# Subconvexity for a twisted double Dirichlet series and non-vanishing of $L$-functions

by

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# Abstract

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We study a double Dirichlet series of the form $\sum_d L(s, \chi_d \chi)\chi'(d)d^{-w}$, where $\chi$ and $\chi'$ are quadratic Dirichlet characters with prime conductors $N$ and $M$ respectively. A functional equation group isomorphic to the dihedral group of order 6 continues the function meromorphically to $\mathbb{C}^2$. A convexity bound at the central point is established to be $(MN)^{3/8+\varepsilon}$ and a subconvexity bound of $(MN(M+N))^{1/6+\varepsilon}$ is proven. This bound is used to prove an upper bound for the smallest positive integer $d$ such that $L(1/2, \chi_{dN})$ does not vanish.

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# Contents

# Chapter 1

# Introduction

The use of multiple Dirichlet series in number theoretic problems, as well as the intrinsic structure they possess, have evolved the subject to be a study in its own right. Indeed, Weyl group multiple Dirichlet series are considered fundamental objects whose structures are intimately linked to their analytic features [2]. In this paper we are interested in the size of a certain type of double Dirichlet series outside the region of absolute convergence, and some of its arithmetic consequences. For ordinary $L$-functions there is the well-known concept of convexity bound as a generic upper bound in the critical strip, and improvements in the direction of the Lindelöf hypothesis often have deep implications. In the situation of multiple Dirichlet series, the notion of convexity is not so obvious to formalize. Nevertheless there is a fairly natural candidate for a "trivial" upper bound of a multiple Dirichlet series, and it is an important problem to improve this estimate. This has first been carried out in [1] for the Dirichlet series defined by

$$Z(s,w) = \zeta^{(2)}(2s+2w-1) \sum_{d \text{ odd}} L^{(2)}(s,\chi_d) d^{-w}$$

at $s = 1/2 + it$, $w = 1/2 + iu$ simultaneously in the $t, u$-aspect, where the $^{(2)}$ superscript denotes that we are removing the Euler factor at 2. Here we study a non-archimedean analogue. We can expand the $L$-function in the summand as a sum indexed by $n$. Instead of twisting by $n^{it}$, $d^{iu}$, we twist by quadratic primitive Dirichlet characters $\chi(n)$, $\chi'(d)$ of conductors $N, M$, respectively. More precisely, we consider

$$Z(s,w;\chi,\chi') := \sum_{\substack{d\geq 1\\ (d,2MN)=1}} \frac{L^{(2MN)}(s,\chi_{d_0}\chi)\chi'(d)P^{(\chi)}_{d_0,d_1}(s)}{d^w}$$

for sufficiently large $\Re s, \Re w$. The $P$ factors are a technical complication necessary in order to construct functional equations, which we accomplish thus: if we expand the $L$-function in the numerator and switch the summation, we can relate it to a similar double Dirichlet series with the arguments and the twisting characters interchanged, with modified correction factors. Next, we have a functional equation obtained by application of the functional equation for Dirichlet $L$-functions, which maps $(s,w)$ to $(1-s, s+w-\frac{1}{2})$. Applying the switch of summation formula to this, we obtain a functional equation mapping $(s,w)$ to $(s+w-\frac{1}{2}, 1-w)$. These continue $Z$ to the complex plane except for the polar lines $s=1$, $w=1$, and $s+w=3/2$. They also generate a group isomorphic to $D_6$, the dihedral group of order 6, which is essentially the Weyl group multiple Dirichlet series of the root system $A_2$. Existence, uniqueness, and construction of the correction polynomials for multiple Dirichlet series arising from $GL(n)$ were studied extensively in [3] and [4]. Interestingly, they are unique in the case of $n$ up to 3.

The notion of convexity is no longer canonical in the case of double Dirichlet series, since our initial bounds for $Z(s,w;\chi,\chi')$ depend on what we know about bounds on their coefficients, and our knowledge here is only partial. Nonetheless, if we assume the Lindelöf hypothesis on average (cf. Theorem 2.2), then through careful choice of initial bounds and functional equation applications (cf. §3.4), we obtain the bound

$$Z(\tfrac{1}{2},\tfrac{1}{2};\chi,\chi') \ll (MN)^{3/8+\varepsilon},$$

which we call the convexity bound. In this work, we present the following subconvexity result.

**Theorem 1.1.** *For quadratic Dirichlet characters $\chi$ and $\chi'$ of conductors $N$ and $M$ which are prime or unity, and for $\varepsilon > 0$, we have the bound*

$$Z(\tfrac{1}{2}, \tfrac{1}{2} + it; \chi, \chi') \ll_\varepsilon (1 + |t|)^{2/\varepsilon} (MN(M+N))^{1/6+\varepsilon}.$$

We first point out that the $t$-aspect bound can be drastically improved, but is only required to be polynomial in $t$ for our applications. For purposes of comparison with the convexity bound, we point out that, via the geometric-arithmetic mean inequality, we have $(MN)^{3/8+\varepsilon} \ll (MN(M+N))^{1/4+\varepsilon}$. This result is comparable to the subconvexity bound obtained by V. Blomer in the archimedean case, particularly $(sw(s+w))^{1/6+\varepsilon}$ for $\Re s = \Re w = \frac{1}{2}$, an improvement upon the convexity bound with 1/6 replaced by 1/4.

A useful arithmetic application of a subconvexity bound for $Z(\frac{1}{2}, \frac{1}{2}+it; \chi, \chi')$ is finding a bound for the least $d$ such that $L(\frac{1}{2}, \chi_{dN})$ does not vanish, where $N$ is a large fixed prime. It is expected that all ordinates of any zeros of $L(s, \chi)$ on the critical line are linearly independent over the rationals, so that in particular it is expected that $L(\frac{1}{2}, \chi)$ is nonzero for any $\chi$. Random matrix theory provides further evidence to this conjecture: The lowest zero in families of $L$-functions (such as $L(s, \chi_d)$) is expected to be distributed like the "smallest" eigenvalue (i.e., closest to unity) of a certain matrix family (depending of the family of $L$-function). Since the corresponding measure vanishes at zero, it suggests that the smallest zero of a Dirichlet $L$-function is "repelled" from the real axis. In the case of quadratic twists of $GL(2)$ automorphic forms which are closely connected to twists of elliptic curves, there is a connection to Waldspurger's theorem [11] which states that $L(\frac{1}{2}, f \times \chi)$ is the square of the Fourier coefficient of a half-integral weight modular form. We focus here on the simpler case of twists of Dirichlet characters. In particular, we have the following theorem.

**Theorem 1.2.** *Let $N$ be an odd prime, and let $D(N) = d$ denote the smallest positive* *integer such that* $L(\frac{1}{2}, \chi_{dN})$ *does not vanish. Then we have* $D(N) \ll N^{2/3+\varepsilon}$.

It is not entirely obvious what should be regarded as the trivial bound in this situation. The most natural approach to non-vanishing would be to prove a lower bound for the first moment $\sum_{d \asymp X} L(\frac{1}{2}, \chi_{dN})$ for $X$ as small as possible in terms of $N$. A straightforward argument produces a main term of size $X \log X$ and an error term of size $O((NX)^{1/2+\varepsilon})$ which suggests the trivial bound $N^{1+\varepsilon}$ for the first non-vanishing twist. This is perhaps unexpectedly weak, since the same bound holds for degree 2 $L$-functions $L(\frac{1}{2}, f \times \chi_d)$, where $f$ is an automorphic form of level $N$. Nevertheless it is not completely obvious how to improve this in the case of Dirichlet characters $\chi_N$. J. Hoffstein and A. Kontorovich [6] sketch an argument for a bound $N^{1/2+\varepsilon}$. While it may be possible to substantiate their argument, the author was not able to complete a proof along these lines. We defer the details to Appendix A. Here we proceed differently according to a further modification of their approach thus: Let $X$ be a large positive number, and $h(y)$ be a smooth function with support on $[1, 2]$. By Mellin inversion, we have

$$\int_{(2)} \tilde{h}(w) Z(\tfrac{1}{2}, w; \chi_N, \psi_1) X^w dw \approx \sum_{0<d<X} L(\tfrac{1}{2}, \chi_{dN}),$$

where $\tilde{h}$ denotes the Mellin transform of $h$, and $\psi_1$ denotes the trivial character. We move the contour to $\Re w = 1/2$, picking up a double pole at $w = 1$. If we apply our subconvexity bound Theorem 1.1 to the resulting integral, then we have

$$\sum_{0<d<X} L(\tfrac{1}{2}, \chi_{dN}) \approx a_N X \log X + b_N X + O(X^{1/2} N^{1/3+\varepsilon}) \tag{1.1}$$

for some coefficients $a_N, b_N$. The idea now is to bound $a_N$ from below in terms of $N$, and choose $X$ so that the main term is greater than the error term. Then it cannot be that $L(\frac{1}{2}, \chi_{dN})$ vanishes for all $0 < d < X$ on the left hand side. We prove the bounds $a_N, b_N \gg N^{-\varepsilon}$ (cf. Theorem 7.1), and thus we can choose $X = N^{2/3+\varepsilon}$. The power of the asymptotic formula (1.1) is that $a_N, b_N$ can be bounded below by finding an asymptotic for $\sum_{d \asymp X} L(\frac{1}{2}, \chi_{dN})$, but now we can take $X$ to be very large.

Of course, in the previous discussion there are significant details suppressed for brevity, which include the aforementioned correction factors and the error term arising from truncation of the $d$-sum. Most nontrivial, however, are the bounds for $a_N, b_N$ given by Theorem 7.1, requiring careful treatment. The techniques used are similar to M. Jutila's in [9], in which an asymptotic formula for

$$\sum_{0<d\leq X} L^k(\tfrac{1}{2}, \chi_d)$$

is proved, where the sum is over fundamental discriminants.

In order to prove the subconvexity bound Theorem 1.1, we follow similar techniques to those used by V. Blomer in [1], but we must deal with some new complications. As mentioned, we need special correction factors in order to obtain functional equations, whereas in the archimedean case, the correction factors are far simpler. Also, the introduction of character twists complicates the functional equations considerably: a pair of twisting characters $(\psi, \psi')$ will not be static under the variable transformations of the functional equations.

We can iterate the functional equations to obtain one under the map $(s, w) \mapsto (1-s, 1-w)$. Using techniques similar to those in the case of $L(s, \chi)$ (cf. [8] Theorem 5.3), we obtain the approximate functional equation

$$Z(\tfrac{1}{2}, \tfrac{1}{2}; \chi, \chi) \approx \sum_{d<X} \frac{L(\frac{1}{2}, \chi_d \chi)\chi'(d)}{d^{1/2}} + \sum_{d \leq M^2 N/X} \frac{L(\frac{1}{2}, \chi_d \chi' \chi_M)\chi\chi_N(d)}{d^{1/2}},$$

(cf. Lemma 4.1). We can further apply the approximate functional equation for $L$-functions,

$$L(\tfrac{1}{2}, \chi_d \chi) \approx \sum_{n \leq (dq)^{1/2}} \frac{\chi(n)}{n^{1/2}}.$$

We hence see that we can roughly express $Z(\frac{1}{2}, \frac{1}{2}; \chi, \chi')$ as a sum of double finite sums of the form

$$S(P, Q; \chi, \chi') := \sum_{d \le P} \sum_{n \le Q} \frac{\chi_d(n)\chi(n)\chi'(d)}{d^{1/2}n^{1/2}}$$

for various character pairs $(\chi, \chi')$. Finally, we apply Heath-Brown's large sieve estimate (cf. Corollary 2.5) to obtain the subconvexity results.

**Notation.** The variable $\varepsilon$ will always denote a sufficiently small positive number, not necessarily the same at each occurrence, and the variable $A$ will denote a sufficiently large positive number, not necessarily the same at each occurrence. The numbers $M$ and $N$ will always denote natural numbers that are either odd primes or unity, possibly equal. For a real function $f$, we denote its Mellin transform by $\tilde{f}$. The trivial character modulo unity will be denoted by $\psi_1$, and the primitive character of conductor 4 shall be denoted $\psi_{-1}$. As for the primitive characters modulo 8, we define $\psi_2$ as the character that is unity at exactly 1 and 7, and we set $\psi_{-2} = \psi_2\psi_{-1}$. If $\chi$ is a character, we use the notation $C_\chi$ to denote its conductor.

# Chapter 2

# Preliminaries

## 2.1 Characters

For a positive integer $d$, we define a character on $(\mathbb{Z}/4d\mathbb{Z})^*$ via the Jacobi symbol by

$$\chi_d(n) = \widetilde{\chi}_n(d) = \left(\frac{d}{n}\right).$$

For odd positive integers $n$ and $d$, we have the following quadratic reciprocity law for the Jacobi-Kronecker symbol (cf. [10], Theorem 4.2.1, page 197).

$$\chi_d(n) = \left(\frac{d}{n}\right) = (-1)^{\frac{d-1}{2}\frac{n-1}{2}}\left(\frac{n}{d}\right) = \begin{cases} \widetilde{\chi}_d(n), & n \equiv 1 \pmod 4; \\ \widetilde{\chi}_d(-n) = \widetilde{\chi}_d(n)\psi_{-1}(d), & n \equiv 3 \pmod 4. \end{cases} \tag{2.1}$$

## 2.2 $L$-function results

Suppose that $\chi$ is a Dirichlet character. We define

$$L^{(P)}(s,\chi) = L(s,\chi)\prod_{p|P}\left(1 - \frac{\chi(p)}{p^s}\right). \tag{2.2}$$

We define the odd sign indicator function of a Dirichlet character $\chi$ by

$$\kappa = \kappa(\chi) = \frac{1}{2}(1 - \chi(-1)). \tag{2.3}$$

We shall make use of the functional equation for Dirichlet $L$-functions. If the character $\chi$ is primitive and quadratic, we have

$$L(s, \chi) = \left(\frac{q}{\pi}\right)^{1/2-s} \frac{\Gamma\left(\frac{1-s+\kappa}{2}\right)}{\Gamma\left(\frac{s+\kappa}{2}\right)} L(1-s, \chi).$$

## 2.3 $L$-functional bounds and approximate functional equation

For a primitive character $\chi$, using an absolute convergence argument for $L(s, \chi)$ in a right half-plane and applying the functional equation, we interpolate via the Phragmén-Lindelöf convexity principle (cf. Theorem 3.11) to obtain

$$L(s, \chi) \ll \begin{cases} [q(1 + |\Im s|)]^{1/2-\Re s}, & \Re s \le -\varepsilon; \\ [q(1 + |\Im s|)]^{(1-\Re s)/2+\varepsilon}, & -\varepsilon < \Re s < 1 + \varepsilon; \\ 1, & \Re s \ge 1 + \varepsilon, \end{cases} \tag{2.4}$$

away from a possible pole at $s = 1$ in the case where $\chi$ is trivial. This bound is known as the convexity bound for Dirichlet $L$-functions.

We shall also need the so-called approximate functional equation for Dirichlet $L$-functions ([8], Theorem 5.3). Particularly, if $\chi$ is a quadratic primitive character modulo odd $q$, $\psi$ is a character with conductor dividing 8, and $d_0$ is odd, squarefree and coprime to $q$, then we have the weighted infinite sum,

$$L(\tfrac{1}{2}, \chi_{d_0}\chi\psi) = 2\sum_{n=1}^{\infty} \frac{(\chi_{d_0}\chi\psi)(n)}{n^{1/2}} G_\kappa\left(\frac{n}{\sqrt{c_0 d_0 q}}\right), \tag{2.5}$$

where

$$\kappa = \kappa(\chi_{d_0}\chi\psi), \quad c_0 = \begin{cases} 1, & d_0 \equiv 1 \,(\mathrm{mod}\ 4), \psi = \psi_1 \,\mathrm{or}\, d_0 \equiv 3 \,(\mathrm{mod}\ 4), \psi = \psi_{-1}; \\ 4, & d_0 \equiv 1 \,(\mathrm{mod}\ 4), \psi = \psi_{-1} \,\mathrm{or}\, d_0 \equiv 3 \,(\mathrm{mod}\ 4), \psi = \psi_1; \\ 8, & \psi = \psi_2 \text{ or } \psi_{-2}, \end{cases} \tag{2.6}$$

and we have the weight function

$$G_\kappa(\xi) = \frac{1}{2\pi i} \int_{(2)} \frac{\Gamma(\frac{1/2+s+\kappa}{2})}{\Gamma(\frac{1/2+\kappa}{2})} \xi^{-s} \frac{ds}{s}, \tag{2.7}$$

which satisfies

$$G_\kappa(\xi) \ \ll \ (1+\xi)^{-A} \tag{2.8}$$

for arbitrary $A > 0$ (cf. [8], Proposition 5.4).

We shall also make use of smooth weight functions.

**Definition 2.1.** We say that $w(x)$ is a smooth weight function if it is a smooth non-negative real function supported on $[1/4, 5/4]$ and unity on $[1/2, 1]$.

It can be shown via sufficiently many applications of integration by parts that

$$\tilde{w}(z) \ll_{A,\Re z} (1+|z|)^{-A} \tag{2.9}$$

for $A \geq 0$, where $\tilde{w}(z)$ is the Mellin transform of $w(x)$.

*Proof.* This is trivial for $A = 0$. We can assume without loss of generality that $A > 5$. If $|z| \leq 2A$, then the claim holds trivially. If we assume that $|z| > 2A$, then by definition, we have

$$\tilde{w}(z) = \int_0^\infty w(x) x^{z-1} dx.$$

We use integration by parts with $u = w(x)$ and $dv = x^{z-1}dx$ to obtain

$$\tilde{w}(z) = -\frac{1}{z}\int_0^\infty x^z w'(x)dx.$$

Performing this multiple times, we thus have

$$\tilde{w}(z) = \frac{(-1)^n}{z(z+1)\cdots(z+n-1)}\int_0^\infty x^{z-1+n}w^{(n)}(x)dx,\ n \in \mathbb{N}$$

which gives

$$|\tilde{w}(z)| \ll_{n,\Re z} |z|^{-n}$$

which holds for $|z| > 2A$ and $n = [A]+1$. $\square$

## 2.4 Short double character sums and $L$-function moments

We shall need the following adaptation of Theorem 2 from [5] which includes a character twist (cf. Appendix B).

**Theorem 2.2.** *Let $\psi$ be a primitive character modulo $j$, and for a positive integer $Q$ define $S(Q)$ to be the set of quadratic primitive Dirichlet characters of conductor at most $Q$. We have*

$$\sum_{\chi \in S(Q)} |L(\sigma + it, \chi\psi)|^4 \ll_\varepsilon \{Q + (Qj(|t|+1))^{2-2\sigma}\}\{Qj(|t|+1)\}^\varepsilon$$

*for any fixed $\sigma \in [1/2, 1]$ and any $\varepsilon > 0$.*

A useful result that rests on this bound is the following estimate on finite $L$-function sums over quadratic characters.

**Theorem 2.3.** *If $\chi$ is a primitive character with conductor $q$ and $X$ is a positive real number, then*

$$\sum_{\substack{d_0 \le X \\ d_0 \text{ odd, squarefree}}} |L(s, \chi_{d_0}\chi)|^4 \ll_\varepsilon (Xq|s|)^{1+\varepsilon}, \ \sigma \ge \tfrac{1}{2}$$

*for all $\varepsilon > 0$.*

An important ingredient for the proof of the subconvexity bound Theorem 1.1 is a large sieve estimate for quadratic characters due to Heath-Brown. In particular we state here Theorem 1 in [5].

**Theorem 2.4.** *Let $P$ and $Q$ be positive integers, and let $(a_n)$ be a sequence of complex numbers. Then*

$$\sum_{m \le P}{}^{*} \left| \sum_{n \le Q}{}^{*} a_n \left(\frac{n}{m}\right) \right|^2 \ll_\varepsilon (PQ)^\varepsilon (P+Q) \sum_{n \le Q}{}^{*} |a_n|^2,$$

*for any $\varepsilon > 0$, where $\sum^*$ denotes that the sum is over odd squarefree numbers.*

Due to the nature of the double Dirichlet series we shall construct, we shall need the following Corollary.

**Corollary 2.5.** *If $(a_m)$, $(b_n)$ are sequences of complex numbers satisfying the bound $a_m, \ b_m \ll m^{-1/2+\varepsilon}$ for some $\varepsilon > 0$, and $P$ and $Q$ are positive real numbers, then*

$$\sum_{\substack{m \le P \\ m \text{ odd}}} \sum_{\substack{n \le Q \\ n \text{ odd}}} a_m b_n \left(\frac{n_0}{m}\right) \ll_\varepsilon (PQ)^\varepsilon (P+Q)^{1/2+\varepsilon},$$

*where we have the composition $n = n_0 n_1^2$ with $n_0$ squarefree, uniformly in $P$ and $Q$, for any $\varepsilon > 0$.*

*Proof.* We use the decomposition $m = m_0 m_1^2$ and $n = n_0 n_1^2$ and separate each sum into two sums accordingly, moving the two squarefree sums to the inside. We use the

Cauchy-Schwarz inequality on the inner sum and apply Theorem 2.4. Finally, we use $(A+B)^r \asymp A^r + B^r$ and the result follows.

In what follows, all of the sums are over odd numbers. We have

$$\sum_{m\le P}\sum_{n\le Q} a_m b_n \left(\frac{n_0}{m}\right) = \sum_{m_1\le P^{1/2}}\sum_{n_1\le Q^{1/2}}\sum_{m_0\le P/m_1^2} a_m \sum_{n_0\le Q/n_1^2} b_n \left(\frac{n_0}{m}\right).$$

We use Cauchy-Schwarz on the penultimate sum to obtain

$$\sum_{m_1\le P^{1/2}}\sum_{n_1\le Q^{1/2}} \left(\sum_{m_0\le P/m_1^2} |a_m|^2\right)^{1/2} \left(\sum_{m_0\le P/m_1^2} \left|\sum_{n_0\le Q/n_1^2} b_n \left(\frac{n_0}{m}\right)\right|^2\right)^{1/2}.$$

We can bound this above by replacing the $m$ in the Legendre symbol by $m_0$, since $(\frac{n_0}{m_1^2})$ vanishes when $(n_0, m_1) > 1$, and is unity otherwise. We now use Theorem 2.4 to bound this above by

$$\sum_{m_1\le P^{1/2}}\sum_{n_1\le Q^{1/2}} \left(\sum_{m_0\le P/m_1^2} |a_m|^2\right)^{1/2} \left(\sum_{n_0\le Q/n_1^2} |b_n|^2\right)^{1/2} \left(\frac{P}{m_1^2}+\frac{Q}{n_1^2}\right)^{1/2} \left(\frac{PQ}{m_1^2 n_1^2}\right)^{\varepsilon/2}.$$

Using the fact that $a_m \ll m^{-1/2+\varepsilon}$ and $b_n \ll n^{-1/2+\varepsilon}$, this is further bounded above by

$$\begin{aligned}
&\sum_{m_1\le P^{1/2}} m_1^{-1+2\varepsilon} \sum_{n_1\le Q^{1/2}} n_1^{-1+2\varepsilon} \left(\sum_{m_0\le P/m_1^2} m_0^{-1+2\varepsilon}\right)^{1/2} \left(\sum_{n_0\le Q/n_1^2} n_0^{-1+2\varepsilon}\right)^{1/2} \\
&\times \left(\frac{P}{m_1^2}+\frac{Q}{n_1^2}\right)^{1/2} \left(\frac{PQ}{m_1^2 n_1^2}\right)^{\varepsilon/2} \\
\ll\ & \sum_{m_1\le P^{1/2}} \frac{1}{m_1} \sum_{n_1\le Q^{1/2}} \frac{1}{n_1} \left(\frac{P^{1/2}}{m_1}+\frac{Q^{1/2}}{n_1}\right)(PQ)^{\varepsilon} \\
\ll\ & (P^{1/2}+Q^{1/2})(PQ)^{\varepsilon} \sum_{m_1\le P^{1/2}} \frac{1}{m_1} \sum_{n_1\le Q^{1/2}} \frac{1}{n_1} \\
\ll\ & (P+Q)^{1/2}(PQ)^{\varepsilon},
\end{aligned}$$

as required. □

## 2.5 Gamma identities

We shall have use for the identity

$$\frac{\Gamma(\frac{2-z}{2})}{\Gamma(\frac{z+1}{2})} = \frac{\Gamma(\frac{1-z}{2})}{\Gamma(\frac{z}{2})} \cot\left(\frac{\pi z}{2}\right), \; z \in \mathbb{C}. \tag{2.10}$$

By Stirling's formula, in particular [8], formula (5.113), for $s \in \mathbb{C}$ with fixed real part and nonzero imaginary part, we have

$$\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})} \ll_{\Re s} (1+|s|)^{1/2-\Re s}, \tag{2.11}$$

away from the poles at the odd positive integers. We also have the cotangent bound,

$$\cot(x+iy) = -i\,\mathrm{sign}(y) + O(e^{-2|y|}), \; \min_{k\in\mathbb{Z}} |z-\pi k| \geq 1/10. \tag{2.12}$$

# Chapter 3

# Structure and analytic properties

## 3.1 Switch of summation formula

The object we would like to study is

$$Z_0(s,w;\chi,\chi') = \sum_{d\geq 1} \frac{L(s,\chi_d\chi)\chi'(d)}{d^w},$$

where $\chi$ and $\chi'$ are quadratic characters with moduli $N$ and $M$ respectively. However, in order to obtain functional equations, we will need to augment this by some correction factors. The exact form of these correction factors (in a more general setting) was determined in [3]. Applying this theory in our case, we have the following theorem, whose proof is contained in [3], but we include here for completeness. We note that the following theorem holds for general Dirichlet characters $\chi$ and $\chi'$, though in our case we are interested in the special case of quadratic twists.

**Theorem 3.1.** *Let $m$ and $d$ be positive integers with $(md, 2MN) = 1$, and write $d = d_0d_1^2$ and $m = m_0m_1^2$, with $d_0$, $m_0$ squarefree and $d_1$, $m_1$ positive, and let $\chi$ and $\chi'$ be characters* *modulo $8MN$. Then the Dirichlet polynomials*

$$P^{(\chi)}_{d_0,d_1}(s) = \prod_{p^\alpha \| d_1} \left[ \sum_{n=0}^{\alpha} \chi(p^{2n}) p^{n-2ns} - \sum_{n=0}^{\alpha-1} (\chi_{d_0}\chi)(p^{2n+1}) p^{n-(2n+1)s} \right],$$

$$Q^{(\chi')}_{m_0,m_1}(w) = \prod_{p^\beta \| m_1} \left[ \sum_{n=0}^{\beta} \chi'(p^{2n}) p^{n-2nw} - \sum_{n=0}^{\beta} (\tilde{\chi}_{m_0}\chi')(p^{2n+1}) p^{n-(2n+1)w} \right]$$

*satisfy the functional equations*

$$P^{(\chi)}_{d_0,d_1}(s) = d_1^{1-2s} \chi(d_1^2) P^{(\bar{\chi})}_{d_0,d_1}(1-s), \tag{3.1}$$

$$Q^{(\chi')}_{m_0,m_1}(w) = m_1^{1-2w} \chi'(m_1^2) Q^{(\bar{\chi}')}_{m_0,m_1}(1-w), \tag{3.2}$$

*and the interchange of summation formula*

$$\sum_{(d,2MN)=1} \frac{L^{(2MN)}(s, \chi_{d_0}\chi)\chi'(d) P^{(\chi)}_{d_0,d_1}(s)}{d^w} = \sum_{(m,2MN)=1} \frac{L^{(2MN)}(w, \tilde{\chi}_{m_0}\chi')\chi(m) Q^{(\chi')}_{m_0,m_1}(w)}{m^s}, \tag{3.3}$$

*for $\Re s, \Re w > 1$.*

*Proof.* Observing that $d_1 = \prod_{p^\alpha \| d_1} p^\alpha$, we have

$$\begin{aligned}
d_1^{1-2s}\chi(d_1^2) P^{(\bar{\chi})}_{d_0,d_1}(1-s) &= \prod_{p^\alpha \| d_1} \left[ \sum_{n=0}^{\alpha} \chi(p^{2\alpha}) \bar{\chi}(p^{2n}) p^{\alpha(1-2s)+n-2n(1-s)} \right. \\
&\quad \left. - \sum_{n=0}^{\alpha-1} (\chi_{d_0}\chi)(p^{2\alpha}) (\chi_{d_0}\bar{\chi})(p^{2n+1}) p^{\alpha(1-2s)+n-(2n+1)(1-s)} \right] \\
&= \prod_{p^\alpha \| d_1} \left[ \sum_{n=0}^{\alpha} \chi(p^{2(\alpha-n)}) p^{(\alpha-n)-(2\alpha-2n)s} \right. \\
&\quad \left. - \sum_{n=0}^{\alpha-1} (\chi_{d_0}\chi)(p^{2(\alpha-n-1)+1}) p^{(\alpha-1-n)-(2\alpha-2n-1)s} \right] \\
&= \prod_{p^\alpha \| d_1} \left[ \sum_{n=0}^{\alpha} \chi(p^{2n}) p^{n-2ns} - \sum_{i=0}^{\alpha-1} (\chi_{d_0}\chi)(p^{2n+1}) p^{n-(2n+1)s} \right] \\
&= P^{(\chi)}_{d_0,d_1}(s),
\end{aligned}$$

as required. The functional equation for $Q$ is proved similarly.

We now prove the switch of summation formula (3.3). In order to do this, we shall write the correction polynomials as sums. It is immediately seen that

$$P^{(\chi)}_{d_0,d_1}(s) = \sum_{n|d_1^2} \frac{\chi_{d_0}(n_0)\chi(n)\mu(n_0)n_1}{n^s}. \tag{3.4}$$

The left-hand side of (3.3) is now

$$\sum_{(d,2MN)=1} \sum_{(m,2MN)=1} \sum_{n|d_1^2} \frac{(\chi_{d_0}\chi)(m)\chi_{d_0}(n_0)\chi'(d)\chi(n)\mu(n_0)n_1}{d^w(mn)^s}, \tag{3.5}$$

where we write $n = n_0 n_1^2$. We can write the right-hand side of (3.3) similarly.

To prove the summation switch equation, we fix positive integers $k$ and $\ell$ and prove that the coefficient of $\ell^{-w}k^{-s}$ is the same on both sides of the equation.

On the left hand side of (3.3), according to (3.5), the coefficient is

$$\sum_{\substack{mn=k \\ n|\ell_1^2}} (\chi_{\ell_0}\chi)(m)\chi_{\ell_0}(n_0)\chi'(\ell)\chi(n)\mu(n_0)n_1 = \chi(k)\chi'(\ell) \sum_{\substack{mn=k \\ n|\ell_1^2}} \chi_{\ell_0}(mn_0)\mu(n_0)n_1.$$

We shall denote the sum factor by $S$. We have

$$\begin{aligned} S &= \sum_{\substack{n|k \\ n|\ell_1^2}} \chi_{\ell_0}\left(\frac{k}{n}n_0\right)\mu(n_0)n_1 \\ &= \sum_{\substack{n|k \\ n|\ell_1^2}} \chi_{\ell_0}\left(\frac{k}{n_1^2}\right)\mu(n_0)n_1. \end{aligned}$$

We now write the sum conditions in terms of $n_0$ and $n_1$. We have thus

$$S = \sum_{\substack{n_1^2|k \\ n_0|k/n_1^2 \\ n_1^2|\ell_1^2 \\ n_0|\ell_1^2/n_1^2}} \chi_{\ell_0}\left(\frac{k}{n_1^2}\right)\mu(n_0)n_1.$$

Separating the sum over $n_0$, we have further

$$S = \sum_{\substack{n_1|k_1 \\ n_1|\ell_1}} \chi_{\ell_0}\left(\frac{k}{n_1^2}\right) n_1 \sum_{\substack{n_0|k_0k_1/n_1 \\ n_0|\ell_1/n_1 \\ \mu^2(n_0)=1}} \mu(n_0).$$

We can remove the squares in the $n_0$ sum condition due to the presence of $\mu(n_0)$ in the summand. Thus,

$$S = \sum_{n_1|(k_1,\ell_1)} \chi_{\ell_0}\left(\frac{k}{n_1^2}\right) n_1 \sum_{n_0|(k_0k_1,\ell_1)/n_1} \mu(n_0).$$

If we apply Möbius inversion to the second sum, we see that it is 1 if $(k_0k_1, \ell_1) = n_1$, and vanishes otherwise. This means that the only $n_1$ term that doesn't vanish is when $n_1 = (k_1, \ell_1)$, as long as $(k_0k_1, \ell_1) = (k_1, \ell_1)$. Thus, the coefficient of $\ell^{-w}k^{-s}$ on the left-hand side of (3.3) is

$$\begin{cases} \chi(k)\chi'(\ell)\chi_{\ell_0}(k_0)\chi_{\ell_0}\left(\frac{k_1}{(k_1,\ell_1)}\right)^2 (k_1,\ell_1), & \text{if } (k_0k_1,\ell_1) = (k_1,\ell_1); \\ 0 & \text{otherwise.} \end{cases}$$

This can be further simplified to

$$\begin{cases} \chi(k)\chi'(\ell)\chi_{\ell_0}(k_0)(k_1,\ell_1), & \text{if } (k_0k_1,\ell_1) = (k_1,\ell_1) \text{ and } (\ell_0, k_1/(k_1,\ell_1)) = 1; \\ 0 & \text{otherwise.} \end{cases}$$

Reflexively, it is shown that the coefficient of $\ell^{-w}k^{-s}$ on the right-hand side of (3.3) is

$$\begin{cases} \chi'(\ell)\chi(k)\tilde{\chi}_{k_0}(\ell_0)(\ell_1,k_1), & \text{if } (\ell_0\ell_1,k_1) = (\ell_1,k_1) \text{ and } (k_0, \ell_1/(\ell_1,k_1)) = 1; \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that the coefficients will be equal as long as they are both non-zero. Thus, it remains to show that they are both non-vanishing simultaneously. That is, we must show that

$$\left[\begin{array}{c}(k_0k_1,\ell_1)=(k_1,\ell_1)\\ \text{and}\\ (\ell_0,\frac{k_1}{(k_1,\ell_1)})=1\end{array}\right] \iff \left[\begin{array}{c}(\ell_0\ell_1,k_1)=(\ell_1,k_1)\\ \text{and}\\ (k_0,\frac{\ell_1}{(\ell_1,k_1)})=1\end{array}\right].$$

We see that this is indeed true after observing that, for $a,b,c\in\mathbb{N}$, we have

$$\left(a,\frac{b}{(b,c)}\right)=1 \iff (b,ac)=(b,c).$$

□

In light of the above theorem, we now define our double Dirichlet series as follows: Let $\chi$ and $\chi'$ be characters modulo $8MN$. Then define

$$Z(s,w;\chi,\chi')=\sum_{(d,2MN)=1}\frac{L^{(2MN)}(s,\chi_{d_0}\chi)\chi'(d_0)P^{(\chi)}_{d_0,d_1}(s)}{d^w}. \tag{3.6}$$

We note at this point that it is easily shown that, for fixed $d=d_0d_1^2$ and $m=m_0m_1^2$ with $\Re s,\Re w\geq\frac{1}{2}$, we have the bounds $|P^{(\chi)}_{d_0,d_1}(s)|\ll d_1^{\varepsilon}$ and $|Q^{(\chi')}_{m_0,m_1}(w)|\ll m_1^{\varepsilon}$. Applying the functional equations (3.1) and (3.2), we therefore have

$$|P^{(\chi)}_{d_0,d_1}(s)|\ll\begin{cases}d_1^{1-2\Re s+\varepsilon}, & \Re s<\frac{1}{2};\\ d_1^{\varepsilon}, & \Re s\geq\frac{1}{2},\end{cases} \qquad |Q^{(\chi')}_{m_0,m_1}(w)|\ll\begin{cases}m_1^{1-2\Re w+\varepsilon}, & \Re w<\frac{1}{2};\\ m_1^{\varepsilon}, & \Re w\geq\frac{1}{2}.\end{cases} \tag{3.7}$$

## 3.2 Functional equations

We recall that $M$ and $N$ are odd prime numbers or unity, possibly equal. From this point on in the paper, we shall use the following notation: Let $\chi$ and $\chi'$ be quadratic primitive characters of squarefree conductors $k$ and $j$ respectively, where $j,k\mid MN$, and if $M=N$, then $j=k=M=N$, and let $\psi,\ \psi'$ be primitive characters with conductors dividing 8. We first derive the following expansion of the regions of absolute convergence of the key series involved.

**Lemma 3.2.** *We have the following two series representations of $Z(s,w;\chi\psi,\chi'\psi')$. We have*

$$Z(s,w;\chi\psi,\chi'\psi') = \sum_{(d,2MN)=1} \frac{L^{(2MN)}(s,\chi_{d_0}\chi\psi)\chi'\psi'(d)P^{(\chi\psi)}_{d_0,d_1}(s)}{d^w}$$

*which is absolutely convergent on the set*

$$R_1^{(1)} := \{\Re s \leq 0, \Re w + \Re s > 3/2\} \cup \{0 < \Re s \leq 1, \Re s/2 + \Re w > 3/2\} \cup \{\Re s, \Re w > 1\},$$

*except for a possible polar line $\{s = 1\}$, and*

$$Z(s,w;\chi\psi,\chi'\psi') = \sum_{(m,2MN)=1} \frac{L^{(2MN)}(w,\tilde{\chi}_{m_0}\chi'\psi')\chi\psi(m)Q^{(\chi'\psi')}_{m_0,m_1}(w)}{m^s}$$

*which is absolutely convergent on the set*

$$R_1^{(2)} := \{\Re w \leq 0, \Re s + \Re w > 3/2\} \cup \{0 < \Re w \leq 1, \Re w/2 + \Re s > 3/2\} \cup \{\Re s, \Re w > 1\},$$

*except for a possible polar line $\{w = 1\}$.*

*Proof.* Applying the bounds for the Dirichlet $L$-function (2.4) and the bounds for the correction polynomials (3.7) to the definition (3.6), aside from the polar line $\{s = 1\}$, we have the following bounds.

$$Z(s,w;\chi,\chi') \ll \begin{cases} [k(1+|\Im s|)]^{1/2-\Re s}, & \Re s \leq -\varepsilon, \Re w > 3/2 + \varepsilon - \Re s; \\ [k(1+|\Im s|)]^{(1-\Re s)/2+\varepsilon} & -\varepsilon < \Re s < 1+\varepsilon, \Re w > 3/2 + \varepsilon - \Re s/2; \\ 1, & \Re s, \Re w \geq 1+\varepsilon. \end{cases} \tag{3.8}$$

Applying the switch of summation (3.3), in the same way, we have

$$Z(s,w;\chi,\chi') \ll \begin{cases} [j(1+|\Im w|)]^{1/2-\Re w}, & \Re w \leq -\varepsilon, \Re s > 3/2+\varepsilon-\Re w; \\ [j(1+|\Im w|)]^{(1-\Re w)/2+\varepsilon}, & -\varepsilon < \Re w < 1+\varepsilon, \Re s > 3/2+\varepsilon-\Re w/2; \\ 1, & \Re s, \Re w \geq 1+\varepsilon, \end{cases} \tag{3.9}$$

except for a possible polar line $\{w = 1\}$. As we take $\varepsilon$ to zero, we obtain the desired sets. □

We now proceed with derivation of functional equations for $Z$. Due to the summation switch formula (3.3), we have

$$Z(s,w;\chi\psi,\chi'\psi') = \sum_{(m,2MN)=1} \frac{L^{(2MN)}(w,\tilde{\chi}_{m_0}\chi'\psi')(\chi\psi)(m)Q_{m_0,m_1}^{(\chi'\psi')}(w)}{m^s}.$$

We can further apply the functional equation for the $Q$ factor (3.2) to obtain

$$Z(s,w;\chi\psi,\chi'\psi') = \sum_{(m,2MN)=1} \frac{L^{(2MN)}(w,\tilde{\chi}_{m_0}\chi'\psi')(\chi\psi)(m)\chi'(m_1^2)Q_{m_0,m_1}^{(\chi'\psi')}(1-w)}{m^s m_1^{2(w-1/2)}}, \tag{3.10}$$

which holds for $(s,w) \in R_1^{(2)}$. The next step is to apply the functional equation for Dirichlet $L$-functions in order to change the $w$ in the $L$-function to $1-w$. This would allow us to switch summation again to obtain $Z$ in its original form, but with a change in variables.

We shall define the following Euler product function: For a character $\chi^\star$ and a positive integer $P$, we define

$$K_P(w;\chi^\star) = \prod_{p|P} \left(1 - \frac{\chi^\star(p)}{p^{1-w}}\right)^{-1} \left(1 - \frac{\chi^\star(p)}{p^w}\right). \tag{3.11}$$

Applying the Dirichlet functional equation along with (2.2) and the above, we now have

$$L^{(2MN)}(w,\tilde{\chi}_{m_0}\chi'\psi') = \pi^{w-\frac{1}{2}}\frac{\Gamma\left(\frac{1-w+\hat{\kappa}'}{2}\right)}{\Gamma\left(\frac{w+\hat{\kappa}'}{2}\right)}K_{2MN}(w;\tilde{\chi}_{m_0}\chi'\psi')(C_{\psi'}jm_0)^{\frac{1}{2}-w}L^{(2MN)}(1-w,\tilde{\chi}_{m_0}\chi'\psi'), \quad (3.12)$$

where $\hat{\kappa}' = \kappa(\tilde{\chi}_{m_0}\chi'\psi')$.

We need to break down some of these parts for further manipulation. Recalling that $(m_0, 2MN) = 1$, if $p$ is prime and does not divide $m_0$, we have

$$K_p(w;\tilde{\chi}_{m_0}\chi^\star) = \frac{\chi^\star(p^2)p - p^2 + \chi_p(m_0)\chi^\star(p)(p^{2-w} - p^{1+w})}{\chi^\star(p^2)p^{2w} - p^2}. \quad (3.13)$$

Indeed,

$$\begin{aligned} K_p(w;\tilde{\chi}_{m_0}\chi^\star) &= \left(\frac{p^{1-w}}{p^{1-w}-(\tilde{\chi}_{m_0}\chi^\star)(p)}\right)\left(\frac{p^w-(\tilde{\chi}_{m_0}\chi^\star)(p)}{p^w}\right) \\ &= \frac{(p-p^{1-w}(\tilde{\chi}_{m_0}\chi^\star)(p))}{(p-p^w(\tilde{\chi}_{m_0}\chi^\star)(p))}\cdot\frac{(p+p^w(\tilde{\chi}_{m_0}\chi^\star)(p))}{(p+p^w(\tilde{\chi}_{m_0}\chi^\star)(p))} \\ &= \frac{p^2+p^{w+1}(\tilde{\chi}_{m_0}\chi^\star)(p)-p^{2-w}(\tilde{\chi}_{m_0}\chi^\star)(p)-p\chi^\star(p^2)}{p^2-\chi^\star(p^2)p^{2w}}. \end{aligned}$$

If for $P \in \mathbb{N}$ and a Dirichlet character $\chi^\star$ we set

$$F_P^{(\chi^\star)}(w) = \begin{cases} 1, & P = 1; \\ \frac{\chi^\star(P^2)P-P^2}{\chi^\star(P^2)P^{2w}-P^2}, & \text{else}, \end{cases} \quad (3.14)$$

$$G_P^{(\chi^\star)}(w) = \begin{cases} 0, & P = 1; \\ \frac{\chi^\star(P)(P^{2-w}-P^{1+w})}{\chi^\star(P^2)P^{2w}-P^2}, & \text{else}, \end{cases} \quad (3.15)$$

then from (3.13) and the definition (3.11) we have the identity

$$K_p(w;\tilde{\chi}_{m_0}\chi^\star) = F_p^{(\chi^\star)}(w) + \chi_p(m_0)G_p^{(\chi^\star)}(w), \quad (3.16)$$

which holds for prime $p$ not dividing $m_0$, or $p = 1$. Noting that $K_P(w; \chi^\star)$ is multiplicative in $P$, and using (3.16) above, we now have the useful expression

$$K_{MN}(w; \tilde{\chi}_{m_0}\chi'\psi') = F_M^{(\chi'\psi')} \cdot F_N^{(\chi'\psi')}(w) + \chi_M(m_0) F_N^{(\chi'\psi')} \cdot G_M^{(\chi'\psi')}(w)$$
$$+ \chi_N(m_0) F_M^{(\chi'\psi')} \cdot G_N^{(\chi'\psi')}(w) + \chi_{MN}(m_0) G_M^{(\chi'\psi')} \cdot G_N^{(\chi'\psi')}(w). \quad (3.17)$$

We note that this holds true even if $M = N$. Indeed, in this case, by the definition (3.11), we have $K_{MN}(w; \tilde{\chi}_{m_0}\chi'\psi') = 1$. This is consistent with (3.17), since according to definitions (3.14) and (3.15) we have $G_N^{(\chi'\psi')}(w) = 0$ and $F_N^{(\chi'\psi')}(w) = 1$.

Next, we see from (2.10) that

$$\frac{\Gamma\left(\frac{1-w+\hat{\kappa}'}{2}\right)}{\Gamma\left(\frac{w+\hat{\kappa}'}{2}\right)} = \frac{\Gamma\left(\frac{1-w}{2}\right)}{\Gamma\left(\frac{w}{2}\right)} \cot\left(\frac{\pi w}{2}\right)^{\hat{\kappa}'}. \quad (3.18)$$

We shall find it useful to remove the dependency of $\hat{\kappa}'$ on $m_0$, or rather, exploit that the dependency is only on its residue modulo 4. Hence, define $\kappa' = \kappa(\chi'\psi')$. Now suppose that $f$ is a function of $\hat{\kappa}'$. By sieving out by congruence classes modulo 4, we see that

$$f(\hat{\kappa}') = \frac{1}{2}(1 + \psi_{-1}(m_0)) f(\kappa') + \frac{1}{2}(1 - \psi_{-1}(m_0)) f(1 - \kappa').$$

Hence, we have

$$\cot\left(\frac{\pi w}{2}\right)^{\hat{\kappa}'} = \frac{1}{2}(1 + \psi_{-1}(m_0)) \cot\left(\frac{\pi w}{2}\right)^{\kappa'} + \frac{1}{2}(1 - \psi_{-1}(m_0)) \cot\left(\frac{\pi w}{2}\right)^{(1-\kappa')}. \quad (3.19)$$

For brevity, for a character $\chi^\star$, we define

$$S(s, w; m, \chi^\star) := \frac{L^{(2MN)}(1 - w, \tilde{\chi}_{m_0}\chi'\psi')\chi^\star(m) Q_{m_0,m_1}^{(\chi'\psi')}(1 - w)}{m^{s+w-1/2}}.$$

Applying the functional equation for Dirichlet $L$-functions (3.12) to the identity (3.10) along with (3.19), and using Lemma 3.2 we now get

$$Z(s,w;\chi\psi,\chi'\psi') = \frac{1}{2}\pi^{w-1/2}\frac{\Gamma\left(\frac{1-w}{2}\right)}{\Gamma\left(\frac{w}{2}\right)}(jC_{\psi'})^{1/2-w}$$
$$\times \sum_{(m,2MN)=1} K_{2MN}(w;\tilde{\chi}_{m_0}\chi'\psi')S(s,w;m,\chi\psi)$$
$$\times\left[(1+\psi_{-1}(m_0))\cot\left(\frac{\pi w}{2}\right)^{\kappa'} + (1-\psi_{-1}(m_0))\cot\left(\frac{\pi w}{2}\right)^{(1-\kappa')}\right], \quad (3.20)$$

for $(s,w) \in R_1^{(2)}$ except for possible polar lines $\{s=1\}$ and $\{w=1\}$. There are two properties of the function $S$ which we shall make use of, stated in the following lemma.

**Lemma 3.3.** *For two characters $\chi^\star$ and $\chi^{\star\star}$ and an integer $m$, the following two properties hold for $(s,w) \in R_1^{(2)}$, except for possible polar lines $\{s=1\}$ and $\{w=1\}$.*

$$\text{(i)} \qquad S(s,w;m,\chi^\star)\chi^{\star\star}(m) = S(s,w;m,\chi^\star\chi^{\star\star}),$$
$$\text{(ii)} \qquad \sum_{(m,2MN)=1} S(s,w;m,\chi^\star) = Z(s+w-\tfrac{1}{2},1-w;\chi^\star,\chi'\psi').$$

□

We can apply the identity (3.17) and Lemma 3.3 to (3.20), and use the bounds (2.11) and (2.12) to come to the following functional equation.

**Theorem 3.4.** *Let $\chi$ and $\chi'$ be primitive Dirichlet characters modulo squarefree $k$ and $j$ respectively, where $j,k \mid MN$, and if $M=N$, then $j=k=M=N$, and let $\psi$ and $\psi'$ be Dirichlet characters modulo 8. There exist functions $a_n^{(\chi',\psi')}(w;\psi^\star)$ for $n \mid MN$ and $\psi^\star$ a Dirichlet character modulo 8 which are holomorphic except for possible poles at the positive integers, and countably many poles on the line $\Re w = 1$, bounded above by $O((16\pi)^{|\Re w|}(1+|w|)^{1/2-\Re w})$ uniformly in $j$ and $k$ away from the poles such that for*

$(s,w) \in R_1^{(2)}$ *away from possible polar lines* $\{s=1\}$ *and* $\{w=1\}$ *we have*

$$Z(s,w;\chi\psi,\chi'\psi') = \\ j^{1/2-w} \sum_{n|MN} A_n^{(\chi')}(w) \sum_{\psi^\star \in \widehat{(\mathbb{Z}/8\mathbb{Z})^*}} a_n^{(\chi',\psi')}(w;\psi^\star) Z(s+w-\tfrac{1}{2}, 1-w; \chi\chi_n\psi\psi^\star, \chi'\psi'),$$

*where* $R_1^{(2)}$ *is defined in Lemma 3.2, we have*

$$\begin{aligned} A_1^{(\chi')}(w) &= F_M^{(\chi')}(w) F_N^{(\chi')}(w), \\ A_M^{(\chi')}(w) &= F_N^{(\chi')}(w) G_M^{(\chi')}(w), \\ A_N^{(\chi')}(w) &= F_M^{(\chi')}(w) G_N^{(\chi')}(w), \\ A_{MN}^{(\chi')}(w) &= G_M^{(\chi')}(w) G_N^{(\chi')}(w), \end{aligned}$$

*and the* $F$ *and* $G$ *functions are defined in (3.14) and (3.15).*

By similar methods, we can obtain a second functional equation under the transformation $(s,w) \mapsto (1-s, s+w-\frac{1}{2})$ by applying the functional equation (3.1) to the definition (3.6), followed by the functional equation for $L$-functions. The result of this similarly lengthy derivation is the following second functional equation.

**Theorem 3.5.** *Let* $\chi$ *and* $\chi'$ *be Dirichlet characters modulo squarefree* $k$ *and* $j$ *respectively, where* $j,k \mid MN$*, and if* $M=N$*, then* $j=k=M=N$*, and let* $\psi$ *and* $\psi'$ *be Dirichlet characters modulo 8. There exist functions* $b_n^{(\chi,\psi)}(s;\psi^\star)$ *for* $n \mid MN$ *and* $\psi^\star$ *a Dirichlet character modulo 8 which are holomorphic except for possible poles at the positive integers, and countably many poles on the line* $\Re s = 1$*, bounded above by* $O((16\pi)^{|\Re s|}(1+|s|)^{1/2-\Re s})$ *uniformly in* $j$ *and* $k$ *away from the poles such that for* $(s,w) \in R_1^{(1)}$ *away from possible* *polar lines $\{s = 1\}$ and $\{w = 1\}$ we have*

$$Z(s, w; \chi\psi, \chi'\psi') = \\ k^{1/2-s} \sum_{n|MN} A_n^{(\chi)}(s) \sum_{\psi^\star \in \widehat{(\mathbb{Z}/8\mathbb{Z})^*}} b_n^{(\chi,\psi)}(s;\psi^\star) Z(1-s, s+w-\tfrac{1}{2}; \chi\psi, \chi'\chi_n\psi'\psi^\star),$$

*where $R_1^{(1)}$ is defined in Lemma 3.2, and the A functions are as in Theorem 3.4.*

*Remark* 3.6. We make note of an analytic subtlety: We note that although the $a_n$, $b_n$, and $A_n$ functions above have poles, they do not contribute poles to $Z(s, w; \chi\psi, \chi'\psi')$; indeed, it will be proven in Proposition 3.10 that the only possible poles of $Z$ are the polar lines $\{s = 1\}$, $\{w = 1\}$, and $\{s + w = 3/2\}$. Looking at (3.20), though the gamma and cotangent factors together have poles at either the even or odd positive integers, and the $K$ factor has countably many poles along the line $\Re w = 1$ (cf. (3.13)), these poles nonetheless do not produce poles on the right-hand side. The equation (3.20) essentially results from the application of the functional equation for $L$-functions (3.12) to the identity (3.10), and subsequently sieving out by congruence classes of $m$ modulo 4. The last step introduces coefficients with poles from the gamma and cotangent factors. This is a manifestation of a phenomenon that is observed in the functional equation for $L$-functions: In (3.12), we know that the $L$-function can only have a pole at $w = 1$, yet on the right-hand side, the gamma function produces poles which are mitigated by the trivial zeros of the $L$-function. It is precisely these poles which appear in the coefficients of (3.20). Additionally, although the $K$ function has poles, these are mitigated by corresponding zeros of the $L$ function due to removal of the Euler factors at primes dividing $2MN$.

It shall be useful to note the following properties of the $A$ coefficients, which follow directly from the definitions.

**Lemma 3.7.** *Let $\chi$ be a Dirichlet character modulo $q$. Then the following properties* *hold.*

*If* $q = MN$, *then*

$$\begin{aligned} A_1^{(\chi)}(w) &= 1, \\ A_M^{(\chi)}(w) = A_N^{(\chi)}(w) &= A_{MN}^{(\chi)}(w) = 0. \end{aligned}$$

*If* $q = M$ *then*

$$\begin{aligned} A_1^{(\chi)}(w) &= F_N^{(\chi)}(w), \\ A_N^{(\chi)}(w) &= G_N^{(\chi)}(w), \\ A_M^{(\chi)}(w) &= A_{MN}^{(\chi)}(w) = 0. \end{aligned}$$

*If* $q = N$ *then*

$$\begin{aligned} A_1^{(\chi)}(w) &= F_M^{(\chi)}(w), \\ A_M^{(\chi)}(w) &= G_M^{(\chi)}(w), \\ A_N^{(\chi)}(w) &= A_{MN}^{(\chi)}(w) = 0. \end{aligned}$$

*Moreover, the following asymptotics hold, if* $P \neq 1$ *and* $(P, q) = 1$.

$$\begin{aligned} F_P^{(\chi)}(w) &\asymp \begin{cases} 1, & \Re w < 1 - \varepsilon; \\ P^{2-2\Re w}, & \Re w > 1 + \varepsilon, \end{cases} \\ G_P^{(\chi)}(w) &\asymp \begin{cases} P^{-\Re w}, & \Re w < \frac{1}{2}; \\ P^{\Re w - 1}, & \frac{1}{2} \leq \Re w < 1 - \varepsilon; \\ P^{1-\Re w}, & \Re w > 1 + \varepsilon. \end{cases} \end{aligned}$$

□

## 3.3 Analytic continuation

We shall make use of the transformations

$$\begin{aligned} \alpha: \ (s,w) &\mapsto (s+w-\tfrac{1}{2}, 1-w), \\ \beta: \ (s,w) &\mapsto (1-s, s+w-\tfrac{1}{2}), \end{aligned}$$

and the functional equations of Theorems 3.4 and 3.5. Indeed, we see that if $Z$ is holomorphic on a set $R \in \mathbb{C}^2$, then by applying the functional equations, it is analytically continued to $R \cup \alpha(R)$ if $R \cap \alpha(R) \neq \emptyset$, and likewise with $\beta$. We shall also make use of the following results from the theory of several complex variables in [7]. We have

**Definition 3.8.** An open set $\Omega$ in $\mathbb{C}^m$ is called a tube if there is an open set $\omega$ in $\mathbb{R}^m$, called the base of $\Omega$, such that $\Omega = \{s \mid \Re s \in \omega\}$.

We denote by $\hat{R}$ the convex hull of a subset $R \subset \mathbb{R}^m$ or $\mathbb{C}^m$. It is easily seen that the convex hull $\hat{\Omega}$ of a tube $\Omega$ with base $\omega$ is a tube with base $\hat{\omega}$.

**Proposition 3.9.** *If $\Omega$ is a connected tube, then any holomorphic function $f$ in $\Omega$ can be extended to a holomorphic function $\hat{f}$ in $\hat{\Omega}$.*

We now prove the following analytic continuation and growth estimate.

**Proposition 3.10.** *Let $\chi$ and $\chi'$ be characters modulo $k$ and $j$ respectively, where $j,k \mid MN$, and if $M = N$ then $j = k = M = N$. The function*

$$\tilde{Z}(s,w;\chi,\chi') = (s-1)(w-1)(s+w-\tfrac{3}{2})Z(s,w;\chi,\chi')$$

*is holomorphic in $\mathbb{C}^2$ and is polynomially bounded in the sense that, given $C_1 > 0$, there exists $C_2 > 0$ such that $\tilde{Z}(s,w;\chi,\chi') \ll [jk(1+|\Im s|)(1+|\Im w|)]^{C_2}$ whenever $|\Re s|, |\Re w| < C_1$.*

*Proof.* Applying Lemma 3.2, we see that the function $Z(s,w;\chi\psi,\chi'\psi')$ continues to a holomorphic function on

$$\begin{aligned} R_1 &:= R_1^{(1)} \cup R_1^{(2)} \\ &= \{\Re s \leq 0, \Re s + \Re w > 3/2\} \\ &\quad \cup \{0 < \Re s < 3/2, \Re w > \min(3/2 - \Re s/2, 3 - 2\Re s)\} \\ &\quad \cup \{\Re s \geq 3/2, \Re s + \Re w > 3/2\}\,. \end{aligned}$$

We define the set

$$R_1' := \{\Re s + \Re w > 3/2, \Re s > 3/2\} \subset R_1$$

and apply the functional equation Theorem 3.5 to the set

$$\beta(R_1') = \{\Re w > 1, \Re s < -1/2\},$$

which is away from the poles of the $A$ and $b$ coefficients. Thus, $Z$ extends meromorphically to

$$R_2 := R_1 \cup \beta(R_1'),$$

with possible poles introduced at $\beta\{w = 1\} = \{s + w = 3/2\}$.

Next, we define

$$R_2' := \{\Re s + \Re w > 3/2, \Re w > 3/2\} \subset R_2$$

and apply the functional equation of Theorem 3.4 to the set

$$\alpha(R_2') = \{\Re s > 1, \Re w < -1/2\},$$

and for the same reason, the poles of the coefficients are excluded. Now $Z$ continues to

$$R_3 := R_2 \cup \alpha(R_2').$$

We further define

$$R_3' := \{\Re s > 1, \Re w < -1/2\} \subset R_3$$

and apply the functional equation of Theorem 3.5 to the set

$$\beta(R_3') = \{\Re s < 0, \Re s + \Re w < 0\},$$

and again $s$ is outside of the region of the poles of the coefficients. This continues $Z$ to

$$R_3 \cup \beta(R_3').$$

Finally, we define

$$R_3'' := \{\Re s < -1/2, \Re w > 1\} \subset R_3$$

and apply the functional equation of Theorem 3.4 to

$$\alpha(R_3'') = \{\Re w < 0, \Re s + \Re w < 0\},$$

and $w$ is away from the poles of the coefficients. This continues $Z$ to

$$R_4 := R_3 \cup \beta(R_3') \cup \alpha(R_3'').$$

We see that $R_4$ is $\mathbb{C}^2$ without the tube defined by a solid polygon with vertices contained within a disc around the origin of radius 2. Applying Proposition 3.9, we see that $\tilde{Z}$ is holomorphic in $\mathbb{C}^2$, since the convex closure of $R_4$ is $\mathbb{R}^2$. We endeavour to show that $\tilde{Z}$ is polynomially bounded as per the Proposition in the annulus $9 < |\Re s|^2 + |\Re w|^2 < 16$. We see that this is true in the region intersecting $R_1$, due to the bounds (3.8) and (3.9). We therefore see that $\tilde{Z}$ is polynomially bounded in the annulus. Due to the maximum modulus principle, we see that $\tilde{Z}$ is polynomially bounded inside the annulus as well. Finally, it is evident that we can now apply the same reasoning to a disc of any size. $\square$

## 3.4 Convexity bound

The notion of convexity is not canonically defined for double Dirichlet series as it is in the case of (single) Dirichlet series; in the latter case, we have a single functional equation which reflects the region of absolute convergence, and interpolating the bounds produces a convexity bound between the two. In the case of double Dirichlet series, things are more complicated. Firstly, our bounds in the region of absolute convergence depend on our knowledge of the bounds on $L(s,\chi)$ on average. Secondly, we have 6 functional equations to choose from to apply to this region. If we assume the Lindelöf hypothesis on average, namely Theorem 2.3, then we can carefully choose a functional equation to apply in order to minimize the resulting convexity bound from application of the Phragmén-Lindelöf convexity principle. Adapting Theorem 5.53 of [8] gives the following theorem.

**Theorem 3.11.** *Suppose $f$ is holomorphic on an open neighbourhood of $a \leq \sigma \leq b$ and $f$ is such that $|f(s)| \leq \exp(|s|^A)$ for some $A \geq 0$ and $a \leq \sigma \leq b$. Suppose also that $f$ is bounded on the boundary of the strip as follows:*

$$|f(a+it)| \leq M_a(1+|t|)^{\alpha},$$
$$|f(b+it)| \leq M_b(1+|t|)^{\beta}$$

*for $t \in \mathbb{R}$. Then we have*

$$|f(\sigma+it)| \leq M_a^{\ell(\sigma)} M_b^{1-\ell(\sigma)} (1+|t|)^{\alpha\ell(\sigma)+\beta(1-\ell(\sigma))}$$

*for $s$ within the strip, where $\ell$ is the linear function such that $\ell(a) = 1$ and $\ell(b) = 0$.*

We shall require some initial bounds. We let $\chi$ and $\chi'$ be quadratic Dirichlet characters with conductors $k$ and $j$ respectively, and $\psi, \psi'$ are characters modulo 8. We first assume that $\Re s = 1/2$ and $\Re w = 1 + \varepsilon$. Writing $d = d_0 d_1^2$, where $d_0$ is squarefree, we apply the identity (2.2) to obtain

$$Z(s,w;\chi\psi,\chi'\psi') = \sum_{\substack{(d_0,2MN)=1\\ \mu^2(d_0)=1}} \frac{L(s,\chi_{d_0}\chi\psi)}{d_0^w} \sum_{(d_1,2MN)=1} \frac{(\chi'\psi')(d_0 d_1^2)}{d_1^{2w}} \prod_{p|2MNd_1} \left(1 - \frac{(\chi_{d_0}\chi\psi)(p)}{p^s}\right) P_{d_0,d_1}^{(\chi\psi)}(s).$$

We can bound the product by

$$\prod_{p|2MNd_1} \left(1 - \frac{\chi_{d_0}\chi\psi(p)}{p^s}\right) = \sum_{n|2MNd_1} \frac{\mu(n)(\chi_{d_0}\chi\psi)(n)}{n^s} \ll \sum_{n|2MNd_1} 1 \ll (MNd_1)^{\varepsilon},$$

and (3.7) gives us $P_{d_0,d_1}^{(\chi\psi)}(s) \ll d_1^{\varepsilon}$, making the outer sum bounded above by $(MN)^{\varepsilon}$. Applying Cauchy-Schwarz to the remaining sum and dividing into dyadic intervals, we see that it is less than or equal to

$$\sum_{n=0}^{\infty} \Bigg( \sum_{\substack{d_0 \sim 2^n\\ (d_0,2MN)=1\\ \mu^2(d_0)=1}} |L(s,\chi_{d_0}\chi\psi)|^4 \Bigg)^{1/4} \Bigg( \sum_{\substack{d_0 \sim 2^n\\ (d_0,2MN)=1\\ \mu^2(d_0)=1}} \frac{1}{d_0^{\frac{4}{3}(1+\varepsilon)}} \Bigg)^{3/4}.$$

Employing the average bound of Theorem 2.3 for the first factor, and applying a trivial bound for the second, we have

$$\begin{aligned} Z(s,w;\chi\psi,\chi'\psi') &\ll (MN)^{\varepsilon} \sum_{n=0}^{\infty} (2^{n+1}k|s|)^{1/4+\varepsilon/4} (2^n)^{-\frac{1}{4}-\varepsilon} \\ &\asymp M^{\varepsilon} k^{1/4+\varepsilon} |s|^{1/4+\varepsilon} \sum_{n=0}^{\infty} 2^{-n\varepsilon}. \end{aligned}$$

Since the last sum is absolutely convergent, we have

$$Z(s,w;\chi\psi,\chi'\psi') \ll (MN)^{\varepsilon} k^{1/4+\varepsilon} (1+|s|)^{1/4+\varepsilon}, \Re s = 1/2, \Re w = 1+\varepsilon, \tag{3.21}$$

and by the switch of summation formula (3.3) we also obtain

$$Z(s,w;\chi\psi,\chi'\psi') \ll (MN)^{\varepsilon} j^{1/4+\varepsilon}(1+|w|)^{1/4+\varepsilon}, \; \Re s = 1+\varepsilon, \Re w = 1/2. \tag{3.22}$$

We use the functional equation Theorem 3.5 with $\Re s = -\varepsilon$ and $\Re w = 1+\varepsilon$ and apply (3.22) on the right-hand side in order to obtain a bound for $Z(s,w;\rho,\rho')$. Looking at the coefficient bounds in Lemma 3.7, we pick up a factor of $N^{1/2+\varepsilon}$. Further, we see that the resulting twisting characters on the right-hand side will have conductors $(k,j) \in \{(N,M),(N,1)\}$, so that we have

$$Z(s,w;\rho,\rho') \ll N^{1/2+\varepsilon} M^{1/4+\varepsilon}(1+|s|)^{1/2+\varepsilon}(1+|s+w|)^{1/4+\varepsilon}, \; \Re s = -\varepsilon, \Re w = 1+\varepsilon. \tag{3.23}$$

Likewise with the functional equation Theorem 3.4 applied to (3.21), we have the symmetric bound

$$Z(s,w;\rho,\rho') \ll M^{1/2+\varepsilon} N^{1/4+\varepsilon}(1+|w|)^{1/2+\varepsilon}(1+|s+w|)^{1/4+\varepsilon}, \; \Re s = 1+\varepsilon, \Re w = -\varepsilon. \tag{3.24}$$

We now interpolate convexly between these bounds along the two diagonal lines in $\mathbb{C}^2$ to obtain the bound

$$Z(s,w;\rho,\rho') \ll [(1+|s|)(1+|w|)(1+|s+w|)]^{1/4+\varepsilon}(MN)^{3/8+\varepsilon}, \; \Re s = \Re w = 1/2,$$

which we henceforth designate as the convexity bound.

# Chapter 4

# Approximate functional equations

## 4.1 A symmetric functional equation

We introduce a succession of applications of the functional equations in the special case of $(s, w) = (1/2, 1/2 - z)$ for some $z \in \mathbb{C}$ with $\Re z > 0$. We recall that $\rho$ and $\rho'$ are primitive quadratic Dirichlet characters of conductors $N$ and $M$ respectively. We first apply Theorem 3.4, which after observing the coefficient properties of Lemma 3.7 gives

$$Z(\tfrac{1}{2}, \tfrac{1}{2} - z; \rho, \rho') = \sum_{\psi^\star \in \widehat{(\mathbb{Z}/8\mathbb{Z})^*}} \Big[ M^z F_N^{(\rho')} a_1^{(\rho', \psi_1)}(\tfrac{1}{2} - z; \psi^\star) Z(\tfrac{1}{2} - z, \tfrac{1}{2} + z; \rho\psi^\star, \rho') + M^z G_N^{(\rho')}(\tfrac{1}{2} - z) a_N^{(\rho', \psi_1)}(\tfrac{1}{2} - z; \psi^\star) Z(\tfrac{1}{2} - z, \tfrac{1}{2} + z; \psi^\star, \rho') \Big]. \quad (4.1)$$

We then apply the functional equation Theorem 3.5 (and again use Lemma 3.7) which further gives

$$
\begin{aligned}
&Z(\tfrac{1}{2},\tfrac{1}{2}-z;\rho,\rho') = \\
&\sum_{\psi^\star,\psi^{\star\star}\in\widehat{(\mathbb{Z}/8\mathbb{Z})^*}} \Big[(MN)^z F_N^{(\rho')}(\tfrac{1}{2}-z)F_M^{(\rho)}(\tfrac{1}{2}-z)c_{1,1}^{(\rho',\rho)}(z;\psi^\star,\psi^{\star\star})Z(\tfrac{1}{2}+z,\tfrac{1}{2};\rho\psi^\star,\rho'\psi^{\star\star}) \\
&+ (MN)^z F_N^{(\rho')}(\tfrac{1}{2}-z)G_M^{(\rho)}(\tfrac{1}{2}-z)c_{1,M}^{(\rho',\rho)}(z;\psi^\star,\psi^{\star\star})Z(\tfrac{1}{2}+z,\tfrac{1}{2};\rho\psi^\star,\psi^{\star\star}) \\
&+ M^z G_N^{(\rho')}(\tfrac{1}{2}-z)F_M^{(\psi_1)}(\tfrac{1}{2}-z)F_N^{(\psi_1)}(\tfrac{1}{2}-z)c_{N,1}^{(\rho',\psi_1)}(z;\psi^\star,\psi^{\star\star})Z(\tfrac{1}{2}+z,\tfrac{1}{2};\psi^\star,\rho'\psi^{\star\star}) \\
&+ M^z G_N^{(\rho')}(\tfrac{1}{2}-z)F_N^{(\psi_1)}(\tfrac{1}{2}-z)G_M^{(\psi_1)}(\tfrac{1}{2}-z)c_{N,M}^{(\rho',\psi_1)}(z;\psi^\star,\psi^{\star\star})Z(\tfrac{1}{2}+z,\tfrac{1}{2};\psi^\star,\psi^{\star\star}) \\
&+ M^z G_N^{(\rho')}(\tfrac{1}{2}-z)F_M^{(\psi_1)}(\tfrac{1}{2}-z)G_N^{(\psi_1)}(\tfrac{1}{2}-z)c_{N,N}^{(\rho',\psi_1)}(z;\psi^\star,\psi^{\star\star})Z(\tfrac{1}{2}+z,\tfrac{1}{2};\psi^\star,\rho'\rho\psi^{\star\star}) \\
&+ M^z G_N^{(\rho')}(\tfrac{1}{2}-z)G_M^{(\psi_1)}(\tfrac{1}{2}-z)G_N^{(\psi_1)}(\tfrac{1}{2}-z)c_{N,MN}^{(\rho',\psi_1)}(z;\psi^\star,\psi^{\star\star})Z(\tfrac{1}{2}+z,\tfrac{1}{2};\psi^\star,\rho\psi^{\star\star})\Big], \quad (4.2)
\end{aligned}
$$

where

$$c_{n,m}^{(\chi,\chi')}(z;\psi^\star,\psi^{\star\star}) = a_n^{(\chi,\psi_1)}(\tfrac{1}{2}-z;\psi^\star)b_m^{(\chi',\psi^\star)}(\tfrac{1}{2}-z;\psi^{\star\star}).$$

We note that, in the case where $M = N \neq 1$, we may not apply the functional equation Theorem 3.5 to $Z(\frac{1}{2}-z,\frac{1}{2}+z;\psi^\star,\rho')$ in (4.1) because Theorem 3.5 is only valid if $j = k = M = N$, but here we have $k = 1 \neq N$. Nonetheless, equation (4.2) still holds: Indeed, in the case where $M = N$, the term with $Z(\frac{1}{2}-z,\frac{1}{2}+z;\psi^\star,\rho')$ of (4.1) vanishes because $G_N^{(\rho')}(\frac{1}{2}-z) = 0$, and so only the first term of (4.2) will remain (note also that $G_M^{(\rho)}(\frac{1}{2}-z) = 0$ in this case).

Looking at (3.14) and (3.15), for a character $\chi^\star$ whose modulus is coprime to $P$, we see that

$$
\begin{aligned}
F_P^{(\chi^\star)}(\tfrac{1}{2}-z) &= \frac{P-P^2}{P^{1-2z}-P^2} = \frac{P^{-1}-1}{P^{-2z-1}-1}, \\
G_P^{(\chi^\star)}(\tfrac{1}{2}-z) &= \frac{\chi^\star(P)(P^{3/2+z}-P^{3/2-z})}{P^{1-2z}-P^2} = P^{z-1/2}\left(\frac{\chi^\star(P)(1-P^{-2z})}{P^{-2z-1}-1}\right). \quad (4.3)
\end{aligned}
$$

Therefore, setting

$$\Phi := \{(\rho,\rho'),\ (\rho,\psi_1),\ (\psi_1,\rho'),\ (\psi_1,\psi_1),\ (\psi_1,\rho'\rho),\ (\psi_1,\rho)\} \tag{4.4}$$

we have

$$Z(\tfrac{1}{2},\tfrac{1}{2}-z;\rho,\rho') = \sum_{\substack{(\chi,\chi')\in\Phi \\ \psi,\psi'\in\widehat{(\mathbb{Z}/8\mathbb{Z})^*}}} \beta_{\psi,\psi'}^{(\chi,\chi')}\omega_{\psi,\psi'}^{(\chi,\chi')}(z)(\gamma_{\psi,\psi'}^{(\chi,\chi')})^z Z(\tfrac{1}{2}+z,\tfrac{1}{2};\chi\psi,\chi'\psi'), \tag{4.5}$$

where we absorb the $F_P^{\chi^\star}(\frac{1}{2}-z)$ factors and parenthetical expression of (4.3) for the $G_P^{\chi^\star}(\frac{1}{2}-z)$ factors, as well as the $c_{n,m}^{(\chi,\chi')}(z;\psi^\star,\psi^{\star\star})$ factors into the $\omega_{\psi,\psi'}^{(\chi,\chi')}(z)$ functions, and collect the remaining factors into the $\beta_{\psi,\psi'}^{(\chi,\chi')}$ and $\gamma_{\psi,\psi'}^{(\chi,\chi')}$ coefficients. Hence we see that for $\Re z > 0$, the $\omega_{\psi,\psi'}^{(\chi,\chi')}(z)$ functions are holomorphic satisfying the bound

$$\omega_{\psi,\psi'}^{(\chi,\chi')}(z) \ll (1+|\Im z|)^{\Re z} \tag{4.6}$$

uniformly in $M$ and $N$. Thus, we obtain the following upper bounds.

Table 4.1: Coefficient upper bounds

| $(\chi,\chi')$ | $M \neq N$ | | $M = N$ | |
|---|---|---|---|---|
| | $\beta_{\psi,\psi'}^{(\chi,\chi')}$ bound | $\gamma_{\psi,\psi'}^{(\chi,\chi')}$ bound | $\beta_{\psi,\psi'}^{(\chi,\chi')}$ bound | $\gamma_{\psi,\psi'}^{(\chi,\chi')}$ bound |
| $(\rho,\rho')$ | 1 | $MN$ | 1 | $N^2$ |
| $(\rho,\psi_1)$ | $M^{-1/2}$ | $M^2N$ | 0 | 0 |
| $(\psi_1,\rho')$ | $N^{-1/2}$ | $MN$ | 0 | 0 |
| $(\psi_1,\psi_1)$ | $M^{-1/2}N^{-1/2}$ | $M^2N$ | 0 | 0 |
| $(\psi_1,\rho'\rho)$ | $N^{-1}$ | $MN^2$ | 0 | 0 |
| $(\psi_1,\rho)$ | $M^{-1/2}N^{-1}$ | $M^2N^2$ | 0 | 0 |

## 4.2 Approximate functional equations

The following lemma essentially takes the preceding functional equation a step further by opening the first sum of $Z$. We have the following lemma.

**Lemma 4.1.** *There exist smooth, rapidly decaying functions $V(\xi;t)$ and $V_{\psi,\psi'}^{(\chi,\chi')}(\xi;t)$ such that for any constant $X>0$ one has*

$$Z(\tfrac{1}{2},\tfrac{1}{2}+it;\rho,\rho') = X^{-it}\sum_{(d,2MN)=1}\frac{L^{(2MN)}(\frac{1}{2},\chi_{d_0}\rho)\rho'(d)P^{(\rho)}_{d_0,d_1}(\frac{1}{2})}{d^{1/2}}V\left(\frac{d}{X};t\right)$$
$$+X^{it}\sum_{\substack{(\chi,\chi')\in\Phi\\ \psi,\psi'\in\widehat{(\mathbb{Z}/8\mathbb{Z})^*}}}\beta_{\psi,\psi'}^{(\chi,\chi')}\sum_{(d,2MN)=1}\frac{L^{(2MN)}(\frac{1}{2},\tilde{\chi}_{d_0}\chi'\psi')\chi\psi(d)Q^{(\chi'\psi')}_{d_0,d_1}(\frac{1}{2})}{d^{1/2}}V_{\psi,\psi'}^{(\chi,\chi')}\left(\frac{dX}{\gamma_{\psi,\psi'}^{(\chi,\chi')}};t\right),$$

*where $\beta_{\psi,\psi}^{(\chi,\chi')}$ and $\gamma_{\psi,\psi'}^{(\chi,\chi')}$ satisfy the bounds listed in Table 4.1, and we have the bounds*

$$\begin{aligned} V_{\psi,\psi'}^{(\chi,\chi')}(\xi;t) &\ll |\xi|^{-B}(1+|t|)^B,\\ V(\xi;t) &\ll |\xi|^{-B} \end{aligned}$$

*uniformly in $\xi$ and $t$ for any number $B>0$.*

*Proof.* Let $B>0$, $H$ be an even, holomorphic function with $H(0)=1$ satisfying the growth estimate

$$H(z) \ll_{\Re z,A} (1+|z|)^{-A}. \tag{4.7}$$

for any $A>0$. We consider the integral

$$I(c,X,t) = \frac{1}{2\pi i}\int_{(1)} X^{cz}\left(\frac{4^{\frac{1}{2}+it+cz}-4}{4^{\frac{1}{2}+it}-4}\right)^2 Z(\tfrac{1}{2},\tfrac{1}{2}+it+cz;\rho,\rho')H(z)\frac{dz}{z}. \tag{4.8}$$

for a real number $c$, a positive real number $X>0$, and a fixed real number $t$. Examining the expression when $c=1$, the fraction cancels the pole of the $Z$ factor at $z=1/2-it$. We apply a shift of the contour to $\Re z=-1$, picking up the pole at $z=0$, whence we obtain

$$I(1,X,t) = Z(\tfrac{1}{2},\tfrac{1}{2}+it;\rho,\rho') + \frac{1}{2\pi i}\int_{(-1)} X^z \left(\frac{4^{\frac{1}{2}+it+z}-4}{4^{\frac{1}{2}+it}-4}\right)^2 Z(\tfrac{1}{2},\tfrac{1}{2}+it+z;\rho,\rho')H(z)\frac{dz}{z}.$$

We now apply a change of variables $z \mapsto -z$, arriving at

$$Z(\tfrac{1}{2},\tfrac{1}{2}+it;\rho,\rho') = I(1,X,t) + I(-1,X,t).$$

Applying the functional equation (4.5) and the switch of summation formula (3.3) and expanding the $Z$ functions using the definition (3.6) gives

$$\begin{aligned} I(-1,X,t) = X^{it} &\sum_{\substack{(\chi,\chi')\in\Phi \\ \psi,\psi'\in\widehat{(\mathbb{Z}/8\mathbb{Z})^*}}} \beta_{\psi,\psi'}^{(\chi,\chi')} \\ &\times \sum_{(m,2MN)=1} \frac{L^{(2MN)}(\frac{1}{2},\tilde{\chi}_{m_0}\chi'\psi')(\chi\psi)(m)Q_{m_0,m_1}^{(\chi'\psi')}(\frac{1}{2})}{m^{1/2}} V_{\psi,\psi'}^{(\chi,\chi')}\left(\frac{mX}{\gamma_{\psi,\psi'}^{(\chi,\chi')}};t\right), \quad (4.9)\end{aligned}$$

where

$$V_{\psi,\psi'}^{(\chi,\chi')}(\xi;t) = \frac{1}{2\pi i}\int_{(1)}\left(\frac{4^{1/2+it-z}-4}{4^{1/2+it}-4}\right)^2 \xi^{-z+it}\omega_{\psi,\psi'}^{(\chi,\chi')}(z-it)H(z)\frac{dz}{z}.$$

We wish to obtain an upper bound for $V_{\psi,\psi'}^{(\chi,\chi')}(\xi;t)$. Moving the contour to $B$ (recalling that there are no poles of $\omega_{\psi,\psi'}^{(\chi,\chi')}$ in this region), and bounding by taking the absolute value of the integrand and using the bound (4.6), we have

$$V_{\psi,\psi'}^{(\chi,\chi')}(\xi;t) \ll |\xi|^{-B}(1+|t|)^B$$

uniformly in $\xi$ and $t$. Changing the summation variable from $m$ to $d$ in (4.9), we obtain the first term in the statement of the lemma.

Looking at $I(1, X, t)$, we have

$$I(1, X, t) = X^{-it} \sum_{(d,2MN)=1} \frac{L^{(2MN)}(\frac{1}{2}, \chi_{d_0}\rho)\rho'(d)P^{(\rho)}_{d_0,d_1}(\frac{1}{2})}{d^{1/2+it}} V\left(\frac{d}{X}; t\right),$$

where

$$V(\xi; t) = \frac{1}{2\pi i} \int_{(1)} \left( \frac{4^{\frac{1}{2}+it+z} - 4}{4^{1/2+it} - 4} \right)^2 \xi^{-z} H(z) \frac{dz}{z}.$$

Also, it is immediate that we have the bound

$$V(\xi; t) \ll |\xi|^{-B}$$

uniformly in $\xi$ and $t$. □

We now wish to truncate the sums above, accruing an error. This is the object of the following Lemma.

**Lemma 4.2.** *Let $A$ be a large positive constant, $t$ be a real number, and $V(\xi; t)$ be a rapidly decaying function in $\xi$ satisfying*

$$V(\xi; t) \ll |\xi|^{-B}(1 + |t|)^B,$$

*uniformly in $\xi$ and $t$ for any number $B > 0$, let $\chi$ be a character modulo $k$, and let $a$ be an arithmetic function satisfying*

$$a(d) \ll d^{\varepsilon}$$

*uniformly in $d$. Then we can truncate the double sum*

$$\sum_{(d,2MN)=1} \frac{L^{(2MN)}(\frac{1}{2}, \chi_{d_0}\chi)\chi'(d)a(d)}{d^{1/2}} V\left(\frac{d}{Y}; t\right)$$

at $d < Y^{1+\varepsilon}$, accruing an error that is bounded above by

$$O((1+|t|)^{2A/\varepsilon} k^{1/4+\varepsilon} Y^{-A}).$$

*Proof.* The $L$-function is bounded asymptotically by $(d_0 k)^{1/4+\varepsilon}$ due to the Phragmén-Lindelöf convexity bound (2.4). The $V$ factor is bounded by its argument to an arbitrarily large power $-B$. Applying this gives an error that is bounded above by

$$(1+|t|)^{2B} \sum_{d > P^{1+\varepsilon}} \frac{(d_0 k)^{1/4+\varepsilon}}{d^{1/2-\varepsilon}} \left(\frac{d}{Y}\right)^{-B}.$$

We apply the bound $d_0 \leq d$ to see that this is bounded above by

$$(1+|t|)^{2B} k^{1/4+\varepsilon} Y^B \sum_{d > P^{1+\varepsilon}} \frac{1}{d^B} \ll k^{1/4+\varepsilon} Y^B (Y^{1+\varepsilon})^{1-B},$$

and the result follows. □

In order to bound $Z(\frac{1}{2}, \frac{1}{2}+it; \rho, \rho')$, by applying a smooth partition of unity as in [1], it now suffices to bound

$$D_{W,a}(Y; t, \chi'\psi', \chi\psi) := \sum_{(d,2MN)=1} \frac{L^{(2MN)}(\frac{1}{2}, \chi_{d_0}\chi'\psi')\chi\psi(d)a(d)}{d^{1/2}} W\left(\frac{d}{Y}; t\right)$$

for $t$ a real number, $\psi, \psi' \in (\widehat{\mathbb{Z}/8\mathbb{Z}})^*$, a smooth function $W$ with support on $[1,2]$ satisfying

$$W(x;t) \ll_B x^{-B}(1+|t|)^B$$

uniformly in $x$ and $t$ for any $B > 0$, an arithmetic function $a$ satisfying the bound $a(d) \ll d^{\varepsilon}$, and the following conditions, according to each of the two sums in Lemma

4.1: either $(\chi, \chi') \in \Phi$ (cf. (4.4)) with conductors $k$ and $j$ respectively, and

$$1 \le Y \le \left( \frac{\gamma_{\psi,\psi'}^{(\chi,\chi')}}{X} \right)^{1+\varepsilon},$$

or $(\chi, \chi') = (\rho', \rho)$ and

$$1 \le Y \le X^{1+\varepsilon}.$$

Expanding according to the Dirichlet functional equation, and further truncating that sum expresses $D_{W,a}(Y; t, \chi', \chi)$ as a double finite character sum, allowing us to apply Heath-Brown's large sieve estimate Corollary 2.5. The result of this is the following lemma.

**Lemma 4.3.** *We have the bound*

$$D_{W,a}(Y; t, \chi', \chi) \ \ll \ (1 + |t|)^{2/\varepsilon} (MN)^{\varepsilon} \left( Y^{1+\varepsilon} + (Yj)^{1/2+\varepsilon} \right)^{1/2+\varepsilon}$$

*uniformly in $t$, $Y$, $j$, and $k$.*

*Proof.* Applying Lemma 4.2 above, we have

$$D_{W,a}(Y; t, \chi', \chi) = \sum_{\substack{(d,2MN)=1 \\ d < Y^{1+\varepsilon}}} \frac{L^{(2MN)}(\frac{1}{2}, \chi_{d_0}\chi)\chi'(d)a(d)}{d^{1/2}} W\left(\frac{d}{Y}; t\right)$$
$$+ O((1 + |t|)^{2/\varepsilon} j^{1/4+\varepsilon} Y^{-1})$$

Applying (2.2) and (2.5), we have

$$D_{W,a}(Y; t, \chi', \chi) = 2 \sum_{\substack{(d,2MN)=1 \\ d < Y^{1+\varepsilon}}} \frac{\chi'(d)}{d^{1/2}} \prod_{p | 2MN} \left( 1 - \frac{(\chi_{d_0}\chi)(p)}{p^{1/2}} \right) a(d) W\left(\frac{d}{Y}; t\right)$$
$$\times \sum_{n=1}^{\infty} \frac{(\chi\chi_{d_0})(n)}{n^{1/2}} G_{\kappa_{\chi'}}\left( \frac{n}{\sqrt{c_0 d_0 j}} \right) + O((1 + |t|)^{2/\varepsilon} j^{1/4+\varepsilon} Y^{-1}), \quad (4.10)$$

where $c_0$ is given in (2.6) and $G_\kappa$ is given in (2.7). Because of the rapid decay of $G_\kappa$ and $W$, we can truncate the $n$-sum at $n < (Yj)^{1/2+\varepsilon}$.

Applying the bounds (2.8) and $a(d) \ll d^\varepsilon$, the error obtained by this is bounded by

$$(MN)^\varepsilon (1+|t|)^B \sum_{d<Y^{1+\varepsilon}} \frac{1}{d^{1/2-\varepsilon}} \left(1+\frac{d}{Y}\right)^{-B} \sum_{n>(Yj)^{1/2+\varepsilon}} \frac{1}{n^{1/2}} \left(1+\frac{n}{\sqrt{c_0 d_0 j}}\right)^{-B}$$

for any large positive number $B$. Indeed, this is bounded above by

$$\begin{aligned} &(MN)^\varepsilon (1+|t|)^B Y^B j^{B/2} \sum_{d<Y^{1+\varepsilon}} d^{-B-1/2+\varepsilon} \sum_{n>(Yj)^{1/2+\varepsilon}} n^{-B-1/2} d^{B/2} \\ \asymp\ &(MN)^\varepsilon (1+|t|)^B Y^B j^{B/2} (Y^{1+\varepsilon})^{-B/2+1/2+\varepsilon} (Yj)^{(1/2+\varepsilon)(-B+1/2)}. \end{aligned}$$

We see this is bounded above by $(MN)^\varepsilon (1+|t|)^B (Yj)^{-\varepsilon B+1}$. We choose $B$ large enough so that $\varepsilon B - 1 \geq 1$, and so we can choose $B = 2/\varepsilon$. Thus, we have

$$\begin{aligned} D_{W,a}(Y;t,\chi',\chi) = 2 &\sum_{\substack{(d,2MN)=1 \\ d<Y^{1+\varepsilon}}} \prod_{p|2MN} \left(1 - \frac{(\chi_{d_0}\chi)(p)}{p^{1/2}}\right) \\ &\times \sum_{n \leq (Yj)^{1/2+\varepsilon}} \frac{(\chi\chi_{d_0})(n)\chi'(d)}{d^{1/2} n^{1/2}} a(d) W\left(\frac{d}{Y};t\right) G_{\kappa_{\chi'}}\left(\frac{n}{\sqrt{c_0 d_0 j}}\right) \\ &\qquad + O((MN)^\varepsilon (1+|t|)^{2/\varepsilon} (Yj)^{-1}) + O((1+|t|)^{2/\varepsilon} j^{1/4+\varepsilon} Y^{-1}). \end{aligned}$$

We wish to apply Heath-Brown's large sieve Corollary 2.5 to the double sum, but to do this, we need to separate the $n$ and $d_0$ dependence on the $G$ function. Hence, we apply Mellin inversion to render the main term above as

$$\begin{aligned} 2 \int_{(\varepsilon)} \int_{(\varepsilon)} \tilde{G}_{\kappa_{\chi'}}(s) j^{s/2} &\sum_{d<Y^{1+\varepsilon}} \prod_{p|2MN} \left(1 - \frac{(\chi_{d_0}\chi)(p)}{p^{1/2}}\right) \\ &\times \sum_{n<(Yj)^{1/2+\varepsilon}} (c_0 d_0)^{s/2} a(d) \frac{(\chi\chi_{d_0})(n)\chi'(d)}{d^{1/2+w} n^{1/2+s}} \tilde{W}(w;t) Y^w \, dw ds. \end{aligned}$$

Looking at the summand, for $\Re s = \varepsilon$, we have

$$(c_0 d_0)^{s/2} a(d) \frac{\chi'(d)}{d^{1/2+w}} \ll d^{-1/2+\varepsilon}$$

and

$$\frac{\chi(n)}{n^{1/2+s}} \ll n^{-1/2+\varepsilon},$$

so that we can now apply Corollary 2.5, and the result follows. $\square$

# Chapter 5

# Proof of Theorem 1.1

We first apply Lemma 4.1 which gives

$$Z(\tfrac{1}{2}, \tfrac{1}{2}+it; \rho, \rho') = X^{it} \sum_{\substack{(\chi,\chi')\in\Phi \\ \psi,\psi'\in\widehat{(\mathbb{Z}/8\mathbb{Z})^*}}} \beta_{\psi,\psi'}^{(\chi,\chi')} D_{V,Q}\left(\frac{\gamma_{\psi,\psi'}^{(\chi,\chi')}}{X}; t, \chi', \chi\right) + X^{-it} D_{V,P}(X; t, \rho, \rho'),$$

where the subscripts for $D$ in the first term are $V = V_{\psi,\psi'}^{(\chi,\chi')}(\xi, t)$ and $Q = Q_{d_0,d_1}^{(\chi'\psi')}(\frac{1}{2})$, and the subscripts for $D$ in the second term are $V = V(\xi, t)$ and $P = P_{d_0,d_1}^{(\chi\psi)}(\frac{1}{2})$. Applying Lemma 4.3 further gives

$$\begin{aligned} Z(\tfrac{1}{2}, \tfrac{1}{2}+it, \rho, \rho') \ll\ & (1+|t|)^{2/\varepsilon}(MN)^{\varepsilon}\left[\left(X + (NX)^{1/2}\right)^{1/2+\varepsilon}\right] \\ & + (1+|t|)^{2/\varepsilon}(MN)^{\varepsilon} \max_{\substack{(\chi,\chi')\in\Phi \\ \psi,\psi'\in\widehat{(\mathbb{Z}/8\mathbb{Z})^*}}} \left[\beta_{\psi,\psi'}^{(\chi,\chi')}\left(\left(\frac{\gamma_{\psi,\psi'}^{(\chi,\chi')}}{X}\right) + \left(\frac{\gamma_{\psi,\psi'}^{(\chi,\chi')} j}{X}\right)^{1/2}\right)^{1/2+\varepsilon}\right]. \end{aligned}$$

Here, $\Phi$ is given by (4.4), and $j$ is the conductor of $\chi'$.

We initially set $X = M^a N^b$ and eventually choose optimized values for $a$ and $b$. We apply the bounds for the $\beta$'s and $\gamma$'s from Table 4.1. We omit the $\varepsilon$ variables in each exponent for compactness. We obtain the following table. We note that only the cases of the first term and $(\chi, \chi') = (\rho, \rho')$ apply in the case of $M = N$.

Table 5.1: Term bounds

| $(\chi, \chi')$ | $\beta^{(\chi,\chi')}_{\psi,\psi'}$ bound | $\gamma^{(\chi,\chi')}_{\psi,\psi'}$ bound | Term bound |
|---|---|---|---|
| First term | Not applicable | | $M^{a/2}N^{b/2} + M^{a/4}N^{b/4+1/4}$ |
| $(\rho, \rho')$ | $1$ | $MN$ | $M^{1/2-a/2}N^{1/2-b/2} + M^{1/2-a/4}N^{1/4-b/4}$ |
| $(\rho, \psi_1)$ | $M^{-1/2}$ | $M^2N$ | $M^{1/2-a/2}N^{1/2-b/2}$ |
| $(\psi_1, \rho')$ | $N^{-1/2}$ | $MN$ | $M^{1/2-a/2}N^{-b/2} + M^{1/2-a/4}N^{-1/4-b/4}$ |
| $(\psi_1, \psi_1)$ | $M^{-1/2}N^{-1/2}$ | $M^2N$ | $M^{1/2-a/2}N^{-b/2}$ |
| $(\psi_1, \rho'\rho)$ | $N^{-1}$ | $MN^2$ | $M^{1/2-a/2}N^{-b/2} + M^{1/2-a/4}N^{-1/4-b/4}$ |
| $(\psi_1, \rho)$ | $M^{-1/2}N^{-1}$ | $M^2N^2$ | $M^{1/2-a/2}N^{-b/2} + M^{-a/4}N^{-1/4-b/4}$ |

From this data, we see that the optimal choice for $X$ is when $a = 2/3$ and $b = 1/3$. In this case, we obtain the subconvexity bound of

$$M^{1/3+\varepsilon}N^{1/6+\varepsilon} + M^{1/6+\varepsilon}N^{1/3+\varepsilon} \asymp (MN(M+N))^{1/6+\varepsilon}$$

(even in the case of $M = N$), compared to the convexity bound of $(MN)^{3/8+\varepsilon}$.

# Chapter 6

# Proof of Theorem 1.2

Here we present an application of our subconvexity bound. Let $h(y)$ be a smooth weight function as in Definition 2.1. Expanding as per (3.6) and by Mellin inversion we have

$$\begin{aligned}\int_{(2)} \tilde{h}(w) Z(\tfrac{1}{2}, w; \chi_N, \psi_1) X^w dw &= \sum_{(d,2N)=1} L^{(2N)}(\tfrac{1}{2}, \chi_{d_0 N}) P^{(\chi_N)}_{d_0,d_1}(\tfrac{1}{2}) \int_{(2)} \tilde{h}(w) (\tfrac{d}{X})^{-w} dw \\ &= \sum_{(d,2N)=1} L^{(2N)}(\tfrac{1}{2}, \chi_{d_0 N}) P^{(\chi_N)}_{d_0,d_1}(\tfrac{1}{2}) h(\tfrac{d}{X}).\end{aligned}$$

We move the contour of the integral on the left-hand side to $\Re w = 1/2$, picking up a residue at $w = 1$ due to the double pole of $Z(\frac{1}{2}, w; \chi_N, \psi_1)$ there. If we write its Laurent expansion as

$$Z(1/2, w; \chi_N, \psi_1) = \frac{\mu_N}{(w-1)^2} + \frac{\nu_N}{(w-1)} + \cdots$$

then we have

$$\begin{aligned}\sum_{(d,2N)=1} L^{(2N)}(\tfrac{1}{2}, \chi_{d_0 N}) P^{(\chi_N)}_{d_0,d_1}(\tfrac{1}{2}) h(d/X) &= [\nu_N \tilde{h}(1) + \mu_N \tilde{h}'(1)] X + \mu_N \tilde{h}(1) X \log X \\ &\quad + \int_{(1/2)} \tilde{h}(w) Z(\tfrac{1}{2}, w; \chi_N, \psi_1) X^w dw.\end{aligned}$$

Our subconvexity bound Theorem 1.1 implies

$$Z(\tfrac{1}{2}, \tfrac{1}{2} + it; \chi_N, \psi_1) \ll (1 + |t|)^{2/\varepsilon} N^{1/3+\varepsilon},$$

which we shall apply to the integral. We thus have

$$S(X; \chi_N) := \sum_{(d,2N)=1} L^{(2N)}(\tfrac{1}{2}, \chi_{d_0 N}) P^{(\chi_N)}_{d_0,d_1}(\tfrac{1}{2}) h(\tfrac{d}{X}) = a_N X \log X + b_N X + O(X^{1/2} N^{1/3+\varepsilon}) \tag{6.1}$$

for certain coefficients $a_N$ and $b_N$. More elementary analysis of $S(X; \chi_N)$ via Theorem 7.1 (cf. Chapter 7) below gives us the lower bounds $a_N, b_N \gg N^{-\varepsilon}$. If we now assume that

$$L^{(2N)}(\tfrac{1}{2}, \chi_{dN}) = L(\tfrac{1}{2}, \chi_{d_0 N}) \prod_{p | 2 d_1 N} (1 - p^{-1/2})$$

vanishes for $d \ll X$, then we get a contradiction as long as $a_N X \log X$ is greater than the error term. We see we can choose $X = N^{2/3+\varepsilon}$, as required.

We essentially combine two asymptotic formulas for $S(X; \chi_N)$: Looking at it elementarily via Theorem 7.1 (cf. Chapter 7) gives us a bad error term, but lower bounds on the coefficients. Looking at it analytically as above allows us to take advantage of the subconvexity bound proven for $Z$ in Theorem 1.1 in order to obtain a smaller error term.

# Chapter 7

# An asymptotic formula for an $L$-function sum

## 7.1 Result

Of particular importance for the smallest nonvanishing quadratic central value of twisted $L$-functions result is an asymptotic formula for the weighted and twisted $L$-function sum given by $S(X;\chi)$ in (6.1). Indeed, we already have an asymptotic formula, but it is important that we have a lower bound on the main term coefficients.

Let $N$ be a natural number, $X$ a large positive real number, and let $\chi$ be a quadratic primitive character modulo $N$. Let $h$ be a smooth weight function as defined in Definition 2.1, and define

$$S(X;\chi) = \sum_{(d,2N)=1} L^{(2N)}(\tfrac{1}{2}, \chi_{d_0}\chi) P^{(\chi)}_{d_0,d_1}(\tfrac{1}{2}) h(d/X).$$

We seek to obtain information on the main term coefficients of the asymptotic formula (6.1). We shall prove the following Theorem.

**Theorem 7.1.** *There exists $\delta > 0$ such that we have the asymptotic formula*

$$S(X;\chi) = a_N X \log X + b_N X + O(N^{3/8+\varepsilon} X^{1-\delta}),$$

*where*

$$N^{-\varepsilon} \ll a_N, b_N \ll N^{\varepsilon},$$

*uniformly in* $N$.

## 7.2 Proof of Theorem 7.1

According to the approximate functional equation for Dirichlet $L$-functions (2.5), we have

$$L(\tfrac{1}{2}, \chi_{d_0}\chi) = 2\sum_{n=1}^{\infty} \frac{(\chi_{d_0}\chi)(n)}{n^{1/2}} G_\kappa\left(\frac{n}{\sqrt{c_0 d_0 N}}\right),$$

where $G_\kappa$ is given in (2.7), and $c_0$ is given in (2.6). From (3.4) we have

$$P^{(\chi)}_{d_0,d_1}(\tfrac{1}{2}) = \sum_{f|d_1^2} \frac{\mu(f_0)(\chi_{d_0}\chi)(f_0)}{f_0^{1/2}},$$

where we write $f = f_0 f_1^2$ with $f_0$ squarefree. We shall also use the expansion

$$1 - \frac{(\chi_{d_0}\chi)(2)}{2^{1/2}} = \sum_{g|2} \frac{\mu(g)(\chi_{d_0}\chi)(g)}{g^{1/2}}.$$

Applying these expressions to $S(X;\chi)$ we have

$$S(X;\chi) = 2\sum_{(d_1,2N)=1}\ \sum_{(d_0,2N)=1}\ \sum_{f_1|d_1}\ \sum_{g|2f_1^2} \frac{\mu(g)(\chi_{d_0}\chi)(g)}{h^{1/2}} \times \sum_{n=1}^{\infty} \frac{(\chi_{d_0}\chi)(n)}{n^{1/2}} G_\kappa\left(\frac{n}{\sqrt{c_0 d_0 N}}\right) h\left(\frac{d_0 d_1^2}{X}\right). \quad (7.1)$$

For a subset $H \subset \mathbb{N}$, we define $S_H(X;\chi)$ to be the same as the expression (7.1) with the added condition in the $n$-sum of $ng \in H$. We use Mellin inversion for $G_\kappa$ to separate the variables, and by moving the $d_0$-sum to the inside, we get

$$S_H(X;\chi) = 2\int_{(1/2+\varepsilon)} N^{s/2} \sum_{(d_1,2N)=1} \sum_{f_1|d_1} \sum_{\substack{g|2f_1^2 \\ (g,N)=1}} \frac{\mu(g)\chi(g)}{g^{1/2}} \sum_{\substack{ng\in H \\ (n,N)=1}} \frac{\chi(n)}{n^{s+1/2}} \times \sum_{(d_0,2N)=1} \tilde{G}_\kappa(s)\chi_{d_0}(ng)c_0^{s/2}d_0^{s/2}h\left(\frac{d_0d_1^2}{X}\right)ds. \quad (7.2)$$

The variables $\kappa$ and $c_0$ depend on the residues of $d_0$ and $N$ modulo 4. Thus, given $\ell \in \{\pm 1\}$, we define $\kappa(\ell)$ and $c_0(\ell)$ to be the corresponding values for $d_0 \equiv \ell \pmod 4$. For convenience, for $\iota \in \{\pm 1\}$, we define $S_H(X;\chi,\iota,\ell)$ to be the same as (7.2) except that $\kappa$ and $c_0$ are replaced with $\kappa(\ell)$ and $c_0(\ell)$, and $\psi_\iota(d_0)$ is multiplied to the summand in the $d_0$-sum.

Observing that $\frac{1}{2}(1\pm\psi_{-1}(d_0))$ is the characteristic function of $d_0 \equiv \pm 1 \pmod 4$, the $d_0$-sum in the integrand of (7.2) is

$$\frac{1}{2}\sum_{(d_0,2N)=1}\sum_{\pm}(1\pm\psi_{-1}(d_0))\tilde{G}_{\kappa(\pm 1)}(s)c_0(\pm 1)^{s/2}\tilde{\chi}_{ng}(d_0){d_0}^{s/2}h\left(\frac{d_0d_1^2}{X}\right),$$

whence we see that

$$S_H(X;\chi) = \frac{1}{2}\sum_{\iota=\pm 1}\sum_{\ell=\pm 1}\operatorname{sgn}(1+\iota+\ell)S_H(X;\chi,\iota,\ell). \quad (7.3)$$

For treatment of the $d_0$-sum, for a positive real number $Y$ and a Dirichlet character $\psi$ we further define

$$T(s;Y,\psi) = \sum_{d_0=1}^{\infty}\psi(d_0)d_0^{s/2}h\left(\frac{d_0}{Y}\right),$$

where the sum is over squarefree $d_0$. With these simplifications, we obtain

$$S_H(X,\chi,\iota,\ell) = 2\int_{(1/2+\varepsilon)} (Nc_0(\ell))^{s/2}\tilde{G}_{\kappa(\ell)}(s) \sum_{(d_1,2N)=1}\sum_{f_1|d_1}\sum_{\substack{g|2f_1^2\\(g,N)=1}} \frac{\mu(g)\chi(g)}{g^{1/2}} \times \sum_{\substack{ng\in H\\(n,N)=1}} \frac{\chi(n)}{n^{s+1/2}} T(s;X/d_1^2,\chi_0^{(2N)}\tilde{\chi}_{ng}\psi_\iota)ds. \quad (7.4)$$

We shall choose $H=\square$ which we use to denote the set of positive squares, and denoting the complement of $H$ in $\mathbb{N}$ by $\bar{H}$, it is clear that

$$S(X;\chi) = S_\square(X;\chi) + S_{\bar{\square}}(X;\chi).$$

In order to estimate the size of $T(s;Y,\psi)$, we observe that there are order $Y$ squarefree numbers up to $Y$. If $\psi$ is a principal character and $\Re s=\varepsilon$, we therefore expect this sum to be roughly of size $Y^{1+\varepsilon}$. If $\psi$ is non-principal, the oscillations will give us a Pólya-Vinogradov type estimate.

Indeed, with this last point in mind, we step back to explain the main idea of the proof: The sum $S_H(X,\chi,\psi_\iota,\ell)$ will hence only be large when $\tilde{\chi}_{ng}\psi_\iota$ is principal, that is, precisely when $H=\square$ and $\iota=1$, and will be small otherwise. To this end, we have the following asymptotic formula.

**Lemma 7.2.** *For $\Re s>0$, we have*

$$T(s;Y,\chi_0^{(m)}) = \frac{\tilde{h}(1+s/2)}{\zeta(2)}\prod_{p|m}\left(1+\frac{1}{p}\right)^{-1} Y^{1+s/2} + U(s;Y,m)$$

*with $U(s;Y,m)$ holomorphic and*

$$U(s;Y,m) \ll |s|^{1/2-\Re s/2+\varepsilon} m^\varepsilon Y^{1/2+\Re s/2+\varepsilon},$$

*uniformly in $Y$ and $\Im s$. Further, if $\psi=\chi_0^{(m)}\tilde{\psi}$, where $\tilde{\psi}$ is a nontrivial quadratic primitive* *character with conductor $c$, then*

$$T(s;Y,\psi) \ll |cs|^{1/2-\Re s/2+\varepsilon} m^{\varepsilon} Y^{1/2+\Re s/2+\varepsilon}$$

*uniformly in $\Im s$, $m$, and $Y$.*

*Proof.* Via Mellin inversion, we have

$$T(s;Y,\chi_0^{(m)}\tilde{\psi}) = \int_{(1+\Re s/2+\varepsilon)} \mathcal{T}_s(z)\tilde{h}(z)Y^z dz,$$

where we have the generating function

$$\mathcal{T}_s(z) \;=\; \sum_{d=1}^{\infty} \frac{\mu^2(d)(\chi_0^{(m)}\tilde{\psi})(d)}{d^{z-s/2}} = \frac{L(z-s/2,\tilde{\psi})}{L(2z-s,\tilde{\psi})} \prod_{p|m} \left(1+\frac{1}{p^{z-s/2}}\right)^{-1}.$$

If $\tilde{\psi}$ is the trivial character then the $L$-function in the numerator is just the zeta function, and therefore has a pole at $z = 1+s/2$. Due to the $1/L(2z-s,\tilde{\psi})$ factor, all other poles lie in $\Re z < \Re s/2$. We move the contour to $(1/2+\Re s/2+\varepsilon)$, and in the case where $\tilde{\psi}$ is trivial, we pick up the residue

$$\frac{\tilde{h}(1+s/2)}{\zeta(2)} \prod_{p|m} \left(1+\frac{1}{p}\right)^{-1} Y^{1+s/2}. \tag{7.5}$$

Due to (2.9) along with the convexity bound

$$L(z-s/2,\tilde{\psi}) \ll [c(1+|\Im z - \Im s/2|)]^{1/2-\Re s/2+\varepsilon}$$

obtained from (2.4), we bound the resulting integral by

$$|cs|^{1/2-\Re s/2+\varepsilon} m^{\varepsilon} Y^{1/2+\Re s/2+\varepsilon}.$$

□

### 7.2.1 Main term

As explained above, the main contribution will come from $S_{\square}(X;\chi,1,\ell)$, which we will bound from below. Looking at the expansion (7.4) with $\iota=\ell=1$ and $H=\square$, since $ng$ is square, and recalling that $g$ is squarefree, we have $n=gm^2$ for $m\in\mathbb{N}$, so the inner sum is

$$\sum_{\substack{ng\in\square\\(n,N)=1}}\frac{1}{n^{s+1/2}}T(s;X/d_1^2,\chi_0^{(2ngN)})=\frac{1}{g^{s+1/2}}\sum_{(m,N)=1}\frac{1}{m^{2s+1}}T(s;X/d_1^2,\chi_0^{(2gmN)}).$$

By Lemma 7.2, the above expression is

$$\frac{1}{g^{s+1/2}}\sum_{(m,N)=1}\frac{1}{m^{2s+1}}\left[\prod_{p|2gmN}\left(1+\frac{1}{p}\right)^{-1}\frac{\tilde{h}(1+s/2)}{\zeta(2)}\left(\frac{X}{d_1^2}\right)^{1+s/2}+U(s;X/d_1^2,2gmN)\right]$$

where $U(s;X/d_1^2,2gmN)$ is holomorphic for $\Re s>0$ and

$$U(s;X/d_1^2,2gmN)\ll|s|^{1/2-\Re s/2+\varepsilon}(gmN)^{\varepsilon}\left(X/d_1^2\right)^{1/2+\Re s/2}.$$

Referring to (7.4), and moving the contour to $(\varepsilon)$, the error term is bounded above by

$$\sum_{(d_1,2N)=1}\int_{(\varepsilon)}|Nc_0(\ell)|^{\Re s/2}|\tilde{G}_{\kappa(\ell)}(s)||s|^{1/2-\Re s/2+\varepsilon}\\ \times\sum_{f_1|d_1}\sum_{\substack{g|2f_1^2\\(g,N)=1}}\frac{1}{g^{s+1}}\sum_{(m,N)=1}\frac{(gmN)^{\varepsilon}}{m^{2s+1}}\left(\frac{X}{d_1^2}\right)^{1/2+\Re s/2}ds. \quad (7.6)$$

Bounding the sums absolutely and using the fact that $\tilde{G}_{\kappa(\ell)}(s)$ decays rapidly in fixed vertical strips, we see that this is bounded above by $N^{\varepsilon}X^{1/2+\varepsilon}$. As for the main term, through a calculation we have the following result.

**Lemma 7.3.** *We have the identity*

$$\sum_{(m,N)=1} \frac{1}{m^{2s+1}} \prod_{p|2gmN} \left(1+\frac{1}{p}\right)^{-1} = \zeta(2s+1)E_0(s)E_1(s;g),$$

*where*

$$\begin{aligned} E_0(s) &= \left(1+\frac{1}{2}-\frac{1}{2^{2s+2}}\right)^{-1} \prod_p \left(1+\frac{1}{p}\right)^{-1}\left(1+\frac{1}{p}-\frac{1}{p^{2s+2}}\right) \\ &\quad \times \prod_{p|N}\left(1-\frac{1}{p^{2s+1}}\right)\left(1+\frac{1}{p}-\frac{1}{p^{2s+2}}\right)^{-1}, \\ E_1(s;g) &= \prod_{\substack{p|g \\ p\neq 2}} \left(1+\frac{1}{p}-\frac{1}{p^{2s+2}}\right)^{-1}. \end{aligned}$$

*Proof.* This is a straightforward but monotonous calculation which we omit. □

Applying this, we now have

$$\begin{aligned} S_\square(X,\chi,1,\ell) = \frac{2}{\zeta(2)}\int_{(\varepsilon)} &\tilde{h}(1+s/2)(c_0(\ell)N)^{s/2}X^{1+s/2}\tilde{G}_{\kappa(\ell)}(s)\zeta(2s+1)E_0(s) \\ &\times \sum_{(d_1,2N)=1} d_1^{-2-s} \sum_{f_1|d_1} \sum_{\substack{g|2f_1^2 \\ (g,N)=1}} \frac{\mu(g)}{g^{s+1}} E_1(s;g)ds + O(N^\varepsilon X^{1/2+\varepsilon}). \end{aligned}$$

Since $E(s;g)$ is multiplicative in $g$, we can further collapse the $g$-sum above into an Euler product. Hence we have

$$\begin{aligned} S_\square(X,\chi,1,\ell) &= \frac{2}{\zeta(2)}\int_{(\varepsilon)} \tilde{h}(1+s/2)(c_0(\ell)N)^{s/2}X^{1+s/2}\tilde{G}_{\kappa(\ell)}(s)\zeta(2s+1)E_0(s)H(s)ds \\ &\quad +O(N^\varepsilon X^{1/2+\varepsilon}), \end{aligned} \tag{7.7}$$

where

$$H(s) = \sum_{(d_1,2N)=1} d_1^{-2-s} \sum_{f_1|d_1} \prod_{p|2f_1} \left(1-p^{-s-1}E_1(s;p)\right).$$

We have the following estimates for $H$.

**Lemma 7.4.** *There exists $K > 0$ such that $1/2 \leq H(0) \leq K$ and $|H'(0)| \leq K$.*

*Proof.* Let $\Re s \geq 0$, and for convenience define

$$H(s; d_1) = \sum_{f_1 | d_1} \prod_{p | 2f_1} \left(1 - p^{-s-1} E_1(s; p)\right).$$

Because $0 < E_1(0; p) \leq 1$, we have $1/2 \leq 1 - p^{-1} E_1(0; p) < 1$, and so taking $d_1 = 1$, we have $H(0) \geq 1/2$. Hence by the same reasoning,

$$|H(0; d_1)| \leq \sum_{n | d_1} 1 \ll d_1^{\varepsilon},$$

and so we see that $H(0)$ is absolutely bounded above. In order to show that $H'(0)$ is absolutely bounded, because we have

$$H'(0) = \sum_{(d_1, 2N)=1} d_1^{-2} (H'(0; d_1) - H(0; d_1) \log d_1),$$

it suffices to show that $H'(0; d_1) \ll d_1^{\varepsilon}$, and for this, it suffices to show that $E_1'(0; p)$ is absolutely bounded, which is easily seen via taking the logarithmic derivative. □

We shall also need the following bounds for the $E_0$ function.

**Lemma 7.5.** *We have the bounds*

$$N^{-\varepsilon} \ll E_0(0), \ E_0'(0) \ll N^{\varepsilon}.$$

*Proof.* We have

$$E_0(0) = K \prod_p \left(1 - \frac{1}{p^2} \left(1 + \frac{1}{p}\right)^{-1}\right) \prod_{p | N} \left(1 - \frac{1}{p}\right) \left(1 + \frac{1}{p} - \frac{1}{p^2}\right)^{-1}$$

for some constant $K$. The infinite product converges to a constant. As for the finite product, we have

$$N^{-\varepsilon} \ll \left(\frac{\varphi(N)}{N}\right)^2 \ll \prod_{p|N}\left(1-\frac{1}{p}\right)\left(1+\frac{1}{p}\right)^{-1} \ll \prod_{p|N}\left(1-\frac{1}{p}\right)\left(1+\frac{1}{p}-\frac{1}{p^2}\right)^{-1} \ll 1,$$

and the result follows.

Next, we treat the derivative. We look at

$$E_0'(0) = E_0(0)\frac{d}{ds}\log E_0(s)|_{s=0}.$$

We have

$$\begin{aligned}\frac{d}{ds}\log E_0(s)|_{s=0} &= \frac{d}{ds}\sum_{p\nmid 2N}\log\left(1-\frac{1}{p^{2s+2}}\left(1+\frac{1}{p}\right)^{-1}\right)+\sum_{p|N}\log\left(1-\frac{1}{p^{2s+1}}\right)\Bigg|_{s=0} \\ &= \sum_{p\nmid 2N}\frac{2\log p}{p^2+p-1}+\sum_{p|N}\frac{2\log p}{p-1},\end{aligned}$$

and it is clear that this is bounded below by a constant and above by $N^{\varepsilon}$, as required. □

We wish to move the contour of integration of (7.7) from $(\varepsilon)$ to $(-\varepsilon)$. In doing so, we pick up a double pole, since $\tilde{G}_{\kappa(\ell)}(s)$ and $\zeta(2s+1)$ have simple poles at $s=0$. The residue of this pole shall be our main term. In order to calculate it, we shall need further analysis of the integrand. We define

$$A(s) = \tilde{h}(1+s/2)N^{s/2}X^{1+s/2}E_0(s)H(s)$$

which is the part of the integrand which is holomorphic at $s=0$. If the Laurent coefficients of $\zeta(2s+1)$ and $\tilde{G}_{\kappa(\ell)}$ (centred at 0) are given by $e_n$ and $g_n$ respectively, then the residue is

$$R := (g_{-1}e_0 + g_0e_{-1})A(0) + g_{-1}e_{-1}A'(0), \tag{7.8}$$

and

$$A(0) = \tilde{h}(1)E_0(0)H(0)X, \tag{7.9}$$

$$\begin{aligned} A'(0) = {} & \frac{1}{2}\tilde{h}'(1)E_0(0)H(0)X + \frac{1}{2}\tilde{h}(1)E_0(0)H(0)(\log N)X \\ & + \frac{1}{2}\tilde{h}(1)E_0(0)H(0)X\log X + \tilde{h}(1)E_0'(0)H(0)X + \tilde{h}(1)E_0(0)H'(0)X. \end{aligned} \tag{7.10}$$

We now prove the following expression of the residue $R$ from the above results.

**Corollary 7.6.** *For sufficiently large $N$, we have*

$$R = a_N X \log X + b_N X,$$

*where*

$$N^{-\varepsilon} \ll a_N, b_N \ll N^{\varepsilon}.$$

*Proof.* The first term comes from the $X \log X$ term in (7.10). We see that $g_{-1} = 1$ due to the definition of $\tilde{G}_\kappa(s)$ given in (2.7), and we also easily see that $e_{-1} = 1/2$, so that the coefficient for $A'(0)$ in (7.8) is positive. Now the bounds for $a_N$ follow from those for $E_0(0)$ and $H(0)$ from Lemmas 7.4 and 7.5, and the fact that $\tilde{h}(1)$ is simply a positive constant.

Next, we prove the bounds for the $X$ term coefficient. The upper bound follows from those for $E_0(0)$ and $H(0)$ in Lemmas 7.4 and 7.5, and again from the fact that $\tilde{h}(1)$ and $\tilde{h}'(1)$ are constants. As for the lower bound, there are two difficulties: First, we do not know if the coefficient for $A(0)$ in (7.8) is negative, which would result in a negative $X$ term contribution from (7.9). Secondly, we do not know whether the $\tilde{h}'(1)$ factor in the first term of (7.10) is positive. Nonetheless, we can simply compare the (positive) $X$ coefficient in the second term of (7.10) to that of the two terms just mentioned, and choose $N$ large enough so that the $\log N$ factor dominates. The lower bound then follows again from Lemmas 7.4 and 7.5. □

Now we just need to bound the remaining integral (7.7) with contour $(-\varepsilon)$. Using the triangle inequality and observing that $\tilde{G}_{\kappa(\ell)}$ decays rapidly in fixed vertical strips, we therefore see that the integral with contour $(-\varepsilon)$ is bounded above by $N^{\varepsilon}X^{1-\varepsilon/2}$.

### 7.2.2 Error term

Using the same method for bounding the error term in the previous section, we can bound $S_{\Box}(X;\chi,-1,\ell)$ from above, since its expansion according to (7.4) will have the factor $T(s;X/d_1^2,\chi_0^{(2ngN)}\psi_{-1})$. By Lemma 7.2 this becomes (7.6). It now remains to bound $S_{\bar{\Box}}(X;\chi,\iota,\ell)$ from above.

According to (7.2) with the integral contour moved to $(3/4+\varepsilon)$, we see that

$$S_{\bar{\Box}}(X;\chi,\iota,\ell) = 2\int_{(3/4+\varepsilon)}(c_0(\ell)N)^{s/2}\tilde{G}_{\kappa(\ell)}(s)\sum_{(d_1,2N)=1}\sum_{f_1|d_1}\sum_{\substack{g|2f_1^2\\(g,N)=1}}\frac{\mu(g)\chi(g)}{g^{1/2}}$$
$$\times\sum_{\substack{ng\in\bar{\Box}\\(n,N)=1}}\frac{\chi(n)}{n^{s+1/2}}T(s;X/d_1^2,\chi_0^{(2N)}\tilde{\chi}_{ng}\psi_{\iota})ds.$$

Applying Lemma 7.2, since $ng$ is not square, the character $\chi_0^{(2N)}\tilde{\chi}_{ng}\psi$ is never principal, so we have

$$T(s;X/d_1^2,\chi_0^{(2N)}\tilde{\chi}_{ng}\psi_{\iota}) \ll (|s|ng)^{1/4+\varepsilon}N^{\varepsilon}(X/d_1^2)^{7/8+\varepsilon}.$$

Now we can absolutely bound the sums and ignore the condition that $ng$ is not square, as we did for (7.6). We note that the $n$-sum will absolutely converge since $\Re s = 3/4+\varepsilon$, as long as we select a small enough $\varepsilon$ in the bound for the $T$-function above. We then arrive at a bound of

$$S_{\bar{\Box}}(X;\chi,\iota,\ell) \ll N^{3/8+\varepsilon}X^{7/8+\varepsilon},$$

whence by (7.3) we have sufficiently bounded $S_{\bar{\Box}}(X;\chi)$.

# Appendix A

# Other methods

It turns out that the double Dirichlet series $Z(\frac{1}{2}, w, \chi_N, \chi_M)$ has a functional equation of conductor $NM$. More precisely, an approximate functional equation shows that one has roughly

$$\operatorname*{res}_{w=1} Z(\tfrac{1}{2}, w, \chi_N, \chi_M) = \sum_{d<(NM)^{1/2}} L(\tfrac{1}{2}, \chi_{dN})\chi_M(d) + (NM)^{-\frac{1}{2}} \sum_{d<(NM)^{1/2}} L(\tfrac{1}{2}, \chi_{dM})\chi_N(d).$$

Hoffstein and Kontorovich now choose $M = 1$ and show that the residue is non-zero. If all twists by $\chi_d$ for $d < N^{1/2}$ would vanish, then the first term on the right hand side vanishes, but unfortunately this does not lead to a contradiction (as claimed in [6]), since there is also a second term. The problem here is that the approximate functional equation is not symmetric. One might be tempted to choose $N = M$, but then the length of both sums is roughly $N$, returning the trivial bound $N$.

# Appendix B

# $L$-function moment upper bound

Here we state and prove an adaptation of a fourth moment $L$-function estimate proven by R. Heath-Brown, specifically Theorem 2 of [5]. This version includes a character twist.

**Theorem B.1.** *Let $\psi$ be a primitive character modulo $j$, and for a positive integer $Q$ define $S(Q)$ to be the set of quadratic primitive Dirichlet characters of conductor at most $Q$. We have*

$$\sum_{\chi\in S(Q)} |L(\sigma+it,\chi\psi)|^4 \ll_\varepsilon \{Q+(Qj(|t|+1))^{2-2\sigma}\}\{Qj(|t|+1)\}^\varepsilon$$

*for any fixed $\sigma\in[1/2,1]$ and any $\varepsilon>0$.*

*Proof.* We follow the proof in [5], adding the character twist $\psi$ where needed.

We define $T := |t|+1$ and $\sigma := \Re s$ for brevity and

$$S_\psi(Q,s) := \sum_{\chi\in S(2Q)-S(Q)} |L(s,\chi\psi)|^4.$$

For $\sigma\in[1/2,1]$, we define $\nu(\sigma)$ to be the infemum of $\nu$ for which

$$S_\psi(Q,s) \ll (Q+(QjT)^{2-2\sigma})(QjT)^\nu.$$

We start with the identity

$$L(s,\chi\psi)^2 = \sum_{n=1}^{\infty} d(n)(\chi\psi)(n)n^{-s}e^{-n/U} - \frac{1}{2\pi i}\int_{(\alpha)} L(w,\chi\psi)^2\Gamma(w-s)U^{w-s}dw,$$

valid for $0 \leq \alpha < \sigma \leq 1$. Applying this, we see that

$$\begin{aligned} S_\psi(Q,s) \ll & \sum_{\chi\in S(2Q)-S(Q)} \left|\sum_{n=1}^{\infty} d(n)(\chi\psi)(n)n^{-s}e^{-n/U}\right|^2 && \text{(B.1)} \\ & + \left(\int_{-\infty}^{\infty} |L(\alpha+i\tau,\chi\psi)|^2|\Gamma(\alpha+i\tau-s)|U^{\alpha-\sigma}d\tau\right)^2 && \text{(B.2)} \end{aligned}$$

We have the bound

$$\Gamma(x+iy) \ll_x e^{-|y|},$$

so that in our case,

$$\Gamma(\alpha-\sigma+i(\tau-t)) \ll_{\alpha,\sigma} e^{-|\tau-t|}.$$

Hence, with the Cauchy-Schwarz inequality, we have further from (B.2) that

$$\begin{aligned} S_\psi(Q,s) \ll & \sum_{\chi\in S(2Q)-S(Q)} \left|\sum_{n=1}^{\infty} d(n)(\chi\psi)(n)n^{-s}e^{-n/U}\right|^2 && \text{(B.3)} \\ & +U^{2(\alpha-\sigma)}\int_{-\infty}^{\infty} S_\psi(Q,\alpha+i\tau)e^{-2|\tau-t|}d\tau && \text{(B.4)} \end{aligned}$$

We set $T_\tau := |\tau|+1$. According to the functional equation for Dirichlet $L$-functions, it is easy to see that

$$|L(\alpha+i\tau,\chi\psi)|^4 \ll (QjT_\tau)^{2-4\alpha}|L(1-\alpha+i\tau,\chi\psi)|^4$$

as long as $\alpha \leq 1/2$. By the definition of $\nu(1-\alpha)$, this gives us

$$S_\psi(Q,\alpha+i\tau) \ll (QjT_\tau)^{2-4\alpha}(Q+(QjT_\tau)^{2\alpha})(QjT_\tau)^{\nu(1-\alpha)+\varepsilon}.$$

Because the integrand of

$$\int_{-\infty}^{\infty} S_\psi(Q, \alpha + i\tau) e^{-2|\tau - t|} d\tau$$

is only substantial when $\tau \sim t$, continuing from (B.4) we see that

$$S_\psi(Q,s) \ll \sum_{\chi \in S(2Q)-S(Q)} \left| \sum_{n=1}^{\infty} d(n)(\chi\psi)(n) n^{-s} e^{-n/U} \right|^2 \tag{B.5}$$

$$+U^{2(\alpha-\sigma)}(QjT)^{2-4\alpha}(Q + (QjT)^{2\alpha})(QjT)^{\nu(1-\alpha)+\varepsilon}. \tag{B.6}$$

We now wish to truncate the inner sum above at $n \le N_0$ for some $N_0$. We have

$$\begin{aligned} \sum_{\chi \in S(2Q)-S(Q)} \left| \sum_{n \ge N_0} d(n)(\chi\psi)(n) n^{-s} e^{-n/U} \right|^2 &\ll \left( \int_{N_0}^{\infty} e^{-x/U} dx \right)^2 \sum_{\substack{\chi \in S(2Q)-S(Q) \\ (\mathrm{cond}(\chi), j)=1}} 1 \\ &\ll QU^2 e^{-2N_0/U}. \end{aligned}$$

We see that the above is asymptotically smaller than the second term of (B.6) if we choose $N_0 = U \log^2 QjT$.

We now wish to bound the first term of (B.6) by dividing the inner sum into $O(\log N_0)$ dyadic intervals $N < n \le 2N$ with $N \ll N_0$. By Corollary 3 of [5], we have

$$\sum_{\chi} \left| \sum_{N < n \le 2N} d(n)(\chi\psi)(n) n^{-s} e^{-n/U} \right|^2 \ll_\varepsilon Q^\varepsilon N^{1-2\sigma+\varepsilon}(Q+N)$$

which gives

$$\begin{aligned} \sum_{\chi} \left| \sum_{n=1}^{\infty} d(n)(\chi\psi)(n) n^{-s} e^{-n/U} \right|^2 &\ll \sum_{i=0}^{\log N_0} \sum_{\chi} \left| \sum_{2^i < n \le 2^{i+1}} d(n)(\chi\psi)(n) n^{-s} e^{-n/U} \right|^2 \\ &\ll Q^\varepsilon \sum_{i=1}^{\log N_0} (2^i)^{1-2\sigma+\varepsilon}(Q + 2^i) \\ &\asymp Q^\varepsilon N_0^\varepsilon (Q N_0^{1-2\sigma} + N_0^{2-2\sigma}). \end{aligned}$$

We see that as long as $1/2 \leq \sigma \leq 1$, we see that this is bounded above by

$$(QjTU)^{\varepsilon}(Q + U^{2-2\sigma}).$$

Incorporating this bound into (B.6), we have

$$\begin{aligned} S_{\psi}(Q,s) &\ll (QjTU)^{\varepsilon}(Q + U^{2-2\sigma}) && \text{(B.7)} \\ &\quad + U^{2(\alpha-\sigma)}(QjT)^{2-4\alpha}(Q + (QjT)^{2\alpha})(QjT)^{\nu(1-\alpha)+\varepsilon} && \text{(B.8)} \end{aligned}$$

for $1/2 \leq \alpha < \sigma \leq 1$.

We first examine the case when $\sigma > 1/2$. Choosing $\alpha = 1 - \sigma$ gives

$$S_{\psi}(Q,s) \ll (QjTU)^{\varepsilon}\left[(Q + U^{2-2\sigma}) + (QjTU^{-1})^{4\sigma-2}(Q + (QjT)^{2-2\sigma})(QjT)^{\nu(\sigma)}\right].$$

We further choose $U = (QjT)^{1+\delta}$, where $0 < \delta < 1$, which gives an upper bound of

$$(QjT)^{3\varepsilon}\left[Q + (QjT)^{2-2\sigma}\right]\left[(QjT)^{\delta} + (QjT)^{\nu(\sigma)-(4\sigma-2)\delta}\right].$$

Since $\varepsilon$ is arbitrary, by the definition of $\nu(\sigma)$, we must have

$$\nu(\sigma) \leq \max\{\delta, \nu(\sigma) - (4\sigma - 2)\delta\}.$$

However, because $(4\sigma - 2)\delta$ is positive, it must be that $\nu(\sigma) \leq \delta$. Further, since $\delta$ is arbitrary, we have $\nu(\sigma) \leq 0$, and the statement of the theorem follows.

In the case of $\sigma = 1/2$, we choose $\alpha = 1/2 - \varepsilon$. We have already shown that $\nu(\beta) \leq 0$ if $1/2 < \beta \leq 1$, and so we have $\nu(1 - \alpha) \leq 0$. Hence, (B.8) gives

$$S_{\psi}(Q,s) \ll (QjTU)^{\varepsilon}(Q + U) + U^{-2\varepsilon}(QjT)^{5\varepsilon}(Q + (QjT)^{1-2\varepsilon}).$$

If we choose $U = QjT$, then we see that $\nu(1/2) \leq 0$, and the statement of the theorem is once again immediate. $\square$

# Bibliography


[1] V. Blomer. Subconvexity for a double Dirichlet series. *Compositio Math.*, 147:355–374, 2011.

[2] Benjamin Brubaker, Daniel Bump, Gautam Chinta, Solomon Friedberg, and Jeffrey Hoffstein. Weyl group multiple Dirichlet series I. *Proceedings of Symposia in Pure Mathematics*, 2006.

[3] D. Bump, S. Friedberg, and J. Hoffstein. *Sums of twisted GL(3) automorphic L-functions, Contributions to automorphic forms, geometry, and number theory.* Johns Hopkins Univ. Press, 2004.

[4] A. Diaconu, D. Goldfeld, and J. Hoffstein. Multiple Dirichlet series and moments of zeta and L-functions. *Compositio Math.*, 139:297–360, 2003.

[5] D. R. Heath-Brown. A mean value estimate for real character sums. *Acta Arith.*, 72:235–275, 1995.

[6] J. Hoffstein and A. Kontorovich. The first non-vanishing quadratic twist of an automorphic L-series. *preprint*, 2010.

[7] L. Hörmander. *An introduction to complex analysis in several variables.* Van Nostrand, Princeton, 1996.

[8] H. Iwaniec and E. Kowalski. *Analytic Number Theory.* American Mathematical Society, Providence, Rhode Island, 2004.

[9] M. Jutila. On the mean value of $L(1/2, \chi)$ for real characters. *Analysis*, 1:149–161, 1981.

[10] R. Mollin. *Fundamental Number Theory with Applications.* CRC Press LLC, 1998.

[11] Jean-Loup Waldspurger. Sur les coefficients de fourier des formes modulaires de poids demi-entier. *Journal de Mathématiques Pures et Appliquées*, 60:675–484, 1981.