbf{s}'}_{i,\mathbf{i}'}(u)T_{v_{\varpi_i},f_{\mathbf{i}'}v_{\varpi_i}}(u)}$$

Again the case $r=1$ holds using Corollary 4.3.6. To prove the inductive step, recall the element S_i from Lemma **??**, which satisfies $[S_i,F_j^{(r)}] = -a_{ij}F_j^{(r+1)}$. Assuming the above formula holds up to r, we wish to conclude the case $r+1$. Bracketing both sides of the formula for r by S_j, we get

$$\underline{u^{m_i}A_i(u)}[S_j,F_{\mathbf{i}}^{(\mathbf{s})}]F_j^{(r)} - 2\underline{u^{m_i}A_i(u)}F_{\mathbf{i}}^{(\mathbf{s})}F_j^{(r+1)} = \sum_{(\mathbf{i}',\mathbf{s}')<(\mathbf{i},\mathbf{s})} Y\cdot[S_j,\underline{G^{\mathbf{s}'}_{i,\mathbf{i}'}(u)T_{v_{\varpi_i},f_{\mathbf{i}'}v_{\varpi_i}}(u)}]$$

since $[S_j, A_i(u)] = 0$. The second term on the left is the one we want to express. The inductive assumption applies to the first term on the left, and for the right hand side we claim that

$$[S_j, \underline{G^{\mathbf{s}'}_{i,\mathbf{i}'}(u)T_{v_{\varpi_i},f_{\mathbf{i}'}v_{\varpi_i}}(u)}] = \sum_{(\mathbf{i}'',\mathbf{s}'')<(\mathbf{i}',\mathbf{s}')} Y \cdot \underline{G^{\mathbf{s}''}_{i,\mathbf{i}''}(u)T_{v_{\varpi_i},f_{\mathbf{i}''}v_{\varpi_i}}(u)}$$

Indeed, this follows by the inductive assumption on d and equation (5.3), since the action of $[S_j,\cdot]$ on products $F^{(\mathbf{s}'')}_{\mathbf{i}''}$ shifts exponents.

(b) The case $f_{\mathbf{j}}v_{\varpi_i} \neq 0$: Choose p maximal such that $i_p = j$ or $i_p \sim j$. Thus

$$F^{(\mathbf{s})}_{\mathbf{i}} = F^{(s_1)}_{i_1} \cdots F^{(s_p)}_{i_p} F^{(r)}_j F^{(s_{p+1})}_{i_{p+1}} \cdots F^{(s_d)}_{i_d}$$

Suppose that $i_p = j$. Since ϖ_i is miniscule, if $f_{i_p}\cdots f_{i_1}v_{\varpi_i} \neq 0$ then $f_j f_{i_p}\cdots f_{i_1}v_{\varpi_i} = 0$. Thus we conclude that $i_p \sim j$.

We will use the following identity, which follows from Definition 4.1.1:

$$[F^{(s+1)}_{i_p}, F^{(r+1)}_j] = [F^{(r+s+1)}_{i_p}, F^{(1)}_j] - \tfrac{1}{2}\sum_{n=1}^{r}(F^{(r+s+1-n)}_{i_p}F^{(n)}_j + F^{(n)}_j F^{(r+s+1-n)}_{i_p}) \tag{5.4}$$

Now, by the inductive assumption in d,

$$\underline{u^{m_i}A_i(u)}F^{(s_1)}_{i_1}\cdots F^{(s_{p-1})}_{i_{p-1}} = \underline{G^{(s_1,\dots,s_{p-1})}_{i,(i_1,\dots,i_{p-1})}(u)T_{v_{\varpi_i},f_{i_{p-1}}\cdots f_{i_1}v_{\varpi_i}}(u)} + \dots$$

Since, $f_j f_{i_p} f_{i_{p-1}}\cdots f_{i_1}v_{\varpi_i} \neq 0$, then since ϖ_i is miniscule, we must have

$$f_{i_p} f_j f_{i_{p-1}}\cdots f_{i_1}v_{\varpi_i} = 0$$

Consider equation (5.4) for $s = s_p$. By the above equation, case (a) applies to all summands of the form $F^{(a)}_j F^{(b)}_{i_p}$. Therefore, modulo lower terms $\underline{u^{m_i}T_{v_{\varpi_i},v_{\varpi_i}}(u)}F^{(\mathbf{r})}_{\mathbf{j}}$ is equal to

$$\underline{u^{m_i}T_{v_{\varpi_i},v_{\varpi_i}}(u)}F^{(s_1+1)}_{i_1}\cdots\left(F^{(s_p+r+1)}_{i_p}F^{(1)}_j - \tfrac{1}{2}\sum_{n=1}^{r}F^{(s_p+r+1-n)}_{i_p}F^{(n)}_j\right)\cdots F^{(s_d+1)}_{i_d}$$

By the definition of p, we can commute all factors $F^{(n)}_k$ to the far right. Now apply the inductive assumption, valid since all exponents $n < r+1$. Modulo lower terms, we get the principal part of

$$\left((u+\tfrac{1}{2}k_p)^{s_p+r} - \tfrac{1}{2}\sum_{n=1}^{s}(u+\tfrac{1}{2}k_p)^{s_p+r-n}(u+\tfrac{1}{2}k)^{n-1}\right)\prod_{q\neq p}(u+\tfrac{1}{2}k_q)^{s_q}T_{v_{\varpi_i},f_{\mathbf{j}}v_{\varpi_i}}(u)$$

where k is defined by $\tilde{f}_j\tilde{f}_{i_d}\cdots\tilde{f}_{i_1}(y_{i,0}) = z^{-1}_{j,k-2}z^{-1}_{i_d,k_d-2}\cdots z^{-1}_{i_1,k_1-2}y_{i,0}$. Now, by the definition of p it follows that

$$k = k_p + 1$$

so the claim holds by the $v = u + \frac{1}{2}k_p$, $c = \frac{1}{2}$ case of the polynomial identity

$$(v-c)^s = v^s - c\sum_{n=1}^{s}v^{s-n}(v-c)^{n-1}$$

□

Lemma 5.3.4. *Assume that* $(\mathbf{i}, \mathbf{s})$ *is such that* $i_a = i_b$ *implies* $s_a = s_b$*, for any* a, b*. Then* $G^{\mathbf{s}}_{i,\mathbf{i}}(u)$ *only depends on* $\tilde{f}_{\mathbf{i}}(y_{i,0})$*, i.e. it is independent of the path taken from* $y_{i,0}$ *to* $\tilde{f}_{\mathbf{i}}(y_{i,0})$ *in the crystal.*

Proof. Write $\tilde{f}_{\mathbf{i}}(y_{i,0}) = y_{i,0} z_{\mathbf{U}}^{-1}$. Such a decomposition is unique, c.f. Remark 2.3.2. From the assumption on $\mathbf{s}$, it is easy to see that $G^{\mathbf{s}}_{i,\mathbf{i}}$ only depends on $\mathbf{U}$:

$$G^{\mathbf{s}}_{i,\mathbf{i}}(u) = u^{m_i} \prod_{(j,k-2)\in \mathbf{U}} (u + \tfrac{1}{2}k)^{s_j}$$

where s_j is defined to be s_a for any $i_a = j$. □

Definition 5.3.5. *Let* ϖ_i *be miniscule and* $\gamma \in V(\varpi_i)$ *a nonzero weight vector. Write* $\gamma = c f_{\mathbf{i}} v_{\varpi_i}$ *for some* $\mathbf{i} = (i_1, \dots, i_d)$ *and* $c \in \mathbb{C}^*$*, and set*

$$\mathbf{s} = (\mu_{i_1} + 1, \mu_{i_2} + 1, \dots, \mu_{i_d} + 1)$$

We define $G_\gamma(u) = G^{\mathbf{s}}_{i,\mathbf{i}}(u)$*. This definition is independent of* $\mathbf{i}$*, by the previous Lemma.*

Note that if $\mu = 0$, then $G_\gamma(u) = u^{m_i}$. In general, $G_\gamma(u)$ is a polynomial of degree $\langle \lambda, \omega_i \rangle - \langle \mu, \gamma \rangle$.

Lemma 5.3.6. *Let* ϖ_i *be miniscule. Suppose that* γ' *lies above* γ *in* $V(\varpi_i)$*, i.e.* $\gamma' \neq 0$ *is proportional to* $e_{j_\ell} \cdots e_{j_1} \gamma$ *for some* $i_1, \dots, i_d$*. Then* $G_{\gamma'}(u)$ *divides* $G_\gamma(u)$*.*

Proof. Proportionality does not affect the definition of $G_\gamma(u)$, so we may assume all scalars are 1. We can write $\gamma' = f_{j_d} \cdots f_{j_{\ell+1}} v_i$ for some $d > \ell$ and indices $j_{\ell+1}, \dots, j_d$, and hence

$$\gamma = f_{j_1} \cdots f_{j_\ell} \gamma' = f_{j_1} \cdots f_{j_\ell} f_{j_d} \cdots f_{j_{\ell+1}} v_i$$

Indeed if $e_j v \neq 0$ for a weight vector $v \in V(\varpi_i)$, then $f_j e_j v = v$ since ϖ_i is miniscule. So $\gamma = f_{\mathbf{i}}(v_i)$ for $\mathbf{i} = (i_1, \dots, i_d) := (j_{\ell+1}, \dots, j_d, j_\ell, \dots, j_1)$, and $G_\gamma(u) = G^{\mathbf{s}}_{i,\mathbf{i}}(u)$ with $\mathbf{s}$ as defined above. Meanwhile, $G_{\gamma'}(u) = G^{\mathbf{s}'}_{i,\mathbf{i}'}(u)$ where $\mathbf{i}' = (i_1, \dots, i_{d-\ell})$ and $\mathbf{s}' = (s_1, \dots, s_{d-\ell})$, so $G_{\gamma'}(u)$ divides $G_\gamma(u)$. □

5.3.2 Partial description of the left ideal

We now are in a position to give a partial description of L_μ^λ.

Definition 5.3.7. *We define*

1. S_μ^λ *to be the set of all coefficients of all series* $\underline{G_\gamma(u) T_{\beta,\gamma}(u)}$*, for* $i \in I$ *miniscule and* $\beta, \gamma \in V(\varpi_i)$ *weight vectors,*
2. $(S_\mu^\lambda)^{\leq}$ *to be the set of all coefficients of all series* $\underline{G_\gamma(u) T_{v_{\varpi_i},\gamma}(u)}$*, for* $i \in I$ *miniscule and* $\gamma \in V(\varpi_i)$ *a weight vector.*

From equations (5.2) and (5.3), we deduce:

Corollary 5.3.8. *There is an equality of* $(Y, Y_\mu^{\leq})$*-bimodules:*

$$Y\{A_i^{(s)} : \varpi_i \text{ miniscule}, s > m_i\} Y_\mu^{\leq} = Y (S_\mu^\lambda)^{\leq}$$

Conjecture 5.3.9. *The subspace* YS_μ^λ *is invariant under the right action of* $\mathbb{C}[A_i^{(r)}]$.

Theorem 5.3.10. *If Conjecture 5.3.9 holds, then*

$$Y\{A_i^{(s)} : \varpi_i \textit{ miniscule}, s > m_i\}Y_\mu = YS_\mu^\lambda$$

Proof. By Corollary 4.3.6, we have

$$[E_i^{(1)}, \underline{G_\gamma(u)T_{\beta,\gamma}(u)}] = \underline{G_\gamma(u)T_{f_i\beta,\gamma}(u)} - \underline{G_\gamma(u)T_{\beta,e_i\gamma}(u)}$$

Note that $e_i\gamma$ lies above γ in the sense of Lemma 5.3.6, hence $G_{e_i\gamma}(u)$ divides $G_\gamma(u)$. By Lemma 5.3.2 the coefficients of

$$\underline{G_\gamma(u)T_{\beta,e_i\gamma}(u)}$$

are linear combinations of the coefficients of $\underline{G_{e_i\gamma}(u)T_{\beta,e_i\gamma}(u)}$. It follows that YS_μ^λ is invariant under right multiplication by the elements $E_i^{(1)}$. The higher modes $E_i^{(r)}$ are obtained from $E_i^{(1)}$ by multiplying by elements of $\mathbb{C}[A_i^{(s)}]$, so if the conjecture holds we conclude that YS_μ^λ is right invariant under $Y_\mu^{\geq}$.

It remains to show that $\underline{G_\gamma(u)T_{\beta,\gamma}(u)}F_k^{(s)} \in YS_\mu^\lambda$ for all $s > \mu_k$, which we prove by induction on $\mathrm{ht}(\varpi_i - \beta)$. The base case when $\beta = v_{\varpi_i}$ follows from Corollary 5.3.8. For the inductive step, write $\beta = \sum_j f_j\beta_j$. Then by Corollary 4.3.6,

$$T_{\beta,\gamma}(u) = \sum_j \left([E_j^{(1)}, T_{\beta_j,\gamma}(u)] + T_{\beta_j,e_j\gamma}(u)\right)$$

Therefore,

$$T_{\beta,\gamma}(u)F_k^{(s)} = \sum_j \left(E_j^{(1)}T_{\beta_j,\gamma}(u) + T_{\beta_j,e_j\gamma}(u)\right) F_k^{(s)} + \sum_j T_{\beta_j,\gamma}(u)E_j^{(1)}F_k^{(s)}$$

Multiply both sides by $G_\gamma(u)$, and take principal parts. Recall also that $G_{e_j\gamma}(u)$ divides $G_\gamma(u)$. For the first sum on the right-hand side, we can apply the inductive assumption since $\mathrm{ht}(\varpi_i - \beta_j) < \mathrm{ht}(\varpi_i - \beta)$. For the second sum, apply the identity

$$E_j^{(1)}F_k^{(s)} = F_k^{(s)}E_j^{(1)} + \delta_{jk}H_k^{(s)}$$

We know from the above that $Y_\mu^{\geq}$ preserves YS_μ^λ, so we are again in the position to apply the inductive assumption. □

If we combine Lemma 5.1.3 with Theorem 5.3.10, then we see that the ideal of the B-algebra of $Y_\mu^\lambda(\mathbf{R})$ contains $\Pi(YS_\mu^\lambda)$ (with equality if $\mathfrak{g}$ is of type A, cf. Corollary 5.3.14). Now we will see that to compute $\Pi(YS_\mu^\lambda)$ we only need "principal" minors.

Proposition 5.3.11. $\Pi(YS_\mu^\lambda)$ *is generated as an ideal of* $\mathbb{C}[H_\bullet^{(\bullet)}]$ *by the coefficients of* $\Pi(\underline{G_\gamma(u)T_{\gamma,\gamma}(u)})$, *running over all minuscule* ϖ_i *and weight vectors* $\gamma \in V(\varpi_i)$.

Proof. We will first rule out several cases using a PBW basis in "FHE" order. We then prove the claim by induction.

$\Pi(YS_\mu^\lambda)$ is spanned by coefficients of series of the form $y = \Pi(x \cdot \underline{G_\gamma(u)T_{\beta,\gamma}(u)})$ where $x \in Y$ and $\beta, \gamma \in V(\varpi_i)$ are weight vectors. Note that $\nu = \mathrm{wt}\,\gamma - \mathrm{wt}\,\beta$ is the weight of $T_{\beta,\gamma}(u)$ with respect to $\hbar$, which lies in the root lattice.

If $\operatorname{ht}\nu > 0$ then $T_{\beta,\gamma}^{(r)}$ contains factors $E_j^{(s)}$ on the right when written in PBW form, so $y = 0$. Assuming $\operatorname{ht}\nu \leq 0$, it suffices to consider $x = FHE$ a PBW monomial of weight $-\nu$. If $F \neq 1$, then $HE \cdot T_{\beta,\gamma}^{(r)}$ contains factors $E_j^{(s)}$ on the right when written in PBW form, so $y = 0$. Since Π is equivariant with respect to left multiplication by H, we may assume $H = 1$.

Working with generators $B_i^{(s)}$ instead of $E_i^{(s)}$, we are reduced to showing that all coefficients of any series

$$\Pi\left(B_{j_1}^{(s_1)} \cdots B_{j_k}^{(s_k)} \underline{G_\gamma(u) T_{\beta,\gamma}(u)}\right)$$

lie in the ideal as claimed, where

$$\nu = \operatorname{wt}\gamma - \operatorname{wt}\beta = -\alpha_{j_1} - \ldots - \alpha_{j_k}$$

We will do this by induction on filtered degree $d = s_1 + \ldots + s_k$, with an inner induction on $k = -\operatorname{ht}\nu$. Note that when all $s_p = 1$, we may push each $B_{i_p}^{(s_p)}$ to the right using Corollary 4.3.6. Applying Π, the result is a sum of terms $\underline{G_\gamma(u) T_{\gamma',\gamma'}(u)}$ with γ' above γ in $V(\varpi_i)$. Since $G_{\gamma'}(u)$ divides $G_\gamma(u)$, the claim follows.

From Proposition 4.2.4 we have that

$$\begin{aligned}(u-v)\{\Delta_{f_j v_{\varpi_j}, v_{\varpi_j}}(u), \Delta_{\beta,\gamma}(v)\} &= c\Delta_{\beta,\gamma}(v)\Delta_{f_j v_{\varpi_j}, v_{\varpi_j}}(u) + \Delta_{f_j\beta,\gamma}(v)\Delta_{v_{\varpi_j}, v_{\varpi_j}}(u) \\ &+ \sum_{\alpha\in\Delta^+}\left(\Delta_{e_\alpha\beta,\gamma}(v)\Delta_{f_\alpha f_j v_{\varpi_j}, v_{\varpi_j}}(u) - \Delta_{f_j v_{\varpi_j}, f_\alpha v_{\varpi_j}}(u)\Delta_{\beta, e_\alpha\gamma}(v)\right)\end{aligned}$$

where $c = (s_j\varpi_j, \operatorname{wt}\beta) - (\varpi_j, \operatorname{wt}\gamma)$. Multiplying by $u^{-1}/(1-u^{-1}v)$ and taking the coefficient of u^{-s} on both sides expresses $\{\Delta_{f_j v_{\varpi_j}, v_{\varpi_j}}^{(s)}, \Delta_{\beta,\gamma}(v)\}$ as a sum, where exponents of terms coming from series in u decrease in degree (i.e. s decreases). This continues to hold if we multiply both sides of the identity by $G_\gamma(v)$ and take principal parts in v.

Because we have a filtered deformation, the lifted version of this identity is true modulo lower filtered terms. All summands lifting those written above lie in YS_μ^λ, and therefore the lower filtered terms also lie in YS_μ^λ. So, the inductive hypothesis applies to these lower terms. Note also that the first and third terms map to zero under Π, while the second and fourth terms have smaller k. □

5.3.3 Calculations in type A

We assume that $\mathfrak{g} = \mathfrak{sl}_n$. In this case, we will prove Conjecture 5.3.9. Since all fundamental weights are miniscule in type A, by Theorem 5.3.10 it will follow that

$$L_\mu^\lambda = YS_\mu^\lambda$$

We will make use of the following explicit formula, given in [Mol07, Example 1.15.8]:

Lemma 5.3.12. *In $Y(\mathfrak{gl}_n)$, for $\mathbf{a}, \mathbf{b} \subset \{1, \ldots, n\}$ of size i and $\mathbf{c}, \mathbf{d} \subset \{1, \ldots, n\}$ of size ℓ,*

$$[Q_{\mathbf{a},\mathbf{b}}(u), Q_{\mathbf{c},\mathbf{d}}(v)] =$$

$$= \sum_{p=1}^{\min\{i,\ell\}} \frac{(-1)^{p-1}p!}{(u-v-i+1)\cdots(u-v-i+p)} \sum_{\substack{i_1<\cdots<i_p \\ j_1<\cdots<j_p}} \left(Q_{\mathbf{a}',\mathbf{b}}(u)Q_{\mathbf{c}',\mathbf{d}}(v) - Q_{\mathbf{c},\mathbf{d}'}(v)Q_{\mathbf{a},\mathbf{b}'}(u)\right)$$

where primes indicate that elements have been exchanged according to the indices $i_1 < \cdots < i_p$ and $j_1 < \cdots < j_p$:

$$\mathbf{a}' = \{a_1, \ldots, c_{j_1}, \ldots, c_{j_p}, \ldots, a_i\} \quad \text{and} \quad \mathbf{c}' = \{c_1, \ldots, a_{i_1}, \ldots, a_{i_p}, \ldots, c_\ell\},$$

$$\mathbf{b}' = \{b_1, \ldots, d_{j_1}, \ldots, d_{j_p}, \ldots, b_i\} \quad \text{and} \quad \mathbf{d}' = \{d_1, \ldots, b_{i_1}, \ldots, b_{i_p}, \ldots, d_\ell\}$$

Proposition 5.3.13. *The subspace YS_μ^λ is invariant under the right action of $(Y_\mu)^{\geq}$. In particular, Conjecture 5.3.9 holds.*

Proof. We will prove that for any weight vectors $\beta, \gamma \in V(\varpi_i)$,

$$\underline{G_\gamma(u)T_{\beta,\gamma}(u)}T^{(r)}_{\delta,v_{\varpi_\ell}} \in YS_\mu^\lambda$$

for all $\delta \in V(\varpi_\ell)$ and $r \geq 1$. Note that the elements $T^{(r)}_{\delta,v_{\varpi_\ell}}$ generate $Y_\mu^{\geq}$.

We will establish this by induction on r. When $r = 1$ the elements $T^{(1)}_{\delta,v_{\varpi_i}}$ lie in the Borel subalgebra $\mathfrak{b} \subset \mathfrak{g} \subset Y$, so we can directly apply Corollary 4.3.6. Since γ remains fixed under this action, the claim holds.

For the inductive step, we will use Lemma 5.3.12. Transporting this identity under the isomorphism ϕ from Corollary 4.3.10, we get

$$[T_{\beta,\gamma}(u), T_{\delta,v_{\varpi_\ell}}(v)] = \sum_{p=1}^{\min\{i,\ell\}} \frac{(-1)^{p-1}p!}{(u-v+\frac{i-\ell}{2}-i+1)\cdots(u-v+\frac{i-\ell}{2}-i+p)} \times$$

$$\times \sum \left(T_{\beta',\gamma}(u)T_{\delta',v_{\varpi_\ell}}(v) - T_{\delta,v_{\varpi'_\ell}}(v)T_{\beta,\gamma'}(u)\right)$$

with primes indicating exchanges as per the Lemma. Expand each of the rational functions $(-1)^{p-1}p!/\cdots$ in the domain $\mathbb{C}[u][[v^{-1}]]$. Multiply both sides by $G_\gamma(u)$, and take the principal part in u. We wish to extract the coefficient of v^{-r} on both sides of the resulting identity.

Consider a term $T_{\beta',\gamma}(u)T_{\delta',v_{\varpi_\ell}}(v)$. When we extract the coefficient of v^{-r}, since we have multiplied by an element of $\mathbb{C}[u][[v^{-1}]]$ on the right-hand side, we get a sum of products $\underline{G_\gamma(u)T_{\beta',\gamma}(u)}T^{(s)}_{\delta',\varpi_\ell}$ where $s < r$. By the inductive hypothesis, these summands all lie in YS_μ^λ.

Consider now a term $T_{\delta,\varpi'_\ell}(v)T_{\beta,\gamma'}(u)$. Since ϖ_ℓ is the highest weight, the element γ' is a weight vector lying above γ in $V(\varpi_i)$, in the sense of Lemma 5.3.6. So $G_{\gamma'}(u)$ divides $G_\gamma(u)$, and therefore these summands already lie in YS_μ^λ. □

By Theorem 5.3.10, it follows that:

Corollary 5.3.14. *For $\mathfrak{g}$ of type A, we have $L_\mu^\lambda = YS_\mu^\lambda$.*

Applying Proposition 5.3.11, we can now give an explicit description of the B-algebra following §5.1.

Corollary 5.3.15. *The B–algebra of $Y_\mu^\lambda(\mathbf{R})$ is the quotient of the ring $\mathbb{C}[H_\bullet^{(\bullet)}]$ by the ideal generated by the coefficients of $\Pi(\underline{G_\gamma(u)T_{\gamma,\gamma}(u)})$, running over all $i \in I$ and weight vectors $\gamma \in V(\varpi_i)$.*

5.3.4 Computing the images $\Pi(T_{\gamma,\gamma}(u))$

In this section we will describe explicitly the images $\Pi(T_{\gamma,\gamma}(u))$, for extremal weight vectors $\gamma \in W\varpi_i$.

Recall that for a locally finite representation V of $\mathfrak{g}$, i.e. one in which every vector lies in a finite-dimensional sub-representation, there are well-defined operators

$$\overline{s}_i = \exp(e_i)\exp(-f_i)\exp(e_i)$$

These operators satisfy the braid relations, and therefore define an action of the braid group $B_\mathfrak{g}$, but this action does not factor through W in general (rather a Tits extension $\widetilde{W}$). However, for $w \in W$ there is still a well-defined operator $\overline{w}$ defined as $\overline{s}_{i_\ell}\cdots\overline{s}_{i_1}$ where $w = s_{i_\ell}\cdots s_{i_1}$ is any reduced expression. If V_η denotes a weight space for $\mathfrak{h}$ on V, then

$$\overline{w}(V_\eta) = V_{w\eta},$$

see for example [Kum02, Lemma 1.3.5].

Lemma 5.3.16. *The adjoint action of $\mathfrak{g}$ on Y under the embedding $U\mathfrak{g} \hookrightarrow Y$ is locally finite.*

Proof. This follows by looking at the action of $\mathfrak{g}$ on the generators of Y. □

By the lemma, we can apply the construction discussed above to the adjoint action of $\mathfrak{g}$ on Y. Note that since $\mathfrak{g}$ acts on Y by derivations, the corresponding operators $\overline{s}_i$ are algebra automorphisms. Under the action of $\mathfrak{g}$, we get a Q–grading $Y = \bigoplus_{\alpha\in Q} Y(\alpha)$ (we use the notation $Y(\alpha)$ to avoid confusion with our notation for shifted Yangians), where

$$\deg F_i^{(r)} = -\alpha_i, \quad \deg H_i^{(r)} = 0, \quad \deg E_i^{(r)} = \alpha_i$$

Lemma 5.3.17. *For any vector $\gamma \in V(\varpi_i)$ and $w \in W$, we have*

$$\overline{w}(T_{\gamma,\gamma}^{(r)}) = T_{\overline{w}\gamma,\overline{w}\gamma}^{(r)}$$

In particular, $\overline{s}_i(A_i(u)) = D_i(u)$ and $\overline{s}_i(A_j(u)) = A_j(u)$ for all $i \neq j$.

Proof. It suffices to consider $w = s_j$. By Corollary 4.3.6, we must find the images of $\overline{s}_j$ under the (integrated) automorphisms $\iota\circ\omega$ and ι of the adjoint group G^{ad}. An SL_2 calculation shows that

$$\iota\circ\omega(\overline{s}_j) = \iota(\overline{s}_j) = \overline{s}_j^{-1}$$

If $\gamma \in V(\varpi_i)$ is a weight vector, then $\overline{s}_j^2 v = (-1)^{\langle \mathrm{wt}\,\gamma, \alpha_j\rangle} v$ by [Kum02, Lemma 1.3.5.]. Therefore

$$\overline{s}_j(T_{\gamma,\gamma}^{(r)}) = T^{(r)}_{\overline{s}_j^{-1}\gamma,\overline{s}_j^{-1}\gamma} = (-1)^{2\langle \mathrm{wt}\,\gamma,\alpha_j\rangle} T^{(r)}_{\overline{s}_j\gamma,\overline{s}_j\gamma} = T^{(r)}_{\overline{s}_j\gamma,\overline{s}_j\gamma}$$

□

To describe the images $\Pi(T_{\gamma,\gamma}(u))$, we introduce some more notation. For an element $w \in W$, the **inversion set** is defined to be

$$\Delta_w = \left\{\beta \in \Delta_+ : w^{-1}\beta \in \Delta_-\right\}$$

If $w = s_{i_d}\cdots s_{i_1}$ is a reduced expression, then [Kum02, Lemma 1.3.14]

$$\Delta_w = \left\{\alpha_{i_d}, s_{i_d}\alpha_{i_{d-1}}, \ldots, s_{i_d}s_{i_{d-1}}\cdots s_{i_2}\alpha_{i_1}\right\} \tag{5.5}$$

Consider a weight $\gamma \in W\varpi_i$. Write $\gamma = w\varpi_i$, and fix a reduced decomposition $w = s_{i_d}\cdots s_{i_1}$. By abuse of notation, we will also denote the corresponding vector $\gamma = \overline{w}v_{\varpi_i} \in V(\varpi_i)$. This is a minor abuse, as this weight space is 1-dimensional.

Recall the fundamental monomial crystal $\mathcal{B}(\varpi_i, 0)$ from §2.3. There is a unique element q_γ of weight γ, and we can write

$$q_\gamma = y_{i,0}z_{\mathbf{U}}^{-1} = \prod_{j,k} y_{j,k}^{b_{j,k}}$$

for some tuple of multisets $\mathbf{U} = (U_i)_{i\in I}$. We will adopt the convention that $U_i(k)$ denotes the multiplicity of the integer k in U_i.

Theorem 5.3.18. *With notation as above,*

$$T_{\gamma,\gamma}(u) = A_i(u)\prod_{j,k} H_j(u+\tfrac{1}{2}k)^{U_j(k-2)} + \sum_{\beta\in\Delta_w} Y(-\beta)Y(\beta)$$

In particular,

$$\Pi(T_{\gamma,\gamma}(u)) = A_i(u)\prod_{j,k} H_j(u+\tfrac{1}{2}k)^{U_j(k-2)}$$

Proof. The second claim follows from the first, by noting that for any $\beta\in\Delta_+$ the space $Y(-\beta)Y(\beta)$ is in the kernel of Π.

By definition, we have $T_{v_{\varpi_i},v_{\varpi_i}}(u) = A_i(u)$, and

$$T_{f_iv_{\varpi_i},f_iv_{\varpi_i}}(u) = D_i(u) = H_i(u)A_i(u) + F_i(u)A_i(u)E_i(u) = H_i(u)A_i(u) + Y(-\alpha_i)Y(\alpha_i)$$

so the claim holds for these elements. We proceed will by induction on the length of w.

Assuming the claim holds for w, let us prove it for s_dw. Observe that the relationship between $H_i(u)$ and $A_j(u)$'s in (4.3) is analogous to that between $z_{i,k-2}^{-1}$ and $y_{j,k}$ in (2.1). In particular, we can rewrite

$$T_{\gamma,\gamma}(u) = X(u)\cdot\prod_k A_d(u+\tfrac{1}{2}k)^{b_{j,k}} + \sum_{\beta\in\Delta_w} Y(-\beta)Y(\beta)$$

for some series $X(u)\in\mathbb{C}[A_j^{(s)} : j\neq d][[u^{-1}]]$. Therefore, by the previous lemma

$$\begin{aligned}
T_{\overline{s}_d\gamma,\overline{s}_d\gamma}(u) &= \overline{s}_d\left(T_{\gamma,\gamma}(u)\right)\\
&= X(u)\cdot\prod_k D_d(u+\tfrac{1}{2}k)^{b_{d,k}} + \sum_{\beta\in\Delta_w} Y(-s_d\beta)Y(s_d\beta)\\
&= X(u)\cdot\prod_k \left(A_d(u+\tfrac{1}{2})H_d(u+\tfrac{1}{2}k)\right)^{b_{d,k}} + Y(-\alpha_d)Y(\alpha_d) + \sum_{\beta\in\Delta_w} Y(-s_d\beta)Y(s_d\beta)\\
&= \left(A_i(u)\prod_{j,k} H_j(u+\tfrac{1}{2}k)\right)\prod_k H_d(u+\tfrac{1}{2}k)^{b_{d,k}} + \sum_{\beta\in\Delta_{s_dw}} Y(-\beta)Y(\beta)
\end{aligned}$$

This agrees with the expression for the monomial $q_{s_d\gamma}$ from Proposition 2.3.3. □

Recall that for a set of parameters $\mathbf{R} = (R_i)_{i\in I}$, we denote $R_i(u) = \prod_{c\in R_i}(u - \frac{1}{2}c)$. By taking into account the expressions (4.2) and (4.3), we get

Corollary 5.3.19. *With notation as above,*

$$\Pi\big(G_\gamma(u)T_{\gamma,\gamma}(u)\big) = \Big(\prod_{j,k} R_j(u+\tfrac{1}{2}k)^{U_j(k-2)}\Big)\Big(\prod_{j,k}\big((u+\tfrac{1}{2}k)^{m_j}A_j(u+\tfrac{1}{2}k)\big)^{b_{j,k}}\Big)$$

5.4 A conjectural description of the B-algebra

Based on Corollary 5.3.15, as well as Corollary 5.3.19, we propose the following definition. For *every* element $q \in \mathcal{B}(\varpi_i, 0)$, we write $q = y_{i,0}z_{\mathbf{U}}^{-1} = \prod_{j,k} y_{j,k}^{b_{j,k}}$, and define an element of $\mathbb{C}[H_\bullet^{(\bullet)}]((u^{-1}))$ by

$$H_q(u) = \Big(\prod_{j,k} R_j(u+\tfrac{1}{2}k)^{U_j(k-2)}\Big)\Big(\prod_{j,k}\Big((u+\tfrac{1}{2}k)^{m_j}A_j(u+\tfrac{1}{2}k)\Big)^{b_{j,k}}\Big) \tag{5.6}$$

Definition 5.4.1.

(a) *The algebra $B^\lambda_\mu(\mathbf{R})$ is defined to be the quotient $\mathbb{C}[H_\bullet^{(\bullet)}]$ by the ideal generated by all coefficients of the principal parts of $\underline{H_q(u)}$, over all* **extremal elements** *$q \in \mathcal{B}(\varpi_i, 0)$ (i.e. having weight in $W\varpi_i$) and $i \in I$.*

(b) *The algebra $\widetilde{B}^\lambda_\mu(\mathbf{R})$ is defined to be the quotient of $\mathbb{C}[H_\bullet^{(\bullet)}]$ by the ideal generated by all coefficients of the principal parts of $\underline{H_q(u)}$, over* **all elements** *$q \in \mathcal{B}(\varpi_i, 0)$ and $i \in I$.*

Remark 5.4.2.

(a) *The series corresponding to the highest weight element $y_{i,0} \in \mathcal{B}(\varpi_i, 0)$ is $H_{y_{i,0}}(u) = u^{m_i}A_i(u)$, and so the elements $A_i^{(r)}$ with $r > m_i$ are in the ideal defining $\widetilde{B}^\lambda_\mu(\mathbf{R})$. Because of this, we may think of $\widetilde{B}^\lambda_\mu(\mathbf{R})$ as a quotient of*

$$\mathbb{C}[A_i^{(s)} : i \in I, 1 \le s \le m_i]$$

(b) *The above definition makes sense for $\lambda \ge \mu$ with $\lambda \in P_+$, and $\mu \in P$* **any** *weight, even though this no longer corresponds to a slice $\mathsf{Gr}^{\overline{\lambda}}_\mu$ in the affine Grassmannian. Rather, this should correspond to a* generalized slice *$\mathcal{W}^\lambda_\mu$ as recently defined by Braverman, Finkelberg and Nakajima [BFN16a]. We believe that the conjecture below should extend to the quantization of this variety.*

In type A there is no distinction between these algebras, since all $V(\varpi_i)$ are miniscule. By Corollary 5.3.15 and Corollary 5.3.19, we have

Corollary 5.4.3. *In type A, $B^\lambda_\mu(\mathbf{R})) = \widetilde{B}^\lambda_\mu(\mathbf{R})$ is isomorphic to the B-algebra for $Y^\lambda_\mu(\mathbf{R})$.*

For general $\mathfrak{g}$, however, it is not clear which (if either) algebra is closest to the true B–algebra $B(Y^\lambda_\mu)$. There are trade-offs:

(1) We will show in Chapter 8 that there is a surjective map from $B^\lambda_\mu(\mathbf{R})$ onto the equivariant cohomology ring of a Nakajima quiver variety. We do not know how to show that the "non-extremal" relations of $\widetilde{B}^\lambda_\mu(\mathbf{R})$ are in the kernel.

(2) The algebra $\widetilde{B}_\mu^\lambda(\mathbf{R})$ has nice combinatorial properties, and in particular we can compute its maximal spectrum. This proof makes use of the fact that we have taken all q in its definition, and we do not know whether it holds for $B_\mu^\lambda(\mathbf{R})$.

The algebras $B_\mu^\lambda(\mathbf{R})$ and $\widetilde{B}_\mu^\lambda(\mathbf{R})$ are naturally filtered, with $\deg H_i^{(r)} = r$. Recall the B–algebra of $\mathcal{O}(\mathcal{G}_\mu^\lambda)$ from §5.2.

Lemma 5.4.4. *There is a surjection*

$$B\big(\mathcal{O}(\mathcal{G}_\mu^\lambda)\big) \longrightarrow \operatorname{gr} B_\mu^\lambda(\mathbf{R}), \quad h_i^{(r)} \mapsto H_i^{(r)}$$

and similarly for $\widetilde{B}_\mu^\lambda(\mathbf{R})$.

Proof. The map $h_i^{(r)} \mapsto H_i^{(r)}$ defines a surjection from $\mathbb{C}[h_i^{(r)}]$ onto $\operatorname{gr} B_\mu^\lambda(\mathbf{R})$, and it is easy to see that the generators of the ideal from Theorem 5.2.4 maps to zero. □

The above lemma provides some evidence that $B_\mu^\lambda(\mathbf{R}) = \widetilde{B}_\mu^\lambda(\mathbf{R})$, since the B–algebra of $\mathcal{O}(\mathcal{G}_\mu^\lambda)$ is generated by extremal elements. Being optimistic, we hope for the following:

Conjecture 5.4.5. *For all $\mathfrak{g}$, we have $B_\mu^\lambda(\mathbf{R}) = \widetilde{B}_\mu^\lambda(\mathbf{R}) \cong B\big(Y_\mu^\lambda(\mathbf{R})\big)$.*

As with the parametric version Y_μ^λ of $Y_\mu^\lambda(\mathbf{R})$ (defined in (4.5)), we can define algebras B_μ^λ and $\widetilde{B}_\mu^\lambda$. These are algebras are quotients of the polynomial ring

$$\mathcal{P}_\mu^\lambda := \mathbb{C}[\hbar, R_i^{(s)}, A_i^{(r)} : i \in I, 1 \leq s \leq \lambda_i, 1 \leq r \leq m_i] \tag{5.7}$$

by ideals as above, but generated using the series

$$H_q(u) = \Big(\prod_{j,k} R_j(u + \tfrac{1}{2}k\hbar)^{U_j(k-2)}\Big)\Big(\prod_{j,k}\big((u+\tfrac{1}{2}k\hbar)^{m_j} A_j(u+\tfrac{1}{2}k\hbar)\big)^{b_{j,k}}\Big) \tag{5.8}$$

By specializing the parameters to $\mathbf{R}\times\{1\} \in \operatorname{Spec}\mathbb{C}[R_i^{(s)}, \hbar]$, we recover the algebras $B_\mu^\lambda(\mathbf{R})$ and $\widetilde{B}_\mu^\lambda(\mathbf{R})$.

Remark 5.4.6. *The algebras $P_\mu^\lambda, B_\mu^\lambda$ and $\widetilde{B}_\mu^\lambda$ are all $\mathbb{Z}_{\geq 0}$–graded, with*

$$\deg R_i^{(s)} = \deg A_i^{(s)} = s, \quad \deg \hbar = 1$$

Conjecture 5.4.7. *For all $\mathfrak{g}$, we have a graded isomorphism $B_\mu^\lambda \cong B\big(Y_\mu^\lambda\big)$*

5.4.1 Coproducts

In §4.1.4, we described a family of conjectural "coproduct" maps,

$$\Delta_{\mu',\mu''}^{\lambda',\lambda''} : Y_\mu^\lambda \longrightarrow Y_{\mu'}^{\lambda'} \otimes_{\mathbb{C}[\hbar]} Y_{\mu''}^{\lambda''}$$

By Lemma 3.3.3 (b), each $\Delta_{\mu',\mu''}^{\lambda',\lambda''}$ would naturally induce a map of B–algebras

$$B\big(Y_\mu^\lambda\big) \longrightarrow B\big(Y_{\mu'}^{\lambda'}\big) \otimes_{\mathbb{C}[\hbar]} B\big(Y_{\mu''}^{\lambda''}\big)$$

We will show now that analogous maps exist for the algebras B_μ^λ and $\widetilde{B}_\mu^\lambda$.

To this end, suppose that $\lambda = \lambda' + \lambda''$ are elements of P_+, and that $\mu = \mu' + \mu''$ are elements of P, such that $\lambda \geq \mu, \lambda' \geq \mu'$ and $\lambda'' \geq \mu''$. Consider the corresponding algebras $\mathcal{P}_\mu^\lambda$, $\mathcal{P}_{\mu'}^{\lambda'}$ and $\mathcal{P}_{\mu''}^{\lambda''}$, and the homomorphism

$$\mathcal{P}_\mu^\lambda \longrightarrow \mathcal{P}_{\mu'}^{\lambda'} \otimes_{\mathbb{C}[\hbar]} \mathcal{P}_{\mu''}^{\lambda''}, \tag{5.9}$$

defined by

$$\begin{aligned} R_i(u) &\longmapsto R_i'(u) \otimes R_i''(u), \\ A_i(u) &\longmapsto A_i'(u) \otimes A_i''(u), \\ \hbar &\longmapsto \hbar \otimes 1 = 1 \otimes \hbar \end{aligned}$$

We have again added primes to distinguish the elements of these algebras (these are not derivatives!).

The next result follows immediately from the definition (5.8).

Lemma 5.4.8. *Under the homomorphism (5.9), for any $q \in \mathcal{B}(\varpi_i, 0)$ we have*

$$H_q(u) \longmapsto H_q'(u) \otimes H_q''(u)$$

Corollary 5.4.9. *The map (5.9) descends to define maps*

$$\Delta_{\mu',\mu''}^{\lambda',\lambda''} : B_\mu^\lambda \longrightarrow B_{\mu'}^{\lambda'} \otimes_{\mathbb{C}[\hbar]} B_{\mu''}^{\lambda''},$$

and analogously for $\widetilde{B}_\mu^\lambda$.

Remark 5.4.10. *(a) As in the case of of the map on algebras Y_μ^λ, we expect that for fixed $\lambda = \lambda' + \lambda''$ the sum*

$$B_\mu^\lambda \longrightarrow \bigoplus_{\mu=\mu'+\mu''} B_{\mu'}^{\lambda'} \otimes_{\mathbb{C}[\hbar]} B_{\mu''}^{\lambda''}$$

should be injective.

(b) The above results go through for the specialized algebras $B_\mu^\lambda(\mathbf{R})$, etc. In this case, tensor products should be taken over $\mathbb{C}$.

5.5 The case of $\mathfrak{sl}_2$

Let $\mathfrak{g} = \mathfrak{sl}_2$, and fix $\lambda \geq 0$ and $\mu \in \mathbb{Z}$ with $\lambda - \mu = 2m \geq 0$. In this case,

$$\mathcal{P}_\mu^\lambda = \mathbb{C}[\hbar, R^{(1)}, \ldots, R^{(\lambda)}, A^{(1)}, \ldots, A^{(m)}]$$

There is a single fundamental weight ϖ, and $W\varpi = \{\varpi, s\varpi\}$. The fundamental crystal is

$$\mathcal{B}(\varpi, 0): \quad y_0 \xrightarrow{\widetilde{f}} y_{-2}^{-1}$$

and we can write $y_{-2}^{-1} = y_0 z_{-2}^{-1}$. Therefore, the series from (5.8) are

$$H_\varpi(u) = u^m A(u), \qquad H_{s\varpi}(u) = \frac{R(u)}{(u-\hbar)^m A(u-\hbar)}$$

The relations $\underline{H_q(u)} = 0$ defining the B–algebra are, respectively, that $A^{(r)} = 0$ for $r > m$ and

$$(u-\hbar)^m + A^{(1)}(u-\hbar)^{m-1} + \ldots + A^{(m)} \text{ divides } u^\lambda + R^{(1)} u^{\lambda-1} + \ldots + R^{(\lambda)}$$

We should interpret "divides" as follows: do polynomial long division, and set the coefficients of the remainder polynomial to zero.

Corollary 5.4.3 applies in this case, of course: the above is a presentation for the B–algebra of $Y_\mu^\lambda(\mathbf{R})$.

Chapter 6

Monomials as highest weights

6.1 Highest weights

In §5.1, for a tuple $J = \big(J_i(u)\big)_{i\in I}$ of series $J_i(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$ we defined corresponding Verma modules $M(J)$ and $M(J,\mathbf{R})$. We have seen that the module $M(J,\mathbf{R})$ is non-zero if and only if J corresponds to a point in $\operatorname{MaxSpec} B\big(Y_\mu^\lambda(\mathbf{R})\big)$. Let us define, then, the **set of highest weights**

$$H_\mu^\lambda(\mathbf{R}) = \operatorname{MaxSpec} B\big(Y_\mu^\lambda(\mathbf{R})\big)$$

Note that if $J \in H_\mu^\lambda(\mathbf{R})$, then each J_i is a rational function (expanded at $u = \infty$) because of the formula relating $H_i(u)$ and $A_i(u)$ and because $A_i^{(s)}\mathbf{1} = 0$ for $s > m_i$. Thus each J_i can be written as a product of linear factors and their inverses. That is, there is a collection of multiplicities $a_{i,k} \in \mathbb{Z}$ for each $k \in \mathbb{C}$ such that

$$J_i(u) = u^{-\mu_i} \prod_k (u - \tfrac{1}{2}k)^{a_{i,k}}.$$

We will prove later, in Proposition 6.2.12, that if $a_{i,k} \neq 0$ then we must have $k \in \mathbb{Z}$ and $i \in I_{\overline{k}}$ (see Section 2.2). Assuming this result, we can define $y(J) = \prod_{i,k} y_{i,k}^{a_{i,k}} \in \mathcal{B}$ to be the Nakajima monomial obtained by converting any factor $u - \frac{1}{2}b$ occurring in $u^{\mu_i}J_i(u)$ into $y_{i,b}$.

This leads us to the main conjecture from [KTW+15]:

Conjecture 6.1.1. *The map $J \mapsto y(J)$ is a bijection between $H_\mu^\lambda(\mathbf{R})$ and $\mathcal{B}(\lambda,\mathbf{R})_\mu$, the set of elements of weight μ in $\mathcal{B}(\lambda,\mathbf{R})$.*

We can reformulate the map $J \mapsto y(J)$ as follows. Suppose we have a Verma module $M_\mu^\lambda(J,\mathbf{R})$ for $Y_\mu^\lambda(\mathbf{R})$. Let us consider the action of $A_i^{(r)}$ on $\mathbf{1}$. Since $A_i^{(r)} = 0$ for $r > m_i$, we can write

$$A_i(u)\mathbf{1} = \prod_{s\in S_i} (1 - \tfrac{1}{2}su^{-1})\mathbf{1}$$

for some multiset S_i. Thus we get a collection of multisets $\mathbf{S} = (S_i)_{i\in I}$ which determine J by

$$J_i(u) = u^{-\mu_i} \prod_{c\in R_i} (u - \tfrac{1}{2}c) \frac{\prod_{j\sim i}\prod_{s\in S_j}(u - \frac{1}{2}s - \frac{1}{2})}{\prod_{s\in S_i}(u - \frac{1}{2}s)(u - \frac{1}{2}s - 1)}$$

From this, it is easy to see that $y(J) = y_{\mathbf{R}} z_{\mathbf{S}}^{-1}$.

6.2 Description of the product monomial crystal

In this section we give a combinatorial characterization of the product monomial crystal, using fundamental monomial crystals.

6.2.1 Monomial data and regularity

Recall the monomial crystal $\mathcal{B}$ from Section 2.2.1. Throughout this section, fix an integral collection of parameters $\mathbf{R}$.

Definition 6.2.1. *A* **R–monomial datum** *is an element $p \in \mathcal{B}$ of the form $p = y_{\mathbf{R}} z_{\mathbf{S}}^{-1}$, where $\mathbf{S} = (S_i)_{i \in I}$ is an integral collections of multisets (in the sense of Section 2.2.1).*

We will now describe the product monomial crystal combinatorially, using conditions indexed by elements of fundamental monomial crystals $\mathcal{B}(\varpi_i, n)$, where i and n have opposite parity.

We fix some notation: we reserve the letter p to denote monomial data

$$p = y_{\mathbf{R}} z_{\mathbf{S}}^{-1} = \prod_{i,k} y_{i,k}^{a_{i,k}} \in \mathcal{B},$$

while we will reserve the letter q to denote an element of a fundamental crystal with opposite parity condition

$$q = y_{i,n} z_{\mathbf{U}}^{-1} = \prod_{j,k} y_{j,k}^{b_{j,k}} \in \mathcal{B}(\varpi_i, n)$$

For a multiset S, let $S(k)$ denote the multiplicity of the element k in S.

Definition 6.2.2. *For p, q as above we set*

$$E_q(p) := \sum_{j,k} U_j(k) R_j(k+1) + \sum_{j,k} b_{j,k} S_j(k-1)$$

Definition 6.2.3. *We call a $\mathbf{R}$–monomial data p* regular *if $E_q(p) \geq 0$ for all $q \in \mathcal{B}(\varpi_i, n)$, where $i \in I_{\overline{n+1}}$.*

6.2.2 Product monomial crystal

In this section we will prove the following theorem, which gives our promised combinatorial characterization of the product monomial crystal.

Theorem 6.2.4. *Consider a $\mathbf{R}$–monomial datum $p = y_{\mathbf{R}} z_{\mathbf{S}}^{-1}$. Then p is regular if and only if it is an element of the product monomial crystal $\mathcal{B}(\lambda, \mathbf{R})$, where $\lambda = \sum_i |\mathbf{R}_i| \varpi_i^{\vee}$.*

We begin with several lemmas.

Lemma 6.2.5. *Consider p, q as in the previous section. Then*

1. $E_q(z_{i,k}^{\pm} p) = E_q(p) \mp b_{i,k+1}$,

2. $E_{z^{\pm}_{i,k}q}(p) = E_q(p) \mp a_{i,k+1}$.

whenever the above are well-defined.

For the proofs of the next two Lemmas, we use the connection of monomial crystals to quiver varieties; see §7.3.3. We do not know purely combinatorial proofs.

Lemma 6.2.6. *Multiplication by $z_{i,k}$ gives a bijection of sets*

$$\begin{pmatrix} \text{Regular } \mathbf{R}\text{–monomial data } p \\ \text{with } a_{i,k} < 0 \text{ and } a_{i,k+2} \geq 0 \end{pmatrix} \xrightarrow{\sim} \begin{pmatrix} \text{Regular } \mathbf{R}\text{–monomial data } p' \\ \text{with } a'_{i,k} \leq 0 \text{ and } a'_{i,k+2} > 0 \end{pmatrix}$$

In particular, the set of regular $\mathbf{R}$–monomial data is invariant under the Kashiwara operators $\tilde{e}_i, \tilde{f}_i$.

Proof. Considering only the conditions on $a_{i,k}$ and $a_{i,k+2}$, this is straightforward. We must show that the property of being regular $\mathbf{R}$–monomial data is preserved in both directions.

Suppose that $p = y_{\mathbf{R}} z_{\mathbf{S}}^{-1}$ has $a_{i,k} < 0$ and $a_{i,k+2} \geq 0$. By Lemma 6.2.5,

$$E_q(z_{i,k}p) = E_q(p) - b_{i,k+1}.$$

If $b_{i,k+1} \leq 0$ this is non-negative because p is regular. If $b_{i,k+1} > 0$ then Lemma 7.3.5 applies, so $q' = z_{i,k-1}^{-b_{i,k+1}} q \in \mathcal{B}(\varpi_j, n)$. Then

$$0 \leq E_{q'}(p) = E_q(p) + b_{i,k+1}a_{i,k} \leq E_q(z_{i,k}p), \tag{6.1}$$

where the last inequality is because $a_{i,k} \leq -1, b_{i,k+1} \geq 1$. In particular, taking $q = y_{i,k-1}$ we get $0 \leq E_q(z_{i,k}p) = S_i(k) - 1$, i.e. that $S_i(k) \geq 1$. This shows that $p' = z_{i,k}p$ is indeed $\mathbf{R}$–monomial data. Equation (6.1) proves regularity.

The proof for the other direction is similar. □

Lemma 6.2.7. *Consider $\mathbf{R}$–monomial data p. If $a_{i,k} \geq 0$ for all pairs (i,k), then*

1. *p is regular,*

2. *$p \in \mathcal{B}(\lambda, \mathbf{R})$, where $\lambda = \sum_i |R_i| \varpi_i^\vee$.*

Proof. For (1), consider first a highest weight element $q = y_{j,n} \in \overline{B}(\varpi_j, n)$. Then $E_q(p) = S_i(n+1) \geq 0$. Every other element of $q' \in \overline{B}(\varpi_j, n)$ can be reached from q by multiplying by a sequence of $z_{i,k}^{-1}$, and by Lemma 6.2.5 multiplying by $z_{i,k}^{-1}$ corresponds to adding $a_{i,k+1}$ to $E_q(p)$. Since all $a_{i,k+1} \geq 0$, we get $E_{q'}(p) \geq 0$ for all q'.

For (2), we define a sequence of elements $p(k) \in \mathcal{B}(\lambda, \mathbf{R})$ for $k_{\min} \leq k \leq k_{\max}$. Here $k_{\max}$ denotes the maximal value of k for which some $R_i(k) > 0$, while $k_{\min}$ denotes the minimal value of k for which some $S_i(k) > 0$. By the construction we will have $p(k_{\min}) = p$, proving the claim.

Define $p(k_{\max}) = \prod_{i,k} y_{i,k}^{R_i(k)}$, so $p(k_{\max}) \in \mathcal{B}(\lambda, \mathbf{R})$ is the element of highest weight. We define the rest of the sequence iteratively. Assuming $p(k+1) \in \mathcal{B}(\lambda, \mathbf{R})$ has been defined, put

$$p(k) = \Big(\prod_i z_{i,k}^{-S_i(k)}\Big) p(k+1)$$

We claim that $p(k) \in \mathcal{B}(\lambda, \mathbf{R})$. From the iterative definition, the exponent of $y_{i,k+2}$ in $p(k+1)$ is

$$R_i(k+2) - S_i(k+2) + \sum_{j \sim i} S_j(k+1) = a_{i,k+2} + S_i(k)$$

By assumption $a_{i,k+2} \geq 0$, so $z_{i,k}^{-S_i(k)} p(k+1) \in \mathcal{B}(\lambda, \mathbf{R})$ by Lemma 7.3.5. Multiplication by $z_{i,k}^{-S_i(k)}$ does not change the exponent of $y_{j,k+2}$ for $j \neq i$, so by applying the same argument for all j it follows that $p(k) \in \mathcal{B}(\lambda, \mathbf{R})$.

The assumption that all $a_{i,k} \geq 0$ forces $S_i(k) = 0$ for $k \geq k_{\max}$. With this in mind it is clear that $p(k_{\min}) = p$, proving that $p \in \mathcal{B}(\lambda, \mathbf{R})$.

□

Proof of Theorem 6.2.4. Consider an equivalence relation on the set of regular $\mathbf{R}$–monomial data, defined by extending the relation from Lemma 6.2.6: define p and p' to be equivalent if p' can be obtained from p by a series of multiplications by some $z_{i,k}$ (or $z_{i,k}^{-1}$), where at each step we had $a_{i,k} < 0$ and $a_{i,k+2} \geq 0$ (resp. $a_{i,k} \leq 0$ and $a_{i,k+2} > 0$). From Corollary 7.3.6, it follows that if some representative of an equivalence class is in $\mathcal{B}(\lambda, \mathbf{R})$, then all representatives are also in $\mathcal{B}(\lambda, \mathbf{R})$.

Similarly, define an equivalence relation on $\mathcal{B}(\lambda, \mathbf{R})$ by extending the relation from Corollary 7.3.6: p and p' are again defined to be equivalent if p' can be obtained from p by a series of multiplications by $z_{i,k}^{\pm 1}$ as above. By Lemma 6.2.6, if some representative of an equivalence class is regular, then all representatives are regular.

In both cases, we claim that every equivalence class contains a representative p^+ satisfying $a_{i,k}^+ \geq 0$ for all i, k. Indeed, starting from p a regular $\mathbf{R}$–monomial data (resp. $p \in \mathcal{B}(\lambda, \mathbf{R})$), choose $a_{i,k} < 0$ with k maximal (assuming some $a_{i,k} < 0$). Then $a_{i,k+2} \geq 0$, so $z_{i,k}p$ is also regular $\mathbf{R}$–monomial data by Lemma 6.2.6 (resp. $z_{i,k}p \in \mathcal{B}(\lambda, \mathbf{R})$ by Corollary 7.3.6). Iterating this argument, we produce an element p^+ in the same equivalence class as claimed.

By Lemma 6.2.7, such an element p^+ is both regular and lies in $\mathcal{B}(\lambda, \mathbf{R})$. We conclude that all regular $\mathbf{R}$–monomial data is in $\mathcal{B}(\lambda, \mathbf{R})$, and vice versa. □

6.2.3 Monomial data and highest weights

Fix $\lambda, \mu, \mathbf{R}$. Our goal now is to relate the product monomial crystal $\mathcal{B}(\lambda, \mathbf{R})_\mu$ to the set of highest weights $H_\mu^\lambda(\mathbf{R})$.

The bridge between the two is the algebra $\widetilde{B}_\mu^\lambda(\mathbf{R})$ from Section §5.4.1. To relate $\operatorname{MaxSpec} \widetilde{B}_\mu^\lambda(\mathbf{R})$ with $\mathcal{B}(\lambda, \mathbf{R})_\mu$, we will make use of the following simple property of principal parts of rational functions:

Lemma 6.2.8. *Suppose that $X(u) \in \mathbb{C}((u^{-1}))$ is the expansion at $u = \infty$ of a rational function $f(u)/g(u)$, where $f, g \in \mathbb{C}[u]$. Then*

$$\underline{X(u)} = 0 \iff g \text{ divides } f$$

As explained in Remark 5.4.2, we may think of $\widetilde{B}_\mu^\lambda(\mathbf{R})$ as a quotient of

$$\mathbb{C}[A_i^{(s)} : i \in I, 1 \leq s \leq m_i]$$

Therefore a point in $\operatorname{MaxSpec} \widetilde{B}_\mu^\lambda(\mathbf{R})$ is equivalent to a tuple of multisets $\mathbf{S} = (S_i)_{i \in I}$ of complex numbers, with $|S_i| = m_i$: such a tuple corresponds to the homomorphism $\mathbb{C}[H_\bullet^{(\bullet)}] \to \mathbb{C}$ given in series form by

$$A_i(u) \mapsto \prod_{s \in S_i} (1 - \tfrac{1}{2} s u^{-1})$$

Lemma 6.2.9. *Let $q \in \mathcal{B}(\varpi_i, 0)$, and write*

$$q = y_{i,0} z_{\mathbf{U}}^{-1} = \prod_{j,k} y_{j,k}^{b_{j,k}}$$

Consider $\mathbf{S} = (S_i)_{i \in I}$ and the corresponding map $\mathbb{C}[H_\bullet^{(\bullet)}] \to \mathbb{C}$, as above. Then the image of $\underline{H_q(u)}$ under this map is zero if and only if there is an inclusion of multisets of $\mathbb{C}$:

$$\bigcup_{\substack{j,k \\ b_{j,k}<0}} (S_j - k)^{-b_{j,k}} \subset \left(\bigcup_{j,k} (R_j - k)^{U_j(k-2)} \right) \cup \left(\bigcup_{\substack{j,k \\ b_{j,k}>0}} (S_j - k)^{b_{j,k}} \right) \tag{6.2}$$

Here, for a multiset X and integer n, X^n denotes the multiset union $\cup_{\ell=1}^n X$. By convention, $X^0 = \emptyset$.

Proof. By the definition of $H_q(u)$, its image under the map corresponding to $\mathbf{S}$ is the rational function

$$\Big(\prod_{j,k} \prod_{c \in R_j} (u - \tfrac{1}{2}c + \tfrac{1}{2}k)^{U_j(k-2)} \Big) \Big(\prod_{j,k} \prod_{s \in S_j} (u - \tfrac{1}{2}s + \tfrac{1}{2}k)^{b_{j,k}} \Big)$$

By Lemma 6.2.8, the principal part of this rational function is zero if and only if its denominator divides its numerator. Since the multisets in (6.2) encode the roots of these polynomials, the principal part is zero if and only if (6.2) holds. □

Since MaxSpec $\widetilde{B}_\mu^\lambda(\mathbf{R})$ is exactly the set of $\mathbf{S}$ for which all $\underline{H_q(u)}$ map to zero, this lemma is the key tool in the following result:

Theorem 6.2.10. *There is a bijection of sets*

$$\operatorname{MaxSpec} \widetilde{B}_\mu^\lambda(\mathbf{R}) \longrightarrow \mathcal{B}(\lambda, \mathbf{R})_\mu$$

defined by $\mathbf{S} \mapsto y_{\mathbf{R}} z_{\mathbf{S}}^{-1}$.

Proof. Firstly, we will show that the image $p := y_{\mathbf{R}} z_{\mathbf{S}}^{-1}$ necessarily lands in $\mathcal{B}$, i.e. that all variables $y_{i,k}$ which appear have $k \in \mathbb{Z}$ and satisfy the parity condition $i \in I_{\overline{k}}$ as per Section 2.2 Consider the element

$$\tilde{f}_i(y_{i,0}) = y_{i,0} z_{i,-2}^{-1} = y_{i,-2}^{-1} \prod_{j \sim i} y_{j,-1} \in \mathcal{B}(\varpi_i, 0)$$

The principal part of the corresponding series $H_{\tilde{f}_i(y_{i,0})}(u)$ must map to zero under $\mathbf{S}$. By Lemma 6.2.9, this is equivalent to the inclusion

$$S_i + 2 \subset R_i \cup \bigcup_{j \sim i} (S_j + 1)$$

It is not hard to see that these containments for all $i \in I$, together with the integrality and parity conditions on $\mathbf{R}$, imply the desired integrality and parity conditions on $\mathbf{S}$.

Next, we will show that the image p is a *regular* element of $\mathcal{B}$. For each $n \in \mathbb{Z}$, there is an isomorphism of crystals $\mathcal{B}(\varpi_i, 0) \xrightarrow{\sim} \mathcal{B}(\varpi_i, n)$ which acts by translation on the variables: $y_{j,k} \mapsto y_{j,k+n}$. For $q \in \mathcal{B}(\varpi_i, 0)$, denote its image by $q_n \in \mathcal{B}(\varpi_i, n)$.

For any fixed q, we claim that the corresponding inclusion of multisets (6.2) is equivalent to the inequalities $E_{q_n}(p) \geq 0$ for all n of opposite parity to i. Indeed, the integers $E_{q_n}(p)$ encode the difference in multiplicity of the number $n-1$ between the multisets appearing on the right-hand and left-hand sides of (6.2). Because $\mathbf{R}$ and $\mathbf{S}$ satisfy the parity conditions, there is an inclusion of multisets (6.2) if and only if these multiplicities are non-negative. By considering all $q \in \mathcal{B}(\varpi_i, 0)$, it follows that p is regular.

Finally, since $p = y_{\mathbf{R}} z_{\mathbf{S}}^{-1}$ is regular, by Theorem 6.2.4 we know that $p \in \mathcal{B}(\lambda, \mathbf{R})$. Since $|S_i| = m_i$ for all i, it follows that $p \in \mathcal{B}(\lambda, \mathbf{R})_\mu$ as claimed. □

This completes the proof of Conjecture 6.1.1 in type A.

Corollary 6.2.11. *For $\mathfrak{g}$ of type A, the map $J \mapsto y(J)$ gives a bijection $H_\mu^\lambda(\mathbf{R}) \cong \mathcal{B}(\lambda, \mathbf{R})_\mu$.*

For general $\mathfrak{g}$, we expect that there exist series $T_{\gamma,\gamma}(u) \in Y[[u^{-1}]]$ such that the coefficients of $\underline{G_\gamma(u) T_{\gamma,\gamma}(u)}$ are in L_μ^λ, such that in $\mathbb{C}[H_\bullet^{(\bullet)}]$ we have

$$\Pi(G_\gamma(u) T_{\gamma,\gamma}(u)) = H_\gamma(u),$$

and such that the coefficients of $\underline{H_\gamma(u)}$ generate $B(Y_\mu^\lambda(\mathbf{R}))$. Exhibiting such elements would prove Conjecture 6.1.1 in general.

By a calculation similar those at the beginning of the proof of Proposition 5.3.3 one can show that the series

$$\Pi(\underline{u^{\mu_i + m_i} T_{f_i v_{\varpi_i}, f_i v_{\varpi_i}}(u)}) = \underline{H_{\tilde{f}_i(y_{i,0})}(u)}$$

are always in the ideal defining the B-algebra $B(Y_\mu^\lambda(\mathbf{R}))$. The next result follows from the argument at the beginning of the proof of the previous theorem.

Proposition 6.2.12. *For general $\mathfrak{g}$, consider a highest weight $J \in H_\mu^\lambda(\mathbf{R})$, and encode the action of $A_i(u)$ in a tuple of multisets $\mathbf{S} = (S_i)_{i \in I}$. Then these satisfy the inclusions of multisets*

$$S_i + 2 \subset R_i \cup \bigcup_{j \sim i} (S_j + 1)$$

In particular, the map $J \mapsto y(J) = y_{\mathbf{R}} z_{\mathbf{S}}^{-1}$ lands in $\mathcal{B}$.

6.2.4 Coproducts and product monomial crystals

In §5.4.1 we defined a coproduct

$$\widetilde{\Delta}_{\mu',\mu''}^{\lambda',\lambda''} : \widetilde{B}_\mu^\lambda(\mathbf{R}) \longrightarrow \widetilde{B}_{\mu'}^{\lambda'} \otimes_{\mathbb{C}[\hbar]} \widetilde{B}_{\mu''}^{\lambda''},$$

which induces a map upon specialization:

$$\widetilde{\Delta}_{\mu',\mu''}^{\lambda',\lambda''} : \widetilde{B}_\mu^\lambda(\mathbf{R}) \longrightarrow \widetilde{B}_{\mu'}^{\lambda'}(\mathbf{R}') \otimes \widetilde{B}_{\mu''}^{\lambda''}(\mathbf{R}'')$$

Proposition 6.2.13. *On the level of maximal ideals, the (specialized) coproduct corresponds to the product map*

$$\mathcal{B}(\lambda', \mathbf{R}')_{\mu'} \times \mathcal{B}(\lambda'', \mathbf{R}'')_{\mu''} \longrightarrow \mathcal{B}(\lambda, \mathbf{R})_{\mu}$$

under the bijection from Theorem 6.2.10.

Remark 6.2.14. *The fact that our coproduct maps corresponds to union of multisets can be understood in terms of the ring Λ of symmetric functions. Working over $\mathbb{C}$ for simplicity, recall that this is the polynomial ring*

$$\Lambda = \mathbb{C}[e_1, e_2, e_3 \ldots]$$

where e_k is the kth elementary symmetric function (in countably many variables). There is a coproduct $\Lambda \longrightarrow \Lambda \otimes \Lambda$ defined by $e_k \mapsto \sum_{i+j=k} e_i \otimes e_j$, see e.g. Macdonald's book [Mac79].

For each m, we can consider the space $\operatorname{Sym}^m \mathbb{A}^1$ of divisors on $\mathbb{A}^1$ of degree m as a closed subscheme of $\operatorname{Spec}\Lambda$, via the map

$$\Lambda \twoheadrightarrow \mathbb{C}[e_1, \ldots, e_m] \cong \mathbb{C}[x_1, \ldots, x_m]^{S_m},$$

taking the e_k to the elementary symmetric functions of the x_i. The coproduct on Λ induces a domimant morphism

$$\operatorname{Sym}^n \mathbb{A}^1 \times \operatorname{Sym}^n \mathbb{A}^1 \longrightarrow \operatorname{Sym}^{n+m} \mathbb{A}^1$$

which corresponds to taking the sum of two divisors. Equivalently, thinking of divisors as multisets, this is the operation of taking their union.

6.3 The case of $\mathfrak{sl}_2$

As in §2.4, elements of the product monomial crystal $\mathcal{B}(\lambda, \mathbf{R})$ have the form $p = y_{\mathbf{R}} z_{\mathbf{S}}^{-1}$. Recall also that the fundamental monomial crystal for $\mathfrak{sl}_2$ is

$$\mathcal{B}(\varpi, 0): \quad y_0 \xrightarrow{\tilde{f}} y_{-2}^{-1}$$

As explained in the proof of Theorem 6.2.10, the regularity conditions from Definition 6.2.3 are equivalent to the containments of multisets given in (6.2). In the $\mathfrak{sl}_2$ case, the conditions (6.2) corresponding to y_0 and $y_{-2}^{-1} = y_0 z_{-2}^{-1}$ are

$$\emptyset \subset \mathbf{S}, \qquad \mathbf{S} + 1 \subset \mathbf{R}$$

Chapter 7

Quiver varieties

7.1 Nakajima quiver varieties

In this section, we will overview the definition of Nakajima quiver varieties. These varieties have been very thoroughly studied, and there are many excellent sources to which we refer the reader for more details, e.g. Nakajima's papers [Nak98], [Nak01a], or Ginzburg's lectures [Gin09] for an overview.

7.1.1 Representations of doubled framed quivers

Fix an orientation of the Dynkin diagram I of $\mathfrak{g}$, that is, a choice of orientation of each edge. Consider the associated quiver Q, which has vertex set I and arrow set Ω corresponding to this orientation. For $a \in \Omega$, denote the source of a by $s(a)$, and the target of a by $t(a)$. If $s(a) = i$ and $t(a) = j$, then we will sometimes write $i \to j$ instead of a.

For each arrow $a \in \Omega$, we will denote by $\overline{a}$ the opposite arrow which goes from $t(a)$ to $s(a)$. Denote by $\overline{\Omega}$ the set consisting of these opposite arrows.

Fix two I-graded vector spaces $V = \bigoplus_{i \in I} V_i$ and $W = \bigoplus_{i \in I} W_i$. Consider the vector space

$$\mathbf{M}(V, W) = \bigoplus_{a \in \Omega \cup \overline{\Omega}} \operatorname{Hom}(V_{s(a)}, V_{t(a)}) \oplus \bigoplus_{i \in I} \operatorname{Hom}(W_i, V_i) \oplus \bigoplus_{i \in I} \operatorname{Hom}(V_i, W_i),$$

We will denote an element of $\mathbf{M}(V, W)$ as a tuple

$$\left((B_a)_{a \in \Omega \cup \overline{\Omega}}, (\eta_i)_{i \in I}, (\varepsilon_i)_{i \in I}\right)$$

or simply $(B_a, \eta_i, \varepsilon_i)$.

Remark 7.1.1. *The space* $\mathbf{M}(V, W)$ *is the set of representations of the doubled framed quiver associated to* Q*, on fixed vector spaces* V, W*. Here* **framed** *means that we have added an additional vertex* i' *for each* $i \in I$*, with an arrow* $i \to i'$*:*

$$\begin{array}{ccccccccc} 1' & & 2' & & 3' & & 4' & & 5' \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ 1 & \longrightarrow & 2 & \longleftarrow & 3 & \longrightarrow & 4 & \longrightarrow & 5 \end{array}$$

This produces a framed quiver Q_{framed}. Meanwhile, **doubled** *means we have added an opposite arrow for every arrow in Q_{framed}. For those arrows $a \in \Omega$, this is just $\overline{a} \in \overline{\Omega}$ as above, while for the remaining arrows $i \to i'$ we add new arrows $i' \to i$.*

We will want to relate $\mathbf{M}(V,W)$, and its associated Nakajima quiver variety, with the Lie algebra $\mathfrak{g}$ corresponding to I. As a start, we relate the graded dimensions of the spaces V, W to elements of the root and weight lattices of $\mathfrak{g}$ in the following (asymmetric) way:

$$\underline{\dim}(V) := (\dim V_i)_{i\in I} \longmapsto \sum_{i\in I} \dim(V_i)\alpha_i \in Q, \tag{7.1}$$
$$\underline{\dim}(W) := (\dim W_i)_{i\in I} \longmapsto \sum_{i\in I} \dim(W_i)\varpi_i \in P$$

By abuse of notation, we will simply identify these elements. More generally, for a subspace $S \subset V$, we will identify

$$\underline{\dim}(S) = \sum_{i\in I} \dim(S_i)\alpha_i \in Q$$

7.1.2 Symplectic structure

There is an important interpretation of $\mathbf{M}(V,W)$ as the cotangent bundle to a vector space. Recall that for any $\mathbb{C}$–vector spaces U_1, U_2, there is a non-degenerate trace pairing

$$\mathrm{Hom}(U_1,U_2) \times \mathrm{Hom}(U_2,U_1) \longrightarrow \mathbb{C}, \qquad (A,B) \mapsto \mathrm{tr}(AB)$$

Now consider the set of representations of the framed quiver Q_{framed} on the vector spaces V and W. This is given by elements of

$$\mathbf{N}(V,W) = \bigoplus_{a\in\Omega} \mathrm{Hom}(V_{s(a)}, V_{t(a)}) \oplus \bigoplus_{i\in I} \mathrm{Hom}(V_i, W_i)$$

The trace pairings allow us to identify

$$\mathbf{M}(V,W) \cong \mathbf{N}(V,W) \oplus \mathbf{N}(V,W)^* \cong T^*\mathbf{N}(V,W)$$

and as a consequence, $\mathbf{M}(V,W)$ has a natural symplectic form. See e.g. [Nak98, §3.2.].

7.1.3 Group actions

There is a natural action of the group $G_V := \prod_{i\in I} GL(V_i)$ on $\mathbf{M}(V,W)$, where an element $g = (g_i)_{i\in I} \in G_V$ acts by

$$g \cdot \big((B_a), (\eta_i), (\varepsilon_i)\big) = \big((g_{t(a)} B_a g_{s(a)}^{-1}), (g_i \eta_i), (\varepsilon_i g_i^{-1})\big) \tag{7.2}$$

This action is Hamiltonian, with moment map [Nak98, Equation (3.4)]

$$\mu : \mathbf{M}(V,W) \longrightarrow \mathfrak{g}_V^* \cong \mathfrak{g}_V, \tag{7.3}$$
$$(B_a, \eta_i, \varepsilon_i) \longmapsto \bigoplus_{i\in I} \Bigg(\sum_{\substack{a\in\Omega,\\ t(a)=i}} B_a B_{\overline{a}} - \sum_{\substack{a\in\Omega,\\ s(a)=i}} B_{\overline{a}} B_a + \eta_i \varepsilon_i \Bigg) \tag{7.4}$$

where we have identified $\mathfrak{g}_V = \bigoplus_{i\in I} \mathfrak{gl}(V_i)$ with $\mathfrak{g}_V^*$ via the trace pairing.

Remark 7.1.2. *In what follows we will focus on the level-set $\mu^{-1}(0)$, although it is possible to work more generally [Gin09]. The equation $\mu = 0$ is known as the ADHM equation, and points $(B_a, \eta_i, \varepsilon_i) \in \mu^{-1}(0)$ are also known as ADHM data, in reference to their connection with gauge theory [Nak98, Section 3].*

Similarly, the group $G_W := \prod_{i\in I} GL(W_i)$ acts on $\mathbf{M}(V,W)$, where an element $g = (g_i)_{i\in I} \in G_W$ acts by

$$g \cdot \big(B_a, \eta_i, \varepsilon_i\big) = \big(B_a, \eta_i g_i^{-1}, g_i\varepsilon_i\big)$$

This action extension extends to an action of $\widetilde{G}_W = G_W \times \mathbb{C}^\times$, by

$$(g,t)\cdot \big(B_a, \eta_i, \varepsilon_i\big) = \big(tB_a, \eta_i g_i^{-1}, t^2 g_i\varepsilon_i\big) \tag{7.5}$$

The actions of $\widetilde{G}_W$ and G_V commute, and the action of $\widetilde{G}_W$ preserves $\mu^{-1}(0)$.

Remark 7.1.3. *There are many other reasonable choices of $\mathbb{C}^\times$–action on $\mathbf{M}(V,W)$. For example, there is an action*

$$(g,t)\cdot (B_a, \eta_i, \varepsilon_i) = (tB_a, t^{k_i}\eta_i g_i^{-1}, t^{\ell_i} g_i\varepsilon_i)$$

where k_i, ℓ_i are integers satisfying $k_i + \ell_i = 2$ (the latter is required to preserve the zero-level of the moment map). However this action is related to the action (7.5) by an automorphism of $\widetilde{G}_W$:

$$\big((g_i)_{i\in I}, t\big) \longmapsto \big((t^{-k_i} g_i)_{i\in I}, t\big)$$

Actions which depend on the orientation Ω have also been studied [Nak01b, §8].

7.1.4 Notation

We will follow a non-standard notation for quiver varieties, which we explain here. Given λ, μ as per usual with $\lambda - \mu = \sum_i m_i\alpha_i$, we take

$$W_i = \mathbb{C}^{\lambda_i}, \quad V_i = \mathbb{C}^{m_i}$$

In other words, we are choosing vector spaces such that

$$\underline{\dim}(V) = \lambda - \mu = \sum_{i\in I} m_i\alpha_i,$$
$$\underline{\dim}(W) = \lambda = \sum_{i\in I} \lambda_i\varpi_i,$$

following the convention of equation (7.1).

Throughout the remainder of this thesis, we will write $\mathbf{M}(\lambda,\mu)$ instead of $\mathbf{M}(V,W)$. However, we will continue to use the notations V, W, G_V and G_W as above.

7.1.5 Hamiltonian reduction and GIT

We proceed towards a key definition of this chapter: Nakajima quiver varieties. To begin, consider the categorical quotient

$$\mathcal{M}_0(\lambda, \mu) := \mu^{-1}(0) /\!\!/ G_m = \operatorname{Spec} \mathbb{C}[\mu^{-1}(0)]^{G_m}$$

This is an affine algebraic variety, and moreover it inherits a Poisson structure as it is a Hamiltonian reduction. However, it is generally singular.

Another quotient can defined using geometric invariant theory (GIT). Consider a character θ of the group G_V. Since $G_V = \prod_{i\in I} GL(V_i)$, any character θ must have the form

$$(g_i)_{i\in I} \longmapsto \prod_{i\in I} \det(g_i)^{-\theta_i}$$

for some $\theta_i \in \mathbb{Z}$. It is useful to identify θ with a weight of $\mathfrak{g}$, according to

$$\theta \longmapsto \sum_{i\in I} \theta_i \varpi_i \in P \tag{7.6}$$

In particular, the important choices $\theta = \pm\rho = \pm \sum_i \varpi_i$ correspond to the characters

$$(g_i)_{i\in I} \longmapsto \prod_{i\in I} \det(g_i)^{\mp 1}$$

Remark 7.1.4. *In Section 7.2 we will make use of this identification of θ as a weight, and in particular will be interested in the corresponding action of the Weyl group W for $\mathfrak{g}$ on the set of characters (where W acts on weights in the usual way).*

Definition 7.1.5. *A point $(B_a, \eta_i, \varepsilon_i) \in \mu^{-1}(0)$ is called θ–***semistable** *if the following condition holds: for any subspace $S = \bigoplus_{i\in I} S_i \subset V$ which is stable under the maps B_a, we have*

$$S_i \subset \operatorname{Ker} \varepsilon_i,\ \forall\, i \in I \quad \Longrightarrow \quad \langle \theta, \underline{\dim} S\rangle \leq 0, \tag{7.7}$$

$$S_i \supset \operatorname{Im} \eta_i,\ \forall\, i \in I \quad \Longrightarrow \quad \langle \theta, \underline{\dim} S\rangle \leq \langle \theta, \underline{\dim} V\rangle \tag{7.8}$$

We will denote the set of θ–semistable points of $\mu^{-1}(0)^{\theta-ss}$.

In the above definition, recall that we are regarding $\theta \in P$ and $\underline{\dim} S \in Q$. The pairing is the usual one between the weight lattice and root lattice, so for example

$$\langle \theta, \underline{\dim} S\rangle = \langle \sum_{i\in I} \theta_i \varpi_i, \sum_{j\in I} \dim(S_j)\alpha_j\rangle = \sum_{i\in I} \theta_i \dim(S_i)$$

Remark 7.1.6. *(a) If θ is dominant regular (i.e. $\theta_i > 0$ for all i), then the above conditions are equivalent to*

$$S_i \subset \operatorname{Ker} \varepsilon_i,\ \forall i \in I \quad \Longrightarrow \quad S = \{0\}$$

(b) Similarly if θ is anti-dominant regular (i.e. $\theta_i < 0$ for all i), then the above conditions are equivalent to

$$S_i \supset \operatorname{Im} \eta_i,\ \forall i \in I \quad \Longrightarrow \quad S = \{0\}$$

The action (7.2) of G_V preserves the set of θ–semistable points. With this in mind, we now come to the main definition of this section.

Definition 7.1.7. *The* **Nakajima quiver variety** *associated to the data λ, μ and θ is defined to be*

$$\mathcal{M}_\theta(\lambda, \mu) := \mu^{-1}(0)^{\theta-ss}/G_V$$

The definition of semistability given above is due to King [Kin94], and is a reformulation for quiver varieties of the more general notion of stability in GIT theory [Gin09, Proposition 5.1.5.]. In particular, $\mathcal{M}_\theta(\lambda, \mu)$ can also be expressed as Proj of the ring of θ–semi-invariants.

The closed points of $\mathcal{M}_\theta(\lambda, \mu)$ are equivalence classes of quiver data, which we'll denote $[B_a, \eta_i, \varepsilon_i]$.

Lemma 7.1.8. *The action of $\widetilde{G}_W$ on $\mathbf{M}(\lambda, \mu)$ descends to an action on $\mathcal{M}_\theta(\lambda, \mu)$. Explicitly, for $(g, t) \in \widetilde{G}_W$ and $[B_a, \eta_i, \varepsilon_i] \in \mathcal{M}_\theta(\lambda, \mu)$, we have*

$$(g, t) \cdot [B_a, \eta_i, \varepsilon_i] = [tB_a, \eta_i g_i^{-1}, t^2 g_i \varepsilon_i]$$

We now discuss the matter of smoothness of the variety $\mathcal{M}_\theta(\lambda, \mu)$. Let us fix some notation. Consider the set of positive roots which lie below the difference $\lambda - \mu = \sum_i m_i \alpha_i$:

$$R_+^V = \{\alpha = \sum_i n_i \alpha_i \in \Delta_+ \ : \ n_i \leq m_i, \ \forall i \in I\}$$

and the set of V**-regular** weights:

$$P^V = \{\theta \in P : \langle \theta, \alpha \rangle \neq 0, \ \forall \alpha \in R_+^V\}$$

In particular, any regular weight is V-regular (recall that θ is regular if $\langle \theta, \alpha \rangle \neq 0$ for all $\alpha \in \Delta_+$).

Theorem 7.1.9 (Theorem 2.8 in [Nak98], see also Theorem 5.2.2 in [Gin09])**.**

For $\theta \in P^V$, the action of G_V on $\mu^{-1}(0)^{\theta-ss}$ is free, and the quotient $\mathcal{M}_\theta(\lambda, \mu)$ is a smooth irreducible algebraic variety of (complex) dimension

$$\dim \mathcal{M}_\theta(\lambda, \mu) = \langle \lambda, \lambda \rangle - \langle \mu, \mu \rangle$$

Moreover, the symplectic structure on $\mathbf{M}(\lambda, \mu)$ induces an algebraic symplectic form on $\mathcal{M}_\theta(\lambda, \mu)$.

7.1.6 Tautological bundles

Recall that for a principal (right) G-bundle $\mathcal{P} \longrightarrow X$, and a representation E of G, there is an associated vector bundle $\mathcal{E}$ on X:

$$\begin{array}{c} \mathcal{P} \times_G E \\ \big\downarrow \pi \\ X \end{array}$$

Explicitly, $\mathcal{E}$ consists of equivalence classes of pairs $[p, v]$ where $p \in \mathcal{P}$ and $v \in L$, with the equivalence relation

$$[pg, v] = [p, gv], \qquad \forall g \in G$$

and the map π takes $[p, v]$ to $[p] \in X \cong \mathcal{P}/G$.

Assume that $\theta \in P^V$. Then $\mu^{-1}(0)^{\theta-ss} \longrightarrow \mathcal{M}_\theta(\lambda, \mu)$ is a principal (left) G_V–bundle, and we can apply the above construction to produce vector bundles on $\mathcal{M}_\theta(\lambda, \mu)$. There is a natural representation of G_V on each graded component $V_i \subset V$. We will also consider each component W_i to be a representation of G_V, with trivial action.

Definition 7.1.10. *The vector bundles $\mathcal{V}_i, \mathcal{W}_i$ on $\mathcal{M}_\theta(\lambda, \mu)$ associated to the representations V_i and W_i are called* **tautological bundles**.

There is an important complex formed from the tautological bundles [Nak98, Section 4]:

$$0 \longrightarrow \mathcal{V}_i \xrightarrow{\phi_i} \mathcal{W}_i \oplus \bigoplus_{j \sim i} \mathcal{V}_j \xrightarrow{\tau_i} \mathcal{V}_i \longrightarrow 0 \tag{7.9}$$

In the fibre over $[B_a, \eta_i, \varepsilon_i]$, these morphisms are induced by the maps of vector spaces

$$\phi_i = \varepsilon_i \oplus \bigoplus_{\substack{a \in \Omega \cup \overline{\Omega}, \\ s(a)=i}} B_a, \qquad \tau_i = \eta_i + \sum_{\substack{a \in \Omega, \\ t(a)=i}} B_a - \sum_{\substack{a \in \Omega, \\ s(a)=i}} B_{\overline{a}} \tag{7.10}$$

The fact that this is a complex follows from the vanishing of the moment map, by looking at the maps (7.10) in each fibre. The next Lemma is a straightforward generalization of [Nak01a, Lemma 2.9.2], and follows from Remark 7.1.6 by again looking at each fibre.

Lemma 7.1.11. *(a) If θ is dominant regular, then ϕ_i is injective.*

(b) If θ is antidominant regular, then τ_i is surjective.

The tautological bundles have $\widetilde{G}_W$–equivariant structures, where $(g, t) \in G_W \times \mathbb{C}^\times$ acts by

$$\begin{aligned} (g,t) \cdot \big[(B_a, \eta_i, \varepsilon_i), v\big] &= \big[(tB_a, \eta_i g_i^{-1}, t^2 g_i \varepsilon_i), v\big], && \text{for } v \in V_i, \\ (g,t) \cdot \big[(B_a, \eta_i, \varepsilon_i), w\big] &= \big[(tB_a, \eta_i g_i^{-1}, t^2 g_i \varepsilon_i), g_i w\big], && \text{for } w \in W_i \end{aligned}$$

The exact sequence (7.9) can be made $\widetilde{G}_W$–equivariant, at the cost of tensoring certain factors by the trivial line bundle q having $\mathbb{C}^\times$–weight 1:

$$0 \longrightarrow \mathcal{V}_i \xrightarrow{\phi_i} q^2 \mathcal{W}_i \oplus \bigoplus_{j - i} q \mathcal{V}_j \xrightarrow{\tau_i} q^2 \mathcal{V}_i \longrightarrow 0 \tag{7.11}$$

7.1.7 Tensor product quiver varieties

Fix a decomposition $\lambda = \lambda' + \lambda''$, where $\lambda, \lambda', \lambda'' \in P_+$. Then we can identify

$$W = W' \oplus W''$$

Define a coweight $\eta : \mathbb{C}^\times \to G_W$ by $\eta(t) = \mathrm{id}_{W'} \oplus t\, \mathrm{id}_{W''}$. This induces a $\mathbb{C}^\times$–action on $\mathcal{M}_\theta(\lambda, \mu)$.

Lemma 7.1.12 ([Nak01b, Lemma 3.2])**.** *Suppose that $\theta \in P_+$ is dominant. Then there is an isomorphism*

$$\bigsqcup_{\mu' + \mu'' = \mu} \mathcal{M}_\theta(\lambda', \mu') \times \mathcal{M}_\theta(\lambda'', \mu'') \xrightarrow{\sim} \mathcal{M}_\theta(\lambda, \mu)^{\mathbb{C}^\times}$$

Let us denote the inclusion

$$\iota^{\lambda',\lambda''}_{\mu',\mu''} : \mathcal{M}_\theta(\lambda',\mu') \times \mathcal{M}_\theta(\lambda'',\mu'') \hookrightarrow \mathcal{M}_\theta(\lambda,\mu)^{\mathbb{C}^\times} \hookrightarrow \mathcal{M}_\theta(\lambda,\mu)$$

Explicitly, for point $\big([B'_a,\eta'_i,\varepsilon'_i],[B''_a,\eta''_i,\varepsilon''_i]\big) \in \mathcal{M}_\theta(\lambda',\mu') \times \mathcal{M}_\theta(\lambda'',\mu'')$. Then its image under $\iota^{\lambda',\lambda''}_{\mu',\mu''}$ is defined by the sum:

$$[B'_a \oplus B''_a, \eta'_i \oplus \eta''_i, \varepsilon'_i \oplus \varepsilon''_i] \in \mathcal{M}_\theta(\lambda,\mu)^{\mathbb{C}^\times}$$

Consider also embedding $G_{W'} \times G_{W''} \times \mathbb{C}^\times \hookrightarrow G_W \times \mathbb{C}^\times$ by $(g',g'',t) \mapsto (g' \oplus g'', t)$. Let $G_{W'} \times G_{W''} \times \mathbb{C}^\times$ act on $\mathcal{M}_\theta(\lambda,\mu)$ via this inclusion.

Lemma 7.1.13.

(a) The inclusion $\iota^{\lambda',\lambda''}_{\mu',\mu''}$ is $G_{W'} \times G_{W''} \times \mathbb{C}^\times$–equivariant.

(b) There are $G_{W'} \times G_{W''} \times \mathbb{C}^\times$–equivariant isomorphisms

$$\iota^*(\mathcal{V}_i) \cong \mathcal{V}'_i \boxtimes \mathcal{V}''_i, \qquad \iota^*(\mathcal{W}_i) \cong \mathcal{W}'_i \boxtimes \mathcal{W}''_i,$$

where $\iota = \iota^{\lambda',\lambda''}_{\mu',\mu''}$.

7.2 Weyl group action

In this section we will consider certain isomorphisms between Nakajima quiver varieties, corresponding to the generators s_i of the Weyl group W:

$$\mathcal{S}_i : \mathcal{M}_\theta(\lambda,\mu) \xrightarrow{\sim} \mathcal{M}_{s_i\theta}(\lambda, s_i\mu)$$

There are several approaches to such isomorphisms that have been studied in the literature. For example, Nakajima used analytic techniques [Nak94]. The approach we follow here is that of Maffei [Maf02], who used the reflection functors of Lusztig [Lus00].

Essentially, the idea is that $\mathcal{S}_i$ will replace the complex

$$0 \longrightarrow V_i \xrightarrow{\phi_i} W_i \oplus \bigoplus_{j-i} V_j \xrightarrow{\tau_i} V_i \longrightarrow 0$$

with the complex

$$0 \longrightarrow \operatorname{Coker}(\phi_i) \xrightarrow{\phi_i\tau_i} W_i \oplus \bigoplus_{j-i} V_j \twoheadrightarrow \operatorname{Coker}(\phi_i) \longrightarrow 0$$

In doing so, we will replace the ADHM data $(B_a, \eta_i, \varepsilon_i)$ by using the components of the maps in the latter complex - this only changes the ADHM near vertex $i \in I$.

Remark 7.2.1. *In fact, the isomorphisms $\mathcal{S}_i$ in the form below appeared in a preprint version of Maffei's paper [Maf02] (available at* `arXiv:math/0003159`*). In the published version, framing vertices are not considered in the same manner.*

7.2.1 Reflection varieties

Consider a dominant weight λ, and a weight μ such that $\lambda - \mu = \sum_i m_i \alpha_i \geq 0$. As in Section 7.1.4, we consider graded vector spaces V, W with $V_i = \mathbb{C}^{m_i}$ and $W_i = \mathbb{C}^{\lambda_i}$, and the associated varieties $\mathbf{M}(\lambda, \mu) = \mathbf{M}(W, V)$, etc.

Consider also the weight $\mu' = s_i \mu$, and assume that $\lambda - \mu' = \sum_i m'_i \alpha_i \geq 0$. Explicitly, the differences $m = \sum_i m_i \alpha_i$ and $m' = \sum_i m'_i \alpha_i$, are related by the "dot action"

$$m' = s_i *_\lambda m := s_i(m - \lambda) + \lambda \tag{7.12}$$

Explicitly, we have $m'_i = \lambda_i - m_i + \sum_{j-i} m_j$. Define the graded vector space $V' = \bigoplus_{i \in I} V'_i$ with $V'_i = \mathbb{C}^{m'_i}$, and the associated varieties $\mathbf{M}(\lambda, s_i \mu) = \mathbf{M}(W, V')$, etc.

The following definition, due originally to Lusztig [Lus00], is key to defining the Weyl group action:

Definition 7.2.2. *Define the* **reflection variety**

$$Z_i(\lambda, \mu) \subset \mu^{-1}(0) \times \mu^{-1}(0) \subset \mathbf{M}(\lambda, \mu) \times \mathbf{M}(\lambda, s_i \mu)$$

to be the subvariety consisting of pairs $\big((B_a, \eta_i, \varepsilon_i), (B'_a, \eta'_i, \varepsilon'_i)\big)$ *such that*

(a) $B_a = B'_a$ *for all* $a \in \Omega \cup \overline{\Omega}$ *such that* $s(a), t(a) \neq i$,

(b) $\eta_j = \eta'_j$ *for all* $j \neq i$,

(c) $\varepsilon_j = \varepsilon'_j$ *for all* $j \neq i$,

(d) we have $\phi_i \tau_i = \phi'_i \tau'_i$.

(e) the following sequence is exact:

$$0 \longrightarrow V_i \xrightarrow{\phi_i} W_i \oplus \bigoplus_{j \sim i} V_j \xrightarrow{\tau'_i} V'_i \longrightarrow 0,$$

Remark 7.2.3. *The above terminology for* $Z_i(\lambda, \mu)$ *is not standard, but seems reasonable given the extisting terminology of reflection functors.*

Maffei proved that the variety $Z_i(\lambda, \mu)$ is well-behaved with respect to stability conditions:

Lemma 7.2.4 ([Maf02])**.** *Let* $\big((B_a, \eta_i, \varepsilon_i), (B'_a, \eta'_i, \varepsilon'_i)\big) \in Z_i(\lambda, \mu)$*, and suppose that* $\theta \in P$ *satisfies* $\theta_i \geq 0$*. Then*

$$(B_a, \eta_i, \varepsilon_i) \text{ is } \theta\text{–semistable} \quad \Longleftrightarrow \quad (B'_a, \eta'_i, \varepsilon'_i) \text{ is } s_i\theta\text{–semistable}$$

We define $Z_i^{\theta-ss}(\lambda, \mu)$ to be the subset of pairs satisfying the conditions of the lemma. Consider the group

$$G_{i,V} = GL(V_i) \times GL(V'_i) \times \prod_{j \neq i} GL(V_j)$$

This group acts in a natural way on the space $Z_i(\lambda, \mu)$, preserving the set $Z_i^{\theta-ss}(\lambda, \mu)$.

Note that there are projections

$$\begin{array}{ccc} & Z_i(\lambda,\mu) & \\ \swarrow^{p_1} & & \searrow^{p_2} \\ \mu^{-1}(0) \subset \mathbf{M}(\lambda,\mu) & & \mu^{-1}(0) \subset \mathbf{M}(\lambda, s_i\mu) \end{array} \tag{7.13}$$

With these preparations, we can state the main theorem of this section:

Theorem 7.2.5 ([Maf02])**.** *Suppose that $\theta_i > 0$. Then the above projections induce isomorphisms of algebraic varieties*

$$\begin{array}{ccc} & Z_i^{\theta-ss}(\lambda,\mu)/G_{i,V} & \\ \overset{\sim}{\swarrow}_{p_1} & & \overset{\sim}{\searrow}_{p_2} \\ \mu^{-1}(0)^{\theta-ss}/G_V & & \mu^{-1}(0)^{s_i\theta-ss}/G_{V'} \end{array}$$

The projections above allow us to define isomorphisms between Nakajima quiver varieties.

Definition 7.2.6. *Let λ, μ, θ and $i \in I$ be as above. Then we define an isomorphism of algebraic varieties*

$$\mathcal{S}_i(\lambda,\mu,\theta) : \mathcal{M}_\theta(\lambda,\mu) \xrightarrow{\sim} \mathcal{M}_{s_i\theta}(\lambda, s_i\mu)$$

in the following way:

(1) if $\theta_i > 0$, then we set $\mathcal{S}_i(\lambda,\mu,\theta) = p_2 \circ p_1^{-1}$, with notation as in (7.2.5),

(2) if $\theta_i < 0$, then the component $(s_i\theta)_i = \langle s_i\theta, \alpha_i\rangle > 0$ and as in (1) there is an isomorphism

$$\mathcal{S}_i(\lambda, s_i\mu, s_i\theta) : \mathcal{M}_{s_i\theta}(\lambda, s_i\mu) \xrightarrow{\sim} \mathcal{M}_\theta(\lambda,\mu)$$

We set $\mathcal{S}_i(\lambda,\mu,\theta) = \mathcal{S}_i(\lambda, s_i\mu, s_i\theta)^{-1}$.

When λ, μ, θ are clear, we will simply write $\mathcal{S}_i = \mathcal{S}_i(\lambda,\mu,\theta)$.

7.2.2 Coxeter relations

Recall that the Weyl group W is generated by the simple reflections s_i with $i \in I$, satisfying that $s_i^2 = 1$ as well as the braid relations

$$s_is_j = s_js_i \ \text{ if } a_{ij} = 0, \quad s_is_js_i = s_js_is_j \ \text{ if } a_{ij} = -1$$

Maffei showed that maps $\mathcal{S}_i$ define an action of the Weyl group. In order to make sense of this statement, we need to restrict θ so that arbitrary compositions of these isomorphisms are well-defined. To this end, we will make the restriction that θ is regular. Indeed, in this case for any $w \in W$ and $i \in I$ we have

$$(w\theta)_i := \langle w\theta, \alpha_i\rangle = \langle \theta, w^{-1}\alpha_i\rangle \neq 0$$

In this case, the map $\mathcal{S}_i^2$ of $\mathcal{M}_\theta(\lambda,\mu)$,

$$\mathcal{M}_\theta(\lambda,\mu) \xrightarrow{\mathcal{S}_i(\lambda,\mu,\theta)} \mathcal{M}_{s_i\theta}(\lambda, s_i\mu) \xrightarrow{\mathcal{S}_i(\lambda,s_i\mu,s_i\theta)} \mathcal{M}_\theta(\lambda,\mu)$$

is the identity, by the above definition.

Theorem 7.2.7 ([Maf02]). *Assume that θ is regular. Then for $i, j \in I$, then the maps $\mathcal{S}_i, \mathcal{S}_j$ satisfy the braid relations:*

(a) if $a_{ij} = 0$, then the following diagram commutes:

$$\begin{array}{ccc} \mathcal{M}_\theta(\lambda,\mu) & \xrightarrow{\mathcal{S}_i} & \mathcal{M}_{s_i\theta}(\lambda, s_i\mu) \\ \downarrow{\scriptstyle \mathcal{S}_j} & & \downarrow{\scriptstyle \mathcal{S}_j} \\ \mathcal{M}_{s_j\theta}(\lambda, s_j\mu) & \xrightarrow{\mathcal{S}_i} & \mathcal{M}_{s_is_j\theta}(\lambda, s_js_i\mu) \end{array}$$

(b) if $a_{ij} = -1$, then the following diagram commutes:

$$\begin{array}{ccccc} \mathcal{M}_\theta(\lambda,\mu) & \xrightarrow{\mathcal{S}_i} & \mathcal{M}_{s_i\theta}(\lambda, s_i\mu) & \xrightarrow{\mathcal{S}_j} & \mathcal{M}_{s_js_i\theta}(\lambda, s_js_i\mu) \\ \downarrow{\scriptstyle \mathcal{S}_j} & & & & \downarrow{\scriptstyle \mathcal{S}_i} \\ \mathcal{M}_{s_j\theta}(\lambda, s_j\mu) & \xrightarrow{\mathcal{S}_i} & \mathcal{M}_{s_is_j\theta}(\lambda, s_is_j\mu) & \xrightarrow{\mathcal{S}_j} & \mathcal{M}_{s_is_js_i\theta}(\lambda, s_is_js_i\mu) \end{array}$$

For an arbitrary element $w \in W$, pick a reduced decomposition $w = s_{i_d} \cdots s_{i_1}$. Consider the composition

$$\mathcal{M}_\theta(\lambda,\mu) \xrightarrow{\mathcal{S}_{i_d}} \mathcal{M}_{s_{i_d}\theta}(\lambda, s_{i_d}\mu) \xrightarrow{\mathcal{S}_{i_{d-1}}} \cdots \xrightarrow{\mathcal{S}_{i_1}} \mathcal{M}_{w^{-1}\theta}(\lambda, w^{-1}\mu) \tag{7.14}$$

Since the maps $\mathcal{S}_i$ satisfy the braid relations in the sense of the previous theorem, we have:

Corollary 7.2.8 ([Maf02]). *Assume that θ is regular. Then for any $w \in W$, the above composition is independent of the choice of reduced expression.*

7.2.3 Equivariance

Recall that there is an action of the group $\widetilde{G}_W = G_W \times \mathbb{C}^*$ on $\mathcal{M}_\theta(\lambda,\mu)$, as in (7.5).

Lemma 7.2.9. *The isomorphism*

$$\mathcal{S}_i : \mathcal{M}_\theta(\lambda,\mu) \xrightarrow{\sim} \mathcal{M}_{s_i\theta}(\lambda, s_i\mu)$$

is $\widetilde{G}_W$–equivariant.

Proof. There is a $\widetilde{G}_W$–action on $Z_i(\lambda,\mu)$ defined by

$$(g,t)\cdot\big((B_a,\eta_i,\varepsilon_i),(B'_a,\eta'_i,\varepsilon'_i)\big) = \big((tB_a, \eta_i g_i^{-1}, t^2 g_i\varepsilon_i),(tB'_a,\eta'_i g_i^{-1}, t^2 g_i \varepsilon'_i)\big)$$

This action commutes with that of $G_{i,V}$, and the projections p_1, p_2 from (7.13) are $\widetilde{G}_W$–equivariant. □

7.2.4 Vector bundles for chamber weights

In this section we will consider certain vector bundles on $\mathcal{M}_\theta(\lambda,\mu)$, which are constructed via pullbacks of the tautological bundles $\mathcal{V}_i$ under the isomorphisms $\mathcal{S}_j$. By Lemma 7.2.9, any vector bundles constructed in this manner will have $\widetilde{G}_W$–equivariant structure.

To avoid confusion, we will denote the tautological bundles over $\mathcal{M}_\theta(\lambda,\mu)$ by $\mathcal{V}_i^{\theta,\mu}$ and $\mathcal{W}_i^{\theta,\mu}$. We have not included λ in our notation, since it will be kept fixed. Note that, as trivial bundles, for any $i,j \in I$ we have $\mathcal{S}_i^* \mathcal{W}_j^{s_i\theta, s_i\mu} \cong \mathcal{W}_j^{\theta,\mu}$.

We begin with the following observation regarding the isomorphism $\mathcal{S}_i : \mathcal{M}_\theta(\lambda,\mu) \longrightarrow \mathcal{M}_{s_i\theta}(\lambda, s_i\mu)$.

Lemma 7.2.10. *Assume that θ is regular.*

(a) For any $i \neq j$, we have $\mathcal{S}_i^* \mathcal{V}_j^{s_i\theta,s_i\mu} \cong \mathcal{V}_j^{\theta,\mu}$.

(b) If $\theta_i = \langle \theta, \alpha_i\rangle > 0$, then there is a $\widetilde{G}_W$–equivariant exact sequence of vector bundles over $\mathcal{M}_\theta(\lambda,\mu)$:

$$0 \longrightarrow \mathcal{V}_i^{\theta,\mu} \xrightarrow{\phi_i} q^2 \mathcal{W}_i^{\theta,\mu} \oplus \bigoplus_{j\sim i} q\mathcal{V}_j^{\theta,\mu} \xrightarrow{\tau_i'} q^2 \mathcal{S}_i^* \mathcal{V}_i^{s_i\theta,s_i\mu} \longrightarrow 0$$

where ϕ_i, τ_i' are is Definition 7.2.2.

Let $i \in I$. Then the stabilizer subgroup

$$W_{I\setminus\{i\}} = \langle s_j : j \neq i \rangle = \mathrm{Stab}_W(\varpi_i)$$

is a parabolic subgroup. In particular, it is known that any coset in $W/W_{I\setminus\{i\}}$ has a unique representative in W of minimal length [Kum02, §1.3.17].

Definition 7.2.11. *Let θ be dominant regular, and $\gamma \in W\varpi_i$ be a chamber weight. Write $\gamma = w\varpi_i$ where $w \in W$ is a minimal representative as above, and choose a reduced expression $w = s_{i_d}\cdots s_{i_1}$. Then we define a vector bundle on $\mathcal{M}_\theta(\lambda,\mu)$, called the* **vector bundle associated to** *γ, as the pull-back*

$$\mathcal{V}_\gamma^{\theta,\mu} := \mathcal{S}_{i_d}^* \mathcal{S}_{i_{d-1}}^* \cdots \mathcal{S}_{i_1}^* \left(\mathcal{V}_i^{w^{-1}\theta, w^{-1}\mu}\right)$$

Note that this is isomorphic to the pull-back of $\mathcal{V}_i^{w^{-1}\theta,w^{-1}\mu}$ under the composition (7.14). By considering the rank of the bundle $\mathcal{V}_i^{w^{-1}\theta,w^{-1}\mu}$ on $\mathcal{M}_{w^{-1}\theta}(\lambda, w^{-1}\mu)$, we see that:

Lemma 7.2.12. *The rank of $\mathcal{V}_\gamma^{\theta,\mu}$ is*

$$\langle \lambda - w^{-1}\mu, \varpi_i\rangle = \langle \lambda - \mu, \varpi_i \rangle + \langle \mu, \varpi_i - \gamma\rangle$$

Remark 7.2.13. *It is not hard to see that for any other $w' \in W$ such that $\gamma = w'\varpi_i$, the vector bundle formed as above using w' is isomorphic to $\mathcal{V}_\gamma$. This follows from part (a) of the above lemma, since we can write $w' = ws_{j_r}\cdots s_{j_1}$ where w is as above and all $j_s \neq i$. We have used minimal length coset representatives in the definition to fix a specific choice of $\mathcal{V}_\gamma$.*

7.2.5 Classes in equivariant K-theory

Our next goal will be to compute the classes of the vector bundles $\mathcal{V}_\gamma$ in $\widetilde{G}_W$–equivariant K-theory. Since $\mathcal{M}(\lambda,\mu)$ is smooth, the **equivariant K-group** $K_0^{\widetilde{G}_W}\big(\mathcal{M}(\lambda,\mu)\big)$ can be defined as the Grothendieck ring of the category of $\widetilde{G}_W$–equivariant vector bundles on $\mathcal{M}(\lambda,\mu)$. Elements consist of isomorphism classes $[\mathcal{E}]$ of $\widetilde{G}_W$–equivariant vector bundles, modulo the relations

$$[\mathcal{E}] = [\mathcal{E}'] + [\mathcal{E}'']$$

whenever there is a $\widetilde{G}_W$–equivariant exact sequence $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$. For an overview of equivariant K-theory, we refer the reader to [CG97, §5].

We will make use of the following result about reduced decompositions.

Lemma 7.2.14. *Let θ be a regular dominant weight and let $w \in W$. If $w = s_{i_d} \cdots s_{i_1}$ is a reduced decomposition for w, then for all $\ell = 1, \dots, d$ we have*

$$\langle s_{i_{\ell+1}} s_{i_{\ell+2}} \cdots s_{i_d} \theta, \alpha_\ell \rangle > 0$$

Proof. Equivalently, we must show that $\langle \theta, s_{i_d} s_{i_{d-1}} \cdots s_{i_\ell} \alpha_{i_{\ell-1}} \rangle > 0$. As in (5.5), the elements $s_{i_d} s_{i_{d-1}} \cdots s_{i_\ell} \alpha_{i_{\ell-1}}$ enumerate the inversion set Δ_w. In particular they are all in Δ_+, and since θ is dominant regular the claim follows. □

Let us introduce some notation. Recall the monomial crystal $\mathcal{B}(\varpi_i, 0)$ from §2.3 (c.f. also §5.4). For any $\gamma \in W\varpi_i$, there is a unique element

$$q_\gamma = y_{i,0} z_{\mathbf{U}}^{-1} = \prod_{j,k} y_{j,k}^{b_{j,k}} \in \mathcal{B}(\varpi_i, 0)$$

of weight γ. We define an element in $K_0^{\widetilde{G}_W}\big(\mathcal{M}_\theta(\lambda,\mu)\big)$ as follows:

$$\kappa^{\theta,\mu}(q_\gamma) := \sum_{j,k} U_j(k-2) q^k [\mathcal{W}_j^{\theta,\mu}] + \sum_{j,k} b_{j,k} q^k [\mathcal{V}_j^{\theta,\mu}] \tag{7.15}$$

We will write q for the class of the bundle q (recall that this is the trivial line bundle with $\mathbb{C}^\times$–weight 1).

The following is the main new result of this chapter.

Theorem 7.2.15. *For all $\gamma \in W\varpi_i$ and $q_\gamma \in \mathcal{B}(\varpi_i, 0)$ as above, we have $[\mathcal{V}_\gamma^{\theta,\mu}] = \kappa^{\theta,\mu}(q_\gamma)$.*

Proof. To simplify notation, we will write $w = s_d \cdots s_1$. For each $0 \le \ell \le d$, define the subexpression $w_\ell := s_d \cdots s_{\ell+1}$ and the weight $\gamma_\ell := s_\ell \cdots s_1 \varpi_i$. Consider the "partial" composition

$$\mathcal{M}_{w_\ell^{-1}\theta}(\lambda, w_\ell^{-1}\mu) \xrightarrow{\mathcal{S}_\ell} \mathcal{M}_{s_\ell w_\ell^{-1}\theta}(\lambda, s_\ell w_\ell^{-1}\mu) \cdots \xrightarrow{\mathcal{S}_1} \mathcal{M}_{w^{-1}\theta}(\lambda, w^{-1}\mu)$$

and define $\mathcal{V}_{\gamma_\ell}^{w_\ell^{-1}\theta, w_\ell^{-1}\mu} := \mathcal{S}_\ell^* \cdots \mathcal{S}_1^*(\mathcal{V}_i^{w^{-1}\theta, w^{-1}\mu})$. By induction on ℓ, we will prove a slightly stronger claim:

$$\big[\mathcal{V}_{\gamma_\ell}^{w_\ell^{-1}\theta, w_\ell^{-1}\mu}\big] = \kappa^{w_\ell^{-1}\theta, w_\ell^{-1}\mu}(q_{\gamma_\ell})$$

When $\ell = 0$, the claim follows from the definitions:

$$\big[\mathcal{V}_{\varpi_i}^{w^{-1}\theta, w^{-1}\mu}\big] = \big[\mathcal{V}_i^{w^{-1}\theta, w^{-1}\mu}\big] = \kappa^{w^{-1}\theta, w^{-1}\mu}(y_{i,0})$$

Denote $q_{\gamma_\ell} = y_{i,0} z_{\mathbf{U}}^{-1} = \prod_{j,k} y_{j,k}^{b_{j,k}}$. Then

$$\begin{aligned}
\left[\mathcal{S}_{\ell+1}^* \mathcal{V}_{\gamma_\ell}^{w_\ell^{-1}\theta, w_\ell^{-1}\mu}\right] &= \mathcal{S}_{\ell+1}^* \left(\sum_{j,k} U_j(k-2) q^k \left[\mathcal{W}_j^{w_\ell^{-1}\theta, w_\ell^{-1}\mu}\right] + \sum_{j,k} b_{j,k} q^k \left[\mathcal{V}_j^{w_\ell^{-1}\theta, w_\ell^{-1}\mu}\right] \right) \\
&= \sum_{\substack{j,k \\ j \neq \ell+1}} U_j(k-2) q^k \left[\mathcal{W}_j^{w_{\ell+1}^{-1}\theta, w_{\ell+1}^{-1}\mu}\right] + \sum_{j,k} b_{j,k} q^k \left[\mathcal{V}_j^{w_{\ell+1}^{-1}\theta, w_{\ell+1}^{-1}\mu}\right]
\end{aligned}$$

$$+ \sum_k U_{\ell+1}(k-2) q^k \left(\left[\mathcal{W}_\ell^{w_{\ell+1}^{-1}\theta, w_{\ell+1}^{-1}\mu}\right] - q^{-2} \left[\mathcal{V}_{\ell+1}^{w_{\ell+1}^{-1}\theta, w_{\ell+1}^{-1}\mu}\right] + q^{-1} \sum_{j \sim \ell+1} \left[\mathcal{V}_j^{w_{\ell+1}^{-1}\theta, w_{\ell+1}^{-1}\mu}\right] \right)$$

For the last line above, we applied part (b) of Lemma 7.2.10, which is valid since $\langle w_\ell^{-1}\theta, \alpha_\ell \rangle > 0$ by the previous lemma. By Proposition 2.3.3, this agrees with the inductive expression for $q_{\gamma_{\ell+1}}$. ☐

7.3 $\mathbb{C}^\times$–actions and the product monomial crystal

Recall that in §2.2, we have a fixed bipartition $I = I_{\overline{0}} \cup I_{\overline{1}}$ of the vertex set I. Consider the orientation of the Dynkin diagram, where we let Ω denote those edges in the quiver which go from $I_{\overline{1}}$ to $I_{\overline{0}}$ and let $\overline{\Omega}$ denote those edges which go from $I_{\overline{0}}$ to $I_{\overline{1}}$.

We now fix once and for all a regular dominant weight θ. Denote the corresponding Nakajima quiver variety by

$$\mathcal{M}(\lambda, \mu) := \mathcal{M}_\theta(\lambda, \mu)$$

This notation is common in the literature on quiver varieties, and is justified: the variety $\mathcal{M}_\theta(\lambda, \mu)$ is the same for all such θ, by part (a) of Remark 7.1.6.

Let $\mathcal{M}(\lambda) = \bigsqcup_\mu \mathcal{M}(\lambda, \mu)$ be the disjoint union of these quiver varieties.

7.3.1 Graded quiver varieties

Fix a set of parameters $\mathbf{R}$ of weight λ. Recall that the group $G_W = \prod_i GL(W_i)$ acts on $\mathcal{M}(V, W)$. Fix homomorphisms $\rho_i : \mathbb{C}^\times \to GL(W_i)$ so that R_i is the set of weights for the action of $\mathbb{C}^\times$ on W_i. This gives us an action of $\mathbb{C}^\times$ on $\mathcal{M}(\lambda, \mu)$ defined by

$$t * [B_a, \eta_i, \varepsilon_i] = \left[t B_a, \eta_i \rho_i(t), t^2 \rho_i(t)^{-1} \varepsilon_i\right]$$

In other words, we are considering the coweight $\mathbb{C}^\times \longrightarrow \widetilde{G}_W = G_W \times \mathbb{C}^\times$ defined by

$$t \longmapsto \left(\rho_i(t)^{-1}, t\right)$$

We let

$$\mathcal{M}(\lambda, \mathbf{R}) = \mathcal{M}(\lambda)^{\mathbb{C}^\times} \quad \text{and} \quad \mathcal{M}(\lambda, \mu, \mathbf{R}) = \mathcal{M}(\lambda, \mu)^{\mathbb{C}^\times}$$

be the fixed points for this action. These varieties are called **graded quiver varieties.**

By [Nak01a, §4.1], each point $x = [B_a, \eta_i, \varepsilon_i] \in \mathcal{M}(\lambda, \mu, \mathbf{R})$ determines maps (unique up to conjugation) $\sigma_i : \mathbb{C}^\times \to GL(V_i)$, such that $\sigma_i(t) \cdot (B_a, \eta_i, \varepsilon_i) = t * (B_a, \eta_i, \varepsilon_i)$. Thus a point x determines a collection $\mathbf{S} = (S_i)_{i\in I}$ of multisets of integers with $|S_i| = m_i$, which are the weights of the above $\mathbb{C}^\times$ action on V_i under σ_i.

Given such a collection $\mathbf{S}$, we let $X(\mathbf{S})$ denote the corresponding subset of $\mathcal{M}(\lambda, \mu, \mathbf{R})$. Put another way, $X(\mathbf{S})$ is the collection of fixed points under $\mathbb{C}^\times$ where the induced action on the fiber of the tautological bundle V_i at that point has spectrum S_i.

Remark 7.3.1. *In Nakajima's papers,* $\mathbf{S}$ *is denoted* ρ *and* $X(\mathbf{S})$ *is denoted* $\mathfrak{Z}(\rho)$ *or* $\mathfrak{M}(\rho)$*. We have changed certain signs in the* $\mathbb{C}^\times$*-actions to match the conventions from previous sections.*

To make this more explicit, denote by $W_i(k)$ and $V_i(k)$ the $\mathbb{C}^\times$-weight spaces of weight $k \in \mathbb{Z}$ under the actions ρ_i, resp. σ_i. Then the multisets R_i, S_i encode dimensions:

$$R_i(k) = \dim W_i(k), \quad S_i(k) = \dim V_i(k).$$

The identity $\sigma_i(t) \cdot (B, \eta, \varepsilon) = t * (B, \eta, \varepsilon)$ is equivalent to

$$B_a(V_{s(a)}(k)) \subset V_{t(a)}(k+1), \quad \eta_i(W_i(k)) \subset V_i(k), \quad \varepsilon(V_i(k)) \subset W_i(k+2).$$

Consider the graded components of the maps ϕ_i, τ_i from (7.10):

$$\phi_i(k) : V_i(k-2) \to W_i(k) \oplus \bigoplus_{s(a)=i} V_{t(a)}(k-1),$$
$$\tau_i(k) : W_i(k) \oplus \bigoplus_{s(a)=i} V_{t(a)}(k-1) \to V_i(k).$$

Since θ is dominant regular, it follows by Lemma 7.1.11 that $\phi_i(k)$ is injective. Thus we get a complex $C_i(k)^p$, $p = -1, 0, 1$ as follows

$$V_i(k-2) \xrightarrow{\phi_i(k)} W_i(k) \oplus \bigoplus_{s(a)=i} V_{t(a)}(k-1) \xrightarrow{\tau_i(k)} V_i(k).$$

7.3.2 Crystal structure

As in Section 2.3.1, given collections of multisets $\mathbf{R}, \mathbf{S}$, we consider

$$y_{\mathbf{R}} z_{\mathbf{S}}^{-1} = \prod_{i\in I, c\in R_i} y_{i,c} \prod_{i \in I, k\in S_i} z_{i,k}^{-1} = \prod_{i,k} y_{i,k}^{a_{i,k}}.$$

A simple computation shows the following result.

Lemma 7.3.2. *On the locus* $X(\mathbf{S})$*, we have* $a_{i,k} = \sum_p (-1)^p \dim C_i(k)^p$.

The next result follows from the work of Nakajima [Nak01a, Theorem 5.5.6]:

Proposition 7.3.3. *For each* $\mathbf{S}$*, if* $X(\mathbf{S})$ *is non-empty, then it is a connected component of* $\mathcal{M}(\lambda, \mu, \mathbf{R})$*. In particular there is an injective map*

$$\pi_0\big(\mathcal{M}(\lambda, \mathbf{R})\big) \longrightarrow \mathcal{B}, \qquad X(\mathbf{S}) \longmapsto y_{\mathbf{R}} z_{\mathbf{S}}^{-1}$$

In [Nak01b], Nakajima studies a different $\mathbb{C}^\times$–action on $\mathcal{M}(\lambda,\mu)$, and constructs a crystal structure on the fundamental group of its fixed-point set. By comparing Nakajima's action with the $\mathbb{C}^\times$–action defined above, in [KTW+15] we proved:

Theorem 7.3.4 ([KTW+15, Proposition 7.7])**.** *The image of the map $\pi_0\big(\mathcal{M}(\lambda,\mathbf{R})\big) \longrightarrow \mathcal{B}$ is the product monomial crystal $\mathcal{B}(\lambda,\mathbf{R})$. There is an isomorphism of crystals*

$$\pi_0\big(\mathcal{M}(\lambda,\mathbf{R})\big) \xrightarrow{\sim} \mathcal{B}(\lambda,\mathbf{R})$$

7.3.3 Some results used earlier

The following two results were used in the proofs of Lemmas 6.2.6 and 6.2.7.

Lemma 7.3.5. *Let $p = y_{\mathbf{R}} z_{\mathbf{S}}^{-1} \in \mathcal{B}(\lambda,\mathbf{R})$, and write $p = \prod_{i,k} y_{i,k}^{a_{i,k}}$.*

1. *If $a_{i,k} > 0$, then $z_{i,k-2}^{-1} p \in \mathcal{B}(\lambda,\mathbf{R})$.*
2. *If $a_{i,k} < 0$, then $z_{i,k} p \in \mathcal{B}(\lambda,\mathbf{R})$.*

Proof. By Proposition 7.3.3, we can find $X(\mathbf{S}) \in \pi_0(\mathcal{M}(W,\mathbf{R}))$ corresponding to p. Fix $[B,\eta,\varepsilon] \in X(\mathbf{S})$. In each case we will produce a point in the appropriate $X(\mathbf{S}')$, proving that it is non-empty, and hence corresponds to an element of $\mathcal{B}(\lambda,\mathbf{R})$.

Case (1): Since $a_{i,k} > 0$, Lemma 7.3.2 gives $\dim \operatorname{Ker} \tau_i(k)/\operatorname{Im}\phi_i(k) > 0$. Choose an embedding $\mathbb{C} \hookrightarrow \operatorname{Ker}\tau_i(k)$ whose image is not contained in $\operatorname{Im}\phi_i(k)$, and extend B and ε to $V_i(k-1)\oplus\mathbb{C}$ as the components of this embedding. Reasoning as in Proposition 4.5 in [Nak94], we see that this extended datum lies in $\mu^{-1}(0)^s$. The dimension of $V_i(k-1)$ has increased by one, which corresponds to multiplying by $z_{i,k-2}^{-1}$.

Case (2): Since $a_{i,k} < 0$, Lemma 7.3.2 implies $\tau_i(k)$ is not surjective. Choose a codimension one subspace $\operatorname{Im}\tau_i(k) \subset V_i'(k+1) \subset V_i(k+1)$, and define (B',η',ε') as the restrictions of (B,η,ε). This decreases $V_i(k+1)$ by 1, so corresponds to multiplying by $z_{i,k}$. ☐

Corollary 7.3.6. *Multiplication by $z_{i,k}$ defines a bijection of sets*

$$\begin{pmatrix} p \in \mathcal{B}(\lambda,\mathbf{R}) \text{ with} \\ a_{i,k} < 0 \text{ and } a_{i,k+2} \geq 0 \end{pmatrix} \xrightarrow{\sim} \begin{pmatrix} p' \in \mathcal{B}(\lambda,\mathbf{R}) \text{ with} \\ a'_{i,k} \leq 0 \text{ and } a'_{i,k+2} > 0 \end{pmatrix}$$

Finally, we provide a proof of Proposition 2.3.3:

Proof of Proposition 2.3.3. The variety $\mathcal{M}(\lambda, w\lambda)$ consists of a single point x. Moreover since $s_i w\lambda < w\lambda$ we have $\langle w\lambda, \alpha_i\rangle > 0$, so by [Nak98, Proposition 4.5 and Corollary 4.6] we can produce from x a point $x' \in \mathcal{M}(\lambda, s_i w\lambda)$. One way is to replace the data near vertex i using the sequence

$$0 \longrightarrow \ker\tau_i \hookrightarrow W_i \oplus \bigoplus_{j\sim i} q\mathcal{V}_j \overset{\phi_i\tau_i}{\twoheadrightarrow} \ker\tau_i \longrightarrow 0$$

It is not hard to see that the spectrum of the $\mathbb{C}^\times$–action changes as claimed. ☐

7.4 The case of $\mathfrak{sl}_2$

For the Dynkin diagram of type A_1, the Nakajima quiver varieties admit a particularly nice description. Fix λ, μ integers, with $\lambda \geq 0$ and $\lambda - \mu = 2m \geq 0$ even. The data in this case consists of a pair of vector spaces $W = \mathbb{C}^\lambda, V = \mathbb{C}^m$ and maps

$$\begin{array}{c} W \\ {\scriptstyle\eta}\big\downarrow\;\big\uparrow{\scriptstyle\varepsilon} \\ V \end{array}$$

satisfying the moment map condition $\eta\varepsilon = 0 \in \mathrm{End}(V)$. The stability condition is an integer θ, and depending on the sign of θ we have:

$$\mathcal{M}_{\theta>0}(\lambda, \mu) \cong T^*\mathsf{Gr}(m, \lambda), \qquad \mathcal{M}_{\theta<0}(\lambda, \mu) \cong T^*\mathsf{Gr}(\lambda, m)$$

where $\mathsf{Gr}(m, \lambda)$ denotes the Grassmannian of m–dimensional subspaces of $\mathbb{C}^\lambda$, and $\mathsf{Gr}(\lambda, m)$ denotes the space of m–dimensional quotients of $\mathbb{C}^\lambda$. For example, if $\theta > 0$ then ε is injective by Remark 7.1.6 and there is a map

$$\mathcal{M}_\theta(\lambda, \mu) \twoheadrightarrow \mathsf{Gr}(m, \lambda), \quad [\eta, \varepsilon] \longmapsto \mathrm{Im}\,\varepsilon \subset \mathbb{C}^\lambda,$$

This is a vector bundle, whose fibre over $\mathrm{Im}\,\varepsilon$ is the space of maps η such that $\eta\varepsilon = 0$.

The action of $G_W = GL(\lambda)$ corresponds to the usual transitive action on $\mathsf{Gr}(m, \lambda)$, induced to $T^*\mathsf{Gr}(m, \lambda)$. The action of $t \in \mathbb{C}^\times$ is by t^{-2} on the cotangent bundle's fibres.

Recall that we have the trivial vector bundle $\mathcal{W}$ over $\mathcal{M}(\lambda, \mu)$, as well as vector bundles $\mathcal{V}_\varpi = \mathcal{V}$ and $\mathcal{V}_{s\varpi}$ corresponding to the Weyl orbit $W\varpi$. These fit into an equivariant exact sequence

$$0 \longrightarrow \mathcal{V} \longrightarrow q^2\mathcal{W} \longrightarrow q^2\mathcal{V}_{s\varpi} \longrightarrow 0$$

Under the identification with $T^*\mathsf{Gr}(m, \lambda)$, $\mathcal{V}$ is the pull-back of the (usual) tautological bundle over $\mathsf{Gr}(m, \lambda)$ under the projection $T^*\mathsf{Gr}(m, \lambda \to \mathsf{Gr}(m, \lambda)$, while $\mathcal{V}_{s\varpi}$ is the quotient bundle $\mathcal{W}/\mathcal{V}$.

Chapter 8

Equivariant cohomology of quiver varieties

8.1 Equivariant cohomology

We will temporarily use G to denote a topological group, locally to this section. For a G–space X, the **equivariant cohomology** ring is defined as

$$H_G^*(X) := H^*(X \times_G E_G)$$

where E_G is a universal space for G: a contractible space on which G acts freely. The map $E_G \longrightarrow B_G := E_G/G$ is a universal principal G–bundle. Here, we have denoted

$$X \times_G E_G = (X \times E_G)/G,$$

the quotient under the natural G action. We will always work with coefficients in $\mathbb{C}$. For an overview of the theory of equivariant cohomology see [GKM98], [Bri98].

An interesting aspect of equivariant cohomology is that $H_G^*(pt)$ has a non-trivial ring structure, and the map $X \to pt$ induces a natural $H_G^*(pt)$–module on $H_G^*(X)$.

For a torus T, there is an isomorphism

$$H_T^*(pt) \cong \mathbb{C}[\mathfrak{h}]$$

where $\mathbb{C}[\mathfrak{h}]$ denote coordinate ring of functions on $\mathfrak{h} = \mathrm{Lie}(T)$. It is a graded isomorphism: $H_T^*(pt)$ is graded by cohomological degree, while $\mathbb{C}[\mathfrak{h}]$ is graded by giving linear functions (i.e elements of $\mathfrak{h}^*$) degree 2.

If G is a reductive group and $T \subset G$ is a maximal torus with associated Weyl group W, then there is an action of W on $H_T^*(pt)$ and graded isomorphism

$$H_G^*(pt) \cong H_T^*(pt)^W$$

which is also known as the equivariant splitting principle.

8.1.1 Chern classes

For any G–equivariant vector bundle $\mathcal{E}$ on X, there are associated **Chern classes**

$$c_1(\mathcal{E}), \dots, c_r(\mathcal{E}) \ \in \ H_G^*(X)$$

where $r = \operatorname{rk} \mathcal{E}$. These satisfy a naturality property: if $f : Y \to X$ is a G–equivariant map, then

$$f^* c_k(\mathcal{E}) = c_k(f^*\mathcal{E})$$

In particular, $c_k(\mathcal{E})$ only depends on the isomorphism class of $\mathcal{E}$.

It will be useful later for us to encode the Chern classes via the Chern series (polynomial)

$$C_{\mathcal{E}}(u) := 1 + c_1(\mathcal{E})u^{-1} + c_2(\mathcal{E})u^{-2} + \ldots \in 1 + u^{-1}H_G^*(X)[[u^{-1}]] \tag{8.1}$$

These series have the crucial property that

$$C_{\mathcal{E}}(u) = C_{\mathcal{E}'}(u) C_{\mathcal{E}'}(u)$$

for any G–equivariant exact sequence $0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0$.

The fact that the Chern class $c_s(\mathcal{E})$ is zero for $s > \operatorname{rk} \mathcal{E}$ will play a key role for us: it will impose certain relations on the generators of our equivariant cohomology rings (more precisely, these generators will come from the Kirwan map). We will refer to such relations as **rank relations**. These can be encoded in series form:

$$\underline{u^{\operatorname{rk} \mathcal{E}} C_{\mathcal{E}}(u)} = 0 \tag{8.2}$$

(recall that we denote by $\underline{X(u)}$ the principal part of a series $X(u)$).

8.2 Equivariant cohomology for $\mathcal{M}(\lambda, \mu)$

8.2.1 Fixing notations

As in the final section of the previous chapter, we will be fixing a dominant weight θ and simply denoting $\mathcal{M}(\lambda, \mu) := \mathcal{M}_\theta(\lambda, \mu)$. We will also denote the vector bundles on $\mathcal{M}(\lambda, \mu)$ from §7.2.4 by $\mathcal{V}_\gamma := \mathcal{V}_\gamma^{\theta, \mu}$.

Recall the groups $G_V = \prod_{i \in I} GL(V_i)$ and $G_W = \prod_{i \in I} GL(W_i)$. We will identify

$$\begin{aligned} H_{G_V}^*(pt) &= \mathbb{C}[A_i^{(s)} : i \in I, 1 \leq s \leq m_i] \\ H_{G_W}^*(pt) &= \mathbb{C}[R_i^{(s)} : i \in I, 1 \leq s \leq \lambda_i], \\ H_{\mathbb{C}^\times}^*(pt) &= \mathbb{C}[\hbar] \end{aligned} \tag{8.3}$$

Putting this all together, we have an isomorphism

$$H_{G_V \times G_W \times \mathbb{C}^\times}^*(pt) \xrightarrow{\sim} P_\mu^\lambda$$

where P_μ^λ is the algebra defined in (5.7). This isomorphism is graded, up to a factor of 2. Indeed, we previously let the generators $R_i^{(s)}$ and $A_i^{(s)}$ to have degree s, and $\hbar$ have degree 1. However, the Chern classes of the left-hand side naturally have even degrees, for example $c_s(\mathcal{W}_i)$ has degree $2s$.

8.2.2 Some cohomological properties of $\mathcal{M}(\lambda, \mu)$

It is known that $H^{odd}\big(\mathcal{M}(\lambda,\mu)\big) = 0$, by [Nak01a, Theorem 7.3.5] ([BPW12, Proposition 2.5] also applies, since $\mathcal{M}(\lambda,\mu)$ is a symplectic resolution). In particular, by [GKM98, Theorem 14.1] this implies that $\mathcal{M}(\lambda,\mu)$ is **equivariantly formal** for $G_W \times \mathbb{C}^\times$(or for any reductive subgroup $H \subset G_W \times \mathbb{C}^\times$). In particular,

$$H^*_{G_W\times\mathbb{C}^\times}\big(\mathcal{M}(\lambda,\mu)\big)$$

is a free module over $H^*_{G_W\times\mathbb{C}^\times}(pt)$.

It is also known that for any closed subgroup H, there is a natural isomorphism

$$H^*_{G_W\times\mathbb{C}^\times}\big(\mathcal{M}(\lambda,\mu)\big)\otimes_{H^*_{G_W\times\mathbb{C}^\times}(pt)} H^*_H(pt) \cong H^*_H\big(\mathcal{M}(\lambda,\mu)\big)$$

This is proven in [Nak01a, §7] for equivariant K–theory, that but the proofs apply to equivariant cohomology as well (see also [SVV14]).

8.2.3 The Kirwan map

Algebraically, we will define the Kirwan map as follows:

Definition 8.2.1. *The* **Kirwan map** Φ *is defined by*

$$H^*_{G_V\times G_W\times\mathbb{C}^\times}(pt) \longrightarrow H^*_{G_W\times\mathbb{C}^\times}\big(\mathcal{M}(\lambda,\mu)\big)$$

$$A_i^{(s)} \longmapsto c_s(\mathcal{V}_i), \quad R_i^{(s)} \longmapsto c_s(\mathcal{W}_i), \quad \hbar \longmapsto \tfrac{1}{2}c_1(q)$$

Remark 8.2.2. *We have introduced a factor of $\frac{1}{2}$ here to match our previous conventions (in particular for B^λ_μ). One way to think of this is via the double cover $\mathbb{C}^\times \to \mathbb{C}^\times$, $z \mapsto z^2$, but we will content ourselves to side-step this issue here.*

Let us describe this map geometrically. Consider the quotient map

$$\mu^{-1}(0)^{ss} \twoheadrightarrow \mathcal{M}(\lambda,\mu)$$

Since the G_V action on $\mu^{-1}(0)^{ss}$ is free, there is a canonical isomorphism

$$H^*_{G_V}(\mu^{-1}(0)^{ss}) \cong H^*(\mathcal{M}(\lambda,\mu)$$

Consider the (G_V–equivariant) embedding $\iota : \mu^{-1}(0)^{ss} \hookrightarrow \mathbf{M}(\lambda,\mu)$. The latter space is contractible, so overall there is an induced map

$$H^*_{G_V}(pt) \cong H^*_{G_V}\big(\mathbf{M}(\lambda,\mu)\big) \xrightarrow{\iota^*} H^*_{G_V}(\mu^{-1}(0)^{ss}) \cong H^*\big(\mathcal{M}(\lambda,\mu)\big)$$

Taking into account the $G_W \times \mathbb{C}^\times$ action, we get the above definition.

Theorem 8.2.3 ([Web15, Corollary 3.7])**.** *The Kirwan map Φ is surjective.*

For general Nakajima quiver varieties outside of finite type ADE, it is not known whether this theorem holds.

8.2.4 Chern characters of the bundles $\mathcal{V}_\gamma$

For each $\gamma \in W\varpi_i$, we constructed a vector bundle $\mathcal{V}_\gamma$ on $\mathcal{M}(\lambda,\mu)$ (Definition 7.2.11). As in §7.2.5 we denote by $q_\gamma \in \mathcal{B}(\varpi_i, 0)$ the unique element of weight γ, and write

$$q_\gamma = y_{i,0} z_{\mathbf{U}}^{-1} = \prod_{j,k} y_{j,k}^{b_{j,k}} \in \mathcal{B}(\varpi_i, 0)$$

In Theorem 7.2.15 we gave an explicit expression for the class $[\mathcal{V}_\gamma] \in K_0^{G_W \times \mathbb{C}^\times}\big(\mathcal{M}(\lambda,\mu)\big)$.

The Chern series (8.1) defines a homomorphism

$$K_0^{G_W \times \mathbb{C}^\times}\big(\mathcal{M}(\lambda,\mu)\big) \longrightarrow 1 + u^{-1} H_G^*(X)[[u^{-1}]],$$

where the left-hand side is considered as an additive group, while the right-hand side is multiplicative.

The effect of tensoring a vector bundle by q can be accounted for in its Chern series:

Lemma 8.2.4. *For any equivariant vector bundle $\mathcal{E}$ on $\mathcal{M}(\lambda,\mu)$ and any $k \in \mathbb{Z}$, we have*

$$C_{q^k \mathcal{E}}(u) = (1 + \tfrac{1}{2} k\hbar u^{-1})^{\operatorname{rk} \mathcal{E}} C_{\mathcal{E}}(u + \tfrac{1}{2} k\hbar)$$

Proof. This follows from an application the splitting principle. □

The following result is now immediate from Theorem 7.2.15:

Corollary 8.2.5. *For $\gamma \in W\varpi_i$ and $q_\gamma \in \mathcal{B}(\varpi_i, 0)$ as above, we have*

$$C_{\mathcal{V}_\gamma}(u) = \prod_{j,k} \Big((1 + \tfrac{1}{2} k\hbar u^{-1})^{\lambda_i} C_{\mathcal{W}_j}(u + \tfrac{1}{2} k\hbar) \Big)^{U_j(k-2)} \prod_{j,k} \Big((1 + \tfrac{1}{2} k\hbar u^{-1})^{m_j} C_{\mathcal{V}_j}(u + \tfrac{1}{2} k\hbar t) \Big)^{b_{j,k}}$$

8.3 The algebras B_μ^λ and $H^*_{G_W \times \mathbb{C}^\times}\big(\mathcal{M}(\lambda,\mu)\big)$

We have now expressed both of these algebras as quotients of

$$\mathcal{P}_\mu^\lambda = \mathbb{C}[\hbar, R_i^{(s)}, A_i^{(r)} : i \in I, 1 \le s \le \lambda_i, 1 \le r \le m_i]$$

On the one hand, in §5.4 we defined B_μ^λ as a quotient of $\mathcal{P}_\mu^\lambda$ by the ideal generated by all coefficients of the series $\underline{H_q(u)}$, for all $i \in I$ and each element $q \in \mathcal{B}(\varpi_i, 0)$ corresponding to an extremal weight $\gamma \in W\varpi_i$. $H_q(u)$ was defined in (5.8).

On the other hand, we have the surjective Kirwan map $\mathcal{P}_\mu^\lambda \twoheadrightarrow H^*_{G_V \times G_W \times \mathbb{C}^\times}(pt)$ as described in §8.2.3. To produce elements in its kernel, we consider the rank relations (8.2) corresponding to the vector bundle $\mathcal{V}_\gamma$:

$$\underline{u^{\operatorname{rk} \mathcal{V}_\gamma} C_{\mathcal{V}_\gamma}(u)} = 0$$

We can now prove the main result of this section:

Theorem 8.3.1. *Under the surjection* $\mathcal{P}_\mu^\lambda \twoheadrightarrow H^*_{G_W\times\mathbb{C}^\times}\big(\mathcal{M}(\lambda,\mu)\big)$*, we have*

$$H_q(u) \longmapsto u^{\operatorname{rk}\mathcal{V}_\gamma} C_{\mathcal{V}_\gamma}(u)$$

for every pair q *and* γ *corresponding as above. In particular, this surjection factors through* B_μ^λ*:*

$$\begin{array}{ccc} \mathcal{P}_\mu^\lambda & & \\ \downarrow & \searrow & \\ B_\mu^\lambda & \dashrightarrow & H^*_{G_W\times\mathbb{C}^\times}\big(\mathcal{M}(\lambda,\mu)\big) \end{array}$$

Proof. Comparing the expression (5.8) for $H_q(u)$ with the expression for $C_{\mathcal{V}_\gamma}(u)$ from Corollary 8.2.5, it is easy to see that these two series are equal up to multiplication by some rational function $\mathbb{C}(u,\hbar)$; we simply need to determine this factor.

For any j,k we have

$$\begin{aligned} R_j(u+\tfrac{1}{2}k\hbar) &= (u+\tfrac{1}{2}k\hbar)^{\lambda_j} + R_j^{(1)}(u+\tfrac{1}{2}k\hbar)^{\lambda_j-1} + \ldots + R_i^{(\lambda_j)} \\ &= (u+\tfrac{1}{2}k\hbar)^{\lambda_j}\Big(1 + R_j^{(1)}(u+\tfrac{1}{2}k\hbar)^{-1} + \ldots + R_j^{(\lambda_j)}(u+\tfrac{1}{2}k\hbar)^{-\lambda_i}\Big) \\ &\longmapsto (u+\tfrac{1}{2}k\hbar)^{\lambda_j} C_{\mathcal{W}_j}(u+\tfrac{1}{2}k\hbar) \\ &= u^{\lambda_j}(1+\tfrac{1}{2}k\hbar u^{-1})^{\lambda_j} C_{\mathcal{W}_j}(u+\tfrac{1}{2}k\hbar) \end{aligned}$$

and also,

$$(u+\tfrac{1}{2}k\hbar)^{m_j} A_j(u+\tfrac{1}{2}k\hbar) \longmapsto u^{m_j}(1+\tfrac{1}{2}k\hbar u^{-1})^{m_j} C_{\mathcal{V}_j}(u+\tfrac{1}{2}k\hbar t)$$

Therefore,

$$H_\gamma(u) \longmapsto u^N C_{\mathcal{V}_\gamma}(u)$$

where $N = \sum_{j,k} U_j(k-2)\lambda_j + \sum_{j,k} b_{j,k} m_j$. By Theorem 7.2.15, N is precisely the *virtual* rank of $\mathcal{V}_\gamma$, and therefore $N = \operatorname{rk}\mathcal{V}_\gamma$. □

Conjecture 8.3.2. *The map from the previous theorem defines an isomorphism of* $\mathcal{P}_\mu^\lambda$*–algebras:*

$$B_\mu^\lambda \cong H^*_{G_W\times\mathbb{C}^\times}\big(\mathcal{M}(\lambda,\mu)\big),$$

which is a degree doubling map.

8.3.1 Compatibilty with coproducts

Fix elements $\lambda = \lambda' + \lambda''$ from P_+, and elements $\mu = \mu' + \mu''$ from P, satisfying $\lambda \geq \mu, \lambda' \geq \mu'$ and $\lambda'' \geq \mu''$.

Recall that in §5.4.1, we defined a homomorphism

$$\Delta^{\lambda',\lambda''}_{\mu',\mu''} : B_\mu^\lambda \longrightarrow B_{\mu'}^{\lambda'} \otimes_{\mathbb{C}[\hbar]} B_{\mu''}^{\lambda''}$$

$$R_i(u) \longmapsto R_i'(u)\otimes R_i''(u), \quad A_i(u) \longmapsto A_i'(u)\otimes A_i''(u), \quad \hbar \longmapsto \hbar\otimes 1$$

In §7.1.7, following Nakajima, we described an inclusion

$$\iota^{\lambda',\lambda''}_{\mu',\mu''} : \mathcal{M}(\lambda',\mu') \times \mathcal{M}(\lambda'',\mu'') \hookrightarrow \mathcal{M}(\lambda,\mu)$$

There is an induced map in $G_{W'} \times G_{W''} \times \mathbb{C}^\times$–equivariant cohomology,

$$H^*_{G_{W'}\times G_{W''}\times\mathbb{C}^\times}\big(\mathcal{M}(\lambda,\mu)\big) \longrightarrow H^*_{G_{W'}\times G_{W''}\times\mathbb{C}^\times}\big(\mathcal{M}(\lambda',\mu')\times\mathcal{M}(\lambda'',\mu'')\big)$$

Using the equivariant Künneth isomorphism, the right-hand side is isomorphic to

$$H^*_{G_{W'}\times\mathbb{C}^\times}\big(\mathcal{M}(\lambda',\mu')\big) \otimes_{H^*_{\mathbb{C}^\times}(pt)} H^*_{G_{W''}\times\mathbb{C}^\times}\big(\mathcal{M}(\lambda'',\mu'')\big)$$

We will refer to the composition of these two maps as a coproduct.

Proposition 8.3.3. *All of the maps from Theorem 8.3.1 intertwine coproducts.*

Proof. Since we are looking at surjections, it suffices to check the claim for the maps out of P^λ_μ. By definition, the map $P^\lambda_\mu \longrightarrow B^\lambda_\mu$ respects coproducts.

For the quiver variety case, consider the Kirwan map $P^\lambda_\mu \longrightarrow H^*_{G_W\times\mathbb{C}^\times}\big(\mathcal{M}(\lambda,\mu)\big)$. The image of $A_i(u)$ is the Chern series $C_{\mathcal{V}_i}(u)$. From Lemma 7.1.13, $\mathcal{V}_i$ pulls back to $\mathcal{V}_i'\boxtimes\mathcal{V}_i''$ on $\mathcal{M}(\lambda',\mu')\times\mathcal{M}(\lambda'',\mu'')$. Under the Künneth isomorphism, the Chern series of this pullback is therefore the tensor product of the Chern series for $\mathcal{V}_i'$ and $\mathcal{V}_i''$. This is precisely the image of $A_i'(u)\otimes A_i''(u)$ under the product of Kirwan maps, proving the claim in this case. □

8.3.2 Specialization of parameters

Let us now consider a set of parameters $\mathbf{R}$. This determines a $\mathbb{C}^\times$–action on $\mathcal{M}(\lambda,\mu)$ as in §7.3.1, via the homomorphism

$$\rho : \mathbb{C}^\times \longrightarrow G_W \times \mathbb{C}^\times, \quad t \longmapsto (\rho_i(t)^{-1}, t)$$

On the level of cohomology, there is an associated restriction homomorphism

$$H^*_{G_W\times\mathbb{C}^\times}\big(\mathcal{M}(\lambda,\mu)\big) \otimes_{H^*_{G_W\times\mathbb{C}^\times}(pt)} H^*_{\mathbb{C}^\times}(pt) \cong H^*_{\mathbb{C}^\times}\big(\mathcal{M}(\lambda,\mu)\big)$$

where on the right-hand side $\mathbb{C}^\times$ acts on $\mathcal{M}(\lambda,\mu)$ by ρ. Since $\mathcal{M}(\lambda,\mu)$ is equivariantly formal, there is an injection corresponding to the inclusion the fixed point locus:

$$H^*_{\mathbb{C}^\times}\big(\mathcal{M}(\lambda,\mu)\big) \hookrightarrow H^*_{\mathbb{C}^\times}\big(\mathcal{M}(\lambda,\mu,\mathbf{R})\big)$$

We make a further specialization, and set $\hbar = 1$. Now, recall from §7.3.2 that the connected components of the fixed point set $\mathcal{M}(\lambda,\mu,\mathbf{R})$ are labelled by monomials $p = y_{\mathbf{R}} z_{\mathbf{S}}^{-1} \in \mathcal{B}(\lambda,\mathbf{R})_\mu$. Consider a component $X(\mathbf{S})$, and choose a point $x \in X(\mathbf{S})$. The inclusions

$$pt = \{x\} \hookrightarrow X(\mathbf{S}) \hookrightarrow \mathcal{M}(\lambda,\mu,\mathbf{R})$$

induce maps

$$H^*\big(\mathcal{M}(\lambda,\mu,\mathbf{R})\big) \longrightarrow H^*\big(X(\mathbf{S})\big) \longrightarrow \mathbb{C}$$

As noted in the proof of [KTW+15, Proposition 8.11], these correspond precisely to the maximal ideals of $H^*\big(\mathcal{M}(\lambda,\mu,\mathbf{R})\big)$: for an algebraic variety Y, the homomorphisms $H^*(Y)\to\mathbb{C}$ are given by $H^*(Y)\to H^*(Y_i)\to\mathbb{C}$, where Y_i ranges over the connected components of Y.

On the other hand, we can specialize B^λ_μ at the point $\mathbf{R}\times\{1\}\in\operatorname{Spec}\mathbb{C}[R_i^{(s)},\hbar]$ and get the algebra $B^\lambda_\mu(\mathbf{R})$. Any $y_\mathbf{R} z_\mathbf{S}^{-1}\in\mathcal{B}(\lambda,\mathbf{R})_\mu$ defines a point $\mathbf{S}\in\operatorname{MaxSpec}\widetilde{B}^\lambda_\mu(\mathbf{R})$ by Theorem 6.2.10, so in particular point in $\operatorname{MaxSpec}B^\lambda_\mu(\mathbf{R})$ via

$$B^\lambda_\mu(\mathbf{R})\twoheadrightarrow\widetilde{B}^\lambda_\mu(\mathbf{R})\longrightarrow\mathbb{C}$$

Proposition 8.3.4 ([KTW+15, Lemma 8.12])**.** *For any $\mathbf{S}$ such that $y_\mathbf{R} z_\mathbf{S}^{-1}\in\mathcal{B}(\lambda,\mathbf{R})_\mu$, the above specializations are compatible, in the sense that the following diagram commutes:*

$$\begin{array}{ccccc}
B^\lambda_\mu & \longrightarrow & B^\lambda_\mu(\mathbf{R}) & \longrightarrow & \mathbb{C}\\
\downarrow & & \downarrow & & \|\\
H^*_{G_W\times\mathbb{C}^\times}\big(\mathcal{M}(\lambda,\mu)\big) & \longrightarrow & H^*\big(\mathcal{M}(\lambda,\mu,\mathbf{R})\big) & \longrightarrow & \mathbb{C}
\end{array}$$

8.3.3 The conjecture of Nakajima

In [KTW+15], we proved Hikita's conjecture for the slices $\mathsf{Gr}^{\overline{\lambda}}_\mu$ and their (expected) symplectic duals $\mathcal{M}(\lambda,\mu)$:

Theorem 8.3.5 ([KTW+15, Theorem 8.1])**.** *For $\lambda\geq\mu\in P_+$, there is an isomorphism of graded $H^*_{G_W}(pt)$–algebras*

$$\mathcal{O}\big((\mathsf{Gr}^{\overline{\lambda}}_\mu)^{\mathbb{C}^\times}\big)\cong H^*\big(\mathcal{M}(\lambda,\mu)\big)$$

In §5.2 we gave a presentation for the B–algebra of $\mathcal{O}(\mathcal{G}^\lambda_\mu)$, or equivalently for $\mathcal{O}\big((\mathcal{G}^\lambda_\mu)^{\mathbb{C}^\times}\big)$. We also proposed in Conjecture 5.2.5 that this algebra is isomorphic to $\mathcal{O}\big((\mathsf{Gr}^{\overline{\lambda}}_\mu)^{\mathbb{C}^\times}\big)$. Based on the above theorem, we will now show that this conjecture is strong enough to imply Nakajima's conjecture:

Theorem 8.3.6. *Let $\lambda\in P_+$. If Conjecture 5.2.5 holds for all $\mu\in P_+$ with $\lambda\geq\mu$, then Conjecture 8.3.2 holds for all $\mu\in P$ with $\lambda\geq\mu$: there is an isomorphism of graded P^λ_μ–algebras*

$$B^\lambda_\mu\xrightarrow{\sim}H^*_{G_W\times\mathbb{C}^\times}\big(\mathcal{M}(\lambda,\mu)\big)$$

In particular, this holds in type A.

Proof. Any closed point $\mathbf{R}\times\{a\}\in\operatorname{Spec}\mathbb{C}[R_i^{(s)},\hbar]$ defines a homomorphism

$$H^*_{G_W\times\mathbb{C}^\times}(pt)\cong\mathbb{C}[R_i^{(s)},\hbar]\longrightarrow\mathbb{C}$$

Since $H^*_{G_W\times\mathbb{C}^\times}\big(\mathcal{M}(\lambda,\mu)\big)$ is equivariantly formal for $G_W\times\mathbb{C}^\times$, it is free of finte rank over $H^*_{G_W\times\mathbb{C}^\times}(pt)$, and the dimension of the base change

$$H^*_{G_W\times\mathbb{C}^\times}\big(\mathcal{M}(\lambda,\mu)\big)\otimes_{H^*_{G_W\times\mathbb{C}^\times}(pt)}\mathbb{C}$$

is independent of the choice of $\mathbf{R}$ and a. In particular it is equal $d=\dim H^*\big(\mathcal{M}(\lambda,\mu)\big)$.

Suppose that $\mu\in P_+$ is dominant. Assuming Conjecture 5.2.5 holds, by the previous theorem we conclude that

$$H^*\big(\mathcal{M}(\lambda,\mu)\big)\cong B\big(\mathcal{O}(\mathcal{G}^\lambda_\mu)\big)\twoheadrightarrow\operatorname{gr}\left(B^\lambda_\mu\otimes_{\mathbb{C}[R_i^{(s)},\hbar]}\mathbb{C}\right),$$

where the surjection on the right is as in Lemma 5.4.4. In particular, the dimension of the fibre of B^λ_μ over any closed point $\mathbf{R} \times \{a\}$ is equal to d. It follows that the dimension of the fibre over *any* point in $\operatorname{Spec} \mathbb{C}[R^{(r)}_i, \hbar]$ is d, using [GW10, Corollary 7.30] and the fact that the set of closed points is very dense [GW10, Proposition 3.35]. It follows that B^λ_μ is locally free of rank d over $\mathbb{C}[R^{(s)}_i, \hbar]$, by [GW10, Corollary 11.18]. As it is finitely presented, B^λ_μ is thus projective over $\mathbb{C}[R^{(s)}_i, \hbar]$, and hence free by the Quillen-Suslin theorem [Lan02, Theorem 3.7, XXI, §4]. Since

$$B^\lambda_\mu \twoheadrightarrow H^*_{G_W \times \mathbb{C}^\times}\big(\mathcal{M}(\lambda, \mu)\big)$$

is a surjective map between free modules of the same rank, it is an isomorphism [GW10, Corollary B.4].

If $\mu \in P$ is not dominant, we cannot immediately apply the previous theorem. Note however that the algebra explicitly presented in Theorem 5.2.4 still makes sense if μ is not dominant – let us denote this algebra by b^λ_μ – and there is still a surjection

$$b^\lambda_\mu \twoheadrightarrow \operatorname{gr}\left(B^\lambda_\mu \otimes_{\mathbb{C}[R^{(s)}_i, \hbar]} \mathbb{C}\right)$$

Moreover, Lemma 8.3.7 below shows that $b^\lambda_\mu \cong b^\lambda_{\tilde{\mu}}$ where $\tilde{\mu}$ is the dominant Weyl translate of μ. In particular, their dimensions are the same. On the other hand, it is known that there is an equality

$$\dim H^*\big(\mathcal{M}(\lambda, \mu)\big) = \dim H^*\big(\mathcal{M}(\lambda, \tilde{\mu})\big),$$

(for example, because the direct sum over μ of these rings is a finite-dimensional representation for $\mathfrak{g}$ [Nak98]). Also, since $\tilde{\mu}$ is dominant the previous theorem applies and $b^\lambda_{\tilde{\mu}} \cong H^*\big(\mathcal{M}(\lambda, \tilde{\mu})\big)$.

Putting this together, there is an isomorphism of vector spaces $b^\lambda_\mu \cong H^*\big(\mathcal{M}(\lambda, \mu)\big)$. With this in mind, the same reasoning from the dominant case now goes through, proving the claim.

□

Lemma 8.3.7. *For any $j \in I$, there is an isomorphism*

$$b^\lambda_\mu \cong b^\lambda_{s_j\mu}, \quad h^{(s)}_\nu \mapsto h^{(s)}_{s_j\nu}$$

Proof. First observe that the map

$$\mathbb{C}[h^{(\bullet)}_\bullet] \longrightarrow \mathbb{C}[h^{(\bullet)}_\bullet], \quad h^{(s)}_\nu \mapsto h^{(s)}_{s_j\nu}$$

is well-defined: it extends the map defined on the generating elements $h^{(s)}_{\varpi_i}$. Now, let $\nu \in W\varpi_i$. Then $h^{(s)}_\nu$ is in the ideal for b^λ_μ for

$$s > \langle \lambda, \tilde{\nu} \rangle - \langle \mu, \nu \rangle = \langle \lambda, \tilde{\nu} \rangle - \langle s_j\mu, s_j\nu \rangle$$

since the pairing is Weyl-invariant. $\tilde{\mu}$ is also the dominant Weyl translate of $s_j\mu$, so the right-hand side is precisely the bound for $h^{(s)}_{s_j\nu}$ to be in the ideal for $b^\lambda_{s_j\nu}$. By the symmetry of this argument, the claim follows. □

8.4 The case of $\mathfrak{sl}_2$

Recall from §7.4 that for $\mathfrak{sl}_2$, there is an isomorphism

$$\mathcal{M}(\lambda, \mu) \cong T^*\mathsf{Gr}(m, \lambda)$$

In this case,

$$P_\mu^\lambda = \mathbb{C}[\hbar, A^{(1)}, \dots, A^{(m)}, R^{(1)}, \dots, R^{(\lambda)}]$$

and the B–algebra is a quotient by relations as in §5.5. By Theorem 8.3.6, we get a presentation for

$$H^*_{GL(\lambda)\times\mathbb{C}^\times}\left(T^*\mathsf{Gr}(m, \lambda)\right)$$

In terms of Chern classes: there is an equivariant exact sequence

$$0 \longrightarrow \mathcal{V} \longrightarrow q^2\mathcal{W} \longrightarrow q^2\mathcal{V}_{s\varpi} \longrightarrow 0$$

and the relations are that

$$c_r(\mathcal{V}) = 0, \text{ for } r > \operatorname{rk}\mathcal{V}, \qquad c_r(\mathcal{V}_{s\varpi}) = 0, \text{ for } r > \operatorname{rk}\mathcal{V}_{s\varpi}$$

Bibliography

[AMR06] D. Arnaudon, A. Molev, and E. Ragoucy, *On the R-matrix realization of Yangians and their representations*, Annales Henri Poincaré **7** (2006), 1269–1325.

[BDG15] M. Bullimore, T. Dimofte, and D. Gaiotto, *The Coulomb branch of* 3d $\mathcal{N} = 4$ *theories*, 2015, arXiv:1503.04817.

[BDGH16] M. Bullimore, T. Dimofte, D. Gaiotto, and J. Hilburn, *Boundaries, mirror symmetry, and symplectic duality in* 3d $\mathcal{N} = 4$ *gauge theory*, 2016, arXiv:1603.08382,.

[BF10] A. Braverman and M. Finkelberg, *Pursuing the double affine Grassmannian, I: Transversal slices via instantons on* A_k*-singularities*, Duke Mathematical Journal **152** (2010), 175–206.

[BFN16a] A. Braverman, M. Finkelberg, and H. Nakajima, *Coulomb branches of* 3d $\mathcal{N} = 4$ *quiver gauge theories and slices in the affine grassmannian*, 2016, arXiv:1604.03625.

[BFN16b] ———, *Towards a mathematical definition of Coulomb branches of 3-dimensional* $\mathcal{N} = 4$ *gauge theories, II*, 2016, arXiv:1601.03586.

[BK05] J. Brundan and A. Kleshchev, *Parabolic presentations of the Yangian* $Y(\mathfrak{gl}_n)$, Communications in Mathematical Physics **254** (2005), no. 1, 191–220.

[BK06] ———, *Shifted Yangians and finite W–algebras*, Advances in Mathematics **200** (2006), 136–195.

[BK08] ———, *Representations of shifted Yangians and finite* W*-algebras*, Memoirs of the American Mathematical Society **196** (2008), no. 918, viii+107.

[BLPW14] T. Braden, A. Licata, N. Proudfoot, and B. Webster, *Quantizations of conical symplectic resolutions II: category* $\mathcal{O}$, 2014, arXiv:1407.0964.

[BPW12] T. Braden, N. Proudfoot, and B. Webster, *Quantizations of conical symplectic resolutions I: local and global structure*, 2012, arXiv:1407.0964.

[Bri98] M. Brion, *Equivariant cohomology and equivariant intersection theory*, Representation Theories and Algebraic Geometry, NATO Science Series, Springer Netherlands, 1998, pp. 1–37.

[CG97] N. Chriss and V. Ginzburg, *Representation theory and complex geometry*, Birkhäuser, 1997.

[CP91] V. Chari and A. Pressley, *Fundamental representations of Yangians and singularities of R-matrices*, Journal für die reine und angewandte Mathematik **417** (1991), 87–128.

[CP95] ———, *A guide to quantum groups*, Cambridge University Press, 1995.

[Dri85] V. Drinfeld, *Hopf algebras and the quantum Yang-Baxter equation*, Doklady Akademii Nauk SSSR (in Russian) **283** (1985), no. 5, 1060–1064.

[Dri87] ———, *A new realization of Yangians and quantum affine algebras*, Doklady Akademii Nauk SSSR (in Russian) **296** (1987), no. 1, 13–17.

[Fal03] G. Faltings, *Algebraic loop groups and moduli spaces of bundles*, Journal of the European Mathematical Society **5** (2003), no. 1, 41–68.

[Fog73] J. Fogarty, *Fixed point schemes*, American Journal of Mathematics **95** (1973), no. 1, 35–51.

[Gin00] V. Ginzburg, *Perverse sheaves on a loop group and Langlands' duality*, 2000, `arXiv:alg-geom/9511007`.

[Gin09] ———, *Lectures on Nakajima's quiver varieties*, 2009, `arXiv:0905.0686`.

[GKLO05] A. Gerasimov, S. Kharchev, D. Lebedev, and S. Oblezin, *On a class of representations of the Yangian and moduli space of monopoles*, Communications in Mathematical Physics **260** (2005), no. 3, 511–525.

[GKM98] M. Goresky, R. Kottwitz, and R. MacPherson, *Equivariant cohomology, Koszul duality, and the localization theorem*, Inventiones Mathematicae **131** (1998), 25–83.

[GL15] S. Gautam and V. Toledano Laredo, *Yangians, quantum loop algebras and abelian difference equations*, Journal of the AMS (2015).

[GW10] U. Görtz and T. Wedhorn, *Algebraic Geometry I*, Vieweg+Teubner Verlag, 2010.

[Hik15] T. Hikita, *An algebro-geometric realization of the cohomology ring of Hilbert scheme of points in the affine plane*, 2015, `arXiv:1501.02430`.

[Jos95] A. Joseph, *Quantum groups and their primitive ideals*, Springler-Verlag, 1995.

[Kam14] J. Kamnitzer, *Categorification of lie algebras*, Séminaire Bourbaki, Astérisque **1072** (2014), 1–22.

[Kas03] M. Kashiwara, *Realizations of crystals*, Combinatorial and geometric representation theory (Seoul, 2001), Comtemporary Mathematics, vol. 325, American Mathematical Society, 2003, pp. 133–139.

[Kin94] A. King, *Moduli of representations of finite dimensional algebras*, Quarterly Journal of Mathematics **45** (1994), no. 4, 515–530.

[KMW16] J. Kamnitzer, D. Muthiah, and A. Weekes, *On a reducedness conjecture for spherical Schubert varieties and slices in the affine Grassmannian*, 2016, `arXiv:1604.00053`.

[KMWY] J. Kamnitzer, D. Muthiah, A. Weekes, and O. Yacobi, *In preparation.*

[KTW+15] J. Kamnitzer, P. Tingley, B. Webster, A. Weekes, and O. Yacobi, *Highest weights for truncated shifted Yangians and product monomial crystals*, 2015, `arXiv:1511.09131`.

[Kum02] S. Kumar, *Kac-Moody groups, their flag varieties and representation theory*, Birkhäuser, 2002.

[KWWY14] J. Kamnitzer, B. Webster, A. Weekes, and O. Yacobi, *Yangians and quantization of slices in the affine Grassmannian*, Algebra and Number Theory **8** (2014), no. 4, 857–893.

[Lan02] S. Lang, *Algebra*, 3 ed., Springer-Verlag, 2002.

[Lev93] S. Levendorski'i, *On PBW bases for Yangians*, Letters in Mathematical Physics **27** (1993), no. 1, 37–42.

[Los12] I. Losev, *Isomorphisms of quantizations via quantizations of resolutions*, Advances in Mathematics **231** (2012), no. 3-4, 1216–1270.

[Los15] ———, *On categories $\mathcal{O}$ for quantized symplectic resolutions*, 2015, `arXiv:1502.00595`.

[Lus00] G. Lusztig, *Quiver varieties and Weyl group actions*, Annales de l'institut Fourier (2000), no. 2, 461–489.

[Mac79] I. Macdonald, *Symmetric functions and Hall polynomials*, Oxford University Press, 1979.

[Maf02] A. Maffei, *A remark on quiver varieties and Weyl groups*, Annali della Scuola Normale Superiore di Pisa, Classe di Scienze **1** (2002), no. 3, 649–686.

[Mol07] A. Molev, *Yangians and classical Lie algebras*, American Mathematical Society, 2007.

[MV03] I. Mirković and M. Vybornov, *On quiver varieties and affine Grassmannians of type A*, Comptes Rendus de l'Académie des Sciences, Series I **336** (2003), 207–212.

[MV07] I. Mirković and K. Vilonen, *Geometric Langlands dualtiy and representations of algebraic groups over commutative rings*, Annals of Mathematics **166** (2007), no. 1, 95–143.

[Nak94] H. Nakajima, *Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras*, Duke Mathematical Journal **76** (1994), no. 2, 365–416.

[Nak98] ———, *Quiver varieties and Kac-Moody algebras*, Duke Mathematical Journal **91** (1998), no. 3, 515–560.

[Nak01a] ———, *Quiver varieties and finite-dimensional representations of quantum affine algebras*, Journal of the American Mathematical Society **14** (2001), 145–238.

[Nak01b] ———, *Quiver varieties and tensor products*, Inventionaes Mathematicae **146** (2001), no. 2, 399–449.

[Nak15a] ———, *Questions on provisional Coulomb branches of 3-dimensiona $\mathcal{N} = 4$ gauge theories*, 2015, `arXiv:1510.03908`.

[Nak15b] ———, *Towards a mathematical definition of Coulomb branches of 3-dimensional $N = 4$ gauge theories, I*, 2015, `arXiv:1503.03676`.

[SVV14] P. Shan, M. Varagnolo, and E. Vasserot, *On the center of quiver-Hecke algebras*, 2014, `arXiv:1411.4392`.

[Var00] M. Varagnolo, *Quiver varieties and yangians*, Letters in Mathematical Physics **53** (2000), 273–283.

[Web11] B. Webster, *Singular blocks of parabolic category $\mathcal{O}$ and finite W–algebras*, Journal of Pure and Applied Algebra **215** (2011), no. 12, 2797–2804.

[Web15] ———, *Centers of KLR algebras and cohomology rings of quiver varieties*, 2015, `arXiv:1504.04401`.

[WWY] B. Webster, A. Weekes, and O. Yacobi, *On a quantum Mirkovíc-Vybornov isomorphism*, *In preparation.*

[Zhu16] X. Zhu, *An introduction to affine Grassmannians and the geometric Satake equivalence*, 2016, `arXiv:1603.05593`.