Rescaled Directed Random Polymer in Random Environment in Dimension $1 + 2$

by

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Abstract

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Directed random polymer in random environment is a statistical physics model. Heuristically, it concerns a long, directed chain of molecules evolving in some random background, and one of the questions one asks is how is long term behaviour of the polymer affected by the strength of the noise, which in the model is controlled by a parameter β called inverse temperature.

In this thesis, we will let β depend on the length of the polymer and consider the model in time-space dimension $1+2$. Also, we will work with continuous time and space, and Gaussian random environment that is white noise in time.

The main result is if we rescale β as $\beta_t = \frac{\alpha_t}{\sqrt{\log t}}$, where $\alpha_t^2 = 4\pi + \frac{\sigma}{\log t}$ for $\sigma \in \mathbb{R}$, then we observe evidence for non-Gaussian fluctuation of the free energy of the random polymer.

Along the way, we give a formula for the exponential moment of the regularized local time of two dimensional Brownian motion starting at 0, and show the two dimensional particles with delta function interaction are correlated.

Dedication

To my parents

I am indebted to my parents. Without you, without your sacrifices, I would not have been here today. Over the years, you have unconditionally supported me, and given me guidance during tough times. I dedicate this thesis to you, my dear father and mother.

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Chapter 1

Introduction

In the discrete time-space lattice $\mathbb{Z}_+ \times \mathbb{Z}^d$, let us place a random variable $\eta(i,x)$ at each site, and run a simple symmetric random walk $S = (S_n)_{n\geq 0}$, starting at 0, in this random environment. During its evolution, the trajectory of the walk will be influenced by the random background: It might go to sites where the noise is favourable, or avoid regions that are not. We define the random polymer to be the simple symmetric random walk weighted by the amount of energy it picks up during its evolution; more precisely, it is the random Gibbs measure,

$$\mu_n(dS) = \frac{1}{Z_n} e^{\beta \sum_{i=1}^n \eta(i,S_i)} P(dS), \tag{1.0.1}$$

where Z_n is a normalizing constant to make it a probability measure. We ask, in the long run, whether or not the polymer will behave like a random walk.

In the model, we have a parameter β that controls the strength of the random environment, called the inverse temperature. For β small, the influence of the environment on the walk is relatively weak, and we expect the polymer to be close to the original random walk. For β large, the walk might be induced to explore high energy regions, and in so doing might become very different from a random walk. Thus, the question is what is the critical value of β at which we see a transition of the polymer from a random walk to something else?

Before we continue, let us fix some notations for the Introduction. Let the simple symmetric random walk $S = (S_n)_{n\geq 0}$ be defined on some probability space $(\Omega, \mathcal{F}, P)$, and the expectation with respect to P be denoted by E. Let the random environment $\eta = (\eta(i,x))_{i\in\mathbb{Z}_+, x\in\mathbb{Z}^d}$ be defined on another probability space $(H, \mathcal{G}, \mathbb{Q})$, that is independent of the random walk, and the expectation with respect to $\mathbb{Q}$ be denoted by $\mathbb{E}$.

1.1 Background - Random Polymer

Directed random polymer in random environment is a statistical physics model introduced by Huse and Henley [17] in the 1980's. During that time, Imbrie and Spencer [18] obtained the first mathematical results on this model for dimensions $d \geq 3$ and Bernoulli environment $\eta(i,x) = \pm\frac{1}{2}$. They showed the second moment of the polymer endpoint is of order n. In other words, for $\beta > 0$ sufficiently small, with probability 1 with respect to the random environment measure $\mathbb{Q}$, as $n \to \infty$,

$$\frac{\mu_n(|S_n|^2)}{n} \to 1.$$

Shortly after, under the same assumptions, using a martingale argument, Bolthausen [6] extended it to a Central Limit Theorem that with probability 1 with respect to $\mathbb{Q}$, as $n \to \infty$,

$$\mu_n\left(\prod_{j=1}^{d}\left(\frac{S_{j,n}}{\sqrt{n}}\right)^{k_j}\right) \to \gamma(k), \tag{1.1.2}$$

where $S_n = (S_{1,n}, \dots, S_{d,n})$, $k = (k_1, \dots, k_d)$, and $\gamma(k)$ are the moments of a centered Gaussian random vector with covariance matrix $1/d$ times the identity matrix.

For the existence of a phase transition in terms of the parameter β for $d \geq 3$, Comets and Yoshida showed it in [10]. Moreover, they did this for general environment with finite cumulant generating function (or logarithm of the moment generating function) and proved the polymer satisfies a Central Limit Theorem up to (but not including) the critical point $\beta_c > 0$. For dimensions $d = 1, 2$, Carmona and Hu [8] showed $\beta_c = 0$ for Gaussian random environment, and Comets, Shiga and Yoshida [9] extended it for general environment with finite cumulant generating function. Also, in both papers, the authors showed the polymer localizes in the sense that if S_n^1, S_n^2 are two independent copies of the polymer path and $\mathbb{1}_0(x)$ is the indicator function at 0, then for $\beta > 0$, there exists $c > 0$ such that with probability 1 with respect to $\mathbb{Q}$,

$$\limsup_{n\to\infty} \mu_n(\mathbb{1}_0(S_n^1 - S_n^2)) \geq c.$$

For $d = 1$, in order to better understand the transition at $\beta_c = 0$, Alberts, Khanin, and Quastel [1] zoomed in around the critical point and investigated in more details as to how the transition takes place, by scaling β with the length of the polymer and setting $\beta = \beta_n := \sigma n^{-1/4}$, where $\sigma \in \mathbb{R}$ is a parameter. Moreoever, they computed the exact distribution of the polymer endpoint in this intermediate scaling regime, as well as the fluctuation of the free energy $\log Z_n$. One way to see this choice of scaling is to consider the second moment of the partition function $Z_n = E\left[e^{\beta \sum_{i=1}^n \eta(i,S_i)}\right]$; and one can show $\mathbb{E}[Z_n^2] = E\left[e^{\beta^2 \sum_{i=1}^n \mathbb{1}_0(S_i^1 - S_i^2)}\right] e^{n\beta^2}$, where S^1, S^2 are two independent copies of the random walk. If $L_n(S) = \sum_{i=1}^n \mathbb{1}_0(S_i)$, local time of the random walk at 0 up to time n, and since $S^1 - S^2$ is another random walk, it is known that for $d = 1$, $\frac{L_n(S^1 - S^2)}{\sqrt{n}} \to |N|$, where N is a centerd normal random variable. Thus, choosing $\beta = \sigma n^{-1/4}$, up to some constant, the second moment of the partition function is of order 1.

In this thesis, we undertake a similar task for dimension $d = 2$, by identifying the correct scaling of β in terms of the length of the polymer, and give some results to demonstrate why it is critical. More precisely, we show if we rescale $\beta = \frac{\alpha_t}{\sqrt{\log t}}$, where $\alpha_t^2 = 4\pi + \frac{\sigma}{\log t}$ for $\sigma \in \mathbb{R}$, then we observe evidence for non-Gaussian fluctuation of the free energy, Theorem 4.0.1. Along the way, we give a formula for the exponential moment of the regularized local time of two dimensional Brownian motion starting at 0, Theorem 3.0.1; and show the two dimensional particles with delta function interaction are correlated, Theorem 4.3.1.

Remark 1.1.1. *Let us also mention the work of Lacoin [21], and the sharper results by Berger and*

Lacoin in [4], where the authors considered the asymptotic of the free energy for small β. In their model, β is fixed and does not depend on the length of the polymer. By first letting the length of the polymer tend to infinity, and then $\beta \to 0$, they computed the limit of the free energy. From [8] and [9] above, for dimension $d = 2$, we know the polymer is localized for $\beta > 0$. The purpose of the work in [4] is to investigate how the polymer transitions from localized to delocalized as $\beta \to 0$. But in this thesis, we address a different question, and we want to identify the critical scaling in terms of the length of the polymer so that we see a transition in the free energy.

1.2 Setup

We will work with continuous time and space in dimension $1+2$. Let $p_1(x) = \frac{1}{2\pi}e^{-\frac{|x|^2}{2}}$ be the density function of two dimensional standard normal random variable; $\xi = (\xi(s,x))_{s\in\mathbb{R}_+, x\in\mathbb{R}^2}$ a centered Gaussian process defined on a probability space $(H, \mathcal{G}, \mathbb{Q})$ with covariance

$$\mathbb{E}\left[\xi(s,x)\xi(t,y)\right] = \delta_0(s-t)p_1\left(\frac{x-y}{\sqrt{2}}\right); \tag{1.2.3}$$

and $B = (B_t)_{t\geq 0}$ a two dimensional Brownian motion[1] defined on another probability space $(\Omega, \mathcal{F}, P)$, starting at 0 and has second moment $E[|B_t|^2] = 2\cdot(2t)$, where ξ, B are independent.

Analogously, if[2] $H_t(B) = \int_0^t ds\ \xi(s, B_s)$, then the *polymer measure in environment* ξ is defined as

$$\mu_t(dB) = \frac{1}{Z_t}e^{\beta H_t(B)}P(dB), \tag{1.2.4}$$

where $Z_t = E[e^{\beta H_t(B)}]$ is the partition function.

Remark 1.2.1. *It is important for the results in this thesis that the random environment ξ has covariance $p_1(x)$ and we use Brownian motion, rather than random walk, for the underlying particle. For example, if we work with the polymer measure (1.0.1) with S a simple symmetric random walk and η a family of standard normals, the argument in Chapter 4 for the fluctuation of the free energy $\log Z_n$ would not work because moments of the partition function Z_n are exponential moments of the collision local time of independent copies of the discrete walk S, and we do not know a discrete version of the Gaussian Correlation Inequality. And the choice of $p_1(x)$ is to reduce the problem to one of the proven cases of the Gaussian Correlation Conjecture.*

1.3 Warning

In the estimates, the constants C, C', C_1, C_2 etc will change from line to line.

---

[1] The Brownian motion we use has generator Laplacian Δ.

[2] By a direct computation of the second moment of $H_t(B)$ with respect to the random environment, we see it is a well-defined random variable because the Gaussian process ξ is only white noise in time.

Chapter 2

Diffusivity at Subcritical Scaling

We want to say at sufficiently high temperature, the second moment of the endpoint of the polymer is of order t.

Remark 2.0.1. *All the Brownian motions we use have generator* Δ*, in particular the second moment is* $E[|B_t|^2] = 2 \cdot (2t)$*. (See Section 1.2.)*

Theorem 2.0.2. *For the polymer measure* μ_t *defined in (1.2.4) with* $\beta = \frac{\alpha_t}{\sqrt{\log t}}$*, where* $\alpha_t \to 0$*, then*

$$\frac{\mu_t(|\frac{B_t}{\sqrt{4}}|^2)}{t} \to 1$$

in probability as $t \to \infty$*.*

Remark 2.0.3. *Since we are in dimension* $d = 2$*, and our Brownian motion has second moment* $E[|B_t|^2] = 2 \cdot (2t)$*, we need to divide* B_t *by* $\sqrt{4}$*.*

Remark 2.0.4. *This is the same result as our previous result in [13] for the discrete model. For example, the polymer is diffusive if* $\beta = \frac{1}{(\log t)^a}$ *for* $a > 1/2$*.*

Define

$$Z_t = E\left[e^{\beta \int_0^t ds \ \xi(s,B_s)}\right], \qquad K_t = E\left[\left|\frac{B_t}{\sqrt{4}}\right|^2 e^{\beta \int_0^t ds \ \xi(s,B_s)}\right]. \tag{2.0.1}$$

Note $\mu_t(|\frac{B_t}{\sqrt{4}}|^2) = \frac{K_t}{Z_t}$. We will show the Theorem by expanding the polymer measure $e^{\beta \int_0^t ds \ \xi(s,B_s)}$ into a series and estimating the second moments of K_t, Z_t.

Remark 2.0.5. *Theorem 2.0.2 is not difficult because to show the polymer is diffusive when* $\alpha_t \to 0$*, it is enough to consider the second moment of* K_t, Z_t*. If* α_t *is constant, say* $\alpha_t^2 < 4\pi$*, which we believe to be the entire subcritical region for this polymer model, one still has the second moment of* K_t *of order t and that of* Z_t *of order* 1*, which suggests the polymer is diffusive. To compute the second moments in the region* $\alpha_t^2 < 4\pi$*, one can alternatively adapt the argument in Chapter 3, where we give a formula for the exponential moment of the regularized local time of two dimensional Brownian motion, or as we will see shortly, equivalently, the second moment of the partition function.*

2.1 Second Moment Formulae

Lemma 2.1.1. *Let Z_t, K_t be defined as in (2.0.1), and $\lambda(\beta) = \frac{\beta^2}{2} p_1(0)$ with $p_1(0) = \frac{1}{2\pi}$. For any fixed $\beta > 0, t > 0$,*

$$\mathbb{E}[Z_t^2] = E\left[e^{\beta^2 \int_0^t ds\ p_1(B_s)} e^{2t\lambda(\beta)}\right],$$
$$\mathbb{E}[K_t^2] = \frac{1}{64} E\left[|B_t^2 - B_t^1|^2 |B_t^2 + B_t^1|^2 e^{\beta^2 \int_0^t ds\ p_1(B_s^1)} e^{2t\lambda(\beta)}\right],$$

where B, B^1, B^2 are independent two dimensional Brownian motions.

Proof. To compute the second moment of Z_t, in general, we note the moments of the partition function are the same as taking independent copies of the underlying Brownian motion in the same random environment, so we use Fubini to first average out the environment.

When we expand the polymer measure $e^{\beta \int_0^t ds\ \xi(s,B_s)}$ into a series, we get

$$\begin{aligned}
\mathbb{E}[Z_t^2] &= \mathbb{E}E\left[e^{\beta \int_0^t ds\ \xi(s,B_s^1)+\xi(s,B_s^2)}\right] \\
&= 1 + \sum_{k=1}^{\infty} \frac{\beta^k}{k!} E\mathbb{E}\left[\int_0^t ds\ \xi(s,B_s^1) + \xi(s,B_s^2)\right]^k \\
&= 1 + \sum_{k=1}^{\infty} \frac{\beta^k}{k!} E\left\{\sum_{\sigma_i \in \{1,2\}} \int_{s_1,\dots,s_k \le t} \mathbb{E}\left[\prod_{j=1}^k \xi(s_j, B_{s_j}^{\sigma_j})\right]\right\} \\
&= 1 + \sum_{k=1}^{\infty} \frac{\beta^{2k}}{(2k)!} E\left\{\sum_{\sigma_i \in \{1,2\}} \int_{s_1,\dots,s_{2k} \le t} \sum \prod \left[\delta_0(s_i - s_j) p_1\left(\frac{B_{s_i}^{\sigma_i} - B_{s_j}^{\sigma_j}}{\sqrt{2}}\right)\right]\right\}.
\end{aligned}$$

In the last equality, to compute the mixed moments of the Gaussian process ξ, we use its covariance (1.2.3) and Wick's formula for the moments of Gaussians, which says for $Z_1, \dots, Z_{2k}$ Gaussians random variables with zero means that $\mathbb{E}\left[Z_1 \cdots Z_{2k}\right] = \sum \prod \mathbb{E}\left[Z_i Z_j\right]$, where the sum is over all $\frac{(2k)!}{2^k k!}$ disjoint partitions of $(1, 2, \dots, 2k)$ into pairs. In particular, we see the sum over k in the second moment of Z_t is nonzero only for even k.

Now, since we integrate over s_i and sum over σ_i, for each partition of $(1, 2, \dots, 2k)$, by change of variables, the integral-sums are equal and without loss of generality, we may group the time variables s_i into cross pairs, say $(s_1, s_{k+1}), (s_2, s_{k+2})$ etc, and σ_i into consecutive pairs as $(\sigma_1, \sigma_2), (\sigma_3, \sigma_4)$ etc. In other words, for each partition,

$$\sum_{\sigma_i \in \{1,2\}} \int_{s_1,\dots,s_{2k}} \prod \left[\delta_0(s_i - s_j) p_1\left(\frac{B_{s_i}^{\sigma_i} - B_{s_j}^{\sigma_j}}{\sqrt{2}}\right)\right] = \sum_{\sigma_i \in \{1,2\}} \int_{s_1,\dots,s_k} \prod_{j=1}^k p_1\left(\frac{B_{s_j}^{\sigma_{2j-1}} - B_{s_j}^{\sigma_{2j}}}{\sqrt{2}}\right).$$

Also, by considering for each pair $(\sigma_{2j-1}, \sigma_{2j})$ whether or not they are the same, the latter is equal to

$$\int_{s_1,\dots,s_k} 2^k \prod_{j=1}^k \left[p_1\left(\frac{B_{s_j}^1 - B_{s_j}^2}{\sqrt{2}}\right) + p_1(0)\right].$$

We conclude

$$\begin{aligned}
\mathbb{E}[Z_t^2] &= 1 + \sum_{k=1}^{\infty} \frac{\beta^{2k}}{k!} E\left\{ \sum_{\sigma_i \in \{1,2\}} \int_{s_1,\dots,s_k \le t} \frac{1}{2^k} \prod_{j=1}^{k} p_1\left(\frac{B_{s_j}^{\sigma_{2j-1}} - B_{s_j}^{\sigma_{2j}}}{\sqrt{2}} \right) \right\} \\
&= 1 + \sum_{k=1}^{\infty} \frac{\beta^{2k}}{k!} E\left\{ \int_{s_1,\dots,s_k \le t} \prod_{j=1}^{k} \left[p_1\left(\frac{B_{s_j}^1 - B_{s_j}^2}{\sqrt{2}} \right) + p_1(0) \right] \right\} \\
&= E\left[e^{\beta^2 \int_0^t ds \, \left(p_1\left(\frac{B_s^1 - B_s^2}{\sqrt{2}} \right) + p_1(0) \right)} \right] \\
&= E\left[e^{\beta^2 \int_0^t ds \, p_1(B_s)} e^{t\beta^2 p_1(0)} \right],
\end{aligned}$$

where the last equality follows since $\frac{B^1 - B^2}{\sqrt{2}}$ is a Brownian motion.

For the second moment of K_t, recall by definition $K_t = \frac{1}{4} E\left[|B_t|^2 e^{\beta \int_0^t ds \, \xi(s,B_s)} \right]$, so similarly,

$$\begin{aligned}
\mathbb{E}[K_t^2] &= \frac{1}{16} \mathbb{E} E\left[|B_t^1|^2 |B_t^2|^2 e^{\beta \int_0^t ds \, \xi(s,B_s^1) + \xi(s,B_s^2)} \right] \\
&= \frac{1}{16} E\left[|B_t^1|^2 |B_t^2|^2 \right] + \frac{1}{16} \sum_{k=1}^{\infty} \frac{\beta^{2k}}{k!} E\left\{ |B_t^1|^2 |B_t^2|^2 \int_{s_1,\dots,s_k \le t} \prod_{j=1}^{k} \left[p_1\left(\frac{B_{s_j}^1 - B_{s_j}^2}{\sqrt{2}} \right) + p_1(0) \right] \right\} \\
&= \frac{1}{16} E\left[|B_t^1|^2 |B_t^2|^2 e^{\beta^2 \int_0^t ds \, p_1\left(\frac{B_s^1 - B_s^2}{\sqrt{2}} \right)} e^{t\beta^2 p_1(0)} \right].
\end{aligned}$$

To do a change of variables, let $W^1 = \frac{B^1 - B^2}{\sqrt{2}}, W^2 = \frac{B^1 + B^2}{\sqrt{2}}$. Then $B^1 = \frac{\sqrt{2}}{2}(W^2 + W^1), B^2 = \frac{\sqrt{2}}{2}(W^2 - W^1)$, and W^1, W^2 are uncorrelated, thus independent, Brownian motions. Now the formula for K_t also follows.

□

2.2 Second Moment Estimates

2.2.1 Second Moment Estimate for Z_t

Lemma 2.2.1. *Let Z_t be as in (2.0.1), and $\lambda(\beta) = \frac{\beta^2}{2} p_1(0)$. For any fixed $\beta > 0, t > 0$, then*

$$\mathbb{E}[Z_t^2 e^{-2t\lambda(\beta)} - 1] \le \sum_{k=1}^{\infty} \frac{\beta^{2k}}{(4\pi)^k} (\log(1+2t))^k.$$

Proof. By Lemma 2.1.1, and by expansion, since

$$\mathbb{E}\left[Z_t^2 e^{-2t\lambda(\beta)} \right] = E\left[e^{\beta^2 \int_0^t ds \, p_1(B_s)} \right] = 1 + \sum_{k=1}^{\infty} \frac{\beta^{2k}}{k!} \left[\int_0^t ds \, p_1(B_s) \right]^k,$$

it is enough to show for each k that

$$E\left[\int_{s_1 < \dots < s_k \le t} \prod_{j=1}^{k} p_1(B_{s_j}) \right] \le \frac{1}{(4\pi)^k} (\log(1+2t))^k. \tag{2.2.2}$$

Since $p_1(x) = p_1(0,x)$, where $p_{2t}(x,y)$ is the transition density function of the Brownian motion, the idea is we can use Gaussian convolution to estimate the expectation iteratively. For $k = 1$,

$$\begin{aligned} E\left[\int_0^t ds_1 p_1(B_{s_1})\right] &= \int_0^t ds_1 \int dx_1\ p_1(x_1)p_{2s_1}(0,x_1) \\ &= \int_0^t ds_1\ p_{1+2s_1}(0,0) \\ &= \frac{1}{4\pi}\int_1^{1+2t} ds_1\ \frac{1}{s_1} \\ &= \frac{1}{4\pi}\log(1+2t). \end{aligned}$$

For $k = 2$, by Gaussian convolution, change of variables and the inequalities $1 + 2(t - s_1) \le 1 + 2t$ and $p_{s_2}(0, x_1) \le \frac{1}{2\pi s_2}$,

$$\begin{aligned} E\left[\int_0^t ds_1 \int_{s_1}^t ds_2\ p_1(B_{s_1})p_1(B_{s_2})\right] &= \int_0^t ds_1 \int_{s_1}^t ds_2 \int dx_1\ p_1(x_1)p_{2s_1}(0,x_1)p_{1+2(s_2-s_1)}(0,x_1) \\ &= \int_0^t ds_1 \int_1^{1+2(t-s_1)} ds_2\ \frac{1}{2}\int dx_1\ p_1(x_1)p_{2s_1}(0,x_1)p_{s_2}(0,x_1) \\ &\le \int_0^t ds_1 \int dx_1\ p_1(x_1)p_{2s_1}(0,x_1)\int_1^{1+2t} ds_2\ \frac{1}{4\pi}\frac{1}{s_2} \\ &= \frac{1}{(4\pi)^2}(\log(1+2t))^2, \end{aligned}$$

The general case is similar, again by Gaussian convolution, change of variables and the inequalities $1 + 2(t - s_{l-1}) \le 1 + 2t$ and $p_{s_l}(0, x_{l-1}) \le \frac{1}{2\pi s_l}$. ☐

2.2.2 Second Moment Estimate for K_t

Lemma 2.2.2. *Let K_t be as in (2.0.1), and $\lambda(\beta) = \frac{\beta^2}{2}p_1(0)$. For any fixed $\beta > 0$, and $t > 0$ large, say $\log t \ge 1000$, then*

$$\mathbb{E}[K_t^2 e^{-2t\lambda(\beta)} - t^2] \le t^2 \sum_{k=1}^{\infty} 100^k \beta^{2k}(\log t)^k.$$

First, we need a formula for the moments of nonzero mean two dimensional normal random variable, and some estimates.

Claim 2.2.3. *Let $N = (N_1, N_2)$ be two dimensional normal with mean x and covariance matrix tI, where I is the identity matrix. Then*

- $E|N|^2 = |x|^2 + 2t,$
- $E|N|^4 = |x|^4 + 8t|x|^2 + 8t^2.$

Proof. It follows by differentiating the characteristic function $Ee^{i\lambda\cdot N} = e^{i\lambda\cdot x}e^{-\frac{1}{2}t|\lambda|^2}$. ☐

Claim 2.2.4. *For $l = 2, 4$, and for T large, say $\log T \ge 1000$,*

$$\int_0^T dt \int dy\ p_1(y)p_t(x,y)|y|^l \le 100\log T.$$

Proof. It follows by first integrating over t and using asymptotic of the incomplete gamma function to estimate small and large y. More precisely, we will use the formula $\int_0^T dt\ \frac{1}{t}e^{-\frac{a}{t}} = \Gamma\left(0, \frac{a}{T}\right)$, where $\Gamma(0,s) = \int_s^\infty dt\ \frac{1}{t}e^{-t}$ is the incomplete gamma function, and the asymptotic $\frac{\Gamma(0,s)}{\log s^{-1}} \to 1$ as $s \to 0$, which implies the bound $\frac{\Gamma(0,s)}{\log s^{-1}} \leq 5$ for $0 \leq s \leq 1/2$. So, by Fubini,

$$\begin{aligned}\int dy\ p_1(y)|y|^l \int_0^T dt\ p_t(x,y) &= \frac{1}{2\pi}\int dy\ p_1(y)|y|^l \Gamma\left(0, \frac{|x-y|^2}{2T}\right)\left(\mathbb{1}_{|x-y|^2\leq T} + \mathbb{1}_{|x-y|^2\geq T}\right)\\ &\leq \frac{1}{2\pi}\int dy\ p_1(y)|y|^l\left[5\log\left(\frac{2T}{|x-y|^2}\right)\mathbb{1}_{|x-y|^2\leq T} + 5\mathbb{1}_{|x-y|^2\geq T}\right]\\ &= \frac{1}{2\pi}\int dy\ p_1(y)|y|^l\\ &\quad\times\left[5\log\left(\frac{2T}{|x-y|^2}\right)\mathbb{1}_{|x-y|^2\leq T}(\mathbb{1}_{|x-y|^2\leq 1} + \mathbb{1}_{1\leq|x-y|^2\leq T}) + 5\mathbb{1}_{|x-y|^2\geq T}\right]\end{aligned}$$

where in the inequality we use the estimate that on $|x-y|^2 \leq T$,

$$\Gamma\left(0, \frac{|x-y|^2}{2T}\right) = \log\left(\frac{2T}{|x-y|^2}\right)\frac{\Gamma\left(0, \frac{|x-y|^2}{2T}\right)}{\log\left(\frac{2T}{|x-y|^2}\right)} \leq 5\log\left(\frac{2T}{|x-y|^2}\right)$$

and on $|x-y|^2 \geq T$, $\Gamma(0,s) \leq 5$ for $s \geq 1/2$. Also, on $1 \leq |x-y|^2 \leq T$, note $\left|\log\left(\frac{1}{|x-y|^2}\right)\right| \leq \log T^2$. Continuing,

$$\int dy\ p_1(y)|y|^l \int_0^T dt\ p_t(x,y) \leq \int dy\ p_1(y)|y|^l\left[10\log T + \log\left(\frac{1}{|x-y|^2}\right)\mathbb{1}_{|x-y|^2\leq 1}\right] =: A + B$$

for T large. To estimate B, for $l = 2, 4$, note $2\pi p_1(y)|y|^l \leq 5$, and using polar coordinates, then

$$\int dy\ p_1(y)|y|^l \log\left(\frac{1}{|x-y|^2}\right)\mathbb{1}_{|x-y|^2\leq 1} \leq \int dy\ \log\left(\frac{1}{|y|^2}\right)\mathbb{1}_{|y|^2\leq 1} \leq 5\cdot 2\pi.$$

Finally, since $\int dy\ p_1(y)|y|^l$ is a finite constant, we conclude for T large that

$$\int dy \int_0^T dt\ p_1(y)p_t(x,y)|y|^l \leq 100\log T.$$

∎

We are ready to show Lemma 2.2.2.

Proof. Since $E\left[|B_t^2 + B_t^1|^2|B_t^2 - B_t^1|^2\right] = 64t^2$, as in the proof of the estimate for $\mathbb{E}[Z_t^2]$, it is enough to show

$$E\left[|B_t^2 + B_t^1|^2|B_t^2 - B_t^1|^2\int_{s_1<\cdots<s_k<t}\prod_{j=1}^k p_1(B_{s_j}^1)\right] \leq t^2 100^k(\log t)^k. \tag{2.2.3}$$

The idea here is also to estimate the expectation iteratively; t^2 comes from the second moments of the two Brownian motions B^1, B^2 at time t and the rest from the transition density functions as before in Lemma 2.2.1. Writing out the expectation, we have

$$
\begin{aligned}
E&\left[|B_t^2+B_t^1|^2|B_t^2-B_t^1|^2\int_{s_1<\cdots<s_k<t}\prod_{j=1}^k p_1(B^1_{s_j})\right]\\
&=\int_{s_1<\cdots<s_k<t}\int_{x_1,\dots,x_k}\prod_{j=1}^k p_1(x_j)p_{2(s_j-s_{j-1})}(x_{j-1},x_j)\int dx\ p_{2(t-s_k)}(x_k,x)\\
&\quad\times\int dy\ \left[|y|^4+2|y|^2|x|^2-4\left\langle y,x\right\rangle^2+|x|^4\right]p_{2t}(0,y)\\
&=\int_{s_1<\cdots<s_k<t}\int_{x_1,\dots,x_k}\prod_{j=1}^k p_1(x_j)p_{2(s_j-s_{j-1})}(x_{j-1},x_j)\int dx\ p_{2(t-s_k)}(x_k,x)\left[32t^2+|x|^4\right].
\end{aligned}
$$

For $32t^2$, by Lemma 2.2.1,

$$
\int_{s_1<\cdots<s_k<t}\int_{x_1,\dots,x_k}\prod_{j=1}^k p_1(x_j)p_{2(s_j-s_{j-1})}(x_{j-1},x_j)\int dx\ p_{2(t-s_k)}(x_k,x)32t^2\le 32t^2\frac{1}{(4\pi)^k}(\log(1+2t))^k.
$$

For $|x|^4$, using Claim 2.2.3,

$$
\begin{aligned}
&\int_{s_1<\cdots<s_k<t}\int_{x_1,\dots,x_k}\prod_{j=1}^k p_1(x_j)p_{2(s_j-s_{j-1})}(x_{j-1},x_j)\int dx\ p_{2(t-s_k)}(x_k,x)|x|^4\\
&\quad=\int_{s_1<\cdots<s_k<t}\int_{x_1,\dots,x_k}\prod_{j=1}^k p_1(x_j)p_{2(s_j-s_{j-1})}(x_{j-1},x_j)\left[|x_k|^4+16(t-s_k)|x_k|^2+32(t-s_k)^2\right].
\end{aligned}
$$

To continue, for $32(t-s_k)^2$, it is as before; for $16(t-s_k)|x_k|^2$ and $|x_k|^4$, we use Claim 2.2.4. For example, first estimating the last term, we get

$$
\int dx_k\ \int_0^{t-s_{k-1}}ds_kp_1(x_k)p_{2s_k}(x_{k-1},x_k)|x_k|^2\le\int dx_k\ \int_0^t ds_kp_1(x_k)p_{2s_k}(x_{k-1},x_k)|x_k|^2\le 100\log t.
$$

In any case, we have at most $100t^2\log t$. Now the rest follows from Lemma 2.2.1. □

2.3 Proof of Theorem 2.0.2

Recall the notations in Theorem 2.0.2: the polymer measure μ_t is defined in (1.2.4) and we choose $\beta=\frac{\alpha_t}{\sqrt{\log t}}$, where $\alpha_t\to 0$ as $t\to\infty$. We want to show as $t\to\infty$,

$$
\frac{\mu_t(|\frac{B_t}{\sqrt{4}}|^2)}{t}\to 1
$$

in probability. To do this, since we can rewrite $\mu_t(|\frac{B_t}{\sqrt{4}}|^2)=\frac{K_t}{Z_t}$, where Z_t,K_t are defined in (2.0.1), we will use Chebyshev's inequality and the second moment estimates in Lemmas 2.2.1 and 2.2.2 with $\beta=\frac{\alpha_t}{\sqrt{\log t}}$ to conclude the Theorem.

Lemma 2.3.1. *As $t\to\infty$, $Z_te^{-t\lambda(\beta)}\to 1$, $\frac{1}{Z_te^{-t\lambda(\beta)}}\to 1$, $\frac{K_te^{-t\lambda(\beta)}}{t}\to 1$ in probability.*

Proof. Fix $\epsilon > 0$. By Chebyshev's inequality,

$$\begin{aligned}
\mathbb{P}(|Z_t e^{-t\lambda(\beta)} - 1| > \epsilon) &\le \epsilon^{-2}\mathbb{E}\left\{\left[Z_t e^{-t\lambda(\beta)} - 1\right]^2\right\} \\
&= \epsilon^{-2}\left\{e^{-2t\lambda(\beta)}\mathbb{E}[Z_t^2] - 2e^{-t\lambda(\beta)}\mathbb{E}[Z_t] + 1\right\} \\
&= \epsilon^{-2}\left\{E\left[e^{\beta^2\int_0^t ds\ p_1(B_s^1)}\right] - 1\right\} \\
&\le \epsilon^{-2}\sum_{k=1}^{\infty}\beta^{2k}\frac{1}{(4\pi)^k}(\log(1+2t))^k,
\end{aligned}$$

where we use $\mathbb{E}[Z_t] = e^{t\lambda(\beta)}$, which follows from the proof of Lemma 2.1.1. Since $\beta = \frac{\alpha_t}{\sqrt{\log t}}$, where $\alpha_t \to 0$, we conclude the probability tends to 0.

Similarly,

$$\begin{aligned}
\mathbb{P}\left(\left|\frac{K_t e^{-t\lambda(\beta)}}{t} - 1\right| > \epsilon\right) &\le \epsilon^{-2}t^{-2}\mathbb{E}\left\{\left[K_t e^{-t\lambda(\beta)} - t\right]^2\right\} \\
&= \epsilon^{-2}t^{-2}\left\{e^{-2t\lambda(\beta)}\mathbb{E}[K_t^2] - 2e^{-t\lambda(\beta)}\mathbb{E}[K_t]t + t^2\right\} \\
&= \epsilon^{-2}t^{-2}\left\{\frac{1}{64}E\left[|B_t^2 + B_t^1|^2|B_t^2 - B_t^1|^2 e^{\beta^2\int_0^t ds\ p_1(B_s^1)}\right] - t^2\right\} \\
&\le \epsilon^{-2}t^{-2}t^2\sum_{k=1}^{\infty}\beta^{2k}100^k(\log t)^k,
\end{aligned}$$

which also tends to 0, and where we use $\mathbb{E}[K_t] = e^{t\lambda(\beta)}t$ (it also follows from the proof of Lemma 2.1.1).

Finally, we saw where $Z_t e^{-t\lambda(\beta)}$ is different from 1 has negligible probability. Since the function $f(x) = \frac{1}{x}$ is continuous for $x > 0$, it follows that $\frac{1}{Z_t e^{-t\lambda(\beta)}} \to 1$ in probability. □

Chapter 3

Exponential Moment of Two Dimensional Regularized Local Time

This chapter is independent of the others. We only need Theorem 3.0.1 for Chapter 4.

Let $B = (B_t)_{t\geq 0}$ be a two dimensional Brownian motion starting at 0, with second moment $E[|B_t|^2] = 2 \cdot (2t)$. (See Section 1.2.)

Theorem 3.0.1. *Let $\beta_\epsilon = \frac{4\pi}{\log \epsilon^{-2}} + \frac{\sigma}{(\log \epsilon^{-2})^2}$, $\sigma \in \mathbb{R}$. Then*

$$\lim_{\epsilon \to 0} \frac{1}{\log \epsilon^{-1}} \int dx\ E_x \left[\delta_0(B_t) e^{\beta_\epsilon \int_0^t ds\ \ \epsilon^{-2} p_1(\epsilon^{-1} B_s)} \right] = \int dx\ \frac{1}{2\pi i} \int_\gamma dz\ e^{tz} \frac{1}{2\pi} \frac{K_0(\sqrt{z}|x|)}{g(z)},$$

where $K_0(z)$ is the modified Bessel function of the second kind[1]*, $g(z) = \log \frac{\sqrt{z}}{2} - \psi(1) - \frac{\sigma}{8\pi} + \nu(p_1)$ with $\nu(p_1) = \int dx\ p_1(x) \int dy \log|x-y| p_1(y)$, and γ is a contour encircling the zero of $g(z)$ and the negative real line*[2]*.*

Note $\epsilon^{-2} p_1(\epsilon^{-1} x) \to \delta_0(x)$, so the Theorem gives a formula for the exponential moment of the regularized local time of two dimensional Brownian motion. Let us say a few words on some past work, and motivation for the particular form of the scaling β_ϵ and why we further divide the expectation by $\log \epsilon^{-1}$.

3.1 Background - Two Dimensional Local Time

For a two dimensional simple symmetric random walk S starting at 0, the local time at 0 is $L_n(S) = \sum_{i=1}^n \mathbb{1}_0(S_i)$. Erdős and Taylor [12] first computed the distribution of $L_n(S)$ and showed in distribution, as $n \to \infty$,

$$\frac{\pi}{\log n} L_n(S) \to X,$$

[1] By definition, $K_0(z) = -\sum_{k=0}^\infty \frac{(z/2)^{2k}}{k!^2} [\log(z/2) - \psi(k+1)]$, where $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ with $\Gamma(x)$ the gamma function.

[2] More generally, we have a similar formula for regularization $V(x)$ that is radial, real-analytic, and decays sufficiently fast at infinity.

where X is an exponential random variable with mean 1. Then, for any $\varphi(n)$ such that $\phi(n) := \frac{n}{\varphi(n)\log n} \to \infty$, Gantert and Zeitouni [15] derived a large deviation principle for $\frac{L_n(S)}{\log n\varphi(n)}$ in the sense that

$$\frac{\log \phi(n)}{\varphi(n)\log n} \log P\left[\frac{L_n(S)}{\varphi(n)\log n} \geq y\right] \to -y.$$

For Brownian motion, Darling and Kac [11] first computed the distribution of the local time of two dimensional Brownian motion B (with generator Δ) starting at 0 in the sense that in distribution, as $t \to \infty$,

$$\frac{4\pi}{\log t}\int_0^t ds\ p_1(B_s) \to X,$$

where X is an exponential random variable with mean 1.

Now, let $t = \epsilon^{-2}$. By Brownian scaling, first, we note the integrand scales to a delta function and it is equal to $\frac{4\pi}{\log \epsilon^{-2}}\int_0^1 ds\ \epsilon^{-2}p_1(\epsilon^{-1}B_s)$. Second, by the Feynman-Kac formula, the exponential moment of the local time of Brownian motion starting at x with regularization $p_1(x)$ is $\int dy\ \left[e^{-t(-\Delta-V_\epsilon)}\delta_y\right](x)$, where $V_\epsilon(x) = \frac{4\pi}{\log \epsilon^{-2}}\epsilon^{-2}p_1(\epsilon^{-1}x)$. And Theorem 3.0.1 is the limit of $\int dy\ \left[e^{-t(-\Delta-V_\epsilon)}\delta_y\right](x)$ for $x = 0$, as $\epsilon \to 0$.

In the Mathematical Physics community, Albeverio et al [2] showed that $-\Delta - V_\epsilon$ has a limit, say $-\hat{\Delta}_0$ in the norm resolvent sense as operators on $L^2(\mathbb{R}^2)$, and the latter has a resolvent kernel $(-\hat{\Delta}_0+z)^{-1}(x,y)$ for $z \in \mathbb{C} - \{\mathbb{R}_+ \cup \{\hat{z}_0\}\}$, $x \neq y$ and $x, y \neq 0$. Here $\hat{z}_0$ is an eigenvalue of the operator $-\hat{\Delta}_0$ in $\mathbb{R}_-$. We see, by heuristics, we have a similar formula to Theorem 3.0.1 for a Brownian motion NOT starting at 0, and in this case we do not need to divide the exponential moment by $\log \epsilon^{-1}$. On the other hand, if the Brownian motion starts at 0, it is not clear how one can adapt the results of [2] without understanding very well the exponential moment $E_0\left[e^{\frac{4\pi}{\log \epsilon^{-2}}\int_0^t ds\ \epsilon^{-2}p_1(\epsilon^{-1}B_s)}\right]$.

Finally, to see the choice of β_ϵ, we saw earlier the local time converges to an exponential random variable with mean 1, so heuristically its exponential moment converges to that of the exponential random variable. But the latter has a singularity, and β_ϵ is precisely this singularity point[3].

Remark 3.1.1. *Bertini and Cancrini [5] also considered the exponential moment of the regularized local time of two dimensional Brownian motion, based on [2], but their Brownian motion does NOT start at* 0*. More precisely, the main result in [5] is for the two dimensional stochastic heat equation, the authors computed the two point correlation function of the solution with $L^2(\mathbb{R}^2)$ initial conditions, from which they deduced the exponential moment for Brownian motion not starting at* 0 *(see (3.14) in Remark following Theorem 3.2 therein, and note the translation by x of their Brownian motion). As mentioned earlier, by heuristics, based on [2], it is not difficult to deduce the formula.*

Remark 3.1.2. *Another reason for working directly with $E_0\left[e^{\frac{4\pi}{\log \epsilon^{-2}}\int_0^t ds\ \epsilon^{-2}p_1(\epsilon^{-1}B_s)}\right]$ is that if we introduce an additional parameter λ in the renormalization $\lambda\frac{4\pi}{\log \epsilon^{-2}}$, by the method in this Chapter, we also have a similar formula for the exponential moment of the regularized local time, though it might or might not require further rescaling depending on the regime. If $\lambda < 1$, we call it the subcritical regime, $\lambda = 1$ critical, and $\lambda > 1$ supercritical. In all these cases, the main contribution is from a single eigenvalue of the operator $\Delta + \frac{4\pi}{\log \epsilon^{-2}}\epsilon^{-2}p_1(\epsilon^{-1}\cdot)$ (see Remark 3.2.2). The main difference is in the*

[3]Strictly speaking, since $\beta_\epsilon = \frac{1}{\log \epsilon^{-2}}\left(4\pi + \frac{\sigma}{\log \epsilon^{-2}}\right)$, only 4π is the singularity point. The reason we also include the second order term $\frac{\sigma}{\log \epsilon^{-2}}$ is to parameterize this singularity point.

subcritical regime, the eigenvalue tends to 0 *in* ϵ*; in the critical regime, the eigenvalue is constant; and in the supercritical regime, the eigenvalue tends to* ∞.

On the other hand, the authors of [2] claimed that for $\lambda > 1$ ($\lambda_1 > 2\pi$ *in their notation, see Theorem 2.4 on page 95 [2])* $-\Delta - \lambda \frac{4\pi}{\log \epsilon^{-2}} \epsilon^{-2} p_1(\epsilon^{-1}\cdot)$ *converges to* $-\Delta$ *in norm-resolvent on* $L^2(\mathbb{R}^2)$*, but this contradicts our result in the critical regime because clearly for* $\lambda > 1$, $E_0\left[e^{\frac{4\pi}{\log \epsilon^{-2}} \int_0^t ds\ \epsilon^{-2} p_1(\epsilon^{-1} B_s)}\right] \leq E_0\left[e^{\lambda \frac{4\pi}{\log \epsilon^{-2}} \int_0^t ds\ \epsilon^{-2} p_1(\epsilon^{-1} B_s)}\right]$*, and if the results of [2] were true in the supercritical regime, the right hand side would be of order 1, but we know from Theorem 3.0.1 that the left hand side is order* $\log \epsilon^{-1}$.

3.2 Key Step

Let $p_{1,\alpha}(x) = c_\alpha^{-1} e^{-\frac{|x|^2}{2}} \mathbb{1}_{|x|\leq \alpha}$, where $c_\alpha = 2\pi(1 - e^{-\alpha^2/2})$ is normalizing constant so that $p_{1,\alpha}(x)$ has mean 1. We will first compute the limit for the exponential moment with $p_{1,\alpha}(x)$.

Theorem 3.2.1. *Let* $\beta_\epsilon = \frac{4\pi}{\log \epsilon^{-2}} + \frac{\sigma}{(\log \epsilon^{-2})^2}$, $\sigma \in \mathbb{R}$*; for* $\delta > 0$ *small, let* $\alpha = \sqrt{2 \log(\log \epsilon^{-1})^{1+2\delta}}$*, then*

$$\lim_{\epsilon \to 0} \frac{1}{\log \epsilon^{-1}} \int dx\ E_x\left[\delta_0(B_t) e^{\beta_\epsilon \int_0^t ds\ \epsilon^{-2} p_{1,\alpha}(\epsilon^{-1} B_s)}\right] = \int dx\ \frac{1}{2\pi i} \int_\gamma dz\ e^{tz} \frac{1}{2\pi} \frac{K_0(\sqrt{z}|x|)}{g(z)}.$$

Remark 3.2.2. *The reason we first compute the formula for* $p_1(x)$ *cut off at* $|x| = \alpha$ *is that at the critical scaling* β_ϵ*, there is a matching condition at* $|x| = \infty$ *(one could first compute the limit with* $V(x) = \frac{1}{\pi} \mathbb{1}_{|x|\leq 1}$ *to see how the formula arises) that contributes to the presence of an eigenvalue of the operator* $\Delta + \beta_\epsilon \epsilon^{-2} p_{1,\alpha}(\epsilon^{-1}\cdot)$*, and it is the reason why the polymer has non-Gaussian fluctation at the critical scaling, see Chapter 4; otherwise it is not clear how to identify the eigenvalue.*

The reason we cut off at $|x| = \alpha = \sqrt{2 \log(\log \epsilon^{-1})^{1+2\delta}}$ *is that when we use Cauchy-Schwarz to deduce the formula for* $p_1(x)$*, the conjugate exponent need be chosen in a way that it does not affect the critical scaling* β_ϵ *and that we remain in the critical regime, rather than supercritical.*

Remark 3.2.3. *To show Theorem 3.2.1, we will solve the corresponding Feynman-Kac or heat equation by Laplace transform in the time variable* t*. The technical difficulty is that we need sharp estimates to identify the singularity, in particular the poles, to facilitate inversion of the Laplace transform, which will be most of the work.*

3.3 Feynman-Kac Equation

Let $u^{\epsilon,\alpha}(t, x) = E_x\left[\delta_0(B_t) e^{\beta_\epsilon \int_0^t ds\ \epsilon^{-2} p_{1,\alpha}(\epsilon^{-1} B_s)}\right]$, and $V(x) = \beta_\epsilon \epsilon^{-2} p_{1,\alpha}(\epsilon^{-1} x)$. Define $V_-(x) = \beta_\epsilon \epsilon^{-2} c_\alpha^{-1} e^{-\frac{|x|^2}{2\epsilon^2}}$, and $V_+(x) = 0$. Now write $V(x) = V_-(x)\mathbb{1}_{|x|\leq \epsilon\alpha} + V_+(x)\mathbb{1}_{|x|>\epsilon\alpha}$.

Lemma 3.3.1. $u^{\epsilon,\alpha}(t, x)$ *satisfies*[4]

$$\begin{cases} \partial_t u(t,x) = \Delta u(t,x) + V_-(x) u(t,x) & \text{for } |x| < \epsilon\alpha \\ \partial_t u(t,x) = \Delta u(t,x) + V_+(x) u(t,x) & \text{for } |x| > \epsilon\alpha \\ u(0,x) = \delta_0(x) \\ u(t,\infty) = 0. \end{cases}$$

[4]For a function $f(x)$, $x \in \mathbb{R}^2$, by $f(\infty) = 0$ we mean $\lim_{x\to\infty} f(x) = 0$.

Proof. From standard texts such as [27] and [20], we know the equation is satisfied if the potential $V(x)$ is continuous and bounded and the inital condition is continuous. In our case, $V(x)$ is bounded, but has only a simple jump discontinuity at $|x| = \epsilon\alpha$. Nevertheless, we will carry out the argument as in [27], and give some estimates which we will also need for the associated resolvent equation in Section 3.4, since proof for the resolvent equation is not immediately available from [20].

It[5] is clear $u^{\epsilon,\alpha}(t,x)$ satisfies the initial condition, and the boundary condition at $|x| = \infty$. To show it satisfies the equation, we take inspiration from [27] (see page 56) by expanding the function into a series and checking the equation each term satisfies.

Let $f(x) = \delta_0(x)$, and $[P_t g](x) = E_x[g(B_t)]$. Suppose $|x| < \epsilon\alpha$ (the other case $|x| > \epsilon\alpha$ is similar). Expanding $u^{\epsilon,\alpha}(t,x)$ into a series, then

$$u^{\epsilon,\alpha}(t,x) = E_x[f(B_t)] + \sum_{k=1}^{\infty} E_x\left[f(B_t) \int_{s_1<\cdots<s_k<t} \prod_{j=1}^{k} V(B_{s_j}) \right].$$

As in [27], using the Markov property and by a change of variables of the time variables, we can rewrite each term in the series as successive applications of the semigroup operator P_t followed by the multiplication operator where we multiply by $V(x)$,

$$\begin{aligned}\int_{s_1<\cdots<s_k<t} E_x\left[V(B_{s_1})\cdots V(B_{s_k})f(B_t)\right] &= \int_{s_k<\cdots<s_1<t} E_x\left[V(B_{t-s_1})\cdots V(B_{t-s_k})f(B_t)\right] \\ &= \int_{s_k<\cdots<s_1<t} \left[P_{t-s_1}VP_{s_1-s_2}V\cdots P_{s_{k-1}-s_k}VP_{s_k}f\right](x).\end{aligned}$$

It follows that

$$\begin{aligned}&\partial_t \int_0^t ds_1 \cdots \int_0^{s_{k-1}} ds_k \ \left[P_{t-s_1}VP_{s_1-s_2}V\cdots P_{s_{k-1}-s_k}VP_{s_k}f\right](x) \\ &\quad= V_-(x)\int_0^t ds_2 \cdots \int_0^{s_{k-1}} ds_k \left[P_{t-s_2}V\cdots P_{s_{k-1}-s_k}VP_{s_k}f\right](x) \\ &\qquad+ \Delta \int_0^t ds_1 \cdots \int_0^{s_{k-1}} ds_k \ \left[P_{t-s_1}VP_{s_1-s_2}V\cdots P_{s_{k-1}-s_k}VP_{s_k}f\right](x).\end{aligned}$$

Now, note the first term on the right hand side is multiplication of the $(k-1)^{th}$ term of the series over k by $V_-(x)$, and summing over k, we have

$$\sum_{k=0}^{\infty} \left[\partial_t - \Delta\right] E_x\left[f(B_t) \int_{s_1<\cdots<s_k<t} \prod_{j=1}^{k} V(B_{s_j}) \right] = \sum_{k=0}^{\infty} V_-(x) E_x\left[f(B_t) \int_{s_1<\cdots<s_k<t} \prod_{j=1}^{k} V(B_{s_j}) \right]. \tag{3.3.1}$$

To conclude $\partial_t u(t,x) = \Delta u(t,x) + V_-(x)u(t,x)$, we need to pull out the differential operators. For this, we use the Fundamental Theorem of Calculus.

[5] In what follows, to be precise, since the initial condition is $\delta_0(x)$, we should first smooth it out, check the equation, and then remove the smoothing.

For $y=(y_1,y_2)\in\mathbb{R}^2$, let

$$F_k(s;y_1,y_2)=E_y\left[f(B_s)\int_{s_1<\dots<s_k<s}\prod_{j=1}^k V(B_{s_j})\right],\qquad G_k(s;y_1,y_2)=\left[\partial_s-\partial_1^2-\partial_2^2\right]F_k(s;y_1,y_2),$$

where ∂_i is derivative in the i^{th} coordinate of the space variable $y=(y_1,y_2)$. Then, using identity (3.3.1) and by Fubini,

$$\begin{aligned}
\int_r^t ds\int_{a_2}^{x_2}dy_2\int_{a_1}^{x_1}dy_1\sum_{k=0}^\infty G_k(s;y_1,y_2) &= \int_r^t ds\int_{a_2}^{x_2}dy_2\int_{a_1}^{x_1}dy_1\sum_{k=0}^\infty V_-(y)F_k(s;y_1,y_2)\\
&=\sum_{k=0}^\infty\int_r^t ds\int_{a_2}^{x_2}dy_2\int_{a_1}^{x_1}dy_1\ V_-(y)F_k(s;y_1,y_2)\\
&=\sum_{k=0}^\infty\int_r^t ds\int_{a_2}^{x_2}dy_2\int_{a_1}^{x_1}dy_1\ G_k(s;y_1,y_2)\\
&=\sum_{k=0}^\infty\int_{a_2}^{x_2}dy_2\int_{a_1}^{x_1}dy_1\ \left[F_k(t;y_1,y_2)-F_k(r;y_1,y_2)\right]\\
&\quad-\sum_{k=0}^\infty\int_{a_2}^{x_2}dy_2\int_r^t ds\ \left[\partial_1F_k(s;x_1,y_2)-\partial_1F_k(s;a_1,y_2)\right]\\
&\quad-\sum_{k=0}^\infty\int_{a_1}^{x_1}dy_2\int_r^t ds\ \left[\partial_2F_k(s;y_1,x_2)-\partial_2F_k(s;y_1,a_2)\right]\\
&=\int_{a_2}^{x_2}dy_2\int_{a_1}^{x_1}dy_1\ \left[u(t;y_1,y_2)-u(r;y_1,y_2)\right]\\
&\quad-\int_{a_2}^{x_2}dy_2\int_r^t ds\ \left[\partial_1u(s;x_1,y_2)-\partial_1u(s;a_1,y_2)\right]\\
&\quad-\int_{a_1}^{x_1}dy_1\int_r^t ds\ \left[\partial_2u(s;y_1,x_2)-\partial_2u(s;y_1,a_2)\right].
\end{aligned}$$

Our conclusion follows by the Fundamental Theorem of Calculus by differentiating in t,x_1,x_2 on both sides, given we justify the last equality, in particular interchanging the partial ∂_i with the sum over k, because recall $u(t;x_1,x_2)=\sum_{k=0}^\infty F_k(t;x_1,x_2)$, and after we differentiate, we have

$$\sum_{k=0}^\infty G_k(t;x_1,x_2)=\partial_tu(t;x_1,x_2)-\partial_1^2u(t;x_1,x_2)-\partial_2^2u(t;x_1,x_2),$$

and

$$\sum_{k=0}^\infty G_k(t;x_1,x_2)=V_-(x)\sum_{k=0}^\infty F_k(t;x_1,x_2)=V_-(x)u(t;x_1,x_2).$$

Before we continue, writing out the expectation,

$$\begin{aligned}
&E_x\left[f(B_t)\int_{s_1<\dots<s_k<t}\prod_{j=1}^k V(B_{s_j})\right]\\
&\qquad=\int_{z^1,\dots,z^k}\int_{s_1<\dots<s_k<t}p_{2s_1}(x,z^1)V(z^1)\prod_{j=2}^k V(z^j)p_{2(s_j-s_{j-1})}(z^{j-1},z^j)p_{2(t-s_k)}(z^k,0).
\end{aligned}$$

The following estimate would suffice because with it we can apply the same type of argument as above using Fubini and the Fundamental Theorem of Calculus. ☐

Claim 3.3.2. *For $i = 1, 2$, $t > 0, x \in \mathbb{R}^2$,*

$$\sum_{k=0}^{\infty} \int_{z^1,\ldots,z^k} \int_{s_1<\cdots<s_k<t} \left| \partial_i p_{2s_1}(x, z^1) V(z^1) \prod_{j=2}^{k} V(z^j) p_{2(s_j-s_{j-1})}(z^{j-1}, z^j) p_{2(t-s_k)}(z^k, 0) \right| < \infty$$

and the function is continuous in t, x.

Proof. It is a direct computation using bound on $V(x)$, Gaussian convolution on the transition density functions and finally Cauchy-Schwarz to estimate the derivative. We do it only for ∂_1.

Let C be a constant such that $V(x) \leq C$. Then

$$\begin{aligned}
&\int_{s_1<\cdots<s_k<t} \int_{z^1,\ldots,z^k} \left| \partial_1 p_{2s_1}(x, z^1) V(z^1) \prod_{j=2}^{k} V(z^j) p_{2(s_j-s_{j-1})}(z^{j-1}, z^j) p_{2(t-s_k)}(z^k, 0) \right| \\
&\quad \leq C^k \int_{s_1<\cdots<s_k<t} \int_{z^1,\ldots,z^k} \left| \frac{x_1 - z_1^1}{2s_1} \right| p_{2s_1}(x, z^1) \prod_{j=2}^{k} p_{2(s_j-s_{j-1})}(z^{j-1}, z^j) p_{2(t-s_k)}(z^k, 0) \\
&\quad = C^k \int_0^t ds_1 \, \frac{(t-s_1)^{k-1}}{(k-1)!} \int dz^1 \, \left| \frac{x_1 - z_1^1}{s_1} \right| p_{2s_1}(x, z^1) p_{2(t-s_1)}(z^1, 0),
\end{aligned}$$

where in the equality we use Gaussian convolution, and the fact that $\int_{s_1}^t ds_2 \cdots \int_{s_{k-1}}^t ds_k = \frac{(t-s_1)^{k-1}}{(k-1)!}$. Continuing, to estimate the integral over z^1, by Cauchy-Schwarz,

$$\begin{aligned}
&\int dz^1 \, \left| \frac{x_1 - z_1^1}{s_1} \right| p_{2s_1}(x, z^1) p_{2(t-s_1)}(z^1, 0) \\
&\quad = \frac{1}{s_1} \int dz_2^1 \, \hat{p}_{2s_1}(x_2, z_2^1) \hat{p}_{2(t-s_1)}(z_2^1, 0) \int dz_1^1 \, \left| x_1 - z_1^1 \right| \hat{p}_{2s_1}(x_1, z_1^1) \hat{p}_{2(t-s_1)}(z_1^1, 0) \\
&\quad \leq \frac{1}{s_1} \hat{p}_{2t}(0, x_2) \left(\int dz_1^1 \, \left| x_1 - z_1^1 \right|^2 \hat{p}_{2s_1}^2(x_1, z_1^1) \right)^{1/2} \left(\int dz_1^1 \, \hat{p}_{2(t-s_1)}^2(z_1^1, 0) \right)^{1/2} \\
&\quad = C' \cdot \frac{1}{s_1} \hat{p}_{2t}(0, x_2) \left(\frac{1}{\sqrt{s_1}} s_1 \right)^{1/2} \left(\frac{1}{\sqrt{t - s_1}} \right)^{1/2},
\end{aligned}$$

where $\hat{p}_{2t}(a, b) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{(b-a)^2}{4t}}$ is the transition density of one dimensional Brownian motion, and C' another constant. Now first part of the Claim follows since $s_1^{-3/4}(t - s_1)^{-1/4}$ is integrable and the integral over s_1 is finite, so is the series when we sum over k; and the second part follows by Dominated Convergence. ☐

3.4 Resolvent Equation

For $z \in \mathbb{C}$ such that $\Re z > \beta_\epsilon \epsilon^{-2} c_\alpha^{-1}$, define $w^{z,\epsilon,\alpha}(x) = \int_0^\infty dt \; e^{-tz} u^{\epsilon,\alpha}(t, x)$.

Lemma 3.4.1. *$w^{z,\epsilon,\alpha}(x)$ is well-defined*[6]*, and it is continuously differentiable at $|x| = \epsilon\alpha$. Also,*

[6] We will see later that we only need z such that $z \in \mathbb{C} - \mathbb{R}_-$ and $z \neq z_{0,\alpha} = 4e^{2(\frac{\sigma}{8\pi} + \psi(1) - \nu(p_{1,\alpha}))}$, where $\nu(p_{1,\alpha}) = \int dx \; p_{1,\alpha}(x) \int dy \; \log|x - y| \; p_{1,\alpha}(y)$.

$w^{z,\epsilon,\alpha}(x)$ *satisfies*[7]

$$\begin{cases} -(\Delta w(x) + (V_-(x) - z)w(x)) = \delta_0(x) & \text{for } |x| < \epsilon\alpha \\ -(\Delta w(x) + (V_+(x) - z)w(x)) = 0 & \text{for } |x| > \epsilon\alpha \\ w(\infty) = 0. \end{cases}$$

Proof. Since $u^{\epsilon,\alpha}(t,x) \le p_{2t}(x,0)e^{\beta_\epsilon \epsilon^{-2} c_\alpha^{-1} t}$, by choice of z, we see $w^{z,\epsilon,\alpha}(x)$ is well-defined. For continuity of the first derivative at $|x| = \epsilon\alpha$, it follows from estimate in Claim 3.3.2 and Dominated Convergence. Also, it is clear that $w^{z,\epsilon,\alpha}(\infty) = 0$.

Let us show $w^{z,\epsilon,\alpha}(x)$ satisfies the equation[8]. Suppose $|x| < \epsilon\alpha$ (the other case $|x| > \epsilon\alpha$ is similar). By Lemma 3.3.1, we know

$$\partial_t u^{\epsilon,\alpha}(t,x) = \Delta u^{\epsilon,\alpha}(t,x) + V_-(x)u^{\epsilon,\alpha}(t,x). \tag{3.4.2}$$

Then by integration by parts,

$$\begin{aligned} \int_0^\infty dt\ e^{-tz}\Delta u^{\epsilon,\alpha}(t,x) &= \int_0^\infty dt\ e^{-tz}\partial_t u^{\epsilon,\alpha}(t,x) - \int_0^\infty dt\ e^{-tz} V_-(x)u^{\epsilon,\alpha}(t,x) \\ &= -\delta_0(x) + z\int_0^\infty dt\ e^{-tz}u^{\epsilon,\alpha}(t,x) - V_-(x)\int_0^\infty dt\ e^{-tz}u^{\epsilon,\alpha}(t,x). \end{aligned}$$

To conclude, it is enough to show

$$\int_0^\infty dt\ e^{-tz}\Delta u^{\epsilon,\alpha}(t,x) = \Delta\int_0^\infty dt\ e^{-tz}u^{\epsilon,\alpha}(t,x),$$

which follows by the Fundamental Theorem of Calculus as is done in Lemma 3.3.1.

More precisely, for $y = (y_1, y_2) \in \mathbb{R}^2$, by (3.4.2) and Fubini,

$$\begin{aligned} &\int_{a_1}^{x_1} dy_1 \int_{a_2}^{x_2} dy_2 \int_0^\infty dt\ e^{-tz}(\Delta - \partial_t)u^{\epsilon,\alpha}(t;y_1,y_2) \\ &\quad = \int_0^\infty dt\ e^{-tz}\int_{a_1}^{x_1} dy_1 \int_{a_2}^{x_2} dy_2\ (\Delta - \partial_t)u^{\epsilon,\alpha}(t;y_1,y_2) \\ &\quad = \int_0^\infty dt\ e^{-zt}\int_{a_2}^{x_2} dy_2\,[\partial_1 u^{\epsilon,\alpha}(t;x_1,y_2) - \partial_1 u^{\epsilon,\alpha}(t;a_1,y_2)] \\ &\qquad + \int_0^\infty dt\ e^{-zt}\int_{a_1}^{x_1} dy_1\,[\partial_2 u^{\epsilon,\alpha}(t;y_1,x_2) - \partial_2 u^{\epsilon,\alpha}(t;y_1,a_2)] \\ &\qquad + \int_0^\infty dt\ e^{-zt}\int_{a_1}^{x_1} dy_1 \int_{a_2}^{x_2} dy_2\ \partial_t u^{\epsilon,\alpha}(t;y_1,y_2), \end{aligned}$$

where ∂_i is derivative in the i^{th} coordinate of the space variable $y = (y_1, y_2)$. Note $\int_{a_1}^{x_1} dy_1 \int_{a_2}^{x_2} dy_2 \int_0^\infty dt\ e^{-tz}\partial_t u^{\epsilon,\alpha}(t;y_1,y_2) = \int_0^\infty dt\ e^{-zt}\partial_t \int_{a_1}^{x_1} dy_1 \int_{a_2}^{x_2} dy_2\ u^{\epsilon,\alpha}(t;y_1,y_2)$, which follows by integration by parts and the initial condition. This implies that if we can justify interchanging the partial ∂_t with the integrals, then we can cancel the integrals of $\partial_t u^{\epsilon,\alpha}(t;y_1,y_2)$ from both sides. Thus by the Fundamental Theorem of Calculus, as is done in Lemma 3.3.1, it is enough to interchange the partials ∂_i with the integrals over y_1, y_2, t, and interchange ∂_t with the integrals over y_1, y_2.

[7] Again, for a function $f(x)$, $x \in \mathbb{R}^2$, by $f(\infty) = 0$ we mean $\lim_{x\to\infty} f(x) = 0$.

[8] More precisely, to check the equation, we need be careful with $\delta_0(x)$. We should again first smooth out the delta function, check the equation, and then remove the smoothing.

To interchange ∂_i, we use estimates in Claim 3.3.2. To interchange ∂_t, it is enough to show $\partial_t u^{\epsilon,\alpha}(t;y_1,y_2)$ is continuous in y_1, y_2. To see this, recall[9] for solution to

$$\begin{cases} \partial_t u(t,x) = \Delta u(t,x) + V(x)u(t,x) \\ u(0,x) = \varphi(x), \end{cases}$$

where $V(x), f(x)$ are continuous bounded functions in $\mathbb{R}^2$, that $\partial_t u(t,x), \partial_{ij} u(t,x)$ are all continuous functions. In our case, though $V(x)$ is not continuous, it is bounded and it has only a simple jump discontinuity, so we can first smooth it, check the equation, and then remove the smoothing.

□

Remark 3.4.2. *On one hand, we see $w^{z,\epsilon,\alpha}(x)$ is a radial solution[10] to the equation in Lemma 3.4.1 such that it is continuously differentiable on $|x| = \epsilon\alpha$, and we are going to show such solution is unique. On the other hand, by definition, $w^{z,\epsilon,\alpha}(x)$ is Laplace transform in t of $u^{\epsilon,\alpha}(t,x)$, so to recover $u^{\epsilon,\alpha}(t,x)$, we invert $w^{z,\epsilon,\alpha}(x)$ in z.*

3.5 Two General Solutions

Proposition 3.5.1. *Two independent radial solutions to*

$$\Delta w(x) + (\gamma e^{-\frac{|x|^2}{\rho}} - z)w(x) = 0 \tag{3.5.3}$$

are

$$\tilde{I}_{z,\rho,\gamma}(|x|) := \sum_{k=0}^{\infty} \sum_{l=1}^{k} \sum_{i_1+\cdots+i_l=k-l} \frac{d_{i_1}\cdots d_{i_l}}{e_{i_1+1}\cdots e_{i_l+\cdots+i_1+l}} |x|^{2k} \tag{3.5.4}$$

and

$$\begin{aligned} \tilde{K}_{z,\rho,\gamma}(|x|) := &-\sum_{k=0}^{\infty} \sum_{l=1}^{k} \sum_{i_1+\cdots+i_l=k-l} \frac{d_{i_1}\cdots d_{i_l}|x|^{2k}}{2^{2k}k!^2} \\ &\times \left[(\log(\sqrt{d_0}|x|/2) f_{i_1,\ldots,i_l}(0) + f'_{i_1,\ldots,i_l}(0)) - \psi(k+1) f_{i_1,\ldots,i_l}(0)\right], \end{aligned} \tag{3.5.5}$$

where[11] $d_0 = z - \gamma$, $d_k = -\gamma\frac{(-1)^k}{\rho^k k!}, e_k = (2k)^2$ for $k \geq 1$, $\psi(k+1) = \frac{\Gamma'(k+1)}{\Gamma(k+1)}$ with $\Gamma(x)$ the gamma function, and $f_{i_1,\ldots,i_l}(\nu) = \frac{2^{2(k-l)}k!(1+\nu)\cdots(k+\nu)}{(i_1+1)(i_1+1+\nu)\cdots(i_l+\cdots+i_1+l)(i_1+\cdots+i_1+l+\nu)}$.

Proof. The argument is the same as solving the usual Bessel equation (equation (3.5.3) without the function $\gamma e^{-\frac{|x|^2}{\rho}}$). The only difference in this case is we also need to solve a certain recurrence relation. Suppose solution to (3.5.3) is radial, and let us solve the equation in four steps.

Step 1: Find the recurrence relation.

[9] See [20] Corollary 4.5 on page 270.

[10] The potential function $p_{1,\alpha}(x)$ is radial, so by expanding $u^{\epsilon,\alpha}(t,x)$ into a series, one can show $u^{\epsilon,\alpha}(t,x)$ itself is radial in x.

[11] Strictly speaking, as defined in the proof $e_k = 2^2k(k+\nu)$; by abuse of notation, we also denote $e_k = (2k)^2$, i.e. for $\nu = 0$.

As is done with the usual Bessel functions, we will solve the more general equation with $\nu > 0$ and then let $\nu \to 0+$,

$$r^2 w''(r) + r w'(r) + (r^2(\gamma e^{-\frac{r^2}{\rho}} - z) - \nu^2) w(r) = 0. \tag{3.5.6}$$

Write $w(r) = \sum_{k=0}^{\infty} \hat{a}_k r^{k+\xi}$. Substituting w, w', w'' into equation (3.5.6), one can check that $\hat{a}_{2k} = \frac{\hat{d}_0 \hat{a}_{2(k-1)} + \hat{d}_2 \hat{a}_{2(k-2)} + \cdots + \hat{d}_{2(k-1)} \hat{a}_0}{\hat{e}_{2k}}$, $\hat{a}_{2k+1} = 0, \xi = \nu$, where $\hat{d}_0 = z - \gamma, \hat{d}_{2k} = -\gamma \frac{(-1)^k}{\rho^k k!}, \hat{e}_{2k} = 2^2 k(k+\nu)$ for $k \geq 1$ and $\hat{a}_0$ to be chosen. To simplify the notations, let $a_k = \hat{a}_{2k}, d_k = \hat{d}_{2k}, e_k = \hat{e}_{2k}$. Then we want to solve

$$a_k = \frac{d_0 a_{k-1} + d_1 a_{k-2} + d_2 a_{k-3} + \cdots + d_{k-2} a_1 + d_{k-1} a_0}{e_k}.$$

Step 2: Solve the recurrence relation.

We want to show for $k \geq 1$,

$$a_k = \sum_{l=0}^{k-1} \sum_{i_1 + \cdots + i_{k-l-1} \leq l} \frac{a_0 d_{i_1} \cdots d_{i_{k-l-1}} d_{l-(i_1+\cdots+i_{k-l-1})}}{e_{i_1+1} e_{i_2+i_1+2} \cdots e_{i_{k-l-1}+\cdots+i_1+(k-l-1)} e_k}.$$

First, let us see what the terms look like. For $k = 1$, we have $l = 0$, and $a_1 = \frac{a_0 d_0}{e_1}$. For $k \geq 2$, and $l = 0$, we have $i_1 = \cdots = i_{k-1} = 0$, and the summand is $\frac{a_0 d_0^{k-1} d_0}{e_1 e_2 \cdots e_{k-1} e_k}$; for $l = k-1$, the indices $i_1, \ldots, i_{k-l-1}$ are vacuous, so the summand is $\frac{a_0 d_{k-1}}{e_k}$.

To solve the recurrence relation, we do it by induction. Suppose it is true for $k \geq 1$. Note for $k \geq 1$, a_k has k terms for $l = 0, 1, \ldots, k-1$, a_{k-1} has $k-1$ terms for $l = 0, 1, \ldots, k-2$ etc. The main idea is we group the terms "diagonally" in the sense that we group $l = 0$ for a_k with itself; we group $l = 1$ for a_k with $l = 0$ for a_{k-1}; $l = 2$ for a_k with $l = 1$ for a_{k-1} and $l = 0$ for a_{k-2} etc. Then we rewrite a_{k+1}

$$\begin{aligned} a_{k+1} = &\left(\frac{d_0}{e_{k+1}} \frac{a_0 d_0^k}{e_1 e_2 \cdots e_k} \right) + \left(\frac{d_0}{e_{k+1}} \sum_{i_1+\cdots+i_{k-2} \leq 1} \frac{a_0 d_{i_1} \cdots d_{i_{k-2}} d_{1-(i_1+\cdots+i_{k-2})}}{e_{i_1+1} \cdots e_{i_{k-2}+\cdots+i_1+(k-2)} e_k} + \frac{d_1}{e_{k+1}} \frac{a_0 d_0^{k-1}}{e_1 \cdots e_{k-2} e_{k-1}} \right) \\ &+ \left(\frac{d_0}{e_{k+1}} \sum_{i_1+\cdots+i_{k-3} \leq 2} \frac{a_0 d_{i_1} \cdots d_{i_{k-3}} d_{2-(i_1+\cdots+i_{k-3})}}{e_{i_1+1} \cdots e_{i_{k-3}+\cdots+i_1+(k-3)} e_k} + \frac{d_1}{e_{k+1}} \sum_{i_1+\cdots+i_{k-3} \leq 1} \frac{a_0 d_{i_1} \cdots d_{i_{k-3}} d_{1-(i_1+\cdots+i_{k-3})}}{e_{i_1+1} \cdots e_{i_{k-3}+\cdots+i_1+(k-3)} e_{k-1}} \right. \\ &\quad \left. + \frac{d_2}{e_{k+1}} \frac{a_0 d_0^{k-2}}{e_1 \cdots e_{k-3} e_{k-2}} \right) + \cdots + \left(\frac{d_0}{e_{k+1}} \frac{a_0 d_{k-1}}{e_k} + \frac{d_1}{e_{k+1}} \frac{a_0 d_{k-2}}{e_{k-1}} + \cdots + \frac{d_{k-1}}{e_{k+1}} \frac{a_0 d_0}{e_1} \right) + \left(\frac{a_0 d_k}{e_{k+1}} \right) \\ &=: c_{0,k+1} + c_{1,k+1} + c_{2,k+1} + \cdots + c_{k-1,k+1} + c_{k,k+1}. \end{aligned}$$

Each $c_{l,k+1}$ is equal to $\sum_{i_1+\cdots+i_{k-l} \leq l} \frac{a_0 d_{i_1} \cdots d_{i_{k-l}} d_{l-(i_1+\cdots+i_{k-l})}}{e_{i_1+1} \cdots e_{i_{k-l}+\cdots+i_1+(k-l)} e_{k+1}}$ for $l = 0, \ldots, k$, based on the possibilities $i_1 + \cdots + i_{k-l} = j$ for $j = 0, \ldots, l$. For example, for $l = 1$,

$$\begin{aligned} c_{1,k+1} &= \frac{d_0}{e_{k+1}} \sum_{i_1+\cdots+i_{k-2} \leq 1} \frac{a_0 d_{i_1} \cdots d_{i_{k-2}} d_{1-(i_1+\cdots+i_{k-2})}}{e_{i_1+1} \cdots e_{i_{k-2}+\cdots+i_1+(k-2)} e_k} + \frac{d_1}{e_{k+1}} \frac{a_0 d_0^{k-1}}{e_1 \cdots e_{k-2} e_{k-1}} \\ &= \sum_{i_1+\cdots+i_{k-1} \leq 1} \frac{a_0 d_{i_1} \cdots d_{i_{k-1}} d_{1-(i_1+\cdots+i_{k-1})}}{e_{i_1+1} \cdots e_{i_{k-1}+\cdots+i_1+(k-1)} e_{k+1}}, \end{aligned}$$

based on the two possibilities, either $i_1 + \cdots + i_{k-1} = 0$ or $i_1 + \cdots + i_{k-1} = 1$. If it is the first possibility, then $i_1 = \cdots = i_{k-1} = 0$, and it is the second term; if it is the second possibility, then it is the first term because we can rewrite

$$\sum_{i_1+\cdots+i_{k-2}\le 1} \frac{a_0 d_{i_1}\cdots d_{i_{k-2}} d_{1-(i_1+\cdots+i_{k-2})}}{e_{i_1+1}\cdots e_{i_{k-2}+\cdots+i_1+(k-2)}e_k} = \sum_{i_1+\cdots+i_{k-2}+i_{k-1}=1} \frac{a_0 d_{i_1}\cdots d_{i_{k-2}} d_{i_{k-1}}}{e_{i_1+1}\cdots e_{i_{k-2}+\cdots+i_1+(k-2)}e_{i_{k-1}+\cdots+i_1+(k-1)}}.$$

Similarly, for $l = 2$,

$$c_{2,k+1} = \frac{d_0}{e_{k+1}} \sum_{i_1+\cdots+i_{k-3}\le 2} \frac{a_0 d_{i_1}\cdots d_{i_{k-3}} d_{2-(i_1+\cdots+i_{k-3})}}{e_{i_1+1}\cdots e_{i_{k-3}+\cdots+i_1+(k-3)}e_k} + \frac{d_1}{e_{k+1}} \sum_{i_1+\cdots i_{k-3}\le 1} \frac{a_0 d_{i_1}\cdots d_{i_{k-3}} d_{1-(i_1+\cdots+i_{k-3})}}{e_{i_1+1}\cdots e_{i_{k-3}+\cdots+i_1+(k-3)}e_{k-1}}$$
$$+\frac{d_2}{e_{k+1}} \frac{a_0 d_0^{k-2}}{e_1\cdots e_{k-3}e_{k-2}} = \sum_{i_1+\cdots+i_{k-2}\le 2} \frac{a_0 d_{i_1}\cdots d_{i_{k-2}} d_{2-(i_1+\cdots+i_{k-2})}}{e_{i_1+1}\cdots e_{i_{k-2}+\cdots+i_1+(k-2)}e_{k+1}},$$

based on the three possbilities, either $i_1 + \cdots + i_{k-2} = 0$, $i_1 + \cdots + i_{k-2} = 1$, or $i_1 + \cdots + i_{k-2} = 2$. In general, we have

$$a_{k+1} = \sum_{l=0}^{k} \sum_{i_1+\cdots+i_{k-l}\le l} \frac{a_0 d_{i_1}\cdots d_{i_{k-l}} d_{l-(i_1+\cdots+i_{k-l})}}{e_{i_1+1}e_{i_2+i_1+2}\cdots e_{i_{k-l}+\cdots+i_1+(k-l)}e_{k+1}}.$$

Step 3: Deduce solutions for (3.5.3).

We see two independent solutions to (3.5.6) are

- $\tilde{I}_{z,\rho,\gamma;\nu}(r) := \sum_{k=0}^{\infty}\sum_{l=1}^{k}\sum_{i_1+\cdots+i_l=k-l} \frac{a_0 d_{i_1}\cdots d_{i_l}}{e_{i_1+1}\cdots e_{i_l+\cdots+i_1+l}} r^{2k+\nu}$,
- $\tilde{K}_{z,\rho,\gamma;\nu}(r) := \frac{\pi}{2}\frac{\tilde{I}_{z,\rho,\gamma;-\nu}(r)-\tilde{I}_{z,\rho,\gamma;\nu}(r)}{\sin(\nu\pi)}$.

So two independent solutions to (3.5.3) are

- $\tilde{I}_{z,\rho,\gamma}(|x|) := \lim_{\nu\to 0}\tilde{I}_{z,\rho,\gamma;\nu}(|x|)$,
- $\tilde{K}_{z,\rho,\gamma}(|x|) := \lim_{\nu\to 0}\tilde{K}_{z,\rho,\gamma;\nu}(|x|)$.

Step 4: Simplify the solutions to (3.5.3).

To see what the second solution $\tilde{K}_{z,\rho,\gamma}(|x|)$ looks like, let us rewrite the summand in $\tilde{I}_{z,\rho,\gamma:\nu}(|x|)$. Recall the definitions $e_k = 2^2 k(k+\nu)$, $d_0 = z-\gamma$. Choosing[12] $a_0 = \frac{(z-\gamma)^{\nu/2}}{2^\nu\Gamma(1+\nu)}$,

$$\begin{aligned}\tilde{I}_{z,\rho,\gamma:\nu}(|x|) &= \sum_{k=0}^{\infty}\sum_{l=1}^{k}\sum_{i_1+\cdots+i_l=k-l} \frac{a_0 d_{i_1}\cdots d_{i_l}|x|^{2k+\nu}}{e_{i_1+1}\cdots e_{i_l+\cdots+i_1+l}} \\ &= \sum_{k=0}^{\infty}\sum_{l=1}^{k}\sum_{i_1+\cdots+i_l=k-l} \frac{d_0^{\nu/2} d_{i_1}\cdots d_{i_l}|x|^{2k+\nu}}{2^{2k+\nu}k!\Gamma(k+\nu+1)} \\ &\quad\times \frac{2^{2(k-l)}k!(1+\nu)\cdots(k+\nu)}{(i_1+1)(i_1+1+\nu)\cdots(i_l+\cdots+i_1+l)(i_1+\cdots+i_1+l+\nu)} \\ &= \sum_{k=0}^{\infty}\sum_{l=1}^{k}\sum_{i_1+\cdots+i_l=k-l} \frac{d_{i_1}\cdots d_{i_l}|x|^{2k}}{2^{2k}k!}\frac{(\sqrt{d_0}|x|/2)^\nu}{\Gamma(k+\nu+1)} f_{i_1,\ldots,i_l}(\nu),\end{aligned}$$

[12]This choice of the constant is to ensure the two solutions $\tilde{I}_{z,\rho,\gamma}, \tilde{K}_{z,\rho,\gamma}$ have the same initial conditions at $r = 0+$ as the usual modified Bessel functions, which we recall are two independent radial solutions to (3.5.3) without the function $\gamma e^{-\frac{|x|^2}{\rho}}$.

where in the second equality we use the identity $(1+\nu)\cdots(k+\nu)\Gamma(\nu+1) = \Gamma(k+1+\nu)$, and in the third equality we use the definition $f_{i_1,\ldots,i_l}(\nu) = \frac{2^{2(k-l)}k!(1+\nu)\cdots(k+\nu)}{(i_1+1)(i_1+1+\nu)\cdots(i_l+\cdots+i_1+l)(i_1+\cdots+i_1+l+\nu)}$. By l'Hopital's rule, we see

$$\tilde{K}_{z,\rho,\gamma}(|x|) = \frac{\pi}{2}\lim_{\nu\to 0}\frac{\frac{d}{d\nu}\tilde{I}_{z,\rho,\gamma;-\nu}(|x|) - \frac{d}{d\nu}\tilde{I}_{z,\rho,\gamma;\nu}(|x|)}{\pi\cos(\nu\pi)}$$

$$= -\sum_{k=0}^{\infty}\sum_{l=1}^{k}\sum_{i_1+\cdots+i_l=k-l}\frac{d_{i_1}\cdots d_{i_l}|x|^{2k}}{2^{2k}k!^2}\left[(\log(\sqrt{d_0}|x|/2)f_{i_1,\ldots,i_l}(0) + f'_{i_1,\ldots,i_l}(0)) - \psi(k+1)f_{i_1,\ldots,i_l}(0)\right],$$

where $\psi(k+1) = \frac{\Gamma'(k+1)}{\Gamma(k+1)}$.

□

Remark 3.5.2. *Key Notations: Set $\rho = 2\epsilon^2, \gamma = \beta_\epsilon\epsilon^{-2}c_\alpha^{-1}$. In this case, we denote*

- $\tilde{I}_{z,\epsilon,\alpha}(r) := \tilde{I}_{z,\rho,\gamma}(r)$,
- $\tilde{K}_{z,\epsilon,\alpha}(r) := \tilde{K}_{z,\rho,\gamma}(r)$.

Remark 3.5.3. *To see $\tilde{I}_{z,\epsilon,\alpha}(r), \tilde{K}_{z,\epsilon,\alpha}(r)$ are well-defined for any $r \geq 0$, one can use the integral equations they satisfy as in Section 3.8.*

3.6 Some Properties of the Solutions

Before we proceed to solve (3.4.1), we need the small value asymptotic of the functions $\tilde{I}_{z,\epsilon,\alpha}(r), \tilde{K}_{z,\epsilon,\alpha}(r)$ and their relation.

Lemma 3.6.1. *For $r \to 0^+$,*

1. $\tilde{I}_{z,\epsilon,\alpha}(r) = 1 + O(r^2)$,
2. $\tilde{I}'_{z,\epsilon,\alpha}(r) = O(r)$,
3. $\tilde{K}_{z,\epsilon,\alpha}(r) = -(\log(r/2) - \psi(1)) + O(r^2\log r)$,
4. $\tilde{K}'_{z,\epsilon,\alpha}(r) = -(1/r + O(r\log r))$.

Proof. This is clear from the definition (3.5.4) of $\tilde{I}_{z,\epsilon,\alpha}(r)$ and (3.5.5) of $\tilde{K}_{z,\epsilon,\alpha}(r)$ by isolating the term for $l = k$, from which we get

$$\tilde{I}_{z,\epsilon,\alpha}(r) = I_0(\sqrt{d_0}r) + \hat{I}_{z,\epsilon,\alpha}(r), \qquad \tilde{K}_{z,\epsilon,\alpha}(r) = K_0(\sqrt{d_0}r) + \hat{K}_{z,\epsilon,\alpha}(r).$$

where $I_0(r) = \sum_{k=0}^{\infty}\frac{r^{2k}}{k!^2 2^{2k}}, K_0(r) = -\sum_{k=0}^{\infty}\frac{r^{2k}}{k!^2 2^{2k}}[\log(r/2) - \psi(k+1)]$ are the modified Bessel functions of first and second kind respectively, and $\hat{I}_{z,\epsilon,\alpha}(r), \hat{K}_{z,\epsilon,\alpha}(r)$ are by definition the remainder terms.

More precisely, from definitions,

$$\begin{aligned}\tilde{I}_{z,\epsilon,\alpha}(r) &= 1 + \sum_{k=1}^{\infty}\frac{d_0^k}{e_1\cdots e_k}r^{2k} + \sum_{k=2}^{\infty}\sum_{l=1}^{k-1}\sum_{i_1+\cdots+i_l=k-l}\frac{d_{i_1}\cdots d_{i_l}}{e_{i_1+1}\cdots e_{i_l+\cdots+i_1+l}}r^{2k}\\ &= \sum_{k=0}^{\infty}\frac{d_0^k}{2^{2k}k!^2}r^{2k} + \sum_{k=2}^{\infty}\sum_{l=1}^{k-1}\sum_{i_1+\cdots+i_l=k-l}\frac{d_{i_1}\cdots d_{i_l}}{2^{2l}(i_1+1)^2\cdots(i_1+\cdots i_l+l)^2}r^{2k}\\ &=: I_0(\sqrt{d_0}r) + \hat{I}_{z,\epsilon,\alpha}(r).\end{aligned}$$

Similarly, by isolating the term for $l = k$, we have

$$\begin{aligned}\tilde{K}_{z,\epsilon,\alpha}(r) = & -\sum_{k=0}^{\infty} \frac{d_0^k}{2^{2k}k!^2} r^{2k} \left[\log(\sqrt{d_0}r/2) - \psi(k+1)\right] \\ & -\sum_{k=2}^{\infty}\sum_{l=1}^{k-1}\sum_{i_1+\cdots+i_l=k-l} \frac{d_{i_1}\cdots d_{i_l}}{2^{2k}k!^2} r^{2k} \left\{\left[\log(\sqrt{d_0}r/2) - \psi(k+1)\right] f_{i_1,\ldots,i_l}(0) + f'_{i_1,\ldots,i_l}(0)\right\} \\ =: & K_0(\sqrt{d_0}r) + \hat{K}_{z,\epsilon,\alpha}(r).\end{aligned}$$

We deduce

$$I_0(\sqrt{d_0}r) = 1 + \frac{d_0 r^2}{4} + O(d_0^2 r^4); \tag{3.6.7}$$

$$\sqrt{d_0}I_0'(\sqrt{d_0}r) = \frac{d_0 r}{2} + \frac{d_0^2 r^3}{16} + O(d_0^3 r^5); \tag{3.6.8}$$

$$\hat{I}_{z,\epsilon,\alpha}(r) = O(r^4); \tag{3.6.9}$$

$$\hat{I}'_{z,\epsilon,\alpha}(r) = O(r^3); \tag{3.6.10}$$

$$-K_0(\sqrt{d_0}r) = (\log(\sqrt{d_0}r/2) - \psi(1)) + O(d_0 r^2 \log(\sqrt{d_0}r)); \tag{3.6.11}$$

$$-\sqrt{d_0}K_0'(\sqrt{d_0}r) = (1/r + O(d_0 r \log(\sqrt{d_0}r))); \tag{3.6.12}$$

$$-\hat{K}_{z,\epsilon,\alpha}(r) = O(r^4 \log r); \tag{3.6.13}$$

$$-\hat{K}'_{z,\epsilon,\alpha}(r) = O(r^3 \log r). \tag{3.6.14}$$

□

Lemma 3.6.2.

$$\tilde{K}_{z,\epsilon,\alpha}(r)\tilde{I}'_{z,\epsilon,\alpha}(r) - \tilde{K}'_{z,\epsilon,\alpha}(r)\tilde{I}_{z,\epsilon,\alpha}(r) = \frac{1}{r}.$$

Proof. This is the Wronskian analogue of the usual modified Bessel functions. Recall $\tilde{I}_{z,\epsilon,\alpha}(r), \tilde{K}_{z,\epsilon,\alpha}(r)$ are two independent solutions to

$$\frac{d}{dr}(rw'(r)) + (r(\gamma e^{-\frac{r^2}{\rho}} - z))w(r) = 0,$$

where $\rho = 2\epsilon^2, \gamma = \beta_\epsilon \epsilon^{-2} c_\alpha^{-1}$. Call $u_1 = \tilde{I}_{z,\epsilon,\alpha}, u_2 = \tilde{K}_{z,\epsilon,\alpha}$. Multiply the equation for u_1 by u_2 and subtract it from the equation for u_2 multiplied by u_1, we get

$$\begin{aligned}0 &= u_2(r)\left[\frac{d}{dr}(ru_1'(r)) + (r(\gamma e^{-\frac{r^2}{\rho}} - z))u_1(r)\right] - u_1(r)\left[\frac{d}{dr}(ru_2'(r)) + (r(\gamma e^{-\frac{r^2}{\rho}} - z))u_2(r)\right] \\ &= u_2(r)(u_1'(r) + ru_1''(r)) - u_1(r)(u_2'(r) + ru_2''(r)) \\ &= -W(u_1(r), u_2(r)) - r(u_1(r)u_2''(r) + u_1'(r)u_2'(r) - u_1'(r)u_2'(r) - u_1''(r)u_2(r)) \\ &= -\frac{d}{dr}(rW(u_1(r), u_2(r))),\end{aligned}$$

where $W(f(r), g(r)) = f(r)g'(r) - f'(r)g(r)$ is the Wronskian of f, g, and it implies

$$W(u_1(r), u_2(r)) = \frac{C}{r}$$

for some constant C independent of r. Finally, to find C,

$$\begin{aligned} C &= \lim_{r\to 0^+} rW(u_1(r), u_2(r)) \\ &= \lim_{r\to 0^+} r(u_1(r)u_2'(r) - u_1'(r)u_2(r)) \\ &= -1, \end{aligned}$$

using the small value asymptotic of $\tilde{I}_{z,\epsilon,\alpha}, \tilde{I}'_{z,\epsilon,\alpha}, \tilde{K}_{z,\epsilon,\alpha}, \tilde{K}'_{z,\epsilon,\alpha}$ by Lemma 3.6.1.

□

3.7 Solution to the Resolvent Equation

Recall the notations $V(x) = \beta_\epsilon \epsilon^{-2} p_{1,\alpha}(\epsilon^{-1}x)$ and $V(x) = V_-(x)\mathbb{1}_{|x|\le\epsilon\alpha} + V_+(x)\mathbb{1}_{|x|>\epsilon\alpha}$.

Proposition 3.7.1. *For $z \in \mathbb{C}$ such that $\Re z > \beta_\epsilon \epsilon^{-2} c_\alpha^{-1}$, solution $w^{z,\epsilon,\alpha}(x)$ to*

$$\begin{cases} -(\Delta w(x) + (V_-(x) - z)w(x)) = \delta_0(x) & \text{for } |x| < \epsilon\alpha \\ -(\Delta w(x) + (V_+(x) - z)w(x)) = 0 & \text{for } |x| > \epsilon\alpha \\ w(\infty) = 0, \end{cases}$$

which is continuous differentiable on $|x| = \epsilon\alpha$, is given by
for $|x| \le \epsilon\alpha$,

$$\begin{aligned} w_-^{z,\epsilon,\alpha}(x) = &\frac{1}{2\pi} \frac{K_0(\sqrt{z}\epsilon\alpha)}{K_0(\sqrt{z}\epsilon\alpha)\epsilon\alpha\tilde{I}'_{z,\epsilon,\alpha}(\epsilon\alpha) - \sqrt{z}\epsilon\alpha K_0'(\sqrt{z}\epsilon\alpha)\tilde{I}_{z,\epsilon,\alpha}(\epsilon\alpha)} \frac{\tilde{I}_{z,\epsilon,\alpha}(|x|)}{\tilde{I}_{z,\epsilon,\alpha}(\epsilon\alpha)} \\ &- \frac{1}{2\pi} \frac{\tilde{I}_{z,\epsilon,\alpha}(|x|)}{\tilde{I}_{z,\epsilon,\alpha}(\epsilon\alpha)} \tilde{K}_{z,\epsilon,\alpha}(\epsilon\alpha) + \frac{1}{2\pi}\tilde{K}_{z,\epsilon,\alpha}(|x|), \end{aligned}$$

and for $|x| \ge \epsilon\alpha$,

$$w_+^{z,\epsilon,\alpha}(x) = \frac{1}{2\pi} \frac{K_0(\sqrt{z}|x|)}{K_0(\sqrt{z}\epsilon\alpha)\epsilon\alpha\tilde{I}'_{z,\epsilon,\alpha}(\epsilon\alpha) - \sqrt{z}\epsilon\alpha K_0'(\sqrt{z}\epsilon\alpha)\tilde{I}_{z,\epsilon,\alpha}(\epsilon\alpha)}.$$

Proof. To solve the equation, we first do it in a ball $B(0, L)$ of radius L centered at the origin with zero boundary condition on $|x| = L$ and then let $L \to \infty$. Let us break down the proof into four steps.
Step 1: Find general solution in a large ball $B(0, L)$.

By Proposition 3.5.1, for $|x| \le \epsilon\alpha$,

$$w_{-,L}^{z,\epsilon,\alpha}(x) = -c_{1,-}\tilde{I}_{z,\epsilon,\alpha}(|x|) - c_{2,-}\tilde{K}_{z,\epsilon,\alpha}(|x|),$$

and for $\epsilon\alpha \le |x| \le L$,

$$w_{+,L}^{z,\epsilon,\alpha}(x) = -c_{1,+}I_0(\sqrt{z}|x|) - c_{2,+}K_0(\sqrt{z}|x|),$$

(note the minus sign in (3.7.1)) for some constants $c_{1,\pm}, c_{2,\pm}$ to be determined subject to appropriate boundary conditions. The second equation has the usual modified Bessel functions because $V_+(x) = 0$.
Step 2: Compute coefficients of solution in Step 1 that satisfies $\delta_0(x)$ at $x = 0$.

To find $c_{1,-}, c_{2,-}$, we need $w^{z,\epsilon,\alpha}_{-,L}(x)$ satisfies $\delta_0(x)$ at $x=0$. Since $\tilde{K}_{z,\epsilon,\alpha}(r) \approx -\log r$ as $r \to 0^+$, it is not hard to find the constant by standard argument.

Define $F_\sigma(x) = c_{1,-}\tilde{I}_{z,\epsilon,\alpha}(\sqrt{|x|^2+\sigma^2}) + c_{2,-}\tilde{K}_{z,\epsilon,\alpha}(\sqrt{|x|^2+\sigma^2})$. We are going to show with $c_{2,-} = -\frac{1}{2\pi}$ and $c_{1,-}$ arbitrary, as $\sigma \to 0$,

$$\Delta F_\sigma(x) + (|x|^2+\sigma^2)^{-1}|x|^2(V_-(\sqrt{|x|^2+\sigma^2}) - z)F_\sigma(x) \to \delta_0(x). \tag{3.7.15}$$

Note we also have

$$\Delta F_\sigma(x) + (|x|^2+\sigma^2)^{-1}|x|^2(V_-(\sqrt{|x|^2+\sigma^2}) - z)F_\sigma(x) \to \Delta F_0(x) + (V_-(x)-z)F_0(x),$$

where $F_0(x) := \lim_{\sigma\to 0} F_\sigma(x) = -w^{z,\epsilon,\alpha}_{-,L}(x)$.

To compute the left hand side of (3.7.15), first, we have

$$\begin{aligned}
&\partial_i^2 \tilde{I}_{z,\epsilon,\alpha}(\sqrt{|x|^2+\sigma^2}) \\
&\quad = \tilde{I}''_{z,\epsilon,\alpha}(\sqrt{|x|^2+\sigma^2})x_i^2(|x|^2+\sigma^2)^{-1} + \tilde{I}'_{z,\epsilon,\alpha}(\sqrt{|x|^2+\sigma^2})\left((|x|^2+\sigma^2)^{-1/2} - x_i^2(|x|^2+\sigma^2)^{-3/2}\right),
\end{aligned}$$

and a similar expression for $\partial_i^2\tilde{K}_{z,\epsilon,\alpha}(\sqrt{|x|^2+\sigma^2})$. Now, adding and subtracting suitable terms so we can use the equation $\tilde{I}_{z,\epsilon,\alpha}(r)$ satisfies to simplify, we have

$$\begin{aligned}
\Delta\tilde{I}_{z,\epsilon,\alpha}(\sqrt{|x|^2+\sigma^2}) &= \left[\tilde{I}''_{z,\epsilon,\alpha}(\sqrt{|x|^2+\sigma^2})(\sqrt{|x|^2+\sigma^2})^2 + \tilde{I}'_{z,\epsilon,\alpha}(\sqrt{|x|^2+\sigma^2})\sqrt{|x|^2+\sigma^2}\right]\frac{|x|^2}{(|x|^2+\sigma^2)^2} \\
&\quad + (\sqrt{|x|^2+\sigma^2})^2\tilde{I}_{z,\epsilon,\alpha}(\sqrt{|x|^2+\sigma^2})(V_-(\sqrt{|x|^2+\sigma^2})-z)\frac{|x|^2}{(|x|^2+\sigma^2)^2} \\
&\quad + \tilde{I}'_{z,\epsilon,\alpha}(\sqrt{|x|^2+\sigma^2})\left(2(|x|^2+\sigma^2)^{-1/2} - |x|^2(|x|^2+\sigma^2)^{-3/2} - |x|^2(|x|^2+\sigma^2)^{-3/2}\right) \\
&\quad - (\sqrt{|x|^2+\sigma^2})^2\tilde{I}_{z,\epsilon,\alpha}(\sqrt{|x|^2+\sigma^2})(V_-(\sqrt{|x|^2+\sigma^2})-z)\frac{|x|^2}{(|x|^2+\sigma^2)^2} \\
&= 2\tilde{I}'_{z,\epsilon,\alpha}(\sqrt{|x|^2+\sigma^2})\left(\sigma^2(|x|^2+\sigma^2)^{-3/2}\right) \\
&\quad - |x|^2(|x|^2+\sigma^2)^{-1}\tilde{I}_{z,\epsilon,\alpha}(\sqrt{|x|^2+\sigma^2})(V_-(\sqrt{|x|^2+\sigma^2})-z),
\end{aligned}$$

where in the second equality we use $\tilde{I}_{z,\epsilon,\alpha}(r)$ satisfies

$$r^2w''(r) + rw'(r) + r^2(V_-(r)-z)w(r) = 0.$$

Thus

$$\begin{aligned}
&\Delta F_\sigma(x) + (|x|^2+\sigma^2)^{-1}|x|^2(V_-(\sqrt{|x|^2+\sigma^2})-z)F_\sigma(x) \\
&\quad = c_{1,-}\tilde{I}'_{z,\epsilon,\alpha}(\sqrt{|x|^2+\sigma^2})2\left(\sigma^2(|x|^2+\sigma^2)^{-3/2}\right) + c_{2,-}\tilde{K}'_{z,\epsilon,\alpha}(\sqrt{|x|^2+\sigma^2})2\left(\sigma^2(|x|^2+\sigma^2)^{-3/2}\right).
\end{aligned}$$

To conclude (3.7.15), let $\Phi(x) = 2(|x|^2+1)^{-3/2}$, so $2\sigma^2(|x|^2+\sigma^2)^{-3/2} = \sigma(\sigma^{-2}\Phi(\sigma^{-1}x))$, and we will show for any φ test function that is smooth with compact support,

$$\int \tilde{I}'_{z,\epsilon,\alpha}(\sqrt{|x|^2+\sigma^2})2\left(\sigma^2(|x|^2+\sigma^2)^{-3/2}\right)\varphi(x) \\ = \int \tilde{I}'_{z,\epsilon,\alpha}(\sqrt{|x|^2+\sigma^2})\sigma(\sigma^{-2}\Phi(\sigma^{-1}x))\varphi(x) \to 0. \tag{3.7.16}$$

And with $c_{2,-} = -\frac{1}{2\pi}$,

$$\int c_{2,-}\tilde{K}'_{z,\epsilon,\alpha}(\sqrt{|x|^2+\sigma^2})2\left(\sigma^2(|x|^2+\sigma^2)^{-3/2}\right)\varphi(x) \\ = \int c_{2,-}\tilde{K}'_{z,\epsilon,\alpha}(\sqrt{|x|^2+\sigma^2})\sigma(\sigma^{-2}\Phi(\sigma^{-1}x))\varphi(x) \to \varphi(0). \tag{3.7.17}$$

For statement (3.7.16), let $M = \max_{x\in \operatorname{supp}(\varphi)} |\tilde{I}'_{z,\epsilon,\alpha}(\sqrt{|x|^2+\sigma^2})|$, since $\tilde{I}'_{z,\epsilon,\alpha}(r)$ is well-defined everywhere. Then $\sigma M \to 0$ as $\sigma \to 0$, and it is clear that

$$\int \tilde{I}'_{z,\epsilon,\alpha}(\sqrt{|x|^2+\sigma^2})\sigma(\sigma^{-2}\Phi(\sigma^{-1}x))\varphi(x) \to 0.$$

For statement (3.7.17), using the small value asymptotic of $\tilde{K}_{z,\epsilon,\alpha}(r)$ by Lemma 3.6.1 and definitions of $c_{2,-}$, $\Phi(x)$,

$$\begin{aligned}
&\int c_{2,-}\tilde{K}'_{z,\epsilon,\alpha}(\sqrt{|x|^2+\sigma^2})2\left(\sigma^2(|x|^2+\sigma^2)^{-3/2}\right)\varphi(x) \\
&= \int c_{2,-}\tilde{K}'_{z,\epsilon,\alpha}(\sigma\sqrt{|x|^2+1})\sigma\Phi(x)\varphi(\sigma x) \\
&= -\int c_{2,-}\sigma\Phi(x)\varphi(\sigma x)\left(\frac{1}{\sigma\sqrt{|x|^2+1}} + O(\sigma\sqrt{|x|^2+1}\log(\sigma\sqrt{|x|^2+1}))\right) \\
&= \frac{1}{\pi}\left[\int \frac{1}{(|x|^2+1)^2}\varphi(\sigma x) + \int \Phi(x)\varphi(\sigma x)O(\sigma^2)\right]
\end{aligned}$$

where the last line tends to $\varphi(0)$ since $\int_0^\infty \frac{r}{(1+r^2)^2} = \frac{1}{2}$.

Step 3: Compute coefficients of solution in Step 1 that satisfies zero boundary condition on $|x| = L$ and is continuously differentiable on $|x| = \epsilon\alpha$.

To find the other constants $c_{1,-}, c_{1,+}, c_{2,+}$, we use the boundary condition $-a_L$ on $|x| = \epsilon\alpha$ and 0 on $|x| = L$, where a_L to be determined. For $w^{z,\epsilon,\alpha}_{-,L}(x)$, to determine $c_{1,-}$, we need $-c_{1,-}\tilde{I}_{z,\epsilon,\alpha}(\epsilon\alpha) + \frac{1}{2\pi}\tilde{K}_{z,\epsilon,\alpha}(\epsilon\alpha) = -a_L$ on $|x| = \epsilon\alpha$, which implies

$$c_{1,-} = \frac{a_L + \frac{1}{2\pi}\tilde{K}_{z,\epsilon,\alpha}(\epsilon\alpha)}{\tilde{I}_{z,\epsilon,\alpha}(\epsilon\alpha)}.$$

For $w^{z,\epsilon,\alpha}_{+,L}(x)$, we need $-c_{1,+}I_0(\sqrt{z}\epsilon\alpha) - c_{2,+}K_0(\sqrt{z}\epsilon\alpha) = -a_L$ on $|x| = \epsilon\alpha$ and $-c_{1,+}I_0(\sqrt{z}L) - c_{2,+}K_0(\sqrt{z}L) = 0$ on $|x| = L$, which gives

$$c_{1,+} = -\frac{a_LK_0(\sqrt{z}L)}{K_0(\sqrt{z}\epsilon\alpha)I_0(\sqrt{z}L) - I_0(\sqrt{z}\epsilon\alpha)K_0(\sqrt{z}L)}, \qquad c_{2,+} = \frac{a_LI_0(\sqrt{z}L)}{K_0(\sqrt{z}\epsilon\alpha)I_0(\sqrt{z}L) - I_0(\sqrt{z}\epsilon\alpha)K_0(\sqrt{z}L)}.$$

Next, matching $\nabla w_{-,L}^{z,\epsilon,\alpha}(x)$ and $\nabla w_{+,L}^{z,\epsilon,\alpha}(x)$ on $|x| = \epsilon\alpha$, and using the definitions of $c_{1,\pm}, c_{2,\pm}$, we need

$$\frac{-a_L \tilde{I}'_{z,\epsilon,\alpha}(\epsilon\alpha)}{\tilde{I}_{z,\epsilon,\alpha}(\epsilon\alpha)} - \frac{1}{2\pi}\frac{\tilde{K}_{z,\epsilon,\alpha}(\epsilon\alpha)\tilde{I}'_{z,\epsilon,\alpha}(\epsilon\alpha) - \tilde{K}'_{z,\epsilon,\alpha}(\epsilon\alpha)\tilde{I}_{z,\epsilon,\alpha}(\epsilon\alpha)}{\tilde{I}_{z,\epsilon,\alpha}(\epsilon\alpha)}$$
$$= \frac{a_L K_0(\sqrt{z}L)}{K_0(\sqrt{z}\epsilon\alpha)I_0(\sqrt{z}L) - I_0(\sqrt{z}\epsilon\alpha)K_0(\sqrt{z}L)}\sqrt{z}I'_0(\sqrt{z}\epsilon\alpha)$$
$$- \frac{a_L I_0(\sqrt{z}L)}{K_0(\sqrt{z}\epsilon\alpha)I_0(\sqrt{z}L) - I_0(\sqrt{z}\epsilon\alpha)K_0(\sqrt{z}L)}\sqrt{z}K'_0(\sqrt{z}\epsilon\alpha),$$

and solving for a_L, we get

$$a_L = -\frac{1}{2\pi}\frac{\tilde{K}_{z,\epsilon,\alpha}(\epsilon\alpha)\tilde{I}'_{z,\epsilon,\alpha}(\epsilon\alpha) - \tilde{K}'_{z,\epsilon,\alpha}(\epsilon\alpha)\tilde{I}_{z,\epsilon,\alpha}(\epsilon\alpha)}{\tilde{I}_{z,\epsilon,\alpha}(\epsilon\alpha)}\frac{1}{\sqrt{z}\frac{K_0(\sqrt{z}L)I'_0(\sqrt{z}\epsilon\alpha) - I_0(\sqrt{z}L)K'_0(\sqrt{z}\epsilon\alpha)}{K_0(\sqrt{z}\epsilon\alpha)I_0(\sqrt{z}L) - I_0(\sqrt{z}\epsilon\alpha)K_0(\sqrt{z}L)} + \frac{\tilde{I}'_{z,\epsilon,\alpha}(\epsilon\alpha)}{\tilde{I}_{z,\epsilon,\alpha}(\epsilon\alpha)}}.$$

Step 4: Conclude solution to (3.7.1) by letting size of the ball $B(0, L)$ tend to ∞.

First, the large value asymptotic[13] for the usual modified Bessel functions are for z such that $|\arg z| < \pi$, and $\nu = 0, 1$,

$$I_\nu(z) = \frac{1}{\sqrt{2\pi z}}e^z\left[1 + O(|z|^{-1})\right], \qquad K_\nu(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}\left[1 + O(|z|^{-1})\right], \tag{3.7.18}$$

which gives $\frac{I_0(\sqrt{z}L)}{K_0(\sqrt{z}L)} \to \infty$, and

$$\lim_{L\to\infty} a_L = -\frac{1}{2\pi}\frac{\tilde{K}_{z,\epsilon,\alpha}(\epsilon\alpha)\tilde{I}'_{z,\epsilon,\alpha}(\epsilon\alpha) - \tilde{K}'_{z,\epsilon,\alpha}(\epsilon\alpha)\tilde{I}_{z,\epsilon,\alpha}(\epsilon\alpha)}{\tilde{I}_{z,\epsilon,\alpha}(\epsilon\alpha)}\frac{1}{-\sqrt{z}\frac{K'_0(\sqrt{z}\epsilon\alpha)}{K_0(\sqrt{z}\epsilon\alpha)} + \frac{\tilde{I}'_{z,\epsilon,\alpha}(\epsilon\alpha)}{\tilde{I}_{z,\epsilon,\alpha}(\epsilon\alpha)}}$$
$$= -\frac{1}{2\pi}K_0(\sqrt{z}\epsilon\alpha)\frac{\tilde{K}_{z,\epsilon,\alpha}(\epsilon\alpha)\tilde{I}'_{z,\epsilon,\alpha}(\epsilon\alpha) - \tilde{K}'_{z,\epsilon,\alpha}(\epsilon\alpha)\tilde{I}_{z,\epsilon,\alpha}(\epsilon\alpha)}{\tilde{I}'_{z,\epsilon,\alpha}(\epsilon\alpha)K_0(\sqrt{z}\epsilon\alpha) - \sqrt{z}K'_0(\sqrt{z}\epsilon\alpha)\tilde{I}_{z,\epsilon,\alpha}(\epsilon\alpha)}.$$

Finally, using Lemma 3.6.2 to simplify, the solution to (3.7.1) is
for $|x| \le \epsilon\alpha$,

$$w_-^{z,\epsilon,\alpha}(x) = \frac{\frac{1}{2\pi}\frac{K_0(\sqrt{z}\epsilon\alpha)}{\epsilon\alpha\tilde{I}'_{z,\epsilon,\alpha}(\epsilon\alpha)K_0(\sqrt{z}\epsilon\alpha) - \sqrt{z}\epsilon\alpha K'_0(\sqrt{z}\epsilon\alpha)\tilde{I}_{z,\epsilon,\alpha}(\epsilon\alpha)} - \frac{1}{2\pi}\tilde{K}_{z,\epsilon,\alpha}(\epsilon\alpha)}{\tilde{I}_{z,\epsilon,\alpha}(\epsilon\alpha)}\tilde{I}_{z,\epsilon,\alpha}(|x|) + \frac{1}{2\pi}\tilde{K}_{z,\epsilon,\alpha}(|x|)$$

and for $|x| \ge \epsilon\alpha$,

$$w_+^{z,\epsilon,\alpha}(x) = \frac{1}{2\pi}\frac{K_0(\sqrt{z}|x|)}{\epsilon\alpha\tilde{I}'_{z,\epsilon,\alpha}(\epsilon\alpha)K_0(\sqrt{z}\epsilon\alpha) - \sqrt{z}\epsilon\alpha K'_0(\sqrt{z}\epsilon\alpha)\tilde{I}_{z,\epsilon,\alpha}(\epsilon\alpha)}$$

because recall $c_{1,+} \to 0$ and $c_{2,+} \to \frac{a}{K_0(\sqrt{z}\epsilon\alpha)}$ as $L \to \infty$, where $a = \lim_{L\to\infty} a_L$.

□

[13]See [22], page 123.

3.8 Key Estimates

Suppose[14] $z \in \mathbb{C} - \mathbb{R}_-$. In preparation for Section 3.9, where we identify the poles in the resolvent solution $w^{z,\epsilon,\alpha}(x)$, we are going to pick out the part that contributes to the poles from the difference $\epsilon\alpha \tilde{I}'_{z,\epsilon,\alpha}(\epsilon\alpha)K_0(\sqrt{z}\epsilon\alpha) - \sqrt{z}\epsilon\alpha K_0'(\sqrt{z}\epsilon\alpha)\tilde{I}_{z,\epsilon,\alpha}(\epsilon\alpha)$ in the denominator of $w^{z,\epsilon,\alpha}(x)$. Since $w^{z,\epsilon,\alpha}(x)$ is the Laplace transform of $u^{\epsilon,\alpha}(t,x)$ in t, with Laplace variable z, we expect the poles should come from only small z. To do this, we need to expand $\tilde{I}'_{z,\epsilon,\alpha}(\epsilon\alpha)$ up to third order terms and $\tilde{I}_{z,\epsilon,\alpha}(\epsilon\alpha)$ up to second order terms. The main work is to control the remainders. But it turns out they satisfy the same differential equation as $\tilde{I}_{z,\epsilon,\alpha}(r)$ but with certain forcing functions, and for small z, we can uniformly bound them. Moreover, we have a Green function that we can use to solve for the remainders. To complete the analysis for small z, we will use integral equations to compare the Green function for the differential equation for $\tilde{I}_{z,\epsilon,\alpha}(r)$ with the one in terms of the usual modified Bessel functions. This would allow us to estimate the Green function for the equation for $\tilde{I}_{z,\epsilon,\alpha}(r)$.

In subsection 3.8.1, we will obtain the estimates for small z, and we will do the analysis for large z in subsection 3.8.2.

3.8.1 Small Asymptotic Estimates

First, we observe a scaling property.

Lemma 3.8.1. $\tilde{I}_{z,\epsilon,\alpha}(\epsilon\alpha) = \tilde{J}_{z,\epsilon,\alpha}(1)$, *where the function* $\tilde{J}_{z,\epsilon,\alpha}(r)$ *is solution to*

$$\begin{cases} w''(r) + r^{-1}w'(r) + (\beta_\epsilon \alpha^2 c_\alpha^{-1} e^{-\frac{\alpha^2}{2}r^2} - z\epsilon^2\alpha^2)w(r) = 0 \\ w(0) = 1, w'(0) = 0. \end{cases} \tag{3.8.19}$$

Proof. Recall $\tilde{I}_{z,\epsilon,\alpha}(r)$ is solution to

$$\begin{cases} w''(r) + r^{-1}w'(r) + (\beta_\epsilon \epsilon^{-2} c_\alpha^{-1} e^{-\frac{r^2}{2\epsilon^2}} - z)w(r) = 0 \\ w(0) = 1, w'(0) = 0. \end{cases}$$

The scaling property is clear from multiplying the equation by r^2 and evaluating it at $\epsilon\alpha$.

More precisely, recall definition

$$\tilde{I}_{z,\epsilon,\alpha}(\epsilon\alpha) = 1 + \sum_{k=1}^{\infty}\sum_{l=1}^{k}\sum_{i_1+\cdots+i_l=k-l}\frac{d_{i_1}\cdots d_{i_l}}{e_{i_1+1}\cdots e_{i_l+\cdots+i_1+l}}(\epsilon\alpha)^{2k},$$

where $e_k = (2k)^2, d_0 = z - \beta_\epsilon\epsilon^{-2}c_\alpha^{-1}, d_k = -\beta_\epsilon\epsilon^{-2}c_\alpha^{-1}\frac{(-1)^k}{k!(2\epsilon^2)^k}$. Let i_{v_s}'s and i_{u_t}'s be such that $\{i_1, \ldots, i_l\} = \{i_{v_1}, \ldots, i_{v_j}, i_{u_1}, \ldots, i_{u_{l-j}}\}$, and suppose $i_{v_1} = \cdots = i_{v_j} = 0$, so $i_1 + \cdots + i_l = i_{u_1} +$

[14] As $w^{z,\epsilon,\alpha}(x)$ is defined in Section 3.4, $z \in \mathbb{C}$ is such that $\Re z > \beta_\epsilon\epsilon^{-2}c_\alpha^{-1}$. In this section, the analysis will be done for all $z \in \mathbb{C} - \mathbb{R}_-$.

$\cdots + i_{u_{l-j}} = k - l$. Then

$$\begin{aligned} d_0^j d_{i_{u_1}} \cdots d_{i_{u_{l-j}}} (\epsilon\alpha)^{2k} &= \frac{(-1)^{k-l}}{2^{k-l}} (z\epsilon^2 - \beta_\epsilon c_\alpha^{-1})^j (-\beta_\epsilon c_\alpha^{-1})^{l-j} \frac{\alpha^{2k}}{i_{u_1}! \cdots i_{u_{l-j}}!} \\ &= (-1)^{k-l} \left(\frac{\alpha^2}{2}\right)^{k-l} (z\epsilon^2\alpha^2 - \beta_\epsilon c_\alpha^{-1}\alpha^2)^j (-\beta_\epsilon c_\alpha^{-1}\alpha^2)^{l-j} \frac{1}{i_{u_1}! \cdots i_{u_{l-j}}!} \\ &=: f_0^j f_{i_{u_1}} \cdots f_{i_{u_{l-j}}}, \end{aligned}$$

where $f_0 = z\epsilon^2\alpha^2 - \beta_\epsilon c_\alpha^{-1}\alpha^2$, $f_k = -\beta_\epsilon \alpha^2 c_\alpha^{-1} \frac{\alpha^{2k}}{k!} \frac{(-1)^k}{2^k}$ for $k \geq 1$. We conclude

$$\tilde{I}_{z,\epsilon,\alpha}(\epsilon\alpha) = 1 + \sum_{k=1}^{\infty} \sum_{l=1}^{k} \sum_{i_1+\cdots+i_l=k-l} \frac{f_{i_1} \cdots f_{i_l}}{e_{i_1+1} \cdots e_{i_l+\cdots+i_1+l}} = \tilde{J}_{z,\epsilon,\alpha}(1),$$

which is solution to (3.8.19).

□

Lemma 3.8.2. *$\tilde{K}_{z,\epsilon,\alpha}(\epsilon\alpha) = \tilde{L}_{z,\epsilon,\alpha}(1)$, where $\tilde{L}_{z,\epsilon,\alpha}(r)$ is solution to*

$$\begin{cases} w''(r) + r^{-1}w'(r) + (\beta_\epsilon \alpha^2 c_\alpha^{-1} e^{-\frac{\alpha^2}{2}r^2} - z\epsilon^2\alpha^2)w(r) = 0 \\ -\lim_{r\to 0^+} w(r)/\log(r/2) = 1, -\lim_{r\to 0^+} rw'(r) = 1. \end{cases} \tag{3.8.20}$$

Remark 3.8.3. *We see the Green function for equations (3.8.19) and (3.8.20) is*

$$G_{z,\epsilon,\alpha}(s,r) := \frac{\tilde{J}_{z,\epsilon,\alpha}(s)\tilde{L}_{z,\epsilon,\alpha}(r) - \tilde{J}_{z,\epsilon,\alpha}(r)\tilde{L}_{z,\epsilon,\alpha}(s)}{\tilde{J}_{z,\epsilon,\alpha}(s)\tilde{L}'_{z,\epsilon,\alpha}(s) - \tilde{J}'_{z,\epsilon,\alpha}(s)\tilde{L}_{z,\epsilon,\alpha}(s)}$$

for $0 \leq s \leq r$. Also, the term in the denominator is the Wronskian, and it can be checked as in Lemma 3.6.2 that it is equal to $-1/s$.

As mentioned in the beginning of this section, for small z, we need estimates for the remainders of $\tilde{I}_{z,\epsilon,\alpha}(\epsilon\alpha)$ from first, second and third order expansion. Let us use the notation $b_{l,k} := \sum_{i_1+\cdots+i_l=k-l} \frac{f_{i_1}\cdots f_{i_l}}{e_{i_1+1}\cdots e_{i_l+\cdots+i_1+l}}$, where $f_0 = z\epsilon^2\alpha^2 - \beta_\epsilon c_\alpha^{-1}\alpha^2$, $f_k = -\beta_\epsilon\alpha^2 c_\alpha^{-1}\frac{\alpha^{2k}}{k!}\frac{(-1)^k}{2^k}$ for $k \geq 1$; and recall $\beta_\epsilon = \frac{4\pi}{\log \epsilon^{-2}} + \frac{\sigma}{(\log\epsilon^{-2})^2}$, $\sigma \in \mathbb{R}$.

Second order expansion of $\tilde{J}_{z,\epsilon,\alpha}(1)$

Proposition 3.8.4. *Let $\alpha = \sqrt{2\log(\log\epsilon^{-1})^{1+2\delta}}$. Suppose z such that $|z\epsilon^2\alpha^2| \leq 0.5$ and $|z\epsilon^2\alpha^2(\beta_\epsilon^{-1}\alpha^4)|$ ≤ 1. Then the quantity is bounded*

$$\beta_\epsilon^{-2}\alpha^{-4}\left(\tilde{J}_{z,\epsilon,\alpha}(1) - 1 - \sum_{k=1}^{\infty} b_{1,k}\right).$$

Proof. We will break up the proof into four steps.

Step 1: Find an equation for the remainder.

Let $\tilde{J}^{(1)}_{z,\epsilon,\alpha}(r) = \beta_\epsilon^{-2}\alpha^{-4}\left(\tilde{J}_{z,\epsilon,\alpha}(r) - 1 - \sum_{k=1}^{\infty} b_{1,k}r^{2k}\right)$ and $V_{z,\epsilon,\alpha}(r) = \beta_\epsilon\alpha^2 c_\alpha^{-1} e^{-\frac{\alpha^2}{2}r^2} - z\epsilon^2\alpha^2$. By direct computation,

$$\begin{aligned}\tilde{J}^{(1)''}_{z,\epsilon,\alpha}(r) + r^{-1}\tilde{J}^{(1)'}_{z,\epsilon,\alpha}(r) + V_{z,\epsilon,\alpha}(r)\tilde{J}^{(1)}_{z,\epsilon,\alpha}(r) &= \beta_\epsilon^{-2}\alpha^{-4}\left(\tilde{J}''_{z,\epsilon,\alpha}(r) + r^{-1}\tilde{J}'_{z,\epsilon,\alpha}(r) + V_{z,\epsilon,\alpha}(r)\tilde{J}_{z,\epsilon,\alpha}(r)\right)\\ &\quad - \beta_\epsilon^{-2}\alpha^{-4}\sum_{k=1}^{\infty} b_{1,k}(2k)^2 r^{2k-2} - \beta_\epsilon^{-2}\alpha^{-4}V_{z,\epsilon,\alpha}(r)\\ &\quad - \beta_\epsilon^{-2}\alpha^{-4}V_{z,\epsilon,\alpha}(r)\sum_{k=1}^{\infty} b_{1,k}r^{2k}.\end{aligned}$$

The first line equals zero by the equation $\tilde{J}_{z,\epsilon,\alpha}(r)$ satisfies. Also, the second line is zero because from definition, $b_{1,k} = \sum_{i_1=k-1}\frac{f_{i_1}}{e_{i_1+1}} = \frac{f_{k-1}}{(2(k-1+1))^2}$, then

$$\begin{aligned}\sum_{k=1}^{\infty} b_{1,k}(2k)^2 r^{2k} &= r^2(f_0 + \sum_{k=1}^{\infty} f_k r^{2k})\\ &= r^2\left[(z\epsilon^2\alpha^2 - \beta_\epsilon\alpha^2 c_\alpha^{-1}) - \beta_\epsilon\alpha^2 c_\alpha^{-1}\sum_{k=1}^{\infty}\frac{(-1)^k r^{2k}}{k!}\left(\frac{\alpha^2}{2}\right)^k\right]\\ &= -r^2 V_{z,\epsilon,\alpha}(r).\end{aligned}$$

Step 2: Show the forcing function in Step 1 for the remainder is bounded.

We want to show for z in the region stated in Proposition 3.8.4, for $0 \le r \le 1$, the function is bounded

$$\beta_\epsilon^{-2}\alpha^{-4}V_{z,\epsilon,\alpha}(r)\sum_{k=1}^{\infty} b_{1,k}r^{2k}.$$

To simplify the sum, as in Step 1,

$$\begin{aligned}\sum_{k=1}^{\infty} b_{1,k}r^{2k} &= \frac{r^2}{4}\left[\frac{f_0}{1^2} + \sum_{k=1}^{\infty}\frac{f_k}{(k+1)^2}r^{2k}\right]\\ &= \frac{r^2}{4}\left[(z\epsilon^2\alpha^2 - \beta_\epsilon\alpha^2 c_\alpha^{-1}) - \beta_\epsilon\alpha^2 c_\alpha^{-1}\sum_{k=1}^{\infty}\frac{(-1)^k r^{2k}}{k!(k+1)^2}\left(\frac{\alpha^2}{2}\right)^k\right]\\ &= \frac{r^2}{4}\left[z\epsilon^2\alpha^2 - \beta_\epsilon\alpha^2 c_\alpha^{-1}\sum_{k=0}^{\infty}\frac{(-1)^k r^{2k}}{k!(k+1)^2}\left(\frac{\alpha^2}{2}\right)^k\right].\end{aligned}$$

Let us observe if $v_k = \frac{(-1)^k}{2^k}$, then for $r \ge 0$, $\sum_{k=0}^{\infty}\frac{v_k}{k!}\frac{1}{(k+1)^2}r^{2k+2} = O(r^2)$. To see this, note $e^{-\frac{r^2}{2}} = \sum_{k=0}^{\infty}\frac{v_k}{k!}r^{2k}$, so we have $\sum_{k=0}^{\infty}\frac{v_k}{k!}\frac{1}{(k+1)^2}r^{2k+2} = 4\int_0^r dt\ t\cdot t^{-2}\int_0^t ds\ se^{-\frac{s^2}{2}}$. By the Mean Value Theorem, then $\int_0^r dt\ t^{-1}\int_0^t ds\ se^{-\frac{s^2}{2}} = \int_0^r dt\ s_t e^{-\frac{s_t^2}{2}} \le r^2$, where $0 \le s_t \le t$. So there exists some constant C such that

$$\begin{aligned}\left|\beta_\epsilon^{-2}\alpha^{-4}V_{z,\epsilon,\alpha}(r)\sum_{k=1}^{\infty} b_{1,k}r^{2k}\right| &\le C\beta_\epsilon^{-2}\alpha^{-4}\left[|z\epsilon^2\alpha^2| + \beta_\epsilon\alpha^2 c_\alpha^{-1}e^{-\frac{\alpha^2}{2}r^2}\right]\left[|z\epsilon^2\alpha^2 r^2| + \beta_\epsilon\alpha^2 r^2 c_\alpha^{-1}\right]\\ &= C\left[|z\epsilon^2\alpha^2\beta_\epsilon^{-1}\alpha^{-2}| + c_\alpha^{-1}e^{-\frac{\alpha^2}{2}r^2}\right]\left[|z\epsilon^2\alpha^2\beta_\epsilon^{-1}\alpha^{-2}r^2| + r^2 c_\alpha^{-1}\right],\end{aligned}$$

which is bounded if $|z\epsilon^2\alpha^2(\beta_\epsilon^{-1}\alpha^4)| \le 1$, $0 \le r \le 1$.

Step 3: Use the Green function to rewrite the remainder and estimate the Green function.

Now we see $\tilde{J}^{(1)}_{z,\epsilon,\alpha}(r)$ is a solution to

$$\begin{cases} w''(r) + r^{-1}w'(r) + V_{z,\epsilon,\alpha}(r)w(r) = -\beta_\epsilon^{-2}\alpha^{-4}V_{z,\epsilon,\alpha}(r)\sum_{k=1}^{\infty} b_{1,k}r^{2k} \\ w(0) = 0, w'(0) = 0. \end{cases} \tag{3.8.21}$$

If $f^{(1)}_{z,\epsilon,\alpha}(r) = -\beta_\epsilon^{-2}\alpha^{-4}V_{z,\epsilon,\alpha}(r)\sum_{k=1}^{\infty} b_{1,k}r^{2k}$, by Remark 3.8.3, then

$$\tilde{J}^{(1)}_{z,\epsilon,\alpha}(r) = \int_0^r ds\ G_{z,\epsilon,\alpha}(s,r) f^{(1)}_{z,\epsilon,\alpha}(s).$$

To conclude Proposition 3.8.4, by Step 2, it is enough to show $G_{z,\epsilon,\alpha}(s,1)$ is bounded for $0 \le s \le 1$ and $|z\epsilon^2\alpha^2| \le 0.5$.

For this, the main idea is to find integral equations for $\tilde{J}_{z,\epsilon,\alpha}(r), \tilde{L}_{z,\epsilon,\alpha}(r)$ so we can compare them with the usual modified Bessel functions. We will adapt the argument in [26] (see Chapter 1 therein), and further divide this step into two parts.

Step 3.1: Find integral equations for $\tilde{J}_{z,\epsilon,\alpha}(r), \tilde{L}_{z,\epsilon,\alpha}(r)$.

Let $M_1(r), M_2(r)$ be independent solutions to $w''(r) + r^{-1}w'(r) - (1 + z\epsilon^2\alpha^2)w(r) = 0$ such that $M_1(0) = 1, M_1'(0) = 0$ and $M_2(r) \sim -\log(r/2), M_2'(r) \sim -1/r$ as $r \to 0^+$. (Note $M_1(r), M_2(r)$ are merely the modified Bessel functions of first and second kind $I_0(\sqrt{1+z\epsilon^2\alpha^2}r), K_0(\sqrt{1+z\epsilon^2\alpha^2}r)$ respectively.) Also, let $W(r) = M_1(r)M_2'(r) - M_1'(r)M_2(r)$ be the Wronskian. It can be checked that $rW(r) = -1$.

Define the operators

$$Lf(r) = f''(r) + r^{-1}f'(r) - (1 + z\epsilon^2\alpha^2)f(r), \qquad Gf(r) = \int_0^r ds\ \tfrac{M_1(s)M_2(r) - M_1(r)M_2(s)}{W(s)} f(s).$$

By integration by parts, and the fact that $LM_1, LM_2 = 0$, we have the following relation

$$(G \circ Lf)(r) = f(r) + \alpha(f)M_1(r) + \beta(f)M_2(r), \tag{3.8.22}$$

where[15] $\alpha(f) = -\frac{f(0)M_2'(0) - f'(0)M_2(0)}{W(0)}$, $\beta(f) = \frac{f(0)M_1'(0) - f'(0)M_1(0)}{W(0)}$. To find an integral equation for $\tilde{J}_{z,\epsilon,\alpha}(r)$, from the differential equation (3.8.19) it satisfies, then

$$\begin{aligned} L\tilde{J}_{z,\epsilon,\alpha}(r) &= \tilde{J}''_{z,\epsilon,\alpha}(r) + r^{-1}\tilde{J}'_{z,\epsilon,\alpha}(r) - (1 + z\epsilon^2\alpha^2)\tilde{J}_{z,\epsilon,\alpha}(r) \\ &= -(\beta_\epsilon\alpha^2c_\alpha^{-1}e^{-\frac{\alpha^2}{2}r^2} - z\epsilon^2\alpha^2)\tilde{J}_{z,\epsilon,\alpha}(r) - (1 + z\epsilon^2\alpha^2)\tilde{J}_{z,\epsilon,\alpha}(r) \\ &= -(\beta_\epsilon\alpha^2c_\alpha^{-1}e^{-\frac{\alpha^2}{2}r^2} + 1)\tilde{J}_{z,\epsilon,\alpha}(r). \end{aligned}$$

Applying the operator G on both sides and by relation (3.8.22), we get

$$\tilde{J}_{z,\epsilon,\alpha}(r) = -\alpha M_1(r) - \beta M_2(r) - \int_0^r ds\ \hat{V}_{z,\epsilon,\alpha}(s)\frac{M_1(s)M_2(r) - M_1(r)M_2(s)}{W(s)}\tilde{J}_{z,\epsilon,\alpha}(s),$$

where $\hat{V}_{z,\epsilon,\alpha}(r) = \beta_\epsilon\alpha^2c_\alpha^{-1}e^{-\frac{\alpha^2}{2}} + 1$, and α, β depend on the initial conditions of $\tilde{J}_{z,\epsilon,\alpha}$. Also, one can use the small value asymptotic of $M_i(r), M_i'(r)$ for $i = 1, 2$ from Lemma 3.6.1 to show $\alpha = -1, \beta = 0$.

[15] At the singularity $r = 0$, it is understood to be the limit $r \to 0^+$.

Thus

$$\tilde{J}_{z,\epsilon,\alpha}(r) = M_1(r) - \int_0^r ds\ \hat{V}_{z,\epsilon,\alpha}(s)\frac{M_1(s)M_2(r) - M_1(r)M_2(s)}{W(s)}\tilde{J}_{z,\epsilon,\alpha}(s). \tag{3.8.23}$$

And similarly,

$$\begin{aligned}\tilde{L}_{z,\epsilon,\alpha}(r) - M_2(r) = &- \int_0^r ds\ \hat{V}_{z,\epsilon,\alpha}(s)\frac{M_1(s)M_2(r) - M_1(r)M_2(s)}{W(s)}M_2(s) \\ &- \int_0^r ds\ \hat{V}_{z,\epsilon,\alpha}(s)\frac{M_1(s)M_2(r) - M_1(r)M_2(s)}{W(s)}\left(\tilde{L}_{z,\epsilon,\alpha}(s) - M_2(s)\right).\end{aligned} \tag{3.8.24}$$

Step 3.2: Estimate the integrands in the integral equations.

We want to show there exists constant C such that for $|z\epsilon^2\alpha^2| \le 0.5$, $0 \le s \le r \le 1$, and $i = 1, 2$,

$$\left|\frac{M_1(s)M_2(r) - M_1(r)M_2(s)}{W(s)}M_i(s)\right| \le C.$$

As mentioned earlier, $M_1(r) = I_0(\sqrt{1+z\epsilon^2\alpha^2}r), M_2(r) = K_0(\sqrt{1+z\epsilon^2\alpha^2}r)$, where $I_0(z), K_0(z)$ are the modified Bessel functions of the first and second kind respectively, and their small value asymptotic can be deduced from the last part of Lemma 3.6.1. We will estimate for $i = 2$, and the proof for $i = 1$ is similar.

Case 1: $|(1 + z\epsilon^2\epsilon^2)r^2| \le 0.5$.

Directly using the small value asymptotic,

$$\begin{aligned}&\left|\frac{M_1(s)M_2(r) - M_1(r)M_2(s)}{W(s)}M_2(s)\right| \\ &\quad \le C\left(\sqrt{r}|\log(\sqrt{1+z\epsilon^2\alpha^2}r)| + \sqrt{s}|\log(\sqrt{1+z\epsilon^2\alpha^2}s)|\right)\sqrt{s}|\log(\sqrt{1+z\epsilon^2\alpha^2}s)|,\end{aligned}$$

which is bounded, because on $|z\epsilon^2\alpha^2| \le 0.5$, $1 + z\epsilon^2\alpha^2$ is bounded away from 0.

Case 2.1: $0.5 \le |(1 + z\epsilon^2\alpha^2)r^2| \le 5$, and $|(1 + z\epsilon^2\alpha^2)s^2| \le 0.5$.

On $0 < a < |z| < b < \infty$, $I_0(z), K_0(z)$ are bounded, so are $I_0(\sqrt{1+z\epsilon^2\alpha^2}r), K_0(\sqrt{1+z\epsilon^2\alpha^2}r)$ on $0.5 \le |(1 + z\epsilon^2\alpha^2)r^2| \le 5$. And to deal with $K_0(\sqrt{1+z\epsilon^2\alpha^2}s)s$, it is the same as Case 1.

Case 2.2: $0.5 \le |(1 + z\epsilon^2\alpha^2)r^2| \le 5$, and $0.5 \le |(1 + z\epsilon^2\alpha^2)s^2| \le 5$.

It is the same as Case 2.1.

Step 4: Conclude the Green function is bounded, hence the remainder and Proposition 3.8.4.

From Step 3, and the integral equation (3.8.23), since $M_1(r)$ is bounded on $|z\epsilon^2\alpha^2| \le 0.5$, by Gronwall's inequality, $\tilde{J}_{z,\epsilon,\alpha}(r) = O(1)$. Similarly, for $\tilde{L}_{z,\epsilon,\alpha}(r) - M_2(r)$, from its integral equation (3.8.24), we have $\tilde{L}_{z,\epsilon,\alpha}(r) - M_2(r) = O(1)$. Thus $G_{z,\epsilon,\alpha}(s,1) = -s\left[(M_2(1) + O(1)) - (M_2(s) + O(1))\right]$, which is bounded.

□

Remark 3.8.5. *In the notations of Proposition 3.8.4, we note* $\tilde{J}^{(1)'}_{z,\epsilon,\alpha}(1)$ *is also bounded, because* $\tilde{J}^{(1)'}_{z,\epsilon,\alpha}(r)$ *satisfies*

$$\begin{cases} w'(r) + r^{-1}w(r) = -V_{z,\epsilon,\alpha}(r)\tilde{J}^{(1)}_{z,\epsilon,\alpha}(r) - f^{(1)}_{z,\epsilon,\alpha}(r) \\ w(0) = 0, \end{cases}$$

so $\tilde{J}^{(1)'}_{z,\epsilon,\alpha}(r) = -\int_0^r ds\ \frac{s}{r}\left(V_{z,\epsilon,\alpha}(s)\tilde{J}^{(1)}_{z,\epsilon,\alpha}(s) + f^{(1)}_{z,\epsilon,\alpha}(s)\right)$.

Remark 3.8.6. *From the proof of Proposition 3.8.4, we also deduce that for* $|z\epsilon^2\alpha^2| \le 0.5$ *and* $0 \le r \le 1$, $\tilde{J}_{z,\epsilon,\alpha}(r) = M_1(r) + O(1), \tilde{L}_{z,\epsilon,\alpha}(r) = M_2(r) + O(1)$.

Remark 3.8.7. *For z in the region stated in Proposition 3.8.4, and by the Proposition, we can rewrite*

$$\begin{aligned}
\tilde{I}_{z,\epsilon,\alpha}(\epsilon\alpha) &= 1 + \sum_{k=1}^{\infty} b_{1,k} + \beta_\epsilon^2\alpha^4\tilde{J}^{(1)}_{z,\epsilon,\alpha}(1) \\
&= 1 + \frac{1}{4}\left[z\epsilon^2\alpha^2 - \beta_\epsilon\alpha^2 c_\alpha^{-1}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(k+1)^2}\left(\frac{\alpha^2}{2}\right)^k\right] + \beta_\epsilon^2\alpha^4\tilde{J}^{(1)}_{z,\epsilon,\alpha}(1) \\
&= 1 + \beta_\epsilon c_\alpha^{-1}\sum_{k=1}^{\infty}\frac{(-1)^k}{k!2k}\left(\frac{\alpha^2}{2}\right)^k + \frac{1}{4}z\epsilon^2\alpha^2 + \beta_\epsilon^2\alpha^4\tilde{J}^{(1)}_{z,\epsilon,\alpha}(1) \\
&= 1 + \beta_\epsilon c_\alpha^{-1}\sum_{k=1}^{\infty}\frac{(-1)^k}{k!2k}a^k + O(\max\{z\epsilon^2\alpha^2, \beta_\epsilon^2\alpha^4\tilde{J}^{(1)}_{z,\epsilon,\alpha}(1)\}),
\end{aligned}$$

where we let $a = \frac{\alpha^2}{2}$.

Third order expansion of $\tilde{J}_{z,\epsilon,\alpha}(1)$

Proposition 3.8.8. *Let* $\alpha = \sqrt{2\log(\log\epsilon^{-1})^{1+2\delta}}$. *Suppose z such that* $|z\epsilon^2\alpha^2| \le 0.5$ *and* $|z\epsilon^2\alpha^2(\beta_\epsilon^{-1}\alpha^4)| \le 1$. *Then the quantity is bounded*

$$\beta_\epsilon^{-3}\alpha^{-6}\left(\tilde{J}_{z,\epsilon,\alpha}(1) - 1 - \sum_{k=1}^{\infty} b_{1,k} - \sum_{k=2}^{\infty} b_{2,k}\right).$$

Proof. As in Proposition 3.8.4, we break up the proof into three steps.

Step 1: Find an equation for the remainder.

Let $\tilde{J}^{(2)}_{z,\epsilon,\alpha}(r) = \beta_\epsilon^{-3}\alpha^{-6}\left(\tilde{J}_{z,\epsilon,\alpha}(r) - 1 - \sum_{k=1}^{\infty} b_{1,k}r^{2k} - \sum_{k=2}^{\infty} b_{2,k}r^{2k}\right)$. By direct computation,

$$\begin{aligned}
\tilde{J}^{(2)''}_{z,\epsilon,\alpha}(r) + r^{-1}\tilde{J}^{(2)'}_{z,\epsilon,\alpha}(r) + V_{z,\epsilon,\alpha}(r)\tilde{J}^{(2)}_{z,\epsilon,\alpha}(r) &= \beta_\epsilon^{-3}\alpha^{-6}\left(\tilde{J}''_{z,\epsilon,\alpha}(r) + r^{-1}\tilde{J}'_{z,\epsilon,\alpha}(r) + V_{z,\epsilon,\alpha}(r)\tilde{J}_{z,\epsilon,\alpha}(r)\right) \\
&\quad - \beta_\epsilon^{-3}\alpha^{-6}\sum_{k=1}^{\infty} b_{1,k}(2k)^2 r^{2k-2} - \beta_\epsilon^{-3}\alpha^{-6}V_{z,\epsilon,\alpha}(r) \\
&\quad - \beta_\epsilon^{-3}\alpha^{-6}\sum_{k=2}^{\infty} b_{2,k}(2k)^2 r^{2k-2} - \beta_\epsilon^{-3}\alpha^{-6}V_{z,\epsilon,\alpha}(r)\sum_{k=1}^{\infty} b_{1,k}r^{2k} \\
&\quad - \beta_\epsilon^{-3}\alpha^{-6}V_{z,\epsilon,\alpha}(r)\sum_{k=2}^{\infty} b_{2,k}r^{2k}.
\end{aligned}$$

The first two lines equal zero. To see the third line is also zero, from definition, $b_{2,k} = \sum_{i=0}^{k-2}\frac{f_i f_{k-2-i}}{(2(i+1))^2(2k)^2}$, so

$$\sum_{k=2}^{\infty} b_{2,k}(2k)^2 r^{2k} = \sum_{k=2}^{\infty}\sum_{i=0}^{k-2} \frac{f_i f_{k-2-i}}{4(i+1)^2} r^{2k}$$
$$= \frac{r^4}{4}\sum_{k=0}^{\infty}\sum_{i=0}^{k} \frac{f_i f_{k-i}}{(i+1)^2} r^{2k}.$$

On the other hand, recall $f_0 = z\epsilon\alpha^2 - \beta_\epsilon \alpha^2 c_\alpha^{-1}$, $f_k = -\beta_\epsilon\alpha^2 c_\alpha^{-1}\frac{(-1)^k}{k!}\left(\frac{\alpha^2}{2}\right)^k$, so $V_{z,\epsilon,\alpha}(r) = -\sum_{k=0}^{\infty} f_k r^{2k}$, then

$$r^2 V_{z,\epsilon,\alpha}(r)\sum_{k=1}^{\infty} b_{1,k} r^{2k} = -\frac{r^4}{4}\sum_{k=0}^{\infty} f_k r^{2k}\sum_{k=0}^{\infty}\frac{f_k}{(k+1)^2} r^{2k}$$
$$= -\frac{r^4}{4}\sum_{k=0}^{\infty}\sum_{i=0}^{k}\frac{f_i f_{k-i}}{(i+1)^2} r^{2k}.$$

Step 2: Show the forcing function in Step 1 for the remainder is bounded.

We want to show for z in the region stated in Proposition 3.8.8, and $0 \le r \le 1$, the function is bounded

$$\beta_\epsilon^{-3}\alpha^{-6} V_{z,\epsilon,\alpha}(r)\sum_{k=2}^{\infty} b_{2,k} r^{2k}.$$

First, simplifying the series,

$$\sum_{k=2}^{\infty} b_{2,k} r^{2k} = \frac{r^4}{2^4}\sum_{k=0}^{\infty}\sum_{i=0}^{k}\frac{f_i f_{k-i}}{(i+1)^2(k+2)^2} r^{2k}$$
$$= \frac{r^4}{2^4}\left[\sum_{k=0}^{\infty}\frac{r^{2k}}{(k+2)^2}\left(\frac{f_0 f_k}{1^2} + \frac{f_k f_0}{(k+1)^2}\right) + \sum_{k=2}^{\infty}\frac{r^{2k}}{(k+2)^2}\sum_{i=1}^{k-1}\frac{f_i f_{k-i}}{(i+1)^2}\right]$$
$$= \frac{r^4}{2^4}\frac{1}{2^2}(z\epsilon^2\alpha^2 - \beta_\epsilon\alpha^2 c_\alpha^{-1})^2 + \frac{r^4}{2^4}(\beta_\epsilon\alpha^2 c_\alpha^{-1})^2\sum_{k=2}^{\infty}\frac{(-1)^k r^{2k}}{(k+2)^2}\left(\frac{\alpha^2}{2}\right)^k\sum_{i=1}^{k-1}\frac{1}{i!(k-i)!}\frac{1}{(i+1)^2}$$
$$-\frac{r^4}{2^4}(z\epsilon^2\alpha^2 - \beta_\epsilon\alpha^2 c_\alpha^{-1})(\beta_\epsilon\alpha^2 c_\alpha^{-1})\sum_{k=1}^{\infty}\frac{(-1)^k r^{2k}}{k!(k+2)^2}\left(\frac{\alpha^2}{2}\right)^k\left(1 + \frac{1}{(k+1)^2}\right)$$
$$= \frac{r^4}{2^4}\left[\frac{1}{2^2}(z\epsilon^2\alpha^2)^2 - (z\epsilon^2\alpha^2)(\beta_\epsilon\alpha^2 c_\alpha^{-1})\sum_{k=0}^{\infty}\frac{(-1)^k r^{2k}}{k!(k+2)^2}\left(\frac{\alpha^2}{2}\right)^k\left(1 + \frac{1}{(k+1)^2}\right)\right]$$
$$+\frac{r^4}{2^4}(\beta_\epsilon\alpha^2 c_\alpha^{-1})^2\sum_{k=0}^{\infty}\frac{(-1)^k r^{2k}}{(k+2)^2}\left(\frac{\alpha^2}{2}\right)^k\sum_{i=0}^{k}\frac{1}{i!(k-i)!}\frac{1}{(i+1)^2}.$$

Next, we need to say for $r \ge 0$, $\sum_{k=0}^{\infty}\frac{(-1)^k r^{2k+4}}{2^k(k+2)^2}\sum_{i=0}^{k}\frac{1}{i!(k-i)!}\frac{1}{(i+1)^2} = O(r^4)$. To see this, let $v_k = \frac{(-1)^k}{2^k}$, so $e^{-\frac{r^2}{2}} = \sum_{k=0}^{\infty}\frac{v_k}{k!}r^{2k}$, one can check

$$\sum_{k=0}^{\infty}\frac{(-1)^k r^{2k+4}}{2^k(k+2)^2}\sum_{i=0}^{k}\frac{1}{i!(k-i)!}\frac{1}{(i+1)^2} = \sum_{k=0}^{\infty}\frac{r^{2k+4}}{(k+2)^2}\sum_{i=0}^{k}\frac{v_i v_{k-i}}{i!(k-i)!}\frac{1}{(i+1)^2}$$
$$= 4\int_0^r dt\ t^{-1}\int_0^t ds\ s e^{-\frac{s^2}{2}} 4\left(\int_0^s du\ u^{-1}\int_0^u dv\ v e^{-\frac{v^2}{2}}\right).$$

By the Mean Value Theorem, the series is $O(r^4)$. The idea for showing $\sum_{k=0}^{\infty} \frac{(-1)^k r^{2k+4}}{k!2^k} \frac{1}{(k+2)^2(k+1)^2} = O(r^4)$ and $\sum_{k=0}^{\infty} \frac{(-1)^k r^{2k+4}}{k!2^k} \frac{1}{(k+2)^2} = O(r^4)$ is also similar. Thus, there exists some constant C such that

$$\begin{aligned}\left|\beta_\epsilon^{-3}\alpha^{-6}V_{z,\epsilon,\alpha}(r)\sum_{k=2}^{\infty} b_{2,k}r^{2k}\right| &\leq C\beta_\epsilon^{-3}\alpha^{-6}\left[|z\epsilon^2\alpha^2| + \beta_\epsilon\alpha^2 c_\alpha^{-1}e^{-\frac{\alpha^2}{2}r^2}\right]\left[|z\epsilon^2\alpha^2 r^2| + \beta_\epsilon\alpha^2 r^2 c_\alpha^{-1}\right]^2 \\ &= C\left[|z\epsilon^2\alpha^2\beta_\epsilon^{-1}\alpha^{-2}| + c_\alpha^{-1}e^{-\frac{\alpha^2}{2}r^2}\right]\left[|z\epsilon^2\alpha^2\beta_\epsilon^{-1}\alpha^{-2}r^2| + r^2 c_\alpha^{-1}\right]^2,\end{aligned}$$

which is bounded if $|z\epsilon^2\alpha^2(\beta_\epsilon^{-1}\alpha^4)| \leq 1$, $0 \leq r \leq 1$.

Step 3: Use the Green function to rewrite the remainder and follow the same steps as in Proposition 3.8.4 to conclude.

We see $\tilde{J}^{(2)}_{z,\epsilon,\alpha}(r)$ is a solution to

$$\begin{cases} w''(r) + r^{-1}w'(r) + V_{z,\epsilon,\alpha}(r)w(r) = -\beta_\epsilon^{-3}\alpha^{-6}V_{z,\epsilon,\alpha}(r)\sum_{k=2}^{\infty} b_{2,k}r^{2k} \\ w(0) = 0, w'(0) = 0. \end{cases} \tag{3.8.25}$$

The rest follows from the same argument as for Proposition 3.8.4.

□

Remark 3.8.9. *As in Remark 3.8.5, $\tilde{J}^{(2)'}_{z,\epsilon,\alpha}(1)$ is bounded.*

Remark 3.8.10. *For z in the region stated in Proposition 3.8.8, in particular as is done in Step 2 of the Proposition and note $\epsilon\alpha\tilde{I}'_{z,\epsilon,\alpha}(\epsilon\alpha) = \tilde{J}'_{z,\epsilon,\alpha}(1)$, we can rewrite*

$$\begin{aligned}\epsilon\alpha\tilde{I}'_{z,\epsilon,\alpha}(\epsilon\alpha) &= \sum_{k=1}^{\infty} b_{1,k}2k + \sum_{k=2}^{\infty} b_{2,k}2k + \beta_\epsilon^3\alpha^6\tilde{J}^{(2)'}_{z,\epsilon,\alpha}(1) \\ &= \frac{1}{2}\left[z\epsilon^2\alpha^2 - \beta_\epsilon\alpha^2 c_\alpha^{-1}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(k+1)}\left(\frac{\alpha^2}{2}\right)^k\right] \\ &\quad + \frac{1}{2^3}\left[\frac{1}{2}(z\epsilon^2\alpha^2)^2 - (z\epsilon^2\alpha^2)(\beta_\epsilon\alpha^2 c_\alpha^{-1})\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(k+2)}\left(\frac{\alpha^2}{2}\right)^k\left(1 + \frac{1}{(k+1)^2}\right)\right] \\ &\quad + \frac{1}{2^3}(\beta_\epsilon\alpha^2 c_\alpha^{-1})^2\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+2)}\left(\frac{\alpha^2}{2}\right)^k\sum_{i=0}^{k}\frac{1}{i!(k-i)!}\frac{1}{(i+1)^2} + \beta_\epsilon^3\alpha^6\tilde{J}^{(2)'}_{z,\epsilon,\alpha}(1) \\ &= \beta_\epsilon c_\alpha^{-1}\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}a^k + (\beta_\epsilon c_\alpha^{-1})^2\sum_{k=2}^{\infty}\frac{(-1)^{k-2}}{2k}a^k\sum_{i=0}^{k-2}\frac{1}{i!(k-2-i)!(i+1)^2} + \beta_\epsilon^3\alpha^6\tilde{J}^{(2)'}_{z,\epsilon,\alpha}(1) \\ &\quad + \frac{z\epsilon^2\alpha^2}{2} + \frac{1}{2^3}\left[\frac{1}{2}(z\epsilon^2\alpha^2)^2 - (z\epsilon^2\alpha^2)(\beta_\epsilon\alpha^2 c_\alpha^{-1})\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(k+2)}a^k\left(1 + \frac{1}{(k+1)^2}\right)\right] \\ &= \beta_\epsilon c_\alpha^{-1}\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}a^k + (\beta_\epsilon c_\alpha^{-1})^2\sum_{k=2}^{\infty}\frac{(-1)^{k-2}}{2k}a^k\sum_{i=0}^{k-2}\frac{1}{i!(k-2-i)!(i+1)^2} \\ &\quad + O(\max\{z\epsilon^2\alpha^2, \beta_\epsilon^3\alpha^6\tilde{J}^{(2)'}_{z,\epsilon,\alpha}(1)\}),\end{aligned}$$

where we let $a = \frac{\alpha^2}{2}$. For the last line, to get the error, as is done in the Proposition, one can show $\sum_{k=0}^{\infty}\frac{(-1)^k}{2^k k!}\frac{\alpha^{2k+4}}{k+2} = O(\alpha^4)$, and $\sum_{k=0}^{\infty}\frac{(-1)^k}{2^k k!}\frac{\alpha^{2k+4}}{(k+2)(k+1)^2} = O(\alpha^4)$; the main point is α^4 is small compared to β_ϵ.

First order expansion of $\tilde{J}_{z,\epsilon,\alpha}(1)$

Proposition 3.8.11. *Let $\alpha = \sqrt{2\log(\log\epsilon^{-1})^{1+2\delta}}$. Suppose z such that $|z\epsilon^2\alpha^2| \le 0.5$ and $|z\epsilon^2\alpha^2(\beta_\epsilon^{-1}\alpha^4)| \ge 1$. Then the quantity is bounded*

$$(z\epsilon^2\alpha^2)^{-1/2}\left(\tilde{J}_{z,\epsilon,\alpha}(1) - 1\right).$$

Proof. Let $\tilde{J}^{(0)}_{z,\epsilon,\alpha}(r) := (z\epsilon^2\alpha^2)^{-1/2}\left(\tilde{J}_{z,\epsilon,\alpha}(r) - 1\right)$. Then it satisfies

$$\begin{cases} w''(r) + r^{-1}w'(r) + V_{z,\epsilon,\alpha}(r)w(r) = -(\beta_\epsilon\alpha^2 c_\alpha^{-1} e^{-\frac{\alpha^2}{2}r^2} - z\epsilon^2\alpha^2)(z\epsilon^2\alpha^2)^{-1/2} \\ w(0) = 0, w'(0) = 0. \end{cases} \tag{3.8.26}$$

The key observation here is $\frac{\beta_\epsilon\alpha^2}{(z\epsilon^2\alpha^2)^{1/2}} = \frac{\beta_\epsilon\alpha^2\beta_\epsilon^{-1/2}\alpha^2}{(z\epsilon^2\alpha^2(\beta_\epsilon^{-1}\alpha^4))^{1/2}} \le \beta_\epsilon^{1/2}\alpha^4$, which is small. Now the Proposition follows as in Propositions 3.8.4 and 3.8.8. ☐

Remark 3.8.12. *As in Remark 3.8.5, $\tilde{J}^{(0)'}_{z,\epsilon,\alpha}(1)$ is bounded.*

3.8.2 Large Asymptotic Estimates

In this subsection, we are going to obtain the large z estimates for $\tilde{J}_{z,\epsilon,\alpha}(r), \tilde{L}_{z,\epsilon,\alpha}(r)$. Again, recall $\beta_\epsilon = \frac{4\pi}{\log\epsilon^{-2}} + \frac{\sigma}{(\log\epsilon^{-2})^2}$.

Proposition 3.8.13. *Let $\alpha = \sqrt{2\log(\log\epsilon^{-1})^{1+2\delta}}$, and suppose z such that $0.5 \le |z\epsilon^2\alpha^2| \le 5$. Then for $0 \le r \le 1$,*

$$\tilde{J}_{z,\epsilon,\alpha}(r) = N_1(r) + O(\beta_\epsilon\alpha^2), \qquad \tilde{L}_{z,\epsilon,\alpha}(r) = N_2(r) + O(\beta_\epsilon\alpha^2).$$

where $N_1(r) = I_0(\sqrt{z\epsilon^2\alpha^2}r), N_2(r) = K_0(\sqrt{z\epsilon^2\alpha^2}r)$.

Proof. As in Section 3.8.1, we estimate from integral equations.

Let $N_1(r), N_2(r)$ be independent solutions to $w''(r) + r^{-1}w'(r) - z\epsilon^2\alpha^2 w(r) = 0$ such that $N_1(0) = 1, N_1'(0) = 0$ and $N_2(r) \sim -\log(r/2), N_2'(r) \sim -1/r$ as $r \to 0^+$. (Note $N_1(r), N_2(r)$ are merely the modified Bessel functions of first and second kind $I_0(\sqrt{z\epsilon^2\alpha^2}r), K_0(\sqrt{z\epsilon^2\alpha^2}r)$ respectively.) Also, let $W(r) = N_1(r)N_2'(r) - N_1'(r)N_2(r)$ be the Wronskian, so $rW(r) = -1$. It then follows

$$\begin{aligned} \tilde{J}_{z,\epsilon,\alpha}(r) - N_1(r) = &-\beta_\epsilon\alpha^2 c_\alpha^{-1}\int_0^r ds\ e^{-\frac{\alpha^2}{2}s^2}\frac{N_1(s)N_2(r) - N_1(r)N_2(s)}{W(s)}N_1(s) \\ &- \beta_\epsilon\alpha^2 c_\alpha^{-1}\int_0^r ds\ e^{-\frac{\alpha^2}{2}s^2}\frac{N_1(s)N_2(r) - N_1(r)N_2(s)}{W(s)}\left(\tilde{J}_{z,\epsilon,\alpha}(s) - N_1(s)\right), \end{aligned} \tag{3.8.27}$$

$$\begin{aligned} \tilde{L}_{z,\epsilon,\alpha}(r) - N_2(r) = &-\beta_\epsilon\alpha^2 c_\alpha^{-1}\int_0^r ds\ e^{-\frac{\alpha^2}{2}s^2}\frac{N_1(s)N_2(r) - N_1(r)N_2(s)}{W(s)}N_2(s) \\ &- \beta_\epsilon\alpha^2 c_\alpha^{-1}\int_0^r ds\ e^{-\frac{\alpha^2}{2}s^2}\frac{N_1(s)N_2(r) - N_1(r)N_2(s)}{W(s)}\left(\tilde{L}_{z,\epsilon,\alpha}(s) - N_2(s)\right). \end{aligned} \tag{3.8.28}$$

Claim 3.8.14. *There exists constant C such that for $0.5 \leq |z\epsilon^2\alpha^2| \leq 5$, $0 \leq s \leq r \leq 1$, and $i = 1, 2$,*

$$\left|\frac{N_1(s)N_2(r) - N_1(r)N_2(s)}{W(s)} N_i(s)\right| \leq C.$$

Proof. It is almost the same as in Step 3.2 of Proposition 3.8.4, except the case $|z\epsilon^2\alpha^2 r^2| \leq 0.5$, where we use the small value asymptotic of $I_0(z), K_0(z)$ in a slightly different way. For example, for $i = 2$, in that case we have

$$\begin{aligned}
&\left|\frac{N_1(s)N_2(r) - N_1(r)N_2(s)}{W(s)} N_2(s)\right| \\
&\quad \leq C\left(\sqrt{r}|\log(\sqrt{z\epsilon^2\alpha^2}r)| + \sqrt{s}|\log(\sqrt{z\epsilon^2\alpha^2}s)|\right)\sqrt{s}|\log(\sqrt{z\epsilon^2\alpha^2}s)| \\
&\quad = C\left(\left|\sqrt{\sqrt{z\epsilon^2\alpha^2}r}\log(\sqrt{z\epsilon^2\alpha^2}r)\right| + \left|\sqrt{\sqrt{z\epsilon^2\alpha^2}s}\log(\sqrt{z\epsilon^2\alpha^2}s)\right|\right) \\
&\qquad \times \left|\sqrt{\sqrt{z\epsilon^2\alpha^2}s}\log(\sqrt{z\epsilon^2\alpha^2}s)\right| |1/\sqrt{z\epsilon^2\alpha^2}|,
\end{aligned}$$

which is bounded, since $0.5 \leq |z\epsilon^2\alpha^2| \leq 5$.

□

Back to the proof of Proposition of 3.8.13. By Gronwall's inequality, $\tilde{J}_{z,\epsilon,\alpha}(r) - N_1(r) = O(\beta_\epsilon\alpha^2)$, and $\tilde{L}_{z,\epsilon,\alpha}(r) - N_2(r) = O(\beta_\epsilon\alpha^2)$. □

Remark 3.8.15. *Moreoever, for $0.5 \leq |z\epsilon^2\alpha^2| \leq 5$, we have $\tilde{J}'_{z,\epsilon,\alpha}(1) = N_1'(1) + O(\beta_\epsilon\alpha^2)$. It follows from the integral equation (3.8.27) given we show $\left|\frac{N_1(s)N_2'(1) - N_1'(1)N_2(s)}{W(s)}\right| \leq C$. But $N_1'(r) = \sqrt{z\epsilon^2\alpha^2} I_0'(\sqrt{z\epsilon^2\alpha^2}r), N_2'(r) = \sqrt{z\epsilon^2\alpha^2}K_0'(\sqrt{z\epsilon^2\alpha^2}r)$, and by the identity[16] $I_0'(z) = I_1(z), K_0'(z) = -K_1(z)$, which are bounded functions on $0 < a \leq |z| \leq b < \infty$, thus on $0.5 \leq |z\epsilon^2\alpha^2| \leq 5$, $I_0'(\sqrt{z\epsilon^2\alpha^2}), K_0'(\sqrt{z\epsilon^2\alpha^2})$ are also bounded.*

Remark 3.8.16. *For $|z\epsilon^2\alpha^2| \leq 0.5$, we also have $\tilde{J}_{z,\epsilon,\alpha}(1) = N_1(1) + O(\beta_\epsilon\alpha^2)$. It also follows from the integral equation (3.8.27) given we show $\left|\frac{N_1(s)N_2(r) - N_1(r)N_2(s)}{W(s)} N_1(s)\right| \leq C$. More precisely, on $|z\epsilon^2\alpha^2| \leq 0.5$, we have $|z\epsilon^2\alpha^2 r^2| \leq 0.5$ for $0 \leq r \leq 1$, so by the small value asymptotic of $I_0(z), K_0(z)$,*

$$\begin{aligned}
&\left|\frac{N_1(s)N_2(r) - N_1(r)N_2(s)}{W(s)} N_1(s)\right| \\
&\quad = s\left|(1 + O(z\epsilon^2\alpha^2 s^2))(-\log(\sqrt{z\epsilon^2\alpha^2}r/2) + O(1)) - (1 + O(z\epsilon^2\alpha^2 r^2))(-\log(\sqrt{z\epsilon^2\alpha^2}s/2) + O(1))\right| \\
&\quad = s\left|-\log(r/2) + \log(s/2) + O(z\epsilon^2\alpha^2\log(\sqrt{z\epsilon^2\alpha^2}))\right|,
\end{aligned}$$

which is bounded.

With same notations as in Proposition 3.8.13, we write

$$\tilde{J}_{z,\epsilon,\alpha}(r) = N_1(r)\left[1 + \left(\tfrac{\tilde{J}_{z,\epsilon,\alpha}(r) - N_1(r)}{N_1(r)}\right)\right], \qquad \tilde{J}'_{z,\epsilon,\alpha}(1) = N_1'(1)\left[1 + \left(\tfrac{\tilde{J}'_{z,\epsilon,\alpha}(1) - N_1'(1)}{N_1'(1)}\right)\right].$$

[16] See [22] page 110.

Proposition 3.8.17. *There exists constant $c_\epsilon \to 0$ as $\epsilon \to 0$ such that for $|z\epsilon^2\alpha^2| \geq 5$ and $0 \leq r \leq 1$,*

$$\left|\frac{\tilde{J}_{z,\epsilon,\alpha}(r) - N_1(r)}{N_1(r)}\right| \leq c_\epsilon, \qquad \left|\frac{\tilde{J}'_{z,\epsilon,\alpha}(1) - N_1'(1)}{N_1'(1)}\right| \leq c_\epsilon.$$

Proof. For the first statement, from integral equation (3.8.27),

$$\begin{aligned}\frac{\tilde{J}_{z,\epsilon,\alpha}(r) - N_1(r)}{N_1(r)} &= -\beta_\epsilon \alpha^2 c_\alpha^{-1} \int_0^r ds\ e^{-\frac{\alpha^2}{2}s^2} \left(\frac{N_1(s)N_2(r) - N_1(r)N_2(s)}{W(s)} \frac{N_1(s)}{N_1(r)}\right) \\ &\quad - \beta_\epsilon \alpha^2 c_\alpha^{-1} \int_0^r ds\ e^{-\frac{\alpha^2}{2}s^2} \left(\frac{N_1(s)N_2(r) - N_1(r)N_2(s)}{W(s)} \frac{N_1(s)}{N_1(r)}\right) \frac{\tilde{J}_{z,\epsilon,\alpha}(s) - N_1(s)}{N_1(s)},\end{aligned} \tag{3.8.29}$$

it is enough to estimate the summand.

Claim 3.8.18. *There exists constant C such that for $|z\epsilon^2\alpha^2| \geq 5$ and $0 \leq s \leq r \leq 1$,*

$$\left|\frac{N_1(s)N_2(r) - N_1(r)N_2(s)}{W(s)} \frac{N_1(s)}{N_1(r)}\right| \leq C.$$

Proof. We break it down into cases.

Case 1: $|z\epsilon^2\alpha^2 r^2| \leq 5$.

As in Claim 3.8.14, we see $\left|\frac{N_1(s)N_2(r)-N_1(r)N_2(s)}{W(s)} N_1(s)\right|$ is bounded, and it remains to show $N_1(r) = I_0(\sqrt{z\epsilon^2\alpha^2}r)$ is bounded away from 0. To see this, it follows by the argument in Lemma 3.11.2.

Case 2.1: $|z\epsilon^2\alpha^2 r^2| \geq 5$, and $|z\epsilon^2\alpha^2 s^2| \leq 0.5$.

In this case,

$$\begin{aligned}&\left|\frac{N_1(s)N_2(r) - N_1(r)N_2(s)}{W(s)} \frac{N_1(s)}{N_1(r)}\right| \\ &\quad = \left|\frac{I_0(\sqrt{z\epsilon^2\alpha^2}s)}{I_0(\sqrt{z\epsilon^2\alpha^2}r)} I_0(\sqrt{z\epsilon^2\alpha^2}s)K_0(\sqrt{z\epsilon^2\alpha^2}r) - I_0(\sqrt{z\epsilon^2\alpha^2}s)K_0(\sqrt{z\epsilon^2\alpha^2}s)\right| s.\end{aligned}$$

By the large value asymptotic (3.7.18), $\frac{K_0(\sqrt{z\epsilon^2\alpha^2}r)}{I_0(\sqrt{z\epsilon^2\alpha^2}r)}$ is bounded. To deal with $K_0(\sqrt{z\epsilon^2\alpha^2}s)s$, by the small value asymptotic of $K_0(z)$, then

$$|K_0(\sqrt{z\epsilon^2\alpha^2}s)s| \leq C|\log(\sqrt{z\epsilon^2\alpha^2}s)|\sqrt{z\epsilon^2\alpha^2}s||1/\sqrt{z\epsilon^2\alpha^2}| \leq C,$$

since $|z\epsilon^2\alpha^2| \geq 5$.

Case 2.2: $|z\epsilon^2\alpha^2 r^2| \geq 5$, and $0.5 \leq |z\epsilon^2\alpha^2 s^2| \leq 5$.

It is similar to Case 2.1.

Case 2.3: $|z\epsilon^2\alpha^2 r^2| \geq 5$, and $|z\epsilon^2\alpha^2 s^2| \geq 5$.

As in Case 2.1, now $I_0(\sqrt{z\epsilon^2\alpha^2}s)K_0(\sqrt{z\epsilon^2\alpha^2}r), I_0(\sqrt{z\epsilon^2\alpha^2}s)K_0(\sqrt{z\epsilon^2\alpha^2}s) \leq C$ for some constant C. □

Back to the proof of Proposition 3.8.17. To show the second statement, from integral equation

(3.8.27), we have

$$\frac{\tilde{J}'_{z,\epsilon,\alpha}(1) - N_1'(1)}{N_1'(1)} = -\beta_\epsilon \alpha^2 c_\alpha^{-1} \int_0^1 ds\ e^{-\frac{\alpha^2}{2}s^2} \left(\frac{N_1(s)N_2'(1) - N_1'(1)N_2(s)}{W(s)} \frac{1}{N_1'(1)} \right) \tilde{J}_{z,\epsilon,\alpha}(s) \mathbb{1}_{|z\epsilon^2\alpha^2 s^2| \le 5}$$
$$- \beta_\epsilon \alpha^2 c_\alpha^{-1} \int_0^1 ds\ e^{-\frac{\alpha^2}{2}s^2} \left(\frac{N_1(s)N_2'(1) - N_1'(1)N_2(s)}{W(s)} \frac{N_1(s)}{N_1'(1)} \right) \frac{\tilde{J}_{z,\epsilon,\alpha}(s)}{N_1(s)} \mathbb{1}_{|z\epsilon^2\alpha^2 s^2| \ge 5}.$$

Now, by the first statement of the Proposition, we know $\frac{\tilde{J}_{z,\epsilon,\alpha}(s)}{N_1(s)}$ is bounded on $|z\epsilon^2\alpha^2 s^2| \ge 5$, and $\tilde{J}_{z,\epsilon,\alpha}(s)$ is bounded on $|z\epsilon^2\alpha^2 s^2| \le 5$. Thus the second statement follows if there exists constant C such that $\left| \frac{N_1(s)N_2'(1) - N_1'(1)N_2(s)}{W(s)} \frac{N_1(s)}{N_1'(1)} \right| \le C$. To see this, recall $N_1'(1) = \sqrt{z\epsilon^2\alpha^2} I_0'(\sqrt{z\epsilon^2\alpha^2}), N_2'(1) = \sqrt{z\epsilon^2\alpha^2} K_0'(\sqrt{z\epsilon^2\alpha^2})$, $I_0'(z) = I_1(z)$, $K_0'(z) = -K_1(z)$, and the estimate follows similarly as in Claim 3.8.18 by the large value asymptotic of $I_0(z), K_0(z), I_1(z), K_1(z)$ (see (3.7.18)).

□

Proposition 3.8.19. *There exists constants C_1, C_2 such that for $|z\epsilon^2\alpha^2| \ge 5$ and $0 \le r \le 1$,*

$$|\tilde{L}_{z,\epsilon,\alpha}(r)| \le C_1 |N_2(r)| |\mathbb{1}_{|z\epsilon^2\alpha^2 r^2| \le 5} + C_2 |N_1(r)| \mathbb{1}_{|z\epsilon^2\alpha^2 r^2| \ge 5}.$$

Proof. For this, we use

$$\frac{\tilde{L}_{z,\epsilon,\alpha}(r)}{N_1(r)} = \frac{N_2(r)}{N_1(r)} - \beta_\epsilon \alpha^2 c_\alpha^{-1} \int_0^r ds\ e^{-\frac{\alpha^2}{2}s^2} \left(\frac{N_1(s)N_2(r) - N_1(r)N_2(s)}{W(s)} \frac{N_1(s)}{N_1(r)} \right) \frac{\tilde{L}_{z,\epsilon,\alpha}(s)}{N_1(s)}.$$

By Claim 3.8.18, $\left| \frac{N_1(s)N_2(r) - N_1(r)N_2(s)}{W(s)} \frac{N_1(s)}{N_1(r)} \right|$ is bounded, whether $|z\epsilon^2\alpha^2 r^2| \le 5$ or $|z\epsilon^2\alpha^2 r^2| \ge 5$. Thus for some constant C,

$$\left| \frac{\tilde{L}_{z,\epsilon,\alpha}(r)}{N_1(r)} \right| \le \left| \frac{N_2(r)}{N_1(r)} \right| + C\beta_\epsilon \alpha^2 \int_0^r ds \left| \frac{\tilde{L}_{z,\epsilon,\alpha}(s)}{N_1(s)} \right|.$$

Next, by the differential form of Gronwall's inequality (since $\left| \frac{\tilde{L}_{z,\epsilon,\alpha}(r)}{N_1(r)} \right| = \frac{d}{dr} \int_0^r ds\ \left| \frac{\tilde{L}_{z,\epsilon,\alpha}(s)}{N_1(s)} \right|$), we see

$$\int_0^r ds \left| \frac{\tilde{L}_{z,\epsilon,\alpha}(s)}{N_1(s)} \right| \le e^{\int_0^r ds\ C\beta_\epsilon \alpha^2} \int_0^r ds\ \left| \frac{N_2(s)}{N_1(s)} \right|.$$

To conclude the Proposition, it remains to bound $\int_0^r ds\ \left| \frac{N_2(s)}{N_1(s)} \right|$.

For $|z\epsilon^2\alpha^2 s^2| \le 5$, we know $|N_2(s)| \le C |\log \sqrt{z\epsilon^2\alpha^2} s| \le C \log(|\sqrt{z\epsilon^2\alpha^2}| s)$, and $N_1(s)$ is bounded below, so

$$\int_0^r ds\ \left| \frac{N_2(s)}{N_1(s)} \right| \mathbb{1}_{|\sqrt{z\epsilon^2\alpha^2}| s \le \sqrt{5}} \le C \frac{1}{|\sqrt{z\epsilon^2\alpha^2}|} \int_0^{\sqrt{5}} ds\ \log s,$$

which is bounded since $|z\epsilon^2\alpha^2| \ge 5$. For $|z\epsilon^2\alpha^2 s^2| \ge 5$, we use the large value asymptotic of $I_0(z), K_0(z)$ to deduce $\frac{N_2(s)}{N_1(s)}$ is bounded. □

3.9 Small Value Asymptotic of the Denominator

Suppose $z \in \mathbb{C} - \mathbb{R}_-$. We will use the small value estimates in Section 3.8.1 to identify the poles in the resolvent solution $w^{z,\epsilon,\alpha}(x)$ for small z. In particular in this section, we focus on the difference $K_0(\sqrt{z}\epsilon\alpha)\epsilon\alpha\tilde{I}'_{z,\epsilon,\alpha}(\epsilon\alpha) - \sqrt{z}\epsilon\alpha K_0'(\sqrt{z}\epsilon\alpha)\tilde{I}_{z,\epsilon,\alpha}(\epsilon\alpha)$ in the denominator of $w^{z,\epsilon,\alpha}(x)$, and use the estimates to expand each term. Recall $\beta_\epsilon = \frac{4\pi}{\log\epsilon^{-2}} + \frac{\sigma}{(\log\epsilon^{-2})^2}$.

Proposition 3.9.1. *For $\alpha = \alpha_\epsilon = \sqrt{2\log(\log\epsilon^{-1})^{1+2\delta}}$, where $\delta > 0$ small, let $A_{\epsilon,\alpha} = [z \mid |z\epsilon^2\alpha^2| \leq 0.5]$ and $B_{\epsilon,\alpha} = [z \mid |z\epsilon^2\alpha^2(\beta_\epsilon^{-1}\alpha^4)| \leq 1]$.*

On $A_{\epsilon,\alpha} \cap B_{\epsilon,\alpha}$, then

$$K_0(\sqrt{z}\epsilon\alpha)\epsilon\alpha\tilde{I}'_{z,\epsilon,\alpha}(\epsilon\alpha) - \sqrt{z}\epsilon\alpha K_0'(\sqrt{z}\epsilon\alpha)\tilde{I}_{z,\epsilon,\alpha}(\epsilon\alpha) = \frac{1}{\log\epsilon^{-1}}(g_\alpha(z) + O(1/\log\epsilon^{-1})),$$

where $g_\alpha(z) = \log\frac{\sqrt{z}}{2} - \psi(1) - \frac{\sigma}{8\pi} + \nu(p_{1,\alpha})$ with zero $z_{0,\alpha} = 4e^{2(\frac{\sigma}{8\pi} - \nu(p_{1,\alpha}) + \psi(1))}$, and $\nu(p_{1,\alpha}) = \int dx\; p_{1,\alpha}(x) \int dy\; \log|x-y| p_{1,\alpha}(y)$.

On $A_{\epsilon,\alpha} \cap B^c_{\epsilon,\alpha}$, then

$$K_0(\sqrt{z}\epsilon\alpha)\epsilon\alpha\tilde{I}'_{z,\epsilon,\alpha}(\epsilon\alpha) - \sqrt{z}\epsilon\alpha K_0'(\sqrt{z}\epsilon\alpha)\tilde{I}_{z,\epsilon,\alpha}(\epsilon\alpha) = 1 + O(\sqrt{z\epsilon^2\alpha^2}\log(\sqrt{z\epsilon^2\alpha^2})).$$

Proof. We break up the proof into two cases.

Case 1: $|z\epsilon^2\alpha^2| \leq 0.5$, and $|z\epsilon^2\alpha^2(\beta_\epsilon^{-1}\alpha^4)| \leq 1$.

First, if $a = \frac{\alpha^2}{2}$, from Remarks 3.8.7 and 3.8.10, we have

- $\tilde{I}_{z,\epsilon,\alpha}(\epsilon\alpha) = 1 + \beta_\epsilon c_\alpha^{-1}\sum_{k=1}^\infty \frac{(-1)^k}{k!2k}a^k + O(\max\{z\epsilon^2\alpha^2, \beta_\epsilon^2\alpha^4\tilde{J}^{(1)}_{z,\epsilon,\alpha}(1)\})$,
- $\epsilon\alpha\tilde{I}'_{z,\epsilon,\alpha}(\epsilon\alpha) = \beta_\epsilon c_\alpha^{-1}\sum_{k=1}^\infty \frac{(-1)^k}{k!}a^k + (\beta_\epsilon c_\alpha^{-1})^2\sum_{k=2}^\infty \frac{(-1)^{k-2}}{2k}a^k\sum_{i=0}^{k-2}\frac{1}{i!(k-2-i)!(i+1)^2}$

  $+O(\max\{z\epsilon^2\alpha^2, \beta_\epsilon^3\alpha^6\tilde{J}^{(2)'}_{z,\epsilon,\alpha}(1)\})$.

Second, let us also write down the small value asymptotic of the modified Bessel functions $K_0(z), K_0'(z)$,

- $K_0(\sqrt{z}\epsilon\alpha) = -\log\frac{\sqrt{z}\epsilon\alpha}{2} + \psi(1) + O(z\epsilon^2\alpha^2\log(\sqrt{z\epsilon^2\alpha^2}))$,
- $\sqrt{z}\epsilon\alpha K_0'(\sqrt{z}\epsilon\alpha) = -1 + O(z\epsilon^2\alpha^2\log(\sqrt{z\epsilon^2\alpha^2}))$.

Third, if $a = \frac{\alpha^2}{2}$, then $c_\alpha = 2\pi(1-e^{-a})$, and from definitions,

- $\beta_\epsilon c_\alpha^{-1} = \left(\frac{1}{\log\epsilon^{-1}} + \frac{\sigma/(8\pi)}{(\log\epsilon^{-1})^2}\right)(1-e^{-a})^{-1}$,
- $\log\epsilon^{-1}\beta_\epsilon c_\alpha^{-1} = \left(1 + \frac{\sigma/(8\pi)}{\log\epsilon^{-1}}\right)(1-e^{-a})^{-1}$,
- $\log\epsilon^{-1}(\beta_\epsilon c_\alpha^{-1})^2 = \left(\frac{1}{\log\epsilon^{-1}} + O\left(\frac{1}{(\log\epsilon^{-1})^2}\right)\right)(1-e^{-a})^{-2}$,
- $\sum_{k=1}^\infty \frac{(-1)^k}{k!}a^k = -(1-e^{-a})$.

Putting them all together, we have[17]

$$K_0(\sqrt{z}\epsilon\alpha)\epsilon\alpha\tilde{I}'_{z,\epsilon,\alpha}(\epsilon\alpha) = -1 + \frac{1}{\log\epsilon^{-1}}\left[\log\frac{\sqrt{z}}{2}(1+O(\beta_\epsilon^2\alpha^6\tilde{J}^{(2)'}_{z,\epsilon,\alpha}(1))) - \psi(1) - \frac{\sigma}{8\pi}\right] + \frac{1}{\log\epsilon^{-1}}\left[\log\alpha + (1-e^{-a})^{-2}\sum_{k=2}^{\infty}\frac{(-1)^{k-2}}{2k}a^k\sum_{i=0}^{k-2}\frac{(-1)^{k-2}}{i!(k-2-i)!(i+1)^2}\right] + O(\max\{z\epsilon^2\alpha^2\log(\sqrt{z\epsilon^2\alpha^2}), \beta_\epsilon^3\alpha^6\tilde{J}^{(2)'}_{z,\epsilon,\alpha}(1)\log(\epsilon\alpha)\}),$$

and

$$-\sqrt{z}\epsilon\alpha K_0'(\sqrt{z}\epsilon\alpha)\tilde{I}_{z,\epsilon,\alpha}(\epsilon\alpha) = 1 + \frac{1}{\log\epsilon^{-1}}(1-e^{-a})^{-1}\sum_{k=1}^{\infty}\frac{(-1)^k}{k!2k}a^k + O(\max\{z\epsilon^2\alpha^2\log(\sqrt{z\epsilon^2\alpha^2}), \beta_\epsilon^2\alpha^4\tilde{J}^{(1)}_{z,\epsilon,\alpha}(1)\}),$$

so

$$K_0(\sqrt{z}\epsilon\alpha)\epsilon\alpha\tilde{I}'_{z,\epsilon,\alpha}(\epsilon\alpha) - \sqrt{z}\epsilon\alpha K_0'(\sqrt{z}\epsilon\alpha)\tilde{I}_{z,\epsilon,\alpha}(\epsilon\alpha) = \frac{1}{\log\epsilon^{-1}}\left[\log\frac{\sqrt{z}}{2}(1+O(\beta_\epsilon^2\alpha^6\tilde{J}^{(2)'}_{z,\epsilon,\alpha}(1))) - \psi(1) - \frac{\sigma}{8\pi} + E(z,\epsilon,\alpha)\right] + \frac{1}{\log\epsilon^{-1}}\left[\log\alpha + (1-e^{-a})^{-2}\sum_{k=2}^{\infty}\frac{(-1)^{k-2}}{2k}a^k\sum_{i=0}^{k-2}\frac{(-1)^{k-2}}{i!(k-2-i)!(i+1)^2} + (1-e^{-a})^{-1}\sum_{k=1}^{\infty}\frac{(-1)^k}{k!2k}a^k\right],$$

where $E(z,\epsilon,\alpha) = \log\epsilon^{-1}\max\{z\epsilon^2\alpha^2\log(\sqrt{z\epsilon^2\alpha^2}), \beta_\epsilon^3\alpha^6\tilde{J}^{(2)'}_{z,\epsilon,\alpha}(1)\log(\epsilon\alpha), \beta_\epsilon^2\alpha^4\tilde{J}^{(1)}_{z,\epsilon,\alpha}(1)\}$, and let us show the error term $E(z,\epsilon,\alpha)$ is small in Case 1. By Propositions 3.8.4 and 3.8.8, $\tilde{J}^{(1)}_{z,\epsilon,\alpha}(1), \tilde{J}^{(2)}_{z,\epsilon,\alpha}(1)$ are bounded for $|z\epsilon^2\alpha^2(\beta_\epsilon^{-1}\alpha^4)| \le 1$, and so is $\tilde{J}^{(2)'}_{z,\epsilon,\alpha}(1)$ by Remark 3.8.9. Also, note

$$\log\epsilon^{-1}z\epsilon^2\alpha^2\log\sqrt{z\epsilon^2\alpha^2} = \alpha^{-4}z\epsilon^2\alpha^2(\beta_\epsilon^{-1}\alpha^4)\log\sqrt{z\epsilon^2\alpha^2(\beta_\epsilon^{-1}\alpha^4)} - z\epsilon^2\alpha^2(\beta_\epsilon^{-1}\alpha^4)\alpha^{-4}\log\sqrt{\beta_\epsilon^{-1}\alpha^4},$$

and from the definitions of α and β_ϵ, it is clear both terms are small. Now it follows every term in the error $E(z,\epsilon,\alpha)$ is small (this is why we expand $\tilde{I}_{z,\epsilon,\alpha}$ up to second order and $\tilde{I}'_{z,\epsilon,\alpha}$ up to third order), and each is at most $1/\log\epsilon^{-1}$ with some factor $\alpha\log\alpha$. Since we only need the error goes to 0, without loss of generality, suppose the error is of order $1/\log\epsilon^{-1}$. Lastly, it remains to identify

$$\log\alpha + (1-e^{-a})^{-2}\sum_{k=2}^{\infty}\sum_{i=0}^{k-2}\frac{(-1)^{k-2}}{i!(k-2-i)!(2k)(i+1)^2}a^k + (1-e^{-a})^{-1}\sum_{k=1}^{\infty}\frac{1}{2}\frac{(-1)^k}{k!k}a^k,$$

which we do in Section 3.10. To finish the Proposition, we need to consider the other case where $|z\epsilon^2\alpha^2(\beta_\epsilon^{-1}\alpha^4)| \ge 1$.

Case 2: $|z\epsilon^2\alpha^2| \le 0.5$, and $|z\epsilon^2\alpha^2(\beta_\epsilon^{-1}\alpha^4)| \ge 1$.

[17] When we cross-multiply, we need be careful with the error terms, and we need to know how big the series are, from $\tilde{I}_{z,\epsilon,\alpha}(\epsilon\alpha), \epsilon\alpha\tilde{I}'_{z,\epsilon,\alpha}(\epsilon\alpha)$. For this, we use the same argument as in Step 2 of Proposition 3.8.8 to show they are small compared to β_ϵ.

In the notations of Proposition 3.8.11, let $\tilde{J}^{(0)}_{z,\epsilon,\alpha}(1) = (z\epsilon^2\alpha^2)^{-1/2}\left(\tilde{J}_{z,\epsilon,\alpha}(1) - 1\right)$. Rewrite

$$\begin{aligned}
&K_0(\sqrt{z}\epsilon\alpha)\epsilon\alpha\tilde{I}'_{z,\epsilon,\alpha}(\epsilon\alpha) - \sqrt{z}\epsilon\alpha K_0'(\sqrt{z}\epsilon\alpha)\tilde{I}_{z,\epsilon,\alpha}(\epsilon\alpha)\\
&\quad = K_0(\sqrt{z\epsilon^2\alpha^2})\sqrt{z\epsilon^2\alpha^2}\tilde{J}^{(0)'}_{z,\epsilon,\alpha}(1) - \sqrt{z\epsilon^2\alpha^2}K_0'(\sqrt{z\epsilon^2\alpha^2}) - \sqrt{z\epsilon^2\alpha^2}K_0'(\sqrt{z\epsilon^2\alpha^2})\sqrt{z\epsilon^2\alpha^2}\tilde{J}^{(0)}_{z,\epsilon,\alpha}(1)\\
&\quad = O(\sqrt{z\epsilon^2\alpha^2}\log(\sqrt{z\epsilon^2\alpha^2})) + 1 + O(\sqrt{z\epsilon^2\alpha^2}),
\end{aligned}$$

where we use the small value asymptotic of $K_0(z), K_0'(z)$ as in Case 1, and by Proposition 3.8.11 and Remark 3.8.12, $\tilde{J}^{(0)}_{z,\epsilon,\alpha}(1), \tilde{J}^{(0)'}_{z,\epsilon,\alpha}(1)$ are bounded.

□

3.10 Identification of the Pole of the Solution

From Section 3.9, we know the main contribution to the denominator$K_0(\sqrt{z}\epsilon\alpha)\epsilon\alpha\tilde{I}'_{z,\epsilon,\alpha}(\epsilon\alpha)$ $-\sqrt{z}\epsilon\alpha K_0'(\sqrt{z}\epsilon\alpha)\tilde{I}_{z,\epsilon,\alpha}(\epsilon\alpha)$ in $w^{z,\epsilon,\alpha}(x)$ is

$$\begin{aligned}
&\log\frac{\sqrt{z}}{2} - \psi(1) - \frac{\sigma}{8\pi}\\
&+\log\alpha + (1-e^{-\frac{\alpha^2}{2}})^{-2}\sum_{k=2}^{\infty}\sum_{i=0}^{k-2}\frac{(-1)^{k-2}}{i!(k-2-i)!(2k)(i+1)^2}\left(\frac{\alpha^2}{2}\right)^k + (1-e^{-\frac{\alpha^2}{2}})^{-1}\sum_{k=1}^{\infty}\frac{1}{2}\frac{(-1)^k}{k!k}\left(\frac{\alpha^2}{2}\right)^k.
\end{aligned}$$

In this section, we are going to identify the second line as

$$\int dx\ p_{1,\alpha}(x)\int dy\ \log|x-y|p_{1,\alpha}(y),$$

which shows the main contribution is indeed $g_\alpha(z) = \log\frac{\sqrt{z}}{2} - \psi(1) - \frac{\sigma}{8\pi} + \nu(p_{1,\alpha})$ as stated in Proposition 3.9.1, and this identification will follow from the next three Lemmas. And it is useful to keep in mind the definition $p_{1,\alpha}(x) = c_\alpha^{-1}e^{-\frac{|x|^2}{2}}\mathbb{1}_{|x|\le\alpha}$, where $c_\alpha = 2\pi(1-e^{-\frac{\alpha^2}{2}})$.

Lemma 3.10.1.

$$\begin{aligned}
&(1-e^{-\frac{\alpha^2}{2}})^2\log\alpha + \sum_{k=0}^{\infty}\sum_{i=0}^{k}\frac{(-1)^k}{i!(k-i)!2(k+2)(i+1)^2}\left(\frac{\alpha^2}{2}\right)^{k+2} - (1-e^{-\frac{\alpha^2}{2}})\sum_{k=0}^{\infty}\frac{(-1)^k}{k!2(k+1)^2}\left(\frac{\alpha^2}{2}\right)^{k+1}\\
&\quad = \sum_{k,l=0}^{\infty}\frac{(-1)^k}{k!}\frac{(-1)^l}{l!}\left(\frac{\alpha^2}{2}\right)^{l+2+k}\left(\frac{1}{l+1}\frac{1}{k+2+l}2\log\alpha - \frac{1}{l+1}\frac{1}{(l+2+k)^2}\right).
\end{aligned}$$

Proof. Let $v_k = \frac{(-1)^k}{2^k}$, so $(1-e^{-\frac{\alpha^2}{2}}) = \frac{\alpha^2}{2}\sum_{k=0}^{\infty}\frac{v_k}{k!}\frac{\alpha^{2k}}{k+1}$. First, let us match the log terms. By definition of multiplication of series,

$$(1-e^{-\frac{\alpha^2}{2}})^2 = \left(\frac{\alpha^2}{2}\right)^2\sum_{n=0}^{\infty}\alpha^{2n}\sum_{m=0}^{n}\frac{v_m v_{n-m}}{m!(n-m)!(m+1)(n-m+1)}. \tag{3.10.30}$$

On the other hand, we have

$$2\sum_{k,l=0}^{\infty}\frac{(-1)^k}{k!}\frac{(-1)^l}{l!}\left(\frac{\alpha^2}{2}\right)^{l+2+k}\frac{1}{l+1}\frac{1}{k+2+l} = 2\left(\frac{\alpha^2}{2}\right)^2\sum_{n=0}^{\infty}\alpha^{2n}\sum_{m=0}^{n}\frac{v_m v_{n-m}}{m!(n-m)!(n+2)(m+1)}, \tag{3.10.31}$$

where we let $n=k+l, m=l$. One can check directly that we have pairwise equality of the two sides (3.10.30) and (3.10.31) in the sense[18] that

$$\frac{1}{(m+1)(n-m+1)}(\mathbb{1}_{m=0}+\mathbb{1}_{m=n}) = 2\frac{1}{(n+2)(m+1)}(\mathbb{1}_{m=0}+\mathbb{1}_{m=n}),$$
$$\frac{1}{(m+1)(n-m+1)}(\mathbb{1}_{m=1}+\mathbb{1}_{m=n-1}) = 2\frac{1}{(n+2)(m+1)}(\mathbb{1}_{m=1}+\mathbb{1}_{m=n-1}),$$

and so on.

Second, to match the rest, it follows by the same argument, and the main point is we also have pairwise equality of the two sides. More precisely, again letting $v_k=\frac{(-1)^k}{2^k}$, so $(1-e^{-\frac{\alpha^2}{2}})=\frac{\alpha^2}{2}\sum_{k=0}^{\infty}\frac{v_k}{k!}\frac{\alpha^{2k}}{k+1}$. On one hand, we have

$$\begin{aligned}
&\sum_{k=0}^{\infty}\sum_{i=0}^{k}\frac{(-1)^k}{i!(k-i)!2(k+2)(i+1)^2}\left(\frac{\alpha^2}{2}\right)^{k+2} - (1-e^{-\frac{\alpha^2}{2}})\sum_{k=0}^{\infty}\frac{(-1)^k}{k!2(k+1)^2}\left(\frac{\alpha^2}{2}\right)^{k+1}\\
&= \frac{\alpha^4}{2^2}\left[\sum_{k=0}^{\infty}\frac{\alpha^{2k}}{2(k+2)}\sum_{i=0}^{k}\frac{v_i v_{k-i}}{i!(k-i)!(i+1)^2} - \sum_{l=0}^{\infty}\frac{v_l}{l!}\frac{\alpha^{2l}}{l+1}\sum_{k=0}^{\infty}\frac{v_k}{k!}\frac{\alpha^{2k}}{2(k+1)^2}\right]\\
&= \frac{\alpha^4}{2^2}\sum_{n=0}^{\infty}\alpha^{2n}\sum_{m=0}^{n}\frac{v_m v_{n-m}}{m!(n-m)!}\left[\frac{1}{2(n+2)(m+1)^2} - \frac{1}{(n-m+1)2(m+1)^2}\right]\\
&= \frac{\alpha^4}{2^2}\sum_{n=0}^{\infty}\alpha^{2n}\sum_{m=0}^{n}\frac{v_m v_{n-m}}{m!(n-m)!}\frac{1}{2(m+1)}\frac{1}{(n+2)(n-m+1)}.
\end{aligned}$$

On the other hand, let $n=k+l, m=l$, we have

$$-\sum_{k,l=0}^{\infty}\frac{(-1)^k}{k!}\frac{(-1)^l}{l!}\left(\frac{\alpha^2}{2}\right)^{l+2+k}\frac{1}{l+1}\frac{1}{(l+2+k)^2} = -\frac{\alpha^4}{2^2}\sum_{n=0}^{\infty}\alpha^{2n}\sum_{m=0}^{n}\frac{v_m v_{n-m}}{m!(n-m)!}\frac{1}{(n+2)^2(m+1)}.$$

We can again check we have pairwise equality in the sense[19] that

$$\frac{1}{(m+1)(n+2)^2}(\mathbb{1}_{m=0}+\mathbb{1}_{m=n}) = \frac{1}{2(m+1)}\frac{1}{(n+2)(n-m+1)}(\mathbb{1}_{m=0}+\mathbb{1}_{m=n}),$$
$$\frac{1}{(m+1)(n+2)^2}(\mathbb{1}_{m=1}+\mathbb{1}_{m=n-1}) = \frac{1}{2(m+1)}\frac{1}{(n+2)(n-m+1)}(\mathbb{1}_{m=1}+\mathbb{1}_{m=n-1}),$$

and so on. □

Lemma 3.10.2.

$$\int dx\ p_{1,\alpha}(x)\int dy\ \log|x-y|p_{1,\alpha}(y) = 2(2\pi)^2 c_\alpha^{-2}\int_0^\alpha dr\ re^{-\frac{r^2}{2}}\int_r^\alpha ds\ se^{-\frac{s^2}{2}}\log s.$$

[18] If n is even, then one also checks $\frac{1}{(m+1)(n-m+1)}\mathbb{1}_{m=n/2} = 2\frac{1}{(n+2)(m+1)}\mathbb{1}_{m=n/2}$.

[19] If n is even, then one also checks $\frac{1}{(m+1)(n+2)^2}\mathbb{1}_{m=n/2} = \frac{1}{2(m+1)}\frac{1}{(n+2)(n-m+1)}\mathbb{1}_{m=n/2}$.

Proof. It is a direct computation (though a bit long) using polar coordinates, the expansion $\log(1+r) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} r^k$ for $|r| < 1$, and the formula $\int_0^{2\pi} d\theta \int_0^{2\pi} d\phi \cos^k(\theta - \phi) = 2(2\pi)^2 2^{-k} \begin{pmatrix} k-1 \\ k/2 \end{pmatrix} \mathbb{1}_{k \text{ even}}$. More precisely, using polar coordinates,

$$\begin{aligned}\int dx\ p_{1,\alpha}(x) \int dy\ \log|x-y| p_{1,\alpha}(y) &= c_\alpha^{-2} \int_0^\alpha dr\ re^{-\frac{r^2}{2}} \int_0^{2\pi} d\theta \int_0^\alpha ds\ se^{-\frac{s^2}{2}} \int_0^{2\pi} d\phi \\ &\quad \times \frac{1}{2}\left\{\log[r^2+s^2] + \log\left[1 - \frac{2rs\cos(\theta-\phi)}{r^2+s^2}\right]\right\} \\ &= c_\alpha^{-2} \int_0^\alpha dr\ re^{-\frac{r^2}{2}} \int_0^{2\pi} d\theta \int_0^\alpha ds\ se^{-\frac{s^2}{2}} \int_0^{2\pi} d\phi \\ &\quad \times \frac{1}{2}\left\{\log[r^2+s^2] + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\left(-\frac{2rs\cos(\theta-\phi)}{r^2+s^2}\right)^k\right\}.\end{aligned}$$

Integrating over θ, ϕ, by induction on k, one can show

$$\int_0^{2\pi} d\theta \int_0^{2\pi} d\phi \cos^k(\theta-\phi) = 2(2\pi)^2 2^{-k} \mathbb{1}_{k=2l} \begin{pmatrix} k-1 \\ k/2 \end{pmatrix}.$$

Next, using the formula $\sum_{l=1}^{\infty} \frac{1}{2l} r^{2l} \begin{pmatrix} 2l-1 \\ l \end{pmatrix} = -\frac{1}{2}\log\left[\frac{1}{2}\left(1+\sqrt{1-4r^2}\right)\right]$ for r such that $1-4r^2 \geq 0$, and summing over k,

$$\begin{aligned}&\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\left(-\frac{2rs}{r^2+s^2}\right)^k 2(2\pi)^2 2^{-k} \mathbb{1}_{k=2l} \begin{pmatrix} k-1 \\ k/2 \end{pmatrix} \\ &\quad = -2(2\pi)^2 \sum_{l=1} \frac{1}{2l}\left(\frac{rs}{r^2+s^2}\right)^{2l} \begin{pmatrix} 2l-1 \\ l \end{pmatrix} \\ &\quad = (2\pi)^2 \left\{\log\left[\frac{1}{2}\left(1 + \frac{\sqrt{(r^2-s^2)^2}}{r^2+s^2}\right)\right]\right\} \\ &\quad = 2(2\pi)^2 \left[\log r \mathbb{1}_{r>s} + \log s \mathbb{1}_{s>r} - \frac{1}{2}\log(r^2+s^2)\right],\end{aligned}$$

where in the last equality we also use $1 + \frac{\sqrt{(r^2-s^2)^2}}{r^2+s^2} = \frac{r^2+s^2+(r^2-s^2)\mathbb{1}_{r>s}+(s^2-r^2)\mathbb{1}_{s>r}}{r^2+s^2} = \frac{2r^2\mathbb{1}_{r>s}+2s^2\mathbb{1}_{s>r}}{r^2+s^2}$. Finally, we conclude

$$\begin{aligned}\int dx\ p_{1,\alpha}(x) \int dy\ \log|x-y| p_{1,\alpha}(y) &= \frac{c_\alpha^{-2}}{2} \int_0^\alpha dr\ re^{-\frac{r^2}{2}} \int_0^\alpha ds\ se^{-\frac{s^2}{2}} \\ &\quad \times \left\{(2\pi)^2 \log[r^2+s^2] + 2(2\pi)^2\left[2\log s \mathbb{1}_{s>r} - \frac{1}{2}\log(r^2+s^2)\right]\right\} \\ &= 2(2\pi)^2 c_\alpha^{-2} \int_0^\alpha dr\ re^{-\frac{r^2}{2}} \int_r^\alpha ds\ se^{-\frac{s^2}{2}} \log s.\end{aligned}$$

□

Lemma 3.10.3.

$$\int_0^\alpha dr\ re^{-\frac{r^2}{2}} \int_r^\alpha ds\ se^{-\frac{s^2}{2}} \log s$$
$$= \frac{1}{2}\sum_{k,l=0}^{\infty} \frac{(-1)^k}{k!}\frac{(-1)^l}{l!}\left(\frac{\alpha^2}{2}\right)^{l+2+k}\left(\frac{1}{l+1}\frac{1}{k+2+l}2\log\alpha - \frac{1}{l+1}\frac{1}{(l+2+k)^2}\right).$$

Proof. It is also a direct (though long) computation by expanding $e^{-\frac{r^2}{2}}$ into a series and using the formula $\int_a^b dx\ x^k \log x = \frac{1}{(1+k)^2}\left[b^{1+k}(-1+(1+k)\log b) - a^{1+k}(-1+(1+k)\log a)\right]$. More precisely, let $a = \frac{\alpha^2}{2}$, by direct computation,

$$\begin{aligned}\int_0^\alpha dr\ re^{-\frac{r^2}{2}} \int_r^\alpha ds\ se^{-\frac{s^2}{2}} \log s &= \frac{1}{2}\int_0^{\alpha^2/2} dr\ e^{-r}\int_r^{\alpha^2/2} ds\ e^{-s}\log(2s)\\ &= \frac{1}{2}\log 2\int_0^a dr\ e^{-r}\int_r^a ds\ e^{-s} + \frac{1}{2}\int_0^a dr\ e^{-r}\int_r^a ds\ e^{-s}\log s\\ &=: A\frac{1}{2}\log 2 + B\frac{1}{2}.\end{aligned}$$

To compute A, B, we expand e^{-r}, e^{-s} into series. For A,

$$\begin{aligned}A &= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\int_0^a dr\ e^{-r}\frac{1}{k+1}(a^{k+1} - r^{k+1})\\ &= \sum_{k,l=0}^{\infty}\frac{(-1)^k}{k!}\frac{(-1)^l}{l!}\left[\frac{1}{l+1}\frac{1}{k+1}a^{1+k}a^{l+1} - \frac{1}{k+1}\frac{1}{k+2+l}a^{l+2+k}\right]\\ &= \sum_{k,l=0}^{\infty}\frac{(-1)^k}{k!}\frac{(-1)^l}{l!}a^{l+2+k}\frac{1}{l+1}\frac{1}{k+2+l}.\end{aligned}$$

For B,

$$\begin{aligned}B &= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\int_0^a dr\ e^{-r}\frac{1}{(1+k)^2}\left[a^{1+k}(-1+(1+k)\log a) - r^{1+k}(-1+(1+k)\log r)\right]\\ &= \sum_{k,l=0}^{\infty}\frac{(-1)^l}{l!}\frac{(-1)^k}{k!}\frac{1}{l+1}a^{l+1}\frac{1}{(1+k)^2}a^{1+k}(-1+(1+k)\log a)\\ &\quad + \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\left[\int_0^a dr\ \frac{1}{(1+k)^2}r^l r^{1+k} - \int_0^a dr\ \frac{1}{1+k}r^l r^{1+k}\log r\right]\\ &= \sum_{k,l=0}^{\infty}\frac{(-1)^l}{l!}\frac{(-1)^k}{k!}\frac{1}{l+1}a^{l+1}\frac{1}{(1+k)^2}a^{1+k}(-1+(1+k)\log a) + \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\\ &\quad\times\left[\frac{1}{(1+k)^2}\frac{1}{l+1+1+k}a^{l+1+1+k} - \frac{1}{1+k}\frac{1}{(l+1+1+k)^2}a^{l+1+1+k}(-1+(l+1+1+k)\log a)\right]\\ &= \sum_{k,l=0}^{\infty}\frac{(-1)^l}{l!}\frac{(-1)^k}{k!}\frac{1}{l+1}a^{k+l+2}\frac{1}{l+2+k}\log a - \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}a^{l+2+k}\frac{1}{l+1}\frac{1}{(l+2+k)^2}.\end{aligned}$$

Finally, recall $\int_0^\alpha dr\ re^{-\frac{r^2}{2}}\int_r^\alpha ds\ se^{-\frac{s^2}{2}}\log s = \frac{1}{2}(A\log 2 + B)$, and the definition $a = \frac{\alpha^2}{2}$, and substitute in the expressions for A, B, we get the Lemma. $\square$

3.11 Large Value Asymptotic of the Denominator

Suppose $z \in \mathbb{C} - \mathbb{R}_-$. In this Section, we are going to finish the analysis for identifying the poles in the resolvent solution $w^{z,\epsilon,\alpha}(x)$. In particular, we will use the large value estimates in Section 3.8.2 to show that for z large, the difference $K_0(\sqrt{z}\epsilon\alpha)\epsilon\alpha\tilde{I}'_{z,\epsilon,\alpha}(\epsilon\alpha) - \sqrt{z}\epsilon\alpha K'_0(\sqrt{z}\epsilon\alpha)\tilde{I}_{z,\epsilon,\alpha}(\epsilon\alpha)$ in the denominator of $w^{z,\epsilon,\alpha}(x)$ is bounded away from 0. Recall $\beta_\epsilon = \frac{4\pi}{\log \epsilon^{-2}} + \frac{\sigma}{(\log \epsilon^{-2})^2}$.

Proposition 3.11.1. *There exists c_ϵ such that for $\alpha = \alpha_\epsilon = \sqrt{2\log(\log \epsilon^{-1})^{1+2\delta}}$ and on $A^c_{\epsilon,\alpha} = [z \mid |z\epsilon^2\alpha^2| \geq 0.5]$, we have*

$$K_0(\sqrt{z}\epsilon\alpha)\epsilon\alpha\tilde{I}'_{z,\epsilon,\alpha}(\epsilon\alpha) - \sqrt{z}\epsilon\alpha K'_0(\sqrt{z}\epsilon\alpha)\tilde{I}_{z,\epsilon,\alpha}(\epsilon\alpha) = 1 + c_\epsilon$$

where $c_\epsilon \to 0$ uniformly on $A^c_{\epsilon,\alpha}$.

Proof. We do it in two cases.
Case 1: $0.5 \leq |z\epsilon^2\alpha^2| \leq 5$.

Rewrite the difference as

$$\begin{aligned}
&K_0(\sqrt{z\epsilon^2\alpha^2})\sqrt{z\epsilon^2\alpha^2}I'_0(\sqrt{z\epsilon^2\alpha^2}) - \sqrt{z\epsilon^2\alpha^2}K'_0(\sqrt{z\epsilon^2\alpha^2})I_0(\sqrt{z\epsilon^2\alpha^2})\\
&-\sqrt{z\epsilon^2\alpha^2}K'_0(\sqrt{z\epsilon^2\alpha^2})\left(\tilde{I}_{z,\epsilon,\alpha}(\epsilon\alpha) - I_0(\sqrt{z\epsilon^2\alpha^2})\right)\\
&+K_0(\sqrt{z\epsilon^2\alpha^2})\left(\epsilon\alpha\tilde{I}'_{z,\epsilon,\alpha}(\epsilon\alpha) - \sqrt{z\epsilon^2\alpha^2}I'_0(\sqrt{z\epsilon^2\alpha^2})\right).
\end{aligned}$$

For the first line $K_0(\sqrt{z\epsilon^2\alpha^2})\sqrt{z\epsilon^2\alpha^2}I'_0(\sqrt{z\epsilon^2\alpha^2})-\sqrt{z\epsilon^2\alpha^2}K'_0(\sqrt{z\epsilon^2\alpha^2})I_0(\sqrt{z\epsilon^2\alpha^2})$, we note it is the Wronskian of I_0, K_0, so it is equal to 1; for the second line $-\sqrt{z\epsilon^2\alpha^2}K'_0(\sqrt{z\epsilon^2\alpha^2})\left(\tilde{I}_{z,\epsilon,\alpha}(\epsilon\alpha) - I_0(\sqrt{z\epsilon^2\alpha^2})\right)$, in the notations of Proposition 3.8.13, the difference is $\tilde{J}_{z,\epsilon,\alpha}(1) - N_1(1)$, and it is uniformly small for z such that $0.5 \leq |z\epsilon^2\alpha^2| \leq 5$. Also, since $0.5 \leq |z\epsilon^2\alpha^2| \leq 5$, $K'_0(\sqrt{z\epsilon^2\alpha^2})$ is bounded, so the second line is small; for the third line $K_0(\sqrt{z\epsilon^2\alpha^2})\left(\epsilon\alpha\tilde{I}'_{z,\epsilon,\alpha}(\epsilon\alpha) - \sqrt{z\epsilon^2\alpha^2}I'_0(\sqrt{z\epsilon^2\alpha^2})\right)$, by Remark 3.8.15 it is also small. We conclude there is constant c_ϵ small such that on $[0.5 \leq |z\epsilon^2\alpha^2| \leq 5]$,

$$K_0(\sqrt{z}\epsilon\alpha)\epsilon\alpha\tilde{I}'_{z,\epsilon,\alpha}(\epsilon\alpha) - \sqrt{z}\epsilon\alpha K'_0(\sqrt{z}\epsilon\alpha)\tilde{I}_{z,\epsilon,\alpha}(\epsilon\alpha) = 1 + c_\epsilon.$$

Case 2: $|z\epsilon^2\alpha^2| \geq 5$.

In this case, rewrite the difference as

$$\begin{aligned}
&K_0(\sqrt{z\epsilon^2\alpha^2})\sqrt{z\epsilon^2\alpha^2}I'_0(\sqrt{z\epsilon^2\alpha^2}) - \sqrt{z\epsilon^2\alpha^2}K'_0(\sqrt{z\epsilon^2\alpha^2})I_0(\sqrt{z\epsilon^2\alpha^2})\\
&-\sqrt{z\epsilon^2\alpha^2}K'_0(\sqrt{z\epsilon^2\alpha^2})I_0(\sqrt{z\epsilon^2\alpha^2})\left(\frac{\tilde{I}_{z,\epsilon,\alpha}(\epsilon\alpha) - I_0(\sqrt{z\epsilon^2\alpha^2})}{I_0(\sqrt{z\epsilon^2\alpha^2})}\right)\\
&+K_0(\sqrt{z\epsilon^2\alpha^2})\sqrt{z\epsilon^2\alpha^2}I'_0(\sqrt{z\epsilon^2\alpha^2})\left(\frac{\epsilon\alpha\tilde{I}'_{z,\epsilon,\alpha}(\epsilon\alpha) - \sqrt{z\epsilon^2\alpha^2}I'_0(\sqrt{z\epsilon^2\alpha^2})}{\sqrt{z\epsilon^2\alpha^2}I'_0(\sqrt{z\epsilon^2\alpha^2})}\right).
\end{aligned}$$

As in Case 1, the first line is 1. And by Proposition 3.8.17, we have

- $\tilde{I}_{z,\epsilon,\alpha}(\epsilon\alpha) = I_0(\sqrt{z\epsilon^2\alpha^2})\,[1 + O(c_\epsilon)]$,
- $\epsilon\alpha\tilde{I}'_{z,\epsilon,\alpha}(\epsilon\alpha) = \sqrt{z\epsilon^2\alpha^2}I'_0(\sqrt{z\epsilon^2\alpha^2})\,[1 + O(c_\epsilon)]$,

where c_ϵ is uniformly small for z such that $|z\epsilon^2\alpha^2| \geq 5$. We see the ratios in the bracket are $O(c_\epsilon)$, and it remains to show the prefactors are bounded. Substituting in the large value asymptotic of $I_0(z), K_0(z)$ (see 3.7.18), this is clear, and we have the same conclusion as in Case 1.

□

Lemma 3.11.2. *The quantity $\tilde{I}_{z,\epsilon,\alpha}(\epsilon\alpha)$ has no zero over z.*

Proof. In the notation of Lemma 3.8.1, $\tilde{I}_{z,\epsilon,\alpha}(\epsilon\alpha) = \tilde{J}_{z,\epsilon,\alpha}(1)$.
Case 1: $|z\epsilon^2\alpha^2| \leq 5$.

Write $\tilde{J}_{z,\epsilon,\alpha}(1) = N_1(1) + (\tilde{J}_{z,\epsilon,\alpha}(1) - N_1(1))$. By Proposition 3.8.13 and Remark 3.8.16, $\tilde{J}_{z,\epsilon,\alpha}(1) - N_1(1) = O(\beta_\epsilon\alpha^2)$, so it remains to show $N_1(1)$ is bounded away from 0. Recall $N_1(1) = I_0(\sqrt{z\epsilon^2\alpha^2})$ and $z \in \mathbb{C} - \mathbb{R}_-$, and let $w = z\epsilon^2\alpha^2$. From definition, $I_0(\sqrt{w}) = J_0(i\sqrt{w})$ where $J_0(z) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k z^{2k}}{2^{2k}k!^2}$, and $J_0(z)$ is Bessel function of the first kind. It is a fact that $J_0(z)$ has only real zeros (see [28], page 482), and the first zero is at $z = 2.40483$ (see Mathematica). Thus[20] $J_0(i\sqrt{w})$ is bounded away from zero for $|w| \leq 5$ since $\sqrt{5} = 2.23607$. By taking $\epsilon > 0$ small, $\tilde{J}_{z,\epsilon,\alpha}(1)$ has no zero in this case.
Case 2: $5 \leq |z\epsilon^2\alpha^2|$.

By Proposition 3.8.17, and the large value asymptotic of $I_0(z)$, we conclude $\tilde{I}_{z,\epsilon,\alpha}(\epsilon\alpha)$ has no zeros.

□

3.12 Contour Deformation

Recall the resolvent solution $w^{z,\epsilon,\alpha}(x)$ is the Laplace transform in t of $u^{\epsilon,\alpha}(t,x)$ (see Sections 3.3 and 3.4). As $w^{z,\epsilon,\alpha}(x)$ is first defined in Section 3.4, z is taken to be $\Re z > \beta_\epsilon\epsilon^{-2}c_\alpha^{-1}$; to recover $u^{\epsilon,\alpha}(t,x)$, it immediately follows that it is the inverse Laplace transform of $w^{z,\epsilon,\alpha}(x)$ along any vertical line that is to the right of $\beta_\epsilon\epsilon^{-2}c_\alpha^{-1}$. But to facilitate taking limit later, we will use a different contour. Recall $z_0 = 4e^{2(\frac{\sigma}{8\pi} - \nu(p_1) + \psi(1))}$ with $\nu(p_1) = \int dx\ p_1(x) \int dy\ \log|x-y| p_1(y)$.

Proposition 3.12.1. *Let $\lambda > \beta_\epsilon\epsilon^{-2}c_\alpha^{-1}$, and γ a contour encircling z_0 and $\mathbb{C} - \mathbb{R}_-$, then*

$$\frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} dz\ e^{tz} w^{z,\epsilon,\alpha}(x) = \frac{1}{2\pi i}\int_\gamma dz\ e^{tz} w^{z,\epsilon,\alpha}(x).$$

where γ is defined to be $(L_{\delta,-} \cup K_\delta \cup L_{\delta,+}) \cup C_{z_0,\delta}$ with $L_{\delta,\pm} = [-r \pm i\delta \mid r > -\delta], K_\delta = [\delta + iy \mid |y| \leq \delta], C_{z_0,\delta} = [z_0 + \delta e^{i\theta} \mid 0 \leq \theta \leq 2\pi]$, and $\delta > 0$ small.

Proof. We do this in two steps. Below, all contours are oriented counterclockwise.
Step 1: Deform into the half strip containing z_0 and $\mathbb{C} - \mathbb{R}_-$, i.e.

$$\frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} dz\ e^{tz} w^{z,\epsilon,\alpha}(x) = \frac{1}{2\pi i}\int_{\gamma_1} dz\ e^{tz} w^{z,\epsilon,\alpha}(x) \tag{3.12.32}$$

where $\gamma_1 = [x \pm i\delta \mid x \leq \lambda] \cup [\lambda + iy \mid |y| \leq \delta]$.

To deform the contour from the left hand side to that of the right hand side in (3.12.32), we consider $w^{z,\epsilon,\alpha}(x)$ in the two quarter circles in the upper and lower half planes, centered at $\lambda \pm i\delta$ and with sides along the two contours. From Propositions 3.9.1, 3.11.1 and Lemma 3.11.2, we know $w^{z,\epsilon,\alpha}(x)$ has pole only at $z_{0,\alpha}$ (where $z_{0,\alpha} \to z_0$ as $\epsilon \to 0$) and it is defined in $\mathbb{C} - \mathbb{R}_-$. Thus, by Cauchy's Theorem, the contour integrals of $w^{z,\epsilon,\alpha}(x)$ over the two quarter circles are zero, and to show (3.12.32), it is enough to

[20] Though $\Im(i\sqrt{w}) > 0$, from definition of $J_0(z)$, it extends continuously to $\Im(i\sqrt{w}) \geq 0$.

show contributions from the arcs of the quarter circles are small as they get bigger. This means we only need to consider the large z asymptotic of $w^{z,\epsilon,\alpha}(x)$. More precisely, by Cauchy's Theorem, we have

$$\frac{1}{2\pi i}\int_{\cup_{i=1}^{3}\gamma_{1,i}} dz\ e^{tz} w^{z,\epsilon,\alpha}(x) = 0,$$

where $\gamma_{1,1} = [\lambda + iy \mid \delta \le y \le L], \gamma_{1,2} = [\lambda + i\delta + Le^{i\theta} \mid \pi/2 \le \theta \le \pi], \gamma_{1,3} = [x + i\delta \mid -L \le x \le \lambda]$, and it is enough to show as $L \to \infty$,

$$\frac{1}{2\pi i}\int_{\gamma_{1,2}} dz\ e^{tz} w^{z,\epsilon,\alpha}(x) \to 0. \tag{3.12.33}$$

(There is a similar statement for the "lower half" of the contour.) Lastly, before we start the proof, note for $z \in \gamma_{1,2}$ that $\cos\theta < 0, \sin\theta > 0$, so $e^{tz} = e^{t(\lambda+i\delta)}e^{tL\cos(\theta)}e^{itL\sin(\theta)} \to 0$; also, $\sqrt{z\epsilon^2\alpha^2} = |z\epsilon^2\alpha^2|^{1/2}e^{i(\frac{\pi}{2} - \frac{1}{2}\arctan(\frac{\delta + L\sin\theta}{-L\cos\theta - \lambda}))}$.

Now, recall from Proposition 3.7.1, $w^{z,\epsilon,\alpha}(x) = w_-^{z,\epsilon,\alpha}(x)\mathbb{1}_{|x|\le\epsilon\alpha} + w_+^{z,\epsilon,\alpha}(x)\mathbb{1}_{|x|\ge\epsilon\alpha}$, where

$$\begin{aligned} w_-^{z,\epsilon,\alpha}(x) = &\frac{1}{2\pi}\frac{K_0(\sqrt{z\epsilon^2\alpha^2})}{K_0(\sqrt{z\epsilon^2\alpha^2})\epsilon\alpha\tilde{I}'_{z,\epsilon,\alpha}(\epsilon\alpha) - \sqrt{z\epsilon^2\alpha^2}K_0'(\sqrt{z\epsilon^2\alpha^2})\tilde{I}_{z,\epsilon,\alpha}(\epsilon\alpha)}\frac{\tilde{I}_{z,\epsilon,\alpha}(|x|)}{\tilde{I}_{z,\epsilon,\alpha}(\epsilon\alpha)} \\ &- \frac{1}{2\pi}\frac{\tilde{I}_{z,\epsilon,\alpha}(|x|)}{\tilde{I}_{z,\epsilon,\alpha}(\epsilon\alpha)}\tilde{K}_{z,\epsilon,\alpha}(\alpha\epsilon) + \frac{1}{2\pi}\tilde{K}_{z,\epsilon,\alpha}(|x|) =: A + B + C \end{aligned}$$

and

$$w_+^{z,\epsilon,\alpha}(x) = \frac{1}{2\pi}\frac{K_0(\sqrt{z}|x|)}{K_0(\sqrt{z\epsilon^2\alpha^2})\epsilon\alpha\tilde{I}'_{z,\epsilon,\alpha}(\epsilon\alpha) - \sqrt{z\epsilon^2\alpha^2}K_0'(\sqrt{z\epsilon^2\alpha^2})\tilde{I}_{z,\epsilon,\alpha}(\epsilon\alpha)}.$$

To show (3.12.33), we want to say each piece of $w^{z,\epsilon,\alpha}(x)$ is small on the arcs of the quarter circles. We will consider only $w_-^{z,\epsilon,\alpha}(x)$ in the upper quarter circle, and write it as $A + B + C$; work for $w_+^{z,\epsilon,\alpha}(x)$ and the lower quarter circle is similar. Since we let $L \to \infty$ (with ϵ, α, x fixed), we may suppose L large such that $|z\epsilon^2\alpha^2| \ge 5$.

For A: By Proposition 3.11.1, note the difference in the denominator is bounded away from 0. Also, for $w_-^{z,\epsilon,\alpha}(x)$, x is such that $|x| \le \epsilon\alpha$. so $s := \frac{|x|}{\epsilon\alpha} \le 1$, and in the notations of Lemma 3.8.1, $\frac{\tilde{I}_{z,\epsilon,\alpha}(|x|)}{\tilde{I}_{z,\epsilon,\alpha}(\epsilon\alpha)} = \frac{\tilde{J}_{z,\epsilon,\alpha}(s)}{\tilde{J}_{z,\epsilon,\alpha}(1)}$. Since z large such that $|z\epsilon^2\alpha^2 s^2| \ge 5$, by Proposition 3.8.17 and the large value asymptotic of $I_0(z)$, then $\left|\frac{\tilde{J}_{z,\epsilon,\alpha}(s)}{\tilde{J}_{z,\epsilon,\alpha}(1)}\right| \le C'\frac{1}{\sqrt{s}}$, for some constant C'. Thus, up to some constant, we see

$$\frac{1}{2\pi i}\int_{\gamma_{1,2}} dz\ e^{tz} A\mathbb{1}_{|x|\le\epsilon\alpha} = \frac{1}{2\pi i}\int_{\gamma_{1,2}} dz\ e^{tz}\frac{K_0(\sqrt{z\epsilon^2\alpha^2})}{1 + c_\epsilon}\mathbb{1}_{|x|\le\epsilon\alpha}.$$

Again, since $|z\epsilon^2\alpha^2|$ is large, by the large value asymptotic of $K_0(z)$, it follows the integral tends to 0.

For B, C: It is enough to consider $\tilde{K}_{z,\epsilon,\alpha}(|x|)$, and in the notations of Lemma 3.8.2, $\tilde{K}_{z,\epsilon,\alpha}(|x|) = \tilde{L}_{z,\epsilon,\alpha}(s)$. By Proposition 3.8.19, if z large enough such that $|z\epsilon^2\alpha^2 s^2| \ge 5$, we have $|\tilde{L}_{z,\epsilon,\alpha}(s)| \le C'|e^{\sqrt{z\epsilon^2\alpha^2}s}|$ for some constant C'. But $e^{\sqrt{z\epsilon^2\alpha^2}s}$ is small compared to e^{tz} on $\gamma_{1,2}$, thus
$\frac{1}{2\pi i}\int_{\gamma_{1,2}} dz\ e^{tz}\tilde{K}_{z,\epsilon,\alpha}(|x|) \to 0$ as $L \to \infty$.

Step 2: Deform into γ, i.e.

$$\frac{1}{2\pi i}\int_{\gamma_1} dz\ e^{tz} w^{z,\epsilon,\alpha}(x) = \frac{1}{2\pi i}\int_{\gamma} dz\ e^{tz} w^{z,\epsilon,\alpha}(x). \tag{3.12.34}$$

To deform γ_1 into γ, first, we take a rectangle to the left of z_0 in the right half plane, whose horizontal sides lie on γ_1, and by Cauchy's Theorem, the contour integral of $w^{z,\epsilon,\alpha}(x)$ on this rectangle is 0. More precisely, by Cauchy's Theorem,

$$\frac{1}{2\pi i}\int_{\cup_{i=1}^4 \gamma_{2,i}} dz\ e^{tz} w^{z,\epsilon,\alpha}(x) = 0,$$

where $\gamma_{2,1} = [z_0 - \delta + iy \mid |y| \le \delta], \gamma_{2,2} = [x + i\delta \ |\delta \le x \le z_0 - \delta], \gamma_{2,3} = [\delta + iy \mid |y| \le \delta], \gamma_{2,4} = [x - i\delta \mid \delta \le x \le z_0 - \delta]$. Now, Step 2 follows if we also deform the resulting box around z_0 into a circle to get γ. □

3.13 Proof of Theorem 3.2.1

Let us restate the Theorem: We want to show for $\beta_\epsilon = \frac{4\pi}{\log \epsilon^{-2}} + \frac{\sigma}{(\log \epsilon^{-2})^2}$, and $\alpha = \sqrt{2\log(\log \epsilon^{-1})^{1+2\delta}}$,

$$\lim_{\epsilon\to 0} \frac{1}{\log \epsilon^{-1}} \int dx\ E_x\left[\delta_0(B_t) e^{\beta_\epsilon \int_0^t ds\ \epsilon^{-2} p_{1,\alpha}(\epsilon^{-1}B_s)}\right] = \int dx\ \frac{1}{2\pi i}\int_\gamma dz\ e^{tz}\frac{1}{2\pi}\frac{K_0(\sqrt{z}|x|)}{g(z)}. \tag{3.13.35}$$

Proof. By the comments in the beginning of Section 3.12, and Proposition 3.12.1, we have

$$E_x\left[\delta_0(B_t) e^{\beta_\epsilon \int_0^t ds\ \epsilon^{-2} p_{1,\alpha}(\epsilon^{-1}B_s)}\right] = \frac{1}{2\pi i}\int_\gamma dz\ e^{tz} w^{z,\epsilon,\alpha}(x).$$

where $w^{z,\epsilon,\alpha}(x)$ has two pieces, depending on $|x| \le \epsilon\alpha$ or $|x| \ge \epsilon\alpha$, and γ is a contour encircling the pole of $w^{z,\epsilon,\alpha}(x)$ and the negative real line. We will show (3.13.35) in two steps.

Step 1: We show (3.13.36)

$$\lim_{\epsilon\to 0} \frac{1}{\log \epsilon^{-1}} \int dx\ \frac{1}{2\pi i}\int_\gamma dz\ e^{tz} w_-^{z,\epsilon,\alpha}(x) \mathbb{1}_{|x|\le\epsilon\alpha} = 0. \tag{3.13.36}$$

Since $\int dx\ \mathbb{1}_{|x|\le\epsilon\alpha} = \pi(\epsilon\alpha)^2 \to 0$, we expect the integral to be small. The argument is almost the same as for Proposition 3.12.1, and we use both small and large z estimates in Section 3.8 to bound each term in $w_-^{z,\epsilon,\alpha}(x)$, which we again write as $w_-^{z,\epsilon,\alpha}(x) = A + B + C$ (see Step 1 of Proposition 3.12.1).

For A: Let $A_{\epsilon,\alpha} = [z \mid |z\epsilon^2\alpha^2| \le 0.5]$ and $B_{\epsilon,\alpha} = [z \mid |z\epsilon^2\alpha^2(\beta_\epsilon^{-1}\alpha^4)| \le 1]$. Using the small value asymptotic Proposition 3.9.1, and large value asymptotic Proposition 3.11.1 for the difference in the denominator of A, we have

$$K_0(\sqrt{z\epsilon^2\alpha^2})\epsilon\alpha\tilde{I}_{z,\epsilon,\alpha}(\epsilon\alpha) - \sqrt{z\epsilon^2\alpha^2}K_0'(\sqrt{z\epsilon^2\alpha^2})\tilde{I}_{z,\epsilon,\alpha}(\epsilon\alpha)$$
$$= \frac{1}{\log \epsilon^{-1}}(g_\alpha(z) + O(1/(\log \epsilon^{-1})))\mathbb{1}_{A_{\epsilon,\alpha}\cap B_{\epsilon,\alpha}} + (1 + O(\sqrt{z\epsilon^2\alpha^2}\log(\sqrt{z\epsilon^2\alpha^2})))\mathbb{1}_{A_{\epsilon,\alpha}\cap B^c_{\epsilon,\alpha}} + (1 + c_\epsilon)\mathbb{1}_{A^c_{\epsilon,\alpha}}$$

Furthermore, the integral for A in terms of these cases becomes

$$\begin{aligned}
&\frac{1}{\log\epsilon^{-1}}\int dx\ \frac{1}{2\pi i}\int_\gamma dz\ e^{tz}A\mathbb{1}_{|x|\le\epsilon\alpha}\\
&\quad=\int dx\ \frac{1}{2\pi i}\int_\gamma dz\ \frac{e^{tz}K_0(\sqrt{z\epsilon^2\alpha^2})\mathbb{1}_{|x|\le\epsilon\alpha}\mathbb{1}_{A_{\epsilon,\alpha}\cap B_{\epsilon,\alpha}}}{g_\alpha(z)+O(1/(\log\epsilon^{-1}))}\frac{\tilde I_{z,\epsilon,\alpha}(|x|)}{\tilde I_{z,\epsilon,\alpha}(\epsilon\alpha)}\\
&\quad+\int dx\ \frac{1}{\log\epsilon^{-1}}\frac{1}{2\pi i}\int_\gamma dz\ \frac{e^{tz}K_0(\sqrt{z\epsilon^2\alpha^2})\mathbb{1}_{|x|\le\epsilon\alpha}\mathbb{1}_{A_{\epsilon,\alpha}\cap B^c_{\epsilon,\alpha}}}{1+O(\sqrt{z\epsilon^2\alpha^2}\log(\sqrt{z\epsilon^2\alpha^2}))}\frac{\tilde I_{z,\epsilon,\alpha}(|x|)}{\tilde I_{z,\epsilon,\alpha}(\epsilon\alpha)}\\
&\quad+\int dx\ \frac{1}{\log\epsilon^{-1}}\frac{1}{2\pi i}\int_\gamma dz\ \frac{e^{tz}K_0(\sqrt{z\epsilon^2\alpha^2})\mathbb{1}_{|x|\le\epsilon\alpha}\mathbb{1}_{A^c_{\epsilon,\alpha}}}{1+c_\epsilon}\frac{\tilde I_{z,\epsilon,\alpha}(|x|)}{\tilde I_{z,\epsilon,\alpha}(\epsilon\alpha)}=:A_1+A_2+A_3,
\end{aligned}$$

and let us show each case is small. For convenience, let us also write down the definition of the contour $\gamma=(L_{\delta,-}\cup K_\delta\cup L_{\delta,+})\cup C_{z_0,\delta}$, where $L_{\delta,\pm}=[-r\pm i\delta\mid r>-\delta], K_\delta=[\delta+iy\mid |y|\le\delta], C_{z_0,\delta}=[z_0+\delta e^{i\theta}\mid 0\le\theta\le 2\pi]$. Below, we only provide the estimates on $L_{\delta,+}$, and work for the other pieces of the contour is similar.

For A1: For $w_-^{z,\epsilon,\alpha}(x)$, x is such that $|x|\le\epsilon\alpha$, so $s:=\frac{|x|}{\epsilon\alpha}\le 1$, and in the notations of Lemma 3.8.1, $\frac{\tilde I_{z,\epsilon,\alpha}(|x|)}{\tilde I_{z,\epsilon,\alpha}(\epsilon\alpha)}=\frac{\tilde J_{z,\epsilon,\alpha}(s)}{\tilde J_{z,\epsilon,\alpha}(1)}$. Since we are on $A_{\epsilon,\alpha}=[z\mid |z\epsilon^2\alpha^2|\le 0.5]$, by Remark 3.8.6, $|\tilde J_{z,\epsilon,\alpha}(s)|\le C'$ for some constant C', and by Remark 3.8.16, $\tilde J_{z,\epsilon,\alpha}(1)=I_0(\sqrt{z\epsilon^2\alpha^2})+O(\beta_\epsilon\alpha^2)=J_0(i\sqrt{z\epsilon^2\alpha^2})+O(\beta_\epsilon\alpha^2)$. And by the argument in Lemma 3.11.2, $J_0(i\sqrt{z\epsilon^2\alpha^2})$ is bounded away from zero for $|z\epsilon^2\alpha^2|\le 5$. Then on $L_{\delta,+}$,

$$\begin{aligned}
|A_1|&\le C'(\epsilon\alpha)^2\int_{-\delta}^\infty dr\left|\frac{e^{tz}K_0(\sqrt{z\epsilon^2\alpha^2})}{g_\alpha(z)+O(1/\log\epsilon^{-1})}\mathbb{1}_{|z\epsilon^2\alpha^2|\le 0.5}\mathbb{1}_{z=-r+i\delta}\right|\\
&=C'\int_{-\delta\epsilon^2\alpha^2}^\infty dr\ \left|\frac{e^{-t(r\epsilon^{-2}\alpha^{-2}-i\delta)}K_0(\sqrt{-r+i\delta\epsilon^2\alpha^2})}{g_\alpha(-r\epsilon^{-2}\alpha^{-2}+i\delta)+O(1/(\log\epsilon^{-1}))}\mathbb{1}_{r\le\sqrt{0.5^2-(\delta\epsilon^2\alpha^2)^2}}\right|.
\end{aligned}$$

It is clear it is small for $\epsilon>0$ small.

For A2: The estimate is similar to A_1.

For A3: On $[0.5\le|z\epsilon^2\alpha^2|\le 5]$, the estimate is similar to A_1; on $[|z\epsilon^2\alpha^2|\ge 5]$, to bound $\frac{\tilde I_{z,\epsilon,\alpha}(|x|)}{\tilde I_{z,\epsilon,\alpha}(\epsilon\alpha)}$, we use Proposition 3.8.17 and the large value asymptotic of $I_0(z)$ to get $\frac{\tilde J_{z,\epsilon,\alpha}(s)}{\tilde J_{z,\epsilon,\alpha}(1)}=C'\frac{\sqrt{\sqrt{z\epsilon^2\alpha^2}}}{\sqrt{\sqrt{z\epsilon^2\alpha^2}s}}e^{-\sqrt{z\epsilon^2\alpha^2}(1-s)}$ for some constant C'. Also, on $L_{\delta,+}$, note $\sqrt{z\epsilon^2\alpha^2}=|z\epsilon^2\alpha^2|^{1/2}e^{i(\frac{\pi}{2}-\frac{1}{2}\arctan(\frac{\delta}{r}))}$, and using the large value asymptotic of $K_0(z)$, we get

$$\begin{aligned}
|A_3|&\le C'(\epsilon\alpha)^2\int_{-\delta}^\infty dr\ \left|e^{tz}\frac{1}{\sqrt{\sqrt{z\epsilon^2\alpha^2}}}e^{-\sqrt{z\epsilon^2\alpha^2}}\sqrt{\sqrt{z\epsilon^2\alpha^2}}\mathbb{1}_{|z\epsilon^2\alpha^2|\ge 5}\mathbb{1}_{z=-r+i\delta}\right|\\
&\le C'(\epsilon\alpha)^2\int_{-\delta}^\infty dr\ e^{-tr},
\end{aligned}$$

which is small.

For B, C: It is enough to consider $\tilde K_{z,\epsilon,\alpha}(|x|)$. In the notations of Lemma 3.8.2, $\tilde K_{z,\epsilon,\alpha}(|x|)=$

$\tilde{L}_{z,\epsilon,\alpha}(s)$. By Remark 3.8.6, Propositions 3.8.13 and 3.8.19, we have

$$\begin{aligned}|\tilde{L}_{z,\epsilon,\alpha}(s)| &= |\tilde{L}_{z,\epsilon,\alpha}(s)| \left(\mathbb{1}_{|z\epsilon^2\alpha^2|\le 0.5} + \mathbb{1}_{0.5\le|z\epsilon^2\alpha^2|\le 5} + \mathbb{1}_{|z\epsilon^2\alpha^2 s^2|\le 5}\mathbb{1}_{|z\epsilon^2\alpha^2|\ge 5} + \mathbb{1}_{|z\epsilon^2\alpha^2 s^2|\ge 5}\mathbb{1}_{|z\epsilon^2\alpha^2|\ge 5}\right) \\ &\le C'|\log(\sqrt{1+z\epsilon^2\alpha^2}s)|\mathbb{1}_{|z\epsilon^2\alpha^2|\le 0.5} + C'''|e^{\sqrt{z\epsilon^2\alpha^2}s}|\mathbb{1}_{|z\epsilon^2\alpha^2|\ge 5}\mathbb{1}_{|z\epsilon^2\alpha^2 s^2|\ge 5} \\ &\quad + C''|\log(\sqrt{z\epsilon^2\alpha^2}s)|\left(\mathbb{1}_{0.5\le|z\epsilon^2\alpha^2|\le 5} + \mathbb{1}_{|z\epsilon^2\alpha^2 s^2|\le 5}\mathbb{1}_{|z\epsilon^2\alpha^2|\ge 5}\right),\end{aligned}$$

for some constants C'', C'', C'''. On $L_{\delta,+}$, and $|z\epsilon^2\alpha^2 s^2| \ge 5$,

$$\begin{aligned}\left| \int dx \int_{-\delta}^{\infty} dr e^{tz}\tilde{K}_{z,\epsilon,\alpha}(|x|)\mathbb{1}_{|x|\le\epsilon\alpha}\mathbb{1}_{|z\epsilon^2\alpha^2 s^2|\ge 5}\mathbb{1}_{z=-r+i\delta}\right| &\le C'(\epsilon\alpha)^2 \int_{-\delta}^{\infty} dr \left| e^{-tr} e^{((r\epsilon^2\alpha^2)^2+(\delta\epsilon^2\alpha^2)^2)^{1/4}}\right| \\ &\le C' \int_{-\delta\epsilon^2\alpha^2}^{\infty} dr\ e^{-tr\epsilon^{-2}\alpha^{-2}} e^{(r^2+1)^{1/4}}.\end{aligned}$$

It is clear it is small. Finally, estimates for the other cases such as $\log(\sqrt{z\epsilon^2\alpha^2}s) = \log(\sqrt{z\epsilon^2\alpha^2}) + \log s$, and $\log(\sqrt{1+z\epsilon^2\alpha^2}s) = \log(\sqrt{1+z\epsilon^2\alpha^2}) + \log s$ are similar to A_1; recall the definition $s = \frac{|x|}{\epsilon\alpha}$.

Step 2: We show (3.13.37)

$$\lim_{\epsilon\to 0}\frac{1}{\log\epsilon^{-1}}\int dx\ \frac{1}{2\pi i}\int_{\gamma} dz\ e^{tz} w_{+}^{z,\epsilon,\alpha}(x)\mathbb{1}_{|x|\ge\epsilon\alpha} = \int dx\ \frac{1}{2\pi i}\int_{\gamma} dz\ e^{tz}\frac{1}{2\pi}\frac{K_0(\sqrt{z}|x|)}{g(z)}. \tag{3.13.37}$$

Recall $\gamma = (L_{\delta,-}\cup K_\delta\cup L_{\delta,+})\cup C_{z_0,\delta}$. As is done in Step 1, breaking up into the small and large z estimates for the difference in the denominator of $w_{+}^{z,\epsilon,\alpha}(x)$, we can rewrite

$$\begin{aligned}&\frac{1}{\log\epsilon^{-1}}\int dx\ \frac{1}{2\pi i}\int_{\gamma} dz\ e^{tz} w_{+}^{z,\epsilon,\alpha}(x)\mathbb{1}_{|x|\ge\epsilon\alpha} \\ &\quad = \int dx\ \frac{1}{2\pi i}\int_{\gamma} dz\ \frac{1}{2\pi}\frac{e^{tz}K_0(\sqrt{z}|x|)\mathbb{1}_{|x|\ge\epsilon\alpha}\mathbb{1}_{A_{\epsilon,\alpha}\cap B_{\epsilon,\alpha}}}{g_\alpha(z)+O(1/(\log\epsilon^{-1}))} \\ &\qquad + \int dx\ \frac{1}{\log\epsilon^{-1}}\frac{1}{2\pi i}\int_{\gamma} dz\ \frac{1}{2\pi}\frac{e^{tz}K_0(\sqrt{z}|x|)\mathbb{1}_{|x|\ge\epsilon\alpha}\mathbb{1}_{A_{\epsilon,\alpha}\cap B^c_{\epsilon,\alpha}}}{1+O(\sqrt{z\epsilon^2\alpha^2}\log(\sqrt{z\epsilon^2\alpha^2}))} \\ &\qquad + \int dx\ \frac{1}{\log\epsilon^{-1}}\frac{1}{2\pi i}\int_{\gamma} dz\ \frac{1}{2\pi}\frac{e^{tz}K_0(\sqrt{z}|x|)\mathbb{1}_{|x|\ge\epsilon\alpha}\mathbb{1}_{A^c_{\epsilon,\alpha}}}{1+c_\epsilon} =: A+B+C.\end{aligned}$$

Also, as in Step 1, we only provide the estimates for $L_{\delta,+}$.

On $L_{\delta,+}$, write $\sqrt{z} = |-r+i\delta|^{1/2}e^{i\left(\frac{\pi}{2}-\frac{1}{2}\arctan(\frac{\delta}{r})\right)} =: r_{\delta,+}^{1/2}e^{i\theta_{\delta,r,+}}$. Note $0 < \theta_{\delta,r,+} < \pi/2$, so $\cos\theta_{\delta,r,+} > 0$. The proof of (3.13.37) is based on Dominated Convergence, given we show $e^{tz}K_0(\sqrt{z}|x|)$ is integrable over $z \in L_{\delta,+}$ and over $x \in \mathbb{R}^2$ (see Claim 3.13.1), whose proof we defer to the end of this Section.

For C: We are on $A^c_{\epsilon,\alpha} = [z \mid |z| \ge 0.5\epsilon^{-2}\alpha^{-2}]$, so $\mathbb{1}_{A^c_{\epsilon,\alpha}} \to 0$ as $\epsilon \to 0$. By Claim 3.13.1 and Dominated Convergence, $C \to 0$.

For B: We are on $B^c_{\epsilon,\alpha} = [z \mid |z| \ge \epsilon^{-2}\alpha^{-2}\beta_\epsilon\alpha^4]$, and we have same conclusion as C.

For A: Let us show A tends to the desired limit. We break up the event $\mathbb{1}_{A_{\epsilon,\alpha}\cap B_{\epsilon,\alpha}}\mathbb{1}_{|x|\ge\epsilon\alpha}$ in terms of $\mathbb{1}$ and other subevents where the contribution is small. More precisely, $\mathbb{1}_{A_{\epsilon,\alpha}\cap B_{\epsilon,\alpha}}\mathbb{1}_{|x|\ge\epsilon\alpha} = \mathbb{1} - \mathbb{1}_{(A_{\epsilon,\alpha}\cap B_{\epsilon,\alpha})^c} - \mathbb{1}_{A_{\epsilon,\alpha}\cap B_{\epsilon,\alpha}}\mathbb{1}_{|x|\le\epsilon\alpha}$. On each of the subevents $A^c_{\epsilon,\alpha}, B^c_{\epsilon,\alpha}$ etc, $A \to 0$ because $z \to \infty$ and $g_\alpha(z) \approx \log z$ for z large, and it follows by similar argument as for showing $B, C \to 0$. It remains to show as $\epsilon \to 0$,

$$\int dx\ \frac{1}{2\pi}\int_\gamma dz\ \left| e^{tz}\frac{1}{2\pi}K_0(\sqrt{z}|x|)\left[\frac{1}{g_\alpha(z)+O(1/\log\epsilon^{-1})}-\frac{1}{g(z)}\right]\right| \to 0.$$

But $g_\alpha(z)\to g(z)$ as $\epsilon\to 0$ due to $p_{1,\alpha}(x)\to p_1(x)$, it also follows from Claim 3.13.1 and Dominated Convergence. □

Claim 3.13.1. *For* $r_{\delta,+}=|-r+i\delta|$, $\theta_{\delta,r,+}=\frac{\pi}{2}-\frac{1}{2}\arctan(\frac{\delta}{r})$, *where* $\delta>0$ *small, we have*

$$\int_0^\infty ds\ s\int_0^\infty dr e^{-tr}|K_0(r_{\delta,+}^{1/2}e^{i\theta_{\delta,r,+}}s)|<\infty.$$

Proof. We break up the integral into cases.

Case 1: $[s\le 5, r_{\delta,+}^{1/2}\le 5]$.

By the small value asymptotic of $K_0(z)$, $s|K_0(r_{\delta,+}^{1/2}e^{i\theta_{\delta,r,+}}s)|$ is bounded, so the integral is finite.

Case 2: $[s\ge 5, r_{\delta,+}^{1/2}\ge 5]$.

We use the large value asymptotic of $K_0(z)$ to get $|K_0(r_{\delta,+}^{1/2}e^{i\theta_{\delta,r,+}}s)|\le C'\frac{1}{\sqrt{r_{\delta,+}^{1/2}s}}e^{-r_{\delta,+}^{1/2}\cos(\theta_{\delta,r,+})s}$ for some C'. Next, by Fubini, and the formula $\int_0^\infty ds\ s^{1/2}e^{-as}=\frac{\sqrt{\pi}}{2}\frac{1}{a^{3/2}}$, we have

$$\int_0^\infty ds\ s\frac{1}{\sqrt{s}}\int_0^\infty dr e^{-tr}e^{-r_{\delta,+}^{1/2}\cos(\theta_{\delta,r,+})s}=2\pi\int_0^\infty dr e^{-tr}\frac{\sqrt{\pi}}{2}\frac{1}{(r_{\delta,+}^{1/2}\cos\theta_{\delta,r,+})^{3/2}}.$$

Since $e^{-tr/2}\frac{1}{(\cos\theta_{\delta,r,+})^{3/2}}\le C''$ for some C'', and $r_{\delta,+}$ is bounded away from 0, the integral is finite.

Case 3: $[s\le 5, r_{\delta,+}^{1/2}\ge 5]$.

We further break it up into two subcases.

Case 3.1: $[sr_{\delta,+}^{1/2}\le 5]$.

As in Case 1, we note $s|K_0(r_{\delta,+}^{1/2}e^{i\theta_{\delta,r,+}}s)|$ is bounded, and it follows the integral is finite because e^{-tr} is integrable.

Case 3.2: $[sr_{\delta,+}^{1/2}\ge 5]$.

We can use the large value asymptotic for $K_0(r_{\delta,+}^{1/2}e^{i\theta_{\delta,r,+}}s)$ and estimate as in Case 2.

Case 4: $[s\ge 5, r_{\delta,+}^{1/2}\le 5]$.

Case 4.1: $[sr_{\delta,+}^{1/2}\le 5]$.

In this case, we note since $r_{\delta,+}^{1/2}$ is bounded away from 0, and $s\ge 5$, so $|K_0(r_{\delta,+}^{1/2}e^{i\theta_{\delta,r,+}}s)|$ itself is bounded. Also, $\int_0^\infty dr\ e^{-tr}\int_0^\infty ds\ s\cdot\mathbb{1}_{5\le s}\cdot\mathbb{1}_{r_{\delta,+}^{1/2}\le 5}\cdot\mathbb{1}_{sr_{\delta,+}^{1/2}\le 5}\le C'\int_0^{50}dr e^{-tr}[r_{\delta,+}^{-1}-1]$. for some constant C', and it is finite because $r_{\delta,+}$ is bounded away from 0.

Case 4.1: $[sr_{\delta,+}^{1/2}\ge 5]$.

Same as Case 2.

□

3.14 Proof of Theorem 3.0.1

Let us restate the Theorem: We want to show for $\beta_\epsilon=\frac{4\pi}{\log\epsilon^{-2}}+\frac{\sigma}{(\log\epsilon^{-2})^2}$,

$$\lim_{\epsilon\to 0}\frac{1}{\log\epsilon^{-1}}\int dx\ E_x\left[\delta_0(B_t)e^{\beta_\epsilon\int_0^t ds\ \epsilon^{-2}p_1(\epsilon^{-1}B_s)}\right]=\int dx\ \frac{1}{2\pi i}\int_\gamma dz\ e^{tz}\frac{1}{2\pi}\frac{K_0(\sqrt{z}|x|)}{g(z)}. \tag{3.14.38}$$

Proof. We do it by upper and lower bounds and show they are the same. Before we continue, let us decompose $p_1(x)$ in terms of its cutoff at $|x| = \alpha$ and the remainder[21]

$$\begin{aligned} p_1(x) &= \frac{1}{2\pi} e^{-\frac{|x|^2}{2}} \mathbb{1}_{|x|\le\alpha} + \frac{1}{2\pi} e^{-\frac{|x|^2}{2}} \mathbb{1}_{|x|\ge\alpha} \\ &= p_{1,\alpha}(x) + d_\alpha p_{1,\alpha}(x) + b_\alpha q_{1,\alpha}(x), \end{aligned} \tag{3.14.39}$$

where $d_\alpha = -e^{-\frac{\alpha^2}{2}}$, $b_\alpha = -d_\alpha$, and $q_{1,\alpha}(x) = e^{\frac{\alpha^2}{2}} p_1(x)\mathbb{1}_{|x|\ge\alpha}$. Note $\int q_{1,\alpha}(x) = 1$.

Step 1: Find upper bound for (3.14.38).

Let $d\mu(x)$ be a positive measure, and we will do a bit more for other use.

Step 1.1: Use Hölder and Cauchy-Schwarz inequality to separate out the three terms in the decomposition of $p_1(x)$.

Let us show for $P_{0,x}$ a two dimensional Brownian bridge from 0 to x at time t, and p, q such that $\frac{1}{p} + \frac{1}{q} = 1$, where $p > 1$, then

$$\int d\mu(x) E_{0,x}\left[e^{\beta_\epsilon \int_0^t ds\ \epsilon^{-2} p_1(\epsilon^{-1}B_s)}\right] \le \left(\int d\mu(x) E_{0,x}\left[e^{p\beta_\epsilon \int_0^t ds\ \epsilon^{-2} p_{1,\alpha}(\epsilon^{-1}B_s)}\right]\right)^{1/p} \tag{3.14.40}$$

$$\times\left\{\left(\int d\mu(x) E_{0,x}\left[e^{2qb_\alpha\beta_\epsilon \int_0^t ds\ \epsilon^{-2} p_{1,\alpha}(\epsilon^{-1}B_s)}\right]\right)^{1/2}\left(\int d\mu(x) E_{0,x}\left[e^{2qb_\alpha\beta_\epsilon \int_0^t ds\ \epsilon^{-2} q_{1,\alpha}(\epsilon^{-1}B_s)}\right]\right)^{1/2}\right\}^{1/q}.$$

It follows by two iterations of the Hölder inequality using the conjugate exponents p, q, one over the Brownian bridge $P_{0,x}$ and the other over the positive measure $d\mu(x)$, to separate $p_{1,\alpha}(x)$ from $d_\alpha p_{1,\alpha}(x)$ and $b_\alpha q_{1,\alpha}(x)$; and then two iterations of Cauchy-Schwarz to separate $d_\alpha p_{1,\alpha}(x)$ from $b_\alpha q_{1,\alpha}(x)$.

Step 1.2: Choose suitable Hölder exponents p, q to compute limit of the upper bound.

For the upper bound, we choose $d\mu(x) = p_2(0,x)dx$, and $p = 1 + \frac{1}{(\log\epsilon^{-1})^{1+\delta}}$. Then

- $q = p(\log\epsilon^{-1})^{1+\delta}$,
- $2qb_\alpha \approx \frac{1}{(\log\epsilon^{-1})^{\delta}}$,
- $p\beta_\epsilon = \frac{2\pi}{\log\epsilon^{-1}} + \frac{\sigma_\epsilon}{(\log\epsilon^{-1})^2}$, where $\sigma_\epsilon = \frac{\sigma}{4} + \frac{2\pi}{(\log\epsilon^{-1})^{\delta}} + \frac{\sigma/4}{(\log\epsilon^{-1})^{1+\delta}}$.

Now, by Theorem 3.2.1, the first integral in (3.14.40) when divided by $\log\epsilon^{-1}$ converges to the desired limit (note for the chosen p, $(1/\log\epsilon^{-1})^{1-1/p} \to 1$), and it remains to show the other two integrals each converges to 1. With $d\mu(x) = p_2(0,x)dx$, the other two integrals are merely the exponential of the local time with regularizations $p_{1,\alpha}(x), q_{1,\alpha}(x)$, i.e. $E_0\left[e^{2qb_\alpha\beta_\epsilon \int_0^t ds\ \epsilon^{-2} p_{1,\alpha}(\epsilon^{-1}B_s)}\right]$ and $E_0\left[e^{2qb_\alpha\beta_\epsilon \int_0^t ds\ \epsilon^{-2} q_{1,\alpha}(\epsilon^{-1}B_s)}\right]$ respectively. To see each tends to 1, we expand the exponential and show each term in the series is small. For this, it is crucial that we have the prefactor qb_α in the exponential moment, which is small. Let us do it for $q_{1,\alpha}(x)$; the other is similar.

Let $T = \epsilon^{-2}$ (keep in mind t is fixed). By Brownian scaling, and then expand,

$$\begin{aligned} E_0\left[e^{2qb_\alpha\beta_\epsilon \int_0^t ds\ \epsilon^{-2} q_{1,\alpha}(\epsilon^{-1}B_s)}\right] &= E_0\left[e^{2qb_\alpha\beta_T \int_0^{Tt} ds\ q_{1,\alpha}(B_s)}\right] \\ &= 1 + \sum_{k=1}^{\infty}(2qb_\alpha\beta_T)^k \int_{x_1,\ldots,x_k}\int_{s_1<\cdots<s_k<Tt}\prod_{j=1}^{k} q_{1,\alpha}(x_j)p_{2(s_j-s_{j-1})}(x_{j-1},x_j). \end{aligned}$$

[21] It is very important that we also normalize the cutoff function so that it has mean 1, otherwise the scaling would not be critical.

where $\beta_T = \frac{4\pi}{\log T} + \frac{\sigma}{(\log T)^2}, qb_\alpha = \frac{1}{(\log\sqrt{T})^\delta}$. Note it is enough to show for T large and any $l \le k$,

$$\int dx_l \int_{s_{l-1}}^{Tt} ds_l q_{1,\alpha}(x_l) p_{2(s_l - s_{l-1})}(x_{l-1}, x_l) \le 100\log(Tt), \tag{3.14.41}$$

because if we iterate, we get each summand in the series is at most $100^k(\log T + \log t)^k$, and when combined with $(2qb_\alpha\beta_T)^k$ and let $T \to \infty$, the whole series is small.

To show (3.14.41), and as is done in Claim 2.2.4,

$$\begin{aligned}\int dx_l \int_{s_{l-1}}^{Tt} ds_l q_{1,\alpha}(x_l) p_{2(s_l-s_{l-1})}(x_{l-1}, x_l) &= \frac{1}{4\pi}\int dx_l q_{1,\alpha}(x_l + x_{l-1})\Gamma\left(0, \frac{|x_l|^2}{4(Tt - s_{l-1})}\right) \\ &\le \int dx_l q_{1,\alpha}(x_l + x_{l-1})\left[\log(4Tt) + \log\left(\frac{1}{|x_l|^2}\right)\mathbb{1}_{|x_l|^2\le 1}\right] \\ &= \int dx_l q_{1,\alpha}(x_l + x_{l-1})\log(4Tt) \\ &\quad + \int_0^\infty dr_l r_l \int_0^{2\pi} d\theta_l\ q_{1,\alpha}(x_l + x_{l-1})\log\left(\frac{1}{|r_l|^2}\right)\mathbb{1}_{|r_l|^2\le 1} \\ &\le 100\log(Tt)\end{aligned}$$

for T large, and in the last inequality, we use $\int dx\ q_{1,\alpha}(x) = 1$, $q_{1,\alpha}(x) \le 1$ for all x, and on $r_l \le 1$, $r_l \log(1/r_1^2) \le 5$.

Step 2: Find lower bound for (3.14.38).

For the lower bound, we use the same decomposition of $p_1(x)$ earlier, and a reverse Hölder inequality in the form of for $p > 1$,

$$\int |fg| d\nu \ge \left(\int |f|^{1/p} d\nu\right)^p \left(\int |g|^{-1/(p-1)} d\nu\right)^{-(p-1)}.$$

Again, we will do a bit more for other use, and let $d\mu(x)$ be a positive measure.

Step 2.1: Use reverse Hölder inequality to separate out the three terms in the decomposition of $p_1(x)$.

Let us show for $P_{0,x}$ a two dimensional Brownian bridge from 0 to x at time t, and $p > 1$,

$$\int d\mu(x) E_{0,x}\left[e^{\beta_\epsilon \int_0^t ds\ \epsilon^{-2} p_1(\epsilon^{-1}B_s)}\right] \tag{3.14.42}$$

$$\ge \left(\int d\mu(x) E_{0,x}\left[e^{\frac{1}{p}\beta_\epsilon \int_0^t ds\ \epsilon^{-2} p_{1,\alpha}(\epsilon^{-1}B_s)}\right]\right)^p \left(\int d\mu(x) E_{0,x}\left[e^{-\frac{1}{p-1} d_\alpha \beta_\epsilon \int_0^t ds\ \epsilon^{-2} p_{1,\alpha}(\epsilon^{-1}B_s)}\right]\right)^{-(p-1)}.$$

From the decomposition (3.14.39) of $p_1(x)$, $q_{1,\alpha}(x) \ge 0$, so it follows by two iterations of the reverse Hölder inequality, one over the Brownian bridge $P_{0,x}$ and the other over the positive measure $d\mu(x)$, to separate out $p_{1,\alpha}(x)$ from $d_\alpha p_{1,\alpha}(x)$.

Step 2.2: Choose suitable Hölder exponent p to compute limit of the lower bound.

For the lower bound, we choose $d\mu(x) = p_2(0, x)dx$, and $p = \frac{1}{1 - \frac{1}{(\log \epsilon^{-1})^{1+\delta}}} > 1$. Then

- $\frac{1}{p-1} = (\log \epsilon^{-1})^{1+\delta} - 1$,
- $(-\frac{1}{p-1})d_\alpha \approx \frac{1}{(\log \epsilon^{-1})^{\delta}}$,
- $\frac{1}{p}\beta_\epsilon = \frac{2\pi}{\log \epsilon^{-1}} + \frac{\sigma_\epsilon}{(\log \epsilon^{-1})^2}$, where $\sigma_\epsilon \to \frac{\sigma}{4}$.

Again, by Theorem 3.2.1, the first integral in (3.14.42) when divided by $\log \epsilon^{-1}$ converges to the desired limit (note for the chosen p, $(1/\log \epsilon^{-1})^{1-p} \to 1$), and the second integral converges to 1, as in Step 1.2.

Now, we can conclude Theorem 3.0.1. □

Chapter 4

Fluctuation of Free Energy at Critical Scaling

Define $W_t = e^{-\lambda(\beta)t}Z_t$, where $\lambda(\beta) = \frac{\beta^2}{2}p_1(0)$ with $p_1(0) = \frac{1}{2\pi}$. We want to say a few words to motivate the main result in this Chapter.

From Lemma 2.1.1 and Theorem 3.0.1, at the scaling $\beta = \beta_t = \frac{\alpha_t}{\sqrt{\log t}}$, where $\alpha_t^2 = 4\pi + \frac{\sigma}{\log t}$ with $\sigma \in \mathbb{R}$, $\mathbb{E}[W_t^2]$ is of order $\log t$, so the second moment of the normalized partition function W_t is not uniformly bounded in t. This is reminiscient of the characterization of strong disorder for the usual directed random polymer model with β fixed, as is given in [10], where it says we have strong disorder if the normalized partition function is not uniformly integrable or equivalently, it converges to zero almost surely with respect to the random environment measure $\mathbb{Q}$ (see Proposition 3.1 therein). By this reasoning, it suggests at the critical scaling we consider here, we might get something that is non-Gaussian; but does it mean the free energy $\log\left(\frac{W_t}{\sqrt{\log t}}\right)$ might have a limit? Note in the sense of distribution, or even as a quenched limit with respect to the environment, if we heuristically carry over the characterization of strong disorder to our setting of intermediate scaling, then the limit would just be trivial because $W_t \to 0$ almost surely - actually, this is what Zygouras et al have shown in a recent work [7], though only in distribution.

On the other hand, we know $\mathbb{E}[W_t^2]$ is of order $\log t$, so most of the mass is concentrated in a small region at large times. Perhaps to say anything nontrivial about the free energy we need to consider it in a large deviation scale, i.e. $F_t(\lambda) = \mathbb{E}\left[e^{\lambda \log\left(\frac{W_t}{\sqrt{\log t}}\right)}\right]$ where $\lambda > 0$. For each sufficiently large λ, say $\lambda \geq 2$, one expects them to be large, and ideally, one would like to know how big each $F_t(\lambda)$ is and be able to compute them, or at least be able to compare the $F_t(\lambda)$ for different λ. Even heuristically $\log\left(\frac{W_t}{\sqrt{\log t}}\right) \approx -\infty$ for large t, we can still consider the asymptotic "shape" of the points $F_t(\lambda)$ (or large deviation of the free energy), for example, whether or not they can be interpolated by a quadratic polynomial, and if not, then it gives evidence for non-Gaussian behaviour at the critical scaling.

In this Chapter, we are going to give evidence for non-Gaussian fluctuation of the normalizef free energy $\log W_t - \frac{1}{2}\log\log t$ (we will call it the "normalized" free energy) by showing the small exponential moments cannot be interpolated by a quadratic polynomial. Let $\Gamma_t(k) = \log \mathbb{E}[e^{k(\log W_t - \frac{1}{2}\log\log t)}]$, and $\Lambda_t^{(p,q,r)}(k)$ a quadratic polynomial in k interpolated through the points $\Gamma_t(k)$ for $k = p, q, r$.

Theorem 4.0.1. *For the polymer measure μ_t defined in (1.2.4) with $\beta = \frac{\alpha_t}{\sqrt{\log t}}$, where $\alpha_t^2 = 4\pi + \frac{\sigma}{\log t}$,* *and $\sigma \in \mathbb{R}$, then*

$$\limsup_{t\to\infty} |\Lambda_t^{(0,1,2)}(3) - \Gamma_t(3)| > 0.$$

Remark 4.0.2. *It says at the critical scaling*[1]*, if we look at the logarithmic zeroth, first, second, and third exponential moments of the normalized free energy, then they do not lie on a quadratic polynomial asymptotically, which would be the case for Gaussian.*

Remark 4.0.3. *We want to emphasize that we do not claim to have shown the free energy has non-Gaussian fluctuation – whose precise meaning at the critical scaling is also not entirely clear, as we have seen earlier – but to give evidence in support of it. Also, in the Theorem, we consider only the small exponential moments of the normalized free energy, what about the higher moments? It turns out it is more complicated. See Remark 4.3.3.*

Remark 4.0.4. *We cannot show Theorem 4.0.1 directly, and we need to use the Gaussian Correlation Inequality to reduce the problem. The main difficulty is we would need to compute the third moment of the partition function, but we can only compute its second moment; so to show the Theorem, we would have to reduce the problem in terms of only second moments.*

Before we continue, let us make a detour on the history of the Gaussian Correlation Conjecture, as well as explain the reason for working with a Gaussian environment $\xi(s,x)$ with covariance $p_1(x)$ for our polymer model.

4.1 Background - Gaussian Correlation Inequality

Initially, the random environment we had in mind for our polymer model was such that once it got averaged out then the particles interact in the potential $V(x) := \frac{1}{\pi}\mathbb{1}_{|x|\leq 1}$ instead of $p_1(x)$, and as the reader may notice, in this case Theorem 3.0.1 would be much simpler. But as we will see later in this Chapter, specifically the First Step for Theorem 4.0.1 (see Section 4.3.1), it is not clear from what is known that it is true for this $V(x)$, and it has to do with the Gaussian Correlation Conjecture.

Conjecture 4.1.1. *For $A, B \subset \mathbb{R}^n$ symmetric convex sets, and μ_n the standard Gaussian measure on $\mathbb{R}^n$, then*

$$\mu_n(A\cap B) \geq \mu_n(A)\mu_n(B).$$

With $V(x)$, the type of inequality we need is for B^1, B^2 independent Brownian motions at time points $s_1 < \cdots < s_k, t_1 < \cdots < t_l$,

$$E\left[\prod_{j=1}^{k} V(B^1_{s_j} + B^2_{s_j}) \prod_{i=1}^{l} V(B^1_{t_i})\right] \geq E\left[\prod_{j=1}^{k} V(B^1_{s_j} + B^2_{s_j})\right] E\left[\prod_{i=1}^{l} V(B^1_{t_i})\right].$$

And one can see it as a correlation inequality for multidimensional normal random variables in some nontrivial convex sets. On the other hand, an important observation here is instead of $V(x) = \frac{1}{\pi}\mathbb{1}_{|x|\leq 1}$ in the integrand of the inequality, we have $p_1(x)$ the density function of two dimensional standard normal random variable, then each product becomes exponential of some nonnegative quadratic form, and in

[1] Though we have not shown this scaling is critical, we believe it is, because in the recent work by Zygouras et al [7], they also show at $\beta_t = \frac{\alpha_t}{\sqrt{\log t}}$, where $\alpha_t^2 < 4\pi$, $\log W_t$ is Gaussian; in particular, the cumulant function of W_t is a quadratic polynomial. Thus it gives further evidence that the scaling we consider is critical.

terms of level sets, the correlation inequality becomes one for multidimensional normals in ellipsoids. It turns out this is one of the proven cases of the Conjecture, and let us conclude with a few words on some of its history.

By a simple change of variable, one can show that Conjecture 4.1.1 is equivalent to

Conjecture 4.1.2. *For $X = (X_1, \dots, X_n)$ mean zero Gaussian vector, $1 \le k \le n$, then*

$$P\left(\max_{1\le i\le n} |X_i| \le 1\right) \ge P\left(\max_{1\le i\le k} |X_i| \le 1\right) P\left(\max_{k+1\le i\le n} |X_i| \le 1\right).$$

Independently, Khatri [19] and Šidák [25] showed Conjecture 4.1.2 for $k = 1$. Then Pitt [23] showed Conjecture 4.1.1 for dimension $n = 2$. The result we need is the work by Schechtman, Schlumprecht, and Zinn [24] who showed Conjecture 4.1.1 for A, B ellipsoids for any dimension n. Finally, up till now, the closest to a full proof of the Conjecture is by Hargé [16], who showed for A ellipsoid and B any convex symmetric set for any dimension n.

Below, we will show Theorem 4.0.1 in a series of steps.

4.2 Higher Moments Formula

Lemma 4.2.1. *Let Z_t be defined as in (2.0.1) and $\lambda(\beta) = \frac{\beta^2}{2} p_1(0)$ with $p_1(0) = \frac{1}{2\pi}$. For any fixed $\beta > 0, t > 0$, $l \in \mathbb{N}$,*

$$\mathbb{E}[Z_t^l] = E\left[e^{\beta^2 \int_0^t ds \ \sum_{1\le i<j\le l} p_1\left(\frac{B_s^i - B_s^j}{\sqrt{2}}\right)} e^{tl\lambda(\beta)}\right],$$

where B^i are independent two dimensional Brownian motions started at 0.

Proof. Since $\mathbb{E}[Z_t^l] = \mathbb{E}E\left[e^{\beta \int_0^t ds \ \sum_{i=1}^l \xi(s, B_s^i)}\right]$, as in Lemma 2.1.1, expanding and using Wick's formula, we have

$$\mathbb{E}[Z_t^l] = 1 + \sum_{k=1}^{\infty} \frac{\beta^{2k}}{k!} E\left\{\sum_{\sigma_i \in \{1,\dots,l\}} \int_{s_1,\dots,s_k \le t} \frac{1}{2^k} \prod_{j=1}^k p_1\left(\frac{B_{s_j}^{\sigma_{2j-1}} - B_{s_j}^{\sigma_{2j}}}{\sqrt{2}}\right)\right\}.$$

And the Lemma follows directly from the equality

$$\sum_{\sigma_i \in \{1,\dots,l\}} \frac{1}{2^k} \prod_{j=1}^k p_1\left(\frac{B_{s_j}^{\sigma_{2j-1}} - B_{s_j}^{\sigma_{2j}}}{\sqrt{2}}\right) = \prod_{j=1}^k \left[\sum_{1\le u<v\le l} p_1\left(\frac{B_{s_j}^u - B_{s_j}^v}{\sqrt{2}}\right) + \frac{l}{2} p_1(0)\right]. \tag{4.2.1}$$

To see this, note we can group the σ_i's into consecutive pairs $(\sigma_1, \sigma_2), (\sigma_3, \sigma_4)$ etc. For each pair, we consider the possibilities whether or not $\sigma_{2j-1} = \sigma_{2j}$, and the number of j's for which they are the same. If they are the same, then we have contribution of $p_1(0)$; if different, then $p_1\left(\frac{B_{s_j}^u - B_{s_j}^v}{\sqrt{2}}\right)$ where $u \ne v$. Now it is a simple combinatorial exercise to check expansion of the right hand side of (4.2.1) is given by the left hand side.

□

4.3 Equivalent Statement of Theorem 4.0.1

Theorem 4.3.1. *Theorem 4.0.1 is equivalent to for* $\beta = \frac{\alpha_t}{\sqrt{\log t}}$*, where* $\alpha_t^2 = 4\pi + \frac{\sigma}{\log t}$ *with* $\sigma \in \mathbb{R}$*,*

$$\limsup_{t\to\infty} \frac{E\left[e^{\beta^2 \int_0^t ds\ p_1\left(\frac{B_s^1 - B_s^2}{\sqrt{2}}\right) + p_1\left(\frac{B_s^1 - B_s^3}{\sqrt{2}}\right) + p_1\left(\frac{B_s^2 - B_s^3}{\sqrt{2}}\right)}\right]}{\left(E\left[e^{\beta^2 \int_0^t ds\ p_1\left(\frac{B_s^1 - B_s^2}{\sqrt{2}}\right)}\right]\right)^3} > 1.$$

Remark 4.3.2. *Here, at the critical scaling, we can see non-Gaussian fluctuation of the free energy - at least for small exponential moments - as particles with delta function interaction being correlated, because by Brownian scaling and setting* $t = \epsilon^{-2}$ *(see Lemma 4.5.3), the integrand scales to a delta function. Also, it is not surprising that we have particles, and no longer polymers, when we take moments of the partition function. By definition, random polymer is a particle evolving in some random environment, and by taking moments of the partition function, we get independent copies of the particle evolving in the same random environment; once we average out the environment, we are left with interaction between the particles.*

Proof. Recall $W_t = e^{-\lambda(\beta)t} Z_t$. It is a direct computation using definition of a quadratic polynomial interpolated through $\Gamma_t(k)$ for $k = 0, 1, 2$, which is $\Lambda_t^{(0,1,2)}(k) = ak^2 + bk + c$, where

- $c = \Gamma_t(0) = 0$,
- $b = \frac{1}{2}(4(\Gamma_t(1) - \Gamma_t(0)) - (\Gamma_t(2) - \Gamma_t(0))) = -\frac{1}{2}\log \mathbb{E}[W_t^2] - \frac{1}{2}\log\log t$,
- $a = (\Gamma_t(1) - \Gamma_t(0)) - b = \frac{1}{2}\log \mathbb{E}[W_t^2]$.

Then

$$\begin{aligned}
|\Lambda_t^{(0,1,2)}(3) - \Gamma_t(3)| &= \left|\frac{9}{2}\log \mathbb{E}[W_t^2] - \frac{3}{2}\log \mathbb{E}[W_t^2] - \frac{3}{2}\log\log t - \log \mathbb{E}\left[e^{3\log W_t} e^{-\frac{3}{2}\log\log t}\right]\right| \\
&= \left|\log\left\{\frac{e^{-3t\lambda(\beta)}\mathbb{E}[Z_t^3]}{\left[e^{-2t\lambda(\beta)}\mathbb{E}[Z_t^2]\right]^3}\right\}\right| \\
&= \left|\log\left\{\frac{E\left[e^{\beta^2 \int_0^t ds\ \sum_{1\le i<j\le 3} p_1\left(\frac{B_s^i - B_s^j}{\sqrt{2}}\right)}\right]}{\left[E\left[e^{\beta^2 \int_0^t ds\ p_1\left(\frac{B_s^1 - B_s^2}{\sqrt{2}}\right)}\right]\right]^3}\right\}\right|,
\end{aligned}$$

where in the third equality we use Lemma 4.2.1. ∎

Remark 4.3.3. *In relation to Theorem 4.0.1, and to show the higher moments analogue, it is not enough to know the particles are correlated, but also how they are correlated and how much. For example, consider* $\Lambda_t^{(1,2,3)}(k)$*, the polynomial interpolated through* $\Gamma_t(k)$ *for* $k = 1, 2, 3$*, and suppose we want to show for* $\beta = \frac{\alpha_t}{\sqrt{\log t}}$*, where* $\alpha_t^2 = 4\pi + \frac{\sigma}{\log t}$*,*

$$\limsup_{t\to\infty} \left|\Lambda_t^{(1,2,3)}(4) - \Gamma_t(4)\right| > 0.$$

If we define $L_t(B,k) = E\left[e^{\beta^2\int_0^t ds\ \sum_{1\le i<j\le k} p_1\left(\frac{B_s^i - B_s^j}{\sqrt{2}}\right)}\right]$, *then the equivalent statement is*

$$\limsup_{t\to\infty} \frac{E\left[e^{L_t(B,4)}\right](E\left[e^{L_t(B,2)}\right])^3}{(E\left[e^{L_t(B,3)}\right])^3} = \frac{\left(\frac{E\left[e^{L_t(B,4)}\right]}{(E\left[e^{L_t(B,3)}\right])^2}\right)^2}{\frac{E\left[e^{L_t(B,4)}\right]}{E\left[e^{L_t(B,3)}\right](E\left[e^{L_t(B,2)}\right])^3}} > 1.$$

Here, we have 4 *particles, and in the ratio, we are comparing two of the three ways we can break up the particles into pairs. The numerator is when we break up the* 6 *pairs into 2 groups of* 3, *and the denominator is when we break them up into 1 group of* 3 *and* 3 *groups of* 2. *Without the square in the numerator, it is clear that the denominator is at least as big as the numerator because the more we break up the particles into smaller pairs, the more independent they are (by definition). Question is if we compensate the numerator by a square, then the top should be strictly bigger than the bottom.*

Remark 4.3.4. *On the other hand, if one is interested in the particles being "truly" correlated, it is not hard to extend and show for any* k, *and* $\beta = \frac{\alpha_t}{\sqrt{\log t}}$, *where* $\alpha_t^2 = 4\pi + \frac{\sigma}{\log t}$, *that*

$$\limsup_{t\to\infty} \frac{E\left[e^{\beta^2\int_0^t ds\ \sum_{1\le i<j\le k} p_1\left(\frac{B_s^i - B_s^j}{\sqrt{2}}\right)}\right]}{\left(E\left[e^{\beta^2\int_0^t ds\ p_1\left(\frac{B_s^1 - B_s^2}{\sqrt{2}}\right)}\right]\right)^{\binom{k}{2}}} > 1.$$

4.4 First Step for Theorem 4.3.1

There are two main steps to showing Theorem 4.3.1, and the first is the following.

Proposition 4.4.1. *For any* $\beta > 0, t > 0$,

$$\frac{E\left[e^{\beta\int_0^t ds\ p_1\left(\frac{B_s^1 - B_s^2}{\sqrt{2}}\right) + p_1\left(\frac{B_s^1 - B_s^3}{\sqrt{2}}\right) + p_1\left(\frac{B_s^2 - B_s^3}{\sqrt{2}}\right)}\right]}{E\left[e^{\beta\int_0^t ds\ p_1\left(\frac{B_s^1 - B_s^2}{\sqrt{2}}\right) + p_1\left(\frac{B_s^1 - B_s^3}{\sqrt{2}}\right)}\right] E\left[e^{\beta\int_0^t ds\ p_1\left(\frac{B_s^2 - B_s^3}{\sqrt{2}}\right)}\right]} \ge 1.$$

Remark 4.4.2. *This is a crucial step, and it is where we use the Gaussian Correlation Inequality to reduce the problem. With* $p_1(x)$, *the inequality becomes one of the proven cases of the Conjecture. As mentioned earlier, the main reason we do this and not show Theorem 4.3.1 directly is we do not know how to compute the third moment of the partition function. First, let us rewrite the exponential functional.*

Claim 4.4.3. *For* F *a nonnegative functional of Brownian motion, and* B^i *independent Brownian motions, then*

$$E\left[F\left(\frac{B^1 - B^2}{\sqrt{2}}\right)F\left(\frac{B^1 - B^3}{\sqrt{2}}\right)F\left(\frac{B^2 - B^3}{\sqrt{2}}\right)\right] = E\left[F(Z^1)F\left(\frac{1}{2}Z^1 + \frac{\sqrt{3}}{2}Z^2\right)F\left(\frac{1}{2}Z^1 - \frac{\sqrt{3}}{2}Z^2\right)\right],$$

where the Z^i *are also independent Brownian motions.*

Proof. Let $Z^1 = \frac{1}{\sqrt{2}}B^1 - \frac{1}{\sqrt{2}}B^2$, and $Z^2 = \frac{2}{\sqrt{3}}\left(\frac{1}{2\sqrt{2}}B^1 - \frac{1}{\sqrt{2}}B^3 + \frac{1}{2\sqrt{2}}B^2\right)$, then we have

$$\frac{1}{2}Z^1 + \frac{\sqrt{3}}{2}Z^2 = \frac{1}{\sqrt{2}}B^1 - \frac{1}{\sqrt{2}}B^3, \qquad -\left(\frac{1}{2}Z^1 - \frac{\sqrt{3}}{2}Z^2\right) = \frac{1}{\sqrt{2}}B^2 - \frac{1}{\sqrt{2}}B^3.$$

Also, $E[Z^1Z^2] = \frac{2}{\sqrt{3}}(\frac{1}{4} - \frac{1}{4}) = 0$, i.e. Z^1, Z^2 are independent Brownian motions.

□

Proof. To show Proposition 4.4.1, we need to consider the finite dimensional distributions of the Brownian motions, and with $p_1(x)$, identify the nonnegative quadratic forms so that we have a correlation inequality for multidimensional normal random variables in ellipsoids. First, using Claim 4.4.3 for $F(B) = e^{\beta\int_0^t ds\ p_1(B_s)}$, the inequality is inequivalent to

$$\frac{E\left[e^{\beta\int_0^t ds\ p_1\left(\lambda_1 B_s^1+\lambda_2 B_s^2\right)+p_1\left(\lambda_1 B_s^1-\lambda_2 B_s^2\right)}e^{\beta\int_0^t ds\ p_1\left(B_s^1\right)}\right]}{E\left[e^{\beta\int_0^t ds\ p_1(\lambda_1 B_s^1+\lambda_2 B_s^2)+p_1(\lambda_1 B_s^1-\lambda_2 B_s^2)}\right]E\left[e^{\beta\int_0^t ds\ p_1(B_s^1)}\right]} \geq 1, \tag{4.4.2}$$

with $\lambda_1 = \frac{1}{2}, \lambda_2 = \frac{\sqrt{3}}{2}$. As we will see in the proof, without loss of generality we may suppose $t = 1, \beta = 1$. Let us break down the proof into four steps.

Step 1: Reduce inequality (4.4.2) in terms of the finite dimensional distributions of the Brownian motions.

Expanding the first exponential, and using the definition $p_1(x) = \frac{1}{2\pi}e^{-\frac{|x|^2}{2}}$, we have

$$\begin{aligned}
E[e^{\int_0^1 ds\ p_1(\lambda_1 B_s^1+\lambda_2 B_s^2)+p_1(\lambda_1 B_s^1-\lambda_2 B_s^2)}] &= 1 + \sum_{k=1}^{\infty}\frac{1}{k!}E\left[\int_0^1 ds\ p_1(\lambda_1 B_s^1 + \lambda_2 B_s^2) + p_1(\lambda_1 B_s^1 - \lambda_2 B_s^2)\right]^k \\
&= 1 + \sum_{k=1}^{\infty}\sum_{\sigma_l=\pm1}\int_{s_1<\cdots<s_k} E\left[\prod_{l=1}^k p_1(\lambda_1 B_{s_l}^1 - \lambda_2\sigma_l B_{s_l}^2)\right] \\
&= 1 + \sum_{k=1}^{\infty}\sum_{\sigma_l=\pm1}\int_{s_1<\cdots<s_k} E\left[\prod_{l=1}^k p_1(-\lambda_1 B_{s_l}^1 + \sigma_l\lambda_2 B_{s_l}^2)\right] \\
&= 1 + \sum_{k=1}^{\infty}\sum_{\sigma_l=\pm1}\int_{s_1<\cdots<s_k} \frac{1}{(2\pi)^k}E\left[e^{-\frac{1}{2}\sum_{l=1}^k|\sigma_l\lambda_2 B_{s_l}^2-\lambda_1 B_{s_l}^1|^2}\right].
\end{aligned}$$

Similarly expanding the second exponential, we have

$$E\left[e^{\int_0^1 ds\ p_1(B_s^1)}\right] = 1 + \sum_{n=1}^{\infty}\int_{t_1<\cdots<t_n}\frac{1}{(2\pi)^n}E\left[e^{-\frac{1}{2}\sum_{i=1}^n|B_{t_i}^1|^2}\right].$$

So, given $s_1 < \cdots < s_k, t_1 < \cdots < t_n$, for inequality (4.4.2), it is enough to show

$$E[e^{-\frac{1}{2}\sum_{l=1}^k|\sigma_l\lambda_2 B_{s_l}^2-\lambda_1 B_{s_l}^1|^2}e^{-\frac{1}{2}\sum_{i=1}^n|B_{t_i}^1|^2}] \geq E[e^{-\frac{1}{2}\sum_{l=1}^k|\sigma_l\lambda_2 B_{s_l}^2-\lambda_1 B_{s_l}^1|^2}]E[e^{-\frac{1}{2}\sum_{i=1}^n|B_{t_i}^1|^2}]. \tag{4.4.3}$$

Step 2: Rewrite the integrands in (4.4.3) in terms of nonnegative quadratic forms.

Suppose the time points are ordered as $t_1 < \cdots < t_n < s_1 < \cdots < s_k$; the other cases are similar by a suitable choice of symmetric, nonnegative definite matrices as we do below. Writing the two Brownian motions B^1, B^2 in terms of independent increments, we have

$$\sum_{l=1}^{k} |\sigma_l \lambda_2 B^2_{s_l} - \lambda_1 B^1_{s_l}|^2 = \sum_{l=1}^{k} \left| \sum_{i=1}^{l+n} \sigma_l \lambda_2 (B^2_{r_i} - B^2_{r_{i-1}}) - \lambda_1 (B^1_{r_i} - B^1_{r_{i-1}}) \right|^2$$
$$= \sum_{l=1}^{k} \frac{|\sum_{i=1}^{l+n} \sigma_l \lambda_2/\lambda_1 (B^2_{r_i} - B^2_{r_{i-1}}) - (B^1_{r_i} - B^1_{r_{i-1}})|^2}{1/\lambda_1^2},$$

where $r_i = t_i$ for $i \le n$, and $r_i = s_{i-n}$ for $n+1 \le i \le l+n$. Let $W_i = \lambda_2/\lambda_1 \frac{B^2_{r_i} - B^2_{r_{i-1}}}{1/\lambda_1}$, $m_i = \frac{B^1_{r_i} - B^1_{r_{i-1}}}{1/\lambda_1}$ and $Y_{l,i} = \sigma_l W_i$ (for $i \ge n+1$, $m_i = 0$), then the sum is further equal to

$$\sum_{l=1}^{k} |\sigma_l \lambda_2 B^2_{s_l} - \lambda_1 B^1_{s_l}|^2 = \sum_{l=1}^{k} |\sum_{i=1}^{l+n} Y_{l,i} - m_i|^2$$
$$= (Y - m)^T A (Y - m),$$

for some symmetric, nonnegative definite $(2(n+1)+\cdots+2(n+k)) \times (2(n+1)+\cdots+2(n+k))$ matrix A[2] (recall B^i are two dimensional Brownian motions), vectors $Y = \big(Y^1_{1,1}, Y^2_{1,1} \ldots, Y^1_{1,n+1}, Y^2_{1,n+1}, Y^1_{2,1}, Y^2_{2,1}, \ldots, Y^1_{2,n+2}, Y^2_{2,n+2}, \ldots, Y^1_{k,1}, Y^2_{k,1}, \ldots, Y^1_{k,n+k}, Y^2_{k,n+k}\big)$, and $m = \big(m^1_1, m^2_1, \ldots, m^1_{n+1}, m^2_{n+1}, m^1_1, m^2_1, \ldots, m^1_{n+2}, m^2_{n+2}, \ldots, m^1_1, m^2_1, \ldots, m^1_{n+k}, m^2_{n+k}\big)$. Similarly, we also rewrite

$$\sum_{i=1}^{n} |B^1_{t_i}|^2 = \sum_{i=1}^{n} \frac{|\sum_{j=1}^{i} \lambda_1^{-1} (B^1_{r_j} - B^1_{r_{j-1}})|^2}{1/\lambda_1^2}$$
$$= \sum_{i=1}^{n} \left| \sum_{j=1}^{i} \lambda_1^{-1} m_j \right|^2$$
$$= m^T F m,$$

for some symmetric, nonnegative definite matrix F[3], and vector $m = \big(m^1_1, m^2_1, \ldots, m^1_{n+1}, m^2_{n+1}, m^1_1, m^2_1, \ldots, m^1_{n+2}, m^2_{n+2}, \ldots, m^1_1, m^2_1, \ldots, m^1_{n+k}, m^2_{n+k}\big)$.

As we see, inequality (4.4.3) is now equivalent to

$$E\left[e^{-\frac{1}{2}\langle A(Y-m),(Y-m)\rangle} e^{-\frac{1}{2}\langle Fm,m\rangle}\right] \ge E\left[e^{-\frac{1}{2}\langle A(Y-m),(Y-m)\rangle}\right] E\left[e^{-\frac{1}{2}\langle Fm,m\rangle}\right]. \tag{4.4.4}$$

Step 3: Simplify the quadratic forms in (4.4.4) by integrating out Y so that they are only

[2]For example, suppose $k = n = 2, m_i = 0$, and B^1, B^2 are one dimensional Brownian motions, then

$$\sum_{l=1}^{2} \left| \sum_{i=1}^{2+l} Y_{l,i} \right|^2 = \sum_{l=1}^{2} |Y_{l,1} + \cdots + Y_{l,2+l}|^2 = |Y_{1,1} + Y_{1,2} + Y_{1,3}|^2 + |Y_{2,1} + Y_{2,2} + Y_{2,3} + Y_{2,4}|^2.$$

And A is a 7×7 matrix with a 3×3 block of 1's on the top left corner, and a 4×4 block of 1's in the bottom right corner, with 0's in the rest of the matrix.

[3]For example, as before, suppose $k = n = 2$, and B^1, B^2 are one dimensional Brownian motions, then

$$\sum_{i=1}^{2} \left| \sum_{j=1}^{i} \lambda_1^{-1} m_j \right|^2 = |\lambda_1^{-1} m_1|^2 + |\lambda_1^{-1} m_1 + \lambda_1^{-1} m_2|^2 = 2\lambda_1^{-2} m_1^2 + 2\lambda_1^{-2} m_1 m_2 + \lambda_1^{-2} m_2^2.$$

And F is a 7×7 matrix with a 2×2 block submatrix $\begin{bmatrix} 2\lambda_1^{-2} & \lambda_1^{-2} \\ \lambda_1^{-2} & \lambda_1^{-2} \end{bmatrix}$ in the top left corner and 0's everywhere else.

in terms of m[4].

To compute $E\left[e^{-\frac{1}{2}\langle A(Y-m),(Y-m)\rangle}\right]$, write $Y = D^{1/2}Z$ for $Z \sim N(0,I)$ and some symmetric nonnegative definite matrix D. Then completing square,

$$\begin{aligned}
&\left\langle A(D^{1/2}z-m),(D^{1/2}z-m)\right\rangle + \langle z,z\rangle \\
&\quad= \left\langle (D^{1/2}AD^{1/2}+I)z,z\right\rangle + \langle Am,m\rangle - 2\left\langle (D^{1/2}AD^{1/2}+I)^{-1/2}(D^{1/2}AD^{1/2}+I)^{1/2}z, D^{1/2}Am\right\rangle \\
&\quad= \left\langle (D^{1/2}AD^{1/2}+I)^{1/2}z,(D^{1/2}AD^{1/2}+I)^{1/2}z\right\rangle \\
&\qquad+ \left\langle (D^{1/2}AD^{1/2}+I)^{-1/2}D^{1/2}Am,(D^{1/2}AD^{1/2}+I)^{-1/2}D^{1/2}Am\right\rangle \\
&\qquad- 2\left\langle (D^{1/2}AD^{1/2}+I)^{1/2}z,(D^{1/2}AD^{1/2}+I)^{-1/2}D^{1/2}Am\right\rangle \\
&\qquad- \left\langle (D^{1/2}AD^{1/2}+I)^{-1/2}D^{1/2}Am,(D^{1/2}AD^{1/2}+I)^{-1/2}D^{1/2}Am\right\rangle + \langle Am,m\rangle \\
&\quad= \left\langle (BB^T+I)^{1/2}z-(BB^T+I)^{1/2}BA^{1/2}m,(BB^T+I)^{1/2}z-(BB^T+I)^{1/2}BA^{1/2}m\right\rangle \\
&\qquad+ \left\langle \left[A-A^{1/2}B^T(BB^T+I)^{-1}BA^{1/2}\right]m,m\right\rangle,
\end{aligned}$$

where in the last line we let $B = D^{1/2}A^{1/2}$.

Thus, integrating over Z, inequality (4.4.4) becomes

$$\begin{aligned}
&E\left[e^{-\frac{1}{2}\left\langle \left[A-A^{1/2}B^T(BB^T+I)^{-1}BA^{1/2}\right]m,m\right\rangle}e^{-\frac{1}{2}\langle Fm,m\rangle}\right] \\
&\geq E\left[e^{-\frac{1}{2}\left\langle \left[A-A^{1/2}B^T(BB^T+I)^{-1}BA^{1/2}\right]m,m\right\rangle}\right]E\left[e^{-\frac{1}{2}\langle Fm,m\rangle}\right].
\end{aligned} \tag{4.4.5}$$

To conclude Proposition 4.4.1, we need to show the matrix $A - A^{1/2}B^T(BB^T+I)^{-1}BA^{1/2}$ is symmetric and nonnegative definite, in which case in terms of level sets, (4.4.5) is a correlation inequality for multidimensional normal random variables in ellipsoids, and this is one of the proven cases of the Gaussian Correlation Conjecture in [24].

Step 4: Show the quadratic forms in (4.4.5) are nonnegative, or show $A - A^{1/2}B^T(BB^T+I)^{-1}BA^{1/2}$ is a symmetric and nonnegative definite matrix.

Rewriting $A - A^{1/2}B^T(BB^T+I)^{-1}BA^{1/2} = A^{1/2}(I - B^T(BB^T+I)^{-1}B)A^{1/2}$, it follows $A - A^{1/2}B^T(BB^T+I)^{-1}BA^{1/2}$ is symmetric, and it is enough to show $I - B^T(BB^T+I)^{-1}B$ is nonnegative definite. To do this, let $C = (BB^T+I)^{-1/2}B$, and consider its singular value decomposition $C = U\Sigma V^T$ for some orthogonal matrices U, V and Σ is a diagonal matrix of singular values of C. Then $I - B^T(BB^T+I)^{-1}B = I - C^TC = V(I-\Sigma^2)V^T$. So $I - \Sigma^2$ is a diagonal matrix of eigenvalues of $I - B^T(BB^T+I)^{-1}B$ and we are going to show the eigenvalues are nonnegative.

Consider the singular value decomposition of $B = X\Gamma Y^T$ for X, Y orthogonal matrices and Γ its diagonal matrix of singular values. Similarly, $BB^T + I = (X\Gamma Y^T)(Y\Gamma X^T) + I = X(I+\Gamma^2)X^T$. Also, $(BB^T+I)^{-1/2} = X(I+\Gamma^2)^{-1/2}X^T$. Thus $C = (BB^T+I)^{-1/2}B = X(I+\Gamma^2)^{-1/2}X^TX\Gamma Y^T = X[(I+\Gamma^2)^{-1/2}\Gamma]Y^T$, which is another singular value decomposition of C with $(I+\Gamma^2)^{-1/2}\Gamma$ a diagonal matrix of singular values. Since the singular values are uniquely determined by the matrix C, we must have $\Sigma = (I+\Gamma^2)^{-1/2}\Gamma$. Now it follows the eigenvalues of $I - \Sigma^2$ are nonnegative, and $I - B^T(BB^T+I)^{-1}B$ is nonnegative definite. □

[4]Equivalently, we want to first integrate out Brownian motion B^2 to get a correlation inequality in terms of only Brownian motion B^1.

4.5 Second Step for Theorem 4.3.1

The second main step to showing Theorem 4.3.1 is

Proposition 4.5.1. *For* $\beta = \frac{\alpha_t}{\sqrt{\log t}}$, *where* $\alpha_t^2 = 4\pi + \frac{\sigma}{\log t}$,

$$\limsup_{t\to\infty} \frac{E\left[e^{\beta^2 \int_0^t ds\ p_1\left(\frac{B_s^1 - B_s^2}{\sqrt{2}}\right) + p_1\left(\frac{B_s^1 - B_s^3}{\sqrt{2}}\right)}\right]}{E\left[e^{\beta^2 \int_0^t ds\ p_1\left(\frac{B_s^1 - B_s^2}{\sqrt{2}}\right)}\right] E\left[e^{\beta^2 \int_0^t ds\ p_1\left(\frac{B_s^1 - B_s^3}{\sqrt{2}}\right)}\right]} > 1.$$

Remark 4.5.2. *Here we have reduced Theorem 4.3.1 to the point where we can relate it to the second moment of the partition function. To show the strict inequality, the critical scaling matters, and we need to use the exponential moment of the regularized local time. The idea is to find an effective lower bound for the ratio, and it turns out we can do this by comparing two bridges with different starting points under the new measure induced by the exponential moment of the local time. Basically, we want to say at the critical scaling, the bridges "feel" effect of the new measure and they are different. In the following subsections, we are going to find an effective lower bound for the ratio and show it is bounded away from 1.*

4.5.1 Second Step 2.1

We rescale the integrand in the ratio of Proposition 4.5.1 to a delta function.

Lemma 4.5.3. *Let* $t = \epsilon^{-2}$, *and* $\beta_\epsilon = \frac{2\pi}{\log \epsilon^{-1}} + \frac{\sigma/4}{(\log \epsilon^{-1})^2}$. *Then*

$$\frac{E\left[e^{\beta^2 \int_0^t ds\ p_1(\frac{B_s^1 - B_s^2}{\sqrt{2}}) + p_1(\frac{B_s^1 - B_s^3}{\sqrt{2}})}\right]}{E\left[e^{\beta^2 \int_0^t ds\ p_1(\frac{B_s^1 - B_s^2}{\sqrt{2}})}\right] E\left[e^{\beta^2 \int_0^t ds\ p_1(\frac{B_s^1 - B_s^3}{\sqrt{2}})}\right]} = \frac{E\left[e^{\beta_\epsilon \int_0^1 ds\ \epsilon^{-2} p_1(\epsilon^{-1}(\frac{B_s^1 - B_s^2}{\sqrt{2}})) + \epsilon^{-2} p_1(\epsilon^{-1}(\frac{B_s^1 - B_s^3}{\sqrt{2}}))}\right]}{E\left[e^{\beta_\epsilon \int_0^1 ds\ \epsilon^{-2} p_1(\epsilon^{-1}(\frac{B_s^1 - B_s^2}{\sqrt{2}}))}\right] E\left[e^{\beta_\epsilon \int_0^1 ds\ \epsilon^{-2} p_1(\epsilon^{-1}(\frac{B_s^1 - B_s^3}{\sqrt{2}}))}\right]}.$$

Proof. It follows immediately by Brownian scaling, and $\beta^2 = \frac{4\pi}{\log t} + \frac{\sigma}{(\log t)^2} = \frac{2\pi}{\log \epsilon^{-1}} + \frac{\sigma/4}{(\log \epsilon^{-1})^2} = \beta_\epsilon$. ☐

Remark 4.5.4. *Let us see the ratio in Proposition 4.5.1 is indeed related to the second moment of the partition function or the exponential moment of the regularized local time. Being explicit with the Brownian motions we integrate over, we write*

$$E\left[e^{\beta_\epsilon \int_0^1 ds\ \epsilon^{-2} p_1(\epsilon^{-1}(\frac{B_s^1 - B_s^2}{\sqrt{2}})) + \epsilon^{-2} p_1(\epsilon^{-1}(\frac{B_s^1 - B_s^3}{\sqrt{2}}))}\right] = E^{(1)}\left[E^{(2)}\left[e^{\beta_\epsilon \int_0^1 ds\ \epsilon^{-2} p_1(\epsilon^{-1}(\frac{B_s^1 - B_s^2}{\sqrt{2}}))}\right]\right]^2,$$

where $E^{(i)}$ *is expectation with respect to Brownian motion* B^i. *If we let*

$$F_\epsilon(B^1) = \frac{E^{(2)}\left[e^{\beta_\epsilon \int_0^1 ds\ \epsilon^{-2} p_1(\epsilon^{-1}(\frac{B_s^1 - B_s^2}{\sqrt{2}}))}\right]}{E^{(1)}\left[E^{(2)}\left[e^{\beta_\epsilon \int_0^1 ds\ \epsilon^{-2} p_1(\epsilon^{-1}(\frac{B_s^1 - B_s^2}{\sqrt{2}}))}\right]\right]}, \tag{4.5.6}$$

then Proposition 4.5.1 is $\limsup_{\epsilon\to 0} E^{(1)}[F_\epsilon^2(B^1)] > 1$*, so it is enough to show*

$$\limsup_{\epsilon\to 0} E^{(1)}\left[F_\epsilon^2(B^1)-1\right] > 0. \tag{4.5.7}$$

Let us recall again our Brownian motion has generator Δ, in particular it has second moment $E[|B_t|^2] = 2\cdot(2t)$. We will show (4.5.7) in the remaining subsections.

4.5.2 Second Step 2.2

We show the difference in (4.5.7) has a lower bound given by two bridges starting at different points.

Lemma 4.5.5.

$$E^{(1)}\left[F_\epsilon^2(B^1)-1\right] \geq \int dx\ p_2(0,x)\left[E^{(1)}_{0,x}\left[F_\epsilon(B^1)-1\right]\right]^2,$$

where $p_{2t}(x,y)$ *is the transition density function of two dimensional Brownian motion, and* $P_{0,x}$ *is a two dimensional Brownian bridge from* 0 *to* x *at time* 1*. Moreover,*

$$E^{(1)}_{0,x}[F_\epsilon(B^1)] = \frac{\int dy\ p_2(0,y)E^{(1)}_{0,\frac{x-y}{\sqrt{2}}}\left[e^{\beta_\epsilon\int_0^1 ds\ \epsilon^{-2}p_1(\epsilon^{-1}B^1_s)}\right]}{\int dy\ p_2(0,y)E^{(1)}_{0,y}\left[e^{\beta_\epsilon\int_0^1 ds\ \epsilon^{-2}p_1(\epsilon^{-1}B^1_s)}\right]}.$$

Proof. To show the first statement, note $E^{(1)}[F_\epsilon(B^1)]=1$, and by Jensen's inequality,

$$\begin{aligned}
E^{(1)}\left[F_\epsilon^2(B^1)-1\right] &= E^{(1)}\left[F_\epsilon(B^1)-1\right]^2\\
&= \int dx\ p_2(0,x)E^{(1)}_{0,x}\left[F_\epsilon(B^1)-1\right]^2\\
&\geq \int dx\ p_2(0,x)\left[E^{(1)}_{0,x}\left[F_\epsilon(B^1)-1\right]\right]^2.
\end{aligned}$$

To show the second statement, using the fact that given B^1, B^2 are Brownian bridges, $\frac{B^1-B^2}{\sqrt{2}}$ is also a Brownian bridge (with suitable start and end points), and similarly so if B^1, B^2 are Brownian motions, then

$$\begin{aligned}
E^{(1)}_{0,x}[F_\epsilon(B^1)] &= \frac{E^{(1)}_{0,x}E^{(2)}\left[e^{\beta_\epsilon\int_0^1 ds\ \epsilon^{-2}p_1\left(\epsilon^{-1}(\frac{B^1_s-B^2_s}{\sqrt{2}})\right)}\right]}{E^{(1)}\left[E^{(2)}\left[e^{\beta_\epsilon\int_0^1 ds\ \epsilon^{-2}p_1\left(\epsilon^{-1}(\frac{B^1_s-B^2_s}{\sqrt{2}})\right)}\right]\right]}\\
&= \frac{\int dy\ p_2(0,y)E^{(1)}_{0,x}E^{(2)}_{0,y}\left[e^{\beta_\epsilon\int_0^1 ds\ \epsilon^{-2}p_1\left(\epsilon^{-1}(\frac{B^1_s-B^2_s}{\sqrt{2}})\right)}\right]}{E^{(1)}\left[e^{\beta_\epsilon\int_0^1 ds\ \epsilon^{-2}p_1(\epsilon^{-1}B^1_s)}\right]}\\
&= \frac{\int dy\ p_2(0,y)E^{(1)}_{0,\frac{x-y}{\sqrt{2}}}\left[e^{\beta_\epsilon\int_0^1 ds\ \epsilon^{-2}p_1\left(\epsilon^{-1}B^1_s\right)}\right]}{\int dy\ p_2(0,y)E^{(1)}_{0,y}\left[e^{\beta_\epsilon\int_0^1 ds\ \epsilon^{-2}p_1(\epsilon^{-1}B^1_s)}\right]}.
\end{aligned}$$

□

Before we continue, let us relate the Feynman-Kac expectation under Brownian bridge with one under Brownian motion.

Lemma 4.5.6. *For $V(x)$ bounded, and $P_{0,x}$ two dimensional Brownian bridge from 0 to x at time 1, then*

$$E_{0,x}\left[e^{\int_0^1 ds\ V(B_s)}\right] = \frac{1}{p_2(0,x)} E_0\left[\delta_x(B_1) e^{\int_0^1 ds\ V(B_s)}\right] = \frac{1}{p_2(0,x)} E_x\left[\delta_0(B_1) e^{\int_0^1 ds\ V(B_s)}\right].$$

Proof. It follows by expanding the exponential, using definition of the transition density function of Brownian bridge, and suitable change of variables to reverse time. More precisely, for the first equality, let $q(s,u;\ t,v)$ be the transition density function of Brownian bridge from 0 at time 0 to x at time 1, so

$$q(s,u;\ t,v) = \frac{p_{2(t-s)}(u,v) p_{2(1-t)}(v,x)}{p_{2(1-s)}(u,x)}.$$

Expanding the exponential,

$$\begin{aligned}
E_{0,x}\left[e^{\int_0^1 ds\ V(B_s)}\right] &= 1 + \sum_{k=1}^{\infty} \int_{s_1<\cdots<s_k\le 1} \int_{x_1,\ldots,x_k} \prod_{j=1}^{k} V(x_j) \frac{p_{2(s_j-s_{j-1})}(x_{j-1},x_j) p_{2(1-s_j)}(x_j,x)}{p_{2(1-s_{j-1})}(x_{j-1},x)} \\
&= 1 + \sum_{k=1}^{\infty} \int_{s_1<\cdots<s_k\le 1} \int_{x_1,\ldots,x_k} \prod_{j=1}^{k} V(x_j) \\
&\quad \times \frac{p_{2(s_1-s_0)}(x_0,x_1) p_{2(1-s_1)}(x_1,x)}{p_{2(1-s_0)}(x_0,x)} \frac{p_{2(s_2-s_1)}(x_1,x_2) p_{2(1-s_2)}(x_2,x)}{p_{2(1-s_1)}(x_1,x)} \\
&\quad \cdots \frac{p_{2(s_{k-1}-s_{k-2})}(x_{k-2},x_{k-1}) p_{2(1-s_{k-1})}(x_{k-1},x)}{p_{2(1-s_{k-2})}(x_{k-2},x)} \frac{p_{2(s_k-s_{k-1})}(x_{k-1},x_k) p_{2(1-s_k)}(x_k,x)}{p_{2(1-s_{k-1})}(x_{k-1},x)} \\
&= 1 + \sum_{k=1}^{\infty} \int_{s_1<\cdots<s_k\le 1} \int_{x_1,\ldots,x_k} \prod_{j=1}^{k} V(x_j) p_{2(s_j-s_{j-1})}(x_{j-1},x_j) \frac{p_{2(1-s_k)}(x_k,x)}{p_2(0,x)} \\
&= \frac{1}{p_2(0,x)} E_0\left[\delta_x(B_1) e^{\int_0^1 ds\ V(B_s)}\right],
\end{aligned}$$

where $s_0 = 0, x_0 = 0$. For the second equality in the Lemma, again by expansion,

$$\begin{aligned}
E_0\left[\delta_x(B_1) e^{\int_0^1 ds\ V(B_s)}\right] &= 1 + \sum_{k=1}^{\infty} \int_{s_1<\cdots<s_k\le 1} \int_{x_1,\ldots,x_k} p_{2s_1}(0,x_1) V(x_1) \\
&\quad \times \prod_{j=2}^{k} p_{2(s_j-s_{j-1})}(x_{j-1},x_j) V(x_j) p_{2(1-s_k)}(x_k,x) \\
&= 1 + \sum_{k=1}^{\infty} \int_{s_1<\cdots<s_k\le 1} \int_{x_1,\ldots,x_k} p_{2(1-s_k)}(x,x_k) V(x_k) \\
&\quad \times \prod_{j=k-1}^{1} p_{2((1-s_j)-(1-s_{j+1}))}(x_{j+1},x_j) V(x_j) p_{2(1-(1-s_1))}(x_1,0) \\
&= 1 + \sum_{k=1}^{\infty} \int_0^1 ds_1 \int_{s_1}^1 ds_2 \cdots \int_{s_{k-1}}^1 ds_k \int_{x_1,\ldots,x_k} p_{2(1-s_k)}(x,x_1) V(x_1) \\
&\quad \times \prod_{j=k-1}^{1} p_{2((1-s_j)-(1-s_{j+1}))}(x_{k-j},x_{k-j+1}) V(x_{k-j+1}) p_{2(1-(1-s_1))}(x_k,0),
\end{aligned}$$

where we change variables in x_j's. Finally, let $r_1 = 1 - s_k, r_2 = 1 - s_{k-1}, \ldots, r_j = 1 - s_{k-j+1}$, then

$$\begin{aligned}
E_0\left[\delta_x(B_1)e^{\int_0^1 ds\ V(B_s)}\right] &= 1 + \sum_{k=1}^{\infty}\int_1^0 -dr_k \int_{r_k}^0 -dr_{k-1}\cdots\int_{r_2}^0 -dr_1 \int_{x_1,\ldots,x_k} p_{2r_1}(x,x_1)V(x_1) \\
&\quad \times \prod_{j=k-1}^{1} p_{2(r_{k-j+1}-r_{k-j})}(x_{k-j},x_{k-j+1})V(x_{k-j+1})p_{2(1-r_k)}(x_k,0) \\
&= 1 + \sum_{k=1}^{\infty}\int_0^1 dr_k \int_0^{r_k} dr_{k-1}\cdots\int_0^{r_2} dr_1 \int_{x_1,\ldots,x_k} p_{2r_1}(x,x_1)V(x_1) \\
&\quad \times \prod_{j=2}^{k} p_{2(r_j-r_{j-1})}(x_{j-1},x_j)V(x_j)p_{2(1-r_k)}(x_k,0) \\
&= 1 + \sum_{k=1}^{\infty}\int_{r_1<\cdots<r_k\le 1}\int_{x_1,\ldots,x_k}\prod_{j=1}^{k} p_{2(r_j-r_{j-1})}(x_{j-1},x_j)V(x_j)p_{2(1-r_k)}(x_k,0) \\
&= E_x\left[\delta_0(B_1)e^{\int_0^1 ds\ V(B_s)}\right].
\end{aligned}$$

□

4.5.3 Second Step 2.3

Lemma 4.5.7. *Let $F_\epsilon(B^1)$ be defined as in (4.5.6). There is a set of x of positive Lebesgue measure such that*

$$\lim_{\epsilon\to 0} E_{0,x}^{(1)}[F_\epsilon(B^1)] \neq 1.$$

Remark 4.5.8. *This is the step where we show the two bridges are different under the new measure induced by the exponential moment of the local time, and combining with the Lemmas in the last two subsections, Proposition 4.5.1 now follows. To understand why the two bridges are different, let us look at Theorem 3.0.1 more carefully. In the formula, the modified Bessel function $K_0(\sqrt{z}|x|)$ is divided by some function $g(z)$ that has a zero on the positive real line. Suppose $g(z)$ is absent. Since $K_0(\sqrt{z}|x|)$ is the Laplace transform of the transition density function $p_{2t}(0,x)$ of the two dimensional Brownian motion, the inverse Laplace transform (the contour integral in the formula) would give us back $p_{2t}(0,x)$. From Step 2.3.1 below, the conclusion in Lemma 4.5.7 would be 1 and the two bridges would be the same. On the other hand, with $g(z)$ in the formula, since it has a zero on the positive real line, if we interpret the contour integral as the transition density function of a new Markov process, then it would induce the two bridges to behave differently.*

Proof. From now on, we will denote the Brownian motion B^1 in the Lemma by B, and we will break down the proof of the Lemma into three more steps.

Step 2.3.1: Compute limit of the two bridges under the exponential moment of the local time, i.e.

$$\lim_{\epsilon\to 0} E_{0,x}[F_\epsilon(B)] = \frac{\int dy\ \frac{p_2(0,y)}{p_2(0,(x-y)/\sqrt{2})}\frac{1}{2\pi i}\int_\gamma dz\ e^z \frac{1}{2\pi}\frac{K_0(\sqrt{z}|(x-y)/\sqrt{2}|)}{g(z)}}{\int dy\ \frac{1}{2\pi i}\int_\gamma dz\ e^z\frac{1}{2\pi}\frac{K_0(\sqrt{z}|y|)}{g(z)}}, \tag{4.5.8}$$

where the notations are the same as in Theorem 3.0.1.

So far, we know

$$E_{0,x}[F_\epsilon(B)] = \frac{\frac{1}{\log \epsilon^{-1}} \int dy \ \frac{p_2(0,y)}{p_2(0,(x-y)/\sqrt{2})} E_{(x-y)/\sqrt{2}} \left[\delta_0(B_1) e^{\beta_\epsilon \int_0^1 ds \ \epsilon^{-2} p_1\left(\epsilon^{-1} B_s\right)}\right]}{\frac{1}{\log \epsilon^{-1}} \int dy \ E_y \left[\delta_0(B_1) e^{\beta_\epsilon \int_0^1 ds \ \epsilon^{-2} p_1(\epsilon^{-1} B_s)}\right]},$$

and the limit of the denominator is given by Theorem 3.0.1, which is done by a matching upper and lower bound. To compute the limit of the numerator, we again do it by upper and lower bound. Since the argument is similar for both, we will deal only with lower bound. For the lower bound, from Step 2 of the proof of Theorem 3.0.1 (see Section 3.14), if $d\mu(y) = p_2(0,y)dy$, we have

$$\begin{aligned}\int d\mu(y) E_{0,\frac{x-y}{\sqrt{2}}} \left[e^{\beta_\epsilon \int_0^1 ds \ \epsilon^{-2} p_1(\epsilon^{-1} B_s)}\right] \geq \left(\int d\mu(y) E_{0,\frac{x-y}{\sqrt{2}}} \left[e^{\frac{1}{p}\beta_\epsilon \int_0^1 ds \ \epsilon^{-2} p_{1,\alpha}(\epsilon^{-1} B_s)}\right]\right)^p \qquad (4.5.9)\\ \times \left(\int d\mu(y) E_{0,\frac{x-y}{\sqrt{2}}} \left[e^{-\frac{1}{p-1} d_\alpha \beta_\epsilon \int_0^1 ds \ \epsilon^{-2} p_{1,\alpha}(\epsilon^{-1} B_s)}\right]\right)^{-(p-1)},\end{aligned}$$

and we take $p = \frac{1}{1-\frac{1}{(\log \epsilon^{-1})^{1+\delta}}}$, so $\frac{1}{p}\beta_\epsilon = \frac{2\pi}{\log \epsilon^{-1}} + \frac{\sigma_\epsilon}{(\log \epsilon^{-1})^2}$, where $\sigma_\epsilon \to \frac{\sigma}{4}$ as $\epsilon \to 0$, and $(-\frac{1}{p-1})d_\alpha \approx \frac{1}{(\log \epsilon^{-1})^\delta}$.

The first term in (4.5.9) goes to the desired limit because

$$\int d\mu(y) E_{0,\frac{x-y}{\sqrt{2}}} \left[e^{\frac{1}{p}\beta_\epsilon \int_0^1 ds \ \epsilon^{-2} p_{1,\alpha}(\epsilon^{-1} B_s)}\right] = 2 \int dy \ \frac{p_2(0, x-\sqrt{2}y)}{p_2(0,y)} E_y \left[\delta_0(B_1) e^{\frac{1}{p}\beta_\epsilon \int_0^1 ds \ \epsilon^{-2} p_{1,\alpha}(\epsilon^{-1} B_s)}\right],$$

and we can see $\frac{p_2(0,x-\sqrt{2}y)}{p_2(0,y)}dy = e^{\frac{|x|^2}{4}} e^{-\frac{1}{4}|\sqrt{2}x-y|^2} dy$ as a finite measure in y, which is better than before, so it remains to show the second term goes to 1. For this, as is done in Section 3.14, we expand the exponential, use Brownian scaling, and show each term is small due to $(-\frac{1}{p-1})d_\alpha \approx \frac{1}{(\log \epsilon^{-1})^\delta}$.

Let $\rho = -\frac{1}{p-1} d_\alpha$, and $q(x,y) = \frac{p_2(0,x-\sqrt{2}y)}{p_2(0,y)}$. Expanding the exponential and by change of variables,

$$\begin{aligned}&\int dy \ p_2(0,y) E_{0,\frac{x-y}{\sqrt{2}}} \left[e^{-\frac{1}{p-1} d_\alpha \beta_\epsilon \int_0^1 ds \ \epsilon^{-2} p_{1,\alpha}(\epsilon^{-1} B_s)}\right]\\ &\quad = 1 + 2\sum_{k=1}^\infty \frac{1}{k!} \int dy \ p_2(0, x-\sqrt{2}y) E_{0,y} \left[\rho \beta_\epsilon \int_0^1 ds \ \epsilon^{-2} p_{1,\alpha}(\epsilon^{-1} B_s)\right]^k\\ &\quad = 1 + 2\sum_{k=1}^\infty \frac{1}{k!} \int dy \ \frac{p_2(0, x-\sqrt{2}y)}{p_2(0,y)} p_2(0,y) E_{0,y} \left[\rho \beta_\epsilon \int_0^1 ds \ \epsilon^{-2} p_{1,\alpha}(\epsilon^{-1} B_s)\right]^k\\ &\quad \leq 1 + 2\max_y q(x,y) \sum_{k=1}^\infty \frac{1}{k!} \int dy \ p_2(0,y) E_{0,y} \left[\rho \beta_\epsilon \int_0^1 ds \ \epsilon^{-2} p_{1,\alpha}(\epsilon^{-1} B_s)\right]^k\\ &\quad = 1 + 2\max_y q(x,y) \sum_{k=1}^\infty \frac{1}{k!} E_0 \left[\rho \beta_\epsilon \int_0^1 ds \ \epsilon^{-2} p_{1,\alpha}(\epsilon^{-1} B_s)\right]^k,\end{aligned}$$

where $q(x,y) = \frac{p_2(0,x-\sqrt{2}y)}{p_2(0,y)} = e^{\frac{1}{4}|x|^2} e^{-\frac{1}{4}|\sqrt{2}x-y|^2}$. Finally, let $T = \epsilon^{-2}$, by Brownian scaling, we see the summand is $O(\rho^k)$, which is small.

Step 2.3.2: Compute the contour integral in (4.5.8), i.e.

$$\frac{1}{2\pi i}\int_\gamma dz\ e^z \frac{1}{2\pi}\frac{K_0(\sqrt{z}|y|)}{g(z)} = \int_0^1 d\tau \int_0^\infty du \frac{(1-\tau)^{u-1}e^{-\kappa u}}{\Gamma(u)}\frac{1}{2\pi\tau}e^{-\frac{|y|^2}{4\tau}}, \tag{4.5.10}$$

where $\kappa = -\log z_0$ and $z_0 = 4e^{2\left(\frac{\sigma}{8\pi}-\nu(p_1)+\psi(1)\right)}$.

Recall $g(z) = \log\frac{\sqrt{z}}{2} - \psi(1) - \frac{\sigma}{8\pi} + \nu(p_1)$, so $g(z) = \frac{1}{2}(\log z - \log z_0) = \frac{1}{2}(\log z + \kappa)$. As in Proposition 3.12.1, we can rewrite

$$\frac{1}{2\pi i}\int_\gamma dz\ e^z\frac{K_0(\sqrt{z}|y|)}{g(z)} = \frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} dz\ e^z\frac{K_0(\sqrt{z}|y|)}{g(z)} = 2\frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} dz\ e^z\frac{K_0(\sqrt{z}|y|)}{\log z+\kappa},$$

where $\lambda > z_0$. Also, as is done in [3] (see page 13), we note $\hat{f}(z) = 4\pi\frac{1}{\log z+\kappa}$ is the Laplace transform of $f(t) = 4\pi\int_0^\infty du\frac{t^{u-1}e^{-\kappa u}}{\Gamma(u)}$ for $t>0$, where $\Gamma(x)$ is the gamma function, and $\hat{g}(z) = \frac{1}{2\pi}K_0(\sqrt{z}|y|)$ is the Laplace transform of $g(t) = \frac{1}{4\pi t}e^{-\frac{|y|^2}{4t}}$ for $t>0$. Thus the contour integral is equal to the convolution $(f*g)(t)$ at $t=1$.

Step 2.3.3: Show the ratio on the right hand side of (4.5.8) is not equal to 1, i.e.

$$\frac{\int dy\ \frac{p_2(0,y)}{p_2(0,(x-y)/\sqrt{2})}\frac{1}{2\pi i}\int_\gamma dz\ e^z\frac{K_0(\sqrt{z}|(x-y)/\sqrt{2}|)}{g(z)}}{\int dy\ \frac{1}{2\pi i}\int_\gamma dz\ e^z\frac{K_0(\sqrt{z}|y|)}{g(z)}} \neq 1.$$

Let $x=0, \sigma=1$. We will compute the top and the bottom separately, which are two explicit numbers, and show their ratio is not 1; and since the ratio is a continuous function of x and σ, Lemma 4.5.7 follows. Recall $\nu(p_1) = \int dx\ p_1(x)\int dy\ \log|x-y|p_1(y)$. By direct computation, $\nu(p_1) = \frac{1}{2}(\psi(1)+2\log 2)$. (See Lemma 3.10.2.) So $z_0 = 4e^{2\left(\frac{1}{8\pi}-\nu(p_1)+\psi(1)\right)} = 4e^{\frac{1}{4\pi}+\psi(1)-\log 4} \approx 0.607965$, and $e^{-\kappa u} = e^{u\log z_0} = z_0^u$. Also, at $x=0$, $\frac{p_2(0,y)}{p_2(0,(x-y)/\sqrt{2})} = e^{-\frac{|y|^2}{8}}$. Now, by (4.5.10), the ratio is

$$\frac{\int dy\ e^{-\frac{|y|^2}{8}}\int_0^1 d\tau\int_0^\infty du\frac{(1-\tau)^{u-1}e^{-\kappa u}}{\Gamma(u)}\frac{1}{\tau}e^{-\frac{|y|^2}{8\tau}}}{\int dy\ \int_0^1 d\tau\int_0^\infty du\frac{(1-\tau)^{u-1}e^{-\kappa u}}{\Gamma(u)}\frac{1}{\tau}e^{-\frac{|y|^2}{4\tau}}}.$$

Since the integrand is nonnegative, by Fubini, we will interchange order of integration, and conclude it is not equal to 1.

Note by Gaussian convolution, $\int dy\ e^{-\frac{|y|^2}{8}}\frac{1}{\tau}e^{-\frac{|y|^2}{8\tau}} = (2\pi)\frac{4}{1+\tau}$, and directly, $\int dy\ \frac{1}{\tau}e^{-\frac{|y|^2}{4\tau}} = (2\pi)2$. Now let $G_\kappa(t) = \int_0^\infty du\frac{(1-t)^{u-1}e^{-\kappa u}}{\Gamma(u)}$ for $0\le t\le 1$, which is integrable because by Fubini, $\int_0^1 dt(1-t)^{u-1} = \frac{1}{u}$ and $u\Gamma(u)\to 1$ as $u\to 0$. After the integration in y, the ratio becomes

$$\frac{\int_0^1 d\tau\ G_\kappa(\tau)(2\pi)\frac{4}{1+\tau}}{\int_0^1 d\tau\ G_\kappa(\tau)(2\pi)2} = \frac{\int_0^1 d\tau\ G_\kappa(\tau)\frac{2}{1+\tau}}{\int_0^1 d\tau\ G_\kappa(\tau)} \neq 1.$$

□

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