Lusztig Slices in the Affine Grassmannian and Nilpotent Matrices

by

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Abstract

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The focus of this dissertation is to present some new results related to an isomorphism of Mirković-Vybornov between Lusztig slices in the affine Grassmannian and varieties of nilpotent matrices. We give an explicit alternate Mirković-Vybornov isomorphism in the GL_n-affine Grassmannian and an explicit isomorphism in the GL_2-affine Grassmannian. We also transport an additive action on the Lusztig slice through the Mirković-Vybornov isomorphism and show that it corresponds to a residual conjugation action. We go on to verify that this additive action matches a previously discovered action on the Slodowy slice.

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Contents

Chapter 1

Introduction

The motivating force behind this dissertation is the connection between lattices and nilpotent matrices. We use the word *lattice* to refer to an element of the GL_n-affine Grassmannian $GL_n(\mathcal{K})/GL_n(\mathcal{O})$, where $\mathcal{K} := \mathbb{C}((z))$ and $\mathcal{O} := \mathbb{C}[[z]]$. The use of the word lattice in this context reflects an analogy with the space $GL_n(\mathbb{Q})/GL_n(\mathbb{Z})$, whose elements are lattices in the original sense of the term.

To be precise in our case of the GL_n-affine Grassmannian, elements are free, rank n, $\mathcal{O}$-modules inside $\mathcal{K}^n$. If we take two lattices L_1 and L_2 in the GL_n-affine Grassmannian satisfying $L_1 \subseteq L_2$, then L_2/L_1 is a finite dimensional $\mathbb{C}$-vector space equipped with a nilpotent operator z. Exploring this relationship between lattices and nilpotents will be our primary goal.

New Results. There are three main results in this thesis. In chapter 4 we find an explicit description of the Mirković-Vybornov isomorphism between

Lusztig slices in the GL_2-affine Grassmannian and subvarieties of nilpotent matrices. Then we find an explicit "alternate" Mirković-Vybornov isomorphism in the case of the GL_n-affine Grassmannian. In chapter 5 we transport a certain $\mathbb{G}_a$-action on the Lusztig slice over to the nilpotent cone using the Mirković-Vybornov isomorphism and show it to be a residual conjugation action. Then we verify that this matches a $\mathbb{G}_a$-action on a Slodowy slice as studied in [Mor14].

Another goal of this thesis is to contribute pedagogically to the literature on the affine Grassmannian. This goal has been achieved if any researcher interested in the affine Grassmannian (whether as the richly-structured space that it is, or because of its connections to representation theory) reads this thesis in order to make a pleasant acquaintance with this very interesting mathematical object.

Historical Context. The affine Grassmannian has surfaced during the study of a variety of subjects by a variety of mathematicians. In the context of the representation theory of $GL_n(\mathbb{F}_q)$ over $\mathbb{C}$, Lusztig [Lus81] was led to consider a subset of the $\overline{\mathbb{F}}_q$-affine Grassmannian and identified it with the variety of nilpotent matrices in $\mathfrak{gl}_n(\overline{\mathbb{F}}_q)$. Lusztig later generalized this result in [Lus90a] (see also [Mag02]) and once again demonstrated the importance of using the affine Grassmannian for studying questions related to nilpotent orbits. See [KLMW06] for how Lusztig's isomorphism is applied to study nilpotent orbit closures. However, it is the closely related work of Mirković-Vybornov in [MV07] and [MV08] which forms the bedrock of this thesis. One of the many results of these papers is what we shall call the *Mirković-Vybornov isomorphism*. It is an isomorphism of varieties between *Lusztig slices* in the affine Grassmannian (see 3.2) and a certain affine space (see 3.3.2) intersected with the nilpotent cone.

The affine Grassmannian has also appeared in the context of geometric representation theory, most notably in the *geometric Satake isomorphism* of Lusztig [Lus], Ginzburg [Gin95], and Mirković-Vilonen [MV07]. Here the theories of intersection cohomology and perverse sheaves are used to relate the geometry of the affine Grassmannian $G(\mathcal{K})/G(\mathcal{O})$ to the representation theory of the Langlands dual group $\breve{G}(\mathbb{C})$. The geometric Satake isomorphism was itself an outgrowth of number theory, specifically the Langlands program, in which the *Satake isomorphism* [Sat63] was studied as an unramified local Langlands correspondence (see also [Gro98]).

The affine Grassmannian is also an object of rich combinatorics in its own right. The study of Mirković-Vilonen cycles and polytopes, which are objects that arise from the affine Grassmannian, lead to an understanding of the rich combinatorics of bases for the highest weight representations $V(\lambda)$ of reductive groups over $\mathbb{C}$, see [Kam10] (and also [Kam09] and [AK04]).

The affine Grassmannian also has close connections to the subject of *loop groups*, which have been studied by many mathematicians from a variety of perspectives (see [PS86] and [Fre07]).

Below you can find more detailed summaries of our goals in each chapter.

Chapter 2. This chapter serves as an introduction to affine Grassmannians Gr_G when the group G is equal to GL_n, SL_n or PGL_n, as well as the interpretation of Gr_G as a space of *lattices*. This chapter also examines the $G(\mathcal{O})$-orbits on the affine Grassmannian and how this relates to the Jordan type of a natural nilpotent operator associated to a lattice. The important relationship of lattice containment is also examined in detail.

Chapter 3. The aim of this chapter is to give the definition of a Lusztig slice and to state the Mirković-Vybornov isomorphism. Roughly speaking, the Mirković-Vybornov isomorphism is a one-to-one correspondence between the Lusztig slice and a special subvariety of nilpotent matrices.

Chapter 4. Two of the three main results of this dissertation are in this chapter. First is an explicit realization of the Mirković-Vybornov isomorphism in the case of GL_2. The Lusztig slice is interpreted as matrices in $GL_2(\mathcal{K})$ with a condition on their determinant. Then we explicitly calculate the action of the nilpotent operator z on a finite dimensional vector space. This yields an explicit Mirković-Vybornov isomorphism. Secondly we have an explicit realization of an "alternate" Mirković-Vybornov isomorphism for GL_n in the case when lattices L are *contained in* the standard lattice. This amounts to representing the Lusztig slice as certain matrices in $GL_n(\mathcal{K})$ and then working out the action of the nilpotent operator z on a finite dimensional vector space. Many examples are given of both of our explicit maps in this chapter.

Chapter 5. The third main result of this dissertation is in this chapter. It concerns a $\mathbb{G}_a$-action on the Lusztig slice that was discovered by Kamnitzer, that is conjectured to give a Hamiltonian reduction between "neighboring" Lusztig slices. We determine the form this action takes on the Mirković-Vybornov slice intersect the nilpotent cone by transporting the $\mathbb{G}_a$-action through the Mirković-Vybornov isomorphism. Then we show that this action coincides with the action on the Slodowy slice corresponding to a certain pyramid that is found in Morgan's thesis [Mor14]. Our result of this chapter says that the actions of $\mathbb{G}_a$ on the Lusztig slice and on the Slodowy slice are identical.

Chapter 6. In this final chapter, we highlight some questions that come out of various parts of this thesis and the future research directions to which they point.

Chapter 2

Affine Grassmannians in Type-A

In this chapter we focus on three reductive groups over $\mathbb{C}$: GL_n, SL_n, and PGL_n. We will write G when we do not want to specify a particular one of the three. We denote the $\mathbb{C}$-algebras $\mathbb{C}((z))$ and $\mathbb{C}[[z]]$ by $\mathcal{K}$ and $\mathcal{O}$, respectively. We take $\mathbb{C}((z))$ to be the field formal Laurent series in one variable, and $\mathbb{C}[[z]]$ the ring of formal power series in one variable.

Throughout this chapter, the symbol V will stand for an n-dimensional vector space over $\mathcal{K}$.

2.1 Coset Definition

Taking the points of G in the $\mathbb{C}$-algebras $\mathcal{K}$ and $\mathcal{O}$, we obtain the group $G(\mathcal{K})$ with the subgroup $G(\mathcal{O})$. We can consider $G(\mathcal{O})$ right-acting on $G(\mathcal{K})$.

Definition 2.1.1 (Affine Grassmannian)**.** The coset space $G(\mathcal{K})/G(\mathcal{O})$ is called the *affine Grassmannian* of G and will be denoted by Gr_G.

2.2 Lattices and Gr_{GL_n}

In this section we will discuss the relationship between lattices and the GL_n-affine Grassmannian.

Definition 2.2.1 (Lattices in V)**.** A *lattice* in V is a free $\mathcal{O}$-submodule L in V satisfying $L \otimes_{\mathcal{O}} \mathcal{K} = V$.

A lattice L may be thought of as a free $\mathcal{O}$-submodule of V that has $\mathcal{O}$-module rank equal to the dimension of V as a $\mathcal{K}$-vector space. Any lattice in V can therefore be expressed as $\mathrm{span}_{\mathcal{O}}(v_1, ..., v_n)$ where $\{v_1, ..., v_n\}$ is a $\mathcal{K}$-basis for V. The collection of all $\mathcal{K}$-bases of V is a $GL_n(\mathcal{K})$-torsor and the subgroup of $GL_n(\mathcal{K})$ fixing a particular lattice is conjugate to $GL_n(\mathcal{O})$. Therefore the coset space $\mathrm{Gr}_{GL_n} = GL_n(\mathcal{K})/GL_n(\mathcal{O})$ gives a non-canonical realization of the collection of all lattices in V. It is non-canonical because there is an arbitrary choice of a "base lattice" represented by the identity coset in $GL_n(\mathcal{K})/GL_n(\mathcal{O})$. If we fix a $\mathcal{K}$-basis of V, say $\{e_1, ..., e_n\}$, which identifies V with $\mathcal{K}^n$, then the collection of lattices in V is unambiguously identified with $GL_n(\mathcal{K})/GL_n(\mathcal{O})$, where the identity coset corresponds to the "standard lattice" $\mathrm{span}_{\mathcal{O}}(e_1, ..., e_n)$. However the choice of the basis $\{e_1, ..., e_n\}$ was arbitrary.

Definition 2.2.2 (The standard lattice)**.** Fix a $\mathcal{K}$-basis of V, say $\{e_1, ..., e_n\}$. We call the lattice $\mathrm{span}_{\mathcal{O}}(e_1, ..., e_n)$ the *standard lattice* and denote it by L_0.

2.3 Unimodular Lattices and Gr_{SL_n}

In this section we will discuss the relationship between unimodular lattices and the SL_n-affine Grassmannian.

If we make a choice of $\mathcal{K}$-basis for V, say $\{e_1, ..., e_n\}$, then the one-dimensional $\mathcal{K}$-vector space $\bigwedge^n V$ develops a distinguished basis, namely $\{e_1 \wedge ... \wedge e_n\}$. Then the collection of $\mathcal{K}$-bases $\{v_1, ..., v_n\}$ of V with the property that $v_1 \wedge ... \wedge v_n = e_1 \wedge ... \wedge e_n$ is a torsor for the group $SL_n(\mathcal{K})$.

Definition 2.3.1 (Unimodular lattices in V)**.** A *unimodular lattice* in V is a lattice L in V having a basis $\{v_1, ..., v_n\}$ with the property $v_1 \wedge ... \wedge v_n = e_1 \wedge ... \wedge e_n$.

A unimodular lattice is a lattice L in V that can be represented by a matrix in $SL_n(\mathcal{K})$. The collection of all unimodular lattices in V can be identified with the coset space $\mathrm{Gr}_{SL_n} = SL_n(\mathcal{K})/SL_n(\mathcal{O})$.

2.4 Lattice Homothety Classes and Gr_{PGL_n}

In this section we will discuss the relationship between lattices up to $\mathcal{K}^*$-scaling and the PGL_n-affine Grassmannian.

The set of $\mathcal{K}$-bases of V up to $\mathcal{K}^*$-scaling is a torsor for $PGL_n(\mathcal{K})$. The group fixing a given $\mathcal{K}^*$-scale-class of a lattice is conjugate to $GL_n(\mathcal{O})/\mathcal{O}^* = PGL_n(\mathcal{O})$. Therefore the collection of lattices in V up to $\mathcal{K}^*$-scaling can be identified with the coset space $\mathrm{Gr}_{PGL_n} = PGL_n(\mathcal{K})/PGL_n(\mathcal{O})$.

Definition 2.4.1 (Homothety classes of lattices in V)**.** The collection of lattices equivalent to a lattice L by $\mathcal{K}^*$-scaling is called its *homothety class*.

The collection of lattices equivalent to L by $\mathcal{K}^*$-scaling can be identified with $\mathcal{K}^*/\mathcal{O}^*$ which can then be identified with $\{z^a L : a \in \mathbb{Z}\}$. Therefore an equivalent definition of the homothety class of L is the collection of lattices of the form $z^a L$ for $a \in \mathbb{Z}$.

2.5 $GL_n(\mathcal{O})$-Orbits on Gr_{GL_n}

Consider the group $GL_n(\mathcal{O})$ left-acting on the affine Grassmannian Gr_{GL_n}. In this section, we will discuss the two equivalent statements 2.5.1 and 2.5.2.

Proposition 2.5.1. The collection of $GL_n(\mathcal{O})$-orbits on Gr_{GL_n} is in bijection with $X_+(GL_n)$, the set of dominant coweights of GL_n. Specifically, our orbit decomposition is

$$\mathrm{Gr}_{GL_n} = \coprod_{\lambda \in X_+} GL_n(\mathcal{O}) L_\lambda$$

where $\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{Z}^n$ is a dominant coweight of GL_n, and $L_\lambda = [z^\lambda]$ is the lattice represented by the matrix

$$z^\lambda = \begin{bmatrix} z^{\lambda_1} & & \\ & \ddots & \\ & & z^{\lambda_n} \end{bmatrix} \in GL_n(\mathcal{K}).$$

□

This statement admits the following equivalent form:

Proposition 2.5.2. Every matrix in $GL_n(\mathcal{K})$ can be reduced by $GL_n(\mathcal{O})$-row and $GL_n(\mathcal{O})$-column operations to a matrix of the form

$$z^\lambda = \begin{bmatrix} z^{\lambda_1} & & \\ & \ddots & \\ & & z^{\lambda_n} \end{bmatrix} \in GL_n(\mathcal{K}),$$

where $\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{Z}^n$ and the entries of λ are in order. □

Clarification. Notice that in 2.5.2 we say that the entries of λ are in order, and in 2.5.1 we say that λ is a dominant coweight of GL_n. In 2.5.1 we are only asserting that dominant coweights are in *bijection* with the set of $GL_n(\mathcal{O})$-orbits on Gr_{GL_n}. This is important because at various points of this thesis we will sometimes prefer to represent a $GL_n(\mathcal{O})$-orbit using L_λ with $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$ and sometimes using L_λ with $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$. We will be very explicit about which choice is being made in each instance.

One way to understand the $GL_n(\mathcal{O})$-orbits on Gr_{GL_n} is to ask: what are the possible "echelon forms" of matrices in $GL_n(\mathcal{K})$ up to $GL_n(\mathcal{O})$-row and $GL_n(\mathcal{O})$-column operations?

Given a matrix in $GL_n(\mathcal{K})$, we can determine the coset in

$$GL_n(\mathcal{O})\backslash GL_n(\mathcal{K})/GL_n(\mathcal{O})$$

to which it belongs. We begin by looking for the entry with a lowest power of z, factoring out the lowest power of z, and then zeroing everything in the corresponding row and column using matrices in $GL_n(\mathcal{O})$. Repeating this process reduces the original matrix to one of the form $\sigma z^{\lambda'}\tau$, where $\lambda' \in \mathbb{Z}^n$, and σ and τ are permutation matrices. Left and right-multiplication by appropriate permutation matrices will reduce to a matrix z^λ where the entries of λ are in order.

Example 2.5.3. Work with the matrix

$$g = \begin{bmatrix} z^2 & 1 \\ z^{-3} & z \end{bmatrix} \in GL_2(\mathcal{K}).$$

We perform $GL_2(\mathcal{O})$-row and $GL_2(\mathcal{O})$-column operations:

$$\begin{bmatrix} z^2 & 1 \\ z^{-3} & z \end{bmatrix} \xrightarrow{C_2 \to -z^4 C_1 + C_2} \begin{bmatrix} z^2 & -z^6+1 \\ z^{-3} & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 - z^5 R_2} \begin{bmatrix} 0 & -z^6+1 \\ z^{-3} & 0 \end{bmatrix} \xrightarrow{R_1 \to (1-z^6)^{-1} R_1} \begin{bmatrix} 0 & 1 \\ z^{-3} & 0 \end{bmatrix},$$

where the last operation was to multiply row one by the power series reciprocal of $1 - z^6$. Swapping rows results in the matrix:

$$z^{(-3,0)} = \begin{bmatrix} z^{-3} & 0 \\ 0 & z^0 \end{bmatrix}.$$

We can either say that the matrix $z^{(-3,0)}$ represents the double coset of $g \in GL_2(\mathcal{K})$, or we can say that the lattice $L_{(-3,0)} = [z^{(-3,0)}]$ represents the $GL_2(\mathcal{O})$-orbit of the lattice $L = [g]$. □

2.6 The Containment Relation

Throughout this thesis we will use $\mathbb{Z}_+$ to denote the positive integers including zero and $\mathbb{Z}_-$ to denote the negative integers including zero.

We know that $GL_n(\mathcal{O})$-orbits on Gr_{GL_n} are of the form $GL_n(\mathcal{O})L_\lambda$ where λ is an element of $\mathbb{Z}^n$ with ordered entries.

If $\lambda \in (\mathbb{Z}_-)^n$ then L_λ contains the standard lattice L_0, and furthermore it can be shown that any lattice $L \in GL_n(\mathcal{O})L_\lambda$ also contains the standard lattice. This and many other statements about lattice containment are corollaries of the following statement.

Proposition 2.6.1. Two lattices L_a and L_b satisfy the containment relation $L_a \subseteq L_b$ precisely when matrices a and b in $GL_n(\mathcal{K})$ representing them satisfy $b^{-1}a \in \mathfrak{gl}_n(\mathcal{O})$.

Proof. Suppose $L_a = \mathrm{span}_{\mathcal{O}}(v_1, ..., v_n)$ and $L_b = \mathrm{span}_{\mathcal{O}}(w_1, ..., w_n)$ where $\{v_1, ..., v_n\}$ and $\{w_1, ..., w_n\}$ are bases for $\mathcal{K}^n$. Note that $a = [v_1 ... v_n]$ and $b = [w_1 ... w_n]$ are in $GL_n(\mathcal{K})$. Notice that $L_a \subseteq L_b$ precisely when there is an $n \times n$ matrix $F = (f_{ij})$ defined over $\mathcal{O}$ satisfying

$$\begin{aligned} v_1 &= f_{11}w_1 + f_{12}w_2 + ... + f_{1n}w_n \\ &\vdots \\ v_n &= f_{n1}w_1 + f_{n2}w_2 + ... + f_{nn}w_n. \end{aligned}$$

This can be expressed as $a = bF^t$. Therefore we see that $L_a \subseteq L_b$ precisely when $b^{-1}a \in \mathfrak{gl}_n(\mathcal{O})$. Of course this condition is well-defined; if $\tilde{a}, \tilde{b} \in GL_n(\mathcal{K})$

also represent L_a and L_b respectively, then $\tilde{a} = ag_1$ and $\tilde{b} = bg_2$, where $g_1, g_2 \in GL_n(\mathcal{O})$, and we can see that the condition $\tilde{b}^{-1}\tilde{a} \in \mathfrak{gl}_n(\mathcal{O})$ is preserved. □

It is important to note that a pair of arbitrary lattices do not necessarily have any containment relations. For example, if $G = GL_2$, neither of the lattices $L_{(-1,1)}$ and L_0 contains the other.

Corollary 2.6.2. A lattice L satisfies $L_0 \subseteq L$ precisely when any matrix $g \in GL_n(\mathcal{K})$ representing L satisfies $g^{-1} \in \mathfrak{gl}_n(\mathcal{O})$.

Corollary 2.6.3. A lattice L satisfies $L \subseteq L_0$ precisely when any matrix $g \in GL_n(\mathcal{K})$ representing L satisfies $g \in \mathfrak{gl}_n(\mathcal{O})$.

Corollary 2.6.4. Let g be a matrix in $GL_n(\mathcal{K})$. Two lattices L_a and L_b satisfy $L_a \subseteq L_b$ precisely when $gL_a \subseteq gL_b$.

Corollary 2.6.5. A lattice L satisfies $L_0 \subseteq L$ precisely when it belongs to an orbit $GL_n(\mathcal{O})L_\lambda$ with $L_0 \subseteq L_\lambda$. Equivalently, $L_0 \subseteq L$ precisley when $\lambda \in (\mathbb{Z}_-)^n$.

Corollary 2.6.6. A lattice L satisfies $L \subseteq L_0$ precisely when it belongs to an orbit $GL_n(\mathcal{O})L_\lambda$ with $L_\lambda \subseteq L_0$. Equivalently, $L \subseteq L_0$ precisely when $\lambda \in (\mathbb{Z}_+)^n$.

Corollary 2.6.7. Let L be an arbitrary lattice. There is an integer d such that $L \subseteq z^d L_0$.

Proof. Let $g \in GL_n(\mathcal{K})$ be a matrix representing L. Since the entries of g are in $\mathcal{K}$, there is an integer d so that $z^{-d}g \in \mathfrak{gl}_n(\mathcal{O})$, and by 2.6.3 this means that $z^{-d}L \subseteq L_0$. Then 2.6.4 implies that $L \subseteq z^d L_0$. □

Corollary 2.6.8. Let L_a and L_b be two lattices. There is an integer d such that $L_a \subseteq z^d L_b$.

Proof. Suppose that L_a and L_b are represented by a and b respectively, with $a, b \in GL_n(\mathcal{K})$. Using 2.6.7 on the lattice represented by $b^{-1}a$ we get some integer d such that $b^{-1}aL_0 \subseteq z^d L_0$. By 2.6.4 we then multiply through by b to find $aL_0 \subseteq z^d bL_0$, which means that $L_a \subseteq z^d L_b$. □

Corollary 2.6.9. Let L be an arbitrary lattice. There are integers d and e such that $z^e L_0 \subseteq L \subseteq z^d L_0$.

Proof. We may write $L = gL_\lambda$ for some $g \in GL_n(\mathcal{O})$ and some $\lambda \in \mathbb{Z}^n$. By taking the integer e to be the maximum entry of λ and d to be the minimum entry of λ we have $z^e L_0 \subseteq L_\lambda \subseteq z^d L_0$. By 2.6.4 and the fact that $gL_0 = L_0$, we may multiply through by g to obtain $z^e L_0 \subseteq L \subseteq z^d L_0$. □

Corollary 2.6.10. Let L_a and L_b be two lattices. There are integers d and e such that $z^e L_a \subseteq L_b \subseteq z^d L_a$.

2.7 $G(\mathcal{O})$-Orbits on Gr_G

Remember that we are using G to denote the reductive groups GL_n, SL_n and PGL_n over $\mathbb{C}$. We have already determined that Gr_{GL_n} represents the collection of $\mathcal{O}$-lattices in an n-dimensional $\mathcal{K}$-vector space, and that every $GL_n(\mathcal{O})$-orbit is of the form $\{gL_\lambda : g \in GL_n(\mathcal{O})\}$ where L_λ is the lattice represented by the matrix $z^\lambda \in GL_n(\mathcal{K})$ and $\lambda \in \mathbb{Z}^n$ has ordered entries. In this section, we will work out the $G(\mathcal{O})$-orbits on Gr_G, showing that the dominant coweights $X_+(G)$ are an appropriate index set for the orbits.

Proposition 2.7.1. The left $G(\mathcal{O})$-orbits on Gr_G can be described as follows:

$$\mathrm{Gr}_G = \coprod_{\lambda \in X_+(G)} G(\mathcal{O})[z^\lambda]$$

where $[z^\lambda]$ represents the coset of z^λ in Gr_G.

Proof. ($G = SL_n$) Take an arbitrary $L \in \mathrm{Gr}_{SL_n}$. The lattice L is represented by a matrix $g \in SL_n(\mathcal{K})$. We apply the algorithm of 2.5.2 to g, except we perform $GL_n(\mathcal{O})$-row and $GL_n(\mathcal{O})$-column operations having determinant one, i.e. $SL_n(\mathcal{O})$-row and $SL_n(\mathcal{O})$-column operations. Using these operations, the matrix g will reduce to one of the form:

$$uz^\lambda = \begin{bmatrix} u_1 z^{\lambda_1} & & \\ & \ddots & \\ & & u_n z^{\lambda_n} \end{bmatrix} \in SL_n(\mathcal{K}),$$

where the entries of λ are in order, and $u_1, ..., u_n$ are in $\mathcal{O}^*$. The lattice represented by uz^λ is the same as the lattice represented by the matrix z^λ. Therefore, $[z^\lambda] \in \mathrm{Gr}_{SL_n}$ is a unimodular lattice that represents the $SL_n(\mathcal{O})$-orbit of $L \in \mathrm{Gr}_{SL_n}$. Hence $z^\lambda \in SL_n(\mathcal{K})$, which implies that $\sum \lambda = 0$. Therefore, it is equivalent to describe λ as a dominant coweight of SL_n.

($G = PGL_n$) Take an arbitrary lattice homothety class, i.e. an arbitrary point $[L] \in \mathrm{Gr}_{PGL_n}$. By 2.5.1 the lattice L can be written as $L = gL_\lambda$ for some $\lambda \in X_+(GL_n)$ and $g \in GL_n(\mathcal{O})$. We therefore can write the following equations on homothety classes:

$$[L] = [gL_\lambda] = g[L_\lambda] = [g][L_\lambda]$$

where the last equality holds because the action of $g \in GL_n(\mathcal{O})$ on $[L_\lambda]$ will depend only on its class $[g] \in PGL_n(\mathcal{O})$. Therefore $[L_\lambda]$ represents the left $PGL_n(\mathcal{O})$-orbit of $[L] \in \mathrm{Gr}_{PGL_n}$. The homothety class of the lattice L_λ satisfies the relation $[L_\lambda] = [z^{\pm 1} L_\lambda] = [z^{\pm 1} z^\lambda] = [z^{\lambda \pm (1,\dots,1)}]$, and therefore λ can be regarded as an element of $\mathbb{Z}^n/(1,\dots,1)$. Therefore, it is equivalent to describe λ as a dominant coweight of PGL_n. □

2.8 Closures of $G(\mathcal{O})$-Orbits

We have that left $G(\mathcal{O})$-orbits on Gr_G are of the form $G(\mathcal{O})[z^\lambda]$ with $\lambda \in X_+(G)$, z^λ the matrix in $G(\mathcal{K})$ from 2.5.1, and $[z^\lambda]$ the point in Gr_G. We will be using the notation $\mathrm{Gr}^\lambda_\circ := G(\mathcal{O})[z^\lambda]$ to denote the orbit of $[z^\lambda]$. The closure of the orbit will be denoted by Gr^λ. We have:

$$\mathrm{Gr}^\lambda := \overline{\mathrm{Gr}^\lambda_\circ} = \coprod_{\nu \le \lambda} \mathrm{Gr}^\nu_\circ.$$

The closure of an orbit $\mathrm{Gr}^\lambda_\circ$ consists of all orbits $\mathrm{Gr}^\nu_\circ$ where ν is a dominant coweight with $\lambda - \nu$ equal to a sum of positive coroots.

Definition 2.8.1 (Valuation)**.** The *valuation* of a element of $\mathcal{K}^*$ is the integer representing its class in $\mathcal{K}^*/\mathcal{O}^* \cong \mathbb{Z}$. Equivalently, the valuation of an element $f \in \mathcal{K}^*$ is the integer a that appears when we write f in the form $f = z^a u$, with $u \in \mathcal{O}^*$.

We will let $\mathrm{val}(f)$ denote the valuation of $f \in \mathcal{K}$, where we adopt the convention that $\mathrm{val}(0) = -\infty$.

Let $L \in \mathrm{Gr}_{GL_n}$ be a lattice, and suppose that L is represented by a matrix $g \in GL_n(\mathcal{K})$. The valuation of the determinant of g, denoted $\mathrm{valdet}(g) \in \mathbb{Z}$, is a well-defined property of the lattice L. In fact, it is a well-defined property of the entire $GL_n(\mathcal{O})$-orbit of L. This is expressed in the following proposition.

Proposition 2.8.2. The function $\mathrm{valdet} : GL_n(\mathcal{K}) \to \mathbb{Z}$ is constant on the orbits $GL_n(\mathcal{O})z^\lambda GL_n(\mathcal{O})$.

Proof. Let $g, h \in GL_n(\mathcal{O})$ and consider $\mathrm{valdet}(gz^\lambda h)$.

$$\begin{aligned}\mathrm{valdet}(gz^\lambda h) &= \mathrm{val}(\det(g)\det(z^\lambda)\det(h)) \\ &= \mathrm{valdet}(g) + \mathrm{valdet}(z^\lambda) + \mathrm{valdet}(h) \\ &= 0 + \sum \lambda + 0 \\ &= \sum \lambda.\end{aligned}$$

□

2.9 Dimension

The following proposition about lattices is an important one to notice.

Proposition 2.9.1. Let L_a and L_b be two lattices in V. If $L_a \subseteq L_b$, then L_b/L_a is a finite dimensional $\mathbb{C}$-vector space that is equipped with a nilpotent operator $z \circlearrowright L_b/L_a$.

Proof. By 2.6.10, there are integers e and d such that $z^e L_a \subseteq L_b \subseteq z^d L_a$. Since $L_a \subseteq L_b$ we can take $e = 1$ and d must be a negative integer. Therefore we have that $L_a \subseteq L_b \subseteq z^d L_a$, and hence $L_b/L_a \subseteq z^d L_a/L_a$. This proves that L_b/L_a is finite dimensional vector space over $\mathbb{C}$, and in addition it shows that multiplication by z is a nilpotent operator $z \circlearrowright L_b/L_a$. □

If L_a and L_b are arbitrary lattices (not necessarily having a containment relation) then the lattice $L_a \cap L_b$ is contained in each of them, and we can define the *relative dimension* of L_a and L_b to be:

$$\dim(L_a, L_b) := \dim(L_b/L_b \cap L_a) - \dim(L_a/L_a \cap L_b) \in \mathbb{Z}.$$

Represent a lattice $L \in \mathrm{Gr}_{GL_n}$ with a matrix $g \in GL_n(\mathcal{K})$. Suppose that L belongs to the $GL_n(\mathcal{O})$-orbit $\mathrm{Gr}^{\lambda}_{\circ}$ for $\lambda \in X_+(GL_n)$. Notice that:

$$\mathrm{valdet}(g) = \mathrm{valdet}(z^{\lambda}) = \sum \lambda = \dim(L, L_0) \in \mathbb{Z}.$$

Chapter 3

Lusztig Slices and Nilpotent Matrices

In this chapter we will study Lusztig slices in the case of $G = GL_n$. Lusztig slices are intriguing finite-dimensional subvarieties of the affine Grassmannian. Via the Mirković-Vybornov isomorphism, they correspond to certain varieties of nilpotent matrices.

3.1 The Subset $G_1[z^{-1}] \subseteq GL_n(\mathcal{K})$

The definition of a Lusztig slice first requires an understanding of the subset $G_1[z^{-1}] \subseteq GL_n(\mathcal{K})$. Before we define this subset, let us warm-up with an interpretation of the group $GL_n(\mathcal{O})$.

Proposition 3.1.1. The group $GL_n(\mathbb{C}[[z]])$ can be interpreted as the set of formal power series with matrix coefficients having invertible constant term:

$$GL_n(\mathbb{C}[[z]]) = \left\{A_0 + A_1 z + A_2 z^2 + \dots \mid A_i \in \mathfrak{gl}_n(\mathbb{C}), A_0 \in GL_n(\mathbb{C})\right\}.$$

Proof. The group $GL_n(\mathbb{C}[[z]])$ is the set of matrices with $\mathbb{C}[[z]]$ coefficients having determinant that is an invertible power series, i.e. a power series with non-zero constant term. Therefore if $g \in GL_n(\mathbb{C}[[z]])$, we have that $\det(g) \in \mathbb{C}[[z]]^*$ and so $\det(g)$ evaluated at $z = 0$ is not equal to zero. Since the determinant is polynomial function in the entries of g we have that $0 \neq \det(g)(0) = \det(g(0)) = \det(A_0)$. In summary, a power series with matrix coefficients $A_0 + A_1 z + A_2 z^2 + \dots$ belongs to $GL_n(\mathbb{C}[[z]])$ exactly when A_0 belongs to $GL_n(\mathbb{C})$. □

Now consider the subset $GL_n(\mathbb{C}[z^{-1}]) \subseteq GL_n(\mathcal{K})$ consisting of all $\mathbb{C}[z^{-1}]$-valued points of GL_n. In other words $GL_n(\mathbb{C}[z^{-1}])$ is the set of matrices over the $\mathbb{C}$-algebra $\mathbb{C}[z^{-1}]$ that have determinant is invertible in $\mathbb{C}[z^{-1}]$, i.e. the matrices over $\mathbb{C}[z^{-1}]$ having a nonzero scalar determinant.

We can now define the subset $G_1[z^{-1}] \subseteq GL_n(\mathcal{K})$.

Definition 3.1.2. We define $G_1[z^{-1}]$ to be the subset of $GL_n(\mathbb{C}[z^{-1}])$ consisting of the elements that map to the identity under the evaluation map $\mathfrak{gl}_n(\mathbb{C}[z^{-1}]) \to \mathfrak{gl}_n(\mathbb{C})$ induced by $z^{-1} = 0$.

There is another perspective from which to look at the subset $G_1[z^{-1}]$.

Proposition 3.1.3. The subset $G_1[z^{-1}] \subseteq GL_n(\mathcal{K})$ can be interpreted as the set of polynomials in the variable z^{-1} with matrix coefficients, having identity constant term and nonzero scalar determinant:

$$G_1[z^{-1}] = \left\{ g = I + A_{-1}z^{-1} + \ldots + A_{-k}z^{-k} \mid k \in \mathbb{N}, A_i \in \mathfrak{gl}_n(\mathbb{C}), \det(g) \in \mathbb{C}^* \right\}.$$

3.2 The Lusztig Slices Gr^λ_μ

We will focus on Lusztig slices arising when $\lambda - \mu$ is a sum of positive coroots, which implies that $\sum \lambda = \sum \mu$, and we will use d to denote the integer $d := \sum \lambda = \sum \mu$. We will also assume that *either* $\lambda, \mu \in (\mathbb{Z}_-)^n$ *or* $\lambda, \mu \in (\mathbb{Z}_+)^n$.

Definition 3.2.1 (Lusztig slice)**.** For arbitrary dominant coweights λ and μ, define the *Lusztig slice* Gr^λ_μ as follows:

$$\mathrm{Gr}^\lambda_\mu := G_1[z^{-1}]L_\mu \cap \mathrm{Gr}^\lambda.$$

Lusztig slices Gr^λ_μ are created by intersecting the space of lattices of the form $G_1[z^{-1}]L_\mu$ with an orbit closure Gr^λ. If $d \in -\mathbb{N}$, Lusztig slices correspond via the Mirković-Vybornov isomorphism with certain subvarieties of the nilpotent cone $\mathcal{N} \subseteq \mathfrak{gl}_{-d}(\mathbb{C})$. If $d \in \mathbb{N}$, Lusztig slices correspond via our alternate Mirkovic-Vybornov isomorphism (see section 4.4) with certain subvarieties of the nilpotent cone $\mathcal{N} \subseteq \mathfrak{gl}_d(\mathbb{C})$.

3.3 The Mirković-Vybornov Isomorphism

In this section we will work in the case of $\lambda, \mu \in (\mathbb{Z}_-)^n$ with $\lambda_1 \leq \ldots \leq \lambda_n$ and $\mu_1 \leq \ldots \leq \mu_n$. Let $d := \sum \lambda = \sum \mu$.

As we have already mentioned, the Mirković-Vybornov isomorphism is about the relationship between the Lusztig slices Gr_μ^λ and certain subvarieties of nilpotent matrices in $\mathfrak{gl}_{-d}(\mathbb{C})$. Consider the (somewhat) natural basis of L_μ/L_0:

$$A = \{[z^{-1}e_1], \ldots, [z^{\mu_1}e_1], [z^{-1}e_2], \ldots, [z^{\mu_2}e_2], \ldots, [z^{-1}e_n], \ldots, [z^{\mu_n}e_n]\}.$$

The idea begins with noticing that lattices L inside Gr_μ^λ satisfy $L \supseteq L_0$ and also admit a distinguished vector space isomorphism:

$$L/L_0 \underset{\pi_L^{-1}}{\overset{\pi_L}{\rightleftarrows}} L_\mu/L_0.$$

Where π_L^{-1} is defined by the condition (for $[z^r e_s] \in A$):

$$\pi_L^{-1}[z^r e_s] = [z^r e_s] + \sum_{\substack{1\leq j\leq n \\ i<\mu_j}} *[z^i e_j].$$

The asterisks $*$ represent the required coefficients. Here is an example.

Example 3.3.1. Let $G = GL_2$, $\lambda = (-3, 0)$, and $\mu = (-2, -1)$. Consider the lattice $L \in \mathrm{Gr}_\mu^\lambda$ represented by the matrix

$$\begin{bmatrix} z^{-2} + z^{-3} & z^{-3} \\ z^{-3} & z^{-1} - z^{-2} + z^{-3} \end{bmatrix}.$$

In this case the basis A of L_μ/L_0 is given by $A = \{[z^{-1}e_1], [z^{-2}e_1], [z^{-1}e_2]\}$. We will now describe the isomorphism $L/L_0 \cong L_\mu/L_0$. The vector in L/L_0 corresponding to $[z^{-1}e_2] \in L_\mu/L_0$ is $w_1 := [z^{-1}e_2] - [z^{-2}e_2] + [z^{-3}e_2] + [z^{-3}e_1]$.

The vector corresponding to $[z^{-2}e_1]$ is $w_2 := [z^{-2}e_1] + [z^{-3}e_1] + [z^{-3}e_2]$. To find the vector corresponding to $[z^{-1}e_1]$ we need to do the calculation $w_3 := zw_2 - w_2$, which yields $w_3 = [z^{-1}e_1] - [z^{-3}e_1] + [z^{-2}e_2] - [z^{-3}e_2]$. □

We may thus represent the operator $\pi_L z \pi_L^{-1} \circlearrowright L_\mu/L_0$ as a nilpotent matrix by using the basis A for the vector space L_μ/L_0. It turns out that the lattices $L \in \mathrm{Gr}_\mu^\lambda$ give rise to nilpotent matrices in an affine subspace $M_\mu \subseteq \mathfrak{gl}_{-d}(\mathbb{C})$, called the *Mirković-Vybornov slice* [MV08, 3.3] (which is defined below). In [MV08, 1.2] Mirković and Vybornov show that the nilpotent matrices inside $M_\mu \cap \overline{\mathbb{O}}_\lambda$ (where $\overline{\mathbb{O}}_\lambda$ denotes the closure of the nilpotent orbit $\mathbb{O}_\lambda$ of type λ) are in a one-to-one correspondence with the lattices in the Lusztig slice Gr_μ^λ.

Definition 3.3.2 (The Mirković-Vybornov slice)**.** Let $\mu \in (\mathbb{Z}_-)^n$ and $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_n$. Let $d := \sum \mu$. Define the *Mirković-Vybornov slice* to be the affine subspace $M_\mu \subseteq \mathfrak{gl}_{-d}(\mathbb{C})$ consisting of $\mu_i \times \mu_j$ block matrices (M_{ij}) where: if $i = j$ then M_{ij} has ones on the super-diagonal and arbitrary entries along the bottom row; if $i > j$ then M_{ij} has μ_i left-justified entries along the bottom row; and if $i < j$ then M_{ij} has entries along the bottom row.

The Mirković-Vybornov slice M_μ is a particular example of an affine space that is transverse to the conjugacy class of nilpotent matrices $\mathbb{O}_\mu \subseteq \mathfrak{gl}_{-d}(\mathbb{C})$ of type μ. The significance of M_μ is that it is a natural receptacle for the nilpotent matrices that come from lattices in the Lusztig slice.

The following is a recollection of [MV08, 1.2 ~ ψ] which we have been referring to as the *Mirković-Vybornov isomorphism*.

Theorem 3.3.3 (Mirković-Vybornov isomorphism)**.** Let $G = GL_n$ and $\lambda, \mu \in (\mathbb{Z}_-)^n$ with $\lambda_1 \leq ... \leq \lambda_n$, $\mu_1 \leq ... \leq \mu_n$ and $d := \sum \lambda = \sum \mu$. There is an isomorphism of algebraic varieties:

$$\begin{aligned} \mathrm{Gr}_\mu^\lambda &\cong M_\mu \cap \overline{\mathbb{O}}_\lambda \\ L &\mapsto [\pi_L z \pi_L^{-1}]_A \end{aligned}$$

between the Lusztig slice Gr_μ^λ and $M_\mu \cap \overline{\mathbb{O}}_\lambda \subseteq \mathcal{N}_{\mathfrak{gl}_{-d}(\mathbb{C})}$. □

We will focus on the case of $\lambda = d\omega_1 = (d, 0, ..., 0)$ in the next chapter. Notice that in this case the Mirković-Vybornov isomorphism specializes to:

$$\mathrm{Gr}_\mu^{d\omega_1} \cong M_\mu \cap \mathcal{N}_{\mathfrak{gl}_{-d}(\mathbb{C})}$$

since $\mathbb{O}_{d\omega_1}$ is the regular nilpotent orbit and its closure is the entire nilpotent cone $\mathcal{N} \subseteq \mathfrak{gl}_{-d}(\mathbb{C})$.

Chapter 4

Explicit Isomorphisms

In this chapter we develop explicit isomorphisms between Lusztig slices and varieties of nilpotent matrices. In 4.3 we give an explicit realization of the Mirković-Vybornov isomorphism $\mathrm{Gr}_\mu^{d\omega_1} \cong M_\mu \cap \mathcal{N}_{\mathfrak{gl}_{-d}(\mathbb{C})}$ in the case of $G = GL_2$, $\mu \in (\mathbb{Z}_-)^2$, and $\mu_1 \leq \mu_2$. In 4.4 we work with lattices *contained in* the standard lattice (rather than with lattices *containing* the standard lattice) and give an explicit realization of an alternate Mirković-Vybornov isomorphism $\mathrm{Gr}_\mu^{d\omega_1} \cong \breve{M}_\mu \cap \mathcal{N}_{\mathfrak{gl}_d(\mathbb{C})}$ for $G = GL_n$, $\mu \in (\mathbb{Z}_+)^n$, $\mu_1 \geq \ldots \geq \mu_n$. Here, $\breve{M}_\mu$ is a transposed version of the Mirković-Vybornov slice.

4.1 The Subset $\mathrm{Gr}_\mu^s \subseteq \mathrm{Gr}_{GL_n}$

In this section μ will *either* be in $(\mathbb{Z}_-)^n$ with $\mu_1 \leq \ldots \leq \mu_n$ *or* in $(\mathbb{Z}_+)^n$ with $\mu_1 \geq \ldots \geq \mu_n$.

We will use Gr_μ to denote the collection of lattices of the form $G_1[z^{-1}]L_\mu$.

Note that lattices in Gr_μ are represented by matrices over the polynomial ring $\mathbb{C}[z^{-1}]$ of the form gz^μ where $g = I + g_{-1}z^{-1} + g_{-2}z^{-2} + \ldots + g_{-k}z^{-k}$ $(g_i \in \mathfrak{gl}_n(\mathbb{C}))$ and $\det(g) \in \mathbb{C}^*$.

For $s \in \mathbb{Z}$, we will define subsets $\mathrm{Gr}_\mu^s \subseteq \mathrm{Gr}_\mu$ that behave like "truncations at valuation s" of Gr_μ.

Definition 4.1.1. For $s \in \mathbb{Z}$ define the subset $\mathrm{Gr}_\mu^s \subseteq \mathrm{Gr}_\mu$ to be those lattices in Gr_μ whose representative matrices in $GL_n(\mathcal{K})$ have entries of minimum valuation equal to s.

The minimum valuation of a lattice is in fact a well-defined property of the lattice. Let $g \in GL_n(\mathcal{K})$ be a representative matrix of a lattice L. The condition that the entries of g have valuation no lower than s is unaffected if we left or right-multiply g by matrices in $GL_n(\mathcal{O})$. Therefore this is a property not only of the lattice L, but is in fact a property of the entire $GL_n(\mathcal{O})$-orbit of L.

It is an important observation that lattices in Gr_μ do not necessarily satisfy any containment relations with the standard lattice, as the following example illustrates.

Example 4.1.2. Work in Gr_{GL_2}. Consider the lattice L in $G_1[z^{-1}]L_{(-1,-1)}$ represented by the matrix:

$$\begin{bmatrix} z^{-1} + z^{-3} & z^{-2} \\ z^{-2} & z^{-1} \end{bmatrix}.$$

We want to illustrate that L has no containment relation with the standard lattice. We proceed by performing $GL_n(\mathcal{O})$-row and $GL_n(\mathcal{O})$-column operations to reduce this matrix to

$$\begin{bmatrix} z^{-3} & 0 \\ 0 & z \end{bmatrix}.$$

By corollaries 2.6.5 and 2.6.6, we may conclude that L neither *contains* nor *is contained in* the standard lattice. □

4.2 The Affine Space $\mathrm{G}^s_\mu \subseteq GL_n(\mathcal{K})$

In this section μ will *either* be in $(\mathbb{Z}_-)^n$ with $\mu_1 \leq \ldots \leq \mu_n$ *or* in $(\mathbb{Z}_+)^n$ with $\mu_1 \geq \ldots \geq \mu_n$.

We are going to exhibit an affine space $\mathrm{G}^s_\mu \subseteq GL_n(\mathcal{K})$ of matrices that represent the lattices in Gr^s_μ.

Definition 4.2.1. Assume $s \in \mathbb{Z}$ and $\mu \in (\mathbb{Z}_-)^n$ with $\mu_1 \leq \ldots \leq \mu_n$. We define $\mathrm{G}^s_\mu \subseteq GL_n(\mathcal{K})$ to be the affine space of matrices of the form:

$$z^\mu + Q,$$

where $Q = (q_{ij})$ is a matrix of polynomials $q_{ij}(z^{-1}) \in \mathbb{C}[z^{-1}]$ of the form:

$$q_{ij}(z^{-1}) = \begin{cases} q^1_{ij} z^{\mu_j - 1} + q^2_{ij} z^{\mu_j - 2} + \ldots + q^{\mu_j - s}_{ij} z^s & i \geq j \\ q^1_{ij} z^{\mu_i - 1} + q^2_{ij} z^{\mu_i - 2} + \ldots + q^{\mu_i - s}_{ij} z^s & i < j \end{cases}.$$

Notice that G_μ^s only makes sense when $s \leq \mu_1$.

Proposition 4.2.2. Assume $\mu \in (\mathbb{Z}_-)^n$ with $\mu_1 \leq \ldots \leq \mu_n$ and $s \leq \mu_1$. Every lattice L in Gr_μ^s admits a matrix representative in the affine space G_μ^s.

Proof. We can represent L with a the matrix of the form $z^\mu + (g_{ij})$ for polynomials g_{ij} in the variable z^{-1} that are of the form:

$$g_{ij}(z^{-1}) = *z^{\mu_j - 1} + \ldots + *z^s.$$

Then an evident algorithm of $GL_n(\mathcal{O})$-column operations can be used to bring the matrix $z^\mu + (g_{ij})$ to a matrix of the form $z^\mu + Q \in \mathrm{G}_\mu^s$. □

Now we are going to give a definition of G_μ^s in the case of $\mu \in (\mathbb{Z}_+)^n$ with $\mu_1 \geq \ldots \geq \mu_n$.

Definition 4.2.3. Assume $s \in \mathbb{Z}_+$ and $\mu \in (\mathbb{Z}_+)^n$ with $\mu_1 \geq \ldots \geq \mu_n$. We define $\mathrm{G}_\mu^s \subseteq GL_n(\mathcal{K})$ to be the affine space of matrices of the form:

$$z^\mu + P,$$

where $P = (p_{ij})$ is a matrix of polynomials $p_{ij}(z) \in \mathbb{C}[z]$ of the form:

$$p_{ij}(z) = \begin{cases} p_{ij}^1 z^{\mu_i - 1} + p_{ij}^2 z^{\mu_i - 2} + \ldots + p_{ij}^{\mu_i - s} z^s & i \geq j \\ p_{ij}^1 z^{\mu_j - 1} + p_{ij}^2 z^{\mu_j - 2} + \ldots + p_{ij}^{\mu_j - s} z^s & i < j \end{cases}.$$

Notice that G^s_μ only makes sense when $s \le \mu_n$.

Proposition 4.2.4. Assume $\mu \in (\mathbb{Z}_+)^n$ with $\mu_1 \ge \ldots \ge \mu_n$ and $s \le \mu_n$. Every lattice L in Gr^s_μ admits a matrix representative in the affine space G^s_μ.

Proof. We can represent L with a the matrix of the form $z^\mu + (g_{ij})$ for polynomials g_{ij} in the variable z that are of the form:

$$g_{ij}(z) = * z^{\mu_i - 1} + \ldots + * z^s.$$

Then an evident algorithm of $GL_n(\mathcal{O})$-column operations can be used to bring the matrix $z^\mu + (g_{ij})$ to a matrix of the form $z^\mu + P \in \mathrm{G}^s_\mu$. □

4.3 Explicit Mirković-Vybornov

Let μ be in $(\mathbb{Z}_-)^n$ with $\mu_1 \le \ldots \le \mu_n$ and $d = \sum \mu$. In this section we give an explicit realization of the Mirković-Vybornov isomorphism $\mathrm{Gr}^\lambda_\mu \cong M_\mu \cap \overline{\mathbb{O}}_\lambda$ in the case of GL_2 and $\lambda = d\omega_1$. We explain a general approach that could potentially be adapted to yield an explicit realization of the isomorphism $\mathrm{Gr}^{d\omega_1}_\mu \cong M_\mu \cap \mathcal{N}_{\mathfrak{gl}_{-d}(\mathbb{C})}$ in the case of GL_n.

The Idea. Let $\mu \in (\mathbb{Z}_-)^n$ with $\mu_1 \le \mu_2 \le \ldots \le \mu_n$ and $d = \sum \mu$. Recall that the affine space $\mathrm{G}^s_\mu \subseteq GL_n(\mathcal{K})$ consists of representatives for lattices in Gr^s_μ. Let $s \le \mu_1$ and consider the determinant map $\det : \mathrm{G}^s_\mu \to \mathcal{K}^*$, which takes values in monic polynomials in z^{-1} of degree d. Lattices $L \in \mathrm{Gr}^s_\mu$ are of the form $[z^\mu + Q]$ (see 4.2.1), where the matrix $z^\mu + Q$ has determinant $\det(z^\mu + Q) = z^d$. Therefore if we descend the map $\det : \mathrm{G}^s_\mu \to \mathcal{K}^*$ to lattices in Gr^s_μ we obtain valdet : $\mathrm{Gr}^s_\mu \to \mathcal{K}^*/\mathcal{O}^* \cong \mathbb{Z}$ that takes only the value $\{d\}$. We will prove that $\mathrm{Gr}^s_\mu = \mathrm{Gr}^{\lambda(s)}_\mu$ for a particular $\lambda(s)$ that depends upon s. In the case of $G = GL_2$ and $s = d$, this result specializes to $\mathrm{Gr}^d_\mu = \mathrm{Gr}^{d\omega_1}_\mu$. What we gain from this is a realization of the Lusztig slice $\mathrm{Gr}^{d\omega_1}_\mu$ as represented by matrices in Gr^d_μ with determinant equal to z^d. We can then map $\mathrm{Gr}^{d\omega_1}_\mu$ to $M_\mu \cap \mathcal{N}_{\mathfrak{gl}_d(\mathbb{C})}$ by using the idea of Mirković-Vybornov (see section 3.3), which is to represent the nilpotent operator $\pi_L z \pi_L^{-1} \circlearrowright L_\mu/L_0$ using the basis A. This yields an explicit description of the Mirković-Vybornov isomorphism.

Proposition 4.3.1. Let $G = GL_n$, $\mu \in \mathbb{Z}^n$ with $\mu_1 \leq \ldots \leq \mu_n$, $d := \sum \mu$, and $s \in \mathbb{Z}$ with $s \leq \mu_1$. We have:

$$\mathrm{Gr}^s_\mu = G_1[z^{-1}]L_\mu \cap \Big(\coprod_{\nu \in \mathcal{W}} \mathrm{Gr}^\nu_\circ \Big),$$

where the set $\mathcal{W}$ consists of all coweights $\nu \in \mathbb{Z}^n$ that satisfy $\nu_1 \leq \ldots \leq \nu_n$, $s \leq \nu_1 \leq \mu_1$, and $\sum \nu = d$.

This in particular implies that:

$$\mathrm{Gr}^s_\mu = \mathrm{Gr}^{\lambda(s)}_\mu,$$

where $\lambda(s) \in \mathbb{Z}^n$ is the unique coweight in $\mathcal{W}$ such that for every $\nu \in \mathcal{W}$, $\lambda(s) - \nu$ is a sum of coroots of the form $\omega_i - \omega_{i-1}$ for $i = 2, \ldots, n$.

Proof. Suppose $L \in \mathrm{Gr}^s_\mu$, and let us write $L = [g]$ where $g \in \mathrm{G}^s_\mu$. We may assume that $\det(g) = z^d$. Suppose the minimal valuation of the entries of g is ν_1. It follows that $L \in \mathrm{Gr}^\nu_\circ$ for some ν with $\leq \nu_1 \leq \ldots \leq \nu_n$ and $s \leq \nu_1 \leq \mu_1$. By 2.9, valdet is well defined on the $GL_n(\mathcal{O})$-orbit of L, and therefore $\sum \nu = d$. Conversely, every such $\mathrm{Gr}^\nu_\circ$ is contained in Gr^s_μ. This completes the proof. ☐

Example 4.3.2. Let $G = GL_3$, $\mu = (-1,-1,-1)$, and $s = -2$. We have:

$$\mathrm{Gr}_{\mu}^{-2} = G_1[z^{-1}]L_{\mu} \cap \left(\mathrm{Gr}_{\circ}^{(-2,-2,1)} \sqcup \mathrm{Gr}_{\circ}^{(-2,-1,0)} \sqcup \mathrm{Gr}_{\circ}^{(-1,-1,-1)}\right).$$

For example, the lattice represented by the matrix

$$\begin{bmatrix} z^{-1} & z^{-2} & 0 \\ 0 & z^{-1} & z^{-2} \\ 0 & 0 & z^{-1} \end{bmatrix}$$

belongs to $G_1[z^{-1}] \cap \mathrm{Gr}_{\circ}^{(-2,-2,1)}$, and the lattice represented by the matrix

$$\begin{bmatrix} z^{-1} & z^{-2} & 0 \\ 0 & z^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$$

belongs to $G_1[z^{-1}] \cap \mathrm{Gr}_{\circ}^{(-2,-1,0)}$. □

Consider the maps $\det : \mathrm{G}_{\mu}^{s} \to \mathbb{C}[z^{-1}]$ and $\mathrm{valdet} : \mathrm{Gr}_{\mu}^{s} \to \mathbb{Z}$.

Corollary 4.3.3 (Of 4.3.1)**.** If $G = GL_2$, $\mu \in \mathbb{Z}^2$ with $\mu_1 \leq \mu_2$, and $s \leq \mu_1$, then $\lambda(s) = (s, d-s)$. In other words,

$$\mathrm{valdet}^{-1}(d) = \mathrm{Gr}_{\mu}^{s} = \mathrm{Gr}_{\mu}^{(s,d-s)}.$$

Corollary 4.3.4 (Of 4.3.1)**.** If $G = GL_2$, $\mu \in (\mathbb{Z})_{-})^2$ with $\mu_1 \leq \mu_2$ and $s = d$, then $\lambda(d) = (d, 0) = d\omega_1$. In other words,

$$\mathrm{valdet}^{-1}(d) = \mathrm{Gr}_{\mu}^{d} = \mathrm{Gr}_{\mu}^{d\omega_1}.$$

This fact leads into our explicit Mirković-Vybornov isomorphism.

Theorem 4.3.5. Let $G = GL_2$ and $\mu \in (\mathbb{Z}_-)^2$ with $\mu_1 \leq \mu_2$. The isomorphism of affine spaces $\mathrm{G}^d_\mu \leftrightarrow M_\mu$ given by

$$z^\mu + Q \quad \longleftrightarrow \quad E_\mu + \begin{bmatrix} & & & & & & & \\ & & & & & & & \\ -q_{22}^{-\mu_1} & & \cdots & -q_{22}^2 & q_{11}^1 & q_{12}^{-\mu_2} & \cdots & q_{12}^2 & q_{12}^1 \\ & & & & & & & \\ & & & & & & & \\ q_{21}^{-\mu_2} & \cdots & q_{21}^1 & & & -q_{11}^{-\mu_2} & \cdots & -q_{11}^2 & q_{22}^1 \end{bmatrix}$$

restricts to the Mirković-Vybornov isomorphism $\mathrm{Gr}^{d\omega_1}_\mu = (\mathrm{valdet})^{-1}(d) \cong M_\mu \cap \mathcal{N}_{\mathfrak{gl}_d(\mathbb{C})}$, where Q is the 2×2 matrix over $\mathbb{C}[z^{-1}]$ given by:

$$\begin{bmatrix} q_{11}^1 z^{\mu_1 - 1} + \ldots + q_{11}^{-\mu_2} z^d & q_{12}^1 z^{\mu_1 - 1} + \ldots + q_{12}^{-\mu_2} z^d \\ q_{21}^1 z^{\mu_1 - 1} + \ldots + q_{21}^{-\mu_2} z^d & q_{22}^1 z^{\mu_2 - 1} + \ldots + q_{22}^{-\mu_1} z^d \end{bmatrix}$$

and E_μ is the nilpotent of type μ in Jordan form.

Proof. Let L be a lattice in $\mathrm{Gr}^{d\omega_1}_\mu$. We may be represent L by the matrix $z^\mu + Q \in \mathrm{G}^d_\mu$. Notice that $\det(z^\mu + Q)$ has the form $z^d + *z^{d-1} + \ldots + *z^{2d}$. The lattice $L = [z^\mu + Q]$ belongs to $\mathrm{Gr}^{d\omega_1}_\mu$ if and only if $\det(z^\mu + Q) = z^d$. So all of the lower coefficients of $\det(z^\mu + Q)$ must be zero. Consider the distinguished isomorphism $L/L_0 \xrightarrow{\pi_L} L_\mu/L_0$ and the distinguished basis $A = \{[z^{-1}e_1], \ldots, [z^{\mu_1}e_1], [z^{-1}e_2], \ldots, [z^{\mu_2}e_2]\}$ for L_μ/L_0 explained in 3.3. Let

$\pi_L^{-1}(A) = \{v_{-1}, ..., v_{\mu_1}, w_{-1}, ..., w_{\mu_2}\}$ be the basis for L/L_0 that corresponds with A via the isomorphism π_L. If we represent the operator $\pi_L z \pi_L^{-1} \circlearrowright L_\mu/L_0$ in the basis A we get a nilpotent matrix $[\pi_L z \pi_L^{-1}]_A$ in the Mirković-Vybornov slice. We need to calculate the matrix $[\pi_L z \pi_L^{-1}]_A$ for the lattice $L = [z^\mu + Q] \in \mathrm{Gr}_\mu^{d\omega_1}$. One can see that v_{μ_1} and v_{μ_2} are the two columns of $z^\mu + Q$, and so the μ_1-st and the last column of $[\pi_L z \pi_L^{-1}]_A$ are evident. Now we need to find the remaining columns of the matrix $[\pi_L z \pi_L^{-1}]_A$. Let $\langle z^{\mu_1 - 1} e_1 \rangle$ and $\langle z^{\mu_2 - 1} e_2 \rangle$ be the functions that extract those coefficients of vectors in L/L_0. To complete the matrix $[\pi_L z \pi_L^{-1}]_A$ we need the coefficients $\langle z^{\mu_1 - 1} e_1 \rangle$ and $\langle z^{\mu_2 - 1} e_2 \rangle$ of the vectors $v_{\mu_1+1}, ..., v_{-1}$ and $w_{\mu_2+1}, ..., w_{-1}$. We have that

$$\det(z^\mu + Q) = z^d + (q_{11}^1 + q_{22}^1) z^{d-1} + \sum_{k=2}^{d} C_k z^{d-k}$$

where the coefficients C_k, for $k = 2, ..., d$, have the form:

$$C_k = \begin{cases} \sum\limits_{\alpha+\beta=k} q_{11}^\alpha q_{22}^\beta & 2\mu_1 - 2 - (d-k) < 0 \\ \sum\limits_{\alpha+\beta=k} q_{11}^\alpha q_{22}^\beta - \sum\limits_{\gamma+\delta=2\mu_1-2-(d-k)} q_{12}^\gamma q_{21}^\delta & 2\mu_1 - 2 - (d-k) \geq 0 \end{cases}.$$

For the vectors $v_{\mu_1+1}, ..., v_{\mu_2-1}$ we have that:

$$\begin{aligned} \langle z^{\mu_1-1} e_1 \rangle v_{\mu_1+k} &= C_{k+1} - q_{22}^{k+1} \\ &= -q_{22}^{k+1} \end{aligned}$$

$$\langle z^{\mu_2-1} e_2 \rangle v_{\mu_1+k} = 0.$$

For the vectors $v_{\mu_2}, \ldots, v_{-1}$ we have that:

$$\begin{aligned}\langle z^{\mu_1-1}e_1\rangle v_{\mu_2+k} &= C_{\mu_2-\mu_1+k+1} - q_{22}^{\mu_2-\mu_1+k+1} \\ &= -q_{22}^{\mu_2-\mu_1+k+1}\end{aligned}$$

$$\begin{aligned}\langle z^{\mu_2-1}e_2\rangle v_{\mu_2+k} &= q_{21}^{k+1} - q_{21}^{k}(q_{11}^{1} + q_{22}^{1}) \\ &= q_{21}^{k+1}\end{aligned}$$

For the vectors $w_{\mu_2+1}, \ldots, w_{-1}$ we have that:

$$\begin{aligned}\langle z^{\mu_1-1}e_1\rangle w_{\mu_2+k} &= q_{12}^{k+1} - q_{12}^{k}(q_{11}^{1} + q_{22}^{1}) \\ &= q_{12}^{k+1} \\ \langle z^{\mu_2-1}e_2\rangle w_{\mu_2+k} &= \langle z^{\mu_2-1}e_2\rangle(zw_{\mu_2+k-1} - w_{\mu_2+k-1}) \\ &= C_{k+1} - q_{11}^{k+1} \\ &= -q_{11}^{k+1}.\end{aligned}$$

Therefore the nilpotent matrix $[\pi_L z \pi_L^{-1}]_A$ is as stated in the theorem. Since the lattice $L = [z^\mu + Q]$ is completely determined by the nilpotent operator $z \circlearrowright L/L_0$ and the form that the map takes clearly has an inverse, we have found a variety isomorphism $\mathrm{Gr}_\mu^{d\omega_1} \cong M_\mu \cap \mathcal{N}_{\mathfrak{gl}_{-d}(\mathbb{C})}$. □

Example 4.3.6. Let $\mu = (-2, -1)$. The isomorphism $\mathrm{Gr}_\mu^{-3\omega_1} \cong M_\mu \cap \mathcal{N}_{\mathfrak{gl}_3(\mathbb{C})}$ takes the form:

$$\left\{ \begin{bmatrix} z^{-2} + az^{-3} & cz^{-3} \\ bz^{-3} & z^{-1} + dz^{-2} + ez^{-3} \end{bmatrix} \; : \; \det = z^{-3} \right\}$$

$$\updownarrow$$

$$\begin{bmatrix} 0 & 1 & 0 \\ -e & a & c \\ b & 0 & d \end{bmatrix} \cap \mathcal{N}_{\mathfrak{gl}_3(\mathbb{C})}.$$

□

Example 4.3.7. Let $\mu = (-2, -2)$. The isomorphism $\mathrm{Gr}_{\mu}^{-4\omega_1} \cong M_\mu \cap \mathcal{N}_{\mathfrak{gl}_4(\mathbb{C})}$ takes the form:

$$\left\{ \begin{bmatrix} z^{-2} + az^{-3} + bz^{-4} & ez^{-3} + fz^{-4} \\ cz^{-3} + dz^{-4} & z^{-2} + gz^{-3} + hz^{-4} \end{bmatrix} \; : \; \det = z^{-4} \right\}$$

$$\updownarrow$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -h & a & f & e \\ 0 & 0 & 0 & 1 \\ d & c & -b & g \end{bmatrix} \cap \mathcal{N}_{\mathfrak{gl}_4(\mathbb{C})}.$$

□

Example 4.3.8. Let $\mu = (-3, -2)$. The isomorphism $\mathrm{Gr}_{\mu}^{-5\omega_1} \cong M_\mu \cap \mathcal{N}_{\mathfrak{gl}_5(\mathbb{C})}$

takes the form:

$$\left\{ \begin{bmatrix} z^{-3} + az^{-4} + bz^{-5} & ez^{-4} + fz^{-5} \\ cz^{-4} + dz^{-5} & z^{-2} + gz^{-3} + hz^{-4} + iz^{-5} \end{bmatrix} : \det = z^{-5} \right\}$$

$$\updownarrow$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -i & -h & a & f & e \\ 0 & 0 & 0 & 0 & 1 \\ d & c & 0 & -b & g \end{bmatrix} \cap \mathcal{N}_{\mathfrak{gl}_5(\mathbb{C})}.$$

□

Example 4.3.9. Let $\mu = (d, 0) = d\omega_1$. Notice that $\mathrm{Gr}^{d\omega_1}_{d\omega_1}$ consists of a single point, $L_{d\omega_1}$. The isomorphism $\mathrm{Gr}^{d\omega_1}_{d\omega_1} \cong M_\mu \cap \mathcal{N}_{\mathfrak{gl}_d(\mathbb{C})}$ takes the form:

$$\{L_{d\omega_1}\}$$
$$\updownarrow$$
$$\{E_{d\omega_1}\}$$

where $E_{d\omega_1}$ is the $d \times d$ regular nilpotent matrix with ones on the super-diagonal.

□

Corollary 4.3.10 (Of 4.3.5)**.** In the case of $G = GL_2$, the set of matrices in G^d_μ having determinant equal to z^d is in one-to-one correspondence with $\mathrm{Gr}^{d\omega_1}_\mu$.

There is an interesting potential relationship between the determinant of matrices in G_μ^d and the characteristic polynomial of matrices in the Mirković-Vybornov slice $M_\mu \subseteq \mathfrak{gl}_{-d}(\mathbb{C})$. Let χ be the map that associates to a matrix $A \in \mathfrak{gl}_d(\mathbb{C})$ its characteristic polynomial $\chi_A(z) = \det(zI - A)$, and let ρ denote the map from 4.3.5. Consider the following diagram.

$$\begin{array}{ccc} G_\mu^d & \xrightarrow{\rho} & M_\mu \\ & {\scriptstyle \det} \searrow \quad \swarrow {\scriptstyle \chi} & \\ & \mathbb{C}[z, z^{-1}] & \end{array}$$

After some experimentation, it appears plausible to have a formula like

$$\det(g) = z^{2d} \chi_{\rho(g)}(-z).$$

However, the following example illustrates a case where $\det(g)$ is comes close to, but does not equal $z^{-6}\chi_{\rho(g)}(-z)$. It would be interesting to find an equation that relates determinants of matrices over $\mathbb{C}[z^{-1}]$ to characteristic polynomials either by using an adapted slice or finding another map similar to ρ. What makes this question interesting is that such a formula *does* exist for the alternate Mirković-Vybornov isomorphism that is discussed in 4.4.

Example 4.3.11. Let $\mu = (-2, -1)$. Then we have:

$$\det\begin{bmatrix} z^{-2}+az^{-3} & cz^{-3} \\ bz^{-3} & z^{-1}+dz^{-2}+ez^{-3} \end{bmatrix} = z^{-3}+(a+d)z^{-4}+(e+ad)z^{-5}+(ae-bc)z^{-6}$$

$$\chi\begin{bmatrix} 0 & 1 & 0 \\ -e & a & c \\ b & 0 & d \end{bmatrix} = z^3-(a+d)z^2+(e+ad)z-(ed+bc)$$

□

4.4 Explicit Alternate Mirković-Vybornov

Let μ be in $(\mathbb{Z}_+)^n$ with $\mu_1 \geq \ldots \geq \mu_n$ and $d = \sum \mu$. In this section, we will give an explicit realization of an alternate Mirković-Vybornov isomorphism $\mathrm{Gr}^\lambda_\mu \cong M_\mu \cap \overline{\mathbb{O}}_\lambda$ in the case of $G = GL_n$ and $\lambda = d\omega_1 = (d, 0, \ldots, 0)$. Note that while Mirković and Vybornov considered lattices *containing* the standard lattice, this section deals with lattices that are *contained in* the standard lattice.

Definition 4.4.1 (Alternate Mirković-Vybornov Slice)**.** Let $\mu \in (\mathbb{Z}_+)^n$ and $\mu_1 \geq \ldots \geq \mu_n$. Let $d = \sum \mu$. Define the *alternate Mirković-Vybornov slice* to be the affine subspace $\breve{M}_\mu \subseteq \mathfrak{gl}_d(\mathbb{C})$ consisting of $\mu_i \times \mu_j$ block matrices $(\breve{M}_{ij})$ where : if $i = j$ then $\breve{M}_{ij}$ has ones on the sub-diagonal and arbitrary entries along the last column; and if $i \neq j$ then $\breve{M}_{ij}$ has entries in the last column but not below row μ_j.

Notice that $\breve{M}_\mu$ is the transpose of the Mirković-Vybornov slice M_μ.

The Idea. Let $\mu \in (\mathbb{Z}_+)^n$ with $\mu_1 \geq \ldots \geq \mu_n$ and $d = \sum \mu$. The same principles from 4.3 will continue to apply. Every lattice in Gr^0_μ is represented by a matrix in G^0_μ and we notice that $\mathrm{G}^0_\mu \subseteq \mathfrak{gl}_n(\mathcal{O})$. By 2.6.3 this implies that every lattice in Gr^0_μ is contained in the standard lattice L_0. Using the idea of 4.3.1 but this time with the condition $\mu_1 \geq \ldots \geq \mu_n$, we can see that $\mathrm{Gr}^0_\mu = \mathrm{Gr}^{d\omega_1}_\mu$. We then proceed by taking $L \in \mathrm{Gr}^{d\omega_1}_\mu$ and examining the action of the nilpotent operator $z \circlearrowright L_0/L$ in the basis $B = \{[e_1], \ldots, [z^{\mu_1 - 1}e_1], \ldots, [e_n], \ldots, [z^{\mu_n - 1}e_n]\}$ for L_0/L. This associates to L a nilpotent matrix. The significance of the alternate slice $\breve{M}_\mu$ introduced in definition 4.4.1 is that it is a natural receptacle for these nilpotent matrices coming from the Lusztig slice $\mathrm{Gr}^{d\omega_1}_\mu$. This idea turns out to give an explicit isomorphism $\mathrm{Gr}^{d\omega_1}_\mu \cong \breve{M}_\mu \cap \mathcal{N}_{\mathfrak{gl}_d(\mathbb{C})}$.

Theorem 4.4.2. Let $G = GL_n$ and $\mu \in (\mathbb{Z}_+)^n$ with $\mu_1 \geq \ldots \geq \mu_n$. The isomorphism of affine spaces $\mathrm{G}^0_\mu \leftrightarrow \breve{M}_\mu$ given by:

$$z^\mu + P \quad \longleftrightarrow \quad \breve{E}_\mu + \begin{bmatrix} \vdots & & \vdots & \vdots \\ 0 & & 0 & -p^2_{ij} \\ 0 & \ldots & 0 & -p^1_{ij} \\ 0 & & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & & 0 & 0 \end{bmatrix}_{\mu_i \times \mu_j}$$

restricts to an isomorphism $\mathrm{Gr}^{d\omega_1}_\mu \cong \breve{M}_\mu \cap \mathcal{N}_{\mathfrak{gl}_d(\mathbb{C})}$, where P is the $n \times n$ matrix over $\mathbb{C}[z]$ introduced in 4.2.3 and given by:

$$\begin{bmatrix} p^1_{11}z^{\mu_1-1}+\ldots+p^{\mu_1}_{11} & p^1_{12}z^{\mu_2-1}+\ldots+p^{\mu_2}_{12} & \ldots & p^1_{1n}z^{\mu_n-1}+\ldots+p^{\mu_n}_{1n} \\ p^1_{21}z^{\mu_2-1}+\ldots+p^{\mu_2}_{21} & p^1_{22}z^{\mu_2-1}+\ldots+p^{\mu_2}_{22} & \ldots & p^1_{2n}z^{\mu_n-1}+\ldots+p^{\mu_n}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p^1_{n1}z^{\mu_n-1}+\ldots+p^{\mu_n}_{n1} & p^1_{n2}z^{\mu_n-1}+\ldots+p^{\mu_n}_{n2} & \ldots & p^1_{nn}z^{\mu_n-1}+\ldots+p^{\mu_n}_{nn} \end{bmatrix},$$

and $\breve{E}_\mu$ is the $d\times d$ nilpotent of type μ with ones on the sub-diagonal and the other matrix is a $d\times d$ matrix that is given by a $\mu_i \times \mu_j$ block description.

Proof. Let L be a lattice in $\mathrm{Gr}^{d\omega_1}_\mu$. We may be represent $L=[z^\mu+P]$ by the matrix $z^\mu+P\in \mathrm{G}^0_\mu$. Let us find the matrix $[z]_B$ of the operator $z \circlearrowright L_0/L$ with respect to the basis $B=\{[e_1],\ldots,[z^{\mu_1-1}e_1],\ldots,[e_n],\ldots,[z^{\mu_n-1}e_n]\}$ for L_0/L. We have that $z[z^ie_j]=[z^{i+1}e_j]$ for $0\le i<\mu_j-1$ and $1\le j\le n$. It remains to determine $z[z^{\mu_j-1}e_j]$ in terms of the basis B. We calculate:

$$\begin{aligned} z[z^{\mu_1-1}e_1] &= [z^{\mu_1}e_1] \\ &= -\sum_{s=1}^{\mu_1} p^s_{11}[z^{\mu_1-s}e_1] - \sum_{s=1}^{\mu_2} p^s_{21}[z^{\mu_2-s}e_2] - \ldots - \sum_{s=1}^{\mu_n} p^s_{n1}[z^{\mu_n-s}e_n] \end{aligned}$$

$$\begin{aligned} z[z^{\mu_2-1}e_2] &= [z^{\mu_2}e_2] \\ &= -\sum_{s=1}^{\mu_2} p^s_{12}[z^{\mu_2-s}e_1] - \sum_{s=1}^{\mu_2} p^s_{22}[z^{\mu_2-s}e_2] - \ldots - \sum_{s=1}^{\mu_n} p^s_{nj}[z^{\mu_n-s}e_n] \end{aligned}$$

$$\vdots$$

$$\begin{aligned} z[z^{\mu_n-1}e_n] &= [z^{\mu_n}e_n] \\ &= -\sum_{s=1}^{\mu_n} p^s_{1n}[z^{\mu_n-s}e_1] - \sum_{s=1}^{\mu_n} p^s_{2n}[z^{\mu_n-s}e_2] - \ldots - \sum_{s=1}^{\mu_n} p^s_{nn}[z^{\mu_n-s}e_n]. \end{aligned}$$

Therefore $[z]_B$ is a nilpotent operator of the form described in the theorem. Since the lattice $L = [z^\mu + P]$ is completely determined by the nilpotent operator $z \circlearrowright L_0/L$ and the form that the map takes clearly has an inverse, we have found a variety isomorphism $\mathrm{Gr}_\mu^{d\omega_1} \cong \breve{M}_\mu \cap \mathcal{N}_{\mathfrak{gl}_d(\mathbb{C})}$. □

Example 4.4.3. Let $\mu = \omega_1 + \ldots + \omega_n = (1, \ldots, 1)$, then $\breve{M}_\mu = \mathfrak{gl}_n(\mathbb{C})$. The isomorphism $\mathrm{Gr}_\mu^{n\omega_1} \cong \mathcal{N}_{\mathfrak{gl}_n(\mathbb{C})}$ takes the form:

$$\left\{ z^{\omega_1 + \ldots + \omega_n} + P : \det = z^n \right\} \longleftrightarrow -P.$$

□

Example 4.4.4. Let $\mu = (2, 1)$. The isomorphism $\mathrm{Gr}_\mu^{3\omega_1} \cong \breve{M}_\mu \cap \mathcal{N}_{\mathfrak{gl}_3(\mathbb{C})}$ takes the form:

$$\left\{ \begin{bmatrix} z^2 + az + b & d \\ c & z + e \end{bmatrix} : \det = z^3 \right\}$$

$$\updownarrow$$

$$\begin{bmatrix} 0 & -b & -d \\ 1 & -a & 0 \\ 0 & -c & -e \end{bmatrix} \cap \mathcal{N}_{\mathfrak{gl}_3(\mathbb{C})}.$$

□

Example 4.4.5. Let $\mu = (2, 2, 1)$. The isomorphism $\mathrm{Gr}_\mu^{5\omega_1} \cong \breve{M}_\mu \cap \mathcal{N}_{\mathfrak{gl}_5(\mathbb{C})}$ takes the form:

$$\left\{ \begin{bmatrix} z^2+az+b & fz+g & k \\ cz+d & z^2+hz+i & l \\ e & j & z+m \end{bmatrix} : \det = z^5 \right\}$$

$$\updownarrow$$

$$\begin{bmatrix} 0 & -b & 0 & -g & -k \\ 1 & -a & 0 & -f & 0 \\ 0 & -d & 0 & -i & -l \\ 0 & -c & 1 & -h & 0 \\ 0 & -e & 0 & -j & -m \end{bmatrix} \cap \mathcal{N}_{\mathfrak{gl}_5(\mathbb{C})}.$$

□

Example 4.4.6. Let $\mu = (d, 0, ..., 0) = d\omega_1$. Notice that $\mathrm{Gr}^{d\omega_1}_{d\omega_1}$ consists of a single point, $L_{d\omega_1}$. The isomorphism takes the form:

$$\{L_{d\omega_1}\} = \left\{ \begin{bmatrix} z^d + a_1 z^{d-1} + \ldots + a_d & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} : \det = z^d \right\}$$

$$\updownarrow$$

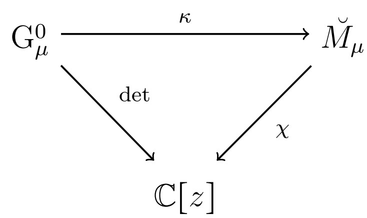
$$\{\breve{E}_{d\omega_1}\} = \left\{ \begin{bmatrix} & & & -a_d \\ 1 & & & \vdots \\ & \ddots & & -a_2 \\ & & 1 & -a_1 \end{bmatrix} : a_1 = a_2 = \ldots = a_d = 0 \right\}$$

where $\breve{E}_{d\omega_1}$ is the $d \times d$ regular nilpotent matrix with ones on the sub-diagonal.

□

Corollary 4.4.7 (Of 4.4.2)**.** In the case of $G = GL_n$, the set of matrices in G^0_μ having determinant equal to z^d is in one-to-one correspondence with $\mathrm{Gr}^{d\omega_1}_\mu$.

In this section, as opposed to 4.3, there is a direct relationship between the determinant of matrices in G^0_μ and the characteristic polynomial of matrices in the alternate Mirković-Vybornov slice $\breve{M}_\mu \subseteq \mathfrak{gl}_d(\mathbb{C})$. Let χ be the map that associates to a matrix $A \in \mathfrak{gl}_d(\mathbb{C})$ its characteristic polynomial $\chi_A(z) = \det(zI - A)$, and let κ denote the isomorphism from 4.4.2. Consider the diagram:



One can verify that the diagram commutes:

$$\det(g) = \chi_{\kappa(g)}(z).$$

Below is an example of this phenomenon.

Example 4.4.8. Let $\mu = (2, 2)$. Then we have:

$$\det\begin{bmatrix} z^2 + az + b & ez + f \\ cz + d & z^2 + gz + h \end{bmatrix}$$

$$= z^4 + (a + g)z^3 + (b + h + ag - ec)z^2 + (ah + gb - de - cf)z + (bh - df)$$

$$\chi\begin{bmatrix} 0 & -b & 0 & -f \\ 1 & -a & 0 & -e \\ 0 & -d & 0 & -h \\ 0 & -c & 1 & -g \end{bmatrix}$$

$$= z^4 - (-a - g)z^3 + (b + h + ag - ec)z^2 - (-bg + cf - ah + ed)z + (bh - df).$$

□

Chapter 5

$\mathbb{G}_a$-Actions

In this section we will introduce a certain $\mathbb{G}_a$-action on the Lusztig slices $\mathrm{Gr}^{\lambda}_{\mu}$. We will focus on the case of $G = GL_n$ and $\lambda, \mu \in (\mathbb{Z}_{-})^n$ with $\mu_1 \leq \ldots \leq \mu_n$ and $\lambda_1 \leq \ldots \leq \lambda_n$.

5.1 The Action $\mathbb{G}_a \circlearrowright \mathrm{Gr}^{\lambda}_{\mu}$

Let α be a simple coroot of GL_n with its corresponding additive subgroup $x_\alpha : \mathbb{G}_a \to N$. For each simple coroot α, define an action $\mathbb{G}_a \circlearrowright \mathrm{Gr}_G$ as follows:

$$a \cdot L = x_\alpha(a) L.$$

Example 5.1.1. The action $\mathbb{G}_a \circlearrowright \mathrm{Gr}_{GL_2}$ is given by:

$$L \to \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} L.$$

□

Example 5.1.2. The two actions $\mathbb{G}_a \circlearrowright \mathrm{Gr}_{GL_3}$ corresponding to the two simple coroots of GL_3 are given by:

$$L \to \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} L \qquad L \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix} L.$$

□

Proposition 5.1.3. This $\mathbb{G}_a$-action restricts to an action $\mathbb{G}_a \circlearrowright \mathrm{Gr}^\lambda_\mu$.

Proof. Clearly $\mathbb{G}_a$ preserves Gr^λ. We must only show that $\mathbb{G}_a$ preserves $G_1[z^{-1}]L_\mu$. To this end, notice that $x_\alpha(a)L_\mu = L_\mu$, and so if $g \in G_1[z^{-1}]$ we have:

$$x_\alpha(a)gL_\mu = x_\alpha(a)gx_\alpha(-a)L_\mu.$$

The proof is completed by noticing that $g \in G_1[z^{-1}]$ implies $x_\alpha(a)gx_\alpha(-a) \in G_1[z^{-1}]$. □

Conjecture 5.1.4. Assume that $\mu + \alpha$ satisfies $\mu_1 + \alpha_1 \leq \ldots \leq \mu_n + \alpha_n$. There is an isomorphism of Poisson varieties:

$$\mathrm{Gr}^\lambda_\mu /\!/_1 \mathbb{G}_a \cong \mathrm{Gr}^\lambda_{\mu+\alpha}.$$

5.2 The Action $\mathbb{G}_a \circlearrowright M_\mu \cap \mathcal{N}_{\mathfrak{gl}_{-d}(\mathbb{C})}$

Let $G = GL_n$ and $\mu \in (\mathbb{Z}_-)^n$ with $\mu_1 \leq \ldots \leq \mu_n$. Consider the Mirković-Vybornov isomorphism from 3.3 where $d := \sum \mu$ and $\lambda = d\omega_1$:

$$\begin{array}{ccc} \mathrm{Gr}^{d\omega_1}_\mu & \xrightarrow{\sim} & M_\mu \cap \mathcal{N}_{\mathfrak{gl}_{-d}(\mathbb{C})} \\ L & \longmapsto & [\pi_L^{-1} z \pi_L]_A, \end{array}$$

where A is the basis $A = \{[z^{-1}e_1], ..., [z^{\mu_1}e_1], ..., [z^{-1}e_n], ..., [z^{\mu_n}e_n]\}$ for L_μ/L_0 and π_L is the distinguished isomorphism $\pi_L : L/L_0 \cong L_\mu/L_0$.

We are going to transport the action $\mathbb{G}_a \circlearrowright \mathrm{Gr}_\mu^{d\omega_1}$ through the Mirković-Vybornov isomorphism. First we will look at an example, and then we will explain the general case. The action $\mathbb{G}_a \circlearrowright M_\mu \cap \mathcal{N}_{\mathfrak{gl}_{-d}(\mathbb{C})}$ turns out to have the form $X \to X^a$, where

$$X^a = h(a)\underline{a}^{-1}X\underline{a}h(a)^{-1}.$$

We first conjugate $X \in M_\mu \cap \mathcal{N}_{\mathfrak{gl}_{-d}(\mathbb{C})}$ by a unipotent matrix $\underline{a} \in GL_{-d}(\mathbb{C})$ induced from the action of $a \in \mathbb{G}_a$ on the vector space L_μ/L_0, and then we conjugate by another unipotent matrix $h(a) \in GL_{-d}(\mathbb{C})$ in order to bring the matrix $\underline{a}^{-1}X\underline{a}$ back into the Mirković-Vybornov slice M_μ.

Here is a complete example.

Example 5.2.1. Let $G = GL_2$, $\mu = (-2, -1)$, and $A = \{[z^{-1}e_1], [z^{-2}e_1], [z^{-1}e_2]\}$ be our basis for L_μ/L_0. By 4.3.6 the isomorphism $\mathrm{Gr}_\mu^{-3\omega_1} \cong M_\mu \cap \mathcal{N}_{\mathfrak{gl}_3(\mathbb{C})}$ is given by:

$$\left\{ \begin{bmatrix} z^{-2} + az^{-3} & cz^{-3} \\ bz^{-3} & z^{-1} + dz^{-2} + ez^{-3} \end{bmatrix} : \ \det = z^{-3} \right\}$$

$$\updownarrow$$

$$\begin{bmatrix} 0 & 1 & 0 \\ -e & a & c \\ b & 0 & d \end{bmatrix} \cap \mathcal{N}_{\mathfrak{gl}_3(\mathbb{C})},$$

which yields the following equations:

$$\begin{aligned} 0 &= a + d \\ 0 &= e + ad \\ 0 &= ae - bc. \end{aligned}$$

Putting everything in terms of b, c and d we have:

$$\left\{ \begin{bmatrix} z^{-2} - dz^{-3} & cz^{-3} \\ bz^{-3} & z^{-1} + dz^{-2} + d^2 z^{-3} \end{bmatrix} : d^3 + bc = 0 \right\}$$

$$\updownarrow$$

$$\left\{ \begin{bmatrix} 0 & 1 & 0 \\ -d^2 & -d & c \\ b & 0 & d \end{bmatrix} : d^3 + bc = 0 \right\}.$$

For $L \in \mathrm{Gr}_\mu^{-3\omega_1}$ we will employ "box-diagrams" to represent the distinguished basis for L/L_0 (here the boxes in row j correspond to $[z^{-3}e_j], [z^{-2}e_j], [z^{-1}e_j]$):

$$\{v_1, v_2, v_3\} = \left\{ \begin{array}{|c|c|c|} \hline -d^2 & 0 & 1 \\ \hline bd & b & 0 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline -d & 1 & 0 \\ \hline b & 0 & 0 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline c & 0 & 0 \\ \hline d^2 & d & 1 \\ \hline \end{array} \right\}.$$

Recall that the action $\mathbb{G}_a \circlearrowright \mathrm{Gr}_\mu^{-3\omega_1}$ is given by:

$$L \to \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} L.$$

The action of $a \in \mathbb{G}_a$ sends this to a basis for aL/L_0, namely the basis:

$$\{w_1, w_2, w_3\} = \left\{ \begin{array}{|c|c|c|} \hline -d^2 + abd & ba & 1 \\ \hline bd & b & 0 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline -d + ab & 1 & 0 \\ \hline b & 0 & 0 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline c + ad^2 & da & a \\ \hline d^2 & d & 1 \\ \hline \end{array} \right\}.$$

However the basis $\{w_1, w_2, w_3\}$ for aL/L_0 is not ready to be identified with the basis A for L_μ/L_0. Instead we must "correct" the basis $\{w_1, w_2, w_3\}$ if we wish to see the action $\mathbb{G}_a \circlearrowright M_\mu \cap \mathcal{N}_{\mathfrak{gl}_3(\mathbb{C})}$. The first correction is $\{w_1, w_2, w_3\} \mapsto \{w_1', w_2', w_3'\} = \{w_1, w_2, w_3 - aw_1\}$. We have:

$$\{w_1', w_2', w_3'\}$$

$$= \left\{ \begin{array}{|c|c|c|} \hline -d^2 + abd & ba & 1 \\ \hline bd & b & 0 \\ \hline \end{array}\ , \begin{array}{|c|c|c|} \hline -d + ab & 1 & 0 \\ \hline b & 0 & 0 \\ \hline \end{array}\ , \begin{array}{|c|c|c|} \hline c + 2ad - a^2bd & ad - a^2b & a \\ \hline d^2 - abd & d - ab & 1 \\ \hline \end{array} \right\}.$$

This corresponds to the conjugation:

$$M_\mu \cap \mathcal{N}_{\mathfrak{gl}_3(\mathbb{C})} \ni X \quad \longmapsto \quad \underline{a}^{-1} X \underline{a}$$

where the matrix $\underline{a}$ is induced from a acting on L_μ/L_0 in the basis A is:

$$\underline{a} = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in GL_3(\mathbb{C}).$$

Notice that the matrix $\underline{a}^{-1} X \underline{a}$ does not necessarily remain in the Mirković-Vybornov slice. We must conjugate it by another unipotent matrix $h(a) \in GL_3(\mathbb{C})$ to bring it back into the slice. This corresponds to performing the remaining corrections:

$$\begin{aligned}\{w_1', w_2', w_3'\} &\longrightarrow \{w_1', w_2', w_3' - *w_2'\} &&=: \{w_1'', w_2'', w_3''\}\\ \{w_1'', w_2'', w_3''\} &\longrightarrow \{w_1'' - *w_2'', w_2'', w_3''\},\end{aligned}$$

resulting in the following basis for aL/L_0:

$$\left\{ \begin{array}{|c|c|c|}\hline -(d-ab)^2 & 0 & 1 \\ \hline b(d-ab) & b & 0 \\ \hline \end{array}, \begin{array}{|c|c|c|}\hline -(d-ab) & 1 & 0 \\ \hline b & 0 & 0 \\ \hline \end{array}, \begin{array}{|c|c|c|}\hline -b^{-1}(d-ab)^3 & 0 & 0 \\ \hline (d-ab)^2 & d-ab & 1 \\ \hline \end{array} \right\}.$$

We have therefore calculated the action $\mathbb{G}_a \circlearrowright M_\mu \cap \mathcal{N}_{\mathfrak{gl}_3(\mathbb{C})}$

$$X \longmapsto h(a)\underline{a}^{-1} X \underline{a} h(a)^{-1}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ -d^2 & -d & c \\ b & 0 & d \end{bmatrix} \longmapsto \begin{bmatrix} 0 & 1 & 0 \\ -(d-ab)^2 & -(d-ab) & -b^{-1}(d-ab)^3 \\ b & 0 & d-ab \end{bmatrix}.$$

□

Definition 5.2.2. Let $U_\alpha \subseteq GL_{-d}(\mathbb{C})$ be the matrix representation of $\mathbb{G}_a$ induced on the vector space L_μ/L_0 (as in 5.2.1) from the action $\mathbb{G}_a \circlearrowright \mathrm{Gr}_{GL_n}$.

Definition 5.2.3. Consider our action $\mathbb{G}_a \circlearrowright \mathrm{Gr}_{GL_n}$. Let $L \in \mathrm{Gr}_\mu^\lambda$ and $\{v_1, ..., v_{-d}\} = \pi_L^{-1}(A)$ be the basis for L/L_0 from the Mirković-Vybornov isomorphism. For $a \in \mathbb{G}_a$, let $\{w_1, ..., w_{-d}\}$ be the resultant basis for aL/L_0. We define $h(a) \in GL_{-d}(\mathbb{C})$ to be the unipotent matrix consisting of all corrections to the basis $\{w_1, ..., w_{-d}\}$ that follow the initial correction $\underline{a} \in GL_{-d}(\mathbb{C})$ when finding a basis for aL/L_0 to identify with the basis A (see example 5.2.1).

The reason the action $\mathbb{G}_a \circlearrowleft M_\mu \cap \mathcal{N}_{\mathfrak{gl}_{-d}(\mathbb{C})}$ has this form is because the isomorphism $\pi_{aL} : aL/L_0 \cong L_\mu/L_0$ is related to $\pi_L : L/L_0 \cong L_\mu/L_0$ through the formula $\pi_{aL} = h(a)\underline{a}^{-1}\pi_L a^{-1}$, where $\underline{a} \in U_\alpha$ and $h(a) \in GL_{-d}(\mathbb{C})$ are the unipotent matrices consisting of the operations needed before we can identify the basis of aL/L_0 with our basis A of L_μ/L_0. Therefore

$$\begin{aligned} X^a &= [\pi_{aL} z \pi_{aL}^{-1}]_A \\ &= h(a)\underline{a}^{-1}[\pi_L z \pi_L^{-1}]_A \underline{a} h(a)^{-1} \\ &= h(a)\underline{a}^{-1} X \underline{a} h(a)^{-1}. \end{aligned}$$

5.3 Equality of $\mathbb{G}_a \circlearrowleft M_\mu \cap \mathcal{N}$ and $\mathbb{G}_a \circlearrowleft S(P_\mu)$

We will work in the case of $G = GL_n$ and with Lusztig slices $\mathrm{Gr}_\mu^{d\omega_1}$ where $\mu \in (\mathbb{Z}_-)^n$ with $\mu_1 \le \ldots \le \mu_n$ and $d := \sum \mu$.

The following conjecture is related to 5.1.4 through the Mirković-Vybornov isomorphism.

Conjecture 5.3.1. Assume that $\mu + \alpha$ satisfies $\mu_1 + \alpha_1 \le \ldots \le \mu_n + \alpha_n$. Then there is an isomorphism of Poisson varieties:

$$(M_\mu \cap \mathcal{N}_{\mathfrak{gl}_{-d}(\mathbb{C})}) /\!\!/_1 \mathbb{G}_a \cong M_{\mu+\alpha} \cap \mathcal{N}_{\mathfrak{gl}_{-d}(\mathbb{C})}.$$

A similar question was studied by Morgan [Mor14]. Instead of the Mirković-Vybornov slice intersected with the nilpotent cone, he considered Slodowy slices $S(P)$ with respect to certain pyramids P (described in [Mor14]). Slodowy slices, like Mirković-Vybornov slices, are examples of affine subspaces of $\mathfrak{gl}_{-d}(\mathbb{C})$ which are transverse to the nilpotent orbits $\mathbb{O}_\mu$. In 5.3.5 we show that Slodowy slices are Poisson isomorphic with Mirković-Vybornov slices. While the reader should see [Mor14, 3.2] for the general definition of a pyramid, it will suffice for us to let a pyramid be a numbered right-justified Young diagram.

In the context of the Mirković-Vybornov isomorphism we were led to consider the basis $A = \{[z^{-1}e_1], ..., [z^{\mu_1}e_1], ..., [z^{-1}e_n], ..., [z^{\mu_n}e_n]\}$ for L_μ/L_0.

Let us depict the basis A for L_μ/L_0 as a numbered right-justified Young diagram. In a similar fashion as [Mor14, 3.4.1] we are going to regard this Young diagram as a pyramid P_μ.

Example 5.3.2. Let $G = GL_3$ and $\mu = (-3,-2,-1)$. The basis for L_μ/L_0 is $A = \{[z^{-1}e_1],[z^{-2}e_1],[z^{-3}e_1],[z^{-1}e_2],[z^2e_2],[z^{-1}e_3]\}$. Depict A as a numbered right-justified Young diagram and regard it as a pyramid:

$$P_\mu = \begin{array}{|c|c|c|} \hline 3 & 2 & 1 \\ \hline & 5 & 4 \\ \cline{2-3} & & 6 \\ \cline{3-3} \end{array}\,.$$

Let α be a simple coroot of GL_n and suppose that μ and $\mu+\alpha$ are in $(\mathbb{Z}_-)^n$ with $\mu_1 \leq ... \leq \mu_n$, and $\mu_1+\alpha_1 \leq ... \leq \mu_n+\alpha_n$. Recall that $a \cdot L = x_\alpha(a)L$ gives an action $\mathbb{G}_a \circlearrowright \mathrm{Gr}_{GL_n}$ which restricts to an action on the Lusztig slices.

In proposition 5.3.5, following [GG02] and a yet-to-be-published paper of Weekes-Yacobi, we show that the Mirković-Vybornov slice M_μ is Poisson isomorphic to a Hamiltonian reduction of $\mathfrak{gl}_{-d}(\mathbb{C})$ by a unipotent group $M(P_\mu)$ given by $(E_\mu + \mathfrak{m}(P_\mu)^\perp)/M(P_\mu)$. Here E_μ is the nilpotent of type μ in Jordan form, P_μ is the pyramid described above, $M(P_\mu)$ is the unipotent Premet group associated to P_μ, and $\mathfrak{m}(P_\mu)$ is its nilpotent Lie algebra. These objects will be defined below. Then in 5.3.6 we show that our action $\mathbb{G}_a \circlearrowright M_\mu \cap \mathcal{N}_{\mathfrak{gl}_{-d}(\mathbb{C})}$ coincides with the action $\mathbb{G}_a \circlearrowright S(P_\mu)$ on the Slodowy slice that is described in [Mor14, 3.4.3].

Clarification. We are going to be numbering the rows and columns of the pyramid P_μ in such a way that the row numbers increase as you move up, and the column numbers increase as you move right to left.

Definition 5.3.3 (The Premet group $M(P_\mu)$)**.** The Premet group $M(P_\mu)$ with respect to the pyramid P_μ is the following unipotent subgroup of $GL_{-d}(\mathbb{C})$,

$$M(P_\mu) := I_{-d} + \mathbb{C}\{E_{lk} : \mathrm{col}(l) > \mathrm{col}(k)\}.$$

Definition 5.3.4 (The Premet Lie algebra $\mathfrak{m}(P_\mu)$)**.** The Premet Lie algebra $\mathfrak{m}(P_\mu)$ with respect to the pyramid P_μ is the following nilpotent Lie subalgebra of $\mathfrak{gl}_{-d}(\mathbb{C})$,

$$\mathfrak{m}(P_\mu) = \mathbb{C}\{E_{lk} : \mathrm{col}(l) > \mathrm{col}(k)\}.$$

Proposition 5.3.5. We have isomorphisms of Poisson varieties:

$$M_\mu \cong (E_\mu + \mathfrak{m}(P_\mu)^\perp)/M(P_\mu) \cong S(P_\mu)$$

Proof. Let $E := E_\mu$ be a nilpotent matrix of type μ in Jordan form and $\{E, F, H\}$ be an $\mathfrak{sl}(\mathbb{C})_2$-triple containing E. Also let $M := M(P_\mu)$ be the Premet group for the pyramid P_μ with lie algebra $\mathfrak{m} := \mathfrak{m}(P_\mu)$ (see [Mor14, 3.4.1]). We will argue that any affine slice of the form $E+C$, where $C \subseteq \mathfrak{gl}_{-d}(\mathbb{C})$ is a subspace that satisfies $\mathfrak{gl}_{-d}(\mathbb{C}) = [\mathfrak{gl}_{-d}(\mathbb{C}), E] \oplus C$ with all the eigenvalues of ad_H on C being ≤ -2, is necessarily Poisson isomorphic to $(E + \mathfrak{m}^\perp)/M$. This will suffice, as one can check that both M_μ and $S(P_\mu)$ are of the form $E + C$ where C satisfies the required conditions (see [MV08, 1.4]). In the case of $S(P_\mu)$ we have that $C = \mathfrak{z}(F)$. Let us consider a general affine subspace $E + C \subseteq \mathfrak{gl}_{-d}(\mathbb{C})$ with C satisfying the conditions. We know that C must be contained within the ≤ -2 weight spaces for ad_H, i.e. $C \subset \bigoplus_{i \leq -2} \mathfrak{g}(i) \subset \mathfrak{m}^\perp$, and therefore the image of the adjoint-action map $M \times (E + C) \to \mathfrak{gl}_{-d}(\mathbb{C})$ is contained in $E+\mathfrak{m}^\perp$. Since $\mathfrak{gl}_{-d}(\mathbb{C}) = [\mathfrak{gl}_{-d}(\mathbb{C}), E] \oplus C$ it follows that $[\mathfrak{m}, E] \cap C = 0$. Since both spaces $\mathfrak{z}(F)$ and C are complimentary to $[\mathfrak{gl}_{-d}(\mathbb{C}), E]$ in $\mathfrak{gl}_{-d}(\mathbb{C})$ they must have the same dimension. Therefore we calculate:

$$\dim \mathfrak{m}^\perp = \dim[\mathfrak{m}, E] + \dim \mathfrak{z}(F) = \dim[\mathfrak{m}, E] + C.$$

This proves that $\mathfrak{m}^\perp = [\mathfrak{m}, E] \oplus C$. The rest of the proof goes through exactly as in [GG02, Lemma 2.1]. □

Theorem 5.3.6. Let α be a simple coroot of GL_n and suppose that μ and $\mu + \alpha$ are in $(\mathbb{Z}_-)^n$ with $\mu_1 \leq \ldots \leq \mu_n$, and $\mu_1 + \alpha_1 \leq \ldots \leq \mu_n + \alpha_n$. Let P_μ be the pyramid associated to μ. Then the action of $\mathbb{G}_a$ on $M_\mu \cap \mathcal{N}_{\mathfrak{gl}_{-d}(\mathbb{C})}$ is equal to the action of $\mathbb{G}_a$ on the Slodowy slice $S(P_\mu)$ as described in [Mor14, 3.4.3].

Proof. We are supposing that α is a simple coroot of GL_n and that μ and

$\mu + \alpha$ satisfy $\mu_1 \leq \ldots \leq \mu_n$ and $\mu_1 + \alpha_1 \leq \ldots \leq \mu_n + \alpha_n$. Therefore, we are in the case where a box moves upwards a single row. Therefore i and j from [Mor14, theorem 3.4.3] are as follows: j is the row of P_μ that has the box which is moving, and $i = j+1$ is the row that it moves to. It is very important to clarify that *we are considering a numbering of rows and columns of P_μ where the row numbers increase as you move up and the column numbers increase as you move right to left.* The additive group that acts on $S(P_\mu)$ predicted in theorem [Mor14, 3.4.3] has Lie algebra $\mathfrak{k} = \mathbb{C}\{E_1\}$ where

$$E_1 = \sum_{\substack{\mathrm{row}(l)=j+1 \\ \mathrm{row}(k)=j \\ \mathrm{col}(k)=\mathrm{col}(l)}} E_{lk}.$$

One can see that the additive group K with Lie algebra $\mathfrak{k}$ is precisely our matrix group $U_\alpha \subseteq GL_{-d}(\mathbb{C})$ defined in 5.2.2. We consider the Premet group $M(P_\mu)$ relative to P_μ which has the Lie algebra $\mathfrak{m}(P_\mu)$ generated as follows:

$$\mathfrak{m}(P_\mu) = \mathbb{C}\{E_{lk} : \mathrm{col}(l) > \mathrm{col}(k)\}.$$

Therefore the Premet group is:

$$M(P_\mu) = I_{-d} + \mathbb{C}\{E_{lk} : \mathrm{col}(l) > \mathrm{col}(k)\}.$$

Let $a \in \mathbb{G}_a$ and consider the lattices L and aL in $\mathrm{Gr}_\mu^{d\omega_1}$ along with the corresponding matrices X and X^a in $M_\mu \cap \mathcal{N}_{\mathfrak{gl}_{-d}(\mathbb{C})}$ via the Mirković-Vybornov isomorphism. Consider the distinguished basis $\{v_1, \ldots, v_{-d}\}$ for L/L_0 as in 5.2.1. Then aL/L_0 has the basis $\{w_1, \ldots, w_{-d}\}$, but it needs to be "corrected" if we are to see the matrix X^a. The initial corrections are of the form:

$$w_l \longrightarrow w_l - a w_k$$

for $\mathrm{row}(l) = j+1$, $\mathrm{row}(k) = j$, and $\mathrm{col}(l) = \mathrm{col}(k)$. Once this correction has been made, the vectors in a basis for aL/L_0 have "leading ones", which means everything in the column with a leading one is zero (see 5.2.1). Therefore the remaining corrections on the basis for aL/L_0 to be performed must be the of the form:

$$w_l \longrightarrow w_l + *w_k$$

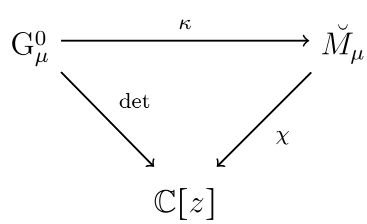
for $\mathrm{col}(l) > \mathrm{col}(k)$. By the considerations of 5.2 we have that $X^a = h(a)\underline{a}^{-1}X\underline{a}h(a)^{-1}$ with $h(a) \in M(P_\mu)$ for every $a \in \mathbb{G}_a$. Since we already mentioned that $K = U_\alpha$, we may conclude that the two actions $\mathbb{G}_a \circlearrowright M_\mu \cap \mathcal{N}_{\mathfrak{gl}_{-d}(\mathbb{C})}$ and $\mathbb{G}_a \circlearrowright S(P_\mu)$ are identical. □

Chapter 6

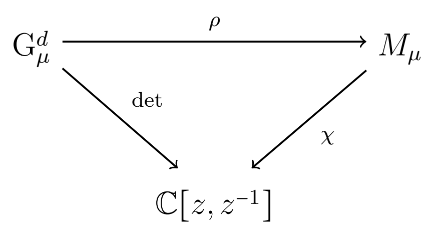
Future Directions

6.1 Characteristic Polynomials

There is an interesting connection between determinants of matrices in G_μ^s and the characteristic polynomials of matrices in these "affine slices". In 4.4, when $\mu \in (\mathbb{Z}_+)^n$ and $s = 0$, we saw that the diagram below was commutative.

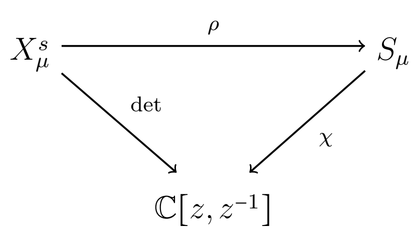


In 4.3 we were in the case of $G = GL_2$, $\mu \in (\mathbb{Z}_-)^2$, and $s = d$, and we had the diagram below.



However, this diagram was not commutative. There was only a vague connection between the determinant of matrices in G^d_μ and the characteristic polynomials of their images under ρ in the Mirković-Vybornov slice.

This makes one curious as to whether there might be some $s \in \mathbb{Z}$, $X^s_\mu \subseteq G^s_\mu$, some affine slice $S_\mu \subseteq \mathfrak{gl}_{-d}(\mathbb{C})$ to the nilpotent orbit of type μ, and an isomorphism ρ, so that the diagram



admits a nice functional relationship between det, ρ, and χ.

Here are two questions potentially connected to these issues.

Question 1. Can 4.3.5 be deduced from 4.4.2?

Question 2. For $G = GL_n$, $\mu \in (\mathbb{Z}_-)^n$, $\mu_1 \leq ... \leq \mu_n$, can we find an explicit Mirković-Vybornov isomorphism $\mathrm{Gr}^{d\omega_1}_\mu \cong M_\mu \cap \mathcal{N}_{\mathfrak{gl}_{-d}(\mathbb{C})}$?

6.2 Explicit Description of Gr_μ^λ

In 4.3 and 4.4 we were able to realize $\mathrm{Gr}_\mu^{d\omega_1}$ as represented by matrices in G_μ^d having determinant equal to z^d. If we want an *explicit* description of Gr_μ^λ in the general case (i.e. for $\lambda \neq d\omega_1$), we would need to extract the subset $\mathrm{Gr}_\mu^\lambda \subseteq \mathrm{Gr}_\mu^s$ by imposing some conditions on matrices in G_μ^s beyond merely $\det = z^d$.

One approach might be to consider extracting the subset $\mathrm{Gr}_\mu^\lambda \subseteq \mathrm{Gr}_\mu^s$ by looking at those matrices in G_μ^s with various conditions on their minors that are adapted to the coweight λ.

6.3 The Reduction $\mathrm{Gr}_\mu^\lambda /\!\!/_1 \mathbb{G}_a \cong \mathrm{Gr}_{\mu+\alpha}^\lambda$

Another goal is to prove conjecture 5.1.4. One can read [KWWY14, 2.6] for a description of the Poisson structure on Gr_μ^λ. Then, given the action $\mathbb{G}_a \circlearrowright \mathrm{Gr}_\mu^\lambda$ introduced in 5.1, one must show that it is Hamiltonian and find a description of the momentum map.

After this is completed, one should try to find a $\mathbb{G}_a$-invariant map of Poisson varieties $\mathrm{Gr}_\mu^\lambda \to \mathrm{Gr}_{\mu+\alpha}^\lambda$ that descends to an isomorphism $\mathrm{Gr}_\mu^\lambda /\!\!/_1 \mathbb{G}_a \cong \mathrm{Gr}_{\mu+\alpha}^\lambda$.

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