# STABILITY OF SINGULARITIES IN GEOMETRIC EVOLUTIONARY PDE

by

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## Abstract

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We study questions of stability of two types of singularities encountered in geometric evolutionary PDE, one in Ricci flow and the other in the context of the Einstein field equations in vacuum.

In the first part of the thesis we introduce certain spherically symmetric singular Ricci solitons and study their stability under the Ricci flow from a dynamical PDE point of view. The solitons in question exist for all dimensions $n+1 \geq 3$, and all have a point singularity where the curvature blows up; their evolution under the Ricci flow is in sharp contrast to the evolution of their smooth counterparts. In particular, the family of diffeomorphisms associated with the Ricci flow "pushes away" from the singularity causing the evolving soliton to open up immediately becoming an incomplete (but non-singular) metric. We study the local-in time stability of this dynamical evolution, under spherically symmetric perturbations of the singular initial metric. We prove a local well-posedness result for the Ricci flow in suitably weighted Sobolev spaces, which in particular implies that the "opening up" of the singularity persists for the perturbations as well.

The second problem we study concerns the backwards-in-time stability of the Schwarzschild singularity from a dynamical PDE point of view. More precisely, considering a spacelike hypersurface $\Sigma_0$ in the interior of the black hole region, tangent to the singular hypersurface $\{r=0\}$ at a single sphere, we study the problem of perturbing the Schwarzschild data on $\Sigma_0$ and solving the Einstein vacuum equations backwards in time. We obtain a local backwards well-posedness result for small perturbations lying in certain weighted Sobolev spaces. The perturbed spacetimes all have a singularity at a "collapsed" sphere on $\Sigma_0$, where the leading asymptotics of the curvature and the metric match those of their Schwarzschild counterparts to a suitably high order. As in the Schwarzschild backward evolution, the pinched initial hypersurface $\Sigma_0$

‘opens up’ instantly, becoming a regular spacelike (cylindrical) hypersurface. This result thus yields classes of examples of non-symmetric vacuum spacetimes, evolving forward-in-time from regular initial data, which form a Schwarzschild type singularity at a collapsed sphere. We rely on a precise asymptotic analysis of the Schwarzschild geometry near the singularity which turns out to be at the threshold that our energy methods can handle.

# Declaration of Originality

The research in the present thesis was conducted at the Department of Mathematics, University of Toronto, in the period between September 2012 and February 2016. The material contained in the thesis is original and a product of both collaborative and independent research. Chapter 1 stemmed from a joint work with Spyros Alexakis and Dezhong Chen, which was published in the scientific journal "Communications in Partial Differential Equations", Volume 40, Issue 12, December 2015, pages 2123-2172. The second chapter has been submitted for publication, a preliminary version of which can be found on the arxiv, http://arxiv.org/abs/1504.04079.

*Dedicated to my parents*

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# Contents

# Chapter 1

# Singular Ricci solitons and their stability under the Ricci flow

## 1.1 Overview

The question of defining solutions of geometric evolution equations with singular initial data is an interesting challenge and has been studied in recent years for a variety of parabolic geometric PDE. For the Ricci flow, a number of solutions have been proposed in various settings. Simon [49] obtained solutions for the Ricci flow for $\mathcal{C}^0$ initial metrics that can be uniformly approximated by smooth metrics with bounded sectional curvature. Koch and Lamm [33] showed existence and uniqueness for the Ricci-DeTurck flow for initial data that are $L^\infty$-close to the Euclidean metric. Angenent, Caputo and Knopf [4] considered initial data of neck-pinch type.[1] They constructed a solution to the flow starting from this singular initial metric, for which the singularity is immediately smoothed out. This can be thought of as a (very weak) notion of surgery in that the method of proof relies on a gluing construction to show the existence of such a solution, but not uniqueness. Cabezas-Rivas and Wilking [6] have obtained solutions of the Ricci flow on open manifolds with nonnegative (and possibly unbounded) complex sectional curvature, using the Cheeger-Gromoll convex exhaustion of such manifolds.

More results have been obtained in the Kähler case and in dimension 2, where the Ricci flow equation reduces to a scalar heat equation; we list a few examples: Chen, Tian and Zhang [10] consider the Kähler-Ricci flow for initial data with $\mathcal{C}^{1,1}$ potentials and construct solutions to the Ricci flow which immediately smooth out. The argument is based on an approximation of the initial potential by smoother ones. Finally, more results have been obtained in dimension 2 (see [29] for a survey): Giesen and Topping [26] (building on earlier work by Topping [51]) have given a construction of Ricci flows on surfaces starting from any (incomplete) initial metric whose curvature is unbounded; these solutions become instantaneously complete and are unique in the *maximally stretched* class that they introduce. More recently yet [27], they constructed

[1] In particular these initial data can form in the evolution of a smooth spherically symmetric initial metric, as demonstrated in [2, 3].

examples of immortal solutions of the flow (on surfaces) which start out with a smooth initial metric, then the supremum of the Gauss curvature becomes infinite for some finite amount of time before becoming finite again.

This paper considers a special class of singular initial metrics and produces examples of Ricci flow whose behavior is different from those listed above. Our initial metrics are *close* to certain singular gradient Ricci solitons that we introduce separately in the first part of this paper. The solitons exist in all dimensions $n+1 \geq 3$. Our main result is that for small enough perturbations of the singular Ricci solitons, the Ricci flow admits a unique solution, up to some time $T > 0$, within a natural class of evolving metrics which stay close (as measured in a certain weighted Sobolev space) to the evolving Ricci solitons. In other words, we obtain a local well-posedness result for the Ricci flow for initial data with the same singularity profile as our Ricci solitons.

The solitons that we introduce (and, in fact, their perturbations that we consider) all have $SO(n+1,\mathbb{R})$-symmetry. In particular, the soliton metric at the initial time $t = 0$ can be written in the form:

$$g_{\text{sol}} = dx^2 + \psi(x)^2 g_{\mathbb{S}^n},$$

where $x \in (0, +\infty)$ for stasy and $x \in (0, \delta), \delta < +\infty$ for non-steady solitons; here $g_{\mathbb{S}^n}$ denotes the canonical metric of the unit $n$-sphere. In all cases the function $\psi(x)$ is a positive smooth function and moreover $\psi(x) \to 0$ as $x \to 0^+$, with leading order behaviour $\psi \sim x^{\frac{1}{\sqrt{n}}}$. In particular, the (incomplete) metric above can be extended to a complete $\mathcal{C}^0$ (in fact $\mathcal{C}^{\frac{1}{\sqrt{n}}}$) metric at $x = 0$, but the extended metric will not be of class $\mathcal{C}^1$. We remark that (in the steady case) our (singular) solitons are complete Riemannian manifolds towards $+\infty$, with an asymptotic profile there that matches the Bryant soliton. For the rest of this introduction we discuss only the steady case.

Our first observation is that the evolution of the singular solitons themselves under the Ricci flow is in sharp contrast with the behavior of their smooth counterparts. As for smooth solitons, there exists an evolution of $g_{\text{sol}}$ under the Ricci flow given by a 1-parameter family of radial[2] diffeomorphisms $\rho_t : (0, +\infty) \times \mathbb{S}^n \to (0, +\infty) \times \mathbb{S}^n$, $t \geq 0$, where $\rho_0 = \text{Id}$. The diffeomorphisms $\rho_t$ are such that the pullback $g(t) = \rho_t^*(g_{\text{sol}})$ solves the Ricci flow

$$\partial_t g(t) = -2\text{Ric}\big(g(t)\big), \qquad g(0) := g_{\text{sol}}.$$

However, the map $\rho_t$ is *not* surjective in this case. In fact, for each $t > 0$, $\rho_t(0, \infty) = (m(t), +\infty)$ where $m(t) > 0$ is non-decreasing in $t$. In other words the flow $\rho_t$ *pushes away* from the singular point $x = 0$. Thus, for each $t > 0$ $(M, g(t))$ can be extended to a smooth manifold with

[2] "Radial" here and furtherdown means that the diffeomorhpism, for each $t \geq 0$, depends only on the parameter $x \in (0, \infty)$.

boundary, where the induced metric on the boundary is that of a round sphere of radius $\lim_{x\to 0^+}\psi(\rho_t(x)) > 0$. One can then visualize the evolving soliton metric $g(t)$ backwards in time: Starting at time $t = 1$ it contains the portion of the original soliton corresponding to $x > m(t)$, and its boundary at $x = m(t)$ shrinks down, as $t \to 0^+$, to a point which yields the singular metric $g_{\text{sol}}$.

The perturbation problem that we consider is still within the spherically symmetric category. In particular, the initial metrics we consider are in the form

$$\tilde{g} = dx^2 + \tilde{\psi}^2(x) g_{\mathbb{S}^n}$$

A loose version of our main result can be written in the following form; the precise statement can be found in Theorem 2.4.6.

**Theorem 1.1.1.** *Let*

$$\xi = \frac{\tilde{\psi}}{\psi} - 1$$

*and assume that*

$$\int_0^1 \frac{\xi^2}{x^{2\alpha}} + \frac{\xi_x^2}{x^{2\alpha-2}} dx + \int_1^{+\infty} \xi^2 + \xi_x^2 dx \ \ll\ 1$$

*for a large enough constant $\alpha$. Then there exists a unique evolving spherically symmetric metric $\tilde{g}(t)$, $t \in [0, T]$, solving the Ricci flow equation*

$$\partial_t \tilde{g}(t) = -2\mathrm{Ric}\big(\tilde{g}(t)\big), \qquad \tilde{g}_0 := \tilde{g}, \qquad \xi(0,t) = 0$$

*and which stays close, measured in a suitable weighted $H^1$-space, to the evolving soliton metric exhibiting the same "opening up" behavior of the initial singularity.*

We remark briefly here on the choice of the weight function $\alpha$: The definition of $\xi$ and the assumption that $\xi$ belongs to the weighted Sobolev space above can be interpreted geometricaly as requiring the initial metric $\tilde{g}$ (encoded in the function $\tilde{\psi}$) and the solition inital metric $g$, ncoded in the function $\psi$ to *agree* asymptoticaly to high order $\alpha$ at $x = 0$. We expand more on this below.

It should be stressed at this point that our work here *does not* have direct bearing on the issue of "flowing through singularities" that form in finite time under the Ricci flow, (as studied, for example, in [4]), at least for closed manifolds. Indeed, it is well known that for such manifolds the minimum of the scalar curvature is a non-decreasing function under the Ricci flow; however the scalar curvature of the solitons we consider (and of their perturbations) converges to $-\infty$ at the singular point ($x = 0$).

While the above solitons have been constructed over the manifolds $\mathbb{R}\times\mathbb{S}^n$, it would perhaps be natural to seek similar examples in the more general cohomogeneity-1 category, studied by

Dancer and Wang, [19, 20, 21].

### 1.1.1 Outline of the ideas

Now, we briefly outline the sections of the paper and the challenges that each addresses. In Section 1.2 we introduce the (singular) spherically symmetric Ricci solitons that we consider. The study of these solitons follows the method presented in [8, Chapter 1], originally developed by R. Bryant. In the class of spherically symmetric metrics, the gradient Ricci soliton equation reduces to a second order ODE system, which can be transformed into a more tractable first order system in parameters $(W, X, Y)$ via a transformation that we review in (A.0.4). Knowledge of the variables $W, X, Y$ in the parameter $y$ allows us to recover the metric component $\psi$ and the gradient $\phi_x$ of the potential function $\phi$ of (A.0.3) in the parameter $x$. In the case of steady solitons, the system (A.0.6) in fact reduces to a $2 \times 2$ system; see §A.0.4. We provide a description of the trajectories in the $X, Y$-plane that correspond to our singular solitons and compare them to the Bryant soliton. In particular, we show there exists a 1-parameter family of singular gradient steady Ricci solitons; they are all singular at $x = 0$ with the leading order asymptotics

$$\psi(x) \sim x^{\frac{1}{\sqrt{n}}} \qquad \phi_x(x) \sim \frac{\sqrt{n}-1}{x}, \qquad n > 1$$

and they are complete towards $x = +\infty$, with the same asymptotic profile as the Bryant soliton.

In Section 1.3 we introduce the perturbation problem we will be studying in the rest of the paper. We consider spherically symmetric initial metrics of the form

$$\tilde{g} = \tilde{\chi}^2(x)dx^2 + \tilde{\psi}^2(x)g_{\mathbb{S}^n}$$

For such initial data, the Ricci flow equation can be written (after a change of variables) in the equivalent form (1.3.4) of a PDE coupled to an ODE. The evolving Ricci soliton metric (defined via the diffeomorphisms $\rho_t$) remains spherically symmetric and is represented by coordinate components $\chi(x,t), \psi(x,t)$, while the stipulated Ricci flow that we wish to solve for corresponds to two functions $\tilde{\chi}(x,t), \tilde{\psi}(x,t)$. Since the singular nature of the initial data do not allow the system (1.3.4) to be attacked directly, we introduce new variables which measure the closeness of $\tilde{\chi}, \tilde{\psi}$ to $\chi, \psi$.

More precisely, we define

$$\zeta = \frac{\tilde{\chi}}{\chi} - 1 \qquad \xi = \frac{\tilde{\psi}}{\psi} - 1.$$

Then the system reduces to (1.3.10), for which the Ricci soliton corresponds to the solution $\zeta = 0, \xi = 0$. The coefficients of this system refer to the variable $\psi$ of the background evolving soliton, expressed with respect to its arc-length parameter $s$. What is critical here is that the coefficients are singular at $(x,t) = (0,0)$; the precise nature of this singularity is essential in our further analysis.

A first challenge appears at this point, which in fact is independent of the singularities of the coefficients. Indeed, it is related to the presence of the second order term $\xi_{ss}$ on the RHS of the first equation in (1.3.10). Since the first equation is only of first order in $\zeta$, this term would *not* make it possible to close the energy estimates for our system. We therefore introduce a new variable defined by

$$\eta = \frac{(\zeta+1)^2}{(\xi+1)^{2n}} - 1.$$

The new system (1.3.14) for $\eta$ and $\xi$ involves only first derivatives of $\xi$ in the evolution equation of $\eta$ and therefore can (in principle) be approached via energy estimates. It is not clear whether there is any geometric significance underlying this change of variables. It is in fact not a priori obvious that such a simplification of the system should have been possible via a change of variables. It is at this point that the spherical symmetry of both the background soliton and of the perturbations that we study is used in an essential way.

Thus, matters are reduced to proving well-posedness of (1.3.14), in the appropriate spaces. We follow the usual approach of performing an iteration[3], by solving a sequence of linear equations for the unknows $(\eta^{m+1}, \xi^{m+1})$ in terms of the known functions $(\eta^m, \xi^m)$ solved for in the previous step, and proving that the sequence $(\eta^m, \xi^m), m \in \mathbb{N}$ converges to a solution $(\eta, \xi)$ of our original system.

We note that the usual approach would be to replace only the highest order terms in the RHSs of (1.3.14) by the unknown function $\xi^{m+1}$ and replace all the lower-order ones by the previously-solved-for $\eta^m, \xi^m$. However in the case at hand this approach would fail for any function space, due to the nature of the singularities in the coefficients. For example, as we will see the coefficient $\frac{\psi_s^2}{\psi^2}$ in the potential terms contains a factor of $\frac{1}{s^2}$, where $s(x,t)$ is the arc-length parameter of the background evolving soliton. It turns out that the leading order in the asymptotic expansion of $s^2$ near $x = 0, t = 0$ is of the form

$$s^2 \sim x^2 + 2(\sqrt{n} - 1)t.$$

Consequently, the best $L^\infty_x$ bound for $\frac{1}{s^2}$ would be $\frac{1}{s^2} \leq \frac{C}{t}$; this would result in an energy estimate of the form $\partial_t \mathcal{E} \leq \mathcal{E} t^{-1}$ which cannot close. The remedy for this problem is to modify the iteration procedure according to (1.4.2). In this linear iteration the unknown functions $\xi^{m+1}, \eta^{m+1}$ at the $(m+1)$-step also appear in certain lower-order terms associated to the most singular coefficients.

Finally, we solve the system (1.4.2) and prove that it defines a contraction mapping in

[3]In reality a contraction mapping argument, although we find it more convenient to phrase our proof in terms of the standard Picard iteration.

certain (time-dependent) weighted Sobolev spaces $H^1_\alpha(s)$ containing all functions

$$u \in H^1(\mathbb{R}_+) \qquad\qquad \int_0^1 \frac{u^2}{(s^2+\sigma t)^\alpha} + \frac{u_s^2}{(s^2+\sigma t)^{\alpha-1}} ds < +\infty,$$

where we note that the weights depend on both the spatial and time variables $x, t$. (We note here that we use the length element $ds$ which corresponds to the arc-length parameter of the background evolving Ricci soliton. In particular $s(x,t) := \rho_t(x)$; thus for all $t > 0$ $s(x,t) > s(0,t) > 0, \forall x > 0$.)

The rather involved estimates in Section 1.4 aim precisely to show that the parameters $\alpha$ and $\sigma > 0$ *can be* chosen in a way to make the estimates close; as we will see, this mostly amounts to controlling the terms in the energy estimate that arise from the most singular coefficients in (1.3.14). We note here that choosing $\alpha$ to be large forces both the initial data and the evolution of the solution to stay close the evolving soliton. Choosing $\sigma$ large allows the evolving solution to 'depart' from the evolving soliton. Thus the challenge is to balance these competing parameters to make the estimates close. We note that it is essential for this 'balancing' to work that we can first close the estimates for the $L^2$ norms, and *after* this has been done we can estimate the $H^1$ norms.

Finally, in Section 1.5 we provide a proof of the existence of solutions to (1.4.2) in the appropriate spaces, using a modification of the Galerkin iteration to this singular PDE-ODE system. This part is included for the sake of completeness, since coupled systems of this singular nature do not appear to have been treated in the literature.

## 1.2 Singular spherically symmetric Ricci Solitons

We will be considering metrics over $M^{n+1} = (0, B) \times \mathbb{S}^n$ (where $B \in \mathbb{R}_+$ or $B = +\infty$), in the form

$$g = dx^2 + \psi^2(x) g_{\mathbb{S}^n}, \tag{1.2.1}$$

where $\psi$ is a positive smooth function and $g_{\mathbb{S}^n}$ denotes the canonical metric on the unit sphere. Our first aim for this section is to obtain such metrics which satisfy the (gradient) Ricci soliton equation

$$Ric(g) + \nabla^2\phi + \lambda g = 0 \qquad\qquad \lambda \in \mathbb{R}, \tag{1.2.2}$$

for a smooth radial potential function $\phi : M \to \mathbb{R}$, and which are singular as $x \to 0^+$. In particular we wish to construct a soliton metric which will extend continuously to $x = 0$ with $\psi(x) \to 0$, as $x \to 0^+$, but will not close smoothly there.

Following known work on the complete case, an approach originally initiated in (unpublished) work of R. Bryant (see Appendix A and [8]), we construct the following singular solutions of the equation (1.2.2).

**Proposition 1.2.1** (Existence of singular Ricci solitons)**.** *For all $\lambda \in \mathbb{R}, n > 1$ there exists a class of spherically symmetric solutions to the gradient Ricci soliton equation (1.2.2) with profile*

$$\psi(x) \sim a x^{\frac{1}{\sqrt{n}}},\ a > 0 \qquad \phi_x(x) \sim \frac{\sqrt{n}-1}{x} \qquad \textit{as } x \to 0^+. \tag{1.2.3}$$

*These solutions are a priori defined for $B = \delta < +\infty$, for some $\delta > 0$ small, such that $\psi, \phi_x$ have a smooth limit, as $x \to \delta^- < +\infty$.*

*In the steady case $\lambda = 0$, the preceding solutions exist up to $B = +\infty$ and their behavior at infinity reads*

$$c x^{\frac{1}{2}} \leq \psi(x) \leq C x^{\frac{1}{2}} \qquad -C(1-\frac{1}{x}) \leq \phi_x(x) \leq -c(1-\frac{1}{x}) \qquad c, C > 0,\ \ x \gg 1. \tag{1.2.4}$$

*Further, the behaviors of the derivatives of the above variables are in each case the derivatives of the corresponding bounds and asymptotics, e.g.,*

$$\psi_x(x) \sim \frac{a}{\sqrt{n}} x^{\frac{1}{\sqrt{n}}-1},\ \textit{as } x \to 0^+ \qquad -\frac{C}{x^2} \leq \phi_{xx}(x) \leq -\frac{c}{x^2},\ x \gg 1$$

*Proof.* See Propositions A.0.9, A.0.11 in Appendix A. □

**Remark 1.2.2.** It is worth noting that for $\lambda = 0$ in dimension five, (i.e., $n = 4$) the soliton metrics and associated diffeomorphisms can in fact be written out explicitly:

$$\psi(x) = a\sqrt{x} \qquad \phi_x(x) = \frac{1}{x} - \frac{6}{a^2}, \qquad x \in (0, +\infty),\ a > 0. \tag{1.2.5}$$

**Remark 1.2.3.** In view of the asymptotics, we conclude that the above Ricci solitons metrics are $\mathcal{C}^0$ extendible at $x = 0$, but singular in $C^1$ norm for all dimensions $n+1 \geq 3$. In particular, one can readily check that the most singular curvature components blow up like $1/x^2$, as $x \to 0^+$.

### 1.2.1 The evolving soliton metric $g(t)$: the action of the diffeomorphisms.

Since the metric $g$ (1.2.1) satisfies the gradient Ricci soliton equation (1.2.2), it admits a Ricci flow

$$\partial_t g(t) = -2Ric\big(g(t)\big) \qquad g(0) = g, \tag{1.2.6}$$

evolving via diffeomorphisms

$$g(t) = \epsilon(t)\rho_t^*(g)$$

up to some time $T > 0$, where $\epsilon(t) := 1 + 2\lambda t > 0$, $t \in [0, T)$, and

$$\rho_t(x, p) = \rho_t(x) \qquad\qquad \rho_0 = id_M$$

is the flow generated by the (time dependent) vector field

$$\frac{1}{\epsilon(t)}\nabla_g\phi.$$

Thus, by definition of the pullback

$$g(t) = \epsilon(t)\big[d(\rho_t(x))^2 + \psi^2(\rho_t(x))g_{\mathbb{S}^n}\big] \tag{1.2.7}$$

We note that since our manifold $(M^{n+1}, g)$ is not complete at $x = 0$, $\rho_t(x)$ is not necessarily defined for all time, but nevertheless it exists locally $t \in (-\varepsilon_x, \varepsilon_x), x > 0$. However, it easily follows from the asymptotics below that for the steady ($\lambda = 0$) solitons the flow exists for all $t \geq 0$.

Suppressing the sphere coordinates corresponding to different points $(x, p)$, $(x, q)$ in $M^{n+1}$, we may consider $\rho_t$ to be a real function in $x$

$$\rho_t : (0, B) \to (0, +\infty)$$

and further we identify the time derivative of $\rho_t$ with the (single) component of $\nabla_g\phi$ in the $\partial_x$ direction, that is,

$$\partial_t\rho_t(x) = \frac{1}{\epsilon(t)}(\nabla_{\partial/\partial x}\phi)_{\rho_t(x)} = \frac{1}{\epsilon(t)}\phi_x(\rho_t(x)). \tag{1.2.8}$$

According to the asymptotics (1.2.3),

$$\partial_t\rho_t(x) \sim \frac{1}{\epsilon(t)}\frac{\sqrt{n}-1}{\rho_t(x)} \tag{1.2.9}$$

which after integrating yields the leading behavior

$$\rho_t^2(x) \sim x^2 + 2(\sqrt{n}-1)t, \qquad\qquad \text{as } x, t \to 0^+. \tag{1.2.10}$$

**Remark 1.2.4.** From the preceding asymptotics it follows that

$$\rho_t\big((0, B)\big) \subseteq (\rho_t(0), +\infty),$$

$\rho_t(0) > 0, t > 0$ non-decreasing, and in particular $\rho_t$ is *not* surjective. A geometric interpretation of the latter is that the flow $\rho_t$ "pushes" the domain away from the singularity at $x = 0$, smoothing out the incomplete metric.

Restricting now on the singular steady solitons, we integrate (1.2.8) once more to arrive at the following estimate at infinity for the flow

$$x - Ct \leq \rho_t(x) \leq x - ct \qquad\qquad x \gg 1 \gg t \geq 0. \tag{1.2.11}$$

In fact, in the steady case $\lambda = 0$ we can give a complete description of the evolution of the singular soliton metrics. Indeed, in this case we derive that there is a critical slice $\{x_{\text{crit}}\} \times \mathbb{S}^n$ of the manifold $M^{n+1} = (0, +\infty) \times \mathbb{S}^n$, which is invariant under $\rho_t(\cdot)$ and moreover an attractor of the flow:

$$\phi_x(x) > 0,\ x \in (0, x_{\text{crit}}) \qquad \phi_x(x_{\text{crit}}) = 0 \qquad \phi_x(x) < 0,\ (x_{\text{crit}}, +\infty) \tag{1.2.12}$$

Whence, for any point $x \in (0, +\infty)$, the integral curve $\rho_t(x)$ will 'reach' $x_{crit}$ as time tends to infinity

$$\lim_{t\to+\infty} \rho_t(x) = x_{\text{crit}} \qquad\qquad \lim_{t\to+\infty} \rho_t\big((0, +\infty)\big) = [x_{\text{crit}}, +\infty).$$

We remark also that the scalar curvature $R$ achieves its maximum at $x_{crit}$, which means that the manifold is deformed in this sense towards higher level sets of scalar curvature.

In order to prove the above picture, it suffices to show that (1.2.12) is valid. From the profiles (1.2.3), (1.2.4) we confirm that $\phi_x$ has a positive sign close to $x = 0$ and is negative near $+\infty$. Hence there exists a point $x_{\text{crit}}$ where $\phi_x(x_{\text{crit}}) = 0$. It remains to show that this is the only zero of $\phi_x$. We recall at this point a general identity for solutions to the gradient Ricci soliton equation (1.2.2) (see for instance [8, Proposition 1.15]).

**Proposition 1.2.5.** *Let $(M^m, g, \nabla\phi)$ be a gradient Ricci soliton, i.e., a solution of the equation (1.2.2). Then the following quantities are constant:*

$$\begin{aligned} &(i)\ \ R + \Delta_g \phi + m\lambda = 0 &&\textit{(tracing)} \\ &(ii)\ \ R + |\nabla_g \phi|^2 + 2\lambda\phi = C_0, && \end{aligned}$$

*where $R$ is the scalar curvature of $(M^m, g)$.*

The fact that the scalar curvature $R$ attains its maximum $(C_0)$ at $x_{crit}$ is an immediate consequence of identity $(ii)$ for $\lambda = 0$.

Subtracting the two identities of the preceding proposition we obtain

$$\Delta_g \phi - |\nabla_g \phi|^2 - 2\lambda\phi + m\lambda = -C_0.$$

Whence, in our context for $\lambda = 0$, the previous equation amounts to

$$\phi_{xx} + \frac{\psi_x}{\psi}\phi_x - \phi_x^2 = -C_0, \tag{1.2.13}$$

*Claim*: $C_0 > 0$. From the asymptotics of $\phi_x$ (1.2.3), (1.2.4), we easily deduce that $\phi$ tends to $-\infty$ at both ends of the manifold $x = 0, +\infty$. This implies that $\phi$ has a global maximum $M$, realized at some point $\tilde{x}$. By (1.2.13) we get $C_0 \geq 0$. However, the constant $C_0$ cannot be zero, otherwise we would have $\phi \equiv M$ (by uniqueness of ODEs), which of course is not possible. Our claim follows.

Thus, every critical point of $\phi$ is a *strict* local maximum. Therefore, $\phi$ can only have one critical point, $x_{\text{crit}} = \tilde{x}$. ∎

## 1.3 The Stability problem

Our main goal in this paper is to prove (local in time) well-posedness of the Ricci flow for spherically symmetric metrics which start out close enough (in certain spaces we construct in §1.3.4) to the soliton metrics (Propositions 1.2.1) we constructed in the previous section. We recall below in §1.3.1 a useful form of the Ricci flow equation for spherically symmetric metrics and then proceed to introduce a transformation of our system into new variables $\zeta, \xi$ (1.3.9). These are designed to capture the closeness of the (putative) evolving solution under the Ricci flow to the evolution of the background Ricci soliton. The resulting system involves a second order parabolic equation in $\xi$ coupled with a transport equation in $\zeta$, both of them having certain singular coefficients. This forces us to study well-posedness of the system in certain weighted Sobolev spaces. Our main result in these variables is stated in Theorem 2.4.6.

However, this is not the system we derive energy estimates with, because of the fact that the transport equation in $\zeta$ contains a second order term in $\xi$, which makes it impossible for such estimates to close. After a further crucial change of variables (§1.3.3), replacing $\zeta$ with a new variable $\eta$, the resulting PDE in $\eta, \xi$ (1.3.14) for which we derive an estimate is of similar nature, except now this problem has been eliminated; the equation of $\eta$ containing only first derivatives of $\xi$.

The singularities in the coefficients of the system are determined fully by the background evolving soliton metric. The precise asymptotics of these coefficients are essential to our further pursuits, so we begin by studying those right after writing down the final system (1.3.14). Next, in §1.3.4 we set up formally the function spaces in which we will be proving our well-posedness result for the system of $\eta, \xi$ and state the final version of our main result very precisely in Theorem 2.4.8. The proof of Theorem 2.4.8 is carried out in the next section §1.4.

One final convention: We will be considering the stability question for all the singular Ricci solitons (see Proposition 1.2.1). Since for $\lambda \neq 0$ our knowledge is restricted only on the bounded interval $(0, \delta)$, we will treat two versions of the resulting PDE problem. One will concern a bounded domain and the other, for the steady case $\lambda = 0$, will regard the whole half-line; i.e., initial domain $x \in (0, B)$, $B = \delta < +\infty$ or $B = +\infty$.

### 1.3.1 Ricci flow in spherical symmetry

Let $\tilde{g}(t)$, $t \in [0, T]$, be a 1-parameter family of smooth spherically symmetric metrics on $M^{n+1} = (0, B) \times \mathbb{S}^n$ ($B = \delta < +\infty$ or $B = +\infty$)

$$\tilde{g}(t) = \tilde{\chi}^2(x,t)dx^2 + \tilde{\psi}^2(x,t)g_{\mathbb{S}^n}, \tag{1.3.1}$$

where $\tilde{\chi}, \tilde{\psi}$ are positive smooth functions, and assume it satisfies the Ricci flow equation

$$\partial_t \tilde{g}(t) = -2Ric\big(\tilde{g}(t)\big) \qquad\qquad t \in [0, T]. \tag{1.3.2}$$

We now let $\tilde{s}(x,t)$ be the radial arc-length parameter for the above metric at any given time $t$, i.e.,

$$d\tilde{s} = \tilde{\chi}(x,t)dx. \tag{1.3.3}$$

Expressing $\tilde{\psi}(\cdot, t)$ relative to the parameter $\tilde{s}$ (and slightly abusing notation), $\tilde{g}(t)$ becomes

$$\tilde{g}(t) = d\tilde{s}^2 + \tilde{\psi}^2(\tilde{s},t)g_{\mathbb{S}^n}.$$

For this type of warped product metrics the Ricci tensor is given by (e.g., [8, §1.3.2])

$$Ric\big(\tilde{g}(t)\big) = -n\frac{\tilde{\psi}_{\tilde{s}\tilde{s}}}{\tilde{\psi}}d\tilde{s}^2 + (n - 1 - \tilde{\psi}\tilde{\psi}_{\tilde{s}\tilde{s}} - (n-1)\tilde{\psi}_{\tilde{s}}^2)g_{\mathbb{S}^n}.$$

Plugging into (1.3.2) we get

$$\begin{cases} 2\tilde{\chi}\tilde{\chi}_t = -2(-n\frac{\tilde{\psi}_{\tilde{s}\tilde{s}}}{\tilde{\psi}})d\tilde{s}^2(\partial_x, \partial_x) = 2n\frac{\tilde{\psi}_{\tilde{s}\tilde{s}}}{\tilde{\psi}}\tilde{\chi}^2 \\ 2\tilde{\psi}\tilde{\psi}_t = -2(n-1-\tilde{\psi}\tilde{\psi}_{\tilde{s}\tilde{s}} - (n-1)\tilde{\psi}_{\tilde{s}}^2) \end{cases}$$

Thus, the Ricci flow equation (1.3.2) reduces to the coupled system

$$\begin{cases} \tilde{\chi}_t = n\frac{\tilde{\psi}_{\tilde{s}\tilde{s}}}{\tilde{\psi}}\tilde{\chi} \\ \tilde{\psi}_t = \tilde{\psi}_{\tilde{s}\tilde{s}} - (n-1)\frac{1-\tilde{\psi}_{\tilde{s}}^2}{\tilde{\psi}} \end{cases} \qquad\qquad t \in [0, T]. \tag{1.3.4}$$

Observe that the first equation involves the evolution of the radial distance function, while the second involves the evolution of the radii of the spheres, at a given radial distance.

Of course, the singular Ricci soliton metrics we studies in the previous section fall in the same framework. Indeed, returning to (1.2.7) we may write

$$g(t) = ds^2 + \psi^2(s,t), g_{\mathbb{S}^n} = \chi^2(x,t)dx^2 + \psi^2(x,t)g_{\mathbb{S}^n}, \tag{1.3.5}$$

where we have set

$$s(x,t) = \sqrt{\epsilon(t)}\,\rho_t(x), \quad s(x,0) = x \qquad\qquad ds = \sqrt{\epsilon(t)}\,\partial_x\rho_t(x)dx \tag{1.3.6}$$

and

$$\chi(x,t) := \sqrt{\epsilon(t)}\,\partial_x\rho_t(x) \qquad\qquad \psi(x,t) := \sqrt{\epsilon(t)}\,\psi(\rho_t(x)). \tag{1.3.7}$$

Note that $\psi(x,0) = \psi(x)$ corresponds to the component of the metric $g$ (1.2.1). Arguing similarly to the case of $\tilde{g}(t)$, it follows that the (1.2.6) is equivalent to

$$\begin{cases} \chi_t = n\frac{\psi_{ss}}{\psi}\chi \\ \psi_t = \psi_{ss} - (n-1)\frac{1-\psi_s^2}{\psi} \end{cases} \qquad \chi(x,0) = 1,\ \psi(x,0) = \psi(x). \tag{1.3.8}$$

### 1.3.2 The main stability result: A transformed system for the Ricci flow of the perturbed metric

The goal is to construct a spherically symmetric Ricci flow (1.3.1), (1.3.2) for the appropriate spherically symmetric perturbed metric $\tilde{g} := \tilde{g}(0)$. We now take a first step towards transforming our system of equations by introducing new variables. Let

$$\zeta = \frac{\tilde{\chi}}{\chi} - 1 \qquad\qquad \xi = \frac{\tilde{\psi}}{\psi} - 1. \tag{1.3.9}$$

The above formulas are defined for all $x \in (0,B), t \in [0,T]$. In particular, these variables measure (in a refined way) the difference between the unknown functions $\tilde{\chi}, \tilde{\psi}$ and the background variables $\chi, \psi$. Note in addition that requiring $\xi = 0$ at the endpoint $x = 0, t = 0$ forces $\tilde{\psi}$ to have the same leading order asymptotics at $x = 0$ as the background component $\psi$.

We next wish to convert (1.3.4) into a system of equations for $\zeta, \xi$, expressing the evolution equations in terms of $t$ and the arc-length parameter $s$ of the background evolving Ricci soliton. We are then forced to deal with the discrepancy between $\tilde{s}, s$. We calculate:

$$\begin{aligned} \partial_{\tilde{s}} &\overset{(1.3.3)}{=} \frac{1}{\tilde{\chi}}\partial_x = \frac{\chi}{\tilde{\chi}}\frac{1}{\chi}\partial_x \overset{(1.3.6),(1.3.7)}{=} \frac{1}{\zeta+1}\partial_s \\ \partial_{\tilde{s}}\partial_{\tilde{s}} &= \frac{1}{\zeta+1}\partial_s(\frac{1}{\zeta+1}\partial_s) = \frac{1}{(\zeta+1)^2}\partial_s\partial_s - \frac{\zeta_s}{(\zeta+1)^3}\partial_s, \end{aligned}$$

and hence we write

$$\begin{aligned} \tilde{\psi}_{\tilde{s}} =& \frac{1}{\zeta+1}\big(\psi(\xi+1)\big)_s \\ \tilde{\psi}_{\tilde{s}\tilde{s}} =& \frac{1}{(\zeta+1)^2}\big(\psi(\xi+1)\big)_{ss} - \frac{\zeta_s}{(\zeta+1)^3}\big(\psi(\xi+1)\big)_s. \end{aligned}$$

Taking time derivatives in (1.3.9) and combining (1.3.4), (1.3.8), we derive the following coupled system in the new variables $\zeta, \xi$.

$$\begin{aligned}
\zeta_t =& n\frac{\psi_{ss}}{\psi}\big[\frac{1}{\zeta+1}-(\zeta+1)\big]+2n\frac{\psi_s}{\psi}\frac{\xi_s}{(\zeta+1)(\xi+1)}+n\frac{\xi_{ss}}{(\zeta+1)(\xi+1)}-n\frac{\psi_s}{\psi}\frac{\zeta_s}{(\zeta+1)^2}\\
&-n\frac{\zeta_s\xi_s}{(\zeta+1)^2(\xi+1)}\\
\xi_t =& (\frac{\psi_{ss}}{\psi}+(n-1)\frac{\psi_s^2}{\psi^2})\big[\frac{\xi+1}{(\zeta+1)^2}-\xi-1\big]+\frac{n-1}{\psi^2}(\xi+1-\frac{1}{\xi+1}) \\
&+2n\frac{\psi_s}{\psi}\frac{\xi_s}{(\zeta+1)^2}+\frac{\xi_{ss}}{(\zeta+1)^2}+(n-1)\frac{\xi_s^2}{(\zeta+1)^2(\xi+1)}-\frac{\psi_s}{\psi}\frac{\zeta_s(\xi+1)}{(\zeta+1)^3}-\frac{\zeta_s\xi_s}{(\zeta+1)^3}
\end{aligned} \tag{1.3.10}$$

Notice that the coefficients of the preceding system are expressed in terms of the components (metric, curvature etc.) of the background soliton, which are of course singular at $x=t=0$. We will elaborate more on the nature of the singularities in the next subsection. We simply mention that this is basically the reason that forces us to study (1.3.10) in non-standard modified spaces. The following version of our main theorem regards the local existence of the system in the variables $\zeta, \xi$ (1.3.9).

**Theorem 1.3.1.** *There exist constants $\alpha, \sigma > 0$ appropriately large, such that the system (1.3.10) is locally well-posed in the (time-dependent) weighted Sobolev space*

$$E(t) := \int_{x=0}^{x=\delta} \frac{u^2}{(s^2+\sigma t)^\alpha}+\frac{u_s^2}{(s^2+\sigma t)^{\alpha-1}}ds+\int_{x=\delta}^{x=B} u^2+u_s^2 ds < +\infty, \tag{1.3.11}$$

*($B=\delta<+\infty, \lambda\neq 0$ or $B=+\infty, \lambda=0$) assuming $E(0)$ sufficiently small and the Dirichlet boundary condition*

$$\xi(x,t)=0, \quad \{x=0,B\}\times[0,T] \tag{1.3.12}$$

We remark that the smallness assumption on $E(0)$ is required to control the smallness in $L^\infty$ of $\eta, \xi$ which appear in the denominators in (1.3.10) by $E(t)$. It could possibly be removed if the initial data lied in a suitably weighted $H^2$ space, combined with an assumption of smallness in $L^\infty$ of $\eta, \xi$.

### 1.3.3 A crucial change of variables: The features of the resulting PDE

Unfortunately, due to the term $n\frac{\xi_{ss}}{(\zeta+1)(\xi+1)}$ in the first equation of (1.3.10) we cannot derive energy estimates in $L^2$ for $\zeta, \xi$. We remedy this problem by replacing the variable $\zeta$ with

$$\eta := \frac{(\zeta+1)^2}{(\xi+1)^{2n}} - 1. \tag{1.3.13}$$

The new system of $\eta, \xi$ reads

$$\begin{aligned}
\eta_t = & -2n(n-1)\Big(\frac{\psi_s^2}{\psi^2}\big[\frac{1}{(\xi+1)^{2n}}-1\big]+2\frac{\psi_s}{\psi}\frac{\xi_s}{(\xi+1)^{2n+1}}+\frac{1-(\xi+1)^{-2}}{\psi^2}+\frac{\xi_s^2}{(\xi+1)^{2n+2}}\Big)\\
& -2n(n-1)\frac{1-(\xi+1)^{-2}}{\psi^2}\eta+2n(n-1)\frac{\psi_s^2}{\psi^2}\eta\\
\xi_t = & \big(\frac{\psi_{ss}}{\psi}+(n-1)\frac{\psi_s^2}{\psi^2}\big)\big[\frac{1}{(\eta+1)(\xi+1)^{2n-1}}-(\xi+1)\big]+\frac{n-1}{\psi^2}(\xi+1-\frac{1}{\xi+1})\\
& +n\frac{\psi_s}{\psi}\frac{\xi_s}{(\eta+1)(\xi+1)^{2n}}+\frac{\xi_{ss}}{(\eta+1)(\xi+1)^{2n}}-\frac{\xi_s^2}{(\eta+1)(\xi+1)^{2n+1}}\\
& -\frac{1}{2}\frac{\psi_s}{\psi}\frac{\eta_s}{(\eta+1)^2(\xi+1)^{2n-1}}-\frac{1}{2}\frac{\eta_s\xi_s}{(\eta+1)^2(\xi+1)^{2n}}
\end{aligned} \tag{1.3.14}$$

It is important that we know the exact leading asymptotics of the coefficients in (1.3.14), as $x, t \to 0^+$. Recall the formulas (1.3.6), (1.3.7)

$$s(x,t) = \sqrt{\epsilon(t)}\rho_t(x) \qquad \psi(s,t) = \sqrt{\epsilon(t)}\,\psi(\frac{s}{\sqrt{\epsilon(t)}})$$

and the profile of the background singular soliton at the two ends $x = 0, B$ (Proposition 1.2.1) to deduce the following estimates:

$$\frac{\psi_s}{\psi} = O(\frac{1}{s}) \qquad \frac{\psi_s^2}{\psi^2} = O(\frac{1}{s^2}) \qquad \frac{\psi_{ss}}{\psi} = O(\frac{1}{s^2}) \qquad x \in (0,B),\ t \in [0,T] \tag{1.3.15}$$

and separately for

$$\frac{1}{\psi^2} = O(\frac{1}{s^{\frac{2}{\sqrt{n}}}}),\ x \ll 1 \qquad \frac{1}{\psi^2} = O(\frac{1}{s}),\ x \gg 1 \qquad t \in [0,T],\ n > 1 \tag{1.3.16}$$

for small $T > 0$. Using the above we also derive

$$\partial_s(\frac{\psi_s}{\psi}) = O(\frac{1}{s^2}) \qquad \partial_s(\frac{\psi_s^2}{\psi^2}) = O(\frac{1}{s^3}) \qquad x \in (0,B),\ t \in [0,T]. \tag{1.3.17}$$

Also, directly from the asymptotics of the flow $\rho_t^2(x)$ (1.2.10),(1.2.11) the arc-length parameter $s$ of the background soliton shows to behave like

$$s^2(x,t) := \epsilon(t)\rho_t^2(x) \sim x^2 + 2(\sqrt{n}-1)t \qquad \text{as } x, t \to 0^+ \tag{1.3.18}$$

and

$$x - Ct \le s \le x - ct \qquad x \gg 1,\ B = +\infty,\ t \in [0,T] \tag{1.3.19}$$

with an evolution estimated employing (1.2.8):

$$\partial_t s = \frac{\lambda}{\epsilon(t)} s + O(\frac{1}{s}),\ x \ll 1 \qquad -C \leq \partial_t s \leq -c,\ x \gg 1,\ B = +\infty \qquad t \in [0,T] \tag{1.3.20}$$

**Remark 1.3.2.** Evidently from the above asymptotics, the best $L^\infty_x$ estimate that one could hope for the ratio $1/s^2$ is of the form

$$\|\frac{1}{s^2}\|_{L^\infty(x)} \leq \frac{C}{t}, \tag{1.3.21}$$

which of course fails to be integrable in $[0,T]$, $T>0$. Note that $1/s^2$ is the leading behavior, suggested from the above estimates, of the most singular coefficients of the potential terms in (1.3.14). This is precisely the reason why the standard Gronwall argument would fail to yield an energy estimate in the usual $H^k$ spaces for the system in question.

It will be useful furtherdown to write the *less* singular coefficients in (1.3.14), namely, $\frac{1}{\psi^2}$ as

$$\frac{1}{\psi^2} =: \frac{A(s,t)}{s}, \qquad \partial_s(\frac{A(s,t)}{s}) = -2\frac{1}{\psi^2}\frac{\psi_s}{\psi} = \frac{A(s,t)}{s}O(\frac{1}{s}), \tag{1.3.22}$$

where setting

$$A(t) := \|A(s,t)\|_{L^\infty(s)}, \qquad \int_0^t A^2(\tau)d\tau = o(\sqrt{t}), \qquad \text{as } t \to 0^+. \tag{1.3.23}$$

As stated in Theorem 2.4.6, the spaces we will be dealing with involve the coordinate vector field $\partial_s$ and the volume form $ds$ of the background soliton metric. The first issue we stress here is the fact that the vector fields $\partial_s, \partial_t$ (the latter is defined so that $\partial_t x = 0$) do not commute. In fact, we find the commutator to be singular:

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{\partial}{\partial s} =& \frac{\partial}{\partial t}\left(\frac{1}{\sqrt{\epsilon(t)}\,\partial_x\rho_t(x)}\frac{\partial}{\partial x}\right) && \text{(by definition of } s \text{ (1.3.6))}\\
=& -\frac{\lambda}{\epsilon(t)^{\frac{3}{2}}}\frac{1}{\partial_x\rho_t(x)}\frac{\partial}{\partial x} - \frac{1}{\sqrt{\epsilon(t)}}\frac{\partial_t(\partial_x\rho_t(x))}{(\partial_x\rho_t(x))^2}\frac{\partial}{\partial x} + \frac{1}{\sqrt{\epsilon(t)}\,\partial_x\rho_t(x)}\frac{\partial}{\partial t}\frac{\partial}{\partial x}\\
=& -\frac{\lambda}{\epsilon(t)}\frac{\partial}{\partial s} - \frac{\partial_x\partial_t\rho_t(x)}{\partial_x\rho_t(x)}\frac{\partial}{\partial s} + \frac{1}{\sqrt{\epsilon(t)}\,\partial_x\rho_t(x)}\frac{\partial}{\partial x}\frac{\partial}{\partial t}\\
=& -\frac{\lambda}{\epsilon(t)}\frac{\partial}{\partial s} - \frac{\partial_x\left[\frac{1}{\epsilon(t)}\phi_x(\rho_t(x))\right]}{\partial_x\rho_t(x)}\frac{\partial}{\partial s} + \frac{\partial}{\partial s}\frac{\partial}{\partial t} && \text{(plugging in (1.2.8))}\\
=& -\frac{\lambda+\phi_{xx}(s)}{\epsilon(t)}\frac{\partial}{\partial s} + \frac{\partial}{\partial s}\frac{\partial}{\partial t}
\end{aligned}$$

Consulting the asymptotics of the second derivative potential function (deduced from Proposition 1.2.1) we conclude that

$$[\partial_t, \partial_s] = O(\frac{1}{s^2})\partial_s \qquad x \in (0, B),\ t \in [0, T]. \tag{1.3.24}$$

We must also calculate the evolution of the volume form $ds$. The derivation is similar:

$$\begin{aligned}\partial_t ds = \partial_t(\sqrt{\epsilon(t)}\,\partial_x \rho_t(x)dx) &= \frac{\lambda}{\sqrt{\epsilon(t)}}\partial_x\rho_t(x)dx + \sqrt{\epsilon(t)}\,\partial_t\partial_x\rho_t(x)dx \\ &= \frac{\lambda}{\epsilon(t)}ds + \sqrt{\epsilon(t)}\,\partial_x\big[\frac{1}{\epsilon(t)}\phi_x(\rho_t(x))\big]dx = \frac{\lambda + \phi_{xx}(s)}{\epsilon(t)}ds,\end{aligned}$$

which as above gives

$$\partial_t ds = O(\frac{1}{s^2})ds. \tag{1.3.25}$$

### 1.3.4 The weighted Sobolev spaces and the final version of the main theorem

As explained the singularities in the coefficients of the system (1.3.14), along with the asymptotic behaviors we have derived force us to study well-posedness in *weighted* Sobolev spaces. The weights will be adapted to the singularity at $x = 0, t = 0$.

**Definition 1.3.3.** Let $\sigma > 0$ (to be determined later). We define the weight

$$\ell^2(x,t) = \begin{cases} s^2 + \sigma t, & (x,t) \in (0,\delta) \times [0,T],\ \lambda \in \mathbb{R} \\ \varphi(s,t), & (x,t) \in [\delta, \delta+1) \times [0,T],\ \lambda = 0,\ B = +\infty \\ 1, & (x,t) \in [\delta+1, +\infty) \times [0,T], \qquad \text{" "} \end{cases} \tag{1.3.26}$$

where $\varphi(\cdot, t)$ is a cut off function interpolating between $\ell^2(\delta, t)$ and 1, for each $t \in [0, T]$.

When we derive the main energy estimates in the next section we will need the following key properties of the weight $\ell$. First, we estimate immediately by Definition 1.3.3 and (1.3.20) how $\ell$ changes along the directions $\partial_s, \partial_t$:

$$\partial_s \ell = O(1) \qquad \partial_t \ell = [\frac{O(1)}{\ell} + \frac{\sigma}{\ell}]\mathbf{1}_{(0,\delta)} + O(1)\mathbf{1}_{[\delta,B)}. \tag{1.3.27}$$

Also, from the asymptotics of $s^2$ (1.3.18),(1.3.19) we obtain the following comparison estimate of the functions $s, \ell$.

$$0 < c \le \frac{\ell^2}{s^2} = \begin{cases} 1 + \frac{2\sigma t}{s^2} \\ \frac{O(1)}{s^2} \end{cases} \le \begin{cases} 1 + \frac{C}{\sqrt{n-1}}\sigma, & x \in (0,\delta) \\ C, & x \in [\delta, +\infty),\ B = +\infty \end{cases} \qquad n > 1. \tag{1.3.28}$$

Now we may proceed to the formal definition of the modified $H^k$ spaces.

**Definition 1.3.4.** For any given $t \in [0, T]$ and $\alpha \geq 1$, we define the weighted space

$$H^k_\alpha[t]: \qquad u \in H^k\big((0, B)\big), \qquad \|u\|^2_{H^k_\alpha[t]} = \int_{x=0}^{x=B} \frac{u^2}{\ell^{2\alpha}} + \cdots + \frac{(\partial^k_s u)^2}{\ell^{2\alpha-2k}} ds < +\infty. \tag{1.3.29}$$

In the case $k = 0$, we denote $H^0_\alpha[t]$ by $L^2_\alpha[t]$. When it is clear, we will suppress $t$ in the notation.

In this spirit, we define the energy

$$\mathcal{E}(u, v; T) = \|u\|^2_{C(0,T;H^1_\alpha)} + \|u\|^2_{L^2(0,T;H^1_{\alpha+1})} + \|v\|^2_{C(0,T;H^1_\alpha)} + \|v\|^2_{L^2(0,T;H^2_{\alpha+1})} \tag{1.3.30}$$

and for brevity let

$$\mathcal{E}_0 = \|\eta_0\|^2_{H^1_\alpha} + \|\xi_0\|^2_{H^1_\alpha}, \tag{1.3.31}$$

where $\eta_0 := \eta(x, 0)$, $\xi_0 := \xi(x, 0)$. We can formulate now a more precise version of our main result regarding the system (1.3.14).

**Theorem 1.3.5.** *There exist $\alpha > 0, \sigma := \sigma(\alpha) > 0$ sufficiently large such that if $\mathcal{E}_0$ is sufficiently small, then the system (1.3.14), subject to*

$$\xi(x, t) = 0 \qquad\qquad \{x = 0, B\} \times [0, T], \tag{1.3.32}$$

*admits a unique solution up to some time $T := T(\mathcal{E}_0, \alpha, \sigma) > 0$ in the spaces*

$$\begin{aligned} \eta &\in C(0, T; H^1_\alpha) \cap L^2(0, T; H^1_{\alpha+1}) & \xi &\in C(0, T; H^1_\alpha) \cap L^2(0, T; H^2_{\alpha+1}) \\ \eta_t &\in C(0, T; L^2_{\alpha-2}) \cap L^2(0, T; H^1_{\alpha-1}) & \xi_t &\in L^2(0, T; L^2_{\alpha-1}) \end{aligned} \tag{1.3.33}$$

*with initial data $\eta_0, \xi_0$.*

We remark here the fact that once we have such a solution to (1.3.14), then we straightforwardly derive that this solution $(\eta, \xi)$ corresponds to a solution of (1.3.4), which in fact will be smooth over $M^{n+1} \times (0, T]$, given the parabolicity of the Ricci flow.

## 1.4 The Contraction Mapping

We will prove Theorem 2.4.8 via an iteration scheme, which is essentially a contraction mapping argument. We note that throughout the subsequent estimates we will use the symbol $C$ to denote a positive constant depending only on $n$. Further, the endpoints of any integration in the spatial variable, unless otherwise indicated, will be the two ends $x = 0, B$.

### 1.4.1 The iteration scheme and the contraction mapping

In order to derive energy estimates, it is very important how we define the Picard iteration for the system (1.3.14). We choose to keep in the unknowns at each step the linear lower order terms in the RHSs which are associated to the most singular coefficients in the system. We construct a sequence $\{\eta^m, \xi^m\}_{m=0}^{\infty}$ in the spaces

$$\begin{aligned} \eta^m &\in C(0,T;H^1_\alpha)\cap L^2(0,T;H^1_{\alpha+1}) \qquad & \xi^m &\in C(0,T;H^1_\alpha)\cap L^2(0,T;H^2_{\alpha+1}) \\ \eta^m_t &\in C(0,T;L^2_{\alpha-2})\cap L^2(0,T;H^1_{\alpha-1}) & \xi^m_t &\in L^2(0,T;L^2_{\alpha-1}), \end{aligned} \tag{1.4.1}$$

satisfying

$$\begin{aligned} \eta^{m+1}_t &= 2n(n-1)\Bigg(\frac{\psi_s^2}{\psi^2}\frac{2n\xi^{m+1}+\sum_{j=2}^{2n}\binom{2n}{j}|\xi^m|^j}{(\xi^m+1)^{2n}} - 2\frac{\psi_s}{\psi}\frac{\xi^{m+1}_s}{(\xi^m+1)^{2n+1}} \\ &\quad - \frac{A(s,t)}{s}\xi^m\frac{\xi^m+2}{(\xi^m+1)^2}(1+\eta^m) - \frac{|\xi^m_s|^2}{(\xi^m+1)^{2n+2}} + \frac{\psi_s^2}{\psi^2}\eta^{m+1}\Bigg) \\ \xi^{m+1}_t &= (\frac{\psi_{ss}}{\psi}+(n-1)\frac{\psi_s^2}{\psi^2})\left[\frac{-\eta^{m+1}-2n(\eta^m+1)\xi^{m+1}}{(\eta^m+1)(\xi^m+1)^{2n-1}} - \frac{\sum_{j=2}^{2n}\binom{2n}{j}|\xi^m|^j}{(\xi^m+1)^{2n-1}}\right] \\ &\quad + (n-1)\frac{A(s,t)}{s}\xi^m\frac{\xi^m+2}{\xi^m+1} + n\frac{\psi_s}{\psi}\frac{\xi^{m+1}_s}{(\eta^m+1)(\xi^m+1)^{2n}} + \frac{\xi^{m+1}_{ss}}{(\eta^m+1)(\xi^m+1)^{2n}} \\ &\quad - \frac{|\xi^m_s|^2}{(\eta^m+1)(\xi^m+1)^{2n+1}} - \frac{1}{2}\frac{\psi_s}{\psi}\frac{\eta^{m+1}_s}{(\eta^m+1)^2(\xi^m+1)^{2n-1}} - \frac{1}{2}\frac{\eta^m_s\xi^m_s}{(\eta^m+1)^2(\xi^m+1)^{2n}}, \end{aligned} \tag{1.4.2}$$

where we set $\eta^0 = \xi^0 = 0$ and initially

$$\eta^{m+1}\Big|_{t=0} = \eta_0 \qquad \xi^{m+1}\Big|_{t=0} = \xi_0 \qquad\qquad m = 0,1,\ldots \tag{1.4.3}$$

Further, $\xi^{m+1}$ is required to verify the Dirichlet boundary condition

$$\xi^{m+1}(x,t) = 0 \qquad\qquad \{x=0,B\}\times[0,T]. \tag{1.4.4}$$

Under the assumptions of Theorem 2.4.8, we show inductively that for sufficiently small $T>0$ (uniform in $m$), the sequence also satisfies the energy estimate

$$\mathcal{E}(\eta^m,\xi^m;T) \le 2\mathcal{E}_0 \qquad\qquad m = 0,1,\ldots \tag{1.4.5}$$

We prove this in Section 1.5.

The main task that we undertake here is to prove Theorem 2.4.8 by showing that the sequence $(\eta^m,\xi^m)_{m\in\mathbb{N}}$ is actually Cauchy in the energy spaces we have introduced.

**Proposition 1.4.1.** *Let*

$$d\eta^{m+1} = \eta^{m+1}-\eta^m, \;\; d\xi^{m+1} = \xi^{m+1}-\xi^m \qquad\qquad m = 0,1,\ldots, \tag{1.4.6}$$

*where $\eta^m, \xi^m$ are the functions constructed above. Then under the assumptions in Theorem*

*2.4.8 on $\alpha, \sigma, \mathcal{E}_0, T$ the following contraction estimate holds:*

$$\mathcal{E}(d\eta^{m+1}, d\xi^{m+1}; T) \leq \frac{1}{2}\mathcal{E}(d\eta^m, d\xi^m; T) \qquad m = 1, 2, \dots, \tag{1.4.7}$$

The previous proposition readily implies Theorem 2.4.8; the iterates $(\eta^m, \xi^m)$ converge to a solution of the system (1.3.14) satisfying the assertions of the theorem.

*Proof.* It is carried out in §1.4.2. □

Some standard pointwise estimates adapted to our weighted norms are needed to proceed.

**Lemma 1.4.2.** *Given functions $\eta^m, \xi^m$, $m \in \mathbb{N}$, in the spaces (1.4.1), the following pointwise bounds are valid:*

$$\|\frac{\eta^m}{\ell^k}\|^2_{L^\infty(x)} \leq C(k+1)\mathcal{E}_0 \qquad \|\frac{\xi^m}{\ell^k}\|^2_{L^\infty(x)} \leq C(k+1)\mathcal{E}_0, \tag{1.4.8}$$

$$\|\frac{\xi^m_s}{\ell^k}\|^2_{L^\infty(x)} \leq C\sqrt{\mathcal{E}_0}\big(\|\frac{\xi^m_{ss}}{\ell^{\alpha-1}}\|_{L^2} + k\|\frac{\xi^m_s}{\ell^\alpha}\|_{L^2}\big), \qquad \int_0^t \|\frac{\xi^m_s}{\ell^k}\|^2_{L^\infty} d\tau \leq C(k+1)\sqrt{T}\mathcal{E}_0, \tag{1.4.9}$$

*for all $k = 0, \dots \alpha - 1$, $\alpha \geq 1$, $t \in [0, T]$. If in addition $\mathcal{E}_0$ is small enough, the following estimates also hold:*

$$\sup_{x\in(0,B)} \big(|\eta^m| + |\xi^m|\big) < \frac{1}{2} \qquad \inf_{x\in(0,B)} (\xi^m + 1)^{-2n} \geq \frac{1}{2}, \tag{1.4.10}$$

We note that (1.4.10) is the first main reason we consider small $\mathcal{E}_0$, which in particular guarantees the parabolicity of the second equation of (1.4.2).

*Proof.* We treat the estimate of $|\frac{\xi^m}{\ell^k}|$. The rest follow easily from the same argument. By the fundamental theorem of calculus we have

$$\Big|\frac{\xi^m\big((s(x,t),t\big)^2}{\ell^{2k}} - \frac{\xi^m\big(s(0,t),t\big)^2}{\ell^{2k}}\Big| \overset{(1.4.4)}{=} \Big|\int_{s(0,t)}^{s(x,t)} 2\frac{\xi^m}{\ell^k}\Big(\frac{\xi^m_s}{\ell^k} - k\frac{\xi^m}{\ell^{k+1}}\ell_s\Big)ds\Big|$$

$$\leq 2\|\frac{\xi^m}{\ell^{k+\frac{1}{2}}}\|_{L^2}\big(\|\frac{\xi^m_s}{\ell^{k-\frac{1}{2}}}\|^2_{L^2} + Ck\|\frac{\xi^m}{\ell^{k+\frac{1}{2}}}\|^2_{L^2}\big) \overset{(1.4.5)}{\leq} C(k+1)\mathcal{E}_0 \qquad (\ell_s = O(1)\ (1.3.27))$$

In the case of $|\frac{\eta^m}{\ell^k}|$, instead of $x = 0$, we choose a reference point $x \in [0, +\infty]$ realizing its infimum, which is controlled by the $L^2$ norm and argue similarly as before. The estimate (1.4.10) follows from (1.4.8) for $k = 0$, provided the initial weighted energy is small enough.

As for (1.4.9), the second part obviously follows from the first by integrating in time and applying C-S, along with the energy estimate (1.4.5). An easy derivation of the first part is obtained by noticing that there exists a reference point $x_0 := x_0(t)$ for which $\xi^m_s(x_0, t) = 0$. Indeed, this is implied by the vanishing of $\xi^m(x, t)$ at the endpoints $x = 0, B$ (1.4.4). The above argument applies directly. □

To write our system for $d\eta^{m+1}, d\xi^{m+1}$ concisely, we introduce generic notation

$$B, D$$

to denote rational functions in $\eta^m, \xi^m$, $m = 0, 1, \ldots$, satisfying the following conditions:

- The denomerators of $B, D$ have non-zero constant terms.
- The constant term in the numerator of $B$ is non-zero, whereas the one in the numerator of $D$ vanishes.

The next lemma is an immediate consequence of the pointwise estimates (1.4.8) and the energy estimate (1.4.5).

**Lemma 1.4.3.** *If $B, D$ are functions as above and $\mathcal{E}_0$ is sufficiently small, then the following estimates hold:*

$$\|B(s,t)\|_{L^\infty(x)} < C \qquad \|\frac{D}{\ell^k}\|^2_{L^\infty(x)} \leq C\mathcal{E}_0, \tag{1.4.11}$$

*where $k = 0, \ldots, \alpha - 1$ and*

$$\|\frac{B_s}{\ell^{\alpha-1}}\|^2_{L^2} + \|\frac{D_s}{\ell^{\alpha-1}}\|^2_{L^2} \leq C\mathcal{E}_0, \tag{1.4.12}$$

*for $0 \leq t \leq T$ and $C$ a positive constant depending on the coefficients of the rational functions $B, D$.*

Consider now the two systems (1.4.2) corresponding to the steps $m+1$ and $m$. We derive a new system for $d\eta^{m+1}, d\xi^{m+1}$ (1.4.6) by subtracting these two systems. Doing so, it is straightforward to check that we arrive at the following system:

$$\begin{aligned} d\eta^{m+1}_t &= \frac{\psi_s^2}{\psi^2} Bd\xi^{m+1} + \frac{\psi_s}{\psi} Bd\xi^{m+1}_s + 2n(n-1)\frac{\psi_s^2}{\psi^2} d\eta^{m+1} + dF_1^m \\ d\xi^{m+1}_t &= (\frac{\psi_{ss}}{\psi} + (n-1)\frac{\psi_s^2}{\psi^2})(Bd\eta^{m+1} + Bd\xi^{m+1}) + \frac{\psi_s}{\psi} Bd\xi^{m+1}_s + \frac{\psi_s}{\psi} Bd\eta^{m+1}_s \\ &\quad + \frac{d\xi^{m+1}_{ss}}{(\eta^m+1)(\xi^m+1)^{2n}} + dF_2^m, \end{aligned} \tag{1.4.13}$$

where

$$\begin{aligned} dF_1^m :&= \frac{\psi_s^2}{\psi^2} Dd\xi^m + \frac{A}{s} B(d\xi^m + d\eta^m) + \frac{\psi_s}{\psi} B\xi^m_s d\xi^m + Bd\xi^m_s(\xi^m_s + \xi^{m-1}_s) \\ &\quad + |\xi^{m-1}_s|^2 d\xi^m B \end{aligned} \tag{1.4.14}$$

and

$$
\begin{aligned}
dF_2^m := & \big(\frac{\psi_{ss}}{\psi} + (n-1)\frac{\psi_s^2}{\psi^2}\big)(Dd\xi^m + Dd\eta^m) + \frac{A}{s}Bd\xi^m + \frac{\psi_s}{\psi}B\xi_s^m(d\eta^m + d\xi^m) \\
& + \xi_{ss}^m B(d\eta^m + d\xi^m) + Bd\xi_s^m(\xi_s^m + \xi_s^{m-1}) + |\xi_s^{m-1}|^2 B(d\eta^m + d\xi^m) \\
& + \frac{\psi_s}{\psi}\eta_s^m B(d\eta^m + d\xi^m) + B(\xi_s^m d\eta_s^m + \eta_s^{m-1} d\xi_s^m) + \eta_s^{m-1}\xi_s^{m-1}B(d\eta^m + d\xi^m)
\end{aligned} \tag{1.4.15}
$$

We note that the terms $d\xi_s^m(\xi_s^m + \xi_s^{m-1})$ and $\xi_{ss}^m B(d\eta^m + d\xi^m)$ are of the most problematic and an additional reason we need to consider small initial energy $\mathcal{E}_0$ in order to close the contraction mapping argument in $H^1_\alpha$.

Similarly to Lemma 1.4.2, we have the following $L^\infty$ estimates for the differences.

**Lemma 1.4.4.** *For every $m \in \mathbb{N}$ and $t \in [0,T]$ the following estimates hold:*

$$
\|\frac{d\xi^m}{\ell^k}\|^2_{L^\infty(s)} \leq C(k+1)\|d\xi^m\|^2_{H^1_{k+1}} \qquad \|\frac{d\eta^m}{\ell^k}\|^2_{L^\infty(s)} \leq C(k+1)\|d\eta^m\|^2_{H^1_{k+1}} \tag{1.4.16}
$$

*and*

$$
\|\frac{d\xi_s^m}{\ell^k}\|^2_{L^\infty(s)} \leq C\|\frac{d\xi_s^m}{\ell^{\alpha-1}}\|_{L^2}\big(\|\frac{d\xi_{ss}^m}{\ell^{\alpha-1}}\|_{L^2} + k\|\frac{d\xi_s^m}{\ell^\alpha}\|_{L^2}\big), \tag{1.4.17}
$$

$k = 0, \ldots, \alpha - 1$.

### 1.4.2 Proof of Proposition 1.4.1: the contraction estimate (2.5.48)

In this subsection we show that the desired contraction estimate (2.5.48) follows from the next proposition, whose proof in turn we divide in three parts occupying the subsequent subsections §1.4.3, §1.4.4, §1.4.5.

**Proposition 1.4.5.** *The following estimates are valid in the time interval $[0,T]$. First, for $d\eta^{m+1}, d\xi^{m+1}$ in $L^2_\alpha$ we have*

$$
\begin{aligned}
& \frac{1}{2}\big(\|d\eta^{m+1}\|^2_{L^2_\alpha[t]} + \|d\xi^{m+1}\|^2_{L^2_\alpha[t]}\big) + \alpha\sigma\int_0^t \big(\|\frac{d\eta^{m+1}}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)} + \|\frac{d\xi^{m+1}}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)}\big)d\tau \\
\leq & C\int_0^t \big(\alpha^2 + \|\xi_s^m\|^2_{L^2}\big)\big(\|d\eta^{m+1}\|^2_{L^2_\alpha[\tau]} + \|d\xi^{m+1}\|^2_{L^2_\alpha[\tau]}\big)d\tau \\
& + C(\alpha^2+\sigma)\int_0^t \big(\|\frac{d\eta^{m+1}}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)} + \|\frac{d\xi^{m+1}}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)}\big)d\tau \\
& + C\big[\mathcal{E}_0\sigma^2 T + (\mathcal{E}_0+1)\sigma\sqrt{T} + \sqrt{T}\mathcal{E}_0^2 + \mathcal{E}_0\big]\mathcal{E}(d\eta^m, d\xi^m; T)
\end{aligned} \tag{1.4.18}
$$

*and second for the first derivatives* $d\eta_s^{m+1}, d\xi_s^{m+1}$ *in* $L^2_{\alpha-1}$

$$\begin{aligned}
&\frac{1}{2}\big(\|d\eta_s^{m+1}\|^2_{L^2_{\alpha-1}[t]}+\|d\xi_s^{m+1}\|^2_{L^2_{\alpha-1}[t]}\big)\\
&+\alpha\sigma\int_0^t\big(\|\frac{d\eta_s^{m+1}}{\ell^\alpha}\|^2_{L^2(0,\delta)}+\|\frac{d\xi_s^{m+1}}{\ell^\alpha}\|^2_{L^2(0,\delta)}\big)d\tau+\frac{1}{6}\int_0^t\|d\xi_{ss}^{m+1}\|^2_{L^2_{\alpha-1}[\tau]}d\tau\\
\leq{}& C\int_0^t\big(\|\xi_s^m\|^2_{L^\infty}+\|\xi_s^{m-1}\|^2_{L^\infty}+\alpha^2\big)\big(\|d\eta_s^{m+1}\|^2_{L^2_{\alpha-1}[\tau]}+\|d\xi_s^{m+1}\|^2_{L^2_{\alpha-1}[\tau]}\big)d\tau\\
&+C\int_0^t\big(\|\frac{\xi_{ss}^m}{\ell^{\alpha-1}}\|_{L^2}+\|\frac{\xi_{ss}^{m-1}}{\ell^{\alpha-1}}\|_{L^2}\big)\|d\xi_s^m\|_{L^\infty}\|d\eta_s^{m+1}\|_{L^2_{\alpha-1}[\tau]}d\tau \qquad (1.4.19)\\
&+C(\alpha^2+\sigma)\int_0^t\big(\|\frac{d\eta_s^{m+1}}{\ell^\alpha}\|^2_{L^2(0,\delta)}+\|\frac{d\xi_s^{m+1}}{\ell^\alpha}\|^2_{L^2(0,\delta)}\big)d\tau\\
&+C\int_0^t\big(\|d\eta^{m+1}\|^2_{L^2_\alpha}+\|d\xi^{m+1}\|^2_{L^2_\alpha}\big)d\tau+C\sigma^2\int_0^t\big(\|\frac{d\eta^{m+1}}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)}+\|\frac{d\xi^{m+1}}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)}\big)d\tau\\
&+C\big[\mathcal{E}_0\sigma^2T+(\mathcal{E}_0+1)\sigma\sqrt{T}+\mathcal{E}_0^2\sqrt{T}+\mathcal{E}_0\big]\mathcal{E}(d\eta^m,d\xi^m;T)
\end{aligned}$$

It is precisely at this point that the significance of the weights we introduced becomes apparent. We wish to close the above energy estimates by applying the standard Gronwall lemma. Unfortunately, this is not possible due to the terms in the RHSs of the estimates in the preceding proposition having larger exponents in the weights (by one) than the ones in the norms differentiated in the LHS, e.g., line three in (1.4.18). We call these terms 'critical'. Estimating the extra weight of the critical terms in $L^\infty(x)$ would not close either, as noted in Remark 1.3.21. We have to keep it in the norms. Thus, the only way to close the estimates is by absorbing these terms into the corresponding critical terms in the LHSs which work in our favor, e.g., (1.4.18) line one. That is where the role of the parameters $\alpha, \sigma$ comes into play:

Clearly, we may choose these parameters appropriately large such that the critical terms in the estimate (1.4.18), line three, are absorbed in the LHS. However, we notice that the critical terms in the estimate (1.4.19), lines five and six, cannot be directly absorbed by the corresponding ones in the estimates (1.4.18), (1.4.19), lines one and two respectively, since $C(\alpha^2+\sigma^2)$ dominates $\alpha\sigma$ ($C$ is large in our setting); see coefficients $\alpha^2, \sigma^2$ in the RHSs of (1.4.18) line three and (1.4.19) line six respectively. In order, to bypass this issue it is crucial that we can close the estimates of $d\eta^{m+1}, d\xi^{m+1}$, before moving on to estimate their derivatives. Since we are able to do that, we can then absorb the critical term in (1.4.19), line five, by choosing $\alpha\sigma > C(\alpha^2+\sigma)$ and use afterwards the already derived estimate of the zeroth order terms to estimate the critical terms in (1.4.19) line six, instead of absorbing them anywhere. This way we can close the estimates for the first order terms $d\eta_s^{m+1}, d\xi_s^{m+1}$ in $L^2_{\alpha-1}$ and obtain the desired contraction estimate (2.5.48) for small $T, \mathcal{E}_0$.

We will use below in the proof the following simple modified version of Gronwall's inequality.

**Lemma 1.4.6.** *Let $f:[a,b]\to\mathbb{R}$ be a continuous function which satisfies:*

$$\frac{1}{2}f^2(t)\le\frac{1}{2}f_0^2+\int_a^t\Psi(\tau)f(\tau)d\tau,\quad t\in[a,b],$$

*where $f_0\in\mathbb{R}$ and $\Psi$ nonnegative continuous in $[a,b]$. Then the estimate*

$$\frac{1}{2}|f(t)|\le\frac{1}{2}|f_0|+\int_a^t\Psi(\tau)d\tau,\quad t\in[a,b]$$

*holds.*

**Proposition 1.4.5 implies the contraction (2.5.48):** Choosing $\alpha,\sigma$ appropriately large such that

$$\alpha\sigma>C(\alpha^2+\sigma)+1,$$

the critical terms on the RHS of (1.4.18), line three, are be absorbed in the LHS. Hence, we may employ the standard (integral form of) Gronwall's inequality, applying the estimate (1.4.9), to close the estimate of the zeroth order terms $d\eta^{m+1},d\xi^{m+1}$:

$$\begin{aligned}&\sup_{[0,T]}\big(\|d\eta^{m+1}\|^2_{L^2_\alpha}+\|d\xi^{m+1}\|^2_{L^2_\alpha}\big)+\int_0^T\big(\|\frac{d\eta_s^{m+1}}{\ell^\alpha}\|^2_{L^2(0,\delta)}+\|\frac{d\xi_s^{m+1}}{\ell^\alpha}\|^2_{L^2(0,\delta)}\big)d\tau\\&\le C\exp\{C(\alpha^2T+\mathcal{E}_0\sqrt{T})\}\big[\mathcal{E}_0\sigma^2T+(\mathcal{E}_0+1)\sigma\sqrt{T}+\sqrt{T}\mathcal{E}_0^2+\mathcal{E}_0\big]\mathcal{E}(d\eta^m,d\xi^m;T)\end{aligned}\tag{1.4.20}$$

We proceed to the estimate of the first derivatives (1.4.19). For the same choice of $\alpha,\sigma$ as above (uniform $C$), we absorb the critical terms in the RHS, line five, involving the first order terms $d\eta_s^{m+1},d\xi_s^{m+1}$. Also, utilizing the preceding estimate (1.4.20) we estimate the zeroth order terms on the RHS of (1.4.19), line six; including the critical terms with a bad sign coefficient of magnitude $\sigma^2$. Thus, we have

$$\begin{aligned}&\frac{1}{2}\big(\|d\eta_s^{m+1}\|^2_{L^2_{\alpha-1}[t]}+\|d\xi_s^{m+1}\|^2_{L^2_{\alpha-1}[t]}\big)\\&+\int_0^t\big(\|\frac{d\eta_s^{m+1}}{\ell^\alpha}\|^2_{L^2(0,\delta)}+\|\frac{d\xi_s^{m+1}}{\ell^\alpha}\|^2_{L^2(0,\delta)}\big)d\tau+\frac{1}{6}\int_0^t\|d\xi_{ss}^{m+1}\|^2_{L^2_{\alpha-1}[\tau]}d\tau\\&\le C\int_0^t\big(\|\xi_s^m\|^2_{L^\infty}+\|\xi_s^{m-1}\|^2_{L^\infty}+\alpha^2\big)\big(\|d\eta_s^{m+1}\|^2_{L^2_{\alpha-1}[\tau]}+\|d\xi_s^{m+1}\|^2_{L^2_{\alpha-1}[\tau]}\big)d\tau\\&+C\int_0^t\big(\|\frac{\xi_{ss}^m}{\ell^{\alpha-1}}\|_{L^2}+\|\frac{\xi_{ss}^{m-1}}{\ell^{\alpha-1}}\|_{L^2}\big)\|d\xi_s^m\|_{L^\infty}\|d\eta_s^{m+1}\|_{L^2_{\alpha-1}[\tau]}d\tau\\&+C\big(\sigma^2e^{C(\alpha^2T+\mathcal{E}_0\sqrt{T})}+1\big)\big[\mathcal{E}_0\sigma^2T+(\mathcal{E}_0+1)\sigma\sqrt{T}+\mathcal{E}_0^2\sqrt{T}+\mathcal{E}_0\big]\mathcal{E}(d\eta^m,d\xi^m;T)\end{aligned}\tag{1.4.21}$$

Employing Lemma 1.4.6 for

$$f^2(t)=\|d\eta_s^{m+1}\|^2_{L^2_{\alpha-1}[t]}+\|d\xi_s^{m+1}\|^2_{L^2_{\alpha-1}[t]}$$

$$\frac{1}{2}f_0^2 = \text{the last term in (1.4.21)}$$
$$\Psi = C\big(\|\xi_s^m\|_{L^\infty}^2 + \|\xi_s^{m-1}\|_{L^\infty}^2 + \alpha^2\big)\big(\|d\eta_s^{m+1}\|_{L^2_{\alpha-1}[\tau]}^2 + \|d\xi_s^{m+1}\|_{L^2_{\alpha-1}[\tau]}^2\big)^{\frac{1}{2}}$$
$$+ C\big(\|\frac{\xi_{ss}^m}{\ell^{\alpha-1}}\|_{L^2} + \|\frac{\xi_{ss}^{m-1}}{\ell^{\alpha-1}}\|_{L^2}\big)\|d\xi_s^m\|_{L^\infty}$$

we obtain

$$\sup_{t\in[0,T]}\big(\|d\eta_s^{m+1}\|_{L^2_{\alpha-1}[t]}^2 + \|d\xi_s^{m+1}\|_{L^2_{\alpha-1}[t]}^2\big) \leq \int_0^T \Psi(\tau)d\tau \tag{1.4.22}$$
$$+ C\big(\sigma^2 e^{C(\alpha^2 T + \mathcal{E}_0\sqrt{T})} + 1\big)\big[\mathcal{E}_0\sigma^2 T + (\mathcal{E}_0+1)\sigma\sqrt{T} + \mathcal{E}_0^2\sqrt{T} + \mathcal{E}_0\big]\mathcal{E}(d\eta^m, d\xi^m; T).$$

Finally, applying C-S and (1.4.5),(1.4.17) we estimate

$$\int_0^T \Psi d\tau \leq C(\alpha^2 T + \mathcal{E}_0\sqrt{T})\sup_{t\in[0,T]}\big(\|d\eta_s^{m+1}\|_{L^2_{\alpha-1}[t]}^2 + \|d\xi_s^{m+1}\|_{L^2_{\alpha-1}[t]}^2\big)^{\frac{1}{2}} \tag{1.4.23}$$
$$+ C\sqrt{\mathcal{E}_0}\cdot\mathcal{E}(d\eta^m, d\xi^m; T)$$

Hence, for $T > 0$ small we absorb the first term in (1.4.23) to the LHS of (1.4.22) and close the estimates of $d\eta_s^{m+1}, d\xi_s^{m+1}$.

From the above estimates we deduce the contraction estimate (2.5.48), provided $T, \mathcal{E}_0$ are sufficiently small. □

### 1.4.3 Proof of Proposistion 1.4.5 I: Estimates for the non-linear terms

We establish some estimates for the functions $dF_1^m, dF_2^m$ (1.4.14),(1.4.15) that we will use in proving the estimates in Proposition 1.4.5.

**Proposition 1.4.7.** *For any function $u \in L^2(0,T;L^2_\alpha)$ and $t \in [0,T]$ the following estimates hold:*

$$\int_0^t \|dF_1^m\|_{L^2_{\alpha-1}}^2 d\tau \leq C\Big(\mathcal{E}_0\sigma^2 T + (\mathcal{E}_0+1)\sigma\sqrt{T} + \sqrt{T}\mathcal{E}_0^2\Big)\mathcal{E}(d\eta^m, d\xi^m; T), \tag{1.4.24}$$

$$\int_0^t \|dF_2^m\|_{L^2_{\alpha-1}}^2 d\tau \leq C\Big(\mathcal{E}_0\sigma^2 T + (\mathcal{E}_0+1)\sigma\sqrt{T} + \sqrt{T}\mathcal{E}_0^2 + \mathcal{E}_0\Big)\mathcal{E}(d\eta^m, d\xi^m; T) \tag{1.4.25}$$

*and*

$$\int_0^t\int \frac{u\cdot\partial_s(dF_1^m)}{\ell^{2\alpha}}dsd\tau \leq \frac{C}{\varepsilon}\sigma\int_0^t \|\frac{u}{\ell^\alpha}\|_{L^2(0,\delta)}^2 d\tau + \frac{C}{\varepsilon}\int_0^t \|\frac{u}{\ell^{\alpha-1}}\|_{L^2}^2 d\tau$$
$$+ \frac{C}{\varepsilon}\int_0^t \big(\|\xi_s^m\|_{L^\infty}^2 + \|\xi_s^{m-1}\|_{L^\infty}^2 + 1\big)\|\frac{u}{\ell^{\alpha-1}}\|_{L^2}^2 d\tau \tag{1.4.26}$$

$$+ C\int_0^t \big(\|\frac{\xi_{ss}^m}{\ell^{\alpha-1}}\|_{L^2} + \|\frac{\xi_{ss}^{m-1}}{\ell^{\alpha-1}}\|_{L^2}\big)\|d\xi_s^m\|_{L^\infty}\|\frac{u}{\ell^{\alpha-1}}\|_{L^2}d\tau \qquad (0<\varepsilon<1)$$
$$+ C\bigg(\mathcal{E}_0\sigma^2 T + \mathcal{E}_0\sigma\sqrt{T} + \varepsilon(\mathcal{E}_0+1) + (\mathcal{E}_0+1)^2\sqrt{T}\bigg)\mathcal{E}(d\eta^m, d\xi^m; T)$$

We remark that the only part that 'does not belong' in the above estimates, is the last summand in (1.4.25) from which we do not gain any smallness in $T$. This term comes from estimating $\xi_{ss}^m B(d\eta^m + d\xi^m)$ in $dF_2^m$ (1.4.15) below.

*Proof.* Recall the leading behavior of the coefficients (1.3.15), (1.3.22). Plugging (1.4.14) in the norm below we estimate:

$$\begin{aligned}
\|dF_1^m\|^2_{L^2_{\alpha-1}} &\leq \|\frac{\psi_s^2}{\psi^2}\frac{Dd\xi^m}{\ell^{\alpha-1}}\|^2_{L^2} + \|\frac{A}{s}\frac{B(d\xi^m+d\eta^m)}{\ell^{\alpha-1}}\|^2_{L^2} + \|\frac{\psi_s}{\psi}B\xi_s^m\frac{d\xi^m}{\ell^{\alpha-1}}\|^2_{L^2} \qquad (1.4.27)\\
&\quad + \|B\frac{d\xi_s^m}{\ell^{\alpha-1}}(\xi_s^m+\xi_s^{m-1})\|^2_{L^2} + \||\xi_s^{m-1}|^2 B\frac{d\xi^m}{\ell^{\alpha-1}}\|^2_{L^2}\\
&\leq C\mathcal{E}_0\|\frac{d\xi^m}{s^2\ell^{\alpha-2}}\|^2_{L^2} \qquad \text{(using the estimate (1.4.11) for the fraction } \tfrac{D}{\ell})\\
&\quad + CA^2(t)\big(\|\frac{d\xi^m}{s\ell^{\alpha-1}}\|^2_{L^2} + \|\frac{d\eta^m}{s\ell^{\alpha-1}}\|^2_{L^2}\big) + C\|\xi_s^m\|^2_{L^\infty}\|\frac{d\xi^m}{s\ell^{\alpha-1}}\|^2_{L^2}\\
&\quad + C\big(\|\xi_s^m\|^2_{L^\infty} + \|\xi_s^{m-1}\|^2_{L^\infty}\big)\|\frac{d\xi_s^m}{\ell^{\alpha-1}}\|^2_{L^2} + C\|\xi_s^m\|^2_{L^\infty}\|d\xi^m\|^2_{L^\infty}\|\frac{\xi_s^{m-1}}{\ell^{\alpha-1}}\|^2_{L^2}
\end{aligned}$$

Employing the comparison estimate $\ell^2/s^2 \leq C\sigma$ (1.3.28) for the first three terms in the RHS of the second inequality above and the $L^\infty$ estimate of $d\xi^m$ (1.4.16) for the last term we obtain

$$\begin{aligned}
\|dF_1^m\|^2_{L^2_{\alpha-1}} &\leq C\big(\mathcal{E}_0\sigma^2 + A^2(t)\sigma + \|\xi_s^m\|^2_{L^\infty}\sigma + \|\xi_s^m\|^2_{L^\infty} + \|\xi_s^{m-1}\|^2_{L^\infty} \qquad (1.4.28)\\
&\quad + \|\xi_s^{m-1}\|^2_{L^\infty}\mathcal{E}_0\big)\big(\|d\xi^m\|^2_{H^1_\alpha} + \|d\eta^m\|^2_{L^2_\alpha}\big)
\end{aligned}$$

After integrating in time and applying (1.3.23),(1.4.9) we arrive at (1.4.24).

Similarly, for the case of $dF_2^m$ plugging in (1.4.15) we derive:

$$\begin{aligned}
&\|dF_2^m\|^2_{L^2_{\alpha-1}} \qquad (1.4.29)\\
&\leq \|(\frac{\psi_{ss}}{\psi} + (n-1)\frac{\psi_s^2}{\psi^2})\frac{D(d\xi^m+d\eta^m)}{\ell^{\alpha-1}}\|^2_{L^2} + \|\frac{A}{s}B\frac{d\xi^m}{\ell^{\alpha-1}}\|^2_{L^2} + \|\frac{\psi_s}{\psi}B\xi_s^m\frac{d\eta^m+d\xi^m}{\ell^{\alpha-1}}\|^2_{L^2}\\
&\quad + \|\xi_{ss}^m B\frac{d\eta^m+d\xi^m}{\ell^{\alpha-1}}\|^2_{L^2} + \|B\frac{d\xi_s^m}{\ell^{\alpha-1}}(\xi_s^m+\xi_s^{m-1})\|^2_{L^2} + \||\xi_s^{m-1}|^2 B\frac{d\eta^m+d\xi^m}{\ell^{\alpha-1}}\|^2_{L^2}\\
&\quad + \|\frac{\psi_s}{\psi}\eta_s^m B\frac{d\eta^m+d\xi^m}{\ell^{\alpha-1}}\|^2_{L^2} + \|B(\xi_s^m\frac{d\eta_s^m}{\ell^{\alpha-1}} + \eta_s^{m-1}\frac{d\xi_s^m}{\ell^{\alpha-1}})\|^2_{L^2} + \|\eta_s^{m-1}\xi_s^{m-1}B\frac{d\eta^m+d\xi^m}{\ell^{\alpha-1}}\|^2_{L^2}\\
&\leq C\mathcal{E}_0\big(\|\frac{d\xi^m}{s^2\ell^{\alpha-2}}\|^2_{L^2} + \|\frac{d\eta^m}{s^2\ell^{\alpha-2}}\|^2_{L^2}\big) \qquad \text{(applying (1.4.11) for } \tfrac{D}{\ell} \text{ and } B)\\
&\quad + CA^2(t)\|\frac{d\xi^m}{s\ell^{\alpha-1}}\|^2_{L^2} + C\|\xi_s^m\|^2_{L^\infty}\big(\|\frac{d\eta^m}{s\ell^{\alpha-1}}\|^2_{L^2} + \|\frac{d\xi^m}{s\ell^{\alpha-1}}\|^2_{L^2}\big)\\
&\quad + C\big(\|d\eta^m\|^2_{L^\infty} + \|d\xi^m\|^2_{L^\infty}\big)\|\frac{\xi_{ss}^m}{\ell^{\alpha-1}}\|^2_{L^2} + C\big(\|\xi_s^m\|^2_{L^\infty} + \|\xi_s^{m-1}\|^2_{L^\infty}\big)\|\frac{d\xi_s^m}{\ell^{\alpha-1}}\|^2_{L^2}
\end{aligned}$$

$$+ C\|\xi_s^{m-1}\|_{L^\infty}^2\big(\|d\eta^m\|_{L^\infty}^2 + \|d\xi^m\|_{L^\infty}^2\big)\|\frac{\xi_s^{m-1}}{\ell^{\alpha-1}}\|_{L^2}^2$$
$$+ C\big(\|\frac{d\eta^m}{s}\|_{L^\infty}^2 + \|\frac{d\xi^m}{s}\|_{L^\infty}^2\big)\|\frac{\eta_s^m}{\ell^{\alpha-1}}\|_{L^2}^2 + C\|\xi_s^m\|_{L^\infty}^2\|\frac{d\eta_s^m}{\ell^{\alpha-1}}\|_{L^2}^2$$
$$+ C\|d\xi_s^m\|_{L^\infty}^2\|\frac{\eta_s^{m-1}}{\ell^{\alpha-1}}\|_{L^2}^2 + C\|\xi_s^{m-1}\|_{L^\infty}^2\big(\|d\eta^m\|_{L^\infty}^2 + \|d\xi^m\|_{L^\infty}^2\big)\|\frac{\eta_s^{m-1}}{\ell^{\alpha-1}}\|_{L^2}^2$$

We employ once more the comparison estimate (1.3.28), the energy estimate of the iterates (1.4.5) and the $L^\infty$ estimates for $d\eta^m, d\xi^m, d\xi_s^m$ to get

$$\begin{aligned} &\|dF_2^m\|_{L^2_{\alpha-1}}^2 && (1.4.30)\\ &\leq C\big(\mathcal{E}_0\sigma^2 + A^2(t)\sigma + \|\xi_s^m\|_{L^\infty}^2\sigma + \sigma\mathcal{E}_0\big)\big(\|d\xi^m\|_{H_\alpha^1}^2 + \|d\eta^m\|_{H_\alpha^1}^2\big)\\ &\quad + C\big(\|\xi_s^m\|_{L^\infty}^2 + \|\xi_s^{m-1}\|_{L^\infty}^2 + \|\xi_s^{m-1}\|_{L^\infty}^2\mathcal{E}_0\big)\big(\|d\xi^m\|_{H_\alpha^1}^2 + \|d\eta^m\|_{H_\alpha^1}^2\big)\\ &\quad + C\|\frac{\xi_{ss}^m}{\ell^{\alpha-1}}\|_{L^2}^2\big(\|d\xi^m\|_{H_\alpha^1}^2 + \|d\eta^m\|_{H_\alpha^1}^2\big) + \mathcal{E}_0\|\frac{d\xi_s^m}{\ell^{\alpha-1}}\|_{L^2}\|\frac{d\xi_{ss}^m}{\ell^{\alpha-1}}\|_{L^2}\end{aligned}$$

Integrating from $0 \leq \tau \leq t$, applying C-S to the last term above and utilizing (1.4.9) we achieve the estimate (1.4.25).[4]

We proceed to the relevant estimates of $\partial_s(dF_1^m)$. This time, to be comprehensive, we plug in each term in the RHS of (1.4.14) at the time and estimate it separately. Recall again the singular orders of the coefficients (1.3.15), (1.3.22) and the ones of their spatial derivatives (1.3.17). Applying C-S to each arising term we have:

$$\begin{aligned} &\int \frac{u}{\ell^{2\alpha-2}}\partial_s\Big[\frac{\psi_s^2}{\psi^2}Dd\xi^m\Big]ds && (1.4.31)\\ &= \int \frac{u}{\ell^{2\alpha-2}}\Big[\partial_s(\frac{\psi_s^2}{\psi^2})Dd\xi^m + \frac{\psi_s^2}{\psi^2}D_sd\xi^m + \frac{\psi_s^2}{\psi^2}Dd\xi_s^m\Big]ds\\ &\leq \|\frac{u}{s\ell^{\alpha-1}}\|_{L^2}^2 + C\mathcal{E}_0\|\frac{d\xi^m}{s^2\ell^{\alpha-2}}\|_{L^2}^2 \qquad \text{(by C-S and the pointwise estimate of } D \text{ (1.4.11), } k=1)\\ &\quad + \|\frac{u}{s\ell^{\alpha-1}}\|_{L^2}^2 + C\|\frac{d\xi^m}{s}\|_{L^\infty}^2\|\frac{D_s}{\ell^{\alpha-1}}\|_{L^2}^2 + \|\frac{u}{s\ell^{\alpha-1}}\|_{L^2}^2 + C\mathcal{E}_0\|\frac{d\xi_s^m}{s\ell^{\alpha-2}}\|_{L^2}^2\\ &\leq C\sigma\|\frac{u}{\ell^\alpha}\|_{L^2(0,\delta)}^2 + C\|\frac{u}{\ell^{\alpha-1}}\|_{L^2(\delta,+\infty)}^2 \qquad \text{(recall def. (1.3.26); estimate } \ell^2/s^2 \text{ (1.3.28))}\\ &\quad + C\mathcal{E}_0\sigma^2\|\frac{d\xi^m}{\ell^\alpha}\|_{L^2}^2 + C\mathcal{E}_0\sigma\|\frac{d\xi_s^m}{\ell^{\alpha-1}}\|_{L^2}^2 \qquad \text{(by (1.4.12) for } D_s \text{ and } d\xi^m\text{-}L^\infty \text{ estimate (1.4.16))}\end{aligned}$$

Similarly, utilizing the estimates on $B$ (1.4.11), (1.4.12) we obtain

$$\begin{aligned} &\int \frac{u}{\ell^{2\alpha-2}}\partial_s\Big[\frac{A}{s}B(d\xi^m + d\eta^m)\Big]ds && (1.4.32)\\ &= \int \frac{u}{\ell^{2\alpha-2}}\Big[\partial_s(\frac{A}{s})B(d\xi^m + d\eta^m) + \frac{A}{s}B_s(d\xi^m + d\eta^m) + \frac{A}{s}B(d\xi_s^m + d\eta_s^m)\Big]ds\end{aligned}$$

[4]The second last term in the RHS of (1.4.30) is the first problematic term that forces us to assume further smallness of the initial energy $\mathcal{E}_0$.

$$
\begin{aligned}
&\leq \|\frac{u}{s\ell^{\alpha-1}}\|_{L^2}^2 + CA^2(t)\big(\|\frac{d\xi^m}{s\ell^{\alpha-1}}\|_{L^2}^2 + \|\frac{d\eta^m}{s\ell^{\alpha-1}}\|_{L^2}^2\big) \\
&\quad \|\frac{u}{s\ell^{\alpha-1}}\|_{L^2}^2 + CA^2(t)\big(\|d\xi^m\|_{L^\infty}^2 + \|d\eta^m\|_{L^\infty}^2\big)\|\frac{B_s}{\ell^{\alpha-1}}\|_{L^2}^2 \\
&\quad + \|\frac{u}{s\ell^{\alpha-1}}\|_{L^2}^2 + CA^2(t)\big(\|\frac{d\xi_s^m}{\ell^{\alpha-1}}\|_{L^2}^2 + \|\frac{d\eta_s^m}{\ell^{\alpha-1}}\|_{L^2}^2\big) \\
&\leq C\sigma\|\frac{u}{\ell^\alpha}\|_{L^2(0,\delta)}^2 + C\|\frac{u}{\ell^{\alpha-1}}\|_{L^2(\delta,+\infty)}^2 + CA^2(t)(\mathcal{E}_0+\sigma+1)\big(\|d\xi^m\|_{H_\alpha^1}^2 + \|d\eta^m\|_{H_\alpha^1}^2\big)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int \frac{u}{\ell^{2\alpha-2}}\partial_s\left[\frac{\psi_s}{\psi}B\xi_s^m d\xi^m\right]ds \qquad (1.4.33)\\
&= \int \frac{u}{\ell^{2\alpha-2}}\left[\partial_s(\frac{\psi_s}{\psi})B\xi_s^m d\xi^m + \frac{\psi_s}{\psi}B_s\xi_s^m d\xi^m + \frac{\psi_s}{\psi}B\xi_{ss}^m d\xi^m + \frac{\psi_s}{\psi}B\xi_s^m d\xi_s^m\right]ds \\
&\leq \|\frac{u}{s\ell^{\alpha-1}}\|_{L^2}^2 + C\|\xi_s^m\|_{L^\infty}^2\|\frac{d\xi^m}{s\ell^{\alpha-1}}\|_{L^2}^2 + C\|\xi_s^m\|_{L^\infty}^2\|d\xi^m\|_{L^\infty}^2\|\frac{B_s}{\ell^{\alpha-1}}\|_{L^2}^2 \\
&\quad + \frac{C}{\varepsilon}\|\frac{u}{s\ell^{\alpha-1}}\|_{L^2}^2 + \varepsilon\|d\xi^m\|_{L^\infty}^2\|\frac{\xi_{ss}^m}{\ell^{\alpha-1}}\|_{L^2}^2 + \|\frac{u}{s\ell^{\alpha-1}}\|_{L^2}^2 + C\|\xi_s^m\|_{L^\infty}^2\|\frac{d\xi_s^m}{\ell^{\alpha-1}}\|_{L^2}^2 \\
&\leq \frac{C}{\varepsilon}\sigma\|\frac{u}{\ell^\alpha}\|_{L^2(0,\delta)}^2 + \frac{C}{\varepsilon}\|\frac{u}{\ell^{\alpha-1}}\|_{L^2(\delta,+\infty)}^2 + C\sigma\|\xi_s^m\|_{L^\infty}^2\|\frac{d\xi^m}{\ell^\alpha}\|_{L^2}^2 \qquad \text{(using (1.3.28))}\\
&\quad + \big(\varepsilon\|\frac{\xi_{ss}^m}{\ell^{\alpha-1}}\|_{L^2}^2 + C\mathcal{E}_0\|\xi_s^m\|_{L^\infty}^2 + C\|\xi_s^m\|_{L^\infty}^2\big)\|d\xi^m\|_{H_\alpha^1}^2
\end{aligned}
$$

The last term to be estimated is a bit more involved. We follow the same plan employing the estimates on $B$ (1.4.11), (1.4.12) and the $L^\infty$ estimates of $d\xi^m, d\xi_s^m$ (1.4.16), (1.4.17).

$$
\begin{aligned}
&\int \frac{u}{\ell^{2\alpha-2}}\partial_s\left[Bd\xi_s^m(\xi_s^m+\xi_s^{m-1}) + |\xi_s^{m-1}|^2 d\xi^m B\right]ds \qquad (1.4.34)\\
&= \int \frac{u}{\ell^{2\alpha-2}}\Big[B_s d\xi_s^m(\xi_s^m+\xi_s^{m-1}) + Bd\xi_{ss}^m(\xi_s^m+\xi_s^{m-1}) + Bd\xi_s^m(\xi_{ss}^m+\xi_{ss}^{m-1}) \\
&\quad + 2\xi_s^{m-1}\xi_{ss}^{m-1}d\xi^m B + |\xi_s^{m-1}|^2 d\xi_s^m B + |\xi_s^{m-1}|^2 d\xi^m B_s\Big]ds \\
&\leq C\big(\|\xi_s^m\|_{L^\infty}^2 + \|\xi_s^{m-1}\|_{L^\infty}^2\big)\|\frac{u}{\ell^{\alpha-1}}\|_{L^2}^2 + \|d\xi_s^m\|_{L^\infty}^2\|\frac{B_s}{\ell^{\alpha-1}}\|_{L^2}^2 \\
&\quad + \varepsilon\|\frac{d\xi_{ss}^m}{\ell^{\alpha-1}}\|_{L^2}^2 + \frac{C}{\varepsilon}\big(\|\xi_s^m\|_{L^\infty}^2 + \|\xi_s^{m-1}\|_{L^\infty}^2\big)\|\frac{u}{\ell^{\alpha-1}}\|_{L^2}^2 \\
&\quad + C\|\frac{u}{\ell^{\alpha-1}}\|_{L^2}\|d\xi_s^m\|_{L^\infty}\big(\|\frac{\xi_{ss}^m}{\ell^{\alpha-1}}\|_{L^2} + \|\frac{\xi_{ss}^{m-1}}{\ell^{\alpha-1}}\|_{L^2}\big) \\
&\quad + \frac{C}{\varepsilon}\|\xi_s^{m-1}\|_{L^\infty}^2\|\frac{u}{\ell^{\alpha-1}}\|_{L^2}^2 + \varepsilon\|d\xi^m\|_{L^\infty}\|\frac{\xi_{ss}^{m-1}}{\ell^{\alpha-1}}\|_{L^2}^2 \\
&\quad + \|\xi_s^{m-1}\|_{L^\infty}^2\|\frac{u}{\ell^{\alpha-1}}\|_{L^2}^2 + C\|\xi_s^{m-1}\|_{L^\infty}^2\|\frac{d\xi_s^m}{\ell^{\alpha-1}}\|_{L^2}^2 + C\|\xi_s^{m-1}\|_{L^\infty}^2\|d\xi^m\|_{L^\infty}^2\|\frac{B_s}{\ell^{\alpha-1}}\|_{L^2}^2 \\
&\leq \frac{C}{\varepsilon}\big(\|\xi_s^m\|_{L^\infty}^2 + \|\xi_s^{m-1}\|_{L^\infty}^2\big)\|\frac{u}{\ell^{\alpha-1}}\|_{L^2}^2 + C\|\frac{u}{\ell^{\alpha-1}}\|_{L^2}\|d\xi_s^m\|_{L^\infty}\big(\|\frac{\xi_{ss}^m}{\ell^{\alpha-1}}\|_{L^2} + \|\frac{\xi_{ss}^{m-1}}{\ell^{\alpha-1}}\|_{L^2}\big) \\
&\quad + C\big[(\mathcal{E}_0+1)\|\xi_s^{m-1}\|_{L^\infty}^2 + \varepsilon\|\frac{\xi_{ss}^m}{\ell^{\alpha-1}}\|_{L^2}^2\big]\|d\xi^m\|_{H_\alpha^1}^2 + \varepsilon\|\frac{d\xi_{ss}^m}{\ell^{\alpha-1}}\|_{L^2}^2 + \mathcal{E}_0\|\frac{d\xi_s^m}{\ell^{\alpha-1}}\|_{L^2}\|\frac{d\xi_{ss}^m}{\ell^{\alpha-1}}\|_{L^2}
\end{aligned}
$$

We remark here that the control of the term $Bd\xi_s^m(\xi_{ss}^m+\xi_{ss}^{m-1})$ in the above estimate, which results to the second term on the RHS of the last inequality, is one of the most delicate that we have to perform[5]; essentially due to the fact that our energies depend on just one derivative in $\eta$. This term also forces us to consider small initial energy $\mathcal{E}_0$ to close the estimates; cf. the last term in the estimate (1.4.23).

Combining (1.4.31)-(1.4.34) we obtain

$$\begin{aligned}
&\int\frac{u\cdot\partial_s(dF_1^m)}{\ell^{2\alpha-2}}ds \qquad (1.4.35)\\
\leq&\frac{C}{\varepsilon}\sigma\|\frac{u}{\ell^\alpha}\|^2_{L^2(0,\delta)}+\frac{C}{\varepsilon}\big(\|\xi_s^m\|^2_{L^\infty}+\|\xi_s^{m-1}\|^2_{L^\infty}+1\big)\|\frac{u}{\ell^{\alpha-1}}\|^2_{L^2(\delta,+\infty)}\\
&+C\|\frac{u}{\ell^{\alpha-1}}\|_{L^2}\|d\xi_s^m\|_{L^\infty}\big(\|\frac{\xi_{ss}^m}{\ell^{\alpha-1}}\|_{L^2}+\|\frac{\xi_{ss}^{m-1}}{\ell^{\alpha-1}}\|_{L^2}\big)\\
&+C\bigg[\mathcal{E}_0\sigma^2+(\mathcal{E}_0+\sigma+1)A^2(t)+\sigma\|\xi_s^m\|^2_{L^\infty}+\varepsilon\|\frac{\xi_{ss}^m}{\ell^{\alpha-1}}\|^2_{L^2}\\
&+(\mathcal{E}_0+1)\big(\|\xi_s^m\|^2_{L^\infty}+\|\xi_s^{m-1}\|^2_{L^\infty}\big)\bigg]\big(\|d\xi^m\|^2_{H^1_\alpha}+\|d\eta^m\|^2_{H^1_\alpha}\big)+\varepsilon\|\frac{d\xi_{ss}^m}{\ell^{\alpha-1}}\|^2_{L^2}\\
&+\mathcal{E}_0\|\frac{d\xi_s^m}{\ell^{\alpha-1}}\|_{L^2}\|\frac{d\xi_{ss}^m}{\ell^{\alpha-1}}\|_{L^2}
\end{aligned}$$

Thus, integrating on $[0,t]$ and employing once more the estimates (1.3.23),(1.4.9) we conclude the desired estimate (1.4.26). This completes the proof Proposition 1.4.7. □

### 1.4.4 Proof of Proposition 1.4.5 II: $L^2_\alpha$ estimates of $d\eta^{m+1},d\xi^{m+1}$

We prove (1.4.18). Let us commence with the $L^2_\alpha$ estimates of $d\eta^{m+1}$. Taking the time derivative of the $L^2_\alpha$ norm of $d\eta^{m+1}$ and using (1.3.27), (1.3.25) we derive

$$\begin{aligned}
\frac{1}{2}\partial_t\|d\eta^{m+1}\|^2_{L^2_\alpha}=&\int\frac{d\eta^{m+1}d\eta_t^{m+1}}{\ell^{2\alpha}}ds-\alpha\int\frac{|d\eta^{m+1}|^2}{\ell^{2\alpha+1}}\partial_t\ell ds+\frac{1}{2}\int\frac{|d\eta^{m+1}|^2}{\ell^{2\alpha}}\partial_t ds\\
\leq&\int\frac{d\eta^{m+1}d\eta_t^{m+1}}{\ell^{2\alpha}}ds-\alpha\sigma\|\frac{d\eta^{m+1}}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)}+C\alpha\|\frac{d\eta^{m+1}}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)} \qquad (1.4.36)\\
&+C\alpha\|\frac{d\eta^{m+1}}{\ell^\alpha}\|^2_{L^2(\delta,+\infty)}+C\|\frac{d\eta^{m+1}}{s\ell^\alpha}\|^2_{L^2}
\end{aligned}$$

As usual, we estimate the last term employing (1.3.28)

$$\|\frac{d\eta^{m+1}}{s\ell^\alpha}\|^2_{L^2}\leq C\sigma\|\frac{d\eta^{m+1}}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)}+C\|\frac{d\eta^{m+1}}{\ell^\alpha}\|^2_{L^2(\delta,+\infty)} \qquad (1.4.37)$$

Recall (1.3.15), (1.3.22) and the pointwise bound of $B$ (1.4.11) to derive

$$\int\frac{d\eta^{m+1}d\eta_t^{m+1}}{\ell^{2\alpha}}ds \qquad \text{(plugging in the RHS of (1.4.13))}$$

[5] In fact, if this term in the equation had been slightly more nonlinear, the overall scheme would break down.

$$
\begin{aligned}
&= \int \frac{d\eta^{m+1}}{\ell^{2\alpha}} \Big[ \frac{\psi_s^2}{\psi^2} B d\xi^{m+1} + \frac{\psi_s}{\psi} B d\xi_s^{m+1} + 2n(n-1)\frac{\psi_s^2}{\psi^2} d\eta^{m+1} + dF_1^m \Big] ds && (1.4.38)\\
&\leq \| \frac{d\eta^{m+1}}{s\ell^\alpha}\|_{L^2}^2 + C\|\frac{d\xi^{m+1}}{s\ell^\alpha}\|_{L^2}^2 + \varepsilon \|\frac{d\xi_s^{m+1}}{\ell^\alpha}\|_{L^2}^2 + \frac{C}{\varepsilon}\|\frac{d\eta^{m+1}}{s\ell^\alpha}\|_{L^2}^2 \\
&\quad + C\|\frac{d\eta^{m+1}}{s\ell^\alpha}\|_{L^2}^2 + \|\frac{d\eta^{m+1}}{\ell^{\alpha+1}}\|_{L^2}^2 + \|dF_1^m\|_{L^2_{\alpha-1}}^2 \\
&\leq \frac{C}{\varepsilon}\sigma \|\frac{d\eta^{m+1}}{\ell^{\alpha+1}}\|_{L^2(0,\delta)}^2 + \frac{C}{\varepsilon}\|\frac{d\eta^{m+1}}{\ell^\alpha}\|_{L^2(\delta,+\infty)}^2 && \text{(employing (1.3.28), } 0<\varepsilon<1)\\
&\quad + C\sigma\|\frac{d\xi^{m+1}}{\ell^{\alpha+1}}\|_{L^2(0,\delta)}^2 + C\|\frac{d\xi^{m+1}}{\ell^\alpha}\|_{L^2(\delta,+\infty)}^2 + \varepsilon\|\frac{d\xi_s^{m+1}}{\ell^\alpha}\|_{L^2}^2 + \|dF_1^m\|_{L^2_{\alpha-1}}^2
\end{aligned}
$$

We proceed to the case of $d\xi^{m+1}$ slightly differently. We control the $L^2_\alpha$ norm of the term $(\eta^m+1)^{\frac{1}{2}} d\xi^{m+1}$ instead. Of course, it is evident from (1.4.10) that it is the same thing as estimating $d\xi^{m+1}$. We should note that it is not needed to go through this procedure if $\mathcal{E}_0$ is small enough, but we wish to provide a more general plan. Similarly to (1.4.36), keeping in mind the pointwise estimate on $\eta^m$ (1.4.10), we deduce

$$
\begin{aligned}
\frac{1}{2}\partial_t \|(\eta^m+1)^{\frac{1}{2}} d\xi^{m+1}\|_{L^2_\alpha}^2 \leq \int \frac{(\eta^m+1) d\xi^{m+1} d\xi_t^{m+1}}{\ell^{2\alpha}} ds - \frac{1}{2}\alpha\sigma \|\frac{d\xi^{m+1}}{\ell^{\alpha+1}}\|_{L^2(0,\delta)}^2 \quad (1.4.39)\\
+ C\alpha \|\frac{d\xi^{m+1}}{\ell^{\alpha+1}}\|_{L^2(0,\delta)}^2 + C\alpha\|\frac{d\xi^{m+1}}{\ell^\alpha}\|_{L^2(\delta,+\infty)}^2 \\
+ C\|\frac{d\xi^{m+1}}{s\ell^\alpha}\|_{L^2}^2 + \|\eta_t^m\|_{L^\infty}\|\frac{d\xi^{m+1}}{\ell^\alpha}\|_{L^2}^2
\end{aligned}
$$

The second last term is controlled via (1.3.28), as in (1.4.37). We estimate the last term from the equation satisfied by $\eta_t^m$, analogous of the first equation in (1.4.2), using the pointwise estimate on the iterates (1.4.8) and the comparison estimate (1.3.28), replacing the singular orders of the coefficients (1.3.15), (1.3.22) with the weights $\ell^k$, $k=1,2$.

$$
\begin{aligned}
&\|\eta_t^m\|_{L^\infty}\|\frac{d\xi^{m+1}}{\ell^\alpha}\|_{L^2}^2 && (1.4.40)\\
&\leq C\big(\sqrt{\mathcal{E}_0}\sigma + \sqrt{\sigma}\|\xi_s^m\|_{L^\infty} + A(t)\sqrt{\sigma\mathcal{E}_0} + \|\xi_s^{m-1}\|_{L^\infty}^2\big)\|\frac{d\xi^{m+1}}{\ell^\alpha}\|_{L^2}^2
\end{aligned}
$$

Moving on to the main term, plugging in the RHS of (1.4.13), we have

$$
\begin{aligned}
&\int \frac{(\eta_m+1) d\xi^{m+1} d\xi_t^{m+1}}{\ell^{2\alpha}} ds && (1.4.41)\\
&= \int \frac{(\eta_m+1) d\xi^{m+1}}{\ell^{2\alpha}} \Big[ (\frac{\psi_{ss}}{\psi} + (n-1)\frac{\psi_s^2}{\psi^2})(Bd\eta^{m+1} + Bd\xi^{m+1}) + \frac{\psi_s}{\psi} B d\xi_s^{m+1} \\
&\quad + \frac{d\xi_{ss}^{m+1}}{(\eta^m+1)(\xi^m+1)^{2n}} + \frac{\psi_s}{\psi} B d\eta_s^{m+1} + dF_2^m \Big] ds \\
&\leq C\|\frac{d\xi^{m+1}}{s\ell^\alpha}\|_{L^2}^2 + C\|\frac{d\eta^{m+1}}{s\ell^\alpha}\|_{L^2}^2 + \varepsilon\|\frac{d\xi_s^{m+1}}{\ell^\alpha}\|_{L^2}^2 + \frac{C}{\varepsilon}\|\frac{d\xi^{m+1}}{s\ell^\alpha}\|_{L^2}^2 && \text{(by (1.4.11) for } B)
\end{aligned}
$$

$$
\begin{aligned}
&+\int \frac{d\xi^{m+1}d\xi^{m+1}_{ss}}{\ell^{2\alpha}(\xi^m+1)^{2n}}ds+\int\frac{\psi_s}{\psi}\frac{Bd\xi^{m+1}d\eta^{m+1}_s}{\ell^{2\alpha}}ds+\int\frac{(\eta_m+1)d\xi^{m+1}dF^m_2}{\ell^{2\alpha}}ds\\
&\leq \frac{C}{\varepsilon}\sigma\|\frac{d\xi^{m+1}}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)}+\frac{C}{\varepsilon}\|\frac{d\xi^{m+1}}{\ell^{\alpha}}\|^2_{L^2(\delta,+\infty)}+\varepsilon\|\frac{d\xi^{m+1}_s}{\ell^\alpha}\|^2_{L^2} \qquad \text{(using (1.3.28))}\\
&+C\sigma\|\frac{d\eta^{m+1}}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)}+C\|\frac{d\eta^{m+1}}{\ell^{\alpha}}\|^2_{L^2(\delta,+\infty)}+\|dF^m_2\|^2_{L^2_{\alpha-1}}\\
&+\int \frac{d\xi^{m+1}d\xi^{m+1}_{ss}}{\ell^{2\alpha}(\xi^m+1)^{2n}}ds+\int\frac{\psi_s}{\psi}\frac{Bd\xi^{m+1}d\eta^{m+1}_s}{\ell^{2\alpha}}ds
\end{aligned}
$$

We treat the last two terms separately integrating by parts. At this point the role of the Dirichlet boundary condition (1.4.4) comes into play.

$$
\begin{aligned}
&\int \frac{d\xi^{m+1}d\xi^{m+1}_{ss}}{\ell^{2\alpha}(\xi^m+1)^{2n}}ds \qquad (1.4.42)\\
&=-\int\frac{|d\xi^{m+1}_s|^2}{\ell^{2\alpha}(\xi^m+1)^{2n}}ds+\int\frac{2n\xi^m_s d\xi^{m+1}d\xi^{m+1}_s}{\ell^{2\alpha}(\xi^m+1)^{2n+1}}ds+2\alpha\int\frac{d\xi^{m+1}d\xi^{m+1}_s}{\ell^{2\alpha+1}(\xi^m+1)^{2n}}\partial_s\ell ds\\
&\leq -\frac{1}{2}\|\frac{d\xi^{m+1}_s}{\ell^\alpha}\|^2_{L^2}+\varepsilon\|\frac{d\xi^{m+1}_s}{\ell^\alpha}\|^2_{L^2}+\frac{C}{\varepsilon}\|\xi^m_s\|^2_{L^\infty}\|\frac{d\xi^{m+1}}{\ell^\alpha}\|^2_{L^2} \qquad \text{(see (1.4.10))}\\
&+\varepsilon\|\frac{d\xi^{m+1}_s}{\ell^\alpha}\|^2_{L^2}+\frac{C\alpha^2}{\varepsilon}\|\frac{d\xi^{m+1}}{\ell^{\alpha+1}}\|^2_{L^2} \qquad (\ell_s=O(1)\ (1.3.27))\\
&\leq (2\varepsilon-\frac{1}{2})\|\frac{d\xi^{m+1}_s}{\ell^\alpha}\|^2_{L^2}+\frac{C}{\varepsilon}\|\xi^m_s\|^2_{L^\infty}\|\frac{d\xi^{m+1}}{\ell^\alpha}\|^2_{L^2}+\frac{C\alpha^2}{\varepsilon}\|\frac{d\xi^{m+1}}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)}+\frac{C\alpha^2}{\varepsilon}\|\frac{d\xi^{m+1}}{\ell^{\alpha}}\|^2_{L^2}
\end{aligned}
$$

Similarly, by (1.3.15), (1.4.11), (1.4.12) we obtain [6]

$$
\begin{aligned}
&\int\frac{\psi_s}{\psi}\frac{Bd\xi^{m+1}d\eta^{m+1}_s}{\ell^{2\alpha}}ds \qquad (1.4.43)\\
&=-\int\partial_s(\frac{\psi_s}{\psi})\frac{Bd\xi^{m+1}d\eta^{m+1}}{\ell^{2\alpha}}ds-\int\frac{\psi_s}{\psi}\frac{B_sd\xi^{m+1}d\eta^{m+1}}{\ell^{2\alpha}}ds-\int\frac{\psi_s}{\psi}\frac{Bd\xi^{m+1}_sd\eta^{m+1}}{\ell^{2\alpha}}ds\\
&+2\alpha\int\frac{\psi_s}{\psi}\frac{Bd\xi^{m+1}d\eta^{m+1}}{\ell^{2\alpha+1}}\partial_s\ell ds\\
&\leq \|\frac{d\eta^{m+1}}{s\ell^\alpha}\|^2_{L^2}+C\|\frac{d\xi^{m+1}}{s\ell^\alpha}\|^2_{L^2}+\frac{C}{\varepsilon}\|\frac{d\eta^{m+1}}{s\ell^\alpha}\|^2_{L^2}+\varepsilon\|\frac{d\xi^{m+1}}{\ell}\|^2_{L^\infty}\|\frac{B_s}{\ell^{\alpha-1}}\|^2_{L^2}\\
&+\varepsilon\|\frac{d\xi^{m+1}_s}{\ell^\alpha}\|^2_{L^2}+\frac{C}{\varepsilon}\|\frac{d\eta^{m+1}}{s\ell^\alpha}\|^2_{L^2}+\|\frac{d\eta^{m+1}}{s\ell^\alpha}\|^2_{L^2}+\alpha^2\|\frac{d\xi^{m+1}}{\ell^{\alpha+1}}\|^2_{L^2}\\
&\leq \frac{C}{\varepsilon}\sigma\|\frac{d\eta^{m+1}}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)}+\frac{C}{\varepsilon}\|\frac{d\eta^{m+1}}{\ell^{\alpha}}\|^2_{L^2(\delta,+\infty)} \qquad \text{(employing (1.3.28), } 0<\varepsilon<1)\\
&+(C\sigma+C\varepsilon\mathcal{E}_0+\alpha^2)\|\frac{d\xi^{m+1}}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)} \qquad \text{(by the } L^\infty \text{ estimate (1.4.16) of } d\xi^m/\ell)\\
&+\varepsilon(1+C\mathcal{E}_0)\|\frac{d\xi^{m+1}_s}{\ell^\alpha}\|^2_{L^2}+C\alpha^2\|\frac{d\xi^{m+1}}{\ell^\alpha}\|^2_{L^2}
\end{aligned}
$$

[6] The possibility to control this next term using an integration by parts to offload the derivative from $d\eta^{m+1}$ is essential in order to close our estimates for the $L^2_\alpha$ norms of $d\xi^{m+1}, d\eta^{m+1}$, without recourse to the higher derivatives.

Putting the above estimates (1.4.36)-(1.4.43) all together we conclude that

$$\begin{aligned}
&\frac{1}{2}\partial_t\big(\|d\eta^{m+1}\|^2_{L^2_\alpha}+\|(\eta^m+1)^{\frac12}d\xi^{m+1}\|^2_{L^2_\alpha}\big)+\frac12\alpha\sigma\big(\|\frac{d\eta^{m+1}}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)}+\|\frac{d\xi^{m+1}}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)}\big)\\
&\leq(4\varepsilon+C\varepsilon\mathcal{E}_0-\frac12)\|\frac{d\xi^{m+1}_s}{\ell^\alpha}\|^2_{L^2}+\frac{C}{\varepsilon}\big(\alpha^2+\|\xi^m_s\|^2_{L^2}\big)\big(\|d\eta^{m+1}\|^2_{L^2_\alpha}+\|d\xi^{m+1}\|^2_{L^2_\alpha}\big) \qquad (1.4.44)\\
&\quad+C(\frac{\alpha^2}{\varepsilon}+\frac{\sigma}{\varepsilon}+\varepsilon\mathcal{E}_0)\big(\|\frac{d\eta^{m+1}}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)}+\|\frac{d\xi^{m+1}}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)}\big)+\|dF_1^m\|^2_{L^2_{\alpha-1}}+\|dF_2^m\|^2_{L^2_{\alpha-1}}
\end{aligned}$$

Choosing $\varepsilon$ small enough, the first term in the RHS of the preceding estimate has a negative sign and hence it can be dropped. Integrating on $[0,t]$ and taking into account the integrated estimates of $dF_1^m, dF_2^m$ in Proposition 1.4.7, we obtain the desired estimate (1.4.18) in Proposition 1.4.5.

### 1.4.5 Proof of Proposition 1.4.5 III: $L^2_{\alpha-1}$ estimates of $d\eta_s^{m+1}, d\xi_s^{m+1}$

In this subsection we prove (1.4.19). Recall the bounds on the derivatives of the weight $\ell$ (1.3.27), the volume form $ds$ (1.3.25) and the commutator $[\partial_s,\partial_t]$ (1.3.24) to obtain

$$\begin{aligned}
\frac12\partial_t\|d\eta_s^{m+1}\|^2_{L^2_{\alpha-1}}&=\int\frac{d\eta_s^{m+1}\partial_t d\eta_s^{m+1}}{\ell^{2\alpha-2}}ds-(\alpha-1)\int\frac{|d\eta_s^{m+1}|^2}{\ell^{2\alpha-1}}\partial_t\ell ds+\frac12\int\frac{|d\eta_s^{m+1}|^2}{\ell^{2\alpha-2}}\partial_t ds\\
&\leq\int\frac{d\eta_s^{m+1}\partial_s d\eta_t^{m+1}}{\ell^{2\alpha-2}}ds-(\alpha-1)\sigma\|\frac{d\eta_s^{m+1}}{\ell^\alpha}\|^2_{L^2(0,\delta)} \qquad (1.4.45)\\
&\quad+C(\alpha-1)\|\frac{d\eta_s^{m+1}}{\ell^\alpha}\|^2_{L^2}+C\|\frac{d\eta_s^{m+1}}{s\ell^{\alpha-1}}\|^2_{L^2}
\end{aligned}$$

As usual, from (1.3.28)

$$\|\frac{d\eta_s^{m+1}}{s\ell^{\alpha-1}}\|^2_{L^2}\leq C\sigma\|\frac{d\eta_s^{m+1}}{\ell^\alpha}\|^2_{L^2(0,\delta)}+C\|\frac{d\eta_s^{m+1}}{\ell^{\alpha-1}}\|^2_{L^2(\delta,+\infty)},$$

In order to estimate the first term in the RHS of the inequality (1.4.45) we plug in $d\eta_t^{m+1}$ from the first equation of (1.4.13) and treat each generated term separately. For all three of the subsequent bounds we apply C-S at each term, using the estimates on the coefficients (1.3.15) and the relevant function $B$ (1.4.11), (1.4.12):

$$\begin{aligned}
&\int\frac{d\eta_s^{m+1}}{\ell^{2\alpha-2}}\partial_s\Big[\frac{\psi_s^2}{\psi^2}Bd\xi^{m+1}\Big]ds \qquad (1.4.46)\\
&=\int\frac{d\eta_s^{m+1}}{\ell^{2\alpha-2}}\Big[\partial_s(\frac{\psi_s^2}{\psi^2})Bd\xi^{m+1}+\frac{\psi_s^2}{\psi^2}B_s d\xi^{m+1}+\frac{\psi_s^2}{\psi^2}Bd\xi_s^{m+1}\Big]ds\\
&\leq\|\frac{d\eta_s^{m+1}}{s\ell^{\alpha-1}}\|^2_{L^2}+C\|\frac{d\xi^{m+1}}{s^2\ell^{\alpha-1}}\|^2_{L^2}+C\|\frac{d\xi^{m+1}}{s}\|^2_{L^\infty}\|\frac{B_s}{\ell^{\alpha-1}}\|^2_{L^2}+C\|\frac{d\xi_s^{m+1}}{s\ell^{\alpha-1}}\|^2_{L^2}\\
&\leq C\sigma\|\frac{d\eta_s^{m+1}}{\ell^\alpha}\|^2_{L^2(0,\delta)}+C\|\frac{d\eta_s^{m+1}}{\ell^{\alpha-1}}\|^2_{L^2(\delta,+\infty)}+C\sigma^2\|\frac{d\xi^{m+1}}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)} \qquad \text{(employing (1.3.28))}\\
&\quad+C\|\frac{d\xi^{m+1}}{\ell^\alpha}\|^2_{L^2(\delta,+\infty)}+C\sigma\|\frac{d\xi_s^{m+1}}{\ell^\alpha}\|^2_{L^2(0,\delta)} \qquad \text{(by the } L^\infty \text{ estimate (1.4.16) on } d\xi^{m+1})
\end{aligned}$$

$$+ C\|\frac{d\xi_s^{m+1}}{\ell^{\alpha-1}}\|^2_{L^2(\delta,+\infty)}$$

Similarly, we obtain

$$\int \frac{d\eta_s^{m+1}}{\ell^{2\alpha-2}}\partial_s\left[\frac{\psi_s}{\psi}Bd\xi_s^{m+1}\right]ds \tag{1.4.47}$$
$$= \int \frac{d\eta_s^{m+1}}{\ell^{2\alpha-2}}\left[\partial_s(\frac{\psi_s}{\psi})Bd\xi_s^{m+1} + \frac{\psi_s}{\psi}B_sd\xi_s^{m+1} + \frac{\psi_s}{\psi}Bd\xi_{ss}^{m+1}\right]ds$$
$$\le \|\frac{d\eta_s^{m+1}}{s\ell^{\alpha-1}}\|^2_{L^2} + C\|\frac{d\xi_s^{m+1}}{s\ell^{\alpha-1}}\|^2_{L^2} + \frac{C}{\varepsilon}\|\frac{d\eta_s^{m+1}}{s\ell^{\alpha-1}}\|^2_{L^2} + \varepsilon\|d\xi_s^{m+1}\|^2_{L^\infty}\|\frac{B_s}{\ell^{\alpha-1}}\|^2_{L^2}$$
$$+ \varepsilon\|\frac{d\xi_{ss}^{m+1}}{\ell^{\alpha-1}}\|^2_{L^2}$$
$$\le \frac{C}{\varepsilon}\sigma\|\frac{d\eta_s^{m+1}}{\ell^{\alpha}}\|^2_{L^2(0,\delta)} + \frac{C}{\varepsilon}\|\frac{d\eta_s^{m+1}}{\ell^{\alpha-1}}\|^2_{L^2(\delta,+\infty)} \qquad \text{(by (1.3.28), } 0<\varepsilon<1)$$
$$+ C\sigma\|\frac{d\xi_s^{m+1}}{\ell^{\alpha}}\|^2_{L^2(0,\delta)} + C(1+\varepsilon\mathcal{E}_0)\|\frac{d\xi_s^{m+1}}{\ell^{\alpha-1}}\|^2_{L^2}$$
$$+ \varepsilon(1+C\mathcal{E}_0)\|\frac{d\xi_{ss}^{m+1}}{\ell^{\alpha-1}}\|^2_{L^2} \qquad \text{(by the } L^\infty \text{ estimate (1.4.17) on } d\xi_s^{m+1}\text{, } k=0\text{, and C-S)}$$

and

$$\int \frac{d\eta_s^{m+1}}{\ell^{2\alpha-2}}\partial_s\left[2n(n-1)\frac{\psi_s^2}{\psi^2}d\eta^{m+1}\right]ds \tag{1.4.48}$$
$$= \int \frac{d\eta_s^{m+1}}{\ell^{2\alpha-2}}\left[2n(n-1)\partial_s(\frac{\psi_s^2}{\psi^2})d\eta^{m+1} + 2n(n-1)\frac{\psi_s^2}{\psi^2}d\eta_s^{m+1}\right]ds$$
$$\le C\|\frac{d\eta_s^{m+1}}{s\ell^{\alpha-1}}\|^2_{L^2} + C\|\frac{d\eta^{m+1}}{s^2\ell^{\alpha-1}}\|^2_{L^2}$$
$$\le C\sigma\|\frac{d\eta_s^{m+1}}{\ell^{\alpha}}\|^2_{L^2(0,\delta)} + C\|\frac{d\eta_s^{m+1}}{\ell^{\alpha-1}}\|^2_{L^2(\delta,+\infty)} + C\sigma^2\|\frac{d\eta^{m+1}}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)} \qquad \text{(by (1.3.28))}$$
$$+ C\|\frac{d\eta^{m+1}}{\ell^{\alpha}}\|^2_{L^2(\delta,+\infty)}$$

We proceed to the case of $d\xi_s^{m+1}$. Similarly to (1.4.45), using in addition the boundary condition (1.4.4) upon integrating by parts we have

$$\frac{1}{2}\partial_t\|d\xi_s^{m+1}\|^2_{L^2_{\alpha-1}} \le \int \frac{d\xi_s^{m+1}\partial_s d\xi_t^{m+1}}{\ell^{2\alpha-2}}ds - (\alpha-1)\sigma\|\frac{d\xi_s^{m+1}}{\ell^{\alpha}}\|^2_{L^2(0,\delta)}$$
$$+ C(\alpha-1)\|\frac{d\xi_s^{m+1}}{\ell^{\alpha}}\|^2_{L^2} + C\|\frac{d\xi_s^{m+1}}{s\ell^{\alpha-1}}\|^2_{L^2}$$
$$= -\int \frac{d\xi_{ss}^{m+1}d\xi_t^{m+1}}{\ell^{2\alpha-2}}ds + (2\alpha-2)\int \frac{d\xi_s^{m+1}d\xi_t^{m+1}}{\ell^{2\alpha-1}}\ell_s ds \tag{1.4.49}$$
$$- (\alpha-1)\sigma\|\frac{d\xi_s^{m+1}}{\ell^{\alpha}}\|^2_{L^2(0,\delta)} + C(\alpha-1)\|\frac{d\xi_s^{m+1}}{\ell^{\alpha}}\|^2_{L^2} + C\|\frac{d\xi_s^{m+1}}{s\ell^{\alpha-1}}\|^2_{L^2}$$

There are two main terms we must estimate here. In both estimates we plug in $d\xi_t^{m+1}$ from

(1.4.13), distributing the singularities in the coefficients (1.3.15) by applying C-S and the usual pointwise estimates. We start first with the term

$$
\begin{aligned}
&(2\alpha-2)\int \frac{d\xi_s^{m+1} d\xi_t^{m+1}}{\ell^{2\alpha-1}}\ell_s ds \qquad (1.4.50)\\
&= (2\alpha-2)\int \frac{d\xi_s^{m+1}}{\ell^{2\alpha-1}}\ell_s\Big[(\frac{\psi_{ss}}{\psi}+(n-1)\frac{\psi_s^2}{\psi^2})(Bd\eta^{m+1}+Bd\xi^{m+1})+\frac{\psi_s}{\psi}Bd\xi_s^{m+1}\\
&\quad +|B|d\xi_{ss}^{m+1}+\frac{\psi_s}{\psi}Bd\eta_s^{m+1}+dF_2^m\Big]ds\\
&\leq \alpha^2\|\frac{d\xi_s^{m+1}}{\ell^\alpha}\|_{L^2}^2 + C\|\frac{d\eta^{m+1}}{s^2\ell^{\alpha-1}}\|_{L^2}^2 + C\|\frac{d\xi^{m+1}}{s^2\ell^\alpha}\|_{L^2}^2 + C\|\frac{d\xi_s^{m+1}}{s\ell^{\alpha-1}}\|_{L^2}^2\\
&\quad +\varepsilon\|\frac{d\xi_{ss}^{m+1}}{\ell^{\alpha-1}}\|_{L^2}^2 + \frac{C}{\varepsilon}\alpha^2\|\frac{d\xi_s^{m+1}}{\ell^\alpha}\|_{L^2}^2 + \alpha^2\|\frac{d\xi_s^{m+1}}{\ell^\alpha}\|_{L^2}^2 + C\|\frac{d\eta_s^{m+1}}{s\ell^{\alpha-1}}\|_{L^2}^2\\
&\quad +(2\alpha-2)\int\frac{d\xi_s^{m+1}dF_1^m}{\ell^{2\alpha-1}}\ell_s ds\\
&\leq \frac{C}{\varepsilon}\alpha^2\|\frac{d\xi_s^{m+1}}{\ell^\alpha}\|_{L^2(0,\delta)}^2 + \frac{C}{\varepsilon}\alpha^2\|\frac{d\xi_s^{m+1}}{\ell^{\alpha-1}}\|_{L^2(\delta,+\infty)}^2 + C\sigma^2\|\frac{d\eta^{m+1}}{\ell^{\alpha+1}}\|_{L^2(0,\delta)}^2 \qquad \text{(by (1.3.28))}\\
&\quad + C\|\frac{d\eta^{m+1}}{\ell^\alpha}\|_{L^2(\delta,+\infty)}^2 + C\sigma^2\|\frac{d\xi^{m+1}}{\ell^{\alpha+1}}\|_{L^2(0,\delta)}^2 + C\|\frac{d\xi^{m+1}}{\ell^\alpha}\|_{L^2(\delta,+\infty)}^2\\
&\quad + C\sigma\|\frac{d\xi_s^{m+1}}{\ell^\alpha}\|_{L^2(0,\delta)}^2 + C\|\frac{d\xi_s^{m+1}}{\ell^{\alpha-1}}\|_{L^2(\delta,+\infty)}^2 + C\sigma\|\frac{d\eta_s^{m+1}}{\ell^\alpha}\|_{L^2(0,\delta)}^2\\
&\quad + C\|\frac{d\eta_s^{m+1}}{\ell^{\alpha-1}}\|_{L^2(\delta,+\infty)}^2 + \varepsilon\|\frac{d\xi_{ss}^{m+1}}{\ell^{\alpha-1}}\|_{L^2}^2 + \|dF_2^m\|_{L^2_{\alpha-1}}^2 \qquad (\ell_s = O(1)\ (1.3.27))
\end{aligned}
$$

and analogously for

$$
\begin{aligned}
&-\int\frac{d\xi_{ss}^{m+1}d\xi_t^{m+1}}{\ell^{2\alpha-2}}ds \qquad (1.4.51)\\
&= -\int\frac{d\xi_{ss}^{m+1}}{\ell^{2\alpha-2}}\Big[(\frac{\psi_{ss}}{\psi}+(n-1)\frac{\psi_s^2}{\psi^2})(Bd\eta^{m+1}+Bd\xi^{m+1})+\frac{\psi_s}{\psi}Bd\xi_s^{m+1}\\
&\quad +\frac{d\xi_{ss}^{m+1}}{(\eta^m+1)(\xi^m+1)^{2n}}+\frac{\psi_s}{\psi}Bd\eta_s^{m+1}+dF_2^m\Big]ds\\
&\leq \varepsilon\|\frac{d\xi_{ss}^{m+1}}{\ell^{\alpha-1}}\|_{L^2}^2 + \frac{C}{\varepsilon}\|\frac{d\xi^{m+1}}{s^2\ell^{\alpha-1}}\|_{L^2}^2 + \frac{C}{\varepsilon}\|\frac{d\eta^{m+1}}{s^2\ell^{\alpha-1}}\|_{L^2}^2 + \frac{C}{\varepsilon}\|\frac{d\xi_s^{m+1}}{s\ell^{\alpha-1}}\|_{L^2}^2\\
&\quad -\int\frac{|d\xi_{ss}^{m+1}|^2}{\ell^{2\alpha-2}(\eta^m+1)(\xi^m+1)^{2n}}ds + \varepsilon\|\frac{d\xi_{ss}^{m+1}}{\ell^{\alpha-1}}\|_{L^2}^2 + \frac{C}{\varepsilon}\|\frac{d\eta_s^{m+1}}{s\ell^{\alpha-1}}\|_{L^2}^2 + \frac{C}{\varepsilon}\|dF_2^m\|_{L^2_{\alpha-1}}^2\\
&\leq 2\varepsilon\|\frac{d\xi_{ss}^{m+1}}{\ell^{\alpha-1}}\|_{L^2}^2 + \frac{C}{\varepsilon}\sigma^2\|\frac{d\xi^{m+1}}{\ell^{\alpha+1}}\|_{L^2(0,\delta)}^2 + \frac{C}{\varepsilon}\|\frac{d\xi^{m+1}}{\ell^\alpha}\|_{L^2(\delta,+\infty)}^2 \qquad \text{(by (1.3.28))}\\
&\quad + \frac{C}{\varepsilon}\sigma^2\|\frac{d\eta^{m+1}}{\ell^{\alpha+1}}\|_{L^2(0,\delta)}^2 + \frac{C}{\varepsilon}\|\frac{d\eta^{m+1}}{\ell^\alpha}\|_{L^2(\delta,+\infty)}^2 + \frac{C}{\varepsilon}\sigma\|\frac{d\xi_s^{m+1}}{\ell^\alpha}\|_{L^2(0,\delta)}^2\\
&\quad + \frac{C}{\varepsilon}\|\frac{d\xi_s^{m+1}}{\ell^{\alpha-1}}\|_{L^2(\delta,+\infty)}^2 + \frac{C}{\varepsilon}\sigma\|\frac{d\eta_s^{m+1}}{\ell^\alpha}\|_{L^2(0,\delta)}^2 + \frac{C}{\varepsilon}\|\frac{d\eta_s^{m+1}}{\ell^{\alpha-1}}\|_{L^2(\delta,+\infty)}^2\\
&\quad -\frac{1}{3}\|\frac{d\xi_{ss}^{m+1}}{\ell^{\alpha-1}}\|_{L^2}^2 + \frac{C}{\varepsilon}\|dF_2^m\|_{L^2_{\alpha-1}}^2 \qquad \text{(from the poinwtise estimate (1.4.10))}
\end{aligned}
$$

**Summary**: Combining (1.4.45)-(1.4.51) we deduce

$$\begin{aligned}
&\frac{1}{2}\partial_t\big(\|d\eta_s^{m+1}\|^2_{L^2_{\alpha-1}} + \|d\xi_s^{m+1}\|^2_{L^2_{\alpha-1}}\big) + (\alpha-1)\sigma\big(\|\frac{d\eta_s^{m+1}}{\ell^\alpha}\|^2_{L^2(0,\delta)} + \|\frac{d\xi_s^{m+1}}{\ell^\alpha}\|^2_{L^2(0,\delta)}\big)\\
&\le C\big(\frac{\alpha^2}{\varepsilon}+\varepsilon\mathcal{E}_0\big)\big(\|d\eta_s^{m+1}\|^2_{L^2_{\alpha-1}} + \|d\xi_s^{m+1}\|^2_{L^2_{\alpha-1}}\big) + \int \frac{d\eta_s^{m+1}\partial_s(dF_1^m)}{\ell^{2\alpha-2}}ds \qquad (1.4.52)\\
&+\frac{C}{\varepsilon}(\alpha^2+\sigma)\big(\|\frac{d\eta_s^{m+1}}{\ell^\alpha}\|^2_{L^2(0,\delta)} + \|\frac{d\xi_s^{m+1}}{\ell^\alpha}\|^2_{L^2(0,\delta)}\big) + (4\varepsilon + C\varepsilon\mathcal{E}_0 - \frac{1}{3})\|\frac{d\xi_{ss}^{m+1}}{\ell^{\alpha-1}}\|^2_{L^2}\\
&+\frac{C}{\varepsilon}\big(\|d\eta^{m+1}\|^2_{L^2_\alpha} + \|d\xi^{m+1}\|^2_{L^2_\alpha}\big) + \frac{C}{\varepsilon}\sigma^2\big(\|\frac{d\eta^{m+1}}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)} + \|\frac{d\xi^{m+1}}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)}\big)\\
&+\frac{C}{\varepsilon}\|dF_2^m\|^2_{L^2_{\alpha-1}}
\end{aligned}$$

Let $\varepsilon > 0$ be small such that

$$4\varepsilon + C\varepsilon\mathcal{E}_0 < \frac{1}{6}.$$

Integrating on $[0,t]$ and invoking the integrated estimates (1.4.25), (1.4.26) for $u = d\eta_s^{m+1}$, we finally arrive at the estimate (1.4.19) in Proposition 1.4.5.

## 1.5 The Linear step in the iteration

In the beginning of §1.4.1 we took for granted that at each step $m+1$, $m = 0, 1, \ldots$, the linear system (1.4.2) possessed a solution with prescribed regularity and energy bounds. We prove these assertions here.

**Definition 1.5.1.** For this section, we will let $f, g, F_1, F_2$ stand for generic functions in the spaces

$$f, g \in L^\infty(0,T;H^1) \qquad F_1 \in L^2(0,T;H^1),\ \ F_2 \in L^2(0,T;L^2) \qquad (1.5.1)$$

satisfying the bounds

$$\frac{1}{2} + \|f\|_{L^\infty_x} < \|f\|_{L^\infty_x} + g(x,t) < C \qquad \|g_s\|^2_{L^\infty(0,T;L^2)} < \varepsilon, \qquad (1.5.2)$$

for appropriate positive constants $c, C$, $\varepsilon$ small, and

$$\int_0^T \|\frac{F_i}{\ell^{\alpha-1}}\|^2_{L^2}dt < +\infty, \qquad i = 1,2 \qquad (1.5.3)$$

$$\int \frac{u\cdot\partial_s F_1}{\ell^{2\alpha-2}}ds \le C\sigma\|\frac{u}{\ell^\alpha}\|^2_{L^2(0,\delta)} + G_1(t)\|\frac{u}{\ell^{\alpha-1}}\|^2_{L^2} + G_2(t),$$

for a.e. $0 \le t \le T$ and the general function $u \in L^2(0,T;L^2_\alpha)$, where $G_1(t), G_2(t)$ are positive $[0,T]$-integrable functions.

Motivated by (1.4.2), we consider the following linear system:

$$\begin{aligned} \eta_t &= \frac{\psi_s^2}{\psi^2} f\xi + \frac{\psi_s}{\psi} f\xi_s + 2n(n-1)\frac{\psi_s^2}{\psi^2}\eta + F_1 \\ \xi_t &= (\frac{\psi_{ss}}{\psi} + (n-1)\frac{\psi_s^2}{\psi^2}) f\cdot(\eta+\xi) + \frac{\psi_s}{\psi} f\xi_s + g\xi_{ss} + \frac{\psi_s}{\psi} f\eta_s + F_2 \\ \eta\Big|_{t=0} &= \eta_0 \qquad \xi\Big|_{t=0} = \xi_0, \qquad \xi = 0, \text{ on } \{x=0,B\}\times[0,T] \end{aligned} \tag{1.5.4}$$

We prove:

**Theorem 1.5.2.** *There exist $\alpha, \sigma$ sufficiently large such that (1.5.4) has a unique solution up to time $T > 0$ in the spaces*

$$\begin{aligned} \eta &\in L^\infty(0,T;H^1_\alpha)\cap L^2(0,T;H^1_{\alpha+1}) \qquad \xi \in L^\infty(0,T;H^1_\alpha)\cap L^2(0,T;H^2_{\alpha+1}) \\ \eta_t &\in L^\infty(0,T;L^2_{\alpha-2})\cap L^2(0,T;H^1_{\alpha-1}) \qquad \xi_t \in L^2(0,T;L^2_{\alpha-1}) \end{aligned} \tag{1.5.5}$$

*Further, the solution satisfies the energy estimate*

$$\mathcal{E}(\eta,\xi;T) \le \widetilde{C}\Big[\mathcal{E}_0 + \sum \int_0^T \|\frac{F_i}{\ell^{\alpha-1}}\|^2_{L^2} dt + \int_0^T G_2(t)dt\Big] =: \widetilde{C}C_0(T), \tag{1.5.6}$$

*for some positive constant $\widetilde{C}$.*

It is easy to see that the linear system (1.4.2) is of the type (1.5.4), if the energy $\mathcal{E}(\eta^m,\xi^m;T)$ is small enough. Taking the latter as an induction hypothesis, Theorem 1.5.2 then implies the existence of $\eta^{m+1},\xi^{m+1}$ satisfying the same assertions, provided $T,\mathcal{E}_0 > 0$ are sufficiently small (uniformly in $m$).

### 1.5.1 Plan of the proof of Theorem 1.5.2

We perform a new iteration for (1.5.4), first solving the first equation (ODE) for $\eta$ (using a previously-soved-for $\tilde{\xi}$)[7] and then plugging $\eta$ into the second (and main) equation of (1.5.4) to solve for the new $\xi$. Let

$$\tilde{\xi} \in L^\infty(0,T;H^1_\alpha)\cap L^2(0,T;H^2_{\alpha+1}) \tag{1.5.7}$$

be a function satisfying

$$\|\tilde{\xi}\|^2_{L^\infty(0,T;H^1_\alpha)} + \|\tilde{\xi}\|^2_{L^2(0,T;H^2_{\alpha+1})} \le \widetilde{C}C_0(T), \tag{1.5.8}$$

[7]This way we avoid some additional problems having to do with the fact that the level of regularity of $\eta$ is lower, by one derivative, than the one we have for $\xi$.

with improved bounds for

$$\int_0^T \|\frac{\tilde{\xi}}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)}dt \leq \frac{\widetilde{C}}{\sigma^2}C_0(T) \qquad \int_0^T \|\frac{\tilde{\xi}_s}{\ell^{\alpha}}\|^2_{L^2(0,\delta)}dt \leq \frac{\widetilde{C}}{\sigma}C_0(T); \tag{1.5.9}$$

$\widetilde{C}$ is some positive constant to be determined later. We consider the system

$$\begin{aligned} \eta_t &= \frac{\psi_s^2}{\psi^2}f\tilde{\xi} + \frac{\psi_s}{\psi}f\tilde{\xi}_s + 2n(n-1)\frac{\psi_s^2}{\psi^2}\eta + F_1 \\ \xi_t &= (\frac{\psi_{ss}}{\psi} + (n-1)\frac{\psi_s^2}{\psi^2})f\cdot(\eta+\xi) + \frac{f}{\psi^2}\xi + \frac{\psi_s}{\psi}f\xi_s + g\xi_{ss} + \frac{\psi_s}{\psi}f\eta_s + F_2 \\ \eta\Big|_{t=0} &= \eta_0 \qquad \xi\Big|_{t=0} = \xi_0, \qquad\qquad \xi = 0, \text{ on } \{x=0,B\}\times[0,T] \end{aligned} \tag{1.5.10}$$

**Claim:** For suitably large $\alpha, \sigma$ the preceding system has a unique solution

$$\begin{aligned} \eta &\in L^\infty(0,T;H^1_\alpha)\cap L^2(0,T;H^1_{\alpha+1}) \qquad \xi \in L^\infty(0,T;H^1_\alpha)\cap L^2(0,T;H^2_{\alpha+1}) \\ \eta_t &\in L^\infty(0,T;L^2_{\alpha-2})\cap L^2(0,T;H^1_{\alpha-1}) \qquad \xi_t \in L^2(0,T;L^2_{\alpha-1}), \end{aligned} \tag{1.5.11}$$

which satisfies the energy estimates

$$\mathcal{E}(\eta,\xi;T) \leq \widetilde{C}C_0(T) \tag{1.5.12}$$

and

$$\int_0^T \|\frac{\xi}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)}dt \leq \frac{\widetilde{C}}{\sigma^2}C_0(T) \qquad \int_0^T \|\frac{\xi_s}{\ell^{\alpha}}\|^2_{L^2(0,\delta)}dt \leq \frac{\widetilde{C}}{\sigma}C_0(T). \tag{1.5.13}$$

Observe that if we can prove this, a standard iteration argument (passing to a subsequence, weak limits etc.) yields a solution $\eta, \xi$ of the original linear problem (1.5.4) in the same space (1.5.11) and satisfying the same estimates as above. This reduces the proof of Theorem 1.5.2 to proving our claim above.

### 1.5.2 A priori estimates for $\eta$

The function $\eta$ given by the (ODE) first equation of (1.5.10) satisfies the following energy estimates for $\alpha, \sigma, \widetilde{C}$ large, $T > 0$ small:

$$\|\eta\|^2_{L^\infty(0,T;H^1_\alpha)} + \|\eta\|^2_{L^\infty(0,T;H^1_{\alpha+1})} \leq \frac{\widetilde{C}}{10}C_0(T) \tag{1.5.14}$$

and

$$\int_0^T \|\frac{\eta}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)}d\tau \leq \frac{C}{\alpha}\frac{\widetilde{C}}{\sigma^2}C_0(T) \qquad \int_0^T \|\frac{\eta_s}{\ell^{\alpha}}\|^2_{L^2(0,\delta)}d\tau \leq \frac{C}{\alpha}\frac{\widetilde{C}}{\sigma}C_0(T). \tag{1.5.15}$$

*Sketch of the argument .* The relevant derivations are the same (and in fact a lot less involved) with the ones in the non-linear case §1.4 (see Proposition 1.4.5). There is a slight difference in the very last argument before closing the estimates, which we present separately here. For example, following §1.4.4, we derive

$$\begin{aligned}&\frac{1}{2}\partial_t\|\eta\|^2_{L^2_\alpha}+\alpha\sigma\|\frac{\eta}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)} \\ \le\, & C(\sigma+\alpha)\|\frac{\eta}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)}+C\alpha\|\eta\|^2_{L^2_\alpha}+C\sigma\|\frac{\tilde{\xi}}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)} \\ &+C\|\frac{\tilde{\xi}}{\ell^{\alpha}}\|^2_{L^2}+C\|\frac{\tilde{\xi}_s}{\ell^{\alpha}}\|^2_{L^2}+\|\frac{F_1}{\ell^{\alpha}}\|^2_{L^2}\end{aligned} \tag{1.5.16}$$

Choosing $\alpha,\sigma$ such that

$$\frac{1}{2}\alpha\sigma>C(\sigma+\alpha)$$

and integrating in time and utilizing (1.5.8), (1.5.9) we have

$$\begin{aligned}&\frac{1}{2}\|\eta\|^2_{L^2_\alpha[t]}+\frac{\alpha\sigma}{2}\int_0^t\|\frac{\eta}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)}d\tau \\ \le\, & \frac{1}{2}\|\eta_0\|^2_{L^2_\alpha}+C\alpha\int_0^t\|\eta\|^2_{L^2_\alpha[\tau]}d\tau+C(\frac{1}{\sigma}+T)\widetilde{C}C_0(T)+\int_0^T\|\frac{F_1}{\ell^\alpha}\|^2_{L^2}d\tau\end{aligned} \tag{1.5.17}$$

The part of (1.5.14), (1.5.15) involving the zeroth order terms follows from (1.5.17) by Gronwall's inequality. □

### 1.5.3 The weak solution $\xi$: A Galerkin-type argument

Now that we have solved the first equation of (1.5.10) for $\eta$ and obtained the required energy estimates, we plug it into the second equation of the system (1.5.10) and solve for $\xi$ via a modified Galerkin method. We initially seek a weak solution

$$\xi\in L^\infty(0,T;L^2_\alpha)\cap L^2(0,T;H^1_{\alpha+1,0}) \qquad \ell^2\xi_t\in L^2(0,T;H^{-1}_{\alpha+1}) \tag{1.5.18}$$

satisfying

$$\begin{aligned}\int_0^T\big(\xi_t,v\big)_{L^2_\alpha}dt=\int_0^T\Big[&\big((\frac{\psi_{ss}}{\psi}+(n-1)\frac{\psi_s^2}{\psi^2})f\cdot(\eta+\xi),v\big)_{L^2_\alpha}+\big(\frac{\psi_s}{\psi}f\xi_s,v\big)_{L^2_\alpha} \\ &-\big(g_s\xi_s,v\big)_{L^2_\alpha}-\big(g\xi_s,v_s\big)_{L^2_\alpha}+2\alpha\big(g\xi_s,v\frac{\ell_s}{\ell}\big)_{L^2_\alpha} \\ &+\big(\frac{\psi_s}{\psi}f\eta_s,v\big)_{L^2_\alpha}+\big(F_2,v\big)_{L^2_\alpha}\Big]dt, \qquad \xi\Big|_{t=0}=\xi_0\end{aligned} \tag{1.5.19}$$

for all

$$v \in L^\infty(0,T;H^1_{\alpha,0}(s)) \cap L^2(0,T;H^1_{\alpha+1}(s)), \tag{1.5.20}$$

where by $\big(\cdot,\cdot\big)_{L^2_\alpha}$ we denote the inner product in $L^2_\alpha$

$$\big(v_1,v_2\big)_{L^2_\alpha} := \int \frac{v_1 v_2}{\ell^{2\alpha}} ds. \tag{1.5.21}$$

and by $H^1_{\alpha,0}$ the closure of compactly supported functions in $H^1_\alpha(0,B)$; $H^{-1}_{\alpha+1}$ being the dual of $H^1_{\alpha+1,0}$. In view of the regularity (1.5.18), $\xi$ is actually continuous in time and hence the initial condition in (1.5.19) makes sense.

Let $\{u_k(x)\}_{k=1}^\infty$ be an orthonormal basis of $L^2(0,B)$, which is also a basis of $H^1_0\big((0,B)\big)$; consisting of smooth, bounded functions. Then for each $t \in [0,T]$ (abusing slightly the notation of the endpoints of integration)

$$w_k(s,t) := \ell^\alpha u_k(s) \qquad\qquad k = 1,2,\dots \tag{1.5.22}$$

is an orthonormal basis of $L^2_\alpha$ and a basis of $H^1_{\alpha,0}$. We note that

$$\begin{aligned}\int_0^T\int_0^B \frac{1}{\ell^2} ds dt &\overset{(1.3.28)}{\leq} C\int_0^T\int_0^B \frac{1}{s^2} ds \leq C\int_0^T \frac{1}{s(0,t)} ds \\ &\overset{(1.3.18)}{\leq} C\int_0^T \frac{1}{\sqrt{t}} dt \leq C\sqrt{T} < +\infty,\end{aligned} \tag{1.5.23}$$

from which it follows that the set

$$\operatorname{span}\big\{d_k(t) w_k(s,t) \;\big|\; t\in[0,T],\; k=1,2\dots\big\}, \tag{1.5.24}$$

$d_k(t)$ smooth, is also dense in $L^2(0,T;H^1_{\alpha+1,0}(s))$. Similarly to (1.5.23), by definition (1.5.22) and (1.3.27), we verify the asymptotics

$$\begin{aligned}\int \frac{w_{k_1} w_{k_2}}{s^2 \ell^{2\alpha}} ds = O(\frac{1}{\sqrt{t}}) \qquad & \int \frac{\partial_s w_{k_1} w_{k_2}}{s \ell^{2\alpha}} ds = O(\frac{1}{\sqrt{t}}) \\ \int \frac{\partial_s w_{k_1} \partial_s w_{k_2}}{\ell^{2\alpha}} ds = O(\frac{1}{\sqrt{t}}) \qquad & \int \frac{\partial_t w_{k_1} w_{k_2}}{\ell^{2\alpha}} ds = O(\frac{1}{\sqrt{t}}),\end{aligned} \tag{1.5.25}$$

without assuming of course any uniformity in the RHSs with respect to the indices $k_1, k_2 \in \{1,2,\dots\}$.

Given $\nu \in \{1,2,\dots\}$, we construct Galerkin approximations of the solution of (1.5.19), which lie in the span of the first $\nu$ basis elements:

$$\xi^\nu := \sum_{k=1}^{\nu} a_k(t) w_k \qquad\qquad a_k(0) := \int \frac{\xi_0 w_k(x,0)}{x^{2\alpha}} dx \tag{1.5.26}$$

solving

$$\begin{aligned}\big(\xi_t^\nu, w_k\big)_{L^2_\alpha} &= \big((\frac{\psi_{ss}}{\psi} + (n-1)\frac{\psi_s^2}{\psi^2})f \cdot (\eta + \xi^\nu), w_k\big)_{L^2_\alpha} + \big(\frac{\psi_s}{\psi} f \xi_s^\nu, w_k\big)_{L^2_\alpha} \\ &- \big(g_s \xi_s^\nu, w_k\big)_{L^2_\alpha} - \big(g \xi_s^\nu, \partial_s w_k\big)_{L^2_\alpha} + 2\alpha\big(g\xi_s^\nu, w_k \frac{\ell_s}{\ell}\big)_{L^2_\alpha} \\ &+ \big(\frac{\psi_s}{\psi} f \eta_s, w_k\big)_{L^2_\alpha} + \big(F_2, w_k\big)_{L^2_\alpha},\end{aligned} \tag{1.5.27}$$

for $t \in [0, T]$ and every $k = 1, \dots, \nu$.

**Proposition 1.5.3** (Galerkin approximations)**.** *For each $\nu = 1, 2, \dots$ there exists a unique function $\xi^\nu$ of the form (1.5.26) satisfying (1.5.27).*

*Proof.* Employing (1.5.25) we see that

$$\big(\xi_t^\nu, w_k\big)_{L^2_\alpha} = a_k'(t) + \sum_{j=1}^{\nu} a_j(t) O(\frac{1}{\sqrt{t}})$$

and also utilizing (1.3.15), (1.5.2)

$$\big((\frac{\psi_{ss}}{\psi} + (n-1)\frac{\psi_s^2}{\psi^2})f \cdot \xi^\nu, w_k\big)_{L^2_\alpha} + \big(\frac{\psi_s}{\psi} f \xi_s^\nu, w_k\big)_{L^2_\alpha} = \sum_{j=1}^{\nu} a_j(t) O(\frac{1}{\sqrt{t}}).$$

Further, by our assumption on $g$ (1.5.1) and (1.5.25) it holds

$$\begin{aligned}-\big(g_s \xi_s^\nu, w_k\big)_{L^2_\alpha} - \big(g \xi_s^\nu, \partial_s w_k\big)_{L^2_\alpha} + 2\alpha\big(g\xi_s^\nu, w_k \frac{\ell_s}{\ell}\big)_{L^2_\alpha} & \qquad (\ell_s = O(1)\ (1.3.27)) \\ = \sum_{j=1}^{\nu} a_k(t) O(1) + \sum_{j=1}^{\nu} a_k(t) O(\frac{1}{\sqrt{t}}), &\end{aligned}$$

Lastly, setting

$$\begin{aligned}d_k(t) &:= \big((\frac{\psi_{ss}}{\psi} + (n-1)\frac{\psi_s^2}{\psi^2})f \cdot \eta, w_k\big)_{L^2_\alpha} + \big(\frac{\psi_s}{\psi} f \eta_s, w_k\big)_{L^2_\alpha} + \big(F_2, w_k\big)_{L^2_\alpha} \\ &\leq C\|\frac{\eta}{s\ell^\alpha}\|^2_{L^2} + \int \frac{1}{s^2} ds + C\|\frac{\eta_s}{\ell^\alpha}\|^2_{L^2} + \int \frac{1}{s^2} ds + C\|\frac{F_2}{\ell^{\alpha-1}}\|^2_{L^2} + \int \frac{1}{\ell^2} ds\end{aligned}$$

we observe that (1.5.27) reduces to a linear first order ODE system of the form

$$a_k'(t) = \sum_{j=1}^{\nu} a_k(t) O(\frac{1}{\sqrt{t}}) + \sum_{j=1}^{\nu} a_k(t) O(1) + d_k(t) \qquad k = 1, \dots, \nu$$

having coefficients which are singular at $t = 0$, but luckily they are all integrable on $[0, T]$. This implies local existence and uniqueness of the system and hence of $\xi^\nu$ at each step $\nu \in \{1, 2, \dots\}$. □

**Proposition 1.5.4** (Energy estimates)**.** *For $\alpha, \sigma, \widetilde{C}$ appropriately large and $T > 0$ small the following estimates hold:*

$$\|\xi^\nu\|^2_{L^\infty(0,T;L^2_\alpha(s))} + \|\xi^\nu_s\|^2_{L^2(0,T;H^1_{\alpha+1,0})} \leq \frac{\widetilde{C}}{10} C_0(T), \tag{1.5.28}$$

$$\int_0^T \|\frac{\xi^\nu}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)} dt \leq \frac{C}{\alpha}\frac{\widetilde{C}}{\sigma^2} C_0(T) \tag{1.5.29}$$

*and*

$$\Big(\int_0^T (\xi^\nu_t, v)_{L^2_\alpha} dt\Big)^2 \leq \frac{\widetilde{C}}{10} C_0(T) \int_0^T \|v\|^2_{H^1_{\alpha+1},0} dt, \tag{1.5.30}$$

*for every $\nu = 1, 2, \ldots$, $v = \sum_{k=1}^\nu d_k(t) w_k$.*

*Proof.* Multiplying the equation (1.5.27) with $a_k(t)$ and summing over $k = 1, \ldots, \nu$, we can then follow the argument in §1.5.2 to prove (1.5.28),(1.5.29). Next, we readily compute using the equation (1.5.27):

$$\begin{aligned}(\xi^\nu_t, v)_{L^2_\alpha} \leq C\big(\|\frac{v}{\ell^{\alpha+1}}\|_{L^2} + \|\frac{v_s}{\ell^\alpha}\|_{L^2}\big)\Big[&\|\frac{\eta}{s^2\ell^{\alpha-1}}\|_{L^2} + \|\frac{\xi^\nu}{s^2\ell^{\alpha-1}}\|_{L^2} + \|\frac{\xi^\nu_s}{s\ell^{\alpha-1}}\|_{L^2} \\ &+ \alpha^2\|\frac{\xi^\nu_s}{\ell^\alpha}\|_{L^2} + \|\frac{\eta_s}{s\ell^{\alpha-1}}\|_{L^2} + \|\frac{F_2}{\ell^{\alpha-1}}\|_{L^2}\Big]\end{aligned}$$

Employing the comparison (1.3.28) and (1.5.14), (1.5.15) along with the already derived (1.5.28), (1.5.29) we arrive at (1.5.30). □

The estimates in Proposition 1.5.4 suffice to pass to a subsequence (applying a diagonal argument due to (1.5.30)), yielding in the limit a weak solution $\xi$ (1.5.18),(1.5.19) verifying the energy bounds

$$\|\xi\|^2_{L^\infty(0,T;L^2_\alpha(s))} + \int_0^T \|\frac{\xi_s}{\ell^\alpha}\|^2_{L^2} dt \leq \frac{\widetilde{C}}{10} C_0(T) \tag{1.5.31}$$

and

$$\int_0^T \|\frac{\xi}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)} dt \leq \frac{C}{\alpha}\frac{\widetilde{C}}{\sigma^2} C_0(T). \tag{1.5.32}$$

Uniqueness follows by the linearity of (1.5.19), since the difference of any two weak solutions satisfies the corresponding estimates with zero initial data and zero inhomogeneous terms.

### 1.5.4 Improved regularity and energy estimates for $\xi$

We now show that $\xi$ is in fact a strong solution of (1.5.10). Let $0 < t_0 < T$ be a fixed positive time. Looking at the second equation of (1.5.10) for $t \in [t_0, T]$, we observe that the coefficients involving $\psi$ and its derivatives are smooth and bounded, while $f, g \in L^\infty(0,T;H^1)$ (1.5.1). Moreover, from §1.5.2 we have $\eta \in L^\infty(0,T;H^1)$ and by assumption $F_i \in L^2(0,T;L^2)$, $i = 1,2$. Hence, by standard theory of parabolic equations the weak solution $\xi$ (1.5.18) of (1.5.10) that we established in §1.5.3, having "initial data" $\xi(s,t_0) \in H^1$ (for a.e. $0 < t_0 < T$), attains interior regularity

$$\xi \in L^\infty(t_0,T;H_0^1) \cap L^2(t_0,T;H^2) \qquad\qquad \xi_t \in L^2(t_0,T;L^2)$$

Since $t_0 \in (0,T)$ is arbitrary, we can improve the regularity of the preceding solution

$$\xi \in L^\infty(0,T;H^1_{\alpha,0}) \cap L^2(0,T;H^2_{\alpha+1}) \qquad\qquad \xi_t \in L^2(0,T;L^2_{\alpha-1}) \tag{1.5.33}$$

by straightforwardly using the second equation in (1.5.10) to derive the desired energy estimates for the higher order terms. Recall that for fixed $t > 0$, the weight $\ell^2$ is bounded above and below (Definition 1.3.3). Thus, it makes sense to (time) differentiate the $L^2_{\alpha-1}$ of $\xi$ and plug in directly the equation (1.5.10) to obtain (as in the non-linear case for $d\xi_s^{m+1}$ §1.4.5):

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\xi_s\|^2_{L^2_{\alpha-1}} + \alpha\sigma\|\frac{\xi_s}{\ell^\alpha}\|^2_{L^2(0,\delta)} + \frac{1}{4}\|\frac{\xi_{ss}}{\ell^{\alpha-1}}\|^2_{L^2} \qquad (1.5.34)\\
&\leq C(\alpha^2+\sigma)\|\frac{\xi_s}{\ell^\alpha}\|^2_{L^2(0,\delta)} + C\alpha^2\|\xi_s\|^2_{L^2_{\alpha-1}}\\
&\quad + C\sigma^2\big(\|\frac{\eta}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)} + \|\frac{\xi}{\ell^{\alpha+1}}\|^2_{L^2(0,\delta)}\big) + C\sigma\big(\|\frac{\eta_s}{\ell^\alpha}\|^2_{L^2(0,\delta)} + \|\frac{\xi_s}{\ell^\alpha}\|^2_{L^2(0,\delta)}\big)\\
&\quad + C\big(\|\eta\|^2_{L^2_\alpha} + \|\xi\|^2_{L^2_\alpha} + \|\eta_s\|^2_{L^2_{\alpha-1}}\big) + C\|\frac{F_2}{\ell^{\alpha-1}}\|^2_{L^2}
\end{aligned}$$

Let $\alpha, \sigma$ large such that $\frac{1}{2}\alpha\sigma > C(\alpha^2+\sigma)$. Invoking (1.5.8), (1.5.9), (1.5.14), (1.5.15), (1.5.31), (1.5.32) upon integrating on $[0,T]$ we deduce

$$\begin{aligned}
&\frac{1}{2}\|\xi_s\|^2_{L^2_{\alpha-1}[t]} + \frac{1}{2}(\alpha-1)\sigma\int_0^t\|\frac{\xi_s}{\ell^\alpha}\|^2_{L^2}d\tau + \frac{1}{4}\int_0^t\|\frac{\xi_{ss}}{\ell^{\alpha-1}}\|^2_{L^2}d\tau \qquad (1.5.35)\\
&\leq \frac{1}{2}\|\partial_x\xi_0\|^2_{L^2_{\alpha-1}} + C\alpha^2\int_0^t\|\xi_s\|^2_{L^2_{\alpha-1}[\tau]}d\tau + C(\frac{1}{\alpha}+T)\widetilde{C}C_0(T) + C\int_0^T\|\frac{F_2}{\ell^{\alpha-1}}\|^2_{L^2}d\tau
\end{aligned}$$

Employing Gronwall's inequality, $t \in [0,T]$, we finally conclude ($T > 0$ small, $\alpha$ large)

$$\|\xi_s\|^2_{L^\infty(0,T;L^2_{\alpha-1}(s))} + \int_0^T\|\frac{\xi_{ss}}{\ell^{\alpha-1}}\|^2_{L^2}d\tau \leq \frac{\widetilde{C}}{10}C_0(T) \tag{1.5.36}$$

and

$$\int_0^T\|\frac{\xi_s}{\ell^\alpha}\|^2_{L^2(0,\delta)}d\tau \leq \frac{\widetilde{C}}{\sigma}C_0(T) \tag{1.5.37}$$

This completes the proof of the *claim* in the outline of the plan §1.5.1 and consequently of Theorem 1.5.2 and the realization of the linear step in the iteration of the non-linear PDE (1.4.2).

# Chapter 2

# On the backward stability of the Schwarzschild black hole singularity

## 2.1 Overview

It is well-known (cf. Birkhoff's theorem [18]) that the only spherically symmetric solution $(\mathcal{M}^{1+3}, g)$ to the Einstein vacuum equations (EVE)

$$\mathrm{Ric}_{ab}(g) = 0, \tag{2.1.1}$$

is the celebrated Schwarzschild spacetime. It was in fact the first non-trivial solution to the Einstein field equations to be discovered [18]. In Kruskal (null) $u, v$ coordinates the maximally extended metric reads

$${}^{S}g = -\Omega^2 dudv + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{2.1.2}$$

where $\Omega^2 = \frac{32M^3}{r} e^{-\frac{r}{2M}}$, $M > 0$, and the radius function $r$ is given implicitly by

$$uv = (1 - \frac{r}{2M})e^{\frac{r}{2M}}. \tag{2.1.3}$$

Here the underlying manifold ${}^{S}\mathcal{M}^{1+3}$ is endowed with the differential structure of $\mathcal{U} \times \mathbb{S}^2$, where $\mathcal{U}$ is the open subset $\{uv < 1\}$ in the $uv$ plane; see Figure 2.1. The spacetime has an essential curvature singularity at $r = 0$, (the future component of) which is contained in the interior of the black hole region, the quadrant $u > 0, v > 0$. In fact, a short computation shows that the Gauss curvature of the $uv$-plane equals

$${}^{S}\mathrm{K} = \frac{2M}{r^3} \tag{2.1.4}$$

and hence the manifold is $C^2$ inextendible past $r = 0$. An interesting feature of this singularity is its spacelike character, that is, it can be viewed as a spacelike hypersurface.

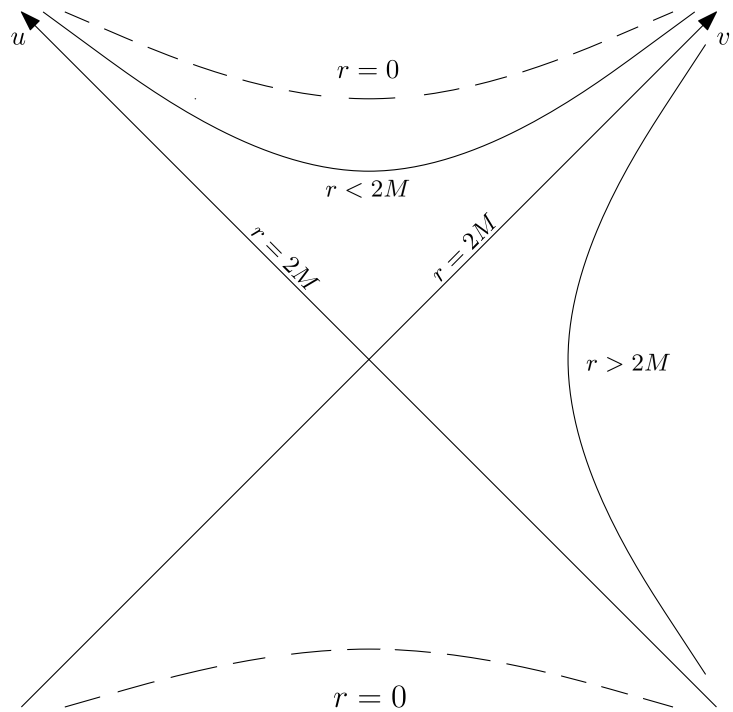


Figure 2.1: The Kruskal plane.

Yet another interesting feature of the Schwarzschild singularity is its unstable nature from the evolutionary dynamical point of view. To illustrate this consider a global spacelike Cauchy hypersurface $\Sigma^3$ in Schwarzschild (Figure 2.2). An initial data set for the EVE consists of a Riemannian metric $\overline{g}$ on $\Sigma$ and a symmetric two tensor $K$ verifying the constraint equations

$$\begin{cases} \overline{\nabla}^j K_{ij} - \overline{\nabla}_i \mathrm{tr}_{\overline{g}} K = 0 \\ \overline{\mathrm{R}} - |K|^2 + (\mathrm{tr}_{\overline{g}} K)^2 = 0 \end{cases}, \tag{2.1.5}$$

where $\overline{\nabla}, \overline{\mathrm{R}}$ are the covariant derivative and scalar curvature intrinsic to $\overline{g}$.

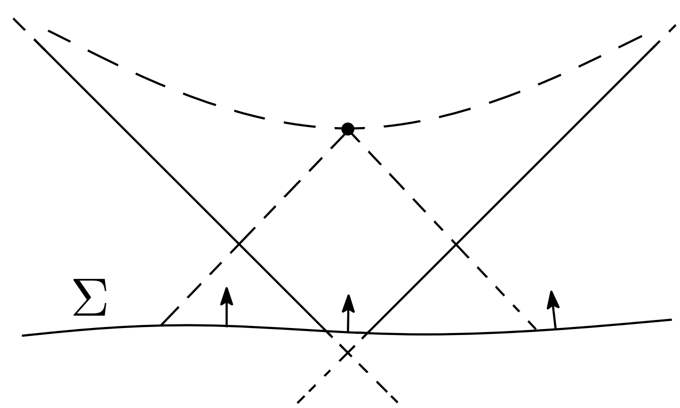


Figure 2.2:

The instability of the Schwarzschild singularity (w.r.t. the forward Cauchy problem) can already be seen by examining the maximal developments of initial data sets on $\Sigma$ arising from the celebrated Kerr [17] (explicit) 2-parameter $\mathcal{K}(a, M)$ family of solutions – of which Schwarzschild is a subfamily ($a = 0$). For $a \neq 0$ the singularity completely disappears and the corresponding (maximal) developments extend smoothly up to (and including) the Cauchy horizons. Moreover, taking $|a| \ll 1$, the 'difference' of the corresponding initial data sets from the Schwarzschild one (with the same $M > 0$), measured in standard Sobolev norms,[1] can be made arbitrarily small.

In fact, the Schwarzschild singularity is conjecturally unstable under *generic* perturbations on $\Sigma$. According to a scenario proposed by Belinskiĭ-Khalatnikov-Lifshitz [7] originally formulated for cosmological singularities, in general, one should expect solutions to exhibit oscillatory behaviour towards the singularity. To our knowledge such behaviour has been rigorously studied only in the spatially homogeneous case for the Euler-Einstein system with Bianchi IX symmetry by Ringström [46]. Nonetheless, the heuristic work of [7] has received a lot of attention over the years, see [45, 30] and the references therein (and [23] for related numerics). On the other hand, there is a growing expectation that, at least in a neighbourhood of subextremal Kerr, the dominant scenario inside the black hole is the formation of Cauchy horizons and (weak) null singularities. This has been supported by rigorous studies on spherically symmetric charged matter models, see works by Poisson and Israel [43], Ori [42] and recently by Dafermos [17].

However, it is not inferred from the existing literature whether the non-oscillatory type of singularity observed in Scwarzschild is an isolated phenomenon for the EVE in some neighbourhood of the Schwarzschild initial data on $\Sigma$ or part of a larger family. A priori it is not clear what to expect, since one might argue that such a special singularity is a mathematical artefact due to spherical symmetry. Therefore, we pose the following question:

*Is there a class of non-spherically symmetric Einstein vacuum spacetimes which develop a first singularity of Schwarzschild type?*

The goal of the present paper is to answer the preceding question in the affirmative. A Schwarzschild type singularity here has the meaning of a first singularity in the vacuum development which has the same geometric blow up profile with Schwarzschild and which can be seen by a foliation of uniformly spacelike hypersurfaces; hence, not contained in a Cauchy horizon. We confine the question to the formation of one singular sphere in the vacuum development in the same manner as in Schwarzschild, where each point on the sphere can be understood as a distinct ideal singular point of the spacetime in the language of TIPs [24]. Ideally one would like to study the forward problem and identify initial data for the EVE on $\Sigma$ (Figure 2.2) that lead to such singularities. Although this is a very interesting problem, we find it far beyond reach at the moment. Instead we study the existence problem backwards-in-time.

More precisely, we adopt the following plan: Let $\Sigma_0^3$ be a spacelike hypersurface in Schwarzschild,

[1] The difference can be defined, for example, component wise for the two pairs of 2-tensors with respect to a common coordinate system and measured in $W^{s,p}$ Sobolev spaces used in the literature [12].

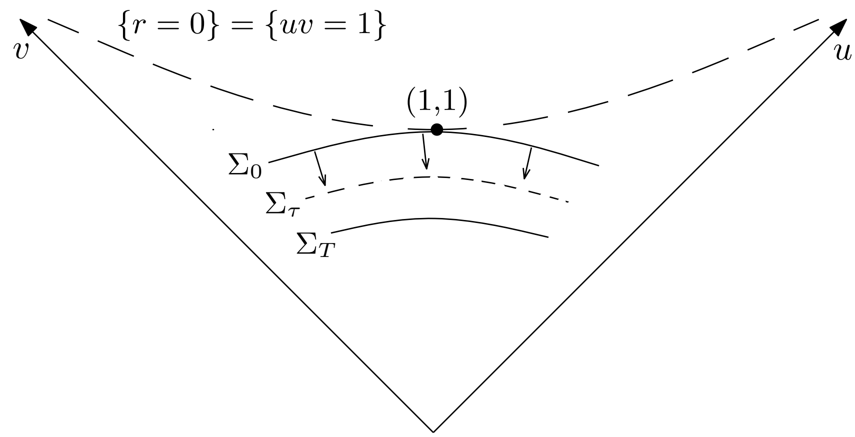


Figure 2.3: The black hole region in Kruskal's extension.

tangent[2] at a single sphere of the singular hyperbola $r = 0$ inside the black hole; Figure 2.3. We assume, without loss of generality,[3] that the tangent sphere is $(u = 1, v = 1)$ in Kruskal coordinates (2.1.2). Consider now initial data sets $(\overline{g}, K)$ on $\Sigma_0$ for the EVE (2.1.1), which have the same singular behaviour to leading order at $(u = 1, v = 1)$ with the induced Schwarzschild initial data set $({}^S\overline{g}, {}^SK)$ on $\Sigma_0$ and solve the EVE backwards, as depicted in the 2-dim Figure 2.3.

Realizing the above plan we thus prove the existence of a class of non-spherically symmetric vacuum spacetimes for which (1) the leading asymptotics of the blow up of curvature and in general of all the geometric quantities (metric, second fundamental form etc.) coincide with their Schwarzschild counterparts, as one approaches the singularity, and (2) the singularity is realized as the limit of uniformly spacelike hypersurfaces, which in the forward direction "pinch off" in finite time at one sphere. Conversely, we visualise the backward evolution of $(\Sigma_0, \overline{g}, K)$ in the following manner: At 'time' $\tau = 0$ the initial slice $\Sigma_0$ is a two ended spacelike (3-dim) hypersurface with a sphere singularity at $(u = 1, v = 1)$. Once $\Sigma_0$ evolves through (2.1.1), it becomes instantaneously a regular spacelike hypersurface $\Sigma_\tau, \tau > 0$ and the singular *pinch* opens up; Figure 2.4.

The main difficulty to overcome in the backward local existence problem is the singularity on $\Sigma_0$, which of course renders it beyond the scope of the classical local existence theorem for the Einstein equations [11], even its latest state of the art improvement by Klainerman-Rodnianski-Szeftel [32], which requires at the very least the curvature of the initial hypersurface to be in $L^2$. For the Schwarzschild initial data set $({}^S\overline{g}, {}^SK)$ on $\Sigma_0$, and hence for perturbed initial data sets $(\overline{g}, K)$ with the same leading order geometry at $(u = 1, v = 1)$, it is not hard to check

[2]The tangency here should be understood with respect to the differential structure of the Kruskal maximal extension induced by the standard $u, v, \theta, \phi$ coordinates (2.1.2).

[3]Recall that the vector field tangent to the $r =$ const. hypersurfaces (Figure 2.1) is Killing and we may hence utilize it to shift $\Sigma_0$ and $(u = 1, v = 1)$ to whichever point on $\{uv = 1\}$ we wish; Figure 2.3.

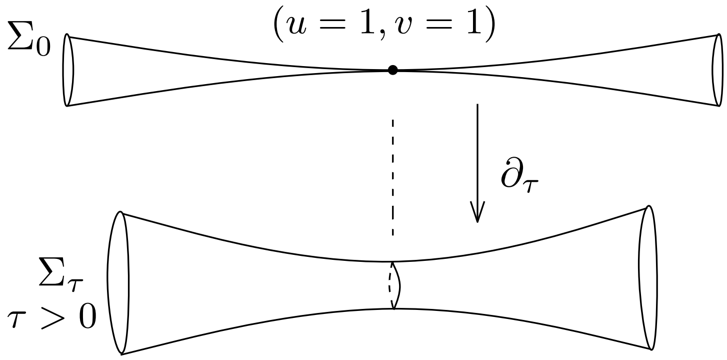


Figure 2.4:

(§2.3) that the initial curvature is at the singular level

$$\overline{\mathrm{R}} \notin L^p(\Sigma_0), \qquad \nabla K \notin L^p(\Sigma_0) \qquad\qquad p \geq \frac{5}{4}. \tag{2.1.6}$$

Thus, we must rely heavily on the background Schwarzschild geometry to control the putative backward evolution. A very useful fact for analysis is the opening up (smoothing out) of the singularity (Figure 2.4) in the backward direction.

To our knowledge, general local existence results for the EVE (2.1.1) with singular initial curvature not in $L^2$ have been achieved only fairly recently by Luk-Rodnianski [35, 36] and Luk [34] for the characteristic initial value problem, where they consider delta curvature singularities and weak null singularities respectively. However, their context is much different from ours and the results do not seem applicable to singularities of Schwarzschild type.

We proceed now to formulate a first version of our main results; for more precise statements, in terms of weighted Sobolev spaces, see Theorems 2.4.6, 2.4.8, 2.6.6.

**Theorem 2.1.1.** *There exists $\alpha > 0$ sufficiently large, such that for every triplet $(\Sigma_0, \overline{g}, K)$ verifying:*
*$(i)$ the constraints (2.1.5),*
*$(ii)$ $\overline{g} = {}^S\overline{g} + r^\alpha \mathcal{O}$, $K = {}^S K + r^{\alpha - \frac{3}{2}} u$, where $\mathcal{O}, u$ are 2-tensors on $\Sigma_0$ bounded in $H^4, H^3$ respectively,*
*$(iii)$ $\|\overline{g} - {}^S\overline{g}\|_{L^\infty(\Sigma_0)} \ll 1$,*
*there exists a $H^4$ local solution $g$ to the Einstein vacuum equations (2.1.1) with initial data $(\overline{g}, K)$, unique up to isometry, in the backward region to $\Sigma_0$, foliated by $\{\Sigma_\tau\}_{\tau \in [0,T]}$ (Figure 2.3); the time of existence $T > 0$ depends continuously on the norms of $\mathcal{O}, u$ and the exponent $\alpha > 0$.*

The fact that non-trivial initial data sets in compliance with Theorem 2.1.1 exist is not at all obvious nor standard. We need to show essentially that for any large parameter $\alpha > 0$, there exist non-spherically symmetric solutions to the constraint equations (2.1.5), having the asymptotics $(ii)$. We construct such solutions using the *conformal method*, which we set up in

§2.6.

**Theorem 2.1.2.** *Let $\alpha > 0$ be sufficiently large, consistent with Theorem 2.1.1. Then for every choice of the transverse, traceless part of the second fundamental form and the mean curvature on $\Sigma_0$, compatible with the assumptions in Theorem 2.1.1, and verifying a reflection symmetry condition, there exists a solution to the constraints (2.1.5) localized near the singular sphere and verifying the asymptotics (ii) above.*

Let us emphasize the fact that the above spacetimes are very special in that they agree with Schwarzschild at the singularity to a high (but finite) order – this is captured by the large exponent $\alpha > 0$ in Theorem 2.1.1 – and therefore are *non-generic*. The need to choose $\alpha$ large may be seen however natural to some extend in view of the instability of the Schwarzschild singularity; from the point of view of the forwards-in-time problem. Indeed, the stable perturbations of the Schwarzschild singularity must form a strict subclass of all perturbed vacuum developments.

We note here that the restrictions imposed on the 'free' data for the constraints in Theorem 2.1.2 go beyond the largeness of the parameter $\alpha$. The sole purpose of this is to overcome some difficulties we encounter particularly for the constraint equations. We discuss this matter further in Section §2.6.6.

### 2.1.1 Method of proof and outline

The largest part of the paper is concerned with the evolutionary part of the problem, i.e., proving Theorem 2.1.1. Due to the singular nature of backward existence problem described above, Figure 2.3, the choice of framework must be carefully considered. The standard wave coordinates approach [11] does not seem to be feasible in our situation; one expects that coordinates would be highly degenerate at the singularity. Also, the widely used CMC gauge condition is not applicable, since the mean curvature of the initial hypersurface $\Sigma_0$ blows up (§2.3). Instead, we find it more suitable to use orthonormal frames and rewrite the EVE one order higher as a quasilinear Yang-Mills hyperbolic system of equations [40, 32], under a Lorenz gauge condition,[4] for the corresponding connection 1-forms. We recall briefly this framework in §2.2.

However, even after expressing the EVE in the above framework, the singular level of initial configurations do not permit a direct energy estimate approach. In addition to (2.1.6), one can see (§2.3) that neither is the second fundamental form in $L^2$

$$K \notin L^2(\Sigma_0). \tag{2.1.7}$$

Note that the latter is at the level of one derivative in the metric. Hence, near the singularity the perturbed spacetimes we wish to construct do not even make sense as weak solutions of

[4] The analogue of a wave gauge for orthonormal frames.

the EVE (2.1.1). Therefore, it is crucial that we use the background Schwarzschild spacetime to recast the evolution equations in a new form having more regular initial data. We do this in §2.4 by considering a new system of equations for the 'difference' between the putative perturbed spacetime and Schwarzschild. The resulting equations have now regular initial data and they are eligible for an energy method, but there is a price to pay. The coefficients of the new system will depend on the Schwarzschild geometry and will necessarily be highly singular at $r = 0$. We compute in §2.3 the precise blow up orders of the Schwarzschild connection coefficients, curvature etc. Nevertheless, the issue of evolving singular initial data has become the more tractable problem of finding appropriate weighted solution spaces for the final singular equations.

In §2.4.2 we introduce the weighted Sobolev spaces which yield the desired flexibility in proving energy estimates. The right weights are given naturally by the singularities in the coefficients of the resulting equations, namely, powers of the Schwarzschild radius function $r$ with a certain analogy corresponding to the order of each term. After stating the general local existence theorems in §2.4.3 and a more precise version of Theorem 2.1.1, we proceed to its proof via a contraction mapping argument which occupies Section 2.5. Therein we derive the main weighted energy estimates by exploiting the asymptotic analysis at $r = 0$ of the Schwarzschild components (§2.3). It is necessary in our result that the power of $r$, $\alpha > 0$, in the weighted norms is sufficiently large; cf. assumption (*ii*) in Theorem 2.1.1. In the estimating process certain *critical* terms are inevitably generated, because of the singularities in the coefficients of the system we are working with; these terms are critical in that they appear with larger weights than the ones in the energy we are trying to control and thus prevent the estimates from closing. The exponent $\alpha > 0$ is then picked sufficiently large such that these critical terms have an overall favourable sign; this allows us to drop the critical terms and close the estimates.

The largeness of $\alpha$ forces the perturbed spacetime to agree asymptotically with Schwarzschild to a high order at the singularity. Although the latter may seem restrictive, it is quite surprising to us that there even exists a suitable choice of $\alpha$ which makes the argument work in the first place. A closer inspection of our method reveals that it is very sensitive with respect to certain asymptotics of the coefficients in the equations that happen to be just borderline to allow an energy-based argument to close. The most important of these are the blow up order of the sectional curvature (2.1.4) and the rate of growth of the Schwarzschild radius function $r$ backwards in time. The latter corresponds to the 'opening up' rate of the *neck pinch* of the singular initial hypersurface $\Sigma_0$, Figure 2.4. In this sense the Schwarzschild singularity is exactly at the threshold that our energy-based method can tolerate.

In the last section, §2.6, we study the constraint equations (2.1.5) in a perturbative manner about the Schwarzschild singular initial data set $({}^S\overline{g}, {}^SK)$ on $\Sigma_0$. We prove Theorem 2.1.2 by employing the inverse function theorem. Following the conformal approach, we prescribe conformal data on $\Sigma_0$ which asymptote to the corresponding Schwarzschild singular data at a high order. Then we prove that the linearized conformal constraint map (about Schwarzschild) is Fredholm in suitable weighted Sobolev spaces,[5] capturing the asymptotics needed for Theorem 2.1.1 to be applied, see Proposition 2.6.5 which we prove in §2.6.2. In the case where $\Sigma_0$ is localized in a neighborhood of its singularity, the weighted elliptic estimates we derive can be improved to yield that the conformal constraint map is actually an isomorphism. It is worth noting that the solutions to the constraints that we produce have singular mean curvature.

### 2.1.2 Final Comments; Possible applications

The understanding of the question of stability of singularities in Einstein's equations and the behaviour of solutions near them is of great significance in the field. However, in general very little is known. In terms of rigorous results, substantial progress has been made in spherical symmetry in the presence of matter [14] [13] [46] [17]. Moreover, certain matter models enjoy the presence of a monotonic quantity, which has been employed to study the stability of singularity formation in the general non-symmetric regime, cf. recent work of Rodnianski-Speck [48] on the FLRW big bang singularity. This is in contrast with the vacuum case of black hole interior and the unstable nature of the Schwarzschild singularity. We hope that the method developed herein can be employed to produce classes of examples of other singular solutions to the Einstein field equations, which until now are only known to exist under very restrictive symmetry assumptions and for which the general stability question may be out of reach.

The idea of constructing singular spacetimes by prescribing a specific singular behaviour and solving for a spacetime 'starting from the singularity' is not new. There exists an extensive literature regarding the construction of cosmological spacetimes exhibiting Kasner type singularities at each point of their 'big bang' hypersurface[6] using Fuchsian techniques [31, 45, 30]. However, the results in this category rely on the undesirable assumption of analyticity [1] and or on various symmetry assumptions, see relevant work on Gowdy spacetimes [44, 47]. Yet, we believe that the usual Fuchsian algorithm cannot be applied to Schwarzschild type singularities due to their more singular nature.[7]

After our treatment of singular initial data containing a single sphere of $\{uv = 1\}$, a reasonable next step would be to study whether the construction of non-spherically symmetric vacuum spacetimes containing an arc of the singular hyperbola (Figure 2.3) is possible or even the whole singularity $r = 0$. Certainly this is a more restrictive question and at first glance not so obvious how to formulate it as a backward initial value problem problem for the EVE. However, we hope that the method developed herein could help approach this direction.

Lastly, one could try to perform a global instead of a local construction by considering a

[5] We note that the spaces we use for the constraint equations differ from those we use for the evolutionary part of the problem.

[6] At each point of the usual singular spacelike hypersurface the spacetime metric approaches asymptotically the metric of a Kasner spacetime, with the Kasner parameters generally varying from point to point, what is called AVTD behaviour [30].

[7] The reason should be understood in an effort to reduce the Einstein equations to Fuchsian type equations for a Schwarzschild type singularity. In this case the singularities in the coefficients of the reduced evolution equations would be stronger than the ones encountered in the literature.

Cauchy hypersurface $\Sigma_0$ extending to spacelike infinity. We expect this follows readily from the work here, but we do not pursue it further. Perhaps a gluing construction could also be achieved.

## 2.2 The Einstein equations as a quasilinear Yang-Mills system

The Einstein vacuum equations (2.1.1), by virtue of the second Bianchi identity, imply the vanishing of the divergence of the Riemann curvature tensor. Decomposing the latter with respect to an orthonormal frame, which satisfies a suitable gauge condition, it results to a quasilinear second order hyperbolic system of equations for the connection 1-forms corresponding to that frame, which bears resemblance to the semilinear Yang-Mills [40]. Recently this formulation of the EVE played a key role in the resolution of the bounded $L^2$ curvature conjecture [32]. In this section we express the EVE (2.1.1) in the above setting, which we are going to use to directly solve the Cauchy problem. This necessitates some technical details which are carried out in Appendix B. Also, to avoid additional computations we write all equations directly in scalar non-tensorial form.[8]
All indices below range from 0 to 3 unless otherwise stated.

### 2.2.1 Cartan formalism

Let $(\mathcal{M}^{1+3}, g)$ be a Lorenzian manifold and let $\{e_0, e_1, e_2, e_3\}$ be an orthonormal frame; $g_{ab} := m_{ab} = \text{diag}(-1, 1, 1, 1)$. Assume also that $\mathcal{M}^{1+3}$ has the differential structure of $\Sigma \times [0, T]$, where each leaf $\Sigma \times \{\tau\} =: \Sigma_\tau$ is a 3-dim spacelike hypersurface. We denote the connection 1-forms associated to the preceding frame by

$$(A_X)_{ij} := g(\nabla_X e_i, e_j) = -(A_X)_{ji}, \tag{2.2.1}$$

where $\nabla$ is the $g$-compatible connection of $\mathcal{M}^{1+3}$. Recall the definition of the Riemann curvature tensor

$$R_{\mu\nu ij} := g(\nabla_{e_\mu}\nabla_{e_\nu} e_i - \nabla_{e_\nu}\nabla_{e_\mu} e_i, e_j). \tag{2.2.2}$$

By the former definition of connection 1-forms, using $m_{ab}$ to raise and lower indices, we write

$$\nabla_{e_a} e_b = (A_a)_b{}^k e_k.$$

Hence, we have

$$\begin{aligned}\nabla_{e_\mu}\nabla_{e_\nu} e_i &= \nabla_{e_\mu}(\nabla_{e_\nu} e_i) - \nabla_{\nabla_{e_\mu} e_\nu} e_i = \nabla_{e_\mu}\big((A_\nu)_i{}^k e_k\big) - (A_\mu)_\nu{}^k (A_k)_i{}^c e_c \\ &= e_\mu (A_\nu)_i{}^k e_k + (A_\nu)_i{}^k (A_\mu)_k{}^d e_d - (A_\mu)_\nu{}^k (A_k)_i{}^c e_c\end{aligned}$$

[8]It will be clear though which are the covariant expressions; see also [32].

Therefore, we get the following expression for the components of the Riemann curvature

$$\begin{aligned} R_{\mu\nu ij} = e_\mu(A_\nu)_{ij} - e_\nu(A_\mu)_{ij} + (A_\nu)_i{}^k(A_\mu)_{kj} - (A_\mu)_i{}^k(A_\nu)_{kj} \\ - (A_\mu)_\nu{}^k(A_k)_{ij} + (A_\nu)_\mu{}^k(A_k)_{ij} \end{aligned} \tag{2.2.3}$$

or setting

$$([A_\mu, A_\nu])_{ij} = (A_\mu)_i{}^k(A_\nu)_{kj} - (A_\nu)_i{}^k(A_\mu)_{kj} \tag{2.2.4}$$

we rewrite

$$(F_{\mu\nu})_{ij} := R_{\mu\nu ij} = e_\mu(A_\nu)_{ij} - e_\nu(A_\mu)_{ij} - ([A_\mu, A_\nu])_{ij} - (A_{[\mu})_{\nu]}{}^k(A_k)_{ij}, \tag{2.2.5}$$

where by standard convention

$$(A_{[\mu})_{\nu]}{}^k(A_k)_{ij} := (A_\mu)_\nu{}^k(A_k)_{ij} - (A_\nu)_\mu{}^k(A_k)_{ij}.$$

In the same manner we compute the covariant derivative of the Riemann tensor:

$$\begin{aligned} \nabla_\sigma R_{\mu\nu ij} &= e_\sigma(F_{\mu\nu})_{ij} - (A_\sigma)_\mu{}^k(F_{k\nu})_{ij} - (A_\sigma)_\nu{}^k(F_{\mu k})_{ij} \\ &\quad - (A_\sigma)_i{}^k(F_{\mu\nu})_{kj} - (A_\sigma)_j{}^k(F_{\mu\nu})_{ik} \\ &= e_\sigma(F_{\mu\nu})_{ij} - (A_\sigma)^k{}_{[\mu}(F_{\nu]k})_{ij} - ([A_\sigma, F_{\mu\nu}])_{ij} \end{aligned} \tag{2.2.6}$$

Recall the transformation law of the above quantities under change of frames: Let $\{\tilde{e}_i\}_0^3$ be an orthonormal frame on $\mathcal{M}^{1+3}$ such that

$$\tilde{e}_a = O_a^k e_k \tag{2.2.7}$$

and let $(\tilde{A}_X)_{ij} := g(\nabla_X \tilde{e}_i, \tilde{e}_j)$ be the corresponding connection 1-forms. Then

$$(\tilde{A}_X)_{ij} = O_i^b O_j^c (A_X)_{bc} + X(O_i^b) O_j^c m_{bc}. \tag{2.2.8}$$

In addition, from (2.2.7) we have

$$\begin{aligned} \nabla_X \tilde{e}_a = X(O_a^k)e_k + O_a^k \nabla_X e_k \\ (\tilde{A}_X)_a{}^d \tilde{e}_d = X(O_a^k)e_k + O_a^k (A_X)_k{}^d e_d \end{aligned}$$

or

$$X(O_a^l) = (\tilde{A}_X)_a{}^d O_d^l - O_a^k (A_X)_k{}^l. \tag{2.2.9}$$

### 2.2.2 $\nabla \times \mathbf{Ric} = 0$

Now we proceed by assuming that the *curl* of the Ricci tensor of the metric $g$ vanishes:

$$\nabla_i R_{\nu j} - \nabla_j R_{\nu i} = 0, \tag{2.2.10}$$

where $R_{ab} := R_{\mu ab}{}^{\mu}$. A direct implication of the (contracted) second Bianchi identity is that the divergence of the Riemann curvature tensor satisfies

$$\nabla^{\mu} R_{ij\nu\mu} = \nabla_i R_{\nu j} - \nabla_j R_{\nu i} = 0. \tag{2.2.11}$$

Thus, it follows from (2.2.6) that

$$e^{\mu}(F_{\mu\nu})_{ij} - (A^{\mu})^{k}{}_{[\mu}(F_{\nu]k})_{ij} - ([A^{\mu}, F_{\mu\nu}])_{ij} = 0 \tag{2.2.12}$$

or by (2.2.5)

$$\begin{aligned} &\boxdot\, (A_{\nu})_{ij} - e^{\mu} e_{\nu}(A_{\mu})_{ij} - e^{\mu}([A_{\mu}, A_{\nu}])_{ij} - e^{\mu}\big((A_{[\mu})_{\nu]}{}^{k}(A_k)_{ij}\big) \\ &= (A^{\mu})^{k}{}_{[\mu}(F_{\nu]k})_{ij} + ([A^{\mu}, F_{\mu\nu}])_{ij}, \end{aligned} \tag{2.2.13}$$

where $\boxdot := -e_0^2 + e_1^2 + e_2^2 + e_3^2$ is the non-covariant box with respect to the frame $e_i$. Since

$$[e_{\mu}, e_{\nu}] = \nabla_{e_{\mu}} e_{\nu} - \nabla_{e_{\nu}} e_{\mu} = (A_{[\mu})_{\nu]}{}^{k} e_k,$$

(2.2.13) takes the equivalent form

$$\begin{aligned} \boxdot(A_{\nu})_{ij} - e_{\nu} e^{\mu}(A_{\mu})_{ij} = (A^{[\mu})_{\nu]}{}^{k} e_k(A_{\mu})_{ij} + e^{\mu}([A_{\mu}, A_{\nu}])_{ij} + e^{\mu}\big((A_{[\mu})_{\nu]}{}^{k}(A_k)_{ij}\big) \\ + (A^{\mu})^{k}{}_{[\mu}(F_{\nu]k})_{ij} + ([A^{\mu}, F_{\mu\nu}])_{ij}, \end{aligned} \tag{2.2.14}$$

$\nu, i, j = 0, 1, 2, 3$. We remark that (2.2.14) is an equation of scalar functions.

### 2.2.3 Choice of gauge

Note that the preceding equation is not of hyperbolic type. We convert (2.2.14) into a quasilinear hyperbolic system of equations by imposing a Lorenz gauge condition on the orthonormal frame $\{e_i\}_0^3$:[9]

$$A^2 = (\mathrm{div} A)_{ij} := \nabla^{\mu}(A_{\mu})_{ij} - (A_{\nabla_{e^{\mu}} e_{\mu}})_{ij} = e^{\mu}(A_{\mu})_{ij} - (A^{\mu})_{\mu}{}^{k}(A_k)_{ij}, \tag{2.2.15}$$

where by $A^2$ we denote some quadratic expression in the connection coefficients $(A_{\nu})_{ij}$ varying in $ij$. This a freedom one has in choosing the frame $e_i$; see Lemma B.0.12. Under (2.2.15), the

[9] A wave type gauge essentially for $e_i$. The Coulomb gauge is another alternative which is used in [32]. We do not employ it here.

equation (2.2.14) becomes the quasilinear second order

$$\begin{aligned}\square(A_\nu)_{ij} =& {(A^{[\mu})_{\nu]}}^k e_k(A_\mu)_{ij} + e^\mu([A_\mu, A_\nu])_{ij} + e^\mu\big({(A_{[\mu})_{\nu]}}^k (A_k)_{ij}\big)\\ &+ {(A^\mu)^k}_{[\mu}(F_{\nu]k})_{ij} + ([A^\mu, F_{\mu\nu}])_{ij} + e_\nu(A^2) + e_\nu\big({(A^\mu)_\mu}^k (A_k)_{ij}\big) \end{aligned} \tag{2.2.16}$$

### 2.2.4 The reduced equations; Initial data for EVE

Following (2.2.11)-(2.2.16) we actually see that the equation

$$\begin{aligned}\nabla_i \mathrm{R}_{\nu j} - \nabla_j \mathrm{R}_{\nu i} + & e_\nu\big(\mathrm{div}A - A^2\big)_{ij} \\ &=: H_{\nu ij} = (\text{LHS of (2.2.16)}) - (\text{RHS of (2.2.16)}),\end{aligned} \tag{2.2.17}$$

holds true for every Lorentzian metric $g$ and orthonormal frame $\{e_i\}_0^3$, without any additional assumptions or gauge condition. We call $H_{\nu ij} = 0$, i.e., the system (2.2.16), the reduced equations. We note that even after the gauge fixing, the reduced equations are not equivalent to the EVE (2.1.1), but only imply the vanishing of the *curl* of the Ricci tensor (2.2.10). However, one may suitably prescribe initial data for (2.2.16) such that they lead to solutions of the EVE and which are consistent with the Lorenz gauge condition (2.2.15).

Now we address the initial value problem for the reduced equations $H_{\nu ij} = 0$ aiming to the EVE. To solve the equation (2.2.16) one needs an equation relating the evolution of the orthormal frame $\{e_i\}_0^3$ to that of the connection 1-forms. Let $\partial_0, \partial_1, \partial_2, \partial_3$ be a reference frame[10] in $\Sigma \times [0, T]$ ($\partial_0$ transversal direction). We express $e_i$ in terms of $\partial_a$:

$$e_i = O_i^a \partial_a \tag{2.2.18}$$

By virtue of the diffeomorphism invariance of the EVE, we may assume that the timelike unit vector of the orthonormal frame $\{e_i\}_0^3$ of the spacetime we solve for is $e_0 = \partial_0$. Doing so we deduce

$$\partial_0(O_i^a) = \mathcal{L}_{e_0}\big(\hat{\partial}_a(e_i)\big) = \mathcal{L}_{\partial_0}(\hat{\partial}_a)e_i + \hat{\partial}_a([\partial_0, e_i]) = O_i^b \mathcal{L}_{\partial_0}(\hat{\partial}_a)\partial_b + \hat{\partial}_a([e_0, e_i]),$$

where $\mathcal{L}$ denotes the Lie derivative and $\hat{\partial}_a$ is the 1-form dual to $\partial_a$. Setting $[\partial_0, \partial_b] =: \Gamma^c_{[0b]}\partial_c$ we rewrite

$$\partial_0(O_i^a) = -O_i^b \Gamma^a_{[0b]} + {(A_{[0})_{i]}}^k O_k^a. \tag{2.2.19}$$

Now we proceed to formulate the necessary and sufficient conditions on the initial data set of the reduced equations (2.2.16), coupled to (2.2.19), such that the corresponding solution yields a solution to the EVE. The following proposition is proved in §B.0.5.

[10]Not orthonormal or coordinates, simply a basis frame.

**Proposition 2.2.1.** *Let $(A_\nu)_{ij}, O^a_i$ be a solution of (2.2.16),(2.2.19), arising from initial configurations subject to*

$$(A_\nu)_{ij}(\tau=0) = -(A_\nu)_{ji}(\tau=0) \qquad \partial_0(A_\nu)_{ij}(\tau=0) = -\partial_0(A_\nu)_{ji}(\tau=0) \qquad (2.2.20)$$
$$O^a_0(\tau=0) = {I_0}^a$$

*and*

$$(divA)_{ij} - A^2 = 0 \iff e^\mu(A_\mu)_{ij} - {(A^\mu)_\mu}^k(A_k)_{ij} - A^2 = 0 \qquad (2.2.21)$$
$$Ric_{ab}(g) = 0 \iff e^\mu(A_\nu)_{i\mu} - e_\nu(A^\mu)_{i\mu} - ([A^\mu, A_\nu])_{i\mu} - {(A^{[\mu})_{\nu]}}^k(A_k)_{i\mu} = 0$$

*on $\Sigma_0$. Then the latter solution corresponds to an Einstein vacuum spacetime $(\mathcal{M}^{1+3}, g)$ and furthermore the frame $\{e_i\}_0^3$ (2.2.18) is $g$-orthonormal, $e_0 = \partial_0$, and satisfies the Lorenz gauge condition (2.2.15).*

**Remark 2.2.2.** Note that the second part of (2.2.21) includes the constraints (2.1.5); $\mathrm{R}_{0b} = \mathrm{R}_{00} - \frac{1}{2}\mathrm{R} = 0$, $b = 1,2,3$, on $\Sigma_0$. The condition (2.2.21) is necessary and sufficient (as we show in B.0.5) to yield the propagation of the gauge and the EVE themselves. Once we have chosen the orthonormal frame initially and the initial data components $(A_0)_{ij}(\tau = 0)$, which correspond to the $\partial_0$ derivative of $\{e_i\}_0^3$, then the rest of the initial data set of (2.2.16) is fixed by the condition (2.2.21), i.e., the Lorenz gauge and the EVE on the initial hypersurface $\Sigma_0$, see Remark B.0.13.

## 2.3 The Schwarzschild components

We fix an explicit Schwarzschild orthonormal reference frame and compute the corresponding connection coefficients, which we then use to find the leading asymptotics of the second fundamental form and curvature of the initial singular hypersurface $\Sigma_0$ in Schwarzschild. Knowing the precise leading blow up behaviour of these quantities is crucial for the study of the backwards well-posedness in the next section. For distinction, we denote Schwarzschild components with an upper left script $^S$.

Let us consider a specific foliation of spacelike hypersurfaces $\Sigma_\tau$, $\tau \in [0, T]$, for the backward problem in a neighbourhood of $(u = 1, v = 1)$; Figure 2.3. For convenience[11] let

$$\Sigma_\tau: \qquad -\frac{1}{2}(u+v) + 1 = \tau \qquad (u,v) \in (1-\epsilon, 1+\epsilon)^2,\ \tau \in [0,T]. \qquad (2.3.1)$$

[11] It is easy to see that the following leading asymptotics we derive are independent of the particular choice of foliation.

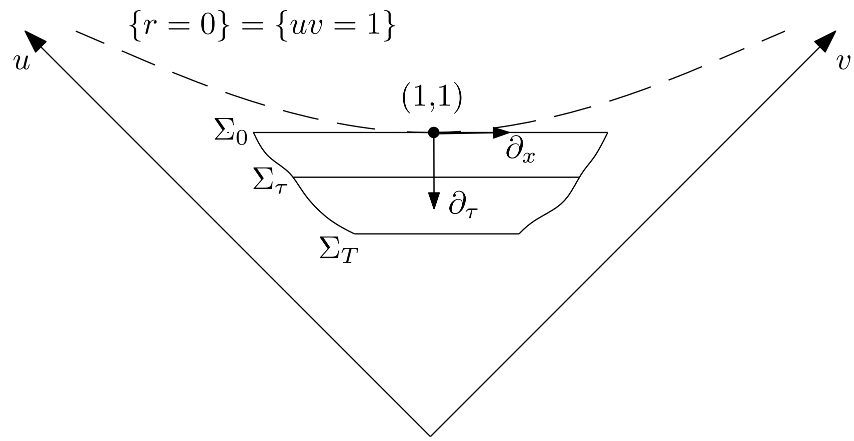


Figure 2.5: The foliation (2.3.1) in the interior of the black hole.

In temporal and spatial coordinates $\tau, x$

$$\partial_\tau := -\partial_u - \partial_v, \qquad \partial_x := \partial_u - \partial_v \qquad x = \frac{1}{2}(u-v), \tag{2.3.2}$$

the metric (2.1.2) takes the form

$$^S g = -\Omega^2 d\tau^2 + \Omega^2 dx^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \qquad \Omega^2 = \frac{32M^3}{r}e^{-\frac{r}{2M}}. \tag{2.3.3}$$

By (2.1.3),(2.3.2) $r$ is related to $\tau, x$ via

$$(1-\tau)^2 - x^2 = (1 - \frac{r}{2M})e^{\frac{r}{2M}}, \tag{2.3.4}$$

from which one can derive the following formulas:

$$\begin{aligned} \partial_\tau r = \frac{\Omega^2}{4M}(1-\tau), \qquad \partial_x r = \frac{\Omega^2}{4M}x \\ \partial_\tau \Omega^2 = -\frac{\Omega^4}{4M}(\frac{1}{r} + \frac{1}{2M})(1-\tau), \qquad \partial_x \Omega^2 = -\frac{\Omega^4}{4M}(\frac{1}{r} + \frac{1}{2M})x \end{aligned} \tag{2.3.5}$$

**Remark 2.3.1.** The above first two identities yield the leading asymptotics:

$$r^2 \sim 16M^2(\frac{x^2}{2} + \tau), \qquad \text{as } \tau, x \to 0. \tag{2.3.6}$$

Directly from the form of the induced metric on $\Sigma_\tau$,

$$^S\overline{g} = \Omega^2 dx^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{2.3.7}$$

we compute the corresponding induced volume form

$$d\mu_{S_{\overline{g}}} = \Omega r^2 \sin\theta dx d\theta d\phi = \left[4\sqrt{2}M^{\frac{3}{2}} r^{\frac{3}{2}} + O(r^2)\right] \sin\theta dx d\theta d\phi \tag{2.3.8}$$

and its rate of change along $\partial_\tau$ using (2.3.5):

$$\partial_\tau d\mu_{S_{\overline{g}}} = \big[\frac{12M^2}{r^2}(1-\tau) + O(\frac{1}{r})\big] d\mu_{S_{\overline{g}}}. \tag{2.3.9}$$

Normalizing, we define the Schwarzschild orthonormal frame

$$\partial_0 = \frac{1}{\Omega}\frac{\partial}{\partial\tau} \qquad \partial_1 = \frac{1}{\Omega}\frac{\partial}{\partial x} \qquad \partial_2 = \frac{1}{r}\frac{\partial}{\partial\theta} \qquad \partial_3 = \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} \tag{2.3.10}$$

and the relative connection coefficients ${}^S(A_\mu)_{ij} = {}^S g({}^S\nabla_{\partial_\mu}\partial_i, \partial_j)$ associated to it. A tedious computation[12] shows that the non-zero components read

$$\begin{aligned}
{}^S(A_0)_{01} &= -\frac{\Omega}{8M}(\frac{1}{r} + \frac{1}{2M})x \\
{}^S(A_1)_{01} &= -\frac{\Omega}{8M}(\frac{1}{r} + \frac{1}{2M})(1-\tau) \\
{}^S(A_2)_{02} = {}^S(A_3)_{03} &= \frac{\Omega}{4M}\frac{1-\tau}{r} \\
{}^S(A_2)_{12} = {}^S(A_3)_{13} &= \frac{\Omega}{4M}\frac{x}{r} \\
{}^S(A_3)_{23} &= \frac{\cot\theta}{r}
\end{aligned} \tag{2.3.11}$$

Recall the (spacetime) divergence formula of the connection 1-forms $X \to {}^S(A_X)_{ij}$

$${}^S(\mathrm{div}A)_{ij} := \partial^\mu {}^S(A_\mu)_{ij} - {}^S(A_{\nabla_{\partial^\mu}\partial_\mu})_{ij} = \partial^\mu {}^S(A_\mu)_{ij} - {}^S(A^\mu)_\mu{}^b {}^S(A_b)_{ij} \tag{2.3.12}$$

Utilizing (2.3.5) and (2.3.11), we check that the first order term in the RHS of (2.3.12) vanishes

$$\partial^\mu {}^S(A_\mu)_{ij} = 0, \tag{2.3.13}$$

leaving

$${}^S(\mathrm{div}A)_{ij} = {}^S(A_3)_{23} {}^S(A_2)_{ij}. \tag{2.3.14}$$

**Remark 2.3.2.** Thus, the orthonormal frame (2.3.10) satisfies a Lorenz gauge type condition (2.2.15).

[12] One may calculate the connection coefficients using the Koszul formula

$${}^S(A_\mu)_{ij} = \frac{1}{2}\Big[{}^S g([\partial_\mu, \partial_i], \partial_j) - {}^S g([\partial_i, \partial_j], \partial_\mu) + {}^S g([\partial_j, \partial_\mu], \partial_i)\Big]$$

**Remark 2.3.3.** Summarizing the above identities and formulas we obtain the following leading asymptotics at $r=0$:

$$\partial_0 \sim \frac{1}{4\sqrt{2}M^{\frac{3}{2}}} r^{\frac{1}{2}}\partial_\tau \qquad \partial_1 \sim \frac{1}{4\sqrt{2}M^{\frac{3}{2}}} r^{\frac{1}{2}}\partial_x$$
$$|{}^S A| \leq \frac{C}{r^{\frac{3}{2}}} \qquad |\partial^{(k)}{}^S A| \leq \frac{C}{r^{(k+1)\frac{3}{2}}}, \tag{2.3.15}$$

where $C$ depends on $M>0$ and $k$. Notice that the latter asymptotics are sharp for $k=0$ and when $\partial^{(k)} = \partial_0^{(k)}$. In fact, the components of the second fundamental form of the slices ${}^S K_{ii} = {}^S(A_i)_{0i}$, $i=1,2,3$, are exactly at this level. In more geometric terms we have (up to constants)

$$|{}^S K| \sim \frac{1}{r^{\frac{3}{2}}} \qquad |\mathrm{tr}_{{}^S\overline{g}}{}^S K| \sim \frac{1}{r^{\frac{3}{2}}} \qquad |{}^S\overline{\mathrm{R}}| \sim \frac{1}{r^2}. \tag{2.3.16}$$

Thus, employing (2.3.8),(2.3.6) for $\tau=0$, we see that both the scalar curvature and the second fundamental of the initial singular hypersurface $\Sigma_0$ are far from being square integrable

$$\int_{\Sigma_0} |{}^S K|^2 d\mu_{{}^S\overline{g}} \sim \int_0^\epsilon \frac{1}{x^3} x^{\frac{3}{2}} dx = \int_0^\varepsilon \frac{1}{x^{\frac{3}{2}}} dx = +\infty$$
$$\int_{\Sigma_0} |{}^S\overline{\mathrm{R}}|^2 d\mu_{{}^S\overline{g}} \sim \int_0^\epsilon \frac{1}{x^4} x^{\frac{3}{2}} dx = \int_0^\epsilon \frac{1}{x^{\frac{5}{2}}} dx = +\infty \tag{2.3.17}$$

The same holds for the mean curvature of $\Sigma_0$. In fact, a similar calculation shows $\mathrm{tr}_{{}^S\overline{g}}{}^S K \notin L^p$, $p \geq \frac{5}{3}$.

**Remark 2.3.4.** The precise leading asymptotics of all computed quantities in this section play a crucial role in the analysis of the backward existence problem and the proofs of the main theorems in the next section. However, exact formulas, like (2.3.11), are not really needed. We could have chosen as well a foliation of the form $\Sigma_\tau : \tau = f(x)$, instead of (2.3.1), i.e., $\tau = \text{const.}$, for some smooth function $f(x)$, $f'(0)=0$. It is easy to see by computing the induced metric and second fundamental form that the leading asymptotics of all relevant quantities of interest remain the same.

## 2.4 The local-in time backwards well-posedness

### 2.4.1 Perturbed spacetime; A transformed system

Let $(\overline{g}, K)$ be a perturbation of the Schwarzschild initial data set $({}^S\overline{g}, {}^S K)$ on $\Sigma_0$, verifying the constraints (2.1.5), and let $\{e_i\}_1^3$ be an orthonormal frame of $(\Sigma_0, \overline{g})$. We fix a reference frame $\{\partial_i\}_0^3$ in $\mathcal{M}^{1+3} = \{\Sigma_\tau\}_{\tau\in[0,T]}$, namely, the Schwarzschild orthonormal frame (2.3.10); Figure

2.5. Let $\{e_i\}_0^3$, $e_0 = \partial_0$, be a frame extension in $\mathcal{M}^{1+3}$ expressed in terms of $\partial_d$ via

$$e_c = O_c^d \partial_d. \tag{2.4.1}$$

Consider now the (unique) metric $g$ for which $e_i$ is orthonormal, $g_{ab} := m_{ab} = \text{diag}(-1,1,1,1)$, and the corresponding connection coefficients $(A_\nu)_{ij} = g(\nabla_{e_\nu} e_i, e_j)$. Then Proposition 2.2.1 asserts that the EVE (2.1.1) for $g$, under the Lorenz gauge condition[13]

$$(\text{div}A)_{ij} = (A_3)_{23}(A_2)_{ij}, \tag{2.4.2}$$

reduce to the system of scalar equations

$$\begin{aligned} \boxdot(A_\nu)_{ij} = {} & (A^{[\mu})_{\nu]}{}^k e_k(A_\mu)_{ij} + e^\mu([A_\mu, A_\nu])_{ij} + e^\mu\big((A_{[\mu})_{\nu]}{}^k (A_k)_{ij}\big) \\ & + (A^\mu)^k{}_{[\mu}(F_{\nu]k})_{ij} + ([A^\mu, F_{\mu\nu}])_{ij} + e_\nu\big((A_3)_{23}(A_2)_{ij}\big) + e_\nu\big((A^\mu)_\mu{}^k (A_k)_{ij}\big) \\ \partial_0(O_c^d) = {} & -O_c^{b\,S}(A_{[0})_{b]}{}^d + (A_{[0})_{c]}{}^k O_k^d, \qquad \nu, i, j, c, d \in \{0,1,2,3\} \end{aligned} \tag{2.4.3}$$

where $\boxdot := -e_0^2 + e_1^2 + e_2^2 + e_3^2$ and ${}^S(A_{[0})_{b]}{}^d = [\partial_0, \partial_b]^d$.

However, the system (2.4.3) has singular initial data in the Schwarzschild background which do not permit an energy approach directly. For this reason we recast the equations in a way that captures the closeness to the Schwarzschild spacetime. Let

$$(u_\nu)_{ij} := (A_\nu)_{ij} - {}^S(A_\nu)_{ij} : \{\Sigma_\tau\}_{\tau\in[0,T]} \to \mathbb{R} \qquad \nu, i, j \in \{0,1,2,3\}, \tag{2.4.4}$$

where the components ${}^S(A_\nu)_{ij}$ are the Schwarzschild connection coefficients corresponding to the frame $\{\partial_i\}_0^3$ (2.3.10) and they are given by (2.3.11). We are going to use these new functions to control the evolution of the perturbed spacetime.

Consider now the analogous system to (2.4.3) satisfied by the Schwarzschild components ${}^S(A_\nu)_{ij}, \partial_c$. In view of the asymptotics (2.3.15), we define $\Gamma_q$ to be a smooth function satisfying the bound

$$|\Gamma_q| \leq \frac{C_q}{r^q} \qquad |\partial^{(k)}\Gamma_q| \leq \frac{C_{q,k}}{r^{q+\frac{3}{2}k}}, \tag{2.4.5}$$

for constants $C_q, C_{q,k}$ depending on $M > 0$. Taking the difference of the two analogous systems we obtain a new system for the functions $(u_\nu)_{ij}, O_c^d - I_c{}^d$ written schematically in the form:

$$\begin{aligned} h^{ab}\partial_a\partial_b(u_\nu)_{ij} = {} & O\Gamma_{\frac{3}{2}}\partial u + O\Gamma_3 u + O\Gamma_{\frac{9}{2}}(O-I) + O\Gamma_3\partial(O-I) \\ & + \Gamma_3 u^2 + Ou\partial u + u^3 + O\partial(O-I)\partial u \\ \partial_0(O_c^d - I_c{}^d) = {} & \Gamma_{\frac{3}{2}}(O-I) + (O-I)u + u, \end{aligned} \tag{2.4.6}$$

[13] We choose now a specific type based on the one satisfied by the Schwarzschild reference frame (2.3.14).

where

$$h^{ab} := m^{cd} O_c^a O_d^b = g^{ab} \tag{2.4.7}$$

and each term in the RHS denotes some algebraic combination of finite number of terms of the depicted type (varying in $\nu, i, j$) where the particular indices do not matter.

**Remark 2.4.1.** Evidently, the systems (2.4.3) and (2.4.6) are equivalent. The benefit is that the assumption on the perturbed spacetime, being close to Schwarzschild, implies that the functions $(u_\nu)_{ij}, O_c^d - I_c{}^d$ are now small and regular. Thus, we have reduced the evolutionary problem to solving the PDE-ODE system of equations (2.4.6). However, the issue of singular initial data in (2.4.3) has become an issue of singularities in the coefficients of the resulting equations (2.4.6), at $\tau = x = 0$, which do not make it possible to apply the energy procedure in standard spaces; see also (2.3.17). These singularities, in large part, are due to the intrinsic curvature blow up and cannot be gauged away; in particular the coefficients $\Gamma_3$ of the potential terms in (2.4.6) correspond to the Schwarzschild curvature (2.1.4). Some of the functions $\Gamma_q$ that appear in (2.4.6), expressed in terms of Schwarzschild connection coefficients (2.3.11) and their derivatives, are less singular than (2.4.5), but representatives of the exact bound do appear in all the terms.

**Remark 2.4.2.** Another crucial asymptotic behaviour that our method heavily depends on is that of the radius function $r$. According to (2.3.6), we observe that the best $L^\infty_{\Sigma_\tau}$ bound one could hope for the ratio $1/r^2$ is of the form

$$\|\frac{1}{r^2}\|_{L^\infty(\Sigma_\tau)} \leq \frac{C}{\tau}, \tag{2.4.8}$$

which obviously fails to be integrable in time $\tau \in [0, T]$, for any $T > 0$. This fact lies at the heart of the difficulty of closing a Gronwall type estimate.

### 2.4.2 The weighted $H^s$ spaces

In order to study the well-posedness of (2.4.6) we introduce certain weighted norms. It turns out that the weights which yield the desired flexibility in obtaining energy estimates are the following.

**Definition 2.4.3.** Given $\alpha > 0$ and $\tau \in [0, T]$, we define the (time dependent) weighted Sobolev space $H^{s,\alpha}[\tau]$, as a subspace of the standard $H^s$ space on $\Sigma_\tau$ with the Schwarzschild induced volume form satisfying:

$$H^{s,\alpha}[\tau] : \quad u \in H^s(\Sigma_\tau), \quad \|u\|^2_{H^{s,\alpha}[\tau]} := \sum_{k \leq s} \int_{\Sigma_\tau} \frac{[\partial^{(k)} u]^2}{r^{2\alpha - 3(k-1)}} d\mu_{S_{\bar{g}}} < +\infty, \tag{2.4.9}$$

where by $\partial^{(k)}$ we denote any order $k$ combination of directional derivatives with respect to the components $\partial_1, \partial_2, \partial_3$ of the Schwarzschild frame (2.3.10). For convenience, we drop $\tau$ from the notation whenever the context is clear.

**Remark 2.4.4.** Observe that the weights in the norm $\|\cdot\|_{H^{s,\alpha}}$ in (2.4.9) blow up only at $\tau = 0, x = 0$. For $\tau > 0$ fixed, the weights are uniformly bounded above by some positive constant $C_\tau$, which becomes infinite as $\tau \to 0^+$. The dependence of the power $2\alpha - 3(k-1)$ on the number $k$ of derivatives corresponds to the singularities in the coefficients of the equation (2.4.6).

**Lemma 2.4.5.** *The weighted $H^{s,\alpha}$ spaces satisfy the properties:*

$$\begin{aligned} H^{s_1,\alpha} &\subset H^{s_2,\alpha} & s_1 &< s_2 \\ r^{-\frac{3}{2}l} u &\in H^{s,\alpha-\frac{3}{2}l}, & \textit{whenever } u &\in H^{s,\alpha} \\ \partial^{(k)} u &\in H^{s-k,\alpha-\frac{3}{2}k} & k \leq s,\ u &\in H^{s,\alpha} \end{aligned} \tag{2.4.10}$$

*Proof.* They are immediate consequences of Definition 2.4.3 and and the fact that

$$|\partial_1(r^{-\frac{3}{2}l})| \leq Clr^{-\frac{3}{2}l-\frac{1}{2}} \qquad |\partial_2(r^{-\frac{3}{2}l})| = |\partial_3(r^{-\frac{3}{2}l})| = 0,$$

cf. (2.3.5), (2.3.10). □

### 2.4.3 Local existence theorems

Let

$$\begin{aligned} \mathcal{E}(u, O; \alpha, T) := \sum_{\nu,i,j=0}^{3} \Bigg[ &\sup_{\tau \in [0,T]} \left( \|(u_\nu)_{ij}\|^2_{H^{3,\alpha}} + \|\partial_0 (u_\nu)_{ij}\|^2_{H^{2,\alpha-\frac{3}{2}}} \right) \\ &+ \int_0^T \left( \|(u_\nu)_{ij}\|^2_{H^{3,\alpha+1}} + \|\partial_0 (u_\nu)_{ij}\|^2_{H^{2,\alpha-\frac{1}{2}}} \right) d\tau \Bigg] \\ &+ \sum_{c,d=0}^{3} \left[ \sup_{\tau \in [0,T]} \|O_c^d - {I_c}^d\|^2_{H^{3,\alpha+\frac{3}{2}}} + \int_0^T \|O_c^d - {I_c}^d\|^2_{H^{3,\alpha+\frac{5}{2}}} d\tau \right] \end{aligned} \tag{2.4.11}$$

be the total weighted energy of the functions $(u_\nu)_{ij}, O_c^d - {I_c}^d$ defined in $\{\Sigma_\tau\}_{\tau \in [0,T]}$ (2.3.1), Figure 2.3.1, the backward domain of dependence of $\Sigma_0$ with respect to the metric $g$ we are solving for. Since the actual domain depends on the unknown solution, it will be fully determined in the end; see Section 2.5. For brevity we denote by

$$\begin{aligned} \mathcal{E}_0 := &\sum_{\nu,i,j \in \{0,1,2,3\}} \left[ \|(u_\nu)_{ij}(\tau = 0)\|^2_{H^{3,\alpha}} + \|\partial_0 (u_\nu)_{ij}(\tau = 0)\|^2_{H^{2,\alpha-\frac{3}{2}}} \right] \\ &+ \sum_{c,d \in \{0,1,2,3\}} \|O_c^d - {I_c}^d\|^2_{H^{3,\alpha+\frac{3}{2}}(\Sigma_0)} \end{aligned} \tag{2.4.12}$$

the energy at the initial singular slice $\Sigma_0$.

The following theorem is our first main local well-posedness result for the system (2.4.6), whose proof occupies Section §2.5.

**Theorem 2.4.6.** *There exist $\alpha > 0$ sufficiently large and $\varepsilon > 0$ small such that if*

$$\mathcal{E}_0 < +\infty \qquad \|O_c^d - I_c{}^d\|_{L^\infty(\Sigma_0)} < \varepsilon, \qquad c, d = 0, 1, 2, 3, \tag{2.4.13}$$

*then the system (2.4.6) admits a unique solution, up to some small time $T = T(\mathcal{E}_0, \alpha) > 0$, in the spaces*

$$\begin{aligned} (u_\nu)_{ij} &\in C([0,T]; H^{3,\alpha}) \cap L^2([0,T]; H^{3,\alpha+1}) \qquad \nu, i, j \in \{0,1,2,3\} \\ \partial_0 (u_\nu)_{ij} &\in C([0,T]; H^{2,\alpha-\frac{3}{2}}) \cap L^2([0,T]; H^{2,\alpha-\frac{1}{2}}) \\ O_c^d - I_c{}^d &\in C([0,T]; H^{3,\alpha+\frac{3}{2}}) \cap L^2([0,T]; H^{3,\alpha+\frac{5}{2}}) \qquad c, d \in \{0,1,2,3\} \end{aligned} \tag{2.4.14}$$

**Remark 2.4.7.** $(i)$ The second part of condition (2.4.13), $\varepsilon > 0$ small, is necessary for the equation (2.4.6) to be hyperbolic, yielding sufficient pointwise control on the $h^{ab}$'s (2.4.7)

$$|h^{bb} - m^{bb}| < \frac{1}{2} \qquad |h^{bc}| \leq C\varepsilon^2, \qquad b, c = 0, 1, 2, 3, \ b \neq c. \tag{2.4.15}$$

It could be obviously replaced by the stronger assumption that $\mathcal{E}_0 < \varepsilon$, since the energy $\mathcal{E}(u, O; \alpha, T)$ controls the $L^\infty$ norm of $u, O$ by standard Sobolev embedding.
$(ii)$ How large the exponent $\alpha$ has to be depends on the coefficients of the system (2.4.6). In the final inequalities in §2.5 $\alpha > 0$ is picked large enough so that certain 'critical' terms can be absorbed in the LHS and the estimates can close.

The above theorem is a local existence result for the system (2.4.6). Imposing now the proper conditions on the initial data set of (2.4.6), the solution (2.4.14) yields a solution of (2.4.3) which in turn corresponds to an Einstein vacuum spacetime (2.1.1).

**Theorem 2.4.8.** *Let $\alpha, \varepsilon$ be such as in Theorem §2.4.6 and let $(\Sigma_0, \overline{g}, K)$ be an initial data set for the Einstein vacuum equations (2.1.1) satisfying the constraints (2.1.5), such that the components*

$$(u_\nu)_{ij} \in H^{3,\alpha}(\Sigma_0) \qquad \nu, i, j = 1, 2, 3, \tag{2.4.16}$$

$$O_c^d - I_c{}^d \in H^{3,\alpha+\frac{3}{2}}(\Sigma_0) \qquad \|O_c^d - I_c{}^d\|_{L^\infty(\Sigma_0)} < \varepsilon \qquad c, d = 1, 2, 3, \tag{2.4.17}$$

*computed with respect to an orthonormal frame $\{e_i\}_1^3$ on $(\Sigma_0, \overline{g})$, and*

$$(u_i)_{0j}(\tau = 0) := K_{ij} - {}^S K_{ij} \in H^{3,\alpha}(\Sigma_0) \qquad i, j = 1, 2, 3. \tag{2.4.18}$$

*Then, there exists a solution $g$ to the EVE (2.1.1) in the backward region to $\Sigma_0$, foliated by $\{\Sigma_\tau\}_{t\in[0,T]}$, with induced initial data set $(\overline{g},K)$ on $\Sigma_0$ and an orthonormal frame extension $\{e_i\}_0^3$ for which the corresponding (spacetime) functions $(u_\nu)_{ij}, O_c^d - I_c^d$ (2.4.4),(2.4.1) lie in the spaces (2.4.14).*
*If in addition $O_c^d - {I_c}^d \in C([0,T];H^{4,\alpha+\frac{3}{2}})$, $c,d = 1,2,3$, then the Einsteinian vacuum development is unique up to isometry.*

The fact that such (non-spherically symmetric) initial data sets $(\Sigma_0,\overline{g},K)$ exist, in compliance with Theorem 2.4.8, is shown in §2.6.

*Proof of Theorem 2.4.8.* We want to invoke Theorem 2.4.6. For this purpose, we prescribe initial data for the system (2.4.6):
($i$) The components (2.4.16), (2.4.17), (2.4.18) are given.
($ii$) Since in the beginning of §2.4.1 we assumed $e_0 = \partial_0$ and since $\{e_i\}_1^3$ is initially tangent to $\Sigma_0$, we set

$$O_0^b(\tau = 0) = {I_0}^b \qquad O_a^0(\tau = 0) = {I_a}^0 \qquad a,b = 0,1,2,3. \tag{2.4.19}$$

($iii$) We (freely) assign[14]

$$(u_0)_{ab}(\tau = 0) := (A_0)_{ab} - {}^S(A_0)_{ab} \in H^{3,\alpha}(\Sigma_0), \qquad a,b = 0,1,2,3. \tag{2.4.20}$$

Once we have prescribed the above, the components $\partial_0(u_\nu)_{ij}(\tau = 0)$ are fixed by the assumption (2.2.21) on the initial data of the original system (2.4.3); see Remark B.0.13. Indeed, subtracting the corresponding Schwarzschild components from (B.0.15),(B.0.16), which obvisouly satisfy the same initial relations, cf. (2.3.14), we obtain schematically:

$$\partial_0(u_\nu)_{ij} = O\partial_a u + \Gamma_{\frac{3}{2}}u + \Gamma_3(O-I) + u^2 \qquad \text{on } \Sigma_0,\ \ a = 1,2,3. \tag{2.4.21}$$

By (2.4.10) and standard Sobolev embedding we conclude that

$$\partial_0(u_\nu)_{ij}(\tau = 0) \in H^{2,\alpha-\frac{3}{2}} \qquad \nu,i,j = 0,1,2,3. \tag{2.4.22}$$

Thus, the assumption (2.4.13) is verified and Theorem 2.4.6 can be invoked. From Proposition 2.2.1 it follows that the solution (2.4.14) of (2.4.6) and hence of (2.4.3) yields indeed an Einstein vacuum spacetime $(\{\Sigma_\tau\}_{\in[0,T]}, g)$.

To prove uniqueness (up to isometry) we rely on the uniqueness statement in Theorem 2.4.6. Suppose there is another Einsteinian vacuum development $(\tilde{\mathcal{M}}^{1+3},\tilde{g})$ of the initial data set $(\Sigma_0,\overline{g},K)$, diffeomorphic to $\{\Sigma_\tau\}_{\tau\in[0,T]}$, satisfying the hypothesis (2.4.16), (2.4.17), (2.4.18); defined by pulling back the relevant quantities through the preceding diffeomorphism, taking

[14]The functions $(u_0)_{ab}(\tau = 0)$ or equivalently $(A_0)_{ab}(\tau = 0)$ fix the $\partial_0$ derivative of the frame $\{e_i\}_0^3$ on $\Sigma_0$; see Lemma B.0.12 and Remark B.0.13.

differences etc. In order to use the uniqueness statement in Theorem 2.4.6, we need the two spacetimes to have the same initial data for the system (2.4.6). The part of the initial data set given by the assumptions in the statement of Theorem 2.4.8 is of course identical for both spacetimes. The remaining components that we want to agree, other than the $(\tilde{u}_0)_{ab}(\tau=0)$'s, as noted in the previous paragraph, can be fixed by condition (2.2.21). Therefore, we get identical initial data components for the system (2.4.6) by constructing a Lorenz gauge frame (2.4.2) $\{\tilde{e}_i\}_0^3$ for $\tilde{g}$, which is initially equal to $\{e_i\}_0^3$ on $\Sigma_0$ and such that $(\tilde{u}_0)_{ab}(\tau=0)=(u_0)_{ab}(\tau=0)$ as well; see Lemma B.0.12. The only assumption to be verified is the well-posedness of the system (B.0.1) for functions in the solution spaces (2.4.14), after taking differences with the equation for the frame $\{e_i\}_0^3$. However, this falls in the category of the system (2.4.6) [in fact simpler, being semilinear] to which Theorem 2.4.6 can be applied. The extra derivative that we have to assume in order to close, $\tilde{O}_c^d - {I_c}^d \in H^{4,\alpha+\frac{3}{2}}$, is due to the div$A$ term in the RHS of (B.0.1). □

## 2.5 Proof of Theorem 2.4.6

Throughout this section we will use the notation $X \lesssim Y$ to denote an inequality between the quantities $X, Y$ of the form $X \leq CY$, where $C$ is an absolute positive constant depending only on the Schwarzschild mass $M > 0$. The same for the standard notation $O(X)$, for a quantity bounded by $|O(X)| \leq CX$, $X > 0$. Furthermore, all the estimates regard only the Schwarzschild region foliated by $\{\Sigma_\tau\}_{\tau\in[0,T]}$; Figure 2.3.

### 2.5.1 Proof outline

We prove Theorem 2.4.6 via a contraction mapping argument. First we establish an energy estimate in the relevant weighted $H^3$ spaces in §2.5.3. Then we obtain a contraction, in §2.5.4, in the corresponding spaces of one derivative less, see (2.5.48), which together with the energy estimate yield the desired solution (2.4.14).

To derive these estimates we have to eliminate some *critical* terms which are generated due to the singularities in the coefficients of the equations, having larger weights than the ones in the norm (2.4.9), and which prevent us from closing (see Propositions 2.5.3,2.5.6). This is where the role of the weights (2.4.9) comes in. The parameter $\alpha > 0$ helps generate critical terms with a favourable sign. Being large enough, but finite, $\alpha$ provides an overall negative sign for the critical terms, hence, rendering them removable from the RHS of the final inequalities. This enables us to close the estimates and complete the proof. The precise asymptotics of the singularities in the coefficients of the equations (2.4.6), at $\tau = x = 0$, and the opening up rate of the radius function $r$ in $\tau > 0$ play a crucial role here.[15]

[15]If we were to tweak the leading orders just by $\epsilon > 0$, the previous procedure would fail no matter how large $\alpha > 0$ is to begin with.

### 2.5.2 Basic estimates

Let $v$ be a scalar function defined on $\Sigma_\tau$, represented by

$$v \circ \psi_\tau : U_\tau \to \mathbb{R}, \tag{2.5.1}$$

where $\psi_\tau : U_\tau \to \Sigma_\tau$ is the $(x, \theta, \phi)$ coordinate chart. We recall some standard inequalities: the classical Sobolev embedding of $H^2(U)$ in $L^\infty(U)$

$$\|v\|_{L^\infty(U)} \lesssim \|v\|_{H^2(U)} \tag{2.5.2}$$

and the interpolation inequality

$$\|v\|_{L^4(U)} \leq C\|v\|^{\frac{1}{4}}_{L^2(U)}\|\nabla v\|^{\frac{3}{4}}_{L^2(U)} \qquad\qquad v \in C^\infty_c(U), \tag{2.5.3}$$

for a bounded domain $U \subset \mathbb{R}^3$ with (piecewise) $C^2$ boundary. In the following proposition $v$ is assumed to be regular enough such that the RHSs make sense.

**Proposition 2.5.1.** *For a general function $v : \Sigma_\tau \to \mathbb{R}$, $\tau \in [0, T]$, with the appropriate regularity, the following inequalities hold:*
*The $L^\infty$ bound*

$$\|\frac{v}{r^k}\|_{L^\infty(\Sigma_\tau)} \lesssim (k+1)^2\|v\|_{H^{2,k+3+\frac{1}{4}}(\Sigma_\tau)} \tag{2.5.4}$$

*and the $L^4$ estimate*

$$\|\frac{v}{r^k}\|_{L^4(\Sigma_\tau)} \lesssim (k+1)^{\frac{3}{4}}\|v\|_{H^{1,k+1+\frac{1}{4}}(\Sigma_\tau)}. \tag{2.5.5}$$

*Proof.* From (2.5.2) we have

$$\begin{aligned}\|\frac{v}{r^k}\|_{L^\infty(\Sigma_\tau)} \overset{(2.5.1)}{=} \|\frac{v}{r^k} \circ \psi_\tau\|_{L^\infty(U_\tau)} &\lesssim \|\frac{v}{r^k} \circ \psi_\tau\|_{H^2(U_\tau)} \\ &\lesssim (k+1)^2\|v\|_{H^{2,k+3+\frac{1}{4}}(\Sigma_\tau)}.\end{aligned}$$
(substituting (2.3.8) and the frame (2.3.10))

We argue similarly in the case of (2.5.5). □

### 2.5.3 Energy estimate in $H^{3,\alpha}$

We set up now the iteration scheme we are going to follow. Let $\{\overline{u}, \overline{O}\} := \{(\overline{u}_\nu)_{ij}, \overline{O}^d_c : \nu, i, j, c, d = 0, 1, 2, 3\}$ be a set of spacetime functions in the solution spaces (2.4.14), verifying $|\overline{O}_c^d - I_c{}^d| < \varepsilon$ initially on $\Sigma_0$. We assume without loss of generality[16]

$$\mathcal{E}(\overline{u},\overline{O};\alpha,T) \leq 2\mathcal{E}_0. \tag{2.5.6}$$

We also assume

$$\|\partial_0(\overline{O}_c^d)\|^2_{H^{2,\alpha}[\tau]} \lesssim \mathcal{E}_0^2 + \mathcal{E}_0 \qquad\qquad \forall \tau \in [0,T], \quad c,d = 0,1,2,3. \tag{2.5.7}$$

*Iteration step*: Consider the following linear version of the system (2.4.6), where we replace the functions $u, O$ in the following specific terms by the corresponding ones from the set $\{\overline{u},\overline{O}\}$:

$$\begin{aligned} \overline{h}^{ab}\partial_a\partial_b(u_\nu)_{ij} &= \overline{O}\Gamma_{\frac{3}{2}}\partial u + \overline{O}\Gamma_3 u + \overline{O}\Gamma_{\frac{9}{2}}(O-I) + \overline{O}\Gamma_3\partial(O-I) \\ &\quad + \Gamma_3\overline{u}^2 + \overline{O}\overline{u}\partial\overline{u} + \overline{u}^3 + \overline{O}\partial(\overline{O}-I)\partial\overline{u} \\ \partial_0(O_c^d - I_c{}^d) &= \Gamma_{\frac{3}{2}}(O-I) + (\overline{O}-I)\overline{u} + u, \end{aligned} \tag{2.5.8}$$

where $\overline{h}^{ab} = m^{cd}\overline{O}_c^a\overline{O}_d^b$. Observe that we kept in the RHS of (2.5.8) the functions $u, O$ attached to the most singular coefficients of the system. This is actually very important to our strategy in order to avoid further complications.

We assume now there exists a solution $(u_\nu)_{ij}, O_c^d - I_c{}^d$ of (2.5.8) lying in the solution space (2.4.14). The existence of such a solution is based mainly on the energy estimate we will derive below and a standard duality argument which we omit.
*Claim*: For a chosen large enough $\alpha > 0$ and $T > 0$ sufficiently small (depending on $\mathcal{E}_0, \alpha$) the following estimate holds

$$\mathcal{E}(u,O;\alpha,T) \leq 2\mathcal{E}_0. \tag{2.5.9}$$

The preceding $H^3$-weighted energy estimate, cf. (2.4.11), will be used in the next subsection to close the contraction argument that yields the existence and uniqueness of the solution (2.4.14) to (2.4.6). Now we begin the proof of (2.5.9):

First note that by the fundamental theorem of calculus, following a $\partial_0$ integral curve and employing (2.5.4), we readily obtain from our initial assumptions and (2.5.7) the pointwise bound

$$\sup_{\tau\in[0,T]} \|\overline{O} - I|\|_{L^\infty(\Sigma_\tau)} \leq \varepsilon + CT\mathcal{E}_0 < 2\varepsilon, \tag{2.5.10}$$

provided $\alpha \geq \frac{1}{2} + 3 + \frac{1}{4}$ and $T < \frac{\varepsilon}{C\mathcal{E}_0}$.

All the more, directly from the ODE in (2.5.8) we deduce the estimate: [applying the bounds

[16] Any assumptions that we make on the functions $\overline{u},\overline{O}$, we must derive for the next set of functions $u, f$ below.

(2.5.4), (2.5.6) to $(\overline{O}-I)\overline{u}$ and employing the asymptotics (2.4.5)]

$$\|\partial_0(O^d_c)\|^2_{H^{2,\alpha}[\tau]} \lesssim \mathcal{E}_0^2 + \|O-I\|^2_{H^{2,\alpha+\frac{3}{2}}[\tau]} + \|u\|^2_{H^{2,\alpha}[\tau]}, \tag{2.5.11}$$

for all $\tau\in[0,T]$, $c,d=0,1,2,3$.[17]

We derive (2.5.9) in the backward domain of dependence of $\Sigma_0$ w.r.t. the metric $(g_{ab})_{\overline{u}} := g_{\overline{u}}(\partial_a,\partial_b)$, $a,b=0,1,2,3$, whose inverse is given by $g^{ab}_{\overline{u}} := \overline{h}^{ab}$; compare to (2.4.7). The boundary of the domain is the backward incoming $g_{\overline{u}}$-null hypersurface $\mathcal{N}^{\overline{u}}$ emanating from $\partial\Sigma_0$ (Figure 2.6). We foliate the domain by the $\tau=$ const. hypersurfaces $\Sigma^{\overline{u}}_\tau$ inside $\mathcal{N}^{\overline{u}}$. Let $\rho$ be the scalar function defined near $\mathcal{N}^{\overline{u}}$ via

$$\rho(\mathcal{C}^{\overline{u}}_\tau) := T-\tau, \tag{2.5.12}$$

where $\mathcal{C}^{\overline{u}}_\tau$ is the cylinder obtained from the flow of $\partial\Sigma^{\overline{u}}_\tau$ backwards along the integral curves of $\partial_0$. Using $\rho$ we may write each leaf of the foliation as

$$\Sigma^{\overline{u}}_\tau = \bigcup_{t^*\in[\tau,T]}\{\rho_\tau = T-t^*\}\bigcup B_\tau \qquad\qquad \tau\in[0,T], \tag{2.5.13}$$

where $\rho_\tau := \rho\big|_{\Sigma^{\overline{u}}_\tau}$ and $B_\tau$ is simply the projection of $\Sigma^{\overline{u}}_T$ onto $\Sigma^{\overline{u}}_\tau$ through the integral curves of $\partial_0$. Since by definition $\rho+\tau-T$ is zero on $\mathcal{N}^{\overline{u}}$, it follows that the $g_{\bar{u}}$-gradient of $\rho+\tau-T$,

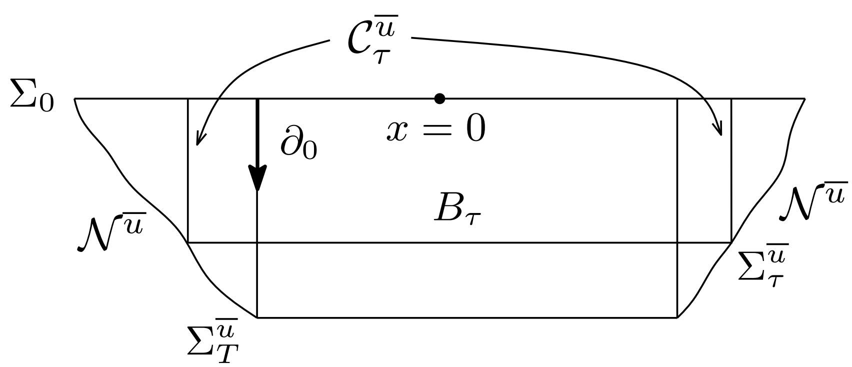


Figure 2.6:

on $\mathcal{N}^{\overline{u}}$, lies on the hypersurface itself and furthermore it is $g_{\bar{u}}$-null, i.e., $\rho$ satisfies the eikonal equation

$$\begin{aligned}\left|\nabla_{g_{\overline{u}}}(\rho+\tau-T)\right|^2_{g_{\overline{u}}} &= \overline{h}^{AB}\partial_A(\rho)\partial_B(\rho) + \Omega^{-2}\overline{h}^{00} + 2\Omega^{-1}\overline{h}^{A0}\partial_A(\rho)\\ &= 0 \qquad\qquad \text{on } \mathcal{N}^{\overline{u}},\end{aligned} \tag{2.5.14}$$

where $A,B=1,2,3$.

[17]This estimate, together with (2.5.9) in the end, imply the analogue of (2.5.7) for the functions $\partial_0(O^d_c)$.

**Remark 2.5.2.** The backward domain of definition of the variables $u, O-I$ depends on $\overline{u}, \overline{O}-I$. For the iteration scheme and the contraction mapping argument in §2.5.4 to be well-defined, all functions involved in the process must have a common domain of definition. To solve this issue is to begin with a slightly 'larger' initial hypersurface $\widetilde{\Sigma}_0 \supset \Sigma_0$ extending $\Sigma_0$ at both ends and to solve at each iteration step for the new variables in a 'smaller' domain contained in the interior of the domain of the previous iterate by shrinking the initial hypersurface $\widetilde{\Sigma}_0$. Since we are also proving a contraction mapping at the same time (see §2.5.4) we can make sure that the shrinking of $\widetilde{\Sigma}_0$ stops at $\Sigma_0$ giving the final backward domain in the limit.

We define the following adapted energy, which controls the part of the total energy (2.4.11) that refers to $u$:

$$E_{s+1,\alpha}[u](\tau) := \frac{1}{2}\sum_{\nu,i,j}\sum_{|J|\leq s}\int_{\Sigma_\tau^{\overline{u}}}\Big[-\overline{h}^{00}\frac{\big[\partial_0(u_\nu)_{ij,J}\big]^2}{r^{2\alpha-3|J|}} + \overline{h}^{AB}\frac{\partial_A(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}}\frac{\partial_B(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}} + \frac{(u_\nu)^2_{ij,J}}{r^{2\alpha-3(|J|-1)}}\Big]d\mu_{S_{\overline{g}}}, \tag{2.5.15}$$

where $(u_\nu)_{ij,J} := \partial^{(J)}(u_\nu)_{ij}$ and $J$ is a spatial multi-index (containing only directions $\partial_1, \partial_2, \partial_3$). It is evident from (2.5.10), $\overline{h}^{ab} = m^{cd}\overline{O}_c^a\overline{O}_d^b$, that $E_{3,\alpha}$ is equivalent to the weighted $H^{3,\alpha}\times H^{2,\alpha-\frac{3}{2}}$ norm of $u$ on $\Sigma_\tau^{\overline{u}}$.

We summarize in the following proposition the main energy estimates derived below.

**Proposition 2.5.3.** *The following two energy estimates hold:*

$$\begin{aligned}&\partial_\tau E_{3,\alpha}[u] + 8M^2e^{-1}(1-\tau)\alpha E_{3,\alpha+1}[u]\\ &\lesssim (\mathcal{E}_0^{\frac{1}{2}} + \mathcal{E}_0 + \alpha^2 + \alpha^3\mathcal{E}_0)E_{3,\alpha}[u] + E_{3,\alpha+1}[u] + \mathcal{E}_0\|O-I\|^2_{H^{3,\alpha+\frac{3}{2}}}\\ &\quad + \|O-I\|^2_{H^{3,\alpha+\frac{5}{2}}} + \alpha^3\mathcal{E}_0^2 + \mathcal{E}_0^3\end{aligned} \tag{2.5.16}$$

$$\begin{aligned}&\frac{1}{2}\partial_\tau\sum_{c,d}\|O_c^d - I_c{}^d\|^2_{H^{3,\alpha+\frac{3}{2}}} + 4M^2e^{-1}(1-\tau)\alpha\sum_{c,d}\|O_c^d - I_c{}^d\|^2_{H^{3,\alpha+\frac{5}{2}}}\\ &\lesssim \|O-I\|^2_{H^{3,\alpha+\frac{5}{2}}} + E_{3,\alpha+1}[u] + \mathcal{E}_0^2,\end{aligned} \tag{2.5.17}$$

*for all* $\tau \in (0,T)$.

The overall energy estimate (2.5.9) follows from Proposition 2.5.3: Adding (2.5.16), (2.5.17) we wish to close the estimate by employing the standard Gronwall lemma. However, this is not possible in general, because of the critical energies in the RHS, having larger weights than the ones differentiated in the LHS, namely, $E_{3,\alpha+1}[u]$, $\|O-I\|^2_{H^{3,\alpha+\frac{5}{2}}}$ instead of $E_{3,\alpha}[u]$, $\|O-I\|^2_{H^{3,\alpha+\frac{3}{2}}}$. It is precisely at this point that the role of the weights we introduced is revealed. Choosing $\alpha > 0$ large enough to begin with, how large depending on the constants in the above inequalities, we absorb the critical terms

$$E_{3,\alpha+1}[u], \|O-I\|^2_{H^{3,\alpha+\frac{5}{2}}}$$

in the LHS and then the standard Gronwall lemma applies to give (2.5.9).

*Proof of (2.5.16).* Let

$$P_{J,\alpha} := \frac{1}{2}\Big[-\overline{h}^{00}\frac{\big[\partial_0(u_\nu)_{ij,J}\big]^2}{r^{2\alpha-3|J|}} + \overline{h}^{AB}\frac{\partial_A(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}}\frac{\partial_B(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}} + \frac{(u_\nu)^2_{ij,J}}{r^{2\alpha+3-3|J|}}\Big], \tag{2.5.18}$$

for any spatial multi-index $J$ with $|J| \leq 2$; recall $(u_\nu)_{ij,J} := \partial^{(J)}(u_\nu)_{ij}$. It follows from (2.5.13) and the coarea formula that

$$\begin{aligned}\partial_\tau \int_{\Sigma^{\overline{u}}_\tau} P_{J,\alpha} d\mu_{S_{\overline{g}}} = &- \int_{\partial\Sigma^{\overline{u}}_\tau} \frac{P_{J,\alpha}}{|^S\overline{\nabla}\rho|} dS + \int_{\Sigma^{\overline{u}}_\tau} \partial_\tau P_{J,\alpha} d\mu_{S_{\overline{g}}} \\ &+ \int_{\Sigma^{\overline{u}}_\tau} P_{J,\alpha}\partial_\tau d\mu_{S_{\overline{g}}},\end{aligned} \tag{2.5.19}$$

where $^S\overline{\nabla}\rho$ stands for the gradient of $\rho$ with respect to the intrinsic connection on $(\Sigma_\tau, {}^S\overline{g})$ and $dS$ is the Schwarzschild induced volume form on $\partial\Sigma^{\overline{u}}_\tau$. Note that the boundary term in (2.5.19) has a favourable sign. Since $\mathcal{N}^{\overline{u}}$ is $g_{\overline{u}}$-incoming null, the sum of all arising boundary terms should have a good sign and therefore can be dropped in the end. Indeed, this is the case and it can be easily seen by keeping track of the few boundary terms that appear below. To analyse the last two terms in (2.5.19), we recall the $\partial_\tau$ differentiation formulas of the radius function $r$ (2.3.5), the estimate on volume form $d\mu_{S_{\overline{g}}}$ (2.3.9) and the commutator relation $[\partial_0, \partial_B] = {}^S(A_{[0)B]})^c\partial_c \overset{(2.3.15)}{=} \Gamma_{\frac{3}{2}}\partial$:

$$\begin{aligned}&\int_{\Sigma^{\overline{u}}_\tau} \partial_\tau P_{J,\alpha} d\mu_{S_{\overline{g}}} + \int_{\Sigma^{\overline{u}}_\tau} P_{J,\alpha}\partial_\tau d\mu_{S_{\overline{g}}} \\ =& -8M^2(1-\tau)\alpha\int_{\Sigma^{\overline{u}}_\tau} e^{-\frac{r}{2M}} P_{J,\alpha+1} d\mu_{S_{\overline{g}}} + \int_{\Sigma^{\overline{u}}_\tau} P_{J,\alpha} O(\frac{1}{r^2}) d\mu_{S_{\overline{g}}} \\ &+ \frac{1}{2}\int_{\Sigma^{\overline{u}}_\tau} \Omega\Big[-\partial_0(\overline{h}^{00})\frac{\big[\partial_0(u_\nu)_{ij,J}\big]^2}{r^{2\alpha-3|J|}} + \partial_0(\overline{h}^{AB})\frac{\partial_A(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}}\frac{\partial_B(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}}\Big] d\mu_{S_{\overline{g}}} \\ &+ \int_{\Sigma^{\overline{u}}_\tau} \Omega\Big[-\overline{h}^{00}\frac{\partial_0(u_\nu)_{ij,J}\partial_0^2(u_\nu)_{ij,J}}{r^{2\alpha-3|J|}} + \overline{h}^{AB}\frac{\partial_A(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}}\frac{\partial_B\partial_0(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}}\Big] d\mu_{S_{\overline{g}}} \\ &+ \int_{\Sigma^{\overline{u}}_\tau} \Omega\overline{h}^{AB}\frac{\partial_A(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}}\frac{\Gamma_{\frac{3}{2}}\partial(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}} d\mu_{S_{\overline{g}}} + \int_{\Sigma^{\overline{u}}_\tau} \Omega\frac{(u_\nu)_{ij,J}\partial_0(u_\nu)_{ij,J}}{r^{2\alpha+3-|J|}} d\mu_{S_{\overline{g}}}\end{aligned} \tag{2.5.20}$$

The first term on the LHS of (2.5.20) is critical having a favourable sign of magnitude $\alpha$. We use this term alone to absorb all arising critical terms in the process. Recall $|\overline{h}| = |\overline{O}^2| \leq 1$, cf.

(2.5.10), and the asymptotics (2.4.5). Also, applying (2.5.4) to $\partial_0\overline{h}$ and (2.5.7) we derive

$$|\Omega\partial_0(h)| \lesssim \mathcal{E}(\overline{u},\overline{O};\alpha,T)^{\frac{1}{2}}, \qquad \Omega \lesssim \frac{1}{r^{\frac{1}{2}}}, \qquad |\Gamma_{\frac{3}{2}}| \lesssim \frac{1}{r^{\frac{3}{2}}}.$$

Hence, by Cauchy's inequality and (2.5.6) we have

$$\begin{aligned}
&\frac{1}{2}\int_{\Sigma_\tau^{\overline{u}}}\Omega\Big[-\partial_0(\overline{h}^{00})\frac{\big[\partial_0(u_\nu)_{ij,J}\big]^2}{r^{2\alpha-3|J|}}+\partial_0(\overline{h}^{AB})\frac{\partial_A(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}}\frac{\partial_B(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}}\Big]d\mu s_{\overline{g}}\\
&+\int_{\Sigma_\tau^{\overline{u}}}\Omega\overline{h}^{AB}\frac{\partial_A(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}}\frac{\Gamma_{\frac{3}{2}}\partial(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}}d\mu s_{\overline{g}}+\int_{\Sigma_\tau^{\overline{u}}}\Omega\frac{(u_\nu)_{ij,J}\partial_0(u_\nu)_{ij,J}}{r^{2\alpha+3-|J|}}d\mu s_{\overline{g}}\\
&\lesssim \mathcal{E}(\overline{u},\overline{O};\alpha,T)^{\frac{1}{2}}E_{3,\alpha}[u]+E_{3,\alpha+1}[u] \qquad (2.5.21)\\
&\lesssim \mathcal{E}_0^{\frac{1}{2}}E_{3,\alpha}[u]+E_{3,\alpha+1}[u]
\end{aligned}$$

For the next term we proceed by integrating by parts[18] (IBP), denoting by $N := \overline{g}^{BB}N_B\partial_B$ the outward unit normal on $\partial\Sigma_\tau^{\overline{u}}$ w.r.t. Schwarzschild metric $\overline{g}$ on $\Sigma_\tau^{\overline{u}}$:

$$\begin{aligned}
&\int_{\Sigma_\tau^{\overline{u}}}\Omega\Big[-\overline{h}^{00}\frac{\partial_0(u_\nu)_{ij,J}\partial_0^2(u_\nu)_{ij,J}}{r^{2\alpha-3|J|}}+\overline{h}^{AB}\frac{\partial_A(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}}\frac{\partial_B\partial_0(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}}\Big]d\mu s_{\overline{g}}\\
&=-\int_{\Sigma_\tau^{\overline{u}}}\Omega\frac{\partial_0(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}}\Big[\overline{h}^{00}\frac{\partial_0^2(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}}+\overline{h}^{AB}\frac{\partial_B\partial_A(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}}\Big]d\mu s_{\overline{g}} \qquad (2.5.22)\\
&+\int_{\partial\Sigma_\tau^{\overline{u}}}\Omega\overline{h}^{AB}\frac{\partial_A(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}}\frac{\partial_0(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}}N_B dS\\
&-\int_{\Sigma_\tau^{\overline{u}}}\Big[\partial_B\big(\frac{\Omega\overline{h}^{AB}}{r^{2\alpha-3|J|}}\big)\partial_A(u_\nu)_{ij,J}+\Omega\overline{h}\frac{\Gamma_{\frac{3}{2}}\partial(u_\nu)_{ij,J}}{r^{2\alpha-3|J|}}\Big]\partial_0(u_\nu)_{ij,J}d\mu s_{\overline{g}}
\end{aligned}$$

It is immediate from the definition of the frame (2.3.10) and (2.3.5) that

$$\big|\partial_1\big(\frac{\Omega}{r^{2\alpha-3}}\big)\big| \lesssim \frac{\alpha}{r^{2\alpha-2}} \qquad \partial_2\big(\frac{\Omega}{r^{\alpha-\frac{3}{2}}}\big)=\partial_3\big(\frac{\Omega}{r^{\alpha-\frac{3}{2}}}\big)=0.$$

Hence, similarly to (2.5.21)

$$\begin{aligned}
&-\int_{\Sigma_\tau^{\overline{u}}}\Big[\partial_B\big(\frac{\Omega\overline{h}^{AB}}{r^{2\alpha-3|J|}}\big)\partial_A(u_\nu)_{ij,J}+\Omega\overline{h}\frac{\Gamma_{\frac{3}{2}}\partial(u_\nu)_{ij,J}}{r^{2\alpha-3|J|}}\Big]\partial_0(u_\nu)_{ij,J}d\mu s_{\overline{g}} \qquad (2.5.23)\\
&\lesssim (\mathcal{E}_0^{\frac{1}{2}}+\alpha^2)E_{3,\alpha}[u]+E_{3,\alpha+1}[u]. \qquad (|J|\leq 2)
\end{aligned}$$

*Remark*: The term in the RHS of the preceding estimate with coefficient $\alpha^2$ is not critical. This is very important otherwise the overall estimates would not close, since the critical term with favourable sign in (2.5.20) is only of magnitude $\alpha$.

[18]We integrate by parts using the spatial part of the Schwarzschild frame $\partial_1,\partial_2,\partial_3$. Doing so we pick up connection coefficients, since it is not covariant IBP.

We proceed to the boundary term in the RHS of (2.5.22). Recall that $\rho$ is constant on $\partial\Sigma^{\overline{u}}_\tau$ (2.5.12), and decreasing in the interior direction of $\Sigma^{\overline{u}}_\tau$. Hence, the outward unit normal $N$ is the Schwarzschild normalized gradient of $\rho$ on $\Sigma^{\overline{u}}_\tau$, $N = \frac{{}^S\overline{\nabla}\rho}{|{}^S\overline{\nabla}\rho|}$. Since $(\overline{h}^{AB})_{A,B=1,2,3}$ is a symmetric positive definite matrix, the following standard inequality holds:

$$\begin{aligned}\left|\overline{h}^{AB}\frac{\partial_A(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}}\Omega N_B\right|^2 &\leq \left(\overline{h}^{AB}\frac{\partial_A(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}}\frac{\partial_B(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}}\right)\left(\Omega^2\overline{h}^{AB}N_AN_B\right)\\ &= \left(\overline{h}^{AB}\frac{\partial_A(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}}\frac{\partial_B(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}}\right)\frac{\Omega^2\overline{h}^{AB}\partial_A(\rho)\partial_B(\rho)}{|{}^S\overline{\nabla}\rho|^2}\\ &= \left(\overline{h}^{AB}\frac{\partial_A(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}}\frac{\partial_B(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}}\right)\frac{-\overline{h}^{00}-2\Omega\overline{h}^{A0}\partial_A(\rho)}{|{}^S\overline{\nabla}\rho|^2} \qquad \text{(by (2.5.14))}\end{aligned}$$

Therefore, we have the bound

$$\begin{aligned}&\int_{\partial\Sigma^{\overline{u}}_\tau}\Omega\overline{h}^{AB}\frac{\partial_A(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}}\frac{\partial_0(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}}N_B dS \qquad (2.5.24)\\ \leq &\int_{\partial\Sigma^{\overline{u}}_\tau}\left|\frac{\partial_0(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}}\right|\sqrt{\frac{-\overline{h}^{00}-2\Omega\overline{h}^{A0}\partial_A(\rho)}{|{}^S\overline{\nabla}\rho|}}\sqrt{\frac{\overline{h}^{AB}}{|{}^S\overline{\nabla}\rho|}\frac{\partial_A(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}}\frac{\partial_B(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}}}dS\\ \leq &\frac{1}{2}\int_{\partial\Sigma^{\overline{u}}_\tau}-\frac{\overline{h}^{00}}{|{}^S\overline{\nabla}\rho_\tau|}\frac{\left[\partial_0(u_\nu)_{ij,J}\right]^2}{r^{2\alpha-3|J|}}-\frac{2\Omega\overline{h}^{A0}\partial_A(\rho)}{|{}^S\overline{\nabla}\rho_\tau|}\frac{\left[\partial_0(u_\nu)_{ij,J}\right]^2}{r^{2\alpha-3|J|}}dS\\ &+\frac{1}{2}\int_{\partial\Sigma^{\overline{u}}_\tau}\frac{\overline{h}^{AB}}{|{}^S\overline{\nabla}\rho|}\frac{\partial_A(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}}\frac{\partial_B(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}}dS\end{aligned}$$

The remaining term to be estimated is the one on first line in the RHS of (2.5.22), which we rewrite

$$\begin{aligned}&-\int_{\Sigma^{\overline{u}}_\tau}\Omega\left[\overline{h}^{00}\frac{\partial_0^2(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}}+\overline{h}^{AB}\frac{\partial_B\partial_A(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}}\right]\frac{\partial_0(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}}d\mu_{S_{\overline{g}}}\\ =&-\int_{\Sigma^{\overline{u}}_\tau}\left(\overline{h}^{ab}\partial_a\partial_b(u_\nu)_{ij,J}\right)\Omega\frac{\partial_0(u_\nu)_{ij,J}}{r^{2\alpha-3|J|}}d\mu_{S_{\overline{g}}} \qquad (2.5.25)\\ &+\int_{\Sigma^{\overline{u}}_\tau}2\Omega\overline{h}^{A0}\frac{\partial_A\partial_0(u_\nu)_{ij,J}\partial_0(u_\nu)_{ij,J}}{r^{2\alpha-3|J|}}+\Omega h\frac{\Gamma_{\frac{3}{2}}\partial(u_\nu)_{ij,J}\partial_0(u_\nu)_{ij,J}}{r^{2\alpha-3|J|}}d\mu_{S_{\overline{g}}}\end{aligned}$$

By taking the $\partial^{(J)}$ derivative ($J$ spatial multi-index $|J|\leq 2$) of the first equation in (2.5.8) and commuting the differentiation in the LHS we obtain the equation

$$\begin{aligned}&\overline{h}^{ab}\partial_a\partial_b(u_\nu)_{ij,J}\\ =&\,\partial^{(J)}\Big[\overline{O}\Gamma_{\frac{3}{2}}\partial u+\overline{O}\Gamma_3 u+\overline{O}\Gamma_{\frac{9}{2}}(O-I)+\overline{O}\Gamma_3\partial(O-I) \qquad (2.5.26)\\ &+\Gamma_3\overline{u}^2+\overline{O}\overline{u}\partial\overline{u}+\overline{u}^3+\overline{O}\partial(\overline{O}-I)\partial\overline{u}\Big]+[\overline{h}^{ab}\partial_a\partial_b,\partial^{(J)}](u_\nu)_{ij},\end{aligned}$$

where the commutator can in turn be written schematically as: [recall (2.3.15),(2.4.5)]

$$\begin{aligned}[\overline{h}^{ab}\partial_a\partial_b,\partial^{(J)}](u_\nu)_{ij} &= \partial^2(\overline{h})\partial^2(u_\nu)_{ij} + \big[\Gamma_{\frac{3}{2}}\partial(\overline{h}) + \Gamma_3\overline{h}\big]\partial^2(u_\nu)_{ij} \\ &+ \partial(\overline{h})\partial^3(u_\nu)_{ij} + \overline{h}\Gamma_{\frac{3}{2}}\partial^3(u_\nu)_{ij} \\ &+ \big[\partial(\overline{h})\Gamma_3 + \overline{h}\Gamma_{\frac{9}{2}}\big]\partial(u_\nu)_{ij} \qquad \text{if } |J|=2 \\ [\overline{h}^{ab}\partial_a\partial_b,\partial^{(J)}](u_\nu)_{ij} &= \partial(\overline{h})\partial^2(u_\nu)_{ij} + \overline{h}\Gamma_{\frac{3}{2}}\partial^2(u_\nu)_{ij} \qquad \text{if } |J|=1 \\ &+ \overline{h}\Gamma_3\partial(u_\nu)_{ij}\end{aligned} \tag{2.5.27}$$

We integrate by parts in the second term on the RHS of (2.5.25) and argue similarly to (2.5.23) to get

$$\begin{aligned}&\int_{\Sigma_\tau^{\overline{u}}} 2\Omega\overline{h}^{A0}\frac{\partial_A\partial_0(u_\nu)_{ij,J}\partial_0(u_\nu)_{ij,J}}{r^{2\alpha-3|J|}} + \Omega\overline{h}\frac{\Gamma_{\frac{3}{2}}\partial(u_\nu)_{ij,J}\partial_0(u_\nu)_{ij,J}}{r^{2\alpha-3|J|}}d\mu_{s_{\overline{g}}} \\ =&\int_{\partial\Sigma_\tau^{\overline{u}}}\Omega\overline{h}^{A0}\frac{\big[\partial_0(u_\nu)_{ij,J}\big]^2}{r^{2\alpha-3|J|}}N_a dS - \int_{\Sigma_\tau^{\overline{u}}}\Omega\partial_A(\overline{h}^{A0})\frac{\big[\partial_0(u_\nu)_{ij,J}\big]^2}{r^{2\alpha-3|J|}}d\mu_{s_{\overline{g}}} \\ &-\int_{\Sigma_\tau^{\overline{u}}}\overline{h}^{A0}\partial_A(\frac{\Omega}{r^{2\alpha-3|J|}})\big[\partial_0(u_\nu)_{ij,J}\big]^2 d\mu_{s_{\overline{g}}} + \int_{\Sigma_\tau^{\overline{u}}}\Omega\overline{h}\Gamma_{\frac{3}{2}}\frac{\big[\partial_0(u_\nu)_{ij,J}\big]^2}{r^{2\alpha-3|J|}}d\mu_{s_{\overline{g}}} \\ &+\int_{\Sigma_\tau^{\overline{u}}}\Omega\overline{h}\frac{\Gamma_{\frac{3}{2}}\partial(u_\nu)_{ij,J}\partial_0(u_\nu)_{ij,J}}{r^{2\alpha-3|J|}}d\mu_{s_{\overline{g}}} \\ \leq&\int_{\partial\Sigma_\tau^{\overline{u}}}\Omega\overline{h}^{A0}\frac{\big[\partial_0(u_\nu)_{ij,J}\big]^2}{r^{2\alpha-3|J|}}N_A dS + C(\mathcal{E}_0^{\frac{1}{2}}+\alpha^2)E_{3,\alpha}[u] + CE_{3,\alpha+1}[u]\end{aligned} \tag{2.5.28}$$

Finally, for the last and main term in the first line of the RHS of (2.5.25) we recall that $|\Omega| \lesssim \frac{1}{r^{\frac{1}{2}}}$ to obtain directly from Cauchy's inequality

$$\begin{aligned}&-\int_{\Sigma_\tau^{\overline{u}}}\big(\overline{h}^{ab}\partial_a\partial_b(u_\nu)_{ij,J}\big)\Omega\frac{\partial_0(u_\nu)_{ij,J}}{r^{2\alpha-3|J|}}d\mu_{s_{\overline{g}}} \\ \lesssim&\|\frac{\overline{h}^{ab}\partial_a\partial_b(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|-\frac{1}{2}}}\|_{L^2}^2 + \|\frac{\partial_0(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|+1}}\|_{L^2}^2 \\ \lesssim&\|\frac{\overline{h}^{ab}\partial_a\partial_b(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|-\frac{1}{2}}}\|_{L^2}^2 + \|\partial_0(u_\nu)_{ij}\|_{H^{2,\alpha-\frac{1}{2}}}^2\end{aligned} \tag{2.5.29}$$

We proceed by plugging the RHS of (2.5.26) into the first term in the last inequality (2.5.29) above and treat each arising group of terms separately. Employing the basic inequalities in Proposition 2.5.1 along with the bounds of $\overline{O},\partial(\overline{O}),\overline{u}$ (2.5.6), (2.5.7) and (2.5.10) we derive:

$$\begin{aligned}&\|\frac{\partial^{(J)}\big[\overline{O}\Gamma_{\frac{3}{2}}\partial u + \overline{O}\Gamma_3 u\big]}{r^{\alpha-\frac{3}{2}|J|-\frac{1}{2}}}\|_{L^2}^2 \\ \lesssim&\|\overline{O}\|_{L^\infty}^2 E_{3,\alpha+1}[u] + \|\partial(\overline{O})\|_{L^\infty}^2 E_{2,\alpha}[u] + \|u\|_{H^{2,\alpha+1}}^2\end{aligned} \tag{2.5.30}$$

$$
\begin{aligned}
&+\|\frac{\partial^{(J)}(\overline{O})\Gamma_{\frac{3}{2}}\partial u}{r^{\alpha-\frac{3}{2}|J|-\frac{1}{2}}}\|^2_{L^2}+\|\frac{\partial^{(J)}(\overline{O})\Gamma_3 u}{r^{\alpha-\frac{3}{2}|J|-\frac{1}{2}}}\|^2_{L^2}\\
&\lesssim \mathcal{E}(\overline{u},\overline{O};\alpha,T)E_{3,\alpha}[u]+E_{3,\alpha+1}[u]+\big(\|u\|^2_{L^\infty}+\|\partial u\|^2_{L^\infty}\big)\|\frac{\partial^{(J)}(\overline{O})}{r^{\alpha-\frac{1}{2}}}\|^2_{L^2}\\
&\qquad\qquad\text{(the last two terms appear only in the case } |J|=2)\\
&\lesssim \mathcal{E}_0E_{3,\alpha}[u]+E_{3,\alpha+1}[u]
\end{aligned}
$$

$$
\begin{aligned}
&\|\frac{\partial^{(J)}\big[\overline{O}\Gamma_{\frac{9}{2}}(O-I)+\overline{O}\Gamma_3\partial(O-I)\big]}{r^{\alpha-\frac{3}{2}|J|-\frac{1}{2}}}\|^2_{L^2} \qquad (2.5.31)\\
&\lesssim \|O-I\|^2_{H^{3,\alpha+\frac{5}{2}}}+\|\partial\overline{O}\|^2_{L^\infty}\|O-I\|^2_{H^{2,\alpha+\frac{3}{2}}}\\
&\quad+\big(\|\frac{O-I}{r^{\frac{3}{2}}}\|^2_{L^\infty}+\|\partial(O-I)\|^2_{L^\infty}\big)\|\frac{\partial^{(J)}(\overline{O})}{r^{\alpha-\frac{1}{2}}}\|^2_{L^2}\\
&\lesssim \|O-I\|^2_{H^{3,\alpha+\frac{5}{2}}}+(\mathcal{E}_0^2+\mathcal{E}_0)\|O-I\|^2_{H^{2,\alpha+\frac{3}{2}}}+\mathcal{E}_0\|O-I\|^2_{H^{3,\alpha+\frac{3}{2}}}\\
&\qquad\text{(we include the last term only when } |J|=2 \text{ and utilize (2.5.7),(2.5.11))}\\
&\quad+\mathcal{E}_0^3+\mathcal{E}_0\|u\|^2_{H^{2,\alpha}}
\end{aligned}
$$

$$
\begin{aligned}
&\|\frac{\partial^{(J)}\big[\Gamma_3\overline{u}^2+\overline{O}\overline{u}\partial\overline{u}+\overline{u}^3+\overline{O}\partial(\overline{O}-I)\partial\overline{u}\big]}{r^{\alpha-\frac{3}{2}|J|-\frac{1}{2}}}\|^2_{L^2} \qquad (2.5.32)\\
&\lesssim \Big[\|\frac{\overline{u}}{r^{\frac{3}{2}}}\|^2_{L^\infty}+\|\frac{\partial\overline{u}}{r^{\frac{3}{2}}}\|^2_{L^\infty}+\|\partial\overline{O}\|^2_{L^\infty}\big(\|\overline{u}\|^2_{L^\infty}+\|\partial\overline{u}\|^2_{L^\infty}\big)\Big]E_{3,\alpha}[\overline{u}]\\
&\quad+\|\overline{u}\|^2_{L^\infty}\|\partial\overline{u}\|^2_{L^\infty}\|\overline{O}-I\|^2_{H^{2,\alpha+\frac{3}{2}}}+\big(\|\overline{u}\|^4_{L^\infty}+\|\overline{u}\|^2_{L^\infty}\|\partial\overline{u}\|^2_{L^\infty}\big)\|\overline{u}\|^2_{H^{2,\alpha}}\\
&\quad+\|\partial\overline{O}\|^2_{L^\infty}E_{3,\alpha}[\overline{u}]+\|\frac{\partial^2(\overline{O}-I)\partial^2\overline{u}}{r^{\alpha-3-\frac{1}{2}}}\|^2_{L^2}+\|\partial\overline{u}\|^2_{L^\infty}\|\overline{O}-I\|^2_{H^{3,\alpha+\frac{3}{2}}}\\
&\quad+\big(\|\partial\overline{O}\|^4_{L^\infty}+\|\partial\overline{O}\|^2_{L^\infty}\|\partial\overline{u}\|^2_{L^\infty}\big)\big(\|\overline{O}-I\|^2_{H^{2,\alpha+\frac{3}{2}}}+E_{2,\alpha}[\overline{u}]\big)\\
&\lesssim \mathcal{E}(\overline{u},\overline{O};\alpha,T)^2+\mathcal{E}(\overline{u},\overline{O};\alpha,T)^3+\|\frac{\partial^2(\overline{O})}{r^{\frac{\alpha}{2}-\frac{3}{2}-\frac{1}{4}}}\|^2_{L^4}\|\frac{\partial^2(\overline{u})}{r^{\frac{\alpha}{2}-\frac{3}{2}-\frac{1}{4}}}\|^2_{L^4}\\
&\lesssim \mathcal{E}_0^2+\mathcal{E}_0^3+\alpha^3\mathcal{E}_0^2 \qquad \text{(employing the } L^4 \text{ estimate (2.5.5))}
\end{aligned}
$$

By (2.5.30)-(2.5.32) we have the following lemma.

**Lemma 2.5.4.** $\partial_0^2(u_\nu)_{ij}\in C([0,T];H^{1,\alpha-3})\cap L^2([0,T];H^{1,\alpha-2})$ *and moreover the following estimate holds:*

$$
\begin{aligned}
&\|\frac{\partial^{(J)}\partial_0^2(u_\nu)_{ij}}{r^{\alpha-\frac{3}{2}|J|-\frac{1}{2}}}\|^2_{L^2[\tau]}\\
&\lesssim \mathcal{E}_0\big(E_{3,\alpha}[u]+\|O-I\|^2_{H^{3,\alpha+\frac{3}{2}}}\big)+E_{3,\alpha+1}[u]+\|O-I\|^2_{H^{3,\alpha+\frac{5}{2}}} \qquad (2.5.33)\\
&\quad+\alpha^3\mathcal{E}_0^2+\mathcal{E}_0^3,
\end{aligned}
$$

*for* $|J| \le 1,\ J \subset \{1,2,3\},\ \tau \in (0,T)$.

*Proof.* The proof follows by solving for $\partial_0^2 (u_\nu)_{ij}$ in the equation (2.5.8) and summing up the above estimates (2.5.30)-(2.5.32). □

To bound the commutator (2.5.27) we treat the cases $|J| = 2$, $|J| = 1$ separately. For $|J| = 1$:

$$\begin{aligned}
&\|\frac{[\overline{h}^{ab}\partial_a\partial_b, \partial^{(J)}](u_\nu)_{ij}}{r^{\alpha-\frac{3}{2}-\frac{1}{2}}}\|^2_{L^2} \qquad (2.5.34)\\
&= \|\frac{\partial(\overline{h})\partial^2(u_\nu)_{ij} + \overline{h}\Gamma_{\frac{3}{2}}\partial^2(u_\nu)_{ij} + \overline{h}\Gamma_3\partial(u_\nu)_{ij}}{r^{\alpha-\frac{3}{2}-\frac{1}{2}}}\|^2_{L^2}\\
&\lesssim \|\partial\overline{h}\|^2_{L^\infty}\|\frac{\partial^2(u_\nu)_{ij}}{r^{\alpha-\frac{3}{2}-\frac{1}{2}}}\|^2_{L^2} + \|\overline{h}\|^2_{L^\infty}\|\frac{\partial^2(u_\nu)_{ij}}{r^{\alpha-\frac{1}{2}}}\|^2_{L^2} + \|\overline{h}\|^2_{L^\infty}\|\frac{\partial(u_\nu)_{ij}}{r^{\alpha+1}}\|^2_{L^2}\\
&\lesssim \mathcal{E}_0\big(E_{3,\alpha}[u] + \|O-I\|^2_{H^{3,\alpha+\frac{3}{2}}}\big) + E_{3,\alpha+1}[u] + \|O-I\|^2_{H^{3,\alpha+\frac{5}{2}}}\\
&\qquad\qquad \text{(employing (2.5.33) in the case } \partial^2(u_\nu)_{ij} = \partial_0^2(u_\nu)_{ij})\\
&\quad + \alpha^3\mathcal{E}_0^2 + \mathcal{E}_0^3
\end{aligned}$$

When $|J| = 2$ we have

$$\begin{aligned}
&\|\frac{[\overline{h}^{ab}\partial_a\partial_b, \partial^{(J)}](u_\nu)_{ij}}{r^{\alpha-3-\frac{1}{2}}}\|^2_{L^2} \qquad (2.5.35)\\
&\lesssim \|\frac{\partial^2(\overline{h})\partial^2(u_\nu)_{ij} + \big[\Gamma_{\frac{3}{2}}\partial(\overline{h}) + \Gamma_3\overline{h}\big]\partial^2(u_\nu)_{ij}}{r^{\alpha-3-\frac{1}{2}}}\|^2_{L^2}\\
&\quad + \|\frac{\partial(\overline{h})\partial^3(u_\nu)_{ij} + \overline{h}\Gamma_{\frac{3}{2}}\partial^3(u_\nu)_{ij} + \big[\partial(\overline{h})\Gamma_3 + \overline{h}\Gamma_{\frac{9}{2}}\big]\partial(u_\nu)_{ij}}{r^{\alpha-3-\frac{1}{2}}}\|^2_{L^2}\\
&\lesssim \|\frac{\partial^2\overline{h}}{r^{\frac{\alpha}{2}-\frac{3}{2}-\frac{1}{4}}}\|^2_{L^4}\|\frac{\partial^2(u_\nu)_{ij}}{r^{\frac{\alpha}{2}-\frac{3}{2}-\frac{1}{4}}}\|^2_{L^4} + \|\partial\overline{h}\|^2_{L^2}\big(E_{3,\alpha}[u] + \|\partial_0^2(u_\nu)_{ij}\|^2_{H^{1,\alpha-3}}\big)\\
&\qquad\qquad \text{(note that term } \partial^3(u_\nu)_{ij} \text{ contains at most two } \partial_0 \text{ derivatives)}\\
&\quad + \|\overline{h}\|^2_{L^\infty}\big(E_{3,\alpha+1}[u] + \|\partial_0^2(u_\nu)_{ij}\|^2_{H^{1,\alpha-2}}\big)\\
&\lesssim \alpha^3\mathcal{E}_0 E_{3,\alpha}[u] + \mathcal{E}_0 E_{3,\alpha}[u] + E_{3,\alpha+1}[u] + \mathcal{E}_0\|O-I\|^2_{H^{3,\alpha+\frac{3}{2}}} + \|O-I\|^2_{H^{3,\alpha+\frac{5}{2}}}\\
&\qquad\qquad \text{(employing the } L^4 \text{ estimate (2.5.5) and (2.5.33))}\\
&\quad + \alpha^3\mathcal{E}_0^2 + \mathcal{E}_0^3
\end{aligned}$$

**Summary**: Incorporating (2.5.20)-(2.5.35) in (2.5.19) we conclude that

$$\begin{aligned}
&\partial_\tau\int_{\Sigma_\tau^{\overline{u}}} P_{J,\alpha}d\mu s_{\overline{g}} + 8M^2e^{-1}(1-\tau)\alpha\int_{\Sigma_\tau^{\overline{u}}} P_{J,\alpha+1}d\mu s_{\overline{g}}\\
&\lesssim (\mathcal{E}_0^{\frac{1}{2}} + \mathcal{E}_0 + \alpha^2 + \alpha^3\mathcal{E}_0)E_{3,\alpha}[u] + E_{3,\alpha+1}[u] + \|O-I\|^2_{H^{3,\alpha+\frac{5}{2}}} \qquad (2.5.36)\\
&\quad + \mathcal{E}_0\|O-I\|^2_{H^{3,\alpha+\frac{3}{2}}} + \alpha^3\mathcal{E}_0^2 + \mathcal{E}_0^3
\end{aligned}$$

Summing over the indices $\nu, i, j$ and $J$, $|J| \leq 2$, we arrive at the desired estimate (2.5.16). □

*Proof of (2.5.17).* Let $J$, $|J| \leq 3$, be a spatial multi-index. Like in the case of (2.5.19), it follows from the coarea formula and the asymptotics (2.3.5),(2.3.9) that

$$\begin{aligned}
&\frac{1}{2}\partial_\tau \| \frac{O^d_{c,J} - I_c{}^d{}_{,J}}{r^{\alpha+3-\frac{3}{2}|J|}} \|^2_{L^2(\Sigma^{\overline{u}}_\tau)} = -\frac{1}{2}\int_{\partial\Sigma^{\overline{u}}_\tau} \frac{(O^d_{c,J} - I_c{}^d{}_{,J})^2}{|{}^S\overline{\nabla}\rho| r^{2\alpha+6-3|J|}} dS \\
&\quad -(\alpha + 3 - \frac{3|J|}{2}) \int_{\Sigma^{\overline{u}}_\tau} \frac{(O^d_{c,J} - I_c{}^d{}_{,J})^2}{r^{2\alpha+7-3|J|}} \partial_\tau r d\mu_{S_{\overline{g}}} \\
&\quad + \int_{\Sigma^{\overline{u}}_\tau} \Omega \frac{(O^d_{c,J} - I_c{}^d{}_{,J})\partial_0(O^d_{c,J})}{r^{2\alpha+6-3|J|}} d\mu_{S_{\overline{g}}} + \frac{1}{2}\int_{\Sigma^{\overline{u}}_\tau} \frac{(O^d_{c,J} - I_c{}^d{}_{,J})^2}{r^{2\alpha+6-3|J|}} \partial_\tau d\mu_{S_{\overline{g}}} \\
&\leq -4M^2 e^{-1}\alpha(1-\tau) \| \frac{O^d_{c,J} - I_c{}^d{}_{,J}}{r^{\alpha+4-\frac{3}{2}|J|}} \|^2_{L^2(\Sigma^{\overline{u}}_\tau)} \\
&\quad + \int_{\Sigma^{\overline{u}}_\tau} \Omega \frac{(O^d_{c,J} - I_c{}^d{}_{,J})\partial_0(O^d_{c,J})}{r^{2\alpha+6-3|J|}} d\mu_{S_{\overline{g}}} + C \| \frac{O^d_{c,J} - I_c{}^d{}_{,J}}{r^{\alpha+4-\frac{3}{2}|J|}} \|^2_{L^2(\Sigma^{\overline{u}}_\tau)},
\end{aligned} \tag{2.5.37}$$

where $O^d_{c,J} - I_c{}^d{}_{,J} := \partial^{(J)}(O^d_c - I_c{}^d)$. By Cauchy's inequality we have ($\Omega \lesssim \frac{1}{r^{\frac{1}{2}}}$)

$$\int_{\Sigma^{\overline{u}}_\tau} \Omega \frac{(O^d_{c,J} - I_c{}^d{}_{,J})\partial_0(O^d_{c,J})}{r^{2\alpha+6-3|J|}} d\mu_{S_{\overline{g}}} \lesssim \| \frac{O^d_{c,J} - I_c{}^d{}_{,J}}{r^{\alpha+4-\frac{3}{2}|J|}} \|^2_{L^2} + \| \frac{\partial_0(O^d_{c,J})}{r^{\alpha+\frac{5}{2}-\frac{3}{2}|J|}} \|^2_{L^2} \tag{2.5.38}$$

Taking the $\partial^{(J)}$ derivative of the ODE in (2.5.8) we obtain

$$\partial_0(O^d_{c,J} - I_c{}^d{}_{,J}) = \partial^{(J)}\Big[\Gamma_{\frac{3}{2}}(O - I) + (\overline{O} - I)\overline{u} + u\Big] + [\partial_0, \partial^{(J)}](O^d_c - I_c{}^d) \tag{2.5.39}$$

The commutator in the RHS of (2.5.39) schematically reads

$$\begin{aligned}
[\partial_0, \partial^{(J)}](O^d_c - I_c{}^d) &= \Gamma_{\frac{3}{2}}\partial(O^d_c - I_c{}^d) && \text{if } |J| = 1 \\
&= \Gamma_3\partial(O^d_c - I_c{}^d) + \Gamma_{\frac{3}{2}}\partial^2(O^d_c - I_c{}^d) && \text{if } |J| = 2 \\
&= \Gamma_{\frac{9}{2}}\partial(O^d_c - I_c{}^d) + \Gamma_3\partial^2(O^d_c - I_c{}^d) && \text{if } |J| = 3 \\
&\quad + \Gamma_{\frac{3}{2}}\partial^3(O^d_c - I_c{}^d),
\end{aligned} \tag{2.5.40}$$

where we note that at most one $\partial_0$ derivative of $O^d_c - I_c{}^d$ appears in the preceding expressions. Hence, we deduce directly from (2.5.39):

$$\begin{aligned}
&\| \frac{\partial_0(O^d_{c,J} - I_c{}^d{}_{,J})}{r^{\alpha+\frac{5}{2}-\frac{3}{2}|J|}} \|^2_{L^2} \\
&\lesssim \|\Gamma_{\frac{3}{2}}(O - I)\|^2_{H^{3,\alpha+1}} + \|(\overline{O} - I)\overline{u}\|^2_{H^{3,\alpha+1}} + +\|u\|^2_{H^{3,\alpha+1}} \\
&\quad + \| \frac{[\partial_0, \partial^{(J)}](O^d_c - I_c{}^d)}{r^{\alpha+\frac{5}{2}-\frac{3}{2}|J|}} \|^2_{L^2}
\end{aligned} \tag{2.5.41}$$

$$
\begin{aligned}
&\lesssim \|O-I\|^2_{H^{3,\alpha+\frac{5}{2}}} + E_{3,\alpha+1}[u] + \|\overline{O}-I\|^2_{H^{3,\alpha+\frac{3}{2}}}\|\overline{u}\|^2_{H^{3,\alpha}} \\
&\qquad\qquad\text{(exmploying Lemma 2.4.5 and applying the } L^\infty \text{ bound on } (\overline{O}-I)\overline{u}) \\
&\quad + \|\frac{\Gamma_{\frac{3}{2}}\partial(O^d_c - I_c{}^d)}{r^{\alpha+\frac{5}{2}-\frac{3}{2}}}\|^2_{L^2} + \|\frac{\Gamma_3\partial(O^d_c - I_c{}^d) + \Gamma_{\frac{3}{2}}\partial^2(O^d_c - I_c{}^d)}{r^{\alpha+\frac{5}{2}-3}}\|^2_{L^2} \\
&\quad + \|\frac{\Gamma_{\frac{9}{2}}\partial(O^d_c - I_c{}^d) + \Gamma_3\partial^2(O^d_c - I_c{}^d) + \Gamma_{\frac{3}{2}}\partial^3(O^d_c - I_c{}^d)}{r^{\alpha+\frac{5}{2}-\frac{9}{2}}}\|^2_{L^2} \\
&\lesssim \|O-I\|^2_{H^{3,\alpha+\frac{5}{2}}} + \mathcal{E}(\overline{u},\overline{O};\alpha,T)^2 + E_{3,\alpha+1}[u]
\end{aligned}
$$

Combining (2.5.37)-(2.5.41) we derive

$$
\begin{aligned}
&\frac{1}{2}\partial_\tau\|\frac{O^d_{c,J} - I_c{}^d{}_{,J}}{r^{\alpha+3-\frac{3}{2}|J|}}\|^2_{L^2(\Sigma^{\overline{u}}_\tau)} + 4M^2e^{-1}\alpha(1-\tau)\|\frac{O^d_{c,J} - I_c{}^d{}_{,J}}{r^{\alpha+4-\frac{3}{2}|J|}}\|^2_{L^2(\Sigma^{\overline{u}}_\tau)} \\
&\lesssim \|O-I\|^2_{H^{3,\alpha+\frac{5}{2}}} + E_{3,\alpha+1}[u] + \mathcal{E}_0^2
\end{aligned}
\tag{2.5.42}
$$

Taking into account the set of indices $c, d$ and $J$, $|J| \le 3$, we complete the proof of (2.5.17) and hence of Proposition 2.5.3. □

### 2.5.4 Contraction mapping in $H^{2,\alpha}$

We proceed to show that the mapping defined via (2.5.8) in the beginning of §2.5.3 is a contraction. Let us consider another set of spacetime functions $(\tilde{u}_\nu)_{ij}, \tilde{O}^d_c, \tilde{\overline{u}}, \tilde{\overline{O}}$ solving the coupled system analogous to (2.5.8). Setting

$$
(du_\nu)_{ij} = (u_\nu)_{ij} - (\tilde{u}_\nu)_{ij},\quad d\overline{u} = \overline{u} - \tilde{\overline{u}},\quad dO^d_c = O^d_c - \tilde{O}^d_c,\quad d\overline{O} = \overline{O} - \tilde{\overline{O}}
\tag{2.5.43}
$$

we obtain schematically the new system of equations (depicting only the types of terms in the RHS suppressing the particular indices)

$$
\begin{aligned}
&\overline{h}^{ab}\partial_a\partial_b(du_\nu)_{ij} \\
&= \overline{O}\Gamma_{\frac{3}{2}}\partial(du) + \overline{O}\Gamma_3 du + \overline{O}\Gamma_{\frac{9}{2}}dO \\
&\quad + d\overline{O}\big[\Gamma_{\frac{3}{2}}\partial\tilde{u} + \Gamma_3\tilde{u} + \Gamma_{\frac{9}{2}}(\tilde{O}-I) + \Gamma_3\partial\tilde{O}\big] \\
&\quad + \overline{O}\Gamma_3\partial(df) + (\overline{O} + \tilde{\overline{O}})d\overline{O}\partial^2(\tilde{u}_\nu)_{ij} + G(d\overline{u}, d\overline{O}),
\end{aligned}
\tag{2.5.44}
$$

where

$$
\begin{aligned}
G(d\overline{u}, d\overline{O}) &= \Gamma_{\frac{3}{2}}d\overline{u}(\overline{u} + \tilde{\overline{u}} + \overline{u}^2 + \tilde{\overline{u}}^2 + \overline{u}\tilde{\overline{u}}) + \overline{O}\overline{u}\partial(d\overline{u}) \\
&\quad + \overline{O}d\overline{u}\partial\tilde{\overline{u}} + d\overline{O}\tilde{\overline{u}}\partial\tilde{\overline{u}} + \overline{O}\partial(\overline{O})\partial(d\overline{u}) \\
&\quad + \overline{O}\partial(d\overline{O})\partial\tilde{\overline{u}} + d\overline{O}\partial(\tilde{\overline{O}})\partial\tilde{\overline{u}}
\end{aligned}
\tag{2.5.45}
$$

and

$$\partial_0(dO^d_c) = \Gamma_{\frac{3}{2}} dO + (\overline{O} - I)d\overline{u} + \tilde{\overline{u}} d\overline{O} + du \tag{2.5.46}$$

Further, we assume that both sets of variables we have introduced are consistent with the energy estimate (2.5.9) we have established in the previous subsection:

$$\mathcal{E}(u, O; \alpha, T),\ \mathcal{E}(\overline{u}, \overline{O}; \alpha, T),\ \mathcal{E}(\tilde{u}, \tilde{O}; \alpha, T),\ \mathcal{E}(\tilde{\overline{u}}, \tilde{\overline{O}}; \alpha, T) \leq 2\mathcal{E}_0. \tag{2.5.47}$$

*Claim*: For large enough $\alpha > 0$ and $T > 0$ is sufficiently small the following contraction holds:

$$E_{2,\alpha}[du] + \sum_{c,d} \|dO^d_c\|^2_{H^{2,\alpha+\frac{3}{2}}} \leq \kappa\big(E_{2,\alpha}[d\overline{u}] + \sum \|d\overline{O}\|^2_{H^{2,\alpha+\frac{3}{2}}}\big), \tag{2.5.48}$$

for some $0 < \kappa < 1$.

**Remark 2.5.5.** We are forced to close the contraction mapping argument in $H^{2,\alpha}$, having one derivative less than the space of the energy estimate (2.5.47), see §2.5.3, as it is common in $2^{\text{nd}}$-order quasilinear hyperbolic PDE [12], because of the problematic term $(\overline{O} + \tilde{\overline{O}})d\overline{O}\partial^2(\tilde{u}_\nu)_{ij}$ in (2.5.44), which is generated from the difference of the top order terms in the LHS.

**Proposition 2.5.6.** *Under the above considerations, the following estimates hold:*

$$\begin{aligned} &\partial_\tau E_{2,\alpha}[du] + 8M^2 e^{-1}(1-\tau)\alpha E_{2,\alpha+1}[du] \\ &\lesssim (\mathcal{E}_0^{\frac{1}{2}} + \mathcal{E}_0 + \alpha^2)E_{2,\alpha}[du] + E_{2,\alpha+1}[du] + \|dO\|^2_{H^{2,\alpha+\frac{5}{2}}} \\ &\quad + (\mathcal{E}_0 + \mathcal{E}_0^2 + \alpha^3\mathcal{E}_0)\big(E_{2,\alpha}[d\overline{u}] + \|d\overline{O}\|^2_{H^{2,\alpha+\frac{3}{2}}}\big) \end{aligned} \tag{2.5.49}$$

$$\begin{aligned} &\frac{1}{2}\partial_\tau \sum_{c,d} \|dO^d_c\|^2_{H^{2,\alpha+\frac{3}{2}}} + 4M^2 e^{-1}(1-\tau)\alpha \sum_{c,d} \|dO^d_c\|^2_{H^{2,\alpha+\frac{5}{2}}} \\ &\lesssim \|dO\|^2_{H^{2,\alpha+\frac{5}{2}}} + E_{2,\alpha+1}[du] + \mathcal{E}_0\big(E_{2,\alpha}[d\overline{u}] + \|d\overline{O}\|^2_{H^{2,\alpha+\frac{3}{2}}}\big), \end{aligned} \tag{2.5.50}$$

*for all $\tau \in (0, T)$.*

Assuming Proposition 2.5.6 we prove the above claim (2.5.48). After summing (2.5.49),(2.5.50), we absorb into the LHS the critical terms

$$E_{2,\alpha+1}[du], \|dO\|^2_{H^{2,\alpha+\frac{5}{2}}},$$

which appear in the RHS of the above inequalities. This is done by picking the parameter $\alpha$ sufficiently large (but finite). The contraction estimate (2.5.48) then follows from Gronwall's inequality for $T > 0$ suitably small.

*Proof of Proposition 2.5.6.* The proof follows exactly the lines of the proof of Proposition 2.5.3. The only notable difference lies in the estimation of the analogous term to (2.5.29), derived in (2.5.30)-(2.5.35). We sketch the argument in the present situation:

Let $J$ denote at most one spatial index, $|J| \leq 1$, either $1, 2$ or $3$. The main term to be estimated is

$$
\begin{aligned}
&-\int_{\Sigma^{\overline{u}}_{\tau}} \partial^{(J)}\Big[\text{RHS of (2.5.44)}\Big]\Omega\frac{\partial_0(du_\nu)_{ij,J}}{r^{2\alpha-3}}d\mu_{S_{\overline{g}}} \qquad (2.5.51)\\
&\lesssim \|\frac{\partial_0(du_\nu)_{ij,J}}{r^{\alpha-\frac12}}\|^2_{L^2} + \|\frac{\partial^{(J)}\big[\text{RHS of (2.5.44)}\big]}{r^{\alpha-2}}\|^2_{L^2}, \qquad (\text{recall } \Omega \lesssim \tfrac{1}{r^{\frac12}})
\end{aligned}
$$

where $(du_\nu)_{ij,J} := \partial^{(J)}(du_\nu)_{ij}$. Plugging in (2.5.44) and using the basic estimates in Proposition 2.5.1, along with the assumption (2.5.47) we obtain

$$
\begin{aligned}
&\|\frac{\partial^{(J)}\big[\text{RHS of (2.5.44)}\big]}{r^{\alpha-2}}\|^2_{L^2} \qquad (2.5.52)\\
\lesssim{}& \|\frac{\partial^{(J)}\big(\overline{O}\Gamma_{\frac32}\partial(du)+\overline{O}\Gamma_3 du+\overline{O}\Gamma_{\frac92}dO\big)}{r^{\alpha-2}}\|^2_{L^2}\\
&+\|\frac{\partial^{(J)}\big(d\overline{O}\big[\Gamma_{\frac32}\partial\tilde{u}+\Gamma_3\tilde{u}+\Gamma_{\frac92}(\tilde{O}-I)+\Gamma_3\partial\tilde{O}\big]\big)}{r^{\alpha-2}}\|^2_{L^2}\\
&+\|\frac{\partial^{(J)}\big(\overline{O}\Gamma_3\partial(dO)\big)}{r^{\alpha-2}}\|^2_{L^2}+\|\frac{\partial^{(J)}\big[(\overline{O}+\tilde{\overline{O}})d\overline{O}\partial^2(\tilde{u}_\nu)_{ij}\big]}{r^{\alpha-2}}\|^2_{L^2}\\
&+\|\frac{\partial^{(J)}G(d\overline{u},d\overline{O})}{r^{\alpha-2}}\|^2_{L^2}\\
\lesssim{}& \|\frac{\partial^{(J)}\partial(du)}{r^{\alpha-\frac12}}\|^2_{L^2}+\|\frac{\partial(du)}{r^{\alpha+1}}\|^2_{L^2}+\|\frac{du}{r^{\alpha+\frac52}}\|^2_{L^2}+\mathcal{E}_0\big(\|\frac{\partial(du)}{r^{\alpha-\frac12}}\|^2_{L^2}+\|\frac{du}{r^{\alpha+1}}\|^2_{L^2}\big)\\
&\qquad (\text{recall the asymptotics (2.4.5)})\\
&+\|\frac{\partial^{(J)}(dO)}{r^{\alpha+\frac52}}\|^2_{L^2}+\|\frac{dO}{r^{\alpha+4}}\|^2_{L^2}+\mathcal{E}_0\|\frac{dO}{r^{\alpha+\frac52}}\|^2_{L^2}\\
&+\mathcal{E}_0\|\frac{\partial^{(J)}(d\overline{O})}{r^{\alpha+\frac32}}\|^2_{L^2}+\|\frac{d\overline{O}}{r^{\frac32}}\|^2_{L^\infty}\big(\|\tilde{u}\|^2_{H^{2,\alpha}}+\|\tilde{O}-I\|^2_{H^{2,\alpha+\frac32}}\big)\\
&+\|\frac{\partial^{(J)}\partial(dO)}{r^{\alpha+1}}\|^2_{L^2}+\|\frac{\partial(dO)}{r^{\alpha+\frac52}}\|^2_{L^2}+\mathcal{E}_0\|\frac{\partial(dO)}{r^{\alpha+1}}\|^2_{L^2}\\
&+\|\frac{\partial^{(J)}\big[(\overline{O}+\tilde{\overline{O}})d\overline{O}\partial^2(\tilde{u}_\nu)_{ij}\big]}{r^{\alpha-2}}\|^2_{L^2}+\|\frac{\partial^{(J)}G(d\overline{u},d\overline{O})}{r^{\alpha-2}}\|^2_{L^2}\\
\lesssim{}& E_{2,\alpha+1}[du]+\mathcal{E}_0E_{2,\alpha}[du]+\|dO\|^2_{H^{2,\alpha+\frac52}}+\mathcal{E}_0\|dO\|^2_{H^{2,\alpha+\frac32}}\\
&+\mathcal{E}_0\|d\overline{O}\|^2_{H^{2,\alpha+\frac32}}+\|\frac{\partial^{(J)}\big[(\overline{O}+\tilde{\overline{O}})d\overline{O}\partial^2(\tilde{u}_\nu)_{ij}\big]}{r^{\alpha-2}}\|^2_{L^2}+\|\frac{\partial^{(J)}G(d\overline{u},d\overline{O})}{r^{\alpha-2}}\|^2_{L^2}
\end{aligned}
$$

We proceed to the problematic term $(\overline{O}+\tilde{\overline{O}})d\overline{O}\partial^2(\tilde{u}_\nu)_{ij}$ which can be controlled only in $H^1$:

$$
\begin{aligned}
&\|\frac{\partial^{(J)}\big[(\overline{O}+\tilde{\overline{O}})d\overline{O}\partial^2(\tilde{u}_\nu)_{ij}\big]}{r^{\alpha-2}}\|^2_{L^2} \qquad (2.5.53)\\
&\lesssim \|\partial^{(J)}(\overline{O}+\tilde{\overline{O}})\|^2_{L^\infty}\|d\overline{O}\|^2_{L^\infty}\|\frac{\partial^2(\tilde{u}_\nu)_{ij}}{r^{\alpha-2}}\|^2_{L^2}+\|\overline{O}+\tilde{\overline{O}}\|^2_{L^\infty}\|\frac{\partial^{(J)}(d\overline{O})}{r^{\frac{\alpha}{2}-1}}\|^2_{L^4}\\
&\cdot\|\frac{\partial^2(\tilde{u}_\nu)_{ij}}{r^{\frac{\alpha}{2}-1}}\|^2_{L^4}+\|\overline{O}+\tilde{\overline{O}}\|^2_{L^\infty}\|\frac{d\overline{O}}{r}\|^2_{L^\infty}\|\frac{\partial^{(J)}\partial^2(\tilde{u}_\nu)_{ij}}{r^{\alpha-2}}\|^2_{L^2}\\
&\lesssim (\mathcal{E}_0^2+\alpha^3\mathcal{E}_0^2)\|d\overline{O}\|^2_{H^{2,\alpha+\frac{3}{2}}} \qquad \text{(employing the } L^4 \text{ estimate (2.5.5))}
\end{aligned}
$$

Finally, plugging in the nonlinearity (2.5.45), we have the bound

$$
\begin{aligned}
&\|\frac{\partial^{(J)}G(d\overline{u},d\overline{O})}{r^{\alpha-2}}\|^2_{L^2} \qquad (2.5.54)\\
&\lesssim \|\frac{\partial^{(J)}\big(\Gamma_{\frac{3}{2}}d\overline{u}(\overline{u}+\tilde{\overline{u}}+\overline{u}^2+\tilde{\overline{u}}^2+\overline{u}\tilde{\overline{u}})\big)}{r^{\alpha-2}}\|^2_{L^2}+\|\frac{\partial^{(J)}\big(\overline{O}\overline{u}\partial(d\overline{u})\big)}{r^{\alpha-2}}\|^2_{L^2}\\
&+\|\frac{\partial^{(J)}\big(\overline{O}d\overline{u}\partial\tilde{\overline{u}}+d\overline{O}\tilde{\overline{u}}\partial\tilde{\overline{u}}+\overline{O}\partial(\overline{O})\partial(d\overline{u})\big)}{r^{\alpha-2}}\|^2_{L^2}\\
&+\|\frac{\partial^{(J)}\big(\overline{O}\partial(d\overline{O})\partial\tilde{\overline{u}}+d\overline{O}\partial(\tilde{\overline{O}})\partial\tilde{\overline{u}}\big)}{r^{\alpha-2}}\|^2_{L^2}\\
&\lesssim (\mathcal{E}_0+\mathcal{E}_0^2)E_{2,\alpha}[d\overline{u}]+(\mathcal{E}_0+1)\|d\overline{u}\|^2_{L^\infty}\|\frac{\partial^{(J)}\partial\tilde{\overline{u}}}{r^{\alpha-2}}\|^2_{L^2}+\|\frac{\partial^2(\overline{O})\partial(d\overline{u})}{r^{\alpha-2}}\|^2_{L^2}\\
&+(\mathcal{E}_0+\mathcal{E}_0^2)\|d\overline{O}\|^2_{H^{2,\alpha+\frac{3}{2}}}+\|\frac{\partial(d\overline{O})\partial^2\tilde{\overline{u}}}{r^{\alpha-2}}\|^2_{L^2}+\|d\overline{O}\|^2_{L^\infty}\mathcal{E}_0^2\\
&\lesssim (\mathcal{E}_0+\mathcal{E}_0^2)\big(E_{2,\alpha}[d\overline{u}]+\|d\overline{O}\|^2_{H^{2,\alpha+\frac{3}{2}}}\big)+\|\frac{\partial^2\overline{O}}{r^{\frac{\alpha}{2}-1}}\|^2_{L^4}\|\frac{\partial(d\overline{u})}{r^{\frac{\alpha}{2}-1}}\|^2_{L^4}\\
&+\|\frac{\partial(d\overline{O})}{r^{\frac{\alpha}{2}-1}}\|^2_{L^4}\|\frac{\partial^2\tilde{\overline{u}}}{r^{\frac{\alpha}{2}-1}}\|^2_{L^4}\\
&\lesssim (\mathcal{E}_0+\mathcal{E}_0^2+\alpha^3\mathcal{E}_0)\big(E_{2,\alpha}[d\overline{u}]+\|d\overline{O}\|^2_{H^{2,\alpha+\frac{3}{2}}}\big) \qquad \text{(by the } L^4 \text{ estimate (2.5.5))}
\end{aligned}
$$

□

## 2.6 The constraint equations in a singular background of unbounded mean curvature

In this section we prove Theorem 2.1.2, our main stability result for the constraint equations (2.1.5) about the Schwarzschild singular initial data. The proof is an application of the inverse function theorem. Although similar results have been achieved in the smooth case and some rough backgrounds (see [15] for a general exposition), to our knowledge, the singular Schwarzschild background (§2.3) eludes the standard references in the literature.

In order to employ the inverse function theorem we derive suitable weighted, elliptic estimates for the linearized conformal constraint map. We show that it is in general Fredholm between the weighted $H^s$ spaces that we work with and an isomorphism in the case where the initial hypersurface $\Sigma_0$ is contained in sufficiently small neighbourhood of its singularity at $x = 0$. The weighted norms that we use to derive our estimates differ slightly from the ones we use for the hyperbolic part of the problem §2.4.2, §2.5. This is due to the fact that different terms in the resulting system have different leading orders. We are forced to take that into account to obtain useful elliptic estimates.

One of the main difficulties to overcome is the unboundedness of the mean curvature $\mathrm{tr}_{\overline{g}}K$ of the perturbation. In fact, one can check (§2.3) that

$$\mathrm{tr}_{\overline{g}}K \notin L^p(\Sigma_0), \qquad p \geq \frac{5}{3}.$$

The blow up orders of the second fundamental form of $\Sigma_0$ and the mean curvature in particular happen to be the most singular of the curvature terms in the equations. A very useful fact that we exploit is that in certain crucial terms they appear with a favourable sign.

The results in the literature of the constraints using the conformal method are mostly restricted to the constant mean curvature (CMC) or 'near CMC' regime [12]. Recently, there have been a number of advances to the case of large mean curvature, 'far from CMC', [28, 37, 22]. However, due to a smallness assumption on one of the variables, these results can be thought of in a sense as 'near CMC' [25, 38]. All the more, they contain certain regularity assumptions which in particular imply that the mean curvature of the initial data set is in $L^\infty$ and therefore do not directly apply to our case. Although our theorem generates initial data sets for the EVE which have unbounded mean curvature, they are also perturbative in the sense that they are 'close' to the corresponding Schwarzschild induced initial data.

### 2.6.1 The conformal approach; Linearization and stability

We wish to construct initial data sets $(\overline{g}, K)$ on $\Sigma_0 = (-\epsilon, \epsilon)_x \times r^2\mathbb{S}^2$ for the EVE, i.e., solutions to the constraints (2.1.5), which are close to the Schwarzschild induced initial data and asymptote to Schwarzschild at a high order towards the singularity $r = 0$, see Theorem 2.1.1. Recall that the Schwarzschild induced metric on $\Sigma_0$ and its second fundamental form are given by

$$\begin{aligned} {}^S\overline{g} &= \Omega^2 dx^2 + r^2 g_{\mathbb{S}^2}, & \Omega^2 &= \frac{32M^3}{r}e^{-\frac{r}{2M}}, \;\; r^2 \sim 8M^2x^2 \\ {}^SK_{11} &= -\frac{1}{2}\frac{\Omega}{4Mr} + l.o.t., & {}^SK_{22} &= {}^SK_{33} = \frac{\Omega}{4Mr}, \end{aligned} \tag{2.6.1}$$

where

$$\partial_1 = \frac{1}{\Omega}\partial_x \qquad \partial_2 = \frac{1}{r}\partial_\theta, \;\; \partial_3 = \frac{1}{r\sin\theta}\partial_\phi. \tag{2.6.2}$$

All derivations below involving spatial indices are carried out using the Schwarzschild frame (2.6.2). We will look for solutions of (2.1.5) of the form

$$\overline{g} = \varphi^{4S}\overline{g} \qquad K_{ij} = \varphi^{-2}(\sigma_{ij} + LW_{ij}) + \frac{1}{3}\varphi^{4S}\overline{g}_{ij}\chi, \tag{2.6.3}$$

where

$$LW_{ij} := {}^S\overline{\nabla}_i W_j + {}^S\overline{\nabla}_j W_i - \frac{2}{3}{}^S\overline{g}_{ij}{}^S\overline{\nabla}^k W_k \tag{2.6.4}$$

The conformal data in this set-up consist of a scalar $\chi$ and a symmetric traceless and transverse (TT) 2-tensor $\sigma$. Then the constraint equations reduce to an elliptic system of equations ([12]) for the conformal factor $\varphi$ and the vector field $W$:

$$\begin{aligned} {}^S\overline{\nabla}^j LW_{ij} - \frac{2}{3}\phi^{6S}\overline{\nabla}_i\chi = 0 \\ -{}^S\overline{\Delta}\phi + \frac{1}{8}{}^S\overline{\mathrm{R}}\phi - \frac{1}{8}|\sigma + LW|^2\phi^{-7} + \frac{1}{12}\chi^2\phi^5 = 0 \end{aligned} \tag{2.6.5}$$

We prefer to analyse the top order term in the first equation of (2.6.5) after commuting derivatives

$${}^S\overline{\nabla}^j LW_{ij} = ({}^S\overline{\Delta}W)_i + \frac{1}{3}{}^S\overline{\nabla}_i({}^S\overline{\nabla}^k W_k) + {}^S\overline{\mathrm{Ric}}_i{}^j W_j \tag{2.6.6}$$

It is easy to see that the Schwarzschild induced initial data on $\Sigma_0$ can be parametrized in this fashion choosing

$${}^S\varphi = 1, \;\; \sigma = 0, \;\; \chi = \mathrm{tr}_{{}^S\overline{g}}{}^S K, \;\; {}^S W = \mathrm{grad}_{{}^S\overline{g}} f(x), \;\; f(x) \sim \frac{a}{\sqrt{|x|}}, \tag{2.6.7}$$

where $f$ is a spherically symmetric function on $\Sigma_0$ solving the ODE[19]

$$\frac{4}{3}\partial_1^3 f + \frac{8}{3}\frac{\Omega}{4Mr}x\cdot\partial_1^2 f + \frac{3}{2}\frac{\Omega^2}{16M^2r^2}x + \frac{1}{2}\frac{\Omega^2}{32M^3r}x = 0, \qquad \partial_1 \sim \sqrt{|x|}\partial_x. \tag{2.6.8}$$

*We now fix* $\chi = \mathrm{tr}_{{}^S\overline{g}}{}^S K, \sigma = 0$:
Setting $Y = W - {}^S W, \eta = \phi - 1$, the linearization of the system (2.6.5) about $Y = 0, \eta = 0$ with inhomogeneous terms $Z, h$ reads

$$\begin{aligned} ({}^S\overline{\Delta}Y)_1 + \frac{1}{3}{}^S\overline{\nabla}_1({}^S\overline{\nabla}^k Y_k) + {}^S\mathrm{R}_{11}Y_1 - 4({}^S\overline{\nabla}_1\mathrm{tr}_{{}^S\overline{g}}{}^S K)\eta = Z_1 \\ ({}^S\overline{\Delta}Y)_i + \frac{1}{3}{}^S\overline{\nabla}_i({}^S\overline{\nabla}^k Y_k) + {}^S\mathrm{R}_{ii}Y_i = Z_i, \;\; i = 2,3 \\ -{}^S\overline{\Delta}\eta + \frac{1}{8}{}^S\overline{\mathrm{R}}\eta + \frac{7}{8}|{}^S LW|^2\eta + \frac{5}{12}(\mathrm{tr}_{{}^S\overline{g}}{}^S K)^2\eta - \frac{1}{2}{}^S LW^{ij\,S}\overline{\nabla}_j Y_i = h \end{aligned} \tag{2.6.9}$$

[19]The first equation in (2.6.5) for $i = 1$ reduces to (2.6.8) in spherical symmetry, whereas the $i = 2,3$ parts of the vector equation for $W$ are automatically satisfied.

Recall (2.3.11) to compute the leading asymptotics, as $x \to 0$, of the (singular Schwarzschild) coefficients of (2.6.9):

$$
\begin{aligned}
{}^S\mathrm{R}_{11} = -\frac{1}{4Mr} + O(1), \qquad {}^S\mathrm{R}_{22} = {}^S\mathrm{R}_{33} = \frac{1}{r^2} + O(\frac{1}{r}), \qquad ({}^S\overline{\nabla}_1 \mathrm{tr}_{{}^S\overline{g}}{}^S K) \sim \frac{c}{r^2} \\
{}^S\overline{\mathrm{R}} = \frac{2}{r^2} + O(\frac{1}{r}), \qquad (\mathrm{tr}_{{}^S\overline{g}}{}^S K)^2 = \frac{9}{4}\frac{\Omega^2}{16M^2r^2}, \qquad \Omega^2 = \frac{32M^3}{r}e^{-\frac{r}{2M}} \qquad (2.6.10) \\
r^2 \sim 8M^2x^2, \qquad {}^S LW_{11} = -\frac{\Omega}{4Mr}, \qquad {}^S LW_{22} = {}^S LW_{33} = \frac{1}{2}\frac{\Omega}{4Mr}.
\end{aligned}
$$

**Remark 2.6.1.** Observe that the most singular coefficients in (2.6.9) are of order $r^{-3}$ and they correspond to the zeroth order terms of the third equation. Fortunately they come with a good sign. This fact plays a crucial role in the analysis below.

In one's effort to derive elliptic estimates for (2.6.9), one encounters an obstruction related to the presence of conformal Killing vector fields on the spheres, which prevent one from obtaining coercive estimates. We choose to overcome this obstruction by imposing a reflection symmetry about the centre of the spheres

$$u(x, \pi - \theta, \phi + \pi) = u(x, \theta, \phi) \qquad ant^*(X) = X, \tag{2.6.11}$$

where $ant$ is the antipodal map $(\theta, \phi) \to (\pi - \theta, \phi + \pi)$ and $ant^*$ its pullback, for all scalar functions $u$ and vector fields $X$ appearing in the system (2.6.5) and its linearization (2.6.9). Notice that we are free to make such an assumption, since it is valid for the background Schwarzschild metric in (2.6.5) which is by consideration (2.6.3) the conformal class of the metrics we are solving for. Hence, the condition (2.6.11) we impose 'eliminates' the spherical conformal Killing vector fields along with the whole odd parity part of the above variables. This assumption however is unwanted and we expect that there is a better way to solve the issue we raised, but we do not pursue it further here.
We make use of the assumption (2.6.11) below only when we employ the following Poincaré inequality for $\mathbb{S}^2$ vector fields.

**Lemma 2.6.2.** *Let $X$ be a vector field on $\mathbb{S}^2$ satisfying*

$$ant^*(X) = X, \qquad ant(\theta, \phi) = (\pi - \theta, \phi + \pi). \tag{2.6.12}$$

*Then*

$$\int_{\mathbb{S}^2} |\nabla X|^2 \geq 5 \int_{\mathbb{S}^2} |X|^2 d\mu_{\mathbb{S}^2}. \tag{2.6.13}$$

*Proof.* The inequality

$$\int_{\mathbb{S}^2} |\nabla X|^2 d\mu_{\mathbb{S}^2} \geq \int_{\mathbb{S}^2} |X|^2 d\mu_{\mathbb{S}^2} \tag{2.6.14}$$

is standard and valid for all spherical vector fields. Moreover, equality in (2.6.14) is achieved if and only if $X$ is conformal Killing. The condition (2.6.12) implies that $X$ is orthogonal to the space of conformal Killing vector fields. In that case, using vector spherical harmonics, we readily deduce that

$$\int_{\mathbb{S}^2} |\nabla X|^2 \geq (\lambda_2 - 1) \int_{\mathbb{S}^2} |X|^2 d\mu_{\mathbb{S}^2} \tag{2.6.15}$$

where $\lambda_2 = 6$ is the second eigenvalue of $-\Delta_{\mathbb{S}^2}$, as required. □

We exploit the fact that the Schwarzschild background is spherically symmetric and split the variables $\phi, \eta, W_1, Y_1, Z_1$ into

$$\begin{aligned} \phi &= \phi_0 + \phi_1 & W_1 &= W_{10} + W_{11} \\ \eta &= \eta_0 + \eta_1 & Y_1 &= Y_{10} + Y_{11} \\ h &= h_0 + h_1 & Z_1 &= Z_{10} + Z_{11}, \end{aligned} \tag{2.6.16}$$

where $\phi_0, \eta_0, W_{10}, Y_{10}, Z_{10}$ are the spherically symmetric parts of the corresponding functions. We also use the notation

$$W^\top := W - W_1\partial_1 \qquad\qquad Y^\top := Y - Y_1\partial_1 \tag{2.6.17}$$

for the tangential part of the vector fields on the spheres.
We proceed now to define the weighted $H^s$ spaces we are going to work with:

$$\begin{aligned} H^{s,\alpha}_{\text{vf-0}} : &\quad v \in H^s \quad \& \quad \sum_{|j|\leq s} \int_{\Sigma_0} \frac{({}^S\overline{\nabla}^{(j)}_1 v)^2}{|x|^{2\alpha - |j| + 1}} d\mu_{{}^S\overline{g}} < +\infty \\ H^{s,\alpha}_{\text{vf-1}} : &\quad v \in H^s \quad \& \quad \sum_{|j|+|k|\leq s} \int_{\Sigma_0} \frac{({}^S\overline{\nabla}^{(j)}_1 \nabla\!\!\!/^{(k)}_l v)^2}{|x|^{2\alpha - 2(|j|+|k|-1)}} d\mu_{{}^S\overline{g}} < +\infty \\ H^{s,\alpha}_{\text{sc}} : &\quad v \in H^s \quad \& \quad \sum_{|j|+|k|\leq s} \int_{\Sigma_0} \frac{({}^S\overline{\nabla}^{(j)}_1 \nabla\!\!\!/^{(k)} v)^2}{|x|^{2\alpha - 3(|j|+|k|-1)}} d\mu_{{}^S\overline{g}} < +\infty, \end{aligned} \tag{2.6.18}$$

where $\nabla\!\!\!/$ denotes the covariant differentiation on the sphere $r^2\mathbb{S}^2$ and $v$ is either a scalar or a vector field satisfying (2.6.11).

**Remark 2.6.3.** The precise ordering of the above derivatives does not matter since the Schwarzschild connection coefficients ${}^S(A_\mu)_{jk} = O(|x|^{-\frac{1}{2}})$, $1 \in \{\mu jk\}$, see (2.3.11). Also, note that we can use either covariant or non-covariant differentiation since ${}^S(A_\mu)_{jk} = O(|x|^{-1})$ for all indices, ${}^S(A_1)_{jk} = 0$, and thus the extra terms arising from the various ${}^S(A_\mu)_{jk}$'s can be incorporated in the norms.

Define the operator

$$\begin{aligned} \Psi(W_{10}-{}^{S}W_{10},W_{11},W^{\top},\phi-1,\chi-\mathrm{tr}_{{}^{S}\overline{g}}{}^{S}K,\sigma):\\ H^{4,\alpha}_{\text{vf-0}}\times\left(H^{4,\alpha}_{\text{vf-1}}\right)^2\times H^{4,\alpha}_{\text{sc}}\times\mathcal{B}_{\chi,\sigma}\to H^{2,\alpha-1}_{\text{vf-0}}\times\left(H^{2,\alpha-2}_{\text{vf-1}}\right)^2\times H^{2,\alpha-3}_{\text{sc}},\\ \Psi=(\text{LHS of the system (2.6.5)}), \end{aligned} \tag{2.6.19}$$

where $\mathcal{B}_{\chi,\sigma}$ can be any of the above spaces of sufficiently high regularity with similar weights.[20]

**Lemma 2.6.4.** *$\Psi$ is well-defined, bounded and $C^1$ (Fréchet).*

*Proof.* Express $\Psi$ as differences of the variables $\phi-1,\chi-\mathrm{tr}_{{}^{S}\overline{g}}{}^{S}K$ etc. The boundedness of $\Psi$ then follows by applying Sobolev embedding to the arising non-linear terms, see (2.5.4), and by controlling the linear terms, which can be read from the linearized system (2.6.9), in the weighted $H^s$ norms (2.6.18) that where carefully defined to for this exact purpose. The same argument actually implies that $\Psi$ is $C^1$. □

By definition we have

$$\begin{aligned} D\Psi_{(W_{10}-{}^{S}W_{10},W_{11},W^{\top},\phi-1)}(\mathbf{0})(Y_{10},Y_{11},Y^{\top},\eta)=:D\Psi(Y_{10},Y_{11},Y^{\top},\eta):\\ H^{4,\alpha}_{\text{vf-0}}\times\left(H^{4,\alpha}_{\text{vf-1}}\right)^2\times H^{4,\alpha}_{\text{sc}}\to H^{2,\alpha-1}_{\text{vf-0}}\times\left(H^{2,\alpha-2}_{\text{vf-1}}\right)^2\times H^{2,\alpha-3}_{\text{sc}},\\ D\Psi=(\text{LHS of (2.6.9)}) \end{aligned} \tag{2.6.20}$$

**Proposition 2.6.5.** *The bounded operator*

$$D\Psi:\left[H^{4,\alpha}_{\textit{vf-0}}\times\left(H^{4,\alpha}_{\textit{vf-1}}\right)^2\times H^{4,\alpha}_{\textit{sc}}\right]\cap H^1_0\to H^{2,\alpha-1}_{\textit{vf-0}}\times\left(H^{2,\alpha-2}_{\textit{vf-1}}\right)^2\times H^{2,\alpha-3}_{\textit{sc}} \tag{2.6.21}$$

*is Fredholm, i.e., it has finite dim kernel and cokernel, for any $\alpha$ sufficiently large, consistent with Theorem 2.4.8. In the case where $\Sigma_0$ is contained in a sufficiently small neighbourhood of $x=0$, $D\Psi$ is in fact an isomorphism.*

We postpone the proof Proposition 2.6.5 for §2.6.2 and proceed to formulate our stability result for the constraints.

**Theorem 2.6.6.** *Let $\alpha$ be sufficiently large, given by Theorem 2.4.8. Also, let $\Sigma_0=(-\epsilon,\epsilon)_x\times r^2\mathbb{S}^2$ be an initial singular hypersurface for $\epsilon$ sufficiently small such that the second part of Proposition 2.6.5 is valid. Then for any $\chi-tr_{{}^{S}\overline{g}}{}^{S}K,\sigma\in H^{3,\alpha}$ with sufficiently small norms, subject to (2.6.11), there exists a solution to the conformal constraint equations (2.6.5) in the spaces*

$$\begin{aligned} (W_{10}-{}^{S}W_{10},W_{11},W^{\top},\phi-1)\in H^{4,\alpha}_{\textit{vf-0}}\times\left(H^{4,\alpha}_{\textit{vf-1}}\right)^2\times H^{4,\alpha}_{\textit{sc}}\\ W-{}^{S}W,\phi-1\in H^1_0 \end{aligned} \tag{2.6.22}$$

[20] Owing to §2.4, $\chi-\mathrm{tr}_{{}^{S}\overline{g}}{}^{S}K,\sigma\in H^{3,\alpha}$ would be fine.

*In particular, the pairs $(\overline{g}, K)$ given by (2.6.3) verify the constraints (2.1.5) and the assumptions of Theorem 2.4.8.*

*Proof.* The main assertion regarding the solution to the conformal constraint equations follows from the inverse function theorem, since $D\Psi$ (2.6.21) is an isomorphism, the level set $\Psi^{-1}(\{0\})$ is the set of solutions to (2.6.5) and $\Psi(0) = 0$. Although the domain of $\Psi$ is slightly different from the space of initial data sets in Theorem 2.4.8, picking $\alpha$ larger than required, we can ensure that the pairs $(\overline{g}, K)$ we construct in this section, given by (2.6.3), satisfy the initial conditions in Theorem 2.4.8. □

### 2.6.2 Proof of Proposition 2.6.5

We derive elliptic estimates for $D\Psi$ in the spaces (2.6.18) defined earlier. The system

$$D\Psi(\eta, Y) = (h, Z). \tag{2.6.23}$$

is by definition (2.6.9). Recall briefly the definitions (2.6.16),(2.6.17) and let

$$\tilde{Y} = Y_{11}\partial_1 + Y^\top \qquad \tilde{Z} = Z_{11}\partial_1 + Z^\top \tag{2.6.24}$$

Then it is easy to see that (2.6.23) reduces to two systems, one for $\tilde{Y}, \eta_1, \tilde{Z}, h_1$: [which we write by replacing the singular coefficients (2.6.10) with their leading orders, recall $r^2 \sim 8M^2x^2$]

$$\begin{aligned} ({}^S\overline{\Delta}\tilde{Y})_1 + \frac{1}{3}{}^S\overline{\nabla}_1({}^S\overline{\nabla}^k\tilde{Y}_k) - \frac{1}{4Mr}\tilde{Y}_1 + O(\frac{1}{|x|^2})\eta_1 = \tilde{Z}_1 \\ ({}^S\overline{\Delta}\tilde{Y})_i + \frac{1}{3}\nabla\!\!\!/({}^S\overline{\nabla}^k\tilde{Y}_k) + \frac{1}{r^2}\tilde{Y}_i = \tilde{Z}_i, \;\; i = 2,3 \\ -{}^S\overline{\Delta}\eta_1 + \frac{b}{|x|^3}\eta_1 + O(\frac{1}{|x|^{\frac{3}{2}}}){}^S\overline{\nabla}_k\tilde{Y}_j = h_1, \;\; b > 0 \end{aligned} \tag{2.6.25}$$

and one for the spherically symmetric parts of the variables $\eta_0, Y_{10}, h_0, Z_{10}$

$$\begin{aligned} \partial_1^2 Y_{10} + O(\frac{1}{|x|^{\frac{1}{2}}})\partial_1 Y_{10} - \frac{1}{4Mr}Y_{10} + O(\frac{1}{|x|^2})\eta_0 = Z_{10} \\ -\partial_1^2\eta_0 + O(\frac{1}{|x|^{\frac{1}{2}}})\partial_1\eta_0 + \frac{b}{|x|^3}\eta_0 + O(\frac{1}{|x|^{\frac{3}{2}}})\partial_1 Y_{10} + O(\frac{1}{|x|^2})Y_{10} = h_0, \;\; b > 0, \end{aligned} \tag{2.6.26}$$

where $O(|x|^a)$ denotes a smooth function, $x \neq 0$, satisfying $\partial_x^{(j)} O(|x|^a) = O(|x|^{a-|j|})$. Recall that $\partial_1 \sim \sqrt{|x|}\partial_x$.

Note that the zeroth order term $-\frac{1}{4Mr}Y_{10}$ in the first equation of (2.6.26) has a favourable sign, but it is one order weaker than the favourable term coming from the sphere Laplacian in the equations, see also (2.6.30) below. This fact forces us to treat the spherically symmetric part of $Y$ separately.

**Proposition 2.6.7.** *[A priori elliptic estimate I] Assume* $h_1 \in H^{2,\alpha-3}_{sc}, \tilde{Z} \in H^{2,\alpha-2}_{vf\text{-}1}$, *and* $\eta_1 \in H^{4,\alpha}_{sc} \cap H^1_0, \tilde{Y} \in H^{4,\alpha}_{vf\text{-}1} \cap H^1_0$ *solving (2.6.25). Then the following estimate holds:*

$$\|\eta_1\|^2_{H^{4,\alpha}_{sc}} + \|\tilde{Y}\|^2_{H^{4,\alpha}_{vf\text{-}1}} \lesssim \|h_1\|^2_{H^{2,\alpha-3}_{sc}} + \|\tilde{Z}\|^2_{H^{2,\alpha-2}_{vf\text{-}1}} + \|\eta_1\|^2_{L^2} + \|\tilde{Y}\|^2_{L^2} \tag{2.6.27}$$

*If in addition* $x \in (-\epsilon, \epsilon)$, $\epsilon > 0$ *sufficiently small (how small depending on the coefficients of the system (2.6.25) and* $\alpha$*), then (2.6.27) can be improved to*

$$\|\eta_1\|^2_{H^{4,\alpha}_{sc}} + \|\tilde{Y}\|^2_{H^{4,\alpha}_{vf\text{-}1}} \lesssim \|h_1\|^2_{H^{2,\alpha-3}_{sc}} + \|\tilde{Z}\|^2_{H^{2,\alpha-2}_{vf\text{-}1}} \tag{2.6.28}$$

*Proof.* We will employ the following inequality for $Y^\top$ that follows from Lemma 2.6.2:[21]

$$\int_{\Sigma_0} \frac{|\overline{\nabla\!\!\!/} Y^\top|^2}{|x|^{2\alpha}} d\mu_{S_{\overline{g}}} \geq 5 \int_{\Sigma_0} \frac{|Y^\top|^2}{r^2|x|^{2\alpha}} d\mu_{S_{\overline{g}}} \tag{2.6.29}$$

and the standard one for $\tilde{Y}_1$

$$\int_{\Sigma_0} \frac{|\overline{\nabla\!\!\!/} \tilde{Y}_1|^2}{|x|^{2\alpha}} d\mu_{S_{\overline{g}}} \geq 2 \int_{\Sigma_0} \frac{\tilde{Y}_1^2}{r^2|x|^{2\alpha}} d\mu_{S_{\overline{g}}}, \tag{2.6.30}$$

which is immediate by definition (2.6.16), (2.6.24). Multiplying (2.6.25) with $\frac{1}{\varepsilon}\frac{\tilde{Y}_i}{x^{2\alpha}}, \frac{\eta_1}{x^{2\alpha}}$ respectively, subtracting the first two equations from the last and integrating by parts we arrive at the inequality

$$\begin{aligned}
&\int_{\Sigma_0} \frac{|{}^S\overline{\nabla}\eta_1|^2}{|x|^{2\alpha}} + \frac{1}{\varepsilon}\sum_{i=1}^{3} \frac{|{}^S\overline{\nabla}\tilde{Y}|^2}{|x|^{2\alpha}} + \frac{1}{\varepsilon}\frac{\tilde{Y}_1^2}{4Mr|x|^{2\alpha}} + \frac{\eta_1^2}{|x|^{2\alpha+3}} d\mu_{S_{\overline{g}}} \\
\leq &\int_{\Sigma_0} C\alpha\frac{|\eta_1\partial_1\eta_1|}{|x|^{2\alpha+\frac{1}{2}}} + \frac{1}{\varepsilon}C\alpha\frac{|\tilde{Y}{}^S\overline{\nabla}_1\tilde{Y}|}{|x|^{2\alpha+\frac{1}{2}}} + \frac{1}{\varepsilon}C\frac{|\eta_1\tilde{Y}_1|}{|x|^{2\alpha+2}} + \frac{1}{\varepsilon}\frac{|Y^\top|^2}{r^2|x|^{2\alpha}} \qquad (\partial_1 \sim \sqrt{|x|}\partial_x) \\
&+ C\frac{|\eta_1{}^S\overline{\nabla}_j\tilde{Y}_i|}{|x|^{2\alpha+\frac{3}{2}}} + \frac{h_1\eta_1}{|x|^{2\alpha}} - \frac{1}{\varepsilon}\frac{\tilde{Z}_i\tilde{Y}^i}{|x|^{2\alpha}} d\mu_{S_{\overline{g}}} \\
\leq &\int_{\Sigma_0} \frac{1}{2}\frac{(\partial_1\eta_1)^2}{|x|^{2\alpha}} + C\alpha^2\frac{\eta_1^2}{|x|^{2\alpha+1}} + \frac{1}{2}\frac{|{}^S\overline{\nabla}_1\tilde{Y}|^2}{|x|^{2\alpha}} + C\alpha^2\frac{|\tilde{Y}|^2}{|x|^{2\alpha+1}} + \frac{C}{\varepsilon}\frac{\eta_1^2}{|x|^{2\alpha+2}} \\
&+ \frac{1}{4\varepsilon}\frac{\tilde{Y}_1^2}{r^2|x|^{2\alpha}} + \frac{1}{\varepsilon}\frac{|Y^\top|^2}{r^2|x|^{2\alpha}} + \frac{1}{2}\frac{\eta_1^2}{|x|^{2\alpha+3}} + C\frac{|{}^S\overline{\nabla}_j\tilde{Y}_i|^2}{|x|^{2\alpha}} \\
&+ \frac{1}{4}\frac{\eta_1^2}{|x|^{2\alpha+3}} + C\frac{h_1^2}{|x|^{2\alpha-3}} + \frac{1}{4\varepsilon}\frac{|\tilde{Y}|^2}{|x|^{2\alpha+2}} + \frac{1}{\varepsilon}\frac{|\tilde{Z}|^2}{|x|^{2\alpha-2}} d\mu_{S_{\overline{g}}}
\end{aligned} \tag{2.6.31}$$

Now the desired estimate at the level of $H^1$ (i.e., the parts of the relevant norms that depend on

[21] We actually only need a constant strictly larger than one in the following two inequalities to make our argument work.

up to one derivative of $\eta_1,\tilde{Y}$) follows by utilizing the inequalities (2.6.29),(2.6.30), comparing the powers in the denominators on both sides and taking $\varepsilon>0$ sufficiently small. If $|x|\ll 1$, then it is easy to see that all weighted $\eta_1,\tilde{Y}$ terms can be absorbed in the LHS. The full $H^{4,\alpha}$ estimate is obtained by using (2.6.25), differentiating the system in the spatial directions and applying a similar procedure. We only derive the second order estimate: Multiply the system (2.6.25) with $\frac{1}{\varepsilon}\frac{{}^S\overline{\nabla}_{22}\tilde{Y}}{|x|^{2\alpha-2}},\frac{{}^S\overline{\nabla}_{22}\eta_1}{|x|^{2\alpha-3}}$, integrate over $\Sigma_0$, subtract the first two equations from the third one and integrate by parts twice to deduce

$$\begin{aligned}
&\int_{\Sigma_0}\frac{|{}^S\overline{\nabla}{}^S\overline{\nabla}_2\eta_1|^2}{|x|^{2\alpha-3}}+\frac{1}{\varepsilon}\sum_{i=1}^3\frac{|{}^S\overline{\nabla}{}^S\overline{\nabla}_2\tilde{Y}|^2}{|x|^{2\alpha-2}}+\frac{1}{\varepsilon}\frac{|{}^S\overline{\nabla}_2\tilde{Y}_1|^2}{4Mr|x|^{2\alpha-2}}+\frac{|{}^S\overline{\nabla}_2\eta_1|^2}{|x|^{2\alpha}}d\mu_{S_{\overline{g}}} \qquad (2.6.32)\\
&\leq\int_{\Sigma_0}C\alpha\frac{|{}^S\overline{\nabla}_{21}\eta_1{}^S\overline{\nabla}_2\eta_1|}{|x|^{2\alpha-3+\frac12}}+\frac{|{}^S\overline{\mathrm{Ric}}{}^S\overline{\nabla}\eta_1{}^S\overline{\nabla}_2\eta_1|}{|x|^{2\alpha-3}}+\frac{C}{\varepsilon}\alpha\frac{|{}^S\overline{\nabla}_{21}\tilde{Y}{}^S\overline{\nabla}_2\tilde{Y}|}{|x|^{2\alpha-2+\frac12}}\\
&+\frac{1}{\varepsilon}\frac{|{}^S\overline{\mathrm{Ric}}{}^S\overline{\nabla}\tilde{Y}{}^S\overline{\nabla}_2\tilde{Y}|}{|x|^{2\alpha-2}}+\frac{1}{\varepsilon}\frac{1}{3}\frac{\overline{\nabla}\!\!\!/({}^S\overline{\nabla}^k\tilde{Y}_k){}^S\overline{\nabla}_{22}\tilde{Y}}{|x|^{2\alpha-2}}\\
&+\frac{1}{\varepsilon}\frac{1}{3}\frac{{}^S\overline{\nabla}_1(\overline{\nabla}\!\!\!/^k\tilde{Y}_k){}^S\overline{\nabla}_{22}\tilde{Y}}{|x|^{2\alpha-2}}+\frac{C}{\varepsilon}\frac{|\eta_1{}^S\overline{\nabla}_{22}\tilde{Y}_1|}{|x|^{2\alpha}}+C\frac{|{}^S\overline{\nabla}_j\tilde{Y}_i{}^S\overline{\nabla}_{22}\eta_1|}{|x|^{2\alpha-\frac32}}\\
&-\frac{h_1{}^S\overline{\nabla}_{22}\eta_1}{|x|^{2\alpha-3}}+\frac{1}{\varepsilon}\frac{\tilde{Z}_i{}^S\overline{\nabla}_{22}\tilde{Y}^i}{|x|^{2\alpha-2}}d\mu_{S_{\overline{g}}}\\
&\leq\int_{\Sigma_0}\frac12\frac{|{}^S\overline{\nabla}_{21}\eta_1|^2}{|x|^{2\alpha-3}}+C\alpha^2\frac{|{}^S\overline{\nabla}_2\eta_1|^2}{|x|^{2\alpha-2}}+C\frac{|{}^S\overline{\nabla}\eta_1|^2}{|x|^{2\alpha-1}}+\frac{1}{2\varepsilon}\frac{|{}^S\overline{\nabla}_{21}\tilde{Y}|^2}{|x|^{2\alpha-2}} \qquad (|{}^S\overline{\mathrm{Ric}}|\sim r^{-2})\\
&+\frac{C}{\varepsilon}\alpha^2\frac{|{}^S\overline{\nabla}_2\tilde{Y}|^2}{|x|^{2\alpha-1}}+\frac{C}{\varepsilon}\frac{|{}^S\overline{\nabla}\tilde{Y}|^2}{|x|^{2\alpha}}+\frac{1}{4\varepsilon}\frac{|{}^S\overline{\nabla}_{22}\tilde{Y}|^2}{|x|^{2\alpha-2}}+\frac{1}{9\varepsilon}\frac{|\overline{\nabla}\!\!\!/{}^S\overline{\nabla}\tilde{Y}|^2}{|x|^{2\alpha-2}}\\
&+\frac{1}{4\varepsilon}\frac{|{}^S\overline{\nabla}_{22}\tilde{Y}|^2}{|x|^{2\alpha-2}}+\frac{1}{9\varepsilon}\frac{|{}^S\overline{\nabla}_1\overline{\nabla}\!\!\!/\tilde{Y}|^2}{|x|^{2\alpha-2}}+C\frac{\eta_1^2}{|x|^{2\alpha+2}}+\frac{1}{20\varepsilon}\frac{|{}^S\overline{\nabla}_{22}\tilde{Y}|^2}{|x|^{2\alpha-2}}\\
&+C\frac{|{}^S\overline{\nabla}_j\tilde{Y}_i|^2}{|x|^{2\alpha}}+\frac12\frac{|{}^S\overline{\nabla}_{22}\eta_1|^2}{|x|^{2\alpha-3}}+C\frac{h_1^2}{|x|^{2\alpha-3}}+\frac{1}{20\varepsilon}\frac{|{}^S\overline{\nabla}_{2j}\tilde{Y}_i|^2}{|x|^{2\alpha-2}}\\
&+\frac{C}{\varepsilon}\frac{|\tilde{Z}|^2}{|x|^{2\alpha-2}}d\mu_{S_{\overline{g}}}
\end{aligned}$$

We obtain a bound for the weighted norms of the second derivatives of the variables, including only one $\partial_1$ derivative, by adding to (2.6.32) its analogue for the terms ${}^S\overline{\nabla}{}^S\overline{\nabla}_3\eta_1,{}^S\overline{\nabla}{}^S\overline{\nabla}_3\tilde{Y}_i$, absorbing the second order terms of the RHS in the LHS and by applying the $H^1$ estimate we derived above to the lower order terms. Finally, in order to bound the corresponding norms of ${}^S\overline{\nabla}_{11}\tilde{Y}_i,{}^S\overline{\nabla}_{11}\eta_1$ as well, we use directly the system (2.6.25) to move the derivatives we have already controlled to the RHS. Then we first take the $\|\frac{\cdot}{|x|^{\alpha-1}}\|_{L^2}$ norm of the first two equations in (2.6.25):

$$\sum_{i=1}^3\|\frac{{}^S\overline{\nabla}_{11}\tilde{Y}}{|x|^{\alpha-1}}\|_{L^2}\lesssim\|\frac{\overline{\nabla}\!\!\!/^2\tilde{Y}}{|x|^{\alpha-1}}\|_{L^2}+\|\frac{{}^S\overline{\nabla}_1\overline{\nabla}\!\!\!/\tilde{Y}}{|x|^{\alpha-1}}\|_{L^2}+\|\frac{{}^S\overline{\nabla}\tilde{Y}}{|x|^{\alpha}}\|_{L^2} \qquad (2.6.33)$$

$$+\|\frac{\tilde{Y}}{|x|^{\alpha+1}}\|_{L^2}+\|\frac{\eta_1}{|x|^{\alpha+1}}\|_{L^2}+\|\frac{\tilde{Z}}{|x|^{\alpha-1}}\|_{L^2}$$

and the $\|\frac{\cdot}{|x|^{\alpha-\frac{3}{2}}}\|_{L^2}$ norm of the third equation of (2.6.25) to infer that

$$\|\frac{{}^S\overline{\nabla}_{11}\eta_1}{|x|^{\alpha-\frac{3}{2}}}\|_{L^2}\leq\|\frac{\not{\nabla}^2\eta_1}{|x|^{\alpha-\frac{3}{2}}}\|_{L^2}+\|\frac{{}^S\overline{\nabla}_1\not{\nabla}\eta_1}{|x|^{\alpha-\frac{3}{2}}}\|_{L^2}+\|\frac{{}^S\overline{\nabla}\eta_1}{|x|^{\alpha}}\|_{L^2} \tag{2.6.34}$$
$$+\|\frac{\eta_1}{|x|^{\alpha+\frac{3}{2}}}\|_{L^2}+\|\frac{{}^S\overline{\nabla}_j\tilde{Y}_i}{|x|^{\alpha}}\|_{L^2}+\|\frac{h_1}{|x|^{\alpha-\frac{3}{2}}}\|_{L^2}$$

□

For the spherically symmetric parts of $\eta, Y$ (2.6.16) we prove the following:

**Proposition 2.6.8.** *[A priori elliptic estimate II] Let $h_0\in H^{2,\alpha-3}_{sc}, Z_{10}\in H^{2,\alpha-1}_{vf\text{-}0}$ and $\eta_0\in H^{4,\alpha}_{sc}\cap H^1_0, Y_{10}\in H^{4,\alpha}_{vf\text{-}0}\cap H^1_0$, all $x$ variable functions solving (2.6.26). Then for $\alpha$ sufficiently large the following estimate holds:*

$$\|\eta_0\|^2_{H^{4,\alpha}_{sc}}+\|Y_{10}\|^2_{H^{4,\alpha}_{vf\text{-}0}}\lesssim\|h_0\|^2_{H^{2,\alpha-3}_{sc}}+\|Z_{10}\|^2_{H^{2,\alpha-1}_{vf\text{-}0}}+\|\eta_0\|^2_{L^2}+\|Y_{10}\|^2_{L^2} \tag{2.6.35}$$

*If in addition $x\in(-\epsilon,\epsilon)$, $\epsilon>0$ sufficiently small, how small depending on the coefficients of the system (2.6.9) and $\alpha$, then in fact*

$$\|\eta_0\|^2_{H^{4,\alpha}_{sc}}+\|Y_{10}\|^2_{H^{4,\alpha}_{vf\text{-}0}}\lesssim\|h_0\|^2_{H^{2,\alpha-3}_{sc}}+\|Z_{10}\|^2_{H^{2,\alpha-1}_{vf\text{-}0}} \tag{2.6.36}$$

*Proof.* We multiply the first equation above with $\frac{\partial_1 Y_{10}}{|x|^{2\alpha-\frac{1}{2}}}$, integrate over $\Sigma_0$ and integrate by parts: [note that the boundary terms are either zero or have a good sign]

$$\int_{\Sigma_0}\alpha\frac{(\partial_1 Y_{10})^2}{|x|^{2\alpha}}d\mu_{S_{\overline{g}}}\leq\int_{\Sigma_0}\alpha\frac{Y^2_{10}}{|x|^{2\alpha+1}}+C\frac{(\partial_1 Y_{10})^2}{|x|^{2\alpha}}+C\frac{|Y_{10}\partial_1 Y_{10}|}{|x|^{2\alpha+\frac{1}{2}}} \tag{2.6.37}$$
$$+C\frac{|\eta_0\partial_1 Y_{10}|}{|x|^{2\alpha+\frac{3}{2}}}+\frac{Z_{10}\partial_1 Y_{10}}{|x|^{2\alpha-\frac{1}{2}}}d\mu_{S_{\overline{g}}}$$
$$\leq\int_{\Sigma_0}(\alpha+C)\frac{Y^2_{i0}}{|x|^{2\alpha+1}}+C\frac{(\partial_1 Y_{10})^2}{|x|^{2\alpha}}$$
$$+\frac{1}{4}\frac{\eta_0^2}{|x|^{2\alpha+3}}+\frac{Z^2_{10}}{|x|^{2\alpha-1}}d\mu_{S_{\overline{g}}}$$

On the other hand, multiplying the second equation in (2.6.26) with $\frac{\eta_0}{|x|^{2\alpha}}$, integrate over $\Sigma_0$ and integrating by parts we have

$$\int_{\Sigma_0}\frac{(\partial_1\eta)^2}{|x|^{2\alpha}}+b\frac{\eta_0^2}{|x|^{2\alpha+3}}d\mu_{S_{\overline{g}}}$$

$$\leq \int_{\Sigma_0} C\alpha\frac{|\eta_0\partial_1\eta_0|}{|x|^{2\alpha+\frac{1}{2}}}+C\frac{|\eta_0\partial_1 Y_{10}|}{|x|^{2\alpha+\frac{3}{2}}}+C\frac{\eta_0 Y_{i0}}{|x|^{2\alpha+2}}+\frac{\eta_0 h_0}{|x|^{2\alpha}}d\mu_{S_{\overline{g}}} \tag{2.6.38}$$
$$\leq \int_{\Sigma_0} \frac{1}{2}\frac{(\partial_1\eta_0)^2}{|x|^{2\alpha}}+C\alpha^2\frac{\eta_0^2}{|x|^{2\alpha+1}}+\frac{1}{2}\frac{\eta_0^2}{|x|^{2\alpha+3}}+C\frac{(\partial_1 Y_{10})^2}{|x|^{2\alpha}}+C\frac{h_0^2}{|x|^{2\alpha-3}}d\mu_{S_{\overline{g}}}$$

Adding (2.6.33), (2.6.34) we employ Hardy's inequality

$$\int_{-\epsilon}^{\epsilon}\frac{Y_{10}^2}{|x|^{2\alpha+1}}dx \leq \frac{1}{\alpha^2}\int_{-\epsilon}^{\epsilon}\frac{(\partial_x Y_{10})^2}{|x|^{2\alpha-1}}dx, \qquad \partial_x \sim \frac{1}{\sqrt{|x|}}\partial_1. \tag{2.6.39}$$

and take advantage of the largeness of $\alpha$ to absorb most terms in the LHS and obtain a weighted $H^1$ estimate for $Y_{10}, \eta_0$. The higher order norms are controlled in turn using the system (2.6.26) and differentiating in $\partial_1$. If in addition $|x| \ll 1$, we deduce the improved estimate (2.6.36) by absorbing in the LHS all the $\eta_0, Y_{10}$ terms appearing in the final inequalities. □

The Propositions 2.6.7, 2.6.8 combined imply that $D\Psi$ (2.6.21) is semi-Fredholm, i.e., it has finite dimensional kernel and closed range. Since similar type of estimates can also be derived for the adjoint operator, it follows that the linearized map is Fredholm. In the case where $|x| \ll 1$, we proved that the estimates can be improved to yield that $D\Psi$ is an isomorphism. This completes the proof of Proposition 2.6.5.

# Appendix A

# Analysis of the singular Ricci solitons

Generally, for metrics of the form (1.2.1) [8, §1.3.2] the Ricci tensor is given by

$$Ric(g) = -n\frac{\psi_{xx}}{\psi}dx^2 + (n-1-\psi\psi_{xx}-(n-1)(\psi_x)^2)g_{\mathbb{S}^n} \tag{A.0.1}$$

and the Hessian of a radial function $\phi$ by

$$\nabla\nabla\phi = \phi_{xx}dx^2 + \psi\psi_x\phi_x g_{\mathbb{S}^n}, \tag{A.0.2}$$

where $\dot{} = \frac{d}{dx}$. Therefore, equation (1.2.2) reduces to a coupled ODE system of the form

$$\begin{cases} n\psi_{xx} - \psi\phi_{xx} = \lambda\psi \\ \psi\psi_{xx} + (n-1)\psi_x^2 - (n-1) - \psi\psi_x\phi_x = \lambda\psi^2. \end{cases} \tag{A.0.3}$$

Following [8, Chapter 1, §5.2], we introduce the transformation

$$W = \frac{1}{-\phi_x + n\frac{\dot\psi}{\psi}}, \qquad X = \sqrt{n}W\frac{\dot\psi}{\psi}, \qquad Y = \frac{\sqrt{n(n-1)}W}{\psi}, \tag{A.0.4}$$

along with a new independent variable $y$ defined via

$$dy = \frac{dx}{W}. \tag{A.0.5}$$

For the above set of variables, the ODE system (A.0.3) becomes

$$\left({}' = \frac{d}{dy}\right)\begin{cases} W' = W(X^2 - \lambda W^2) \\ X' = X^3 - X + \frac{Y^2}{\sqrt{n}} + \lambda(\sqrt{n} - X)W^2 \\ Y' = Y(X^2 - \frac{X}{\sqrt{n}} - \lambda W^2) \end{cases} \tag{A.0.6}$$

We readily check (see also [8, §1.5.2]) that the equilibrium points of the above system are

$$(0,0,0) \qquad\qquad (0,\pm 1,0) \qquad\qquad (0,\frac{1}{\sqrt{n}},\pm\sqrt{1-\frac{1}{n}}).$$

and also $(\pm\frac{1}{\sqrt{\lambda n}},\frac{1}{\sqrt{n}},0)$, when $\lambda>0$.

In the present article we are concerned with the trajectories emanating from the equilibrium point $(0,1,0)$, for all $\lambda\in\mathbb{R}$ (in our primary analysis). The linearization of (A.0.6) at $(0,1,0)$ takes the diagonal form

$$\begin{pmatrix} W \\ X-1 \\ Y \end{pmatrix}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1-\frac{1}{\sqrt{n}} \end{pmatrix} \begin{pmatrix} W \\ X-1 \\ Y \end{pmatrix} \tag{A.0.7}$$

Note that for $n>1$, all eigenvalues (diagonal entries) are positive, which implies that $(0,1,0)$ is a source of the system. Whence, if a trajectory of (A.0.6) is initially ($y=0$) close to $(0,1,0)$, i.e.,

$$|(W(0),X(0)-1,Y(0))|<\varepsilon,$$

for $\varepsilon>0$ sufficiently small (indicated by the RHS of (A.0.6)), then standard ODE theory (e.g., see [16]) yields the estimate

$$|(W(y),X(y)-1,Y(y))|\leq\sqrt{3}\varepsilon e^{\mu y} \qquad\qquad y\leq 0, \tag{A.0.8}$$

for some $0<\mu<1-\frac{1}{\sqrt{n}}$ (least eigenvalue).[1] We will show that these trajectories correspond to an essential singularity of the original metric (1.2.1) at $x=0$.

### A.0.3 Asymptotics at $x=0$

We will be considering solutions of the system (A.0.6), with $(W(0),X(0),Y(0))$ sufficiently close to the equilibrium point $(0,1,0)$ and with $Y(0),W(0)>0$. (The reflection-symmetric trajectories over $\{Y=0\}$ and $\{W=0\}$ are easily seen to correspond to the same metric, while the trajectories with $Y(0)=0$ do not to correspond to Riemannian metrics.)

We proceed to derive the asymptotic behavior, as $y\to-\infty$, of the variables $W,X,Y$. Changing back to $x$, using (A.0.5), we determine the desired asymptotic behavior of the unknown functions in the original system (A.0.3), as $x\to 0^+$. The final estimates will confirm that $x=0$ is actually a singular point of the metric $g$, where in fact the curvature blows up.

**Proposition A.0.9.** *The above initial conditions for the system (A.0.6) furnish trajectories* $(W,X,Y)$, $y\in(-\infty,0]$, *which correspond to solutions* $(\psi,\phi_x)$ *of the system (A.0.3) defined*

[1] The latter estimate improves as the initial conditions approach the equilibrium point $(0,1,0)$; in other words one can pick $\mu$ closer to the eigenvalue $1-\frac{1}{\sqrt{n}}$ by taking $\varepsilon$ sufficiently small.

*locally for $x \in (0,\delta)$, $\delta > 0$, verifying the asymptotics:*

$$W = x + O(x^{2\mu+1}), \qquad X = 1 + O(x^{\mu}), \qquad Y = \frac{\sqrt{n(n-1)}}{a} x^{1-\frac{1}{\sqrt{n}}} + O(x^{2\mu+1-\frac{1}{\sqrt{n}}}),$$
$$\psi = a x^{\frac{1}{\sqrt{n}}} + O(x^{\frac{2\mu+1}{\sqrt{n}}}),\ a > 0 \qquad \frac{\psi_x}{\psi} = \frac{1}{\sqrt{n}}\frac{1}{x} + O(x^{\mu-1}), \tag{A.0.9}$$
$$\phi_x = \frac{\sqrt{n}-1}{x} + O(x^{\mu-1}),\ \ \frac{\psi_{xx}}{\psi} = -\frac{\sqrt{n}-1}{n}\frac{1}{x^2} + O(x^{\mu-2}),\ \ \phi_{xx} = -\frac{\sqrt{n}-1}{x^2} + O(x^{\mu-2})$$

*Proof.* Let $X(y) = 1 + g(y)$. Plugging into the equation of $W'$ in (A.0.6) we obtain

$$W(y) = W(0)\exp\big\{y + \int_0^y W(z)g(z)(2+g(z))dz - \lambda \int_0^y W^3(z)dz\big\},$$

where according to (A.0.8), for $y \le 0$,

$$\Big|\int_0^y W(z)g(z)(2+g(z))dz\Big| \le 3\varepsilon^2(2+\sqrt{3}\varepsilon)\frac{1-e^{2\mu y}}{2\mu}$$

and

$$\Big| -\lambda \int_0^y W^3(z)dz\Big| \le |\lambda| \cdot 3\sqrt{3}\varepsilon^3 \frac{1-e^{3\mu y}}{3\mu}.$$

Thus,

$$\begin{aligned} W(y) =& C_1 e^y + W(0)e^y\Big[\exp\big\{\int_0^y W(z)g(z)(2+g(z))dz - \lambda\int_0^y W^3(z)dz\big\} \\ &- \exp\big\{-\int_{-\infty}^0 W(z)g(z)(2+g(z))dz + \lambda\int_{-\infty}^0 W^3(z)dz\big\}\Big], \end{aligned}$$

where $C_1 = W(0)\exp\big\{-\int_{-\infty}^0 W(z)g(z)(2+g(z))dz + \lambda\int_{-\infty}^0 W^3(z)dz\big\} > 0$. Using (A.0.8) again, we readily estimate the second term as above

$$W(y) = C_1 e^y + O(e^{(2\mu+1)y}) \qquad\qquad y \le 0.$$

Similarly, from the equation of $Y'$ (A.0.6) we obtain

$$Y(y) = C_2 e^{(1-\frac{1}{\sqrt{n}})y} + O(e^{(2\mu+1-\frac{1}{\sqrt{n}})y}) \qquad\qquad y \le 0.$$

for an appropriate positive ($Y(0) > 0$) constant $C_2$. As for $X$, directly from (A.0.8) we have the bound

$$X = 1 + g(y) = 1 + O(e^{\mu y}) \qquad\qquad y \le 0,$$

which we can retrieve from the equation of $X'$ by integrating on $(-\infty, y)$ and using (A.0.8), along with the previously derived estimates for $W, Y$.

Recall the transformation (A.0.4) to derive asymptotics for the variables in (A.0.3): ($y \leq -M$, $M > 0$ large)

$$\begin{aligned}
\psi =& \frac{\sqrt{n(n-1)}W}{Y} = \frac{\sqrt{n(n-1)}(C_1 e^y + O(e^{(2\mu+1)y}))}{C_2 e^{(1-\frac{1}{\sqrt{n}})y} + O(e^{(2\mu+1-\frac{1}{\sqrt{n}})y})} = \frac{\sqrt{n(n-1)}C_1}{C_2} e^{\frac{1}{\sqrt{n}}y} + O(e^{(2\mu+\frac{1}{\sqrt{n}})y}) \\
\frac{\psi_x}{\psi} =& \frac{X}{\sqrt{n}W} = \frac{1 + O(e^{\mu y})}{\sqrt{n}(C_1 e^y + O(e^{(2\mu+1)y}))} = \frac{1}{\sqrt{n}C_1} e^{-y} + O(e^{(\mu-1)y}) \\
\phi_x =& n\frac{\psi_x}{\psi} - \frac{1}{W} = \frac{\sqrt{n}}{C_1} e^{-y} + nO(e^{(\mu-1)y}) - \frac{1}{C_1 e^y + O(e^{(2\mu+1)y})} = \frac{\sqrt{n}-1}{C_1} e^{-y} + O(e^{(\mu-1)y}).
\end{aligned}$$

Also, going back to the second equation of (A.0.3) and dividing both sides by $\psi^2$ yields

$$\begin{aligned}
\frac{\psi_{xx}}{\psi} =& -(n-1)\frac{\psi_x^2}{\psi^2} + \frac{n-1}{\psi^2} + \frac{\psi_x}{\psi}\phi_x + \lambda \\
=& -(n-1)\big[\frac{1}{\sqrt{n}C_1} e^{-y} + O(e^{(\mu-1)y})\big]^2 + \frac{n-1}{\big[\frac{\sqrt{n(n-1)}C_1}{C_2} e^{\frac{1}{\sqrt{n}}y} + O(e^{(2\mu+\frac{1}{\sqrt{n}})y})\big]^2} \\
&+ \big[\frac{1}{\sqrt{n}C_1} e^{-y} + O(e^{(\mu-1)y})\big]\big[\frac{\sqrt{n}-1}{C_1} e^{-y} + O(e^{(\mu-1)y})\big] + \lambda \\
=& -\frac{\sqrt{n}-1}{n}\frac{e^{-2y}}{C_1^2} + O(e^{(\mu-2)y}).
\end{aligned}$$

Furthermore, the first equation of (A.0.3) gives

$$\phi_{xx} = n\frac{\psi_{xx}}{\psi} - \lambda = -(\sqrt{n}-1)\frac{e^{-2y}}{C_1^2} + nO(e^{(\mu-2)y}) + \lambda = -(\sqrt{n}-1)\frac{e^{-2y}}{C_1^2} + O(e^{(\mu-2)y}).$$

Having derived asymptotics, as $y \to -\infty$, for all the unknown functions appearing in the problem, we would like to derive corresponding asymptotics in the independent variable $x$ that we started with. For that we recall (A.0.5) and normalize so that $x \to 0^+$ as $y \to -\infty$ to deduce

$$x = \int W dy = \int C_1 e^y + O(e^{(2\mu+1)y}) dy = C_1 e^y + O(e^{(2\mu+1)y}) \qquad (y \leq 0),$$

Hence, it follows

$$C_1 e^y = x + O(x^{2\mu+1}),$$

for $y \leq -M$, $M > 0$ large. Going back to each of the above estimates, we confirm the rest of the asymptotics in Proposition A.0.9 for $a = \frac{\sqrt{n(n-1)}C_1^{1-\frac{1}{\sqrt{n}}}}{C_2} > 0$. □

**Remark A.0.10.** One could also consider the trajectories which emanate from the other equilibrium $(0, -1, 0)$ of (A.0.6) (also a source). These can be seen to correspond to solitons with profile

$$\psi(x) \sim x^{-\frac{1}{\sqrt{n}}} \qquad \phi_x(x) = -\frac{1+\sqrt{n}}{x}, \qquad \text{as } x \to 0^+.$$

They are in fact defined for all dimensions $n+1 \geq 2$, and in the steady case ($\lambda = 0$), dim $n+1 = 2$, can be explicitly written out as:

$$\psi(x) = \frac{1}{x} \qquad \phi_x(x) = -\frac{2}{x}, \qquad x \in (0, +\infty).$$

Notice that these metrics are also singular at $x = 0$, but their evolution under the Ricci flow (through diffeomorphisms) is almost the opposite from the metrics we obtain near the equilibrium at $(0,1,0)$; see §1.2.1. In particular, they remain singular for all time. However, these solitons are beyond the scope of this paper.

### A.0.4 The steady singular solitons; asymptotics at $x = +\infty$

In the steady case, $\lambda = 0$, we can push the domain of the solutions considered in Proposition A.0.9 all the way up to $+\infty$. A very useful tool in the analysis of the trajectories of (A.0.6) is the Lyapunov function [8, §1.4.3]

$$L = X^2 + Y^2, \qquad (L-1)' = X^2(L-1), \tag{A.0.10}$$

which implies that the unit disk is a stable region of the critical point $(0,0)$. Further, it follows from (A.0.10) that the equation of $W'$ in (A.0.6) is actually redundant, reducing the system to

$$\begin{cases} X' = X^3 - X + \frac{Y^2}{\sqrt{n}} \\ Y' = Y(X^2 - \frac{X}{\sqrt{n}}) \end{cases} \tag{A.0.11}$$

We remark that the unique trajectory emanating from the equilibrium point $(\frac{1}{\sqrt{n}}, \sqrt{1-\frac{1}{n}})$ and converging (as $y \to +\infty$) to the origin $(0,0)$ corresponds to the well-known Bryant soliton (see [8]).

The source considered in (A.0.7) corresponds to the point $(1,0)$. Thus, if we consider solutions of (A.0.6) with initial point $(X(0), Y(0))$ satisfying $X^2(0) + Y^2(0) < 1$, $Y(0) > 0$ and lying close enough to $(1,0)$, we easily conclude that the trajectory $(X(y), Y(y))$ approaches the origin $(0,0)$, as $y \to +\infty$ (at an exponential rate). Whence it exists for all $y \in (-\infty, +\infty)$. In fact, these trajectories emanating from $(1,0)$ translate back to Ricci soliton metrics of the form (1.2.1), which exist (and are smooth) for all $x \in (0, +\infty)$ and have the leading behavior described in Proposition A.0.9 at $x = 0$.

One can easily see that the set of all such trajectories fills up the domain in the unit disc bounded by the Bryant soliton trajectory $\big($which emanates from $(\frac{1}{\sqrt{n}}, \sqrt{1-\frac{1}{n}})\big)$ and the positive $X$-axis. The asymptotics of these trajectories at $+\infty$ are easily seen to matching those of the one corresponding of the Bryant soliton. This has to do with the Lyapunov function (A.0.10) and the uniform convergence of the trajectories at the origin $(0,0)$, as $y \to +\infty$. Following [8, Chapter 1, §4] we arrive at the next proposition.

**Proposition A.0.11.** *The soliton metrics corresponding to the $(X,Y)$-orbits above are complete towards $x=+\infty$ and satisfy the asymptotics*

$$cx^{\frac{1}{2}} \leq \psi \leq Cx^{\frac{1}{2}} \qquad cx^{-\frac{1}{2}} \leq \dot{\psi} \leq Cx^{-\frac{1}{2}} \qquad -Cx^{-\frac{3}{2}} \leq \ddot{\psi} \leq -cx^{-\frac{3}{2}}, \tag{A.0.12}$$

*for $x > M$ large, retrieving from (A.0.3) the asymptotics of the derivatives of $\phi$*

$$-C < \phi_x < -c \qquad\qquad -Cx^{-2} \leq \phi_{xx} \leq -cx^{-2}. \tag{A.0.13}$$

# Appendix B

# Changing frames freedom; Propagating identities; Retrieving the EVE from the reduced equations

Given a spacetime $(\mathcal{M}^{1+3}, g)$ and an orthonormal frame $\{e_i\}_0^3$, one may change to a Lorenz gauge frame $\{\tilde{e}_i\}_0^3$ by solving the following semilinear system of equations, which is derived by taking the divergence of (2.2.9):

$$\begin{aligned}
\Box_g(O_a^l) &= (\text{div}\tilde{A})_a{}^d O_d^l + \tilde{A}\partial O + A\partial O + O_a^k(\text{div}A)_k{}^l \\
&= \tilde{A}^2 O + \tilde{A}\partial O + A\partial O + O\text{div}A && \text{(by (2.2.15) for } \tilde{A}) \\
&= O^5A^2 + O^3(\partial O)^2 + AO^4\partial O + AO^2\partial O && \text{(from (A.0.4))} \\
&\quad + O(\partial O)^2 + A\partial O + O\text{div}A,
\end{aligned}$$

where the terms without indices in the RHS stand for an algebraic expression of a finite number terms of the depicted type.

**Lemma B.0.12.** *If the above system (which we write schematically as)*

$$\begin{aligned}
\Box_g(O_i^b) = O^5A^2 + O^3(\partial O)^2 + AO^4\partial O + AO^2\partial O \\
+ O(\partial O)^2 + A\partial O + O\,divA.
\end{aligned} \tag{B.0.1}$$

*is well-posed in a certain solution space, then there exists a unique orthonormal frame*

$$\tilde{e}_i = O_i^b e_b \tag{B.0.2}$$

*with $O_i^b$ lying in that particular space, which is identical to $\{e_i\}_0^3$ on the initial hypersurface $\Sigma_0$, verifies the Lorenz gauge condition (2.2.15) and such that the connection coefficients $(\tilde{A}_0)_{ij} := g(\nabla_{\tilde{e}_0}\tilde{e}_i, \tilde{e}_j)$, $i < j$, are equal to a priori assigned functions on $\Sigma_0$; within the corresponding* *space of one order of regularity less than* $O_i^b$.

*Proof.* It suffices to show that the initial data for (B.0.1) is uniquely determined by the assertions. We set

$$O_i^b(\tau = 0) := I_i{}^b \qquad \text{(i.e., } \tilde{e}_i = e_i \text{ on } \Sigma_0\text{)}. \tag{B.0.3}$$

Let

$$\tilde{e}_0(O_i^b)(\tau = 0) = e_0(O_i^b)(\tau = 0) =: h_i^b, \qquad h_i^b m_{bj} = -h_j^b m_{bi}. \tag{B.0.4}$$

Then the transition formula (A.0.4) for $X = \tilde{e}_0$ reads

$$(\tilde{A}_0)_{ij}(\tau = 0) = (A_0)_{ij}(\tau = 0) + h_i^b m_{bj}. \tag{B.0.5}$$

Thus, the components $(\tilde{A}_0)_{ij}$ can be freely prescribed initially by choosing $h_i^b$ in (B.0.4) accordingly. □

### B.0.5 Proof of proposition 2.2.1

We will leave the reader to fill in the details for the fact that the solution $(A_\nu)_{ij}, O_i^a$ of (2.2.16),(2.2.19) corresponds to a spacetime $(\mathcal{M}^{1+3}, g)$. This is a consequence of the necessary initial assumption (2.2.20). One such immediate consequence follows from (2.2.19) for $i = 0$:

$$\partial_0(O_0^a) = -O_0^b \Gamma^a_{[0b]}, \qquad O_0^a(\tau = 0) - I_0{}^a = 0, \tag{B.0.6}$$

which implies $O_0^a = I_0{}^a$ and hence $e_0 = \partial_0$ everywhere, since $\Gamma^a_{[00]} = 0$. The set of functions $O_i^a$ defines the orthonormal frame $\{e_i\}_0^3$ in $\mathcal{M}^{1+3}$ through (2.2.18) and hence completely determines the metric $g$. What remains to be verified is that the connection coefficients of $\{e_i\}_0^3$ are indeed the $(A_\nu)_{ij}$'s of the given solution. In other words, we have to show that the connection $D$ induced by the solution set $(A_\nu)_{ij}$,

$$D_{e_\nu} e_i := (A_\nu)_i{}^k e_k, \tag{B.0.7}$$

is the Levi-Civita connection $\nabla$ of the metric $g$. Formally, one cannot take this for granted. It has to be retrieved from the equations (2.2.16),(2.2.19) and the initial assumption (2.2.20). For example, the compatibility of $D$ with respect to $g$ is encoded in the skew-symmetry of the $(A_\nu)_{ij}$'s

$$D(g) = 0, \qquad \text{iff} \qquad (A_\nu)_{ij} + (A_\nu)_{ji} = 0, \tag{B.0.8}$$

which also has to be verified, since it is a priori valid only initially (2.2.20). The way to do this is by deriving the following new system of equations from (2.2.16) for the symmetric sums:

$$\begin{aligned}\boxdot\big((A_\nu)_{ij}+(A_\nu)_{ji}\big) &= (A^{[\mu})_{\nu]}{}^k e_k\big((A_\mu)_{ij}+(A_\mu)_{ji}\big) + e^\mu\big((A_{[\mu})_{\nu]}{}^k\big[(A_k)_{ij}+(A_k)_{ji}\big]\big)\\ &\quad + e_\nu\big(A\big[(A)_{ij}+(A)_{ji}\big]\big) + e_\nu\big((A^\mu)_\mu{}^k\big[(A_k)_{ij}+(A_k)_{ij}\big]\big), \qquad \text{(B.0.9)}\end{aligned}$$

where we have assumed that the sum $(A^2)_{ij}+(A^2)_{ji}$ corresponding to the term $A^2$ in the gauge condition (2.2.15) can be expressed as $A\big[(A)_{ij}+(A)_{ji}\big]$. Since (B.0.9) has zero initial data (2.2.20), the symmetric sums are zero everywhere and hence the skew-symmetry (B.0.8) propagates.[1]

*Proof of proposition 2.2.1; EVE and Lorenz gauge.* Recall (2.2.17) and the reduced equations $H_{\nu ij}=0$. By assumption $(A_\nu)_{ij}$ is a solution of (2.2.16), i.e., the RHS of (2.2.17) vanishes. Taking the divergence of (2.2.17) with respect to the index $\nu$, the first part of the LHS of (2.2.17), corresponding to the curl of the Ricci tensor, vanishes and we are left with

$$\begin{aligned}&\square_g\big(\mathrm{div}A - A^2\big)_{ij} \qquad \text{(B.0.10)}\\ &\quad = (A^\nu)_i{}^c e_\nu\big(\mathrm{div}A - A^2\big)_{cj} + (A^\nu)_j{}^c e_\nu\big(\mathrm{div}A - A^2\big)_{ic}.\end{aligned}$$

The Lorenz gauge condition is valid initially (2.2.21). If the $e_0$ derivative of $\big(\mathrm{div}A - A^2\big)_{ij}$ is zero as well on $\Sigma_0$, then the Lorenz gauge is valid in all of $\mathcal{M}^{1+3}=\Sigma\times[0,T]$.[2] This is in fact implied by (2.2.17), putting $\nu=0$ we have

$$e_0\big(\mathrm{div}A - A^2\big)_{ij} = \nabla_j \mathrm{R}_{0i} - \nabla_i \mathrm{R}_{0j} = 0 \qquad \text{on } \Sigma_0 \qquad \text{(B.0.11)}$$

by virtue of the vanishing of $\mathrm{R}_{ab}(\tau=0)$ (2.2.21) and the (twice contracted) second Bianchi identity, $\nabla^a \mathrm{R}_{ab}=\frac{1}{2}\mathrm{R}$, to replace if necessary a transversal derivative with tangential ones to $\Sigma_0$.

On the other hand, taking the $\nabla^i$ divergence of (2.2.17) and commuting derivatives we obtain

$$\begin{aligned}\square_g \mathrm{R}_{\nu j} = \nabla^i\nabla_j \mathrm{R}_{\nu i} &= \frac{1}{2}\nabla_j\nabla_\nu \mathrm{R} + \mathrm{R}^i{}_j{}^c{}_\nu \mathrm{R}_{ci} + \mathrm{R}^i{}_j{}^c{}_i \mathrm{R}_{\nu c}\\ &= \mathrm{R}^i{}_j{}^c{}_\nu \mathrm{R}_{ci} + \mathrm{R}_j{}^c \mathrm{R}_{\nu c}, \qquad \text{(B.0.12)}\end{aligned}$$

where we employed again the twice contracted second Bianchi identity and the fact that the

[1] This follows by a basic a priori energy estimate for linear systems like (B.0.9), which in the singular Schwarzschild background is derived in §2.5.3 for the more involved quasilinear system (2.4.3).

[2] Note however that the term $e_0\big(\mathrm{div}A - A^2\big)_{ij}$ is of second order in $A$ and hence not at the level of initial data for (2.2.16), which we are allowed to prescribe. If zero initially, this should be a consequence of the geometric nature of the equations.

scalar curvature R vanishes everywhere: [contracting $\{\nu j\}$ in (2.2.17)]

$$0 = \nabla_i \mathrm{R} - \frac{1}{2}\nabla_i \mathrm{R} = \frac{1}{2}\nabla_i \mathrm{R}, \qquad \mathrm{R}\big|_{\Sigma_0} = 0. \tag{B.0.13}$$

Now that we know the Lorenz gauge is valid, the identities (2.2.21) and (2.2.17) $i = 0$ imply

$$\mathrm{R}_{\nu j} = 0, \qquad \nabla_0 \mathrm{R}_{\nu j} = \nabla_j \mathrm{R}_{\nu 0}, \qquad \text{on } \Sigma_0. \tag{B.0.14}$$

Utilizing the second Binachi identity $\nabla^a \mathrm{R}_{ab} = \frac{1}{2}\mathrm{R} = 0$ once more, we conclude that $\nabla_0 \mathrm{R}_{\nu j}$ vanishes and hence the initial data set of (B.0.12) is the trivial one. Thus, the initial condition $\mathrm{R}_{\nu j}(\tau = 0) = 0$ (2.2.21) propagates and the spacetime $(\mathcal{M}^{1+3}, g)$ obtained from the solution of (2.2.16) verifies the EVE (2.1.1). □

**Remark B.0.13.** Given the frame $\{e_i\}_0^3$ initially on $\Sigma_0$, and once the components $(A_0)_{ij}(\tau = 0)$ have been chosen,[3] then the initial data set of (2.2.13) is fixed by condition (2.2.21), i.e., the EVE and Lorenz gauge on $\Sigma_0$. Indeed, the components $(A_\nu)_{ij}(\tau = 0)$, $\nu, i, j = 1, 2, 3$, are determined uniquely by the orthonormal frame $\{e_i\}_1^3$ on $(\Sigma_0, \overline{g})$. The $(A_i)_{0j}(\tau = 0)$'s correspond to the components of second fundamental form $K_{ij}$ of $\Sigma_0$, which is given by the solution to the constraints (2.1.5), included in (2.2.21). Moreover, the expression of (2.2.21) in terms of $A$, for $\nu, i = 1, 2, 3$, reads (schematically)

$$\begin{aligned} e_0(A_\nu)_{0i} &= e_\nu(A) + A^2 \\ e_0(A_0)_{ij} &= \sum_{\mu=1}^{3} e_\mu(A_\mu)_{ij} + A^2 \qquad \text{on } \Sigma_0. \end{aligned} \tag{B.0.15}$$

Hence, the LHS functions are expressed in terms of already determined components. Finally, the rest components $e_0(A_\nu)_{ij}(\tau = 0)$, $\nu, i, j = 1, 2, 3$, are fixed by the algebraic property of the Riemann tensor

$$\begin{aligned} \mathrm{R}_{0\nu ij} &= \mathrm{R}_{ij0\nu} \\ e_0(A_\nu)_{ij} - e_\nu(A_0)_{ij} - ([A_\mu, A_\nu])_{ij} - {(A_{[\mu})_{\nu]}}^k (A_k)_{ij} &= \\ e_i(A_j)_{0\nu} - e_j(A_i)_{0\nu} - ([A_i, A_j])_{0\nu} - {(A_{[i})_{j]}}^k (A_k)_{0\nu} & \end{aligned}$$

or

$$e_0(A_\nu)_{ij} = e_\nu(A_0)_{ij} + e_i(A_j)_{0j} - e_j(A_i)_{0\nu} + A^2, \qquad \text{on } \Sigma_0, \tag{B.0.16}$$

since all the terms in the RHS have been accounted for. Notice that the definition of Riemann curvature was implicitly used in deriving (B.0.12) upon commuting covariant derivatives.

[3]The $(A_0)_{ij}$'s are not fixed by the Lorenz gauge condition; cf. Lemma B.0.12. They correspond to the $\partial_0$ derivative of the frame components $e_i$, which we can freely assign initially.

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